Notes on Path integral formulation of diffusion model

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I. PATH INTEGRAL FORMULATION OF THE FORWARD AND TIME-REVERSED PROCESS

Flow model and diffusion model can be understood as the time evolution of the initial noise toward the target data distribution either through stochastic differential equation. In addition, stochastic process can be formulated with path integral. In this note, we will formulate the diffusion model using path integral to gain insight into the machine learning process.

TO-DO:

Using Onsager-Machlup function for forward process and reverse process

Alternatively, there is another approach using Martin-Siggia-Rose path inegral.

We start with the forward process with a slightly different notation.

$$dx(t) = f(t, x(t))\ dt + g(t)\ dW_t \tag{1.1}$$

where $f(t, x(t))$ is the drift term and $g(t)$ is the scale of the noise. Let's denote the noise term as ξ_t

A discretized version is,

$$\Delta x(t) = f(t, x(t))\ \Delta t + g(t)\sqrt{\Delta t} v_t \tag{1.2}$$

where v_t is drawn from $N(0, 1)$. We assume there is no correlation between noises from different time steps.

$$\langle v_t v_{t'} \rangle = \delta_{t,t'} \tag{1.3}$$

a different perspective of this process is that, probability distribution at time $t + \Delta t$ is given by,

$$\begin{aligned} &P(t + \Delta t, x(t + \Delta t)) \\ =&\frac{1}{\sqrt{2\pi}} \int dv_t e^{-\frac{v_t^2 \Delta_t}{2g(t)^2}} \int dx(t)\ P(t, x(t))\ \delta\Big(x(t) - x_{sol}(t)\Big) \end{aligned} \tag{1.4}$$

where we have assumed that we know $P(t, x(t))$, we also have $x_{sol}(t)$ to be the solution of the equation of motion.

$$\delta\Big(x(t + \Delta t) - x(t) - f(t, x(t))\Delta t + g(t)\sqrt{\Delta t} v_t\Big) \tag{1.5}$$