

Homework 3

龙肖灵

Xiaoling Long

Student ID.:81943968

email:longxl@shanghaitech.edu.cn

October 3, 2017

Problem 1 (3/10, Every subspace has a complement):

Let \mathcal{V} be a vector space and \mathcal{S} a subspace of \mathcal{V} . Show that \mathcal{S} has a complement in \mathcal{V} , i.e., that there exists a subspace \mathcal{T} of \mathcal{V} , such that $\mathcal{V} = \mathcal{S} \oplus \mathcal{T}$

Proof. \mathcal{S} is a subspace of \mathcal{V} , Let \mathcal{T} be a subspace of \mathcal{V} and

$$\mathcal{S} \cap \mathcal{T} = \{0\}$$

Let $s \in \mathcal{S}$ and $t \in \mathcal{T}$. Consider the set,

$$\mathcal{A} = \{as + bt | a, b \in \mathcal{F}\}$$

There are two case.

1. $\exists v \in \mathcal{V}$ and $v \in \mathcal{S}$. Now let $B = 0$, we get

$$\mathcal{A} = \{as | a \in \mathcal{F}\}$$

In this case, we can know that $\mathcal{A} = \mathcal{S}$

2. $\exists v_1, v_2 \in \mathcal{V}$ and $v_1, v_2 \notin \mathcal{S}$. Let $v_1 = a_1s + b_1t$ and $v_2 = a_2s + b_2t$ and $b_i \neq 0, i = 1, 2$

$$\begin{aligned} m_1v_1 + m_2v_2 &= m_1(a_1s + b_1t) + m_2(a_2s + b_2t) \\ &= (m_1a_1 + m_2a_2)s + (m_1b_1 + m_2b_2)t \\ &= ms + nt \end{aligned}$$

Where is $m, n, m_i \in \mathcal{F}, (i = 1, 2)$ so $m_1v_1 + m_2v_2 \in \mathcal{A}$

Let subspace $\mathcal{T} = \{v \in \mathcal{V} | v \notin \mathcal{S}, v \neq 0\}$. We can say that set \mathcal{A} can cover whole space of \mathcal{V} .

We have

$$\mathcal{V} = \mathcal{S} + \mathcal{T}$$

And so anything subspace \mathcal{S} has a complement in \mathcal{V} which called \mathcal{T} and $\mathcal{T} = \{v \in \mathcal{V} | v \notin \mathcal{S}, v \neq 0\}$. Done.

□

Problem 2 (4/10, Behavior of basis over direct sums):

Let \mathcal{V} be a vector space.

1. Let \mathfrak{B} be a basis for \mathcal{V} . Suppose that there exist subsets $\mathfrak{B}_1, \mathfrak{B}_2$ of \mathfrak{B} , such that $\mathfrak{B} = (\mathfrak{B}_1 \cup \mathfrak{B}_2)$ and $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \emptyset$. Then show that $\mathcal{V} = \text{span}(\mathfrak{B}_1) \oplus \text{span}(\mathfrak{B}_2)$.
2. Let $\mathcal{V} = \mathcal{S} \oplus \mathcal{T}$, and let \mathfrak{B}_1 be a basis for \mathcal{S} and \mathfrak{B}_2 be a basis for \mathcal{T} . Show that $\mathfrak{B}_1 \cup \mathfrak{B}_2$ is a basis for \mathcal{V} .

Proof. There are 2 questions.

1. Let $\mathfrak{B} = \{b_i | i = 1, 2, \dots, n\}$, and $\mathfrak{B}_1 = \{b_i | i = 1, 2, \dots, k, k < n\}$, $\mathfrak{B}_2 = \{b_i | i = k+1, k+2, \dots, n\}$ which satisfies $\mathfrak{B} = (\mathfrak{B}_1 \cup \mathfrak{B}_2)$ and $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \emptyset$.

$$\forall v \in \mathcal{V}$$

We have that,

$$v = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

and

$$v_1 = a_1 b_1 + a_2 b_2 + \dots + a_k b_k \in \text{span}(\mathfrak{B}_1)$$

$$v_2 = a_{k+1} b_{k+1} + a_{k+2} b_{k+2} + \dots + a_n b_n \in \text{span}(\mathfrak{B}_2)$$

We get that $v = v_1 + v_2$. So,

$$\mathcal{V} = \mathfrak{B}_1 + \mathfrak{B}_2$$

To prove independence of \mathfrak{B}_1 and \mathfrak{B}_2 , by contrary, suppose there exists $u_1 \in \text{span}(\mathfrak{B}_1)$ and $u_2 \in \text{span}(\mathfrak{B}_2)$, satisfying that

$$r_1 u_1 + r_2 u_2 = 0 \quad r_1, r_2 \in F$$

And r_i aren't all 0. Then we write as

$$u_1 = a_1 b_1 + a_2 b_2 + \dots + a_k b_k$$

$$u_2 = a_{k+1} b_{k+1} + a_{k+2} b_{k+2} + \dots + a_n b_n$$

$$r_1(a_1 b_1 + a_2 b_2 + \dots + a_k b_k) + r_2(a_{k+1} b_{k+1} + \dots + a_n b_n) = 0$$

$$\rightarrow m_1 b_1 + m_2 b_2 + \dots + m_k b_k + m_{k+1} b_{k+1} + \dots + m_n b_n = 0$$

This is contradicting the independence of basis. So

$$\mathfrak{B}_1 \cap \mathfrak{B}_2 = \{0\}$$

2. Let $\mathfrak{B}_1 = \{s_i | i \in I_1\}$ and $\mathfrak{B}_2 = \{t_j | j \in I_2\}$. We have that,

$$\forall v \in \mathcal{V} \quad \exists s \in \mathcal{S} \quad \exists t \in \mathcal{T} \quad v = s + t$$

$$\forall s \in \mathcal{S} \rightarrow s = \sum_i a_i s_i \quad i \in I_1$$

$$\forall t \in \mathcal{T} \rightarrow t = \sum_j b_j t_j \quad j \in I_2$$

So we can say that $\forall v \in \mathcal{V}$, $v = a_i s_i + b_j t_j$. So $\mathfrak{B}_1 \cap \mathfrak{B}_2$ is the spanning set for \mathcal{V}

Suppose that, there exists a set of nonzero coefficients a_i and b_j , let

$$\sum_i a_i s_i + \sum_j b_j t_j = 0$$

Let $s = \sum_i a_i s_i$ and $t = \sum_j b_j t_j$, then we get $s = -t$ which is contradicting $\mathcal{S} \cap \mathcal{T} = \{0\}$. So \mathfrak{B}_1 and \mathfrak{B}_2 are linearly independent. So $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \{0\}$ and that $\mathfrak{B}_1 \cup \mathfrak{B}_2$ is a basis for \mathcal{V} .

Done. □

Problem 3 (3/10, Characterization of a basis):

Prove Theorem 1.7 in Roman by proving that $1) \Rightarrow 4) \Rightarrow 3) \Rightarrow 2) \Rightarrow 1)$.

(In class, I proved it by showing that $1) \Rightarrow 2) \Rightarrow 3) \Rightarrow 4) \Rightarrow 1)$).

Proof. We do four steps to finish this problem.

1. $1) \Rightarrow 4)$

Suppose that 1) holds and \mathcal{S} isn't the maximal linearly independent set, let $\exists k \in \mathcal{V}$ is linearly independent to \mathcal{S} . So we cannot find k be a linear combination of \mathcal{S} which is contradicting the fact that \mathcal{S} is a spanning set of \mathcal{V} .

Hence 1) implies 4).

2. $4) \Rightarrow 3)$

Suppose that 4) holds and \mathcal{S} isn't the minimal spanning set, and let $\mathcal{S} - \mathcal{S}'$ be the minimal spanning set. But we know that \mathcal{S} is the maximal linearly independent set. we cannot write as

$$\forall s' \in \mathcal{S}' \quad s' = \sum_i a_i s_i \quad s_i \in \mathcal{S}, a_i \in \mathcal{F}$$

Every element of \mathcal{S}' cannot be a linear combination of the elements of \mathcal{S} , contradicting the assume. Hence 4) implies 3).

3. $3) \Rightarrow 2)$

Suppose that 3) holds and

$$0 \neq v = a_1 s_1 + \dots + a_n s_n = b_1 s_1 + \dots + b_n s_n$$

Where the $a_i \neq b_i$. By subtracting, we can get that,

$$(a_1 - b_1)s_1 + \dots + (a_n - b_n)s_n = 0$$

So \mathcal{S} isn't the linearly independent set. It means that

$$\exists s \in \mathcal{S} \quad s = \sum_{i \neq j} a_i s_i$$

It contradicts that \mathcal{S} is the minimal spanning set. Hence 3) implies 2).

4. $2) \Rightarrow 1)$

Supposed that 2) hold and that \mathcal{S} is not linearly independent or soans \mathcal{V} . Not linearly independence means 0 is a linear combination of vectors in \mathcal{S} . And not spans \mathcal{V} totally contradicts that every nonzero vector $v \in \mathcal{V}$ is a linear combination of vectors in \mathcal{S} . Hence $2) \Rightarrow 1)$.

Done. □