Anhui University

Semester 2, 2007-2008 Final Examination (Paper B)

Model Answer and Referee Criterion for *Numerical Analysis*

In question 1-6, please choose the correct answer (only one is correct)
1. (5 marks) Word "MATLAB" comes form
(A) Mathematics Laboratory
(B) Matrix Laboratory
(C) Mathematica Laboratory
(D) Maple Laboratory
Answer. (B)

2. (5 marks) The matrix

$$\begin{pmatrix} 4 & -1 & 1 \\ 4 & -8 & 1 \\ -2 & 1 & 5 \end{pmatrix}$$

is
(A) a strictly diagonally dominant matrix;
(B) not a strictly diagonally dominant matrix;
(C) a singular matrix;
(D) a matrix whose determinant is equal to zero.
Answer. (A)

3. (5 marks) The computational complexity of Gaussian elimination for solving linear equation systems $AX = B$ (A is $N \times N$ matrix) is

(A) $O(N)$, (B) $O(N^2)$, (C) $O(N^3)$, (D) $O(N^4)$.

Answer. (C)

4. (5 marks) Assume that $f(x)$ is defined on $[a, b]$, which contains equally spaced nodes $x_k = x_0 + hk$. Additionally, assume that $f''(x)$ is continuous on $[a, b]$. If we use Lagrange interpolation polynomial

$$P_1(x) = \sum_{k=0}^{1} f(x_k) L_{1,k}(x)$$

to approximate $f(x)$, then the error $E_1(x)$ is

(A) $O(1)$ (B) $O(h)$ (C) $O(h^2)$ (D) $O(h^3)$

Answer. (C)

5. (5 marks) Degree 4 Chebyshev Polynomial $T_4(x)$ is

(A) an odd function, (B) an even function
(C) not odd or even function, (D) both odd and even function

Answer. (B)

6. (5 marks) Padé Approximation is
(A) A rational polynomial approximation
(B) A polynomial approximation
(C) A triangular polynomial approximation
(D) A linear function approximation

Answer. (A)

7. (15 marks) Use the false position method to find the root of $x\sin(x)-1=0$ that is located in the interval $[0,2]$ (the function $\sin(x)$ is evaluated in radians).
Solution. Starting with $a_0=0$ and $b_0=2$, we have $f(0)=-1$ and $f(2)=0.8186$, so a root lies in the interval $[0,2]$.

$\cdots$ 2 marks

Using formula

$$c_n = b_n - \frac{f(b_n)(b_n-a_n)}{f(b_n)-f(a_n)},$$

$\cdots$ 7 marks

we get

$$c_0 = 2 - \frac{0.8186(2-0)}{0.8186-(-1)} = 1.0997 \quad \text{and} \quad f(c_0) = -0.0200$$

The function changes sign on the interval $[c_0,b_0]=[1.0997,2]$, so we squeeze from the left and set $a_1=c_0$ and $b_1=b_0$. The formula produces the next approximation:

$$c_1 = 2 - \frac{0.8186(2-1.0997)}{0.8186-(-0.0200)} = 1.1212$$

and

$$f(c_1) = 0.0098.$$

$\cdots$ 11 marks

Next $f(x)$ changes sign on $[a_1,c_1]=[1.0997,1.1212]$, and the next decision is to squeefrom the right and set $a_2=a_1$ and $b_2=c_1$. Hence we get $c_2=1.1141$ and

$$f(c_2) = 0.0000$$

$\cdots$ 15 marks

8. (15 marks) In the following linear equation systems

$$\begin{aligned} 4x - y + z &= 7 \\ 4x - 8y + z &= -21 \\ -2x + y - 5z &= 15, \end{aligned}$$

start with $\boldsymbol{P}_0 = (1, 2, 2)$, and use Gauss-Seidel iteration to find $\boldsymbol{P}_k$ for $k = 1, 2$. Will Gauss-Seidel iteration converge to the solution?
Solution. The Gauss-Seidel iteration is

$$x_{k+1} = \frac{7 + y_k - z_k}{4}$$

$$y_{k+1} = \frac{21 + 4x_{k+1} + z_k}{8}$$

$$z_{k+1} = \frac{15 + 2x_{k+1} - y_{k+1}}{5}$$

$\cdots$ 7 marks

Substitute $y_0 = 2$ and $z_0 = 2$ into the first equation of the systems above and obtain

$$x_1 = \frac{7 + 2 - 2}{4} = 1.75.$$

Then substitute $x_1 = 1.75$ and $z_0 = 2$ into the second equation and get

$$y_1 = \frac{21 + 4(1.75) + 2}{8} = 3.75.$$

Finally, substitute $x_1 = 1.75$ and $y_1 = 3.75$ into the third equation to get

$$z_1 = \frac{15 + 2(1.75) - 3.75}{5} = 2.95.$$

Using the same reason, we get

$$x_2 = 1.95, \quad y_2 = 3.96, \quad \text{and} \quad z_2 = 2.98$$

$\cdots$ 13 marks

Since the exact solution of the linear equation systems is $\boldsymbol{P} = (2, 4, 3)$, Gauss-Seidel iteration $\boldsymbol{P}_k$ will converge to the solution.

$\cdots$ 15 marks

9. (15 marks) Consider $y = f(x) = \cos(x)$ over $[0.0, 1.2]$. Use the three nodes $x_0 = 0.0, x_1 = 0.6$, and $x_2 = 1.2$ to construct a quadratic interpolation polynomial $P_2(x)$.
Solution. The quadratic Lagrange interpolation polynomial for $f(x)$ is

$$P_2(x) = \sum_{k=0}^{2} y_k L_{2,k}(x)$$

$$= y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

$\cdots$ 7 marks

Using $x_0 = 0.0, x_1 = 0.6, x_2 = 1.2$ and $y_0 = \cos(0.0) = 1, y_1 = \cos(0.6) = 0.8253$, and $y_2 = \cos(1.2) = 0.3623$ in the formula above produces

$$P_2(x) = 1.0 \frac{(x - 0.6)(x - 1.2)}{(0.0 - 0.6)(0.0 - 1.2)} + 0.8253 \frac{(x - 0.0)(x - 1.2)}{(0.6 - 0.0)(0.6 - 1.2)}$$

$$+0.3623\frac{(x-0.0)(x-0.6)}{(1.2-0.0)(1.2-0.6)}$$
$$=1.3888(x-0.6)(x-1.2)-2.2925(x-0.0)(x-1.2)$$
$$+0.5032(x-0.0)(x-0.6).$$

$\cdots$ 15 marks

10. (15 marks) Consider $f(x) = 2 + \sin(2\sqrt{x})$. Use the composite Simpson rule with 11 sample points to compute an approximation to the integral of $f(x)$ taken over $[1, 6]$.

Solution. To generate 11 sample points, we must use $M = 5$ and $h = (6 - 1)/10 = 1/2$. Using formula

$$S(f,h) = \frac{h}{3}(f(a)+f(b)) + \frac{2h}{3}\sum_{k=1}^{M-1} f(x_{2k}) + \frac{4h}{3}\sum_{k=1}^{M} f(x_{2k-1}),$$

$\cdots$ 7 marks

the computation is

$$S(f,\frac{1}{2}) = \frac{1}{6}(f(1)+f(6)) + \frac{1}{3}(f(2)+f(3)+f(4)+f(5))$$
$$+\frac{2}{3}(f(\frac{3}{2})+f(\frac{5}{2})+f(\frac{7}{2}+f(\frac{9}{2})+(\frac{11}{2})$$
$$=\frac{1}{6}(2.9092+1.0173)$$
$$+\frac{1}{3}(2.3080+1.6830+1.2431+1.0287)$$
$$+\frac{2}{3}(2.6381+1.9793)+1.4353+1.1083+1.0002)$$
$$=\frac{1}{6}(3.9166+)+\frac{1}{3}(6.2630)+\frac{2}{3}(8.1613)$$
$$=0.6544+2.0876+5.4408=8.1830.$$

$\cdots$ 15 marks

11. (10 marks) Assume that $g \in C[a, b]$. If the range of the mapping $y = g(x)$ satisfies $y \in [a, b]$ for all $x \in [a, b]$, then g has a fixed point in $[a, b]$.

Proof. If $g(a) = a$ or $g(b) = b$, the assertion is true. Otherwise , the values of $g(a)$ and $g(b)$ must satisfy $g(a) \in (a, b]$ and $g(b) \in [a, b)$. The function $f(x) = x - g(x)$ has the property that

$$f(a) = a - g(a) < 0 \quad \text{and} \quad f(b) = b - g(b) > 0.$$

$\cdots$ 5 marks

Now apply **Zero Value Theorem** to $f(x)$, and conclude that there exists a number P with $P \in (a, b)$ so that $f(P) = 0$. Therefore, $P = g(P)$ and P is the desired fixed point of $g(x)$.

$\cdots$ 10 marks