Matrix Analysis Homework 9

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For A symmetric $n \times n$ matrix, we assume the following ordering on its eigenvalues:

$$\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$$

Problem 1 : *(Finish the proof of the second part of the Weyl-II theorem)*
Let A,B be $n \times n$ symmetric matrices. Prove that $\lambda_j(A)+\lambda_k(B) \leq \lambda_{j+k-n}(A+B), \forall j,k=1,\cdots,n$.

Proof.
We already know that $\lambda_{j+k-1} \leq \lambda_j(A) + \lambda_k(B)$.

$$\begin{aligned} -\lambda_{n-(j+k-1)+1}(A+B) &= \lambda_{j+k-1}(-A-B) \\ -\lambda_{n-j-k+2}(A+B) &\leq \lambda_j(-A) + \lambda_k(-B) \\ &= -\lambda_{n-j+1}(A) - \lambda_{n-k+1}(B) \end{aligned}$$

And let $j' = n-j+1$, $k' = n-k+1$, we can get that

$$n - j - k + 2 = j' + k' - n$$

Finally, we can get that

$$\lambda_{j'+k'}(A+B) \geq \lambda_{j'}(A) + \lambda_{k'}(B),\ \forall j', k' = 1, \cdots, n$$

Done. □

Problem 2 :*(Finish the proof of the second part of the Interlacing-II theorem)*
Let A be $n \times n$ symmetric matrix. Let B be an $r \times r$ principal submatrix of A, obtained by deleting rows/columns $i_1, \cdots, i_{n-r}$. Show that $\lambda_k(B) \leq \lambda_k(A), \forall k = 1, \cdots, r$.

Proof.
Similarly, we already have that $\lambda_{n+k-r}(A) \leq \lambda_k(B)$.

$$\begin{aligned} -\lambda_{n-(n+k-r)+1}(A) &= \lambda_{n+k-r}(-A) \\ -\lambda_{r-k+1}(A) &\leq \lambda_k(-B) \\ &= -\lambda_{r-k+1}(B) \end{aligned}$$

So, let $k' = r - k + 1$ and when $k = 1, \cdots, r$, $k' = r, \cdots, 1$ respectively. So we can get that

$$\lambda_{k'}(B) \leq \lambda_{k'}(A),\ \forall k' = 1, \cdots, r$$

Done. □

Problem 3 :*(Finish the proof of the second part of the variational characterization of sums of eigenvalues)*
Let A be $n \times n$ symmetric matrix. Prove that $\sum_{i=n-k+1}^{n} \lambda_i(A) = \min_{U \in \mathbb{R}^{n \times k}, U^T U = I_k} trace(U^T A U)$.

Proof.
Similarly, we have that U^TAU is part of V^TAV. Still by **Interlacing-II** , we get another inequality as follow.

$$\lambda_i(U^TAU) \geq \lambda_{n+i-k}(V^TAV) = \lambda_{n+i-k}(A) \tag{1}$$

Sum 1 for $i = 1, \cdots, k$, we can get that

$$trace(U^TAU) = \sum_{i=1}^{k} \lambda_i(U^TAU) \geq \sum_{i=1}^{k} \lambda_{n+i-k}(A) = \sum_{j=n-k+1}^{n} \lambda_j(A)$$

Done. □

Problem 4 :
Let A be an $n \times n$ symmetric matrix. Prove that $\lambda_n(A) \leq a_{ii} \leq \lambda_1(A), \forall i = 1, \cdots, n$.

Proof.
Let $U = e_i,\ n = 1, \cdots, n$, so $dim(U) = n \times 1$ and $U^TU = 1$. By *Interlacing-II*, we have

$$\sum_{i=n-k+1}^{n} \lambda_i(A) \leq trace(U^TAU) \leq \sum_{i=1}^{k} \lambda_i(A)$$

Now $k = 1$, and $U^TAU = a_{ii}$, Finally, we get that, $\lambda_n(A) \leq a_{ii} \leq \lambda_1(A),\ i = 1, \cdots, n$.
Done. □

Problem 5 :
Let A, B be $n \times n$ symmetric matrices. Prove that $\sum_{i=1}^{k}(\lambda_i(A) + \lambda_i(B)) \geq \sum_{i=1}^{k} \lambda_i(A + B),\ \forall k = 1, \cdots, n$.
Hint: Use the variational characterization of the sum of eigenvalues.

Proof.
By *the variational characterization of the sum of eigenvalues.*

$$\begin{aligned}
\sum_{i=1}^{k} \lambda_i(A+B) &= \max_{\substack{U:n\times k \\ U^TU=I_k}} (U^T(A+B)U) \\
&= \max_{\substack{U:n\times k \\ U^TU=I_k}} (U^TAU + U^TBU) \\
&\leq \max_{\substack{U:n\times k \\ U^TU=I_k}} (U^TAU) + \max_{\substack{U:n\times k \\ U^TU=I_k}} (U^TBU) \\
&= \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B)
\end{aligned}$$

Finally, we get that $\sum_{i=1}^{k}(A + B) \leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k}(B)$.
Done. □