第零章、一些数学准备

$ 0.1 球坐标下拉普拉斯算符的表示

球坐标与直角坐标之间有如下的变换关系

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta, \tag{1}$$

或

$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \mathrm{arctg}\frac{\sqrt{x^2+y^2}}{z}, \quad \varphi = \mathrm{arctg}\frac{y}{x}. \tag{2}$$

先计算

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial x} \\
&= \frac{x}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial r} + \frac{\frac{x}{z\sqrt{x^2+y^2}}}{1+\frac{x^2+y^2}{z^2}}\frac{\partial}{\partial \theta} + \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)\frac{\partial}{\partial \varphi} \\
&= \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{r\sin\theta\cos\varphi}{r\sin\theta}\cdot r\cos\theta\frac{\partial}{\partial \theta} - \frac{r\sin\theta\sin\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial \varphi} \\
&= \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial \theta} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial \varphi}.
\end{aligned} \tag{3}$$

因此，我们有

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2} \\
=\ & \left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial \theta} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial \varphi}\right) \\
\times\ & \left(\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial \theta} - \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial \varphi}\right) \\
=\ & \sin^2\theta\cos^2\varphi\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\cos\theta\cos^2\varphi\frac{\partial}{\partial \theta}\left(\cos\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\sin\varphi\frac{\partial}{\partial \varphi}\left(\sin\varphi\frac{\partial}{\partial \varphi}\right) \\
+\ & \sin\theta\cos\theta\cos^2\varphi\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) - \sin\theta\cos\varphi\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \varphi}\right) \\
+\ & \frac{1}{r}\cos\theta\cos^2\varphi\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\cos\theta\cos\varphi\sin\varphi\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \varphi}\right) \\
-\ & \frac{1}{r}\frac{\sin\varphi}{\sin\theta}\sin\theta\frac{\partial}{\partial \varphi}\left(\cos\varphi\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\frac{\sin\varphi}{\sin\theta}\cos\theta\frac{\partial}{\partial \varphi}\left(\cos\varphi\frac{\partial}{\partial \theta}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sin^2\theta\cos^2\varphi\frac{\partial^2}{\partial r^2} - \frac{1}{r^2}\cos^2\varphi\cos\theta\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2}\cos^2\varphi\cos^2\theta\frac{\partial^2}{\partial\theta^2} \\
&+ \frac{1}{r^2\sin^2\theta}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi} + \frac{1}{r^2\sin^2\theta}\sin^2\varphi\frac{\partial^2}{\partial\varphi^2} \\
&- \frac{1}{r^2}\sin\theta\cos\theta\cos^2\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\sin\theta\cos\theta\cos^2\varphi\frac{\partial^2}{\partial r\partial\theta} \\
&+ \frac{1}{r^2}\cos\varphi\sin\varphi\frac{\partial}{\partial\varphi} - \frac{1}{r}\cos\varphi\sin\varphi\frac{\partial^2}{\partial r\partial\varphi} \\
&+ \frac{1}{r}\cos^2\theta\cos^2\varphi\frac{\partial}{\partial r} + \frac{1}{r}\sin\theta\cos\theta\cos^2\varphi\frac{\partial^2}{\partial r\partial\theta} \\
&+ \frac{1}{r^2}\cos\theta\cos\varphi\sin\varphi\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi} - \frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\cos\varphi\sin\varphi\frac{\partial^2}{\partial\theta\partial\varphi} \\
&+ \frac{1}{r}\sin^2\varphi\frac{\partial}{\partial r} - \frac{1}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial r\partial\varphi} \\
&+ \frac{1}{r^2}\frac{\sin^2\varphi\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} - \frac{1}{r^2}\frac{\sin\varphi\cos\varphi}{\sin\theta}\cos\theta\frac{\partial^2}{\partial\varphi\partial\theta}. \qquad (4)
\end{aligned}
$$

再计算 $\frac{\partial^2}{\partial y^2}$。同理，我们有

$$
\begin{aligned}
\frac{\partial}{\partial y} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial y} + \frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial y} \\
&= \frac{y}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial r} + \frac{\frac{y}{z\sqrt{x^2+y^2}}}{1+\frac{x^2+y^2}{z^2}}\frac{\partial}{\partial\theta} + \frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x}\frac{\partial}{\partial\varphi} \\
&= \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{r\sin\theta\sin\varphi}{r\sin\theta}\cdot r\cos\theta\frac{\partial}{\partial\theta} + \frac{r\sin\theta\cos\varphi}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi} \\
&= \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi} \qquad (5)
\end{aligned}
$$

以及

$$
\begin{aligned}
&\frac{\partial^2}{\partial y^2} \\
&= \left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \\
&\times \left(\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\sin\varphi\frac{\partial}{\partial\theta} + \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}\right) \\
&= \sin^2\theta\sin^2\varphi\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\cos\theta\sin^2\varphi\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\cos\varphi\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\varphi}\right) \\
&+ \sin\theta\sin^2\varphi\cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\right) + \sin\theta\sin\varphi\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial\varphi}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{r}\cos\theta\sin^2\varphi\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\cos\theta\sin\varphi\cos\varphi\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\\
&+ \quad \frac{1}{r}\frac{\cos\varphi}{\sin\theta}\sin\theta\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\cos\varphi}{\sin\theta}\cos\theta\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\theta}\right)\\
&= \quad \sin^2\theta\sin^2\varphi\frac{\partial^2}{\partial r^2}-\frac{1}{r^2}\cos\theta\sin^2\varphi\sin\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2}\cos^2\theta\sin^2\varphi\frac{\partial^2}{\partial\theta^2}\\
&- \quad \frac{1}{r^2\sin^2\theta}\cos\varphi\sin\varphi\frac{\partial}{\partial\varphi}+\frac{1}{r^2\sin^2\theta}\cos^2\varphi\frac{\partial^2}{\partial\varphi^2}\\
&- \quad \frac{1}{r^2}\sin\theta\sin^2\varphi\cos\theta\frac{\partial}{\partial\theta}+\frac{1}{r}\sin\theta\sin^2\varphi\cos\theta\frac{\partial^2}{\partial r\partial\theta}\\
&- \quad \frac{1}{r^2}\sin\varphi\cos\varphi\frac{\partial}{\partial\varphi}+\frac{1}{r}\sin\varphi\cos\varphi\frac{\partial^2}{\partial r\partial\varphi}\\
&+ \quad \frac{1}{r}\cos^2\theta\sin^2\varphi\frac{\partial}{\partial r}+\frac{1}{r}\cos\theta\sin\theta\sin^2\varphi\frac{\partial^2}{\partial r\partial\theta}\\
&- \quad \frac{1}{r^2}\cos\theta\sin\varphi\cos\varphi\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\varphi}+\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\sin\varphi\cos\varphi\frac{\partial^2}{\partial\theta\partial\varphi}\\
&+ \quad \frac{1}{r}\cos^2\varphi\frac{\partial}{\partial r}+\frac{1}{r}\cos\varphi\sin\varphi\frac{\partial^2}{\partial r\partial\varphi}\\
&+ \quad \frac{1}{r^2}\frac{\cos^2\varphi\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2}\frac{\cos\varphi\sin\varphi}{\sin\theta}\cos\theta\frac{\partial^2}{\partial\varphi\partial\theta}.
\end{aligned}\tag{6}
$$

因此，我们有

$$
\begin{aligned}
&\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\\
&= \quad \sin^2\theta\frac{\partial^2}{\partial r^2}-\frac{2}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta}+\frac{1}{r^2}\cos^2\theta\frac{\partial^2}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\\
&+ \quad \frac{2}{r}\sin\theta\cos\theta\frac{\partial^2}{\partial r\partial\theta}+\frac{1}{r}\cos^2\theta\frac{\partial}{\partial r}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta}.
\end{aligned}\tag{7}
$$

最后，我们计算 $\frac{\partial^2}{\partial z^2}$ 。先计算 $\frac{\partial}{\partial z}$ 。

$$
\begin{aligned}
\frac{\partial}{\partial z} &= \frac{\partial}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial}{\partial\theta}\frac{\partial\theta}{\partial z}+\frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}\frac{\partial}{\partial r}-\frac{\frac{\sqrt{x^2+y^2}}{z^2}}{1+\frac{x^2+y^2}{z^2}}\frac{\partial}{\partial\theta}+0\\
&= \cos\theta\frac{\partial}{\partial r}-\frac{r\sin\theta}{r^2}\frac{\partial}{\partial\theta}=\cos\theta\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}.
\end{aligned}\tag{8}
$$

由此，我们得到

$$
\frac{\partial^2}{\partial z^2} = \left(\cos\theta\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial}{\partial r}-\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right)
$$

$$
\begin{aligned}
&= \cos^2\theta\frac{\partial^2}{\partial r^2} - \cos\theta\frac{\partial}{\partial r}\left(\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right) \\
&- \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial r}\right) + \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}\right) \\
&= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\cos\theta\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{r}\cos\theta\sin\theta\frac{\partial^2}{\partial r\partial\theta} \\
&+ \frac{1}{r}\sin^2\theta\frac{\partial}{\partial r} - \frac{1}{r}\cos\theta\sin\theta\frac{\partial^2}{\partial r\partial\theta} + \frac{1}{r^2}\sin\theta\cos\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2}\sin^2\theta\frac{\partial^2}{\partial\theta^2} \\
&= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{2}{r^2}\cos\theta\sin\theta\frac{\partial}{\partial\theta} - \frac{2}{r}\cos\theta\sin\theta\frac{\partial^2}{\partial r\partial\theta} \\
&+ \frac{1}{r}\sin^2\theta\frac{\partial}{\partial r} + \frac{1}{r^2}\sin^2\theta\frac{\partial^2}{\partial\theta^2}. \qquad (9)
\end{aligned}
$$

将这一结果与 (7) 式相加后，我们得到

$$
\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \\
&= \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \qquad (10)
\end{aligned}
$$

\$ 0.2 柱坐标下拉普拉斯算符的表示

首先，柱坐标与直角坐标之间有如下的变换关系

$$
x = \rho\cos\varphi, \quad y = \rho\sin\varphi, \quad z = z, \qquad (11)
$$

或

$$
x^2 + y^2 = \rho^2, \quad \frac{y}{x} = \tan\varphi, \quad z = z. \qquad (12)
$$

因此，我们有

$$
\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial}{\partial\rho}\frac{\partial\rho}{\partial x} + \frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial x} + \frac{\partial}{\partial z}\frac{\partial z}{\partial x} = \frac{x}{\rho}\frac{\partial}{\partial\rho} - \frac{y}{\rho^2}\frac{\partial}{\partial\varphi}, \\
\frac{\partial}{\partial y} &= \frac{\partial}{\partial\rho}\frac{\partial\rho}{\partial y} + \frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial y} + \frac{\partial}{\partial z}\frac{\partial z}{\partial y} = \frac{y}{\rho}\frac{\partial}{\partial\rho} + \frac{x}{\rho^2}\frac{\partial}{\partial\varphi}. \qquad (13)
\end{aligned}
$$

故我们得到

$$
\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}
$$

$$
\begin{aligned}
&= \left(\frac{x}{\rho}\frac{\partial}{\partial\rho}-\frac{y}{\rho^2}\frac{\partial}{\partial\varphi}\right)\left(\frac{x}{\rho}\frac{\partial}{\partial\rho}-\frac{y}{\rho^2}\frac{\partial}{\partial\varphi}\right)+\left(\frac{y}{\rho}\frac{\partial}{\partial\rho}+\frac{x}{\rho^2}\frac{\partial}{\partial\varphi}\right)\left(\frac{y}{\rho}\frac{\partial}{\partial\rho}+\frac{x}{\rho^2}\frac{\partial}{\partial\varphi}\right)\\
&= \frac{x}{\rho}\frac{\partial}{\partial\rho}\left(\frac{x}{\rho}\frac{\partial}{\partial\rho}\right)-\frac{y}{\rho^2}\frac{\partial}{\partial\varphi}\left(\frac{x}{\rho}\frac{\partial}{\partial\rho}\right)-\frac{x}{\rho}\frac{\partial}{\partial\rho}\left(\frac{y}{\rho^2}\frac{\partial}{\partial\varphi}\right)+\frac{y}{\rho^2}\frac{\partial}{\partial\varphi}\left(\frac{y}{\rho^2}\frac{\partial}{\partial\varphi}\right)\\
&+ \frac{y}{\rho}\frac{\partial}{\partial\rho}\left(\frac{y}{\rho}\frac{\partial}{\partial\rho}\right)+\frac{x}{\rho^2}\frac{\partial}{\partial\varphi}\left(\frac{y}{\rho}\frac{\partial}{\partial\rho}\right)+\frac{y}{\rho}\frac{\partial}{\partial\rho}\left(\frac{x}{\rho^2}\frac{\partial}{\partial\varphi}\right)+\frac{x}{\rho^2}\frac{\partial}{\partial\varphi}\left(\frac{x}{\rho^2}\frac{\partial}{\partial\varphi}\right). \qquad (14)
\end{aligned}
$$

注意到 $\frac{x}{\rho}=\cos\varphi$ 和 $\frac{y}{\rho}=\sin\varphi$，上式又可被改写为

$$
\begin{aligned}
&\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\\
&= \cos\varphi\frac{\partial}{\partial\rho}\left(\cos\varphi\frac{\partial}{\partial\rho}\right)-\frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}\left(\cos\varphi\frac{\partial}{\partial\rho}\right)-\cos\varphi\frac{\partial}{\partial\rho}\left(\frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)\\
&+ \frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}\left(\frac{\sin\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)+\sin\varphi\frac{\partial}{\partial\rho}\left(\sin\varphi\frac{\partial}{\partial\rho}\right)+\frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\rho}\right)\\
&+ \sin\varphi\frac{\partial}{\partial\rho}\left(\frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)+\frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}\left(\frac{\cos\varphi}{\rho}\frac{\partial}{\partial\varphi}\right)\\
&= \cos^2\varphi\frac{\partial^2}{\partial\rho^2}+\frac{\sin^2\varphi}{\rho}\frac{\partial}{\partial\rho}-\frac{\sin\varphi\cos\varphi}{\rho}\frac{\partial^2}{\partial\varphi\partial\rho}+\frac{\cos\varphi\sin\varphi}{\rho^2}\frac{\partial}{\partial\varphi}\\
&- \frac{\cos\varphi\sin\varphi}{\rho}\frac{\partial^2}{\partial\rho\partial\varphi}+\frac{1}{\rho^2}\left(\cos\varphi\sin\varphi\frac{\partial}{\partial\varphi}+\sin^2\varphi\frac{\partial^2}{\partial\varphi^2}\right)\\
&+ \sin^2\varphi\frac{\partial^2}{\partial\rho^2}+\frac{\cos^2\varphi}{\rho}\frac{\partial}{\partial\rho}+\frac{\sin\varphi\cos\varphi}{\rho}\frac{\partial^2}{\partial\varphi\partial\rho}-\frac{\cos\varphi\sin\varphi}{\rho^2}\frac{\partial}{\partial\varphi}\\
&+ \frac{\cos\varphi\sin\varphi}{\rho}\frac{\partial^2}{\partial\rho\partial\varphi}+\frac{1}{\rho^2}\left(-\cos\varphi\sin\varphi\frac{\partial}{\partial\varphi}+\cos^2\varphi\frac{\partial^2}{\partial\varphi^2}\right)\\
&= \frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}. \qquad (15)
\end{aligned}
$$

由此表达式，我们最后得到柱坐标下拉普拉斯算符的表示

$$
\begin{aligned}
\nabla^2 &\equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}\\
&= \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}. \qquad (16)
\end{aligned}
$$

$ 0.3 有关 δ- 函数的一些知识

$ 0.3.1 δ- 函数的各种表达形式

Dirac 引入的 δ- 函数的定义由下式给出

$$\delta(x)=\begin{cases}\infty, & x=0;\\ 0, & x\neq 0.\end{cases} \tag{17}$$

除此之外，更为重要的条件是

$$\int_{-\infty}^{\infty}\delta(x)\,dx=1. \tag{18}$$

在数学上， δ- 函数可以通过所谓分布 (Distribution) 理论严格化。它实际上是一个泛函。

在实际计算中，为了方便起见， δ- 函数常常用某些函数的极限形式来表达。在这里，我们给出其最常用的几种表达方式。

(1) 首先，我们有

$$\lim_{\sigma\to 0}\frac{1}{\sqrt{2\pi\sigma}}\exp\left(-\frac{x^2}{2\sigma}\right)=\delta(x). \tag{19}$$

实际上，当 $\sigma\to 0$ 时， δ- 函数的定义式显然是满足的。又由于

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma}}\exp\left(-\frac{x^2}{2\sigma}\right)\,dx &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma}}\exp\left[-\left(\frac{x}{\sqrt{2\sigma}}\right)^2\right]\,dx\\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-\tilde{x}^2)\,d\tilde{x}=\frac{1}{\sqrt{\pi}}\sqrt{\pi}=1.\end{aligned} \tag{20}$$

(2) 其次，我们有

$$\lim_{\alpha\to\infty}\frac{\sin\alpha x}{\pi x}=\delta(x). \tag{21}$$

为了证明这一表达式，我们注意到，当 $\alpha\to\infty$ 时，这一极限形式地满足 δ-函数的定义。但是，为了证明它的积分等于 1 ，我们需要做一些准备工作。

首先，我们注意到积分公式

$$I=\int_0^{\infty}e^{-\gamma x}\cos\beta x\,dx=\frac{\gamma}{\beta^2+\gamma^2},\quad \gamma>0 \tag{22}$$

成立。这是由于连续利用分步积分公式，我们有

$$I = \int_0^{\infty}e^{-\gamma x}\cos\beta x\,dx=e^{-\gamma x}\frac{\sin\beta x}{\beta}\bigg|_0^{\infty}+\frac{\gamma}{\beta}\int_0^{\infty}e^{-\gamma x}\sin\beta x\,dx$$

$$
\begin{aligned}
&= \frac{\gamma}{\beta}\int_0^\infty e^{-\gamma x}\sin\beta x\,dx = -\frac{\gamma}{\beta^2}e^{-\gamma x}\cos\beta x\Big|_0^\infty - \frac{\gamma^2}{\beta^2}\int_0^\infty e^{-\gamma x}\cos\beta x\,dx \\
&= \frac{\gamma}{\beta^2} - \frac{\gamma^2}{\beta^2}\,I. \qquad (23)
\end{aligned}
$$

移项后，我们有

$$
\left(1+\frac{\gamma^2}{\beta^2}\right)\,I = \frac{\gamma}{\beta^2}. \qquad (24)
$$

将此式的两边同除以 $\left(1+\frac{\gamma^2}{\beta^2}\right)$ 后，我们即可得到公式 (22) 。

现在，我们将公式 (22) 两边的变量 β 从 0 积分到 α 。我们得到

$$
\begin{aligned}
&\int_0^\alpha d\beta\left(\int_0^\infty e^{-\gamma x}\cos\beta x\,dx\right) = \int_0^\infty dx\,e^{-\gamma x}\left(\int_0^\alpha d\beta\cos\beta x\right) \\
&= \int_0^\infty dx\,e^{-\gamma x}\,\frac{\sin\alpha x}{x} = \int_0^\alpha d\beta\frac{\gamma}{\beta^2+\gamma^2} = \arctan\frac{\alpha}{\gamma}. \qquad (25)
\end{aligned}
$$

因此，我们有

$$
\lim_{\gamma\to 0}\int_0^\infty dx\,e^{-\gamma x}\frac{\sin\alpha x}{x} = \int_0^\infty dx\,\frac{\sin\alpha x}{x} = \lim_{\gamma\to 0}\arctan\frac{\alpha}{\gamma} = \arctan\infty = \frac{\pi}{2}. \qquad (26)
$$

现在，我们可以完成我们的证明了。我们有

$$
\int_{-\infty}^\infty \frac{\sin\alpha x}{\pi x}\,dx = 2\int_0^\infty \frac{\sin\alpha x}{\pi x}\,dx = \frac{2}{\pi}\cdot\frac{\pi}{2} = 1. \qquad (27)
$$

因此，命题得证。

(3) 接下来，我们有

$$
\frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx}\,dk = \delta(x). \qquad (28)
$$

事实上，直接的计算给出

$$
\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^\infty e^{ikx}\,dk = \lim_{\alpha\to\infty}\frac{1}{2\pi}\int_{-\alpha}^\alpha e^{ikx}\,dk = \lim_{\alpha\to\infty}\frac{1}{2\pi}\,\frac{e^{ikx}}{ix}\Big|_{-\alpha}^\alpha \\
&= \lim_{\alpha\to\infty}\frac{1}{\pi}\frac{e^{i\alpha x}-e^{-i\alpha x}}{2ix} = \lim_{\alpha\to\infty}\frac{1}{\pi}\frac{\sin\alpha x}{x} = \delta(x). \qquad (29)
\end{aligned}
$$

(4) 最后，我们有

$$
\lim_{\epsilon\to 0^+}\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2} = \delta(x). \qquad (30)
$$

首先，当 $x \neq 0$ 时，上式趋向于零。而当 $x = 0$ 时，上式为 ∞ 。其次，我们有

$$\lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{\epsilon}{x^2 + \epsilon^2} \, dx = \lim_{\epsilon \to 0^+} \frac{1}{\pi} \arctan \frac{x}{\epsilon} \Big|_{-\infty}^{\infty} = \frac{1}{\pi} \left[\arctan \infty - \arctan(-\infty) \right] = 1. \quad (31)$$

因此，上式成立。

$ 0.3.2 δ- **函数的一些性质**

(1) δ- 函数是偶函数。即我们有

$$\delta(-x) = \delta(x). \quad (32)$$

(2) 对于任何连续函数 $f(x)$ ，下面的等式

$$\int_{-\infty}^{\infty} f(x) \delta(x) \, dx = f(0) \quad (33)$$

成立。

(3) 对于任何连续函数 $f(x)$ ，下面的等式

$$\int_{-\infty}^{\infty} f(x) \delta(x - a) \, dx = f(a) \quad (34)$$

成立。

(4) $\delta(ax) = \frac{1}{|a|} \delta(x)$ 。

这是由于，对于任何连续函数 $f(x)$ ，利用 δ- 函数是偶函数这一事实，我们有

$$\int_{-\infty}^{\infty} f(x) \delta(ax) \, dx = \int_{-\infty}^{\infty} f(x) \delta(|a| x) \, dx. \quad (35)$$

现在令 $|a| x = x'$ ，我们有

$$\int_{-\infty}^{\infty} f(x) \delta(ax) dx = \frac{1}{|a|} \int_{-\infty}^{\infty} f\left(\frac{x'}{|a|} \right) \delta(x') dx' = \frac{1}{|a|} f(0) = \int_{-\infty}^{\infty} f(x) \left(\frac{1}{|a|} \delta(x) \right) dx. \quad (36)$$

因此，上式成立。

(5) 考虑一个二次以上可导的函数 $\varphi(x)$ 。设 $\{x_i\}$ 为其单零点的集合。即在任一点 x_i 处，我们有

$$\varphi(x_i) = 0, \quad \varphi'(x_i) \neq 0. \quad (37)$$

那么，我们有

$$\delta(\varphi(x))=\sum_i^N\frac{\delta(x-x_i)}{|\varphi'(x_i)|}. \tag{38}$$

按照定义，δ- 函数仅在 $\varphi(x)=0$ 处不为零，因此，对于任何连续函数 $f(x)$，我们有

$$\int_{-\infty}^{\infty}f(x)\delta(\varphi(x))\,dx=\sum_i^N\int_{x_i-\epsilon_i}^{x_i+\epsilon_i}f(x)\delta(\varphi(x))\,dx\equiv\sum_i^N F_i. \tag{39}$$

下面，我们取某一个积分值 F_i 为例。

由于 $\varphi'(x_i)\neq 0$，我们总可以将 ϵ_i 取得到如此之小，使得 $\varphi(x)$ 在区间 $(x_i-\epsilon_i,\,x_i+\epsilon_i)$ 上是单调的。因此，我们可以引入新的变量 $u=\varphi(x)$，使得

$$u_1=\varphi(x_i-\epsilon_i),\quad u_2=\varphi(x_i)=0,\quad u_3=\varphi(x_i+\epsilon_i). \tag{40}$$

特别是当 $\varphi'(x_i)>0$ 时，我们有

$$u_{\max}=u_3,\quad u_{\min}=u_1. \tag{41}$$

而当 $\varphi'(x_i)<0$ 时，我们又有

$$u_{\max}=u_1,\quad u_{\min}=u_3. \tag{42}$$

利用这些记号，我们可以将 F_i 改写成

$$\begin{aligned}F_i &= \int_{x_i-\epsilon_i}^{x_i+\epsilon_i}f(x)\delta(\varphi(x))\,dx=\int_{u_{\min}}^{u_{\max}}f(\varphi^{-1}(u))\delta(u)\,\frac{du}{|\varphi'(\varphi^{-1}(u))|}\\ &= \frac{f(\varphi^{-1}(u_2))}{|\varphi'(\varphi^{-1}(u_2))|}=\frac{f(x_i)}{|\varphi'(x_i)|}.\end{aligned} \tag{43}$$

因此，积分 (39) 可以被写作

$$\begin{aligned}\int_{-\infty}^{\infty}f(x)\delta(\varphi(x))\,dx=\sum_i^N\int_{x_i-\epsilon_i}^{x_i+\epsilon_i}f(x)\delta(\varphi(x))\,dx=\sum_i^N\frac{f(x_i)}{|\varphi'(x_i)|}\\ = \int_{-\infty}^{\infty}f(x)\sum_{i=1}^N\left(\frac{\delta(x-x_i)}{|\varphi'(x_i)|}\right)\,dx.\end{aligned} \tag{44}$$

这样，我们就证明了我们上述公式的正确性。

$ 0.4 **有关矢量分析的一些知识**

(1) 验证恒等式

$$\nabla(\mathbf{A}\cdot\mathbf{v}) = (\mathbf{A}\cdot\nabla)\mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{v}) + \mathbf{v}\times(\nabla\times\mathbf{A}). \tag{45}$$

首先，按照定义，我们有

$$\begin{aligned}
\mathbf{A}\times(\nabla\times\mathbf{v}) = \mathbf{A}\times & \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\
= \ & \mathbf{A}\times\left[\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\mathbf{k}\right] \\
= \ & \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ \left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right) & \left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right) & \left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right) \end{vmatrix} \\
= \ & \left[A_y\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)-A_z\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\right]\mathbf{i} \\
+ \ & \left[A_z\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)-A_x\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\right]\mathbf{j} \\
+ \ & \left[A_x\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)-A_y\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial y}\right)\right]\mathbf{k}.
\end{aligned} \tag{46}$$

我们再计算 $(\mathbf{A}\cdot\nabla)\mathbf{v}$ 。

$$\begin{aligned}
(\mathbf{A}\cdot\nabla)\mathbf{v} &= \left(A_x\frac{\partial}{\partial x}+A_y\frac{\partial}{\partial y}+A_z\frac{\partial}{\partial z}\right)(v_x\mathbf{i}+v_y\mathbf{j}+v_z\mathbf{j}) \\
&= \left(A_x\frac{\partial v_x}{\partial x}+A_y\frac{\partial v_x}{\partial y}+A_z\frac{\partial v_x}{\partial z}\right)\mathbf{i}+\left(A_x\frac{\partial v_y}{\partial x}+A_y\frac{\partial v_y}{\partial y}+A_z\frac{\partial v_y}{\partial z}\right)\mathbf{j} \\
&+ \left(A_x\frac{\partial v_z}{\partial x}+A_y\frac{\partial v_z}{\partial y}+A_z\frac{\partial v_z}{\partial z}\right)\mathbf{k}.
\end{aligned} \tag{47}$$

因此，我们有

$$\begin{aligned}
& (\mathbf{A}\cdot\nabla)\mathbf{v}+\mathbf{A}\times(\nabla\times\mathbf{v}) \\
= \ & \left(A_x\frac{\partial v_x}{\partial x}+A_y\frac{\partial v_x}{\partial y}+A_z\frac{\partial v_x}{\partial z}\right)\mathbf{i}
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \left(A_x\frac{\partial v_y}{\partial x}+A_y\frac{\partial v_y}{\partial y}+A_z\frac{\partial v_y}{\partial z}\right)\mathbf{j}\\
&+ \quad \left(A_x\frac{\partial v_z}{\partial x}+A_y\frac{\partial v_z}{\partial y}+A_z\frac{\partial v_z}{\partial z}\right)\mathbf{k}\\
&+ \quad \left[A_y\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)-A_z\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\right]\mathbf{i}\\
&+ \quad \left[A_z\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)-A_x\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\right]\mathbf{j}\\
&+ \quad \left[A_x\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)-A_y\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial y}\right)\right]\mathbf{k}\\
&= \quad \left(A_y\frac{\partial v_y}{\partial x}+A_z\frac{\partial v_z}{\partial x}+A_x\frac{\partial v_x}{\partial x}\right)\mathbf{i}+\left(A_x\frac{\partial v_x}{\partial y}+A_z\frac{\partial v_z}{\partial y}+A_y\frac{\partial v_y}{\partial y}\right)\mathbf{j}\\
&+ \quad \left(A_x\frac{\partial v_x}{\partial z}+A_y\frac{\partial v_y}{\partial z}+A_z\frac{\partial v_z}{\partial z}\right)\mathbf{k}. \qquad (48)
\end{aligned}
$$

将上式中的 $\mathbf{A}$ 与 $\mathbf{v}$ 对换，我们即可得到

$$
\begin{aligned}
&\quad\ (\mathbf{v}\cdot\nabla)\mathbf{A}+\mathbf{v}\times(\nabla\times\mathbf{A})\\
&= \quad \left(v_y\frac{\partial A_y}{\partial x}+v_z\frac{\partial A_z}{\partial x}+v_x\frac{\partial A_x}{\partial x}\right)\mathbf{i}+\left(v_x\frac{\partial A_x}{\partial y}+v_z\frac{\partial A_z}{\partial y}+v_y\frac{\partial A_y}{\partial y}\right)\mathbf{j}\\
&+ \quad \left(v_x\frac{\partial A_x}{\partial z}+v_y\frac{\partial A_y}{\partial z}+v_z\frac{\partial A_z}{\partial z}\right)\mathbf{k}. \qquad (49)
\end{aligned}
$$

两式相加后，我们得到

$$
\begin{aligned}
&\quad\ (\mathbf{A}\cdot\nabla)\mathbf{v}+\mathbf{A}\times(\nabla\times\mathbf{v})+(\mathbf{v}\cdot\nabla)\mathbf{A}+\mathbf{v}\times(\nabla\times\mathbf{A})\\
&= \quad \left(A_x\frac{\partial v_x}{\partial x}+A_y\frac{\partial v_y}{\partial x}+A_z\frac{\partial v_z}{\partial x}+v_x\frac{\partial A_x}{\partial x}+v_y\frac{\partial A_y}{\partial x}+v_z\frac{\partial A_z}{\partial x}\right)\mathbf{i}\\
&+ \quad \left(A_x\frac{\partial v_x}{\partial y}+A_y\frac{\partial v_y}{\partial y}+A_z\frac{\partial v_z}{\partial y}+v_x\frac{\partial A_x}{\partial y}+v_y\frac{\partial A_y}{\partial y}+v_z\frac{\partial A_z}{\partial y}\right)\mathbf{j}\\
&+ \quad \left(A_x\frac{\partial v_x}{\partial z}+A_y\frac{\partial v_y}{\partial z}+A_z\frac{\partial v_z}{\partial z}+v_x\frac{\partial A_x}{\partial z}+v_y\frac{\partial A_y}{\partial z}+v_z\frac{\partial A_z}{\partial z}\right)\mathbf{k}\\
&= \quad \left(\mathbf{i}\frac{\partial}{\partial x}(\mathbf{A}\cdot\mathbf{v})+\mathbf{j}\frac{\partial}{\partial y}(\mathbf{A}\cdot\mathbf{v})+\mathbf{k}\frac{\partial}{\partial z}(\mathbf{A}\cdot\mathbf{v})\right)\\
&= \quad \nabla(\mathbf{A}\cdot\mathbf{v}). \qquad (50)
\end{aligned}
$$

恒等式得证。