Matrix Analysis Homework 6

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October 25, 2017

Problem 1 :

Let $A, B \in \mathbb{R}^{n\times n}$ be square matrices. Show that
$max\{dim\mathcal{N}(A), dim\mathcal{N}(B)\} \leq dim\mathcal{N}(AB) \leq dim\mathcal{N}(A) + dim\mathcal{N}(B)$

Proof. Divided into two parts.

1) $dim\mathcal{N}(AB) \leq dim\mathcal{N}(A) + dim\mathcal{N}(B)$:

$$\begin{aligned} dim\mathcal{N}(AB) &= n - rank(AB) \\ &\leq n - (rank(A) + rank(B) - n) \\ &= n - rank(A) + n - rank(B) \\ &= dim\mathcal{N}(A) + dim\mathcal{N}(B) \end{aligned}$$

2) $max\{dim\mathcal{N}(A), dim\mathcal{N}(B)\} \leq dim\mathcal{N}(AB)$:

$$\begin{aligned} dim\mathcal{N}(AB) &= n - rank(AB) \\ &\geq n - min(rank(A), rank(B)) \\ &= max(dim\mathcal{N}(A), dim\mathcal{N}(B)) \end{aligned}$$

So we can get that, $max\{dim\mathcal{N}(A), dim\mathcal{N}(B)\} \leq dim\mathcal{N}(AB) \leq dim\mathcal{N}(A) + dim\mathcal{N}(B)$.
Done. □

Problem 2 :

Let $u, v \in \mathbb{R}^{n\times 1}$ be two column vectors such that $u^T v \neq 1$. Let $\mathcal{I}$ be the $n \times n$ identity matrix. Compute a basis for the nullspace of $A := \mathcal{I} - uv^T$. Or if $\mathcal{N}(A) = 0$, prove that A is invertible, and compute A^{-1}

Proof.

1. We take one element $x \in \mathcal{N}(A)$, Then we can get $uv^T x = x$. So x is a eigenvector when $\lambda = 1$. And we know that $rank(uv^T) = 1$ and $\sum_i \lambda_i = trace(uv^T) = u^T v$. λ just have two values whatever the dimension of uv^T is.

$$\lambda = 0 \quad or \quad \lambda = u^T v \neq 1$$

So there doesn't exist this eigenvector. So x just can equal to 0. It means $\mathcal{N}(A) = 0$. the nullspace of A is empty set.

2. $\mathcal{N}(A) = 0 \Rightarrow dim\mathcal{N}(A) = 0 \Rightarrow rank(A) = n - dim\mathcal{N}(A) = n \Rightarrow A$ is invertible. Let $B := uv^T$, then we can get that $BB = uv^T uv^T = v^T uuv^T = v^T uB$. Let $A^{-1} = \alpha I + \beta B$.

$$
\begin{aligned}
(I - B)(\alpha I + \beta B) &= I \\
\alpha I + \beta IB - \alpha BI - \beta BB &= I \\
\alpha I + (\beta - \alpha - \beta v^T u)B &= I \\
\Rightarrow \alpha &= 1 \\
(\beta - \alpha - \beta v^T u) &= 0 \\
\Rightarrow \beta &= \frac{1}{1 - v^T u}
\end{aligned}
$$

So, $A^{-1} = I + \frac{uv^T}{1-v^T u}$. Let verify in another direction.

$$
\begin{aligned}
A^{-1}A &= (I + \frac{uv^T}{1 - v^T u}) \times (I - uv^T) \\
&= I + \frac{uv^T}{1 - v^T u} - uv^T - \frac{uv^T uv^T}{1 - v^T u} \\
&= I + \frac{v^T uuv^T}{1 - v^T u} - \frac{uv^T uv^T}{1 - v^T u} \\
&= I \quad (uv^T uv^T = v^T uuv^T)
\end{aligned}
$$

Done. ☐

Problem 3 :

Let $A \in \mathbb{R}^{m \times n}$ be matrix with $rank(A) = r$. Prove in two different ways that there exist matrices $B \in \mathbb{R}^{m \times r}, C \in \mathbb{R}^{r \times n}$, such that $A = BC$. You may only use arguments that we have developed in the class so far.
Hint: For the first proof, try to write the linear transformation $\tau_A : \mathbb{R}^n \to \mathbb{R}^m$ as a composition of two linear transformations ϕ, ψ in the form $\tau_A : \mathbb{R}^n \xrightarrow{\phi} \mathbb{R}^r \xrightarrow{\psi} \mathbb{R}^m$. For the second proof, B should contain in its columns the basis for some space (which space?).

Proof. We can show that based on propoties of linear transformation and matrix mutiplication.

1) Let $\tau_A : \mathbb{R}^n \to \mathbb{R}^m$. There exists $\phi_C : \mathbb{R}^n \to \mathbb{R}^r$ and $\psi_B : \mathbb{R}^r \to \mathbb{R}^m$, such that $\tau_A = \psi_B \phi_C$. P, Q, R are the ordered basis for $\mathbb{R}^n, \mathbb{R}^r, \mathbb{R}^m$ respectively.
We know that $A = [\tau_A]_{P,R} = [\psi_B \phi_C]_{P,R} = [\psi_B]_{Q,R}[\phi_C]_{P,Q} = BC$ from Theorem 2.15 in Roman. So we can get that $A = BC$.
2) Same as problem 1 in Homework 5. Let $B = (B_1, \cdots, B_r) \in \mathbb{R}^{m \times r}$ be a basis for $\mathbb{R}^r$. And we know $rankA = r$, supose $A = (A_1, \cdots, A_n)$. So,

$$A_i = \sum_{j=0}^{r} B_j c_{i,j}$$

Let

$$C = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r,1} & c_{r,2} & \cdots & c_{r,n} \end{pmatrix}$$

So we can get that $A = BC$.

Done. ☐

Problem 4 :

Let $\| \cdot \| : \mathbb{R}^{m \times n} \to \mathbb{R}$ be a matrix norm induced by the vector norm $\| \cdot \|$.Let A be an invertible matrix. Prove that $\|A^{-1}\| = (min_{\|x\|=1} \|Ax\|)^{-1}$.

Proof. To prove that, we must know $\left\| \frac{a}{\|a\|} \right\| = 1, if a \neq 0$.

$$\begin{aligned}
\|A^{-1}\| &= max_{\|x\|=1} \|A^{-1}x\| \\
&= max_{y \neq 0} \left\| A^{-1} \frac{y}{\|y\|} \right\| \\
&= max_{y \neq 0} \left\| \frac{A^{-1}y}{\|y\|} \right\| \\
&= max_{y \neq 0} \frac{\|A^{-1}y\|}{\|y\|} \\
&= max_{y \neq 0} \frac{\|A^{-1}y\|}{\|AA^{-1}y\|} \\
&= \frac{1}{min_{y \neq 0} \left\| \frac{A(A^{-1}y)}{\|A^{-1}y\|} \right\|} \\
&= \frac{1}{min_{\|x\|=1} \|Ax\|}
\end{aligned}$$

Finally, we get that $\|A^{-1}\| = \frac{1}{min_{\|x\|=1}\|Ax\|}$.
Done. □

Problem 5 :

There is something wrong with statement (5.2.13) p.283 in Meyer. What is it, and why is it wrong? How can you fix it?

Proof. The condition should be $VV^* = I$. We want to show that $\|AV\|_2 = \|A\|_2$.

$$\begin{aligned}
\|AV\|_2^2 &= \|(AV)^*\|_2^2 \\
&= \|V^*A^*\|_2^2 \\
&= max_{\|x\|_2=1} \|V^*A^*x\|_2^2 \\
&= max_{\|x\|_2=1} x^*AVV^*A^*x \\
\|A\|_2^2 &= \|A^*\|_2^2 \\
&= max_{\|x\|_2=1} \|A * x\|_2^2 \\
&= max_{\|x\|_2=1} x^*AA^*x
\end{aligned}$$

So just $VV^* = I$ can get $\|AV\|_2 = \|A\|_2$. So we must change the condition to $VV^* = I$. □

Problem 6 :

Given a vector norm $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ define the dual norm $\|\cdot\|^D : \mathbb{R}^n \to \mathbb{R}$ by $\|x\|^D := max_{\|y\|=1} |x^T y|$. Show that the dual norm is a norm. show that
$\| \cdot \|_2^D = \| \cdot \|_2, \| \cdot \|_1^D = \| \cdot \|_\infty, \| \cdot \|_\infty^D = \| \cdot \|_1$.

Proof.

1) $\| \cdot \|_2^D = \| \cdot \|_2$:
   (a) if $x = 0$, $\|0\|_2^D = \|0\|_2$.
   (b) if $x \neq 0$, we know that $cos\theta = \frac{<x|y>}{\|x\|_2 \|y\|_2}$, and $\|y\|_2 = 1$, we can get that $< x|y >= \|x\|_2 cos\theta$. When $y = \alpha x$, $< x|y >$ can get the max. So let $y = \frac{x}{\|x\|}$.

$$\|x\|_2^D = max_{\|y\|_2=1} |x^T y| = \left| x^T \frac{x}{\|x\|_2} \right| = \|x\|_2$$

2) $\|\cdot\|_1^D = \|\cdot\|_\infty$:
$\|x\|_1^D = max_{\|y\|_1=1}|x^Ty|$. When $y = e_i$, such that $x_i = maxx_i$, euqation get the maximum value. So $\|x\|_1^D = max|x_i|$. Hence,

$$\|\cdot\|_1^D = \|\cdot\|_\infty$$

3) $\|\cdot\|_\infty^D = \|\cdot\|_1$:
$\|x\|_\infty^D = max_{\|y\|_\infty=1}|x^Ty|$. When $x_i < 0$,$y_i = -1$ and $x_i > 0$,$y_i = 1$, $\|x\|_\infty^D$ get the maximum. And $\|x\|_\infty^D = |x^Ty| = \sum_i |x_i|$. Hence,

$$\|\cdot\|_\infty^D = \|\cdot\|_1$$

□