Homework 1

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Problem 1 (4/10, *A little group theory drill*):

Let G be a group with group operation $*$ and identity element e. Suppose that $\forall g \in G$ we have that $g * g = e$. Show that G is an abelian group.

Proof. We can know that,

$$\forall a, b, g \in G$$

We have

$$\begin{aligned} a * b \in G \text{ and } \ g * g^{-1} = e \\ \to \quad g = g^{-1} \\ \to \quad a * b = (a * b)^{-1} \\ = b^{-1} * a^{-1} \\ = b * a \end{aligned}$$

Finally, we get that

$$a * b = b * a$$

So, G is an abelian group.
Done. □

Problem 2 (4/10, *Behavior of identity element and inverses under group homomorphisms*):

Let G, H be groups with group operations $*_G$, $*_H$ and identity elements e_G, e_H respectively. Let $\phi : G \to H$ be a group homomorphism. Show that $\phi(e_G) = e_H$ and that $\forall g \in G$ we have that $\phi(g^{-1}) = (\phi(g))^{-1}$.

Proof. First question.
Because $\phi : G \to H$ is a group homomorphism.
We can say that

$$\forall g \in G, \exists h \in H \quad \phi(g) = h$$

Then,

$$\begin{aligned}\phi(g) &= \phi(g *_G e_G) \\ &= \phi(g) *_H \phi(e_G) \\ &= h *_H \phi(e_G) \\ &= h\end{aligned}$$

So we have

$$h *_H \phi(e_G) = h$$

Then,

$$h^{-1} *_H h *_H \phi(e_G) = h^{-1} *_H h$$

$$\forall h \in H \quad h *_H h^{-1} = h^{-1} *_H h = e_H$$

Finally,

$$\phi(e_G) = e_H$$

Second question.
We know

$$\phi(e_G) = e_H$$

and

$$\forall g \in G \quad g * g^{-1} = e_G$$

So,

$$\begin{aligned}\phi(e_G) = \phi(g *_G g^{-1}) &= \phi(g) *_H \phi(g^{-1}) \\ &= e_H\end{aligned}$$

We can do

$$(\phi(g))^{-1} *_H \phi(g) *_H \phi(g^{-1}) = (\phi(g))^{-1} *_H e_H$$

$$\rightarrow (\phi(g))^{-1} = \phi(g^{-1})$$

Done. $\square$

Problem 3 (2/10, *Group isomorphism is a transitive relation*):

Let G, H, K be groups such that G is isomorphic to H, and H is isomorphic to K. Show that G is isomorphic to K.

Proof. Show that:
Let:$\phi_G, \phi_H, \psi_H, \psi_K \ : \ G \rightarrow H, H \rightarrow K, H \rightarrow G, K \rightarrow H$ be group

homomorsim respectively.

$$\begin{aligned}
\forall g \in G \quad \exists h \in H \quad \phi_G(g) = h \\
\forall h \in H \quad \exists k \in K \quad \phi_H(h) = k \\
\forall k \in K \quad \exists h \in H \quad \psi_K(k) = h \\
\forall h \in H \quad \exists g \in G \quad \psi_H(h) = g \\
\to \forall g \in G \quad \exists k \in K \quad \phi_H(\phi_G(g)) = k \\
\& \quad \forall k \in K \quad \exists g \in G \quad \psi_H(\psi_K(k)) = g
\end{aligned}$$

So we can find a function ϕ let

$$\forall g \in G \quad \exists k \in K \quad \phi(g) = k$$

There also is a function ψ let

$$\forall k \in K \quad \exists g \in G \quad \psi(k) = g$$

So, G is isomorphic to K.
Done. $\square$