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observed, the corresponding dot of the grid is located and circled with a pencil. The rotor is turned to the assumed latitude and the value of h is read to the nearest minute by eye interpolation between the constant altitude curves. The value of z is obtained likewise and read to the nearest degree. In observing a body in the solar system the required dot is penciled in at its proper position on the grid and h and z obtained as before. The surface of the grid is matte-finished to permit marking and erasing.

The above procedure conforms to the standard method for determining the line of position. The computer may also be used for identifying a star from its approximate altitude and azimuth.

4. **Accuracy.** The device is amazingly accurate. The navigator wants the azimuth only to the nearest degree but desires the altitude to the nearest few minutes. H.O. 218 gives the altitude to the nearest minute of arc while H.O. 214 gives the altitude to the nearest tenth of a minute. Bearing in mind that one minute of arc amounts to one nautical mile and the additional fact that the sextant reading, with which the computed altitude is to be compared, is probably in error by a few minutes it appears that this computer is sufficiently accurate for position fixing in the air and in surface craft if an error of a very few miles can be tolerated. The computer was used in long over-ocean flying with results essentially as stated above.

THE JEEP PROBLEM: A MORE GENERAL SOLUTION

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1. **Introduction.** In this MONTHLY, January, 1947, N. J. Fine discusses the problem of advancing a single jeep a given distance beyond a base of supply. He finds where the dumps must be established in order that the amount of gasoline used is a minimum. This problem is one of a more general class whose solution I wish to discuss in detail here.

Now a solution which yields the minimum amount of gasoline for a given distance is also a solution which yields the maximum distance for a given initial amount, and conversely. It so happens that a solution for the latter conditions appears much easier to obtain. This is therefore the type of problem to be treated first.

Before formulating the various problems in detail, it is necessary to define the special terms that will be used. A group of jeeps traveling together will be referred to as a *caravan*. A *station* is defined as a point where it is necessary to establish a dump, or where the number of jeeps in the caravan is changed. The *home station* is the fixed base from which all jeeps originally set out. A *stage* is defined as the distance between two successive stations. In general, the different stages will be of different lengths. And lastly, the letter d will be used throughout to indicate the number of miles a jeep can travel on a full load of gasoline. Consequently, all distances will be given in miles.

There are possible many final dispositions of the jeeps in a caravan. Only two will be considered here: (1) a jeep may be abandoned at a station without its returning any distance, or (2) a jeep may be returned the whole distance to the home station. Other dispositions may be treated by the general methods presently to be outlined.

2. **Special problems.** Suppose m jeeps fully loaded with gasoline set out from the home station. Their object is to advance one of their number to the greatest possible distance away from the home station. Three cases will be considered: (*a*) none return, (*b*) all return, and (*c*) all but one return to the home station.

(*a*) *None return.* It is readily apparent that at some point gasoline must be transferred to the other jeeps and the empty ones abandoned. Otherwise the caravan would advance one of their number no farther than one jeep could advance alone. Where that transfer is made and how many are left at each station are the points to be determined.

When the caravan has advanced a distance of d/m miles, exactly one load of gasoline has been consumed. If one jeep is emptied of its load, the other $m-1$ can proceed fully loaded; the empty one can be abandoned on the spot.

If a jeep is abandoned before this distance is reached, the other jeeps can not carry all the gasoline on hand. They could proceed fully loaded as above but from a point closer to the home station. Ultimately then the farthest advance would be less than if the station were established at a distance of d/m.

On the other hand, if the now empty jeep were to be driven farther than this point, it must consume gasoline carried by the other jeeps. This procedure would correspondingly reduce the supply of gasoline available for the others. With a smaller supply, they would ultimately reach a shorter distance.

Therefore the most advantageous length for the first stage is d/m, at the end of which the first jeep is abandoned. Likewise, for the remaining $m-1$ jeeps, the second stage would be $d/(m-1)$, and so on. The last stage would be d since it would be traversed by a single jeep fully loaded. The total distance this last jeep has advanced would be d times the series

$$1 + 1/2 + 1/3 + \cdots + 1/m.$$

It has been tacitly assumed above that m jeeps partly loaded would still consume more gasoline than $m-1$ jeeps fully loaded. To simplify the solutions in the problems which follow, it is necessary in all of them to make the stronger assumption that a jeep consumes gasoline at exactly the same rate whether lightly or fully loaded.

(*b*) *All jeeps return.* In this case, it would require just as much gasoline for the return trip as for the outward trip. The caravan would advance until half a load had been used up; the other half load would be put into a dump for use on the return trip. The single jeep now empty could take enough from the dump to return at once or it could wait until the others returned. The same amount of gasoline would be used either way.

Since half as much gasoline would be available for the outward trip, each stage would be half as long as before. Hence the greatest advance would be d times the series

$$1/2 + 1/4 + 1/6 + \cdots + 1/2m.$$

Now it would be possible to establish dumps between the stations indicated above but it would be unnecessary. There would be no saving in doing so unless a jeep used less gasoline the lighter its load, contrary to our assumption. Hereafter, therefore, we shall be concerned with only the minimum number of stations needed for the solution.

(c) *All jeeps but one return.* Over the first stage, there will be m trips outward bound and $m-1$ return. If the first station is established at a distance $d/(2m-1)$, this traffic will consume exactly one load of gasoline. To prove that this is the proper distance we argue as before. If the first station is established closer, not all the gasoline will be consumed. If it is established at a greater distance, gasoline will be consumed unnecessarily.

By repeating the argument, the second stage would be $d/(2m-3)$ in length, and so on. The last stage would again be d. Hence the total distance advanced by the last jeep would be d times the series

$$1 + 1/3 + 1/5 + \cdots + 1/(2m - 1).$$

(d) *Equivalence for one jeep.* So far we have considered a caravan of jeeps. The problem as treated by Mr. Fine involved only one jeep. We wish now to establish the equivalence between the travel of one jeep and that of m jeeps.

A single jeep can return to the home station or it can remain at the point of its greatest advance. Assume the former. Supplied with m loads of gasoline, it must shuttle on the first stage between the home station and the first dump. Each time, it departs from the home station fully loaded, deposits in a dump all but enough to return to the home station, and reaches there with an empty tank for its next load. Thus the jeep makes $2m$ trips over the first stage. The consumption of gasoline is exactly the same as if m different jeeps transported the same amount over the first stage. This is the same as case (b). Consequently, the minimum number of stations and their location needed to advance a single jeep to a maximum distance, and return, are identical in number and location with the stations of case (b).

If the jeep remains at the far end of its advance, the first stage is traversed $2m-1$ times. It should now be obvious that this situation corresponds to case (c).

3. General principles of solution. Several general principles governing the solution of this type of problem should now either be evident or else be easily established. These principles apply with equal force to the special problems above.

(I) The first of these is that, if the travel on any stage is begun with fully loaded jeeps, the length of this stage is such that all the travel along it, both outward and return, will consume exactly one load of gasoline.

Let m be the number of fully loaded jeeps (or trips of one jeep) which are outward bound. Let k be the number of jeeps which return, $0 \leqq k \leqq m$. Since there will be $m+k$ trips along this stage, the principle requires that the length of the stage be $d/(m+k)$. The proof goes the same as before: a shorter stage will not use all the gasoline; a longer stage will require too much.

It follows from this principle that, if there is to be a minimum number of stations, a stage does not end until the supply of gasoline to be transported beyond that point is an integral number of loads.

(II) Thus, if a fractional part of a jeep-load is to be consumed, it must be used on the first stage of the journey.

Or, thought of in another way, it will require an extra jeep to carry the fraction of a load. Naturally, this extra jeep will be sent back, or abandoned, as soon as possible; that is, as soon as the fractional load is used up.

(III) It is to be observed in passing that the number of terms in the series is equal to the number of loads of gasoline.

A fractional load will add an irregular term at the end. Other general principles will be developed in the section which follows.

4. The inverse problem. The inverse problem is that in which the distance is given and the number of stations and the amount of gasoline are each to be a minimum. Since the location of the stations for the greatest advance on a given supply of gasoline is the same as the location of the stations for going a given distance on a minimum amount, there remains only to fit the terms of the proper series to the distance to be covered in order to determine the number of loads needed.

(*a*) When the manner by which the jeeps are advanced leads us to a divergent series, as in the problems above, any distance from the home station can be reached. By Principle III, the number of loads of gasoline needed is exactly equal to the number of the terms of the series required to equal the distance. If the given distance is such that it is more than the sum of m terms of the series but less than $m+1$ terms, a fractional load must be used on the first short stage (Principle II).

(*b*) The problem of reaching a certain point has an interesting variation when it is required to arrive at the point with a given amount of gasoline.

Since the method of transporting gasoline as outlined in Section 3 enables one to reach the greatest distance on a given supply, it is the most economical method for transporting to that point the amount on hand at any moment. This amount can be shown by a table or a broken line graph or other means (not including the amount needed for the scheduled return trips). We can then select the place where the amount available, above that needed for return, is equal to the amount it is required to deliver. From that point measure backwards the required distance and count the number of stations included. This is the number of loads required to transport the given amount to the given distance. A fraction of an interval will require a fraction of a load for the first stage. The problem can be complicated by varying the disposition of the jeeps or the number that are to remain with the delivered gasoline.

The result above can be summarized into a general principle.

(IV) The amount of gasoline delivered unconsumed must be saved from the final stage, or stages, of the journey.

5. **Conclusions.** The conclusions reached by Mr. Fine hold here whenever the corresponding series is harmonic. The locations of the stations are the same whether the distance traveled is to be a maximum or the amount of gasoline used is to be a minimum. In addition, the number of stations established can be made a minimum, in which case their locations are unique.

The number of variations upon these problems is almost endless. One could have rendezvous points where jeeps are to assemble. One could consider the delivery of a certain number of jeeps to another supply station by having caravans meet halfway. Still another variation would be to have tank-trucks accompany the jeeps. Most of such problems can be worked by the general principles developed here.

The application of these problems is found, as Mr. Fine suggested, in polar regions or other places where there is no local supply. It is to be noted that the first solution (Section 3, Problem (*a*)) is exactly the one adopted for a space rocket with a multiple charge.

MATHEMATICAL NOTES

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SUBSERIES OF A MONOTONE DIVERGENT SERIES

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1. **Introduction.** There is a class of theorems in analysis in which the proof depends on showing that some quantity is greater than the sum of a series which is either known or proved to be divergent. In studying such theorems it is natural to ask to what extent they can be weakened by taking fewer terms. If the series is merely divergent, then at most a finite number may be neglected. If, however, the series is also monotone, then more terms may be neglected. The following theorem shows how many must be kept in order to still be able to prove theorems of this class.