Homework 3

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Problem 1 (3/10, Every subspace has a complement):

Let $\mathcal{V}$ be a vector space and $\mathcal{S}$ a subspace of $\mathcal{V}$. Show that $\mathcal{S}$ has a complement in $\mathcal{V}$, i.e.,that there exists a subspace $\mathcal{T}$ of $\mathcal{V}$, such that $\mathcal{V} = \mathcal{S} \oplus \mathcal{T}$

Proof. $\mathcal{S}$ is a subspace of $\mathcal{V}$, Let $\mathcal{T}$ be a subspace of $\mathcal{V}$ and

$$\mathcal{S} \cap \mathcal{T} = \{0\}$$

Let $s \in \mathcal{S}$ and $t \in \mathcal{T}$. Consider the set,

$$\mathcal{A} = \{as + bt | a, B \in \mathcal{F}\}$$

There are two case.

1. $\exists v \in \mathcal{V}$ and $v \in \mathcal{S}$. Now let $B = 0$, we get

$$\mathcal{A} = \{as | a \in \mathcal{F}\}$$

   In this case, we can know that $\mathcal{A} = \mathcal{S}$

2. $\exists v_1, v_2 \in \mathcal{V}$ and $v_1, v_2 \notin \mathcal{S}$. Let $v_1 = a_1 s + b_1 t$ and $v_2 = a_2 s + b_2 t$ and $b_i \neq 0, i = 1, 2$

$$\begin{aligned} m_1 v_1 + m_2 v_2 &= m_1(a_1 s + b_1 t) + m_2(a_2 s + b_2 t) \\ &= (m_1 a_1 + m_2 a_2)s + (m_1 b_1 + m_2 b_2)t \\ &= ms + nt \end{aligned}$$

   Where is $m, n, m_i \in \mathcal{F}$,$(i = 1, 2)$ so $m_1 v_1 + m_2 v_2 \in \mathcal{A}$

Let subspace $\mathcal{T} = \{v \in \mathcal{V} | v \notin \mathcal{S}, v \neq 0\}$. We can say that set $\mathcal{A}$ can cover whole space of $\mathcal{V}$. We have

$$\mathcal{V} = \mathcal{S} + \mathcal{T}$$

And so anything subspace $\mathcal{S}$ has a complement in $\mathcal{V}$ which called $\mathcal{T}$ and $\mathcal{T} = \{\forall v \in \mathcal{V} | v \notin \mathcal{S}, v \neq 0\}$. Done.

□

Problem 2 (4/10, Behavior of basis over direct sums):

Let $\mathcal{V}$ be a vector space.

1. Let $\mathfrak{B}$ be a basis for $\mathcal{V}$. Suppose that there exist subsets $\mathfrak{B}_1, \mathfrak{B}_2$ of $\mathfrak{B}$, such that $\mathfrak{B} = (\mathfrak{B}_1 \cup \mathfrak{B}_2)$ and $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \emptyset$. Then show that $\mathcal{V} = span(\mathfrak{B}_1) \oplus span(\mathfrak{B}_2)$.
2. Let $\mathcal{V} = \mathcal{S} \oplus \mathcal{T}$, and let $\mathfrak{B}_1$ be a basis for $\mathcal{S}$ and $\mathfrak{B}_2$ be a basis for $\mathcal{T}$. Show that $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \emptyset$ and that $\mathfrak{B}_1 \cup \mathfrak{B}_2$ is a basis for $\mathcal{V}$.

Proof. There are 2 questions.

1. Let $\mathfrak{B} = \{b_i|i = 1, 2, ..., n\}$, and $\mathfrak{B}_1 = \{b_i|i = 1, 2, ..., k\ k < n\}$,$\mathfrak{B}_2 = \{b_i|i = k+1, k+2, ..., n\}$ which satisfies $\mathfrak{B} = (\mathfrak{B}_1 \cup \mathfrak{B}_2)$ and $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \emptyset$.

$$\forall v \in \mathcal{V}$$

We have that,
$$v = a_1b_1 + a_2b_2 + ... + a_nb_n$$
and
$$v_1 = a_1b_1 + a_2b_2 + ... + a_kb_k \in \text{span}(\mathfrak{B}_1)$$
$$v_2 = a_{k+11}b_{k+1} + a_{k+2}b_{K+2} + ... + a_nb_n \in \text{span}(\mathfrak{B}_2)$$
We get that $v = v_1 + v_2$.So,
$$\mathcal{V} = \text{span}(\mathfrak{B}_1) + \text{span}(\mathfrak{B}_2)$$
To prove independence of $\mathfrak{B}_1$ and $\mathfrak{B}_2$, by contrary, suppose there exists $u_1 \in \text{span}(\mathfrak{B}_1)$ and $u_2 \in \text{span}(\mathfrak{B}_2)$, satisfying that
$$r_1u_1 + r_2u_2 = 0 \quad r_1, r_2 \in F$$
And r_i aren't all 0 . Then we write as
$$\begin{aligned} u_1 &= a_1b_1 + a_2b_2 + ... + a_kb_k \\ u_2 &= a_{k+1}b_{k+1} + a_{k+2}b_{k+2} + ... + a_nb_n \\ r_1(a_1b_1 + a_2b_2 + ... + a_kb_k) &+ r_2(a_{k+1}b_{k+1} + ... + a_nb_n) = 0 \\ \to \quad m_1b_1 + m_2b_2 + ... + m_kb_k &+ m_{k+1}b_{k+1} + ... + m_nb_n = 0 \end{aligned}$$
This is contradicting the independence of basis. So
$$\text{span}(\mathfrak{B}_1) \cap \text{span}(\mathfrak{B}_2) = \{0\}$$

2. Let $\mathfrak{B}_1 = \{s_i|i \in I_1\}$ and $\mathfrak{B}_2 = \{t_j|j \in I_2\}$. We have that,
$$\forall v \in \mathcal{V} \ \ \exists s \in \mathcal{S} \ \ \exists t \in \mathcal{T} \ \ v = s + t$$
$$\forall s \in \mathcal{S} \to s = \sum_i a_is_i \quad i \in I_1$$
$$\forall t \in \mathcal{T} \to t = \sum_j b_jt_j \quad j \in I_2$$
So we can say that $\forall v \in \mathcal{V}$, $v = a_is_i + b_jt_j$.So $\mathfrak{B}_1 \cup \mathfrak{B}_2$ is the spanning set for $\mathcal{V}$
Supose that, there exists a set of nonzero coefficents a_i and b_j , let
$$\sum_i a_is_i + \sum_j b_jt_j = 0$$
Let $s = \sum_i a_is_i$ and $t = \sum_j b_jt_j$, then we get $s = -t$ which is contradicting $\mathcal{S} \cap \mathcal{T} = \{0\}$. So $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are linearly independent. So $\mathfrak{B}_1 \cap \mathfrak{B}_2 = \emptyset$ and that $\mathfrak{B}_1 \cup \mathfrak{B}_2$ is a basis for $\mathcal{V}$.

Done. □

Problem 3 (3/10, Characterization of a basis):

Prove Theorem 1.7 in Roman by proving that 1)⇒ 4)⇒ 3)⇒ 2)⇒ 1).
(In class, I proved it by showing that 1)⇒ 2)⇒ 3)⇒ 4)⇒ 1)).

Proof. We do four steps to finish this problem.

1. 1)$\Rightarrow$ 4)
   Suppose that 1) holds and $\mathcal{S}$ isn't the maximal linearly independent set, let $\exists k \in \mathcal{V}$ is linearly independent to $\mathcal{S}$. So we cannot find k be a linear combination of $\mathcal{S}$ which is constradicting the fact that $\mathcal{S}$ is a spanning set of $\mathcal{V}$.
   Hence 1) implies 4).
2. 4)$\Rightarrow$ 3)
   Suppose that 4) holds and $\mathcal{S}$ isn't the minimal spanning set, and let $\mathcal{S} - \mathcal{S}'$ be the minimal spanning set. But we know that $\mathcal{S}$ is the maximal linearly independent set. we cannot write as
   $$\forall s' \in \mathcal{S}' \quad s' = \sum_i a_i s_i \quad s_i \in \mathcal{S}, a_i \in \mathcal{F}$$
   Every element of $\mathcal{S}'$ cannot be a linear combination of the elements of $\mathcal{S}$, contradicting the assume. Hence 4) implies 3).
3. 3)$\Rightarrow$ 2)
   Suppose that 3) holds and
   $$0 \neq v = a_1 s_1 + ... + a_n s_n = b_1 s_1 + ... + b_n s_n$$
   Where the $a_i \neq b_i$. By subtracting, we can get that,
   $$(a_1 - b_1)s_1 + ... + (a_n - b_n)s_n = 0$$
   So $\mathcal{S}$ isn't the linearly independent set. It means that
   $$\exists s \in \mathcal{S} \quad s = \sum_{i \neq j} a_i s_i$$
   It constradicts that $\mathcal{S}$ is the minimal spanning set.Hence 3) implies 2).
4. 2)$\Rightarrow$ 1)
   Supposed that 2) hold and that $\mathcal{S}$ is not linearly independent or soans $\mathcal{V}$. Not linearly independence means 0 is a linear combination of vectors in $\mathcal{S}$. And not spans $\mathcal{V}$ totally contradicts that every nonzero vector $v \in \mathcal{V}$ is a linear combination of vectors in $\mathcal{S}$. Hence 2) $\Rightarrow$ 1).

Done. $\square$