Matrix Analysis Homework 7

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Problem 1 :

Let A,B be positive-semidefinite matrices. Show that all eigenvalues of AB are non-negative. Is it true that AB is positive-semidefinite? Justify your answer.

Proof.

i) Pick an eigenpair (x, λ) of AB. We have that

$$\begin{aligned} ABx &= \lambda x \\ x^T B^T (ABx) &= \lambda x^T B^T x \\ (Bx)^T A(Bx) &= \lambda x^T Bx \\ (x')^T Ax' &= \lambda x^T Bx \end{aligned}$$

And we know that A and B are positive-semidefinite matrix, for all $x \in R^n$ and $x' \in R^n$, such that $x^T Bx \geq 0$ and $(x')^T Ax' \geq 0$. So $\lambda x^T Bx \geq 0 \Rightarrow \lambda \geq 0$. So, all eigenvalues of AB are non-negative.

ii) Not always true that AB is positive-semidefinite. $(AB)^T = B^T A^T = BA \neq AB$. Since there isn't enought condition to ensure that AB is a symmetric matrix.

Done. □

Problem 2 :

Let A,B be positive-semidefinite matrices. Show that $|||A - B|||_2 \leq \max\{|||A|||_2, |||B|||_2\}$.

Proof.

Let C be a positive-semidefinite matrix, pick an eigenpair (x, λ) of C. Then,

$$\begin{aligned} Cx &= \lambda x \\ C^T Cx &= \lambda Cx \\ C^T Cx &= \lambda\lambda x \\ C^T Cx &= \lambda^2 x \end{aligned}$$

So the eigenvalues of $C^T C$ equal the eigenvalues of C power 2. And all eigenvalues of C are non-negative, so $\|C\|_2 = \sqrt{\lambda_{\max}(C^T C)} = \max|\lambda(C)|$. Since $(A - B)^T = A - B$, so there are two cases

i) $|\lambda_{\max}(A - B)| \geq |\lambda_{\min}(A - B)|$:

$$|||A - B|||_2 = \max_{\|x\|_2 = 1} x^T (A - B)x = \max_{\|x\|_2} (x^T Ax - x^T Bx) \leq \max_{\|x\|_2} x^T Ax = \lambda_{\max}(A) = |||A|||_2$$

ii) $|\lambda_{\max}(A-B)| < |\lambda_{\min}(A-B)|$:

$$|||A-B|||_2 = \max_{\|x\|_2=1} x^T(B-A)x = \max_{\|x\|_2}(x^TBx - x^TAx) \leq \max_{\|x\|_2} x^TBx = \lambda_{\max}(B) = |||B|||_2$$

Finally, we can say that

$$|||A-B|||_2 \leq \max\{|||A|||_2, |||B|||_2\}$$

Done. $\square$

Problem 3 :

Show that the set of all positive-semidefinite matrices of size $n \times n$ is a convex cone of $\mathbb{R}^{n\times n}$ (you need to read up the definition of "convex cone").

Proof. Terminology: A cone C is a convex cone if $\alpha x + \beta y$ belongs to C, for any positive scalars α, β, and any x, y in C.

For all A and B in the set of all positive-semidefinite matrices $\mathcal{C}$ of size $n \times n$. We have that for all $x \in R^n$, such that $x^TAx \geq 0$ and $x^TBx \geq 0$.

For any positive scalars α and β.

$$\begin{aligned} x^TAx \geq 0 \quad & x^TBx \geq 0 \\ \alpha x^TAx \geq 0 \quad & \beta x^TBx \geq 0 \\ x^T\alpha Ax \geq 0 \quad & x^T\beta Bx \geq 0 \\ x^T\alpha Ax + x^T\beta Bx & \geq 0 \\ x^T(\alpha A + \beta B)x & \geq 0 \end{aligned}$$

It means that for all $x \in R^n$for any positive scalars α and β, for any A, B in $\mathcal{C}$, $x^T(\alpha A + \beta B)x \geq 0$ alwways remains. So $\alpha A + \beta B$ belongs to $\mathcal{C}$.

So $\mathcal{C}$ is a convex cone of $R^{n\times n}$.

Done. $\square$

Problem 4 :

Let $A \in \mathbb{R}^{n\times n}$ be positive-semidefinite. Let $B \in \mathbb{R}^{n\times k}$ be any matrix. Show that the matrix B^TAB is positive-semidefinite.

Proof.

Since $A \geq 0$, so we can say that $\exists C \in R_{n\times n}$ such that $A = C^TC$. So we can do these as following,

$$\begin{aligned} x^TB^TABx &= x^TB^TC^TCBx \\ &= (CBx)^TCBx \\ &= \|CBx\|_2 \\ &\geq 0 \end{aligned}$$

Finally, we get that $x^TB^{TAB}x \geq 0$. So the matrix B^TAB is positive-simidefinite. $\square$

Problem 5 :

Let $A \in \mathbb{R}^{n\times n}$ be positive-semidefinite. Show that for every distinct $i \neq j$ we have that $a_{ii}a_{jj} \geq a_{ij}^2$.

Proof.

Since $A \geq 0$, so we know that every principal minor of A is greater than or equal to 0. So When we remove all columns and rows whose indices not equal to i and j. The principal will be $A(i,j) = \begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix}$. And $det(A(i,j)) \geq 0$, then we can get that $a_{ii}a_{jj} - a_{ij}a_{ji} \neq 0$. That $A \geq 0$ implies that $A = A^T$ which means that $a_{ij} = a_{ji}$. Finally we can get that

$$a_{ii}a_{jj} \geq a_{ij}^2$$

Done. $\square$

Problem 6 :

Let $A \in \mathbb{R}^{n\times n}$ be symmetric, strictly diagonally dominant, and suppose that $a_{ii} > 0$. Prove that A is positive-definite.

Proof.
From *Gerschgorin Circles theorem*, all $\lambda_i \in \sigma A$ are in the $i-th$ Gerschgorin Circle defined by

$$|z - a_{ii}| \leq \sum_{\substack{j=1\\j\neq i}}^{n} |a_{ij}|$$

And A is strict diagonally dominant. It means that $a_{ii} > \sum_{\substack{j=1\\j\neq i}}^{n} |a_{ij}|$ and $a_{ii} > 0$.

$$\begin{aligned}
|z - a_{ii}| &\leq \sum_{\substack{j=1\\j\neq i}}^{n} |a_{ij}| \\
\Rightarrow -\sum_{\substack{j=1\\j\neq i}}^{n} |a_{ij}| &< z - a_{ii} < \sum_{\substack{j=1\\j\neq i}}^{n} |a_{ij}| \\
\Rightarrow a_{ii} - \sum_{\substack{j=1\\j\neq i}}^{n} |a_{ij}| &< z < a_{ii} + \sum_{\substack{j=1\\j\neq i}}^{n} |a_{ij}|
\end{aligned}$$

It implies that all elements in the circle are greater than 0. It also implies that $\lambda_i > 0$.
And we also can get these

$$\begin{aligned}
Ax &= \lambda x \\
x^T Ax &= \lambda x^T x \\
&= \lambda \|x\|_2
\end{aligned}$$

For all $x^T x \neq 0$, we always get that $x^T Ax = \lambda \|x\|_2 > 0$, and A also is symmetric matrix.
So A is positive-definite.
Done. $\square$