CS240 Algorithm Design and Analysis: Homework #1

Due on October 10, 2017

Probelm 1

Textbook Chapter 2 Exercise 4.

Problem 2

The following program computes the n^{th} power of integer x.

```
function exp(x,n)
  res = 1
  for i = 1 to n
    res *= x
return res
```

(a) For some function f that you should choose, give a bound of the form $O(f(n))$ on the running time of this algorithm on an input of size n (i.e., a bound on the number of operations performed by the algorithm).

(b) The algorithm above is quite simple. Can you do better? Design another algorithm and show that its running time is $O(g(n))$, where $lim_{n\to\infty}\frac{g(n)}{f(n)} = 0$.

Problem 3

During the National Day holiday, scenic areas are always full of people. Bravely, you are planning to visit such a scenic area in the coming holiday. In that place there are n famous scenic spots, all of which you want to visit. But in order to manage the traffic flow, roads between these spots can be only got through in one direction. Now you get the information of m roads and their traffic directions from the tourist map. Suppose there is not a cycle in this map. Design an $O(m+n)$ algorithm to find if there exists such a path that you can visit all the n spots exactly once and if so, output such a path. You can start at whichever spot.

Problem 4

Suppose a scenic area has n famous scenic spots with m undirected roads between them, and the n spots are connected. Because of the complaints from tourists during the holiday, now the officers want to add a public toilet in one of the n scenic spots. Such a spot will close during the construction period, which means tourists are forbidden to visit the spot. How can you find such a spot without which the tourists can still visit all the other $n-1$ spots starting from any of them (i.e., they are still connected). Give an $O(m+n)$ algorithm to achieve the goal.

Problem 5

Let $G = (V, E)$ be an n-node undirected graph containing two nodes s and t such that the distance between s and t is strictly greater than $n/2$. Show that there must exist some node v, not equal to either s or t, such that deleting v from G destroys all s-t paths. In other words, the graph obtained from G by deleting v contains no path from s to t. Give an algorithm with running time $O(m + n)$ to find such a node v.

Problem 6

Assume you are a kind grandparent and going to give your grandchildren some pieces of cake. However, you cannot satisfy a child unless the size of the piece he receives is no less than his expected cake size. Different children may have different expected sizes. Meanwhile, you cannot give each child more than one piece. For example, if the children's expected sizes are $[1, 3, 4]$ and you have two pieces of cake with sizes $[1, 2]$, then you could only make one child satisfied. Given the children's expected sizes and the sizes of the cake pieces that you have, how can you make the most children satisfied? Prove that your algorithm is correct.

Problem 7

Describe an efficient algorithm that, given a set $\{x_1, x_2, ..., x_n\}$ of points on the real line, determines the smallest set of unit-length closed intervals that contain all of the points. Prove that your algorithm is correct.

Problem 8

Suppose we have an undirected graph $G = (V, E)$ with costs $c_e >= 0$ on the edges $e \in E$. All edge costs are distinct. Assume you are given a minimum-cost spanning tree T in G. Now assume that a new edge is added to G, connecting two nodes v, $w \in V$ with cost c. Give an efficient algorithm in $O(|V|)$ time to test if T remains the minimum-cost spanning tree. Please note any assumptions you make about what data structure is used to represent the tree T and the graph G.