Anhui University

Semester 1, 2008-2009 Final Examination

Model Answer and Referee Criterion for *Numerical Analysis* (Paper A)

1. (15 marks) Consider the function $f(x) = xe^{-x}$.
 (a) Find the Newton-Raphson formula $p_k = g(p_{k-1})$.
 (b) If $p_0 = 0.2$, then find p_1, p_2, p_3, and p_4. What is $\lim_{n\to\infty} p_k$?

Solution. **(a)** From Newton-Raphson iteration formula, we have

$$\begin{aligned} p_k = g(p_{k-1}) = p_{k-1} - \frac{f(p_{k-1})}{f'(p_{k-1})} \\ = p_{k-1} - \frac{p_{k-1}e^{-p_{k-1}}}{(1-p_{k-1})e^{-p_{k-1}}} \\ = p_{k-1} - \frac{p_{k-1}}{1-p_{k-1}} \end{aligned}$$

......7 marks

(b) Starting with $p_0 = 0.2$, we compute

$$\begin{aligned} p_1 &= 0.2 - \frac{0.2}{1-0.2} = -0.0500 \\ p_2 &= -0.0500 - \frac{-0.0500}{1-0.0500} \approx -0.0024 \\ p_3 &= -0.0024 - \frac{-0.0024}{1-0.0024} \approx 0.0000 \\ p_4 &\approx 0.0000 \end{aligned}$$

......14 marks

Easily, we know $\lim_{k\to\infty} p_k = 0$ ..

2. (15 marks) For the linear equation system

$$\begin{cases} 20x_1 + 2x_2 + 3x_3 = 24 \\ x_1 + 8x_2 + x_3 = 12 \\ 2x_1 - 3x_2 + 15x_3 = 30 \end{cases}$$

(a) Starting with $P_0 = (0,0,0)$, and use Gauss-Seidel iteration to find P_1, P_2, P_3
 (b) Prove this Gauss-Seidel iteration is convergent.

Solution. **(a)** From Gauss-Seidel iteration rule, we have

$$x_1^{(k+1)} = \frac{24 - 2x_2^{(k)} - 3x_3^{(k)}}{20}$$

$$x_2^{(k+1)} = \frac{12 - x_1^{(k+1)} - x_3^{(k)}}{8}$$
$$x_3^{(k+1)} = \frac{30 - 2x_1^{(k+1)} + 3x_2^{(k+1)}}{15}$$

Hence, we have

$$P_1 = (1.2000, 1.3500, 2.1100)$$
$$P_2 = (0.7485, 1.1427, 2.1287)$$
$$P_3 = (0.7664, 1.1381, 2.1254)$$

. . . . . . 9 marks

(b) Since the coefficient matrix of the linear equation systems is

$$A = \begin{pmatrix} 20 & 2 & 3 \\ 1 & 8 & 1 \\ 20 & -3 & 15 \end{pmatrix},$$

it is a strictly diagonally dominant matrix. From the iteration theorem, we know this Gauss-seidel iteration converges to the solution.

. . . . . . 15 marks

3. (15 marks) Let $f(x) = x^x$. Find the quadratic Lagrange polynomial $P_2(x)$ using the nodes $x_0 = 1$, $x_1 = 1.25$, and $x_2 = 1.5$ to approximate $f(1.35)$.

Solution. From the nodes $(1, f(1))$, $(1.25, f(1.25))$, and $(1.5, f(1.5))$, we construct the Lagrange coefficients polynomials

$$L_{2,0}(x) = \frac{(x-1.25)(x-1.5)}{(1-1.25)(1-1.5)}$$

$$L_{2,1}(x) = \frac{(x-1)(x-1.5)}{(1.25-1.5)(1.25-1.5)}$$

$$L_{2,2}(x) = \frac{(x-1)(x-1.25)}{(1.5-1)(1.5-1.25)}$$

. . . . . . 6 marks

Hence, the Lagrange interpolating polynomial

$$P_2(x) = L_{2,0}(x) \times f(1) + L_{2,1}(x) \times f(1.25) + L_{2,2}(x) \times f(1.5)$$
$$= 1.5496x^2 - 2.1998x + 1.6502$$

. . . . . . 14 marks

Then, $f(1.35) \approx P_2(1.35) \approx 1.5046$.

. . . . . . 15 marks

4. (15 marks) Use the least-squares method to find the linear fit $y = Ax + B$

for the data points $(-1, 10)$, $(0, 9)$, $(1, 7)$, $(2, 5)$, $(3, 4)$, $(4, 3)$, $(5, 0)$, and $(6, -1)$.

Solution. Easily, we can compute $\sum_{k=1}^{8} x_k^2 = 92$, $\sum_{k=1}^{8} x_k = 20$, $\sum_{k=1}^{8} x_k y_k = 25$, and $\sum_{k=1}^{8} = 37$. From the least-squares line theorem we have the normal equation system

$$\begin{cases} 92A + 20B = 25 \\ 20A + 8B = 37 \end{cases}$$

. . . . . . 9 marks

The solution of the system above is $A \approx -1.6071$ and $B \approx 8.6429$. Hence, the least-squares line is $y = -1.6071x + 8.6429$

. . . . . . 15 marks

5. (15 marks) Using the composite Simpson rule with $M = 4$ to approximate the integral

$$\int_0^1 \frac{4}{1+x^2} dx$$

and compare the result with the exact value of this integral.

Solution. First of all, the interval $[0, 1]$ should be subdivided into eight subintervals. We obtain nine integral nodes and the step is $\frac{1}{8} = 0.125$. From the composite Simpson rule, we have

$$S(f, h) = \frac{h}{3}(f(a) + f(b)) + \frac{2h}{3} \sum_{k=1}^{3} f(x_{2k}) + \frac{4h}{3} \sum_{k=1}^{4} f(x_{2k-1})$$

$$= \frac{1}{24}(f(0)+f(1)) + \frac{1}{12}\Big(f(\frac{2}{8})+f(\frac{4}{8})+f(\frac{6}{8})\Big) + \frac{1}{6}\Big(f(\frac{1}{8})+f(\frac{3}{8})+f(\frac{5}{8})+f(\frac{7}{8})\Big)$$

$$= \frac{1}{24}(4+2) + \frac{1}{12}\Big(3.7647+3.2000+2.56\Big) + \frac{1}{6}\Big(3.9385+3.5068+2.8764+2.2655\Big)$$

$$= 3.1415$$

. . . . . . 12 marks

The exact value of the integral is $\pi \approx 3.1415$.

. . . . . . 15 marks

6. (15 marks) Using the three-points Gauss-Legendre rule to approximate

$$\int_{-1}^{1} \frac{dx}{2+x} = \ln(3) \approx 1.0986$$

Solution. Gauss-Legendre Three-Point Rule is

$$\int_{-1}^{1} f(x)dx \approx G_3(f) = \frac{5f(-\sqrt{3/5}) + 8f(0) + 5f(\sqrt{3/5})}{9}.$$

......7 marks

So

$$\int_{-1}^{1} \frac{dx}{2+x} \approx \frac{5\frac{1}{2-\sqrt{3/5}} + 8 \times \frac{1}{2} + 5\frac{1}{2+\sqrt{3/5}}}{9} = 1.0978$$

$$\int_{-1}^{1} \frac{dx}{2+x} = \ln(3) - \ln(1) \approx 1.0986$$

......15 marks

7. (10 marks) Assume that Newton-Raphson iteration produces a sequence $\{p_n\}_{n=0}^{\infty}$ that converges to the root p of the function $f(x)$. Prove that if p is a double root of the equation $f(x) = 0$, the convergence is linear and

$$|E_{n+1}| \approx \frac{1}{2}|E_n| \quad \text{for} \quad n \quad \text{sufficiently large.}$$

Proof. Since p is a double root of $f(x) = 0$, $f(x)$ can be written in the form

$$f(x) = (x-p)^2 g(x),$$

where $g(p) \neq 0$, Then,

$$\begin{aligned} f'(x) &= 2(x-p)g(x) + (x-p)^2 g'(x) \\ &= (x-p)[2g(x) + (x-p)g'(x)] \end{aligned}$$

......3 marks

From Newton-Raphson Iteration rule, we have

$$\begin{aligned} p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)} &= p_n - \frac{(p_n-p)^2 g(p_n)}{(p_n-p)[2g(p_n) + (p_n-p)g'(p_n)]} \\ &= p_n - \frac{(p_n-p)g(p_n)}{2g(p_n) + (p_n-p)g'(p_n)} \end{aligned}$$

......5 marks

Then,

$$\frac{p_{n+1}-p}{p_n-p} = 1 - \frac{g(p_n)}{2g(p_n) + (p_n-p)g'(p_n)}$$

So,

$$\lim_{n\to\infty} \frac{E_{n+1}}{E_n} = \lim_{n\to\infty} [1 - \frac{g(p_n)}{2g(p_n) + (p_n-p)g'(p_n)}] = 1 - \frac{1}{2} \neq 0$$

That is to say this convergence is linear, and when n is sufficiently large, we have

$$|E_{n+1}| \approx \frac{1}{2}|E_n|$$

......10 marks