Matrix Analysis Homework 4

龙肖灵
Xiaoling Long
Student ID.:81943968
email:longxl@shanghaitech.edu.cn

October 11, 2017

Problem 1 (*Theorem 2.1.1 in Roman*):
Prove that the set $\mathcal{L}(\mathcal{V}, \mathcal{W})$ of all linear transformations from vector space $\mathcal{V}$ to a vector space $\mathcal{W}$ is itself a vector space.

Proof. We should show that this set fit all properties of a vector space.Let $v \in \mathcal{V}, w \in \mathcal{W}$.

1) **Closure of addition** $\forall \tau, \sigma \in \mathcal{L}(\mathcal{V}, \mathcal{W})$. We can get that,

$$[\tau + \sigma](v) = \tau(v) + \sigma(v) \in \mathcal{W}$$

So, $\mathcal{L}(\mathcal{V}, \mathcal{W})$ is closure of addition.

2) **Commutativity of addition** $\forall \tau, \sigma \in \mathcal{L}(\mathcal{V}, \mathcal{W})$.

$$[\tau + \sigma](v) = \tau(v) + \sigma(v) \qquad = \sigma(v) + \tau(v) = [\sigma + \tau](v)$$

So, we can say that,

$$\tau + \sigma = \sigma + \tau$$

3) **Associativity of additon** $\forall \tau, \sigma, \phi \in \mathcal{L}(\mathcal{V}, \mathcal{W})$. We have that,

$$\begin{aligned}
[\tau + (\sigma + \phi)](v) &= \tau(v) + [\sigma + \phi](v) \\
&= \tau(v) + \sigma(v) + \phi(v) \in \mathcal{W} \\
&= [\tau + \sigma](v) + \phi(v) \\
&= [(\tau + \sigma) + \phi](v)
\end{aligned}$$

Finally, we get

$$\tau + (\sigma + \phi) = (\tau + \sigma) + \phi$$

4) **Existence of zero** Let $0_{\mathcal{L}(\mathcal{V},\mathcal{W})}$ maps all $v \in \mathcal{V}$ to $0 \in \mathcal{W}$. Then, $\forall \tau \in \mathcal{L}(\mathcal{V}, \mathcal{W})$,

$$[\tau + 0_{\mathcal{L}(\mathcal{V},\mathcal{W})}](v) = \tau(v) + 0_{\mathcal{L}(\mathcal{V},\mathcal{W})}(v) = \tau(v)$$

5) **Existence of additive inverses** $\forall \tau \in \mathcal{L}(\mathcal{V}, \mathcal{W})$.

$$[\tau + (-\tau)](v) = \tau(v) + (-\tau(v)) = 0 = 0_{\mathcal{L}(\mathcal{V},\mathcal{W})}(v)$$

6) **Scalar multiplication** For all scalars $a, b \in \mathcal{F}$ and for all linear transformation $\tau, \sigma \in$

$\mathcal{L}(\mathcal{V}, \mathcal{W})$,

$$\begin{aligned}
a[\tau+\sigma](v) &= a[\tau(v)+\sigma(v)] \\
&= a\tau(v) = a\sigma(v) \\
&= [a\tau + a\sigma](v) \\
\Rightarrow a(\tau+\sigma) &= a\tau + a\sigma \\
[(a+b)\tau](v) &= (a+b)\tau(v) \\
&= a\tau(v) + b\tau(v) \\
&= [a\tau + b\tau](v) \\
\Rightarrow (a+b)\tau &= a\tau + b\tau \\
[(ab)\tau](v) &= (ab)[\tau(v)] \\
&= a[b\tau(v)] \\
\Rightarrow (ab)\tau &= a(b\tau)
\end{aligned}$$

All properities of vector space fit for the set $\mathcal{L}(\mathcal{V}, \mathcal{W})$. So, it a a vector space. □

Problem 2 (*Theorem 2.5 in Roman*):

Prove that a linear transformation $\tau \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ is an isomorphism if and only if there is a basis $\mathfrak{B}_{\mathcal{V}}$ for $\mathcal{V}$ for which $\tau\mathfrak{B}_{\mathcal{V}}$ is a basis for $\mathcal{W}$. Prove that in this case, τ maps any basis of $\mathcal{V}$ to a basis of $\mathcal{W}$.

Proof. First of all, sufficiency of this.
$\forall v \in \mathcal{V}$ has an unique linear combination of the vectors in $\mathfrak{B}_{\mathcal{V}}$. Let $\mathfrak{B}_{\mathcal{V}} = \{v_i \mid i \in I\}$, we have that,

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

Then we have that,

$$\tau(v) = a_1\tau(v_1) + a_2\tau(v_2) + \cdots + a_n\tau(v_n)$$

$\tau\mathfrak{B}_{\mathcal{V}}$ is a basis for $\mathcal{W}$. So $\forall v \in \mathcal{V}, \exists!\ w \in \mathcal{W}, \text{s.t. } \tau(v) = w$. So τ is injective. And $\forall w \in \mathcal{W}$ also has an unique linear combination of the vectors in $\tau(\mathfrak{B}_{\mathcal{V}})$. So all vectors in $\mathcal{W}$ have $w = \tau(v)$. So τ is surjective. So τ is an isomorphism.
For necessity of this. τ is isomorphism means τ is injective and surjective. So, ker(τ)= $\{0\}$, and there doesn't exist a linear combination whose coefficients aren't all 0. For all $w \in \mathcal{W}$, we have an unique $v \in \mathcal{V}$, $w = \tau(v)$. So,

$$w = \tau(v) = \tau(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = a_1\tau(v_1) + a_2\tau(v_2) + \cdots + a_n\tau(v_n)$$

Only when all coefficients are 0, $v = 0$ and $w = \tau(v) = 0$. So $\tau(\mathfrak{B}_{\mathcal{V}})$ is linearly independent. And all w has a linear combination of $\tau(v_i) \in \tau(\mathfrak{B}_{\mathcal{V}})$, so it is a spanning set for $\mathcal{W}$. So $\tau\mathfrak{B}_{\mathcal{V}}$ is a basis for $\mathcal{W}$.

All basis $\mathfrak{B}$ of $\mathcal{V}$ has simillar properities. Since τ is isomorphism, τ is injective and surjective. For all $w = \tau(v)$ can write as an unique linear combination with coefficients. And only when all coefficients are 0, $w = 0$. So $\tau(\mathfrak{B})$ is linearly independent and is a spanning set. Hence, τ maps any basis of $\mathcal{V}$ to a basis of $\mathcal{W}$. □

Problem 3 (*Corollary 2.9 in Roman*):

Let $\tau \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ be a linear transformation from vector space $\mathcal{V}$ to vector space $\mathcal{W}$, where $\mathcal{V}, \mathcal{W}$ are both finite dimensional vector spaces with $dim(\mathcal{V}) = dim(\mathcal{W})$. Prove that τ is injective if and only if it is surjective.

Proof. Prove injective($\Rightarrow$).
τ is surjective. So, im(τ)= $\mathcal{W}$. From theorem 2.8, we know that

$$\begin{aligned}
dim\,(ker\,(\tau)) + dim\,(im\,(\tau)) &= dim\,(\mathcal{V}) \\
dim\,(ker\,(\tau)) + dim\,(\mathcal{W}) &= dim\,(\mathcal{V})
\end{aligned}$$

And we know that $dim(\mathcal{V}) = dim(\mathcal{W})$. So we can get that,

$$dim\,(ker\,(\tau)) = 0$$

So $ker\,(\tau)$ must be 0. So τ is injective.

Prove surjective($\Leftarrow$).
We have an injective τ. So, $ker\,(\tau) = \{0\}$ and $dim\,(ker\,(\tau)) = 0$. Also we have

$$dim\,(ker\,(\tau)) + dim\,(im\,(\tau)) = dim\,(\mathcal{V})$$

So we can say $dim\,(im\,(\tau)) = dim\,(\mathcal{V}) = dim\,(\mathcal{W})$.
And we know that $im\,(\tau) \subseteq \mathcal{W} \Rightarrow im\,(\tau) = \mathcal{W}$. Hence τ is surjective.
Done. $\square$