Homework n

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Problem 1 (*Theorem 2.1.1 in Roman*):
Prove that the set $\mathcal{L}(\mathcal{V}, \mathcal{W})$ of all linear transformations from vector space $\mathcal{V}$ to a vector space $\mathcal{W}$ is itself a vector space.

Proof. We should show that this set fit all properties of a vector space.Let $v \in \mathcal{V}, w \in \mathcal{W}$.

1) **Closure of addition** $\forall \tau, \sigma \in \mathcal{L}(\mathcal{V}, \mathcal{W})$. We can get that,

$$[\tau + \sigma](v) = \tau(v) + \sigma(v) \in \mathcal{W}$$

So, $\mathcal{L}(\mathcal{V}, \mathcal{W})$ is closure of addition.

2) **Commutativity of addition** $\forall \tau, \sigma \in \mathcal{L}(\mathcal{V}, \mathcal{W})$.

$$[\tau + \sigma](v) = \tau(v) + \sigma(v) \qquad = \sigma(v) + \tau(v) = [\sigma + \tau](v)$$

So, we can say that,

$$\tau + \sigma = \sigma + \tau$$

3) **Associativity of additon** $\forall \tau, \sigma, \phi \in \mathcal{L}(\mathcal{V}, \mathcal{W})$. We have that,

$$\begin{aligned}[\tau + (\sigma + \phi)](v) &= \tau(v) + [\sigma + \phi](v) \\ &= \tau(v) + \sigma(v) + \phi(v) \in \mathcal{W} \\ &= [\tau + \sigma](v) + \phi(v) \\ &= [(\tau + \sigma) + \phi](v)\end{aligned}$$

Finally, we get

$$\tau + (\sigma + \phi) = (\tau + \sigma) + \phi$$

4) **Existence of zero** Let $0_{\mathcal{L}(\mathcal{V},\mathcal{W})}$ maps all $v \in \mathcal{V}$ to $0 \in \mathcal{W}$. Then, $\forall \tau \in \mathcal{L}(\mathcal{V}, \mathcal{W})$,

$$[\tau + 0_{\mathcal{L}(\mathcal{V},\mathcal{W})}](v) = \tau(v) + 0_{\mathcal{L}(\mathcal{V},\mathcal{W})}(v) = \tau(v)$$

5) **Existence of additive inverses** $\forall \tau \in \mathcal{L}(\mathcal{V}, \mathcal{W})$.

$$[\tau + (-\tau)](v) = \tau(v) + (-\tau(v)) = 0 = 0_{\mathcal{L}(\mathcal{V},\mathcal{W})}(v)$$

6) **Scalar multiplication** For all scalars $a, b \in \mathcal{F}$ and for all linear transformation $\tau, \sigma \in$

$\mathcal{L}(\mathcal{V}, \mathcal{W})$,

$$\begin{aligned}
a[\tau+\sigma](v) &= a[\tau(v)+\sigma(v)] \\
&= a\tau(v) = a\sigma(v) \\
&= [a\tau + a\sigma](v) \\
\Rightarrow a(\tau+\sigma) &= a\tau + a\sigma \\
[(a+b)\tau](v) &= (a+b)\tau(v) \\
&= a\tau(v) + b\tau(v) \\
&= [a\tau + b\tau](v) \\
\Rightarrow (a+b)\tau &= a\tau + b\tau \\
[(ab)\tau](v) &= (ab)[\tau(v)] \\
&= a[b\tau(v)] \\
\Rightarrow (ab)\tau &= a(b\tau)
\end{aligned}$$

All properities of vector space fit for the set $\mathcal{L}(\mathcal{V}, \mathcal{W})$. So, it a a vector space. □

Problem 2 (*Theorem 2.5 in Roman*):
Prove that a linear transformation $\tau \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ is an isomorphism if and only if there is a basis $\mathfrak{B}_{\mathcal{V}}$ for $\mathcal{V}$ for which $\tau\mathfrak{B}_{\mathcal{V}}$ is a basis for $\mathcal{W}$. Prove that in this case, τ maps any basis of $\mathcal{V}$ to a basis of $\mathcal{W}$.

Proof. First of all, sufficiency of this.
$\forall v \in \mathcal{V}$ has an unique linear combination of the vectors in $\mathfrak{B}_{\mathcal{V}}$. Let $\mathfrak{B}_{\mathcal{V}} = \{v_i \mid i \in I\}$, we have that,

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

Then we have that,

$$\tau(v) = a_1\tau(v_1) + a_2\tau(v_2) + \cdots + a_n\tau(v_n)$$

$\tau\mathfrak{B}_{\mathcal{V}}$ is a basis for $\mathcal{W}$. So $\forall v \in \mathcal{V}, \exists!\ w \in \mathcal{W}, \text{s.t. } \tau(v) = w$. So τ is injective. And $\forall w \in \mathcal{W}$ also has an unique linear combination of the vectors in $\tau(\mathfrak{B}_{\mathcal{V}})$. So all vectors in $\mathcal{V}$ have $w = \tau(v)$. So τ is surjective. So τ is an isomorphism. For necessity of this. □

Problem 3 (*Corollary 2.9 in Roman*):
Let $\tau \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ be a linear transformation from vector space $\mathcal{V}$ to vector space $\mathcal{W}$, where $\mathcal{V}, \mathcal{W}$ are both finite dimensional vector spaces with $dim(\mathcal{V}) = dim(\mathcal{W})$. Prove that τ is injective if and only if it is surjective.

Proof. aaa □