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1.

(1) $\gamma'(t) = (-a\sin t,\ a\cos t,\ b)$

则 $|\gamma'(t)| = \sqrt{a^2+b^2}$

弧长参数 $s = \int_0^t |r'(u)|\,du = \sqrt{a^2+b^2}\,t$.

$\Rightarrow \gamma(s) = \left(a\cos\frac{s}{\sqrt{a^2+b^2}},\ a\sin\frac{s}{\sqrt{a^2+b^2}},\ \frac{b}{\sqrt{a^2+b^2}}s\right)$

记 $c := \sqrt{a^2+b^2}$

则有 $t(s) := \gamma'(s) = \left(-\frac{a}{c}\sin\frac{s}{c},\ \frac{a}{c}\cos\frac{s}{c},\ \frac{b}{c}\right)$

$\dot t(s) := \gamma''(s) = \left(-\frac{a}{c^2}\cos\frac{s}{c},\ -\frac{a}{c^2}\sin\frac{s}{c},\ 0\right)$

故而 $\gamma$ 的曲率 $k(s) = |\dot t(s)| = \frac{a}{c^2} = \frac{a}{a^2+b^2}$

法向量 $n(s) = \left(-\cos\frac{s}{c},\ -\sin\frac{s}{c},\ 0\right)$

$\Rightarrow b(s) = t(s) \wedge n(s)$

$= \left(\frac{b}{c}\sin\frac{s}{c},\ -\frac{b}{c}\cos\frac{s}{c},\ \frac{a}{c}\right)$

$\dot b(s) = \left(\frac{b}{c^2}\cos\frac{s}{c},\ \frac{b}{c^2}\sin\frac{s}{c},\ 0\right) = -\frac{b}{c^2}\,n(s)$

故而 $\gamma$ 的挠率 $\tau(s) = \frac{b}{c^2} = \frac{b}{a^2+b^2}$

(2) 法曲率 $k_n(\gamma'(t_0)) = \cancel{k_n}\ \dot t(s)$,

注意到 曲面法向 $r_u \wedge r_v = (-a\sin u,\ a\cos u,\ 0) \wedge (0,0,b)$

$= (ab\cos u,\ ab\sin u,\ 0)$

$\Rightarrow n = (\cos u, \sin u, 0)\cdot \mathrm{sgn}(b)$

故法曲率 $k_n(\gamma'(t_0)) = \langle \dot{t}(s), n \rangle = \frac{a}{a^2+b^2}\mathrm{sgn}(b)$ $\frac{a}{a^2+b^2}$

(3) $\gamma$是测地线.

因为$\gamma$的测地曲率 $k_g^2 = k^2 - k_n^2 = 0$.

2. (1) 我们有

$$r_u = (-a\cosh v \sin u,\ a\cosh v\cos u,\ 0)$$

$$r_v = (a\sinh v\cos u,\ a\sinh v\sin u,\ a)$$

故有 $E = \langle r_u, r_u\rangle = a^2\cosh^2 v$

$F = \langle r_u, r_v\rangle = 0$

$G = \langle r_v, r_v\rangle = a^2\sinh^2 v + a^2 = a^2(1+\sinh^2 v)$

$= a^2\cosh^2 v.$

故有 $e_1 = \dfrac{r_u}{a\cosh v}$, $e_2 = \dfrac{r_v}{a\cosh v}$ 和 $e_3 = e_1\wedge e_2$ 给出M的一个正交活动标架

(2) 由于 $\omega^1, \omega^2$ 为 $e_1, e_2$ 的对偶一形式知

$\omega^1 = a\cosh v\, du$

$\cancel{\omega^2 = a\cosh v\, dv}$

故有

$d\omega^1 = a\sinh v\, dv\wedge du = -a\sinh v\, du\wedge dv$

$d\omega^2 = \cancel{a\sinh v\, du\wedge dv}\ 0$

由此知 $\cancel{\omega_1^2 = -a\sinh v\, du + a\sinh v\, dv}$

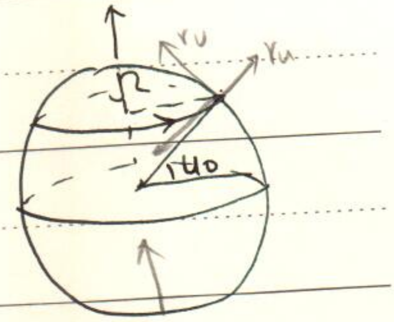
$\cancel{= a\sinh v(dv - du)}$

$d\omega^1 = -\dfrac{a\sinh v}{a^2\cosh^2 v}\omega^1\wedge\omega^2 = \omega^2\wedge\omega_2^1$

$\cancel{d\omega^2 = \dfrac{a\sinh v}{a^2\cosh^2 v}\omega^1\wedge\omega^2 = \omega^1\wedge\omega_1^2}$

$\Rightarrow \omega_1^2 = \dfrac{a\sinh v}{a^2\cosh^2 v}(-\omega^1 + \cancel{\omega^2}) = \dfrac{a\sinh v}{a\cosh v}(-du + \cancel{dv})$

3. 应用 Gauss-Bonnet 公式，有




$$\iint_{\Omega} K\, dv + \oint_C k_g\, ds = 2\pi$$

而 $K = \frac{1}{a^2}$， $\iint_{\Omega} K\, dv = \frac{1}{a^2} \cdot 2\pi a \cdot (a + a\sin u_0)$

用下球冠

$$= 2\pi\,(1 + \sin u_0)$$

$$\Rightarrow \oint_C k_g\, ds = 2\pi - \cancel{\oint} 2\pi(1+\sin u_0)$$

$$= -2\pi \sin u_0 .$$

4. (1) 设曲面之主曲率为 $k_1, k_2$.

由曲面为极小曲面得 $k_1 + k_2 = 0$

故有 $K = k_1 k_2 = k_1 \cdot (-k_1) = -k_1^2 \le 0$

(2) 因在此情况下 有 $k_1 k_2 \equiv 0$ 且 $k_1 + k_2 \equiv 0$

故而 $k_1 = k_2 = 0$

故曲面为全脐点曲面.

因为全脐点曲面为平面或球面知，在我们之情况，曲面是平面.

5. (1) $\mathcal{W}(r_u) = -n_u$, $\mathcal{W}(r_v) = -n_v$

(2) 设 $\mathcal{W}$ 在 $\{r_u, r_v\}$ 下的矩阵表示为

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

则有 $$-\begin{pmatrix} n_u \\ n_v \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} r_u \\ r_v \end{pmatrix}$$

由 $$L = -\langle n_u, r_u\rangle = \langle a r_u + b r_v, r_u\rangle = aE + bF$$

$$M = -\langle n_u, r_v\rangle = aF + bG$$

$$M = -\langle n_v, r_u\rangle = cE + dF$$

$$N = -\langle n_v, r_v\rangle = cF + dG$$

$$\Rightarrow \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

因此 $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}.$$

(3). $$\frac{D}{dv}\frac{D}{du} r_u = \frac{D}{dv}\left(r_{uu} - \langle r_{uu}, n\rangle n\right)$$

$$= \frac{D}{dv}\left(r_{uu} - Ln\right) = r_{uuv} - L_v n - L n_v - \langle r_{uuv} - L_v n - L n_v, n\rangle n$$

$$= r_{uuv} - L_v n - L n_v - \langle r_{uuv}, n\rangle n + L_v n + L\langle n_v, n\rangle n$$

$$= r_{uuv} - L n_v - \langle r_{uuv}, n\rangle n$$

类似地, 有

$$\frac{D}{du}\frac{D}{dv} r_u = r_{uvu} - M n_u - \langle r_{uvu}, n\rangle n$$

应用 $r_{uuv}=r_{uvu}$ 有

$$\frac{D}{dv}\frac{D}{du}r_u-\frac{D}{du}\frac{D}{dv}r_v=-Ln_v+Mn_u$$

$$=(-M,\ L)\begin{pmatrix}-n_u\\ -n_v\end{pmatrix}$$

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$$=(-M,L)\,W\begin{pmatrix}r_u\\ r_v\end{pmatrix}$$

应用(2)得 $\frac{D}{dv}\frac{D}{du}r_u-\frac{D}{du}\frac{D}{dv}r_v=(-M\quad L)\begin{pmatrix}L & M\\ M & N\end{pmatrix}\begin{pmatrix}E & F\\ F & G\end{pmatrix}^{-1}\begin{pmatrix}r_u\\ r_v\end{pmatrix}$

$$=(0\quad LN-M^2)\frac{1}{EG-F^2}\begin{pmatrix}G & -F\\ -F & E\end{pmatrix}\begin{pmatrix}r_u\\ r_v\end{pmatrix}$$

$$=\left(0\quad \frac{LN-M^2}{EG-F^2}\right)\begin{pmatrix}Gr_u-Fr_v\\ -Fr_u+Er_v\end{pmatrix}$$

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$$=K\cdot(Er_v-Fr_u)$$

$$=K\left(|r_u|^2r_v-\langle r_u,r_v\rangle r_u\right)$$

$$=K|r_u|^2\left(r_v-\left\langle r_v,\frac{r_u}{|r_u|}\right\rangle\frac{r_u}{|r_u|}\right)$$

$$=K|r_u|^2(r_v)^{\perp}$$

$$=K|r_u|\cdot|(r_v)^{\perp}|\cdot\left(|r_u|\cdot\frac{(r_v)^{\perp}}{|(r_v)^{\perp}|}\right)$$ 2

$$=K|r_u\wedge r_v|\cdot R_{\frac{\pi}{2}}(r_u)$$