Matrix Analysis Homework 5

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Problem 1 :

Let $\sigma : U \to V, \tau : V \to W$ be linear transformations of finite dimensional vector spaces. Let B be an ordered basis for U and D an ordered basis for W. Prove that the representation of $\tau\circ\sigma : U \to W$ with respect to B, D is given by matrix multiplication.

Note: Prove this directly by computing this representation; the one line proof given in Roman is of course correct (it uses the commutativity of the diagrams) but it will not be accepted as an answer.

Proof. Let the odered bais $\mathcal{B} = (b_1, \cdots, b_n)$ for $\mathcal{U}$,the odered bais $\mathcal{D} = (d_1, \cdots, d_m)$ for $\mathcal{W}$ and $\mathcal{C} = (c_1, \cdots, c_l)$ be an ordered basis for $\mathcal{V}$.

We can know that, every linear transformation can map to a matrix over $\mathcal{F}$. So we can define that,

$$e_i \in \mathcal{F}^l$$
$$\sigma(b_{j\in I_B}) = \sum_{i\in I_C} \alpha_{j,i} c_i$$
$$\mu(\sigma) = A_{l\times n} = ([\sigma(b_1)]_C, \cdots, [\sigma(b_n)]_C)$$
$$\mu(\tau) = A_{m\times l} = ([\tau(c_1)]_D, \cdots, [\tau(c_l)]_D)$$

So we will show that

$$\mu(\tau\circ\sigma) = \mu(\tau)\times\mu(\sigma) = A_{m\times l}\times A_{l\times n} = ([\tau\circ\sigma(b_1)]_D, \cdots, [\tau\circ\sigma(b_n)]_D)$$

$$\begin{aligned} A_{m\times l}\times A_{l\times n} &= ([\tau(c_1)]_D, \cdots, [\tau(c_l)]_D) \times ([\sigma(b_1)]_C, \cdots, [\sigma(b_n)]_C) \\ &= (\sum_{i\in I_C} [\tau(c_i)]_D\, e_i^T\, [\sigma(b_1)]_C, \cdots, \sum_{i\in I_C} [\tau(c_i)]_D\, e_i^T\, [\tau(b_n)]_C) \end{aligned}$$

$$\begin{aligned} \sum_{i\in I_C} [\tau(c_i)]_D\, e_i^T\, [\sigma(b_j)]_C &= [\tau(c_1)]_D\, e_1^T (\alpha_{j,1}, \cdots, \alpha_{j,l})^T + \cdots + [\tau(c_l)]_D\, e_l^T (\alpha_{j,1}, \cdots, \alpha_{j,l})^T \\ &= \alpha_{j,1}[\tau(c_1)]_D + \cdots + \alpha_{j,l}[\tau(c_l)]_D \end{aligned}$$

ϕ_D and τ is a linear transformation.

$$\sum_{i\in I_C} [\tau(c_i)]_D\, e_i^T\, [\sigma(b_j)]_C = [\alpha_{j,1}\tau(c_1) + \cdots + \alpha_{j,l}\tau(c_l)]_D = [\tau\circ\sigma(b_j)]_D$$
$$A_{m\times l}\times A_{l\times n} = ([\tau\circ\sigma(b_1)]_D, \cdots, [\tau\circ\sigma(b_n)]_D)$$

Done. □

Problem 2 :

Consider the map $\mu : \mathcal{L}(V, W) \to F^{m\times n}$ defined in class (and also in Theorem 2.15 in Roman). Describe explicitly the inverse mapping $\lambda : F^{m\times n} \to \mathcal{L}(V, W)$ and explicitly prove that $\lambda\circ\mu(\tau) = \tau, \forall\tau \in \mathcal{L}(V, W)$, as well as that $\mu\circ\lambda(A) = A, \forall A \in F^{m\times n}$.

Proof. These two problem means that first define a mapping and prove it's isomorphism.

1. Describe: Let V and W be finite-dimensional vector space over F, with ordered bases $\mathcal{B} = (b_1, \cdots, b_n)$ and $\mathcal{C} = (c_1, \cdots, c_m)$, respectively. Every matrix $A \in F^{m\times n}$ can define a linear transformation $\mathcal{L}(V, W)$, which is an change of basis matrix(P65 in Roman) from $\mathcal{B}$ to $\mathcal{C}$.
2. Proof: ϕ_B and ϕ_C both are isomorphism. Based on the diagram, we know that, $\mu(\tau) = \phi_B^{-1}\tau\phi_C$ and $\lambda(A) = \phi_B A\phi_C^{-1}$. So we can say that,

$$\begin{aligned}\lambda \circ (\mu(\tau)) &= \lambda(\phi_B^{-1}\tau\phi_C)\\ &= \phi_B\phi_B^{-1}\tau\phi_C\phi_C^{-1}\\ &= \tau\end{aligned}$$

$$\begin{aligned}\mu \circ \lambda(A) &= \mu(\phi_B A\phi_C^{-1})\\ &= \phi_B^{-1}\phi_B A\phi_C^{-1}\phi_C\\ &= A\end{aligned}$$

Actually, μ is isomophism as well. So if $\lambda = \mu^{-1}$. Statement can always stand.

□

Problem 3 :
i) Let $V = S \oplus T$. Show that $\rho_{S,T} + \rho_{T,S} = id_{\mathcal{L}(\mathcal{V},\mathcal{V})}$. This decomposition of the identity map is called a resolution of the identity.
ii) Show that $im(\rho_{S,T}) = S$ and $ker(\rho_{S,T}) = T$.
iii) Show that for an element $v \in V$ we have that $v \in im(\rho_{S,T}) \Leftrightarrow \rho_{S,T}(v) = v$.

Proof.

i) $\forall v = s + t \in \mathcal{V}$, we have that,

$$\begin{aligned}[\rho_{S,T} + \rho_{T,S}](v) &= \rho_{S,T}(s+t) + \rho_{T,S}(s+t)\\ &= s+t\\ &= v\\ &= id_{\mathcal{L}(\mathcal{V},\mathcal{V})}(v)\end{aligned}$$

So we can say that $\rho_{S,T} + \rho_{T,S} = id_{\mathcal{L}(\mathcal{V},\mathcal{V})}$.

ii) For all $v = s + t \in \mathcal{V}$ and, we have $\rho_{S,T}(v) = s \in S \quad \Rightarrow \quad im(\rho_{S,T}) \subset \mathcal{S}$.
And for all $s \in \mathcal{S}$, we still have that $\rho_{S,T}(s) = s \quad \Rightarrow \quad \mathcal{S} \subset im(\rho_{S,T})$.
Hence, we must have

$$im(\rho_{S,T}) = S$$

For all $v = s + t \in \mathcal{V}$ and $v \in ker(\rho_{S,T})$, we have $\rho_{S,T}(v) = s = 0$, so $v = s + t = t \in \mathcal{T} \quad \Rightarrow ker(\rho_{S,T}) \subset \mathcal{T}$.
And for all $t \in \mathcal{T}$, we still have that $\rho_{S,T}(t) = 0 \quad \Rightarrow \quad \mathcal{T} \subset ker(\rho_{S,T})$.
So

$$ker(\rho_{S,T}) = \mathcal{T}$$

iii) For all $v = s + t \in im(\rho_{S,T} = \mathcal{S}) \quad \rightarrow v = s$,

$$\rho_{S,T}(v) = s = v$$

And we know that $\forall v = s + t \in \mathcal{V} \rightarrow \rho_{S,T}(v) = s = v$. So we can say that $v \in \mathcal{S}$. Hence,

$$v \in im(\rho_{S,T})$$

Done. □

Problem 4 :

Consider the setting of Example 2.5 in Roman. Verify all claims in this example, i.e., prove that
i) $\rho_{D,X}\rho_{D,Y} = \rho_{D,Y} \neq \rho_{D,X} = \rho_{D,Y}\rho_{D,X}$
ii) $\rho_{Y,X}\rho_{X,D} = 0$
iii) $\rho_{X,D}\rho_{Y,X}$ is not a projection.

Proof.

i) Let $v = (x, y)$. Based on $X, Y \to v = (x, 0) + (0, y)$. Based on $D, X \to v = (x - y, 0) + (y, y)$ and based on $D, Y \to v = (x, x) + (0, y - x)$.

$$\begin{aligned}
[\rho_{D,X}\rho_{D,Y}](v) &= \rho_{D,X}(\rho_{D,Y}(v)) \\
&= \rho_{D,X}(\rho_{D,Y}((x, x) + (0, y - x))) \\
&= \rho_{D,X}((x, x) + (0, 0)) \\
&= (x, x) \\
\rho_{D,Y}(v) &= \rho_{D,Y}((x, x) + (0, y - x)) \\
&= (x, x) \\
[\rho_{D,Y}\rho_{D,X}](v) &= \rho_{D,Y}(\rho_{D,X}(v)) \\
&= \rho_{D,Y}(\rho_{D,X}((y, y) + (x - y, 0)) \\
&= \rho_{D,Y}((y, y) + (0, 0)) \\
&= (y, y) \\
\rho_{D,X}(v) &= \rho_{D,X}((y, y) + (x - y, 0)) \\
&= (y, y)
\end{aligned}$$

So we can get that

$$\rho_{D,X}\rho_{D,Y} = \rho_{D,Y} \neq \rho_{D,X} = \rho_{D,Y}\rho_{D,X}$$

ii) For all $v = (x, y)$ we always have that

$$[\rho_{Y,X}\rho_{X,D}](v) = \rho_{Y,X}(\rho_{X,D}((y, y)) = \rho_{Y,X}((x - y, 0) + (0, 0)) = (0, 0) = 0$$

So $\rho_{Y,X}\rho_{X,D}$ map all v to 0. Hence, $\rho_{Y,X}\rho_{X,D} = 0$.

iii) We do same things, for all $v = (x, y)$, we can get that,

$$\begin{aligned}
[\rho_{X,D}\rho_{Y,X}](v) &= \rho_{X,D}(\rho_{Y,X}((x + y))) \\
&= \rho_{X,D}((0, y)) \\
&= \rho_{X,D}((y, y) + (-y, 0)) \\
&= (-y, 0) \\
[\rho_{X,D}\rho_{Y,X}\rho_{X,D}\rho_{Y,X}](v) &= \rho_{X,D}\rho_{Y,X}((-y, 0)) \\
&= \rho_{X,D}((0, 0)) \\
&= 0 \neq [\rho_{X,D}\rho_{Y,X}](v)
\end{aligned}$$

So $\rho_{X,D}\rho_{Y,X}$ is not a projection.

□

Problem 5 :

Let $\rho, \sigma \in \mathcal{L}(V, V)$ be projections. Show that if ρ, σ commute, i.e., that $\rho\sigma = \sigma\rho$, then $\rho\sigma$ is a projection. In that case, show that $im(\rho\sigma) = im(\rho) \cap im(\sigma)$ and $ker(\rho\sigma) = ker(\rho) + ker(\sigma)$.

Proof.

1) $\rho\sigma\rho\sigma = \sigma\rho\rho\sigma = \sigma\rho\sigma = \rho\sigma\sigma = \rho\sigma$. So, $\rho\sigma$ is idempotent. Hence, it is a projection.

2) Let ρ be $\rho_{S,T}$ and σ be $\sigma_{A,B}$. For all $v = s + t = a + b \in \mathcal{V}$, so we have that,

$$\begin{aligned} &\rho_{S,T}\sigma_{A,B}(v) = \rho_{S,T}(a) \in im(\rho) \\ &\rho_{S,T}\sigma_{A,B}(v) = \sigma_{A,B}\rho_{S,T}(v) = \sigma_{A,B}(s) \in im(\sigma) \\ \Rightarrow \quad & im(\rho\sigma) \subset im(\rho) \cap im(\sigma) \end{aligned}$$

And for all $v \in im(\rho) \cap im(\sigma)$, we have that

$$\rho(v) = v \quad \sigma(v) = v$$

So we can say that $\rho\sigma(v) = v$, then we can get $im(\rho) \cap im(\sigma) \subset im(\rho\sigma)$.
Hence, $im(\rho\sigma) = im(\rho) \cap im(\sigma)$.

3) For all $v \in ker(\rho)$, we have that $\rho_{S,T}\sigma_{A,B}(v) = \sigma_{A,B}\rho_{S,T}(v) = \sigma_{A,B}(0) = 0$.
And for all $v \in ker(\sigma)$, we have that $\rho_{S,T}\sigma_{A,B}(v) = \rho_{S,T}(0) = 0$,
So, $ker(\rho) + ker(\sigma) \subset ker(\rho\sigma)$.
Based on defination of projection, let for all $v = a + b = s + t \in ker(\rho\sigma)$, So we have that

$$\begin{aligned} \rho_{S,T}\sigma_{A,B}(v) &= \rho_{S,T}(a) \\ \to a \in ker(\rho_{S,T}) &= T \\ \rho_{S,T}\sigma_{A,B}(v) &= \sigma_{A,B}\rho_{S,T}(v) \\ &= \sigma_{A,B}(s) \\ \to s \in ker(\sigma_{A,B}) &= B \end{aligned}$$

However we can we can re-write $v = a + b = s + t = b + t$. So $ker(\rho\sigma) \subset ker(\rho) + ker(\sigma)$.
Hence,

$$ker(\rho\sigma) = ker(\rho) + ker(\sigma)$$

□