Matrix Analysis Homework 7

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Problem 1 :

Let $\mathcal{U}$ be a subspace of $\mathbb{R}^m$ of dimension s, and let $U = [u_1, \cdots, u_s] \in \mathbb{R}^{m\times s}$ be an orthonormal basis for $\mathcal{U}$. Then $P_{\mathcal{U}} = UU^T$ is the orthogonal projection of $\mathbb{R}^m$ onto $\mathcal{U}$. Similarly, let $\mathcal{V}$ be a subspace of $\mathbb{R}^n$ of dimension s, let $V = [v_1, \cdots, v_s]$ be an orthonormal basis for $\mathcal{V}$. Now, let $\mathcal{W}$ be the subspace of $\mathbb{R}^{m\times n}$ spanned by all matrices of the form $u_i x^T$, $y v_j^T$, where $i, j = 1, \cdots, s$ and x, y vary freely in $\mathbb{R}^n$,$\mathbb{R}^m$ respectively.

i) What is the dimension of $\mathcal{W}$?

ii) Find a formula for the orthogonal projection of $\mathbb{R}^{m\times n}$ onto $\mathcal{W}$ in terms of the orthogonal projections P_U,P_V of $\mathbb{R}^m$ onto $\mathcal{U}$, and $\mathbb{R}^n$ onto $\mathcal{V}$ respectively.

iii) Find a formula for the orthogonal projection onto $\mathcal{W}^{\perp}$.

Solution.

i) Extend the basis $U_{ex} = [u_1, u_2, \cdots, u_s, u_{s+1}, \cdots, u_m]$ to be the orthonormal basis for $\mathbb{R}^m$, and $V_{ex} = [v_1, v_2, \cdots, v_s, v_{s+1}, \cdots, v_n]$ to be the orthonormal basis for $\mathbb{R}^n$. So x and y can be linear combination of V_{ex} and U_{ex} respectively. We can get that, $x = a_1v_1 + a_2v_2 + \cdots, a_nv_n$ and $y = b_1u_1 + b_2u_2 + \cdots + b_mu_m$. For all $w \in \mathcal{W}$, we have that

$$\begin{aligned}
w &= \sum_i^s \alpha_i u_i x^T + \sum_j^s \beta_j y v_j^T \\
&= \sum_i^s \alpha_i u_i (a_1v_1 + a_2v_2 + \cdots, a_nv_n)^T + \sum_j^s \beta_j (b_1u_1 + b_2u_2 + \cdots + b_mu_m) v_j^T \\
&= \sum_k^n \sum_i^s \alpha_i a_k u_i v_k^T + \sum_l^m \sum_j^s \beta_j b_l u_l v_j^T \\
:&= s + t
\end{aligned}$$

So $\mathcal{W} = S + T$.And $u_i v_k^T \quad i = 1, \cdots, s \textit{ and } k = 1, \cdots, n$ are the spanning set for S. Now prove there are linearly independent. By contradiction, suppose they are linearly dependent.

We have not all $\xi_{i,j} = 0$,

$$\begin{aligned}
\sum_i^s \sum_j^n \xi_{i,j} u_i v_j^T &= 0 \\
\sum_j^n (\sum_i^s \xi_{i,j} u_i) v_j^T &= 0 \\
\sum_j^n U_j v_j^T &= 0 \\
\iff \sum_j^n U_{j,i} v_j^T &= 0 \quad \forall i
\end{aligned}$$

Because that all v_i are linearly independent. All $U_{j,i} = 0$, which means $\sum_i^s \xi_{i,j} u_i = 0$. So all coeffients $\xi_{i,j} = 0$(Contradiction). So $u_i v_k^T \quad i = 1, \cdots, s \ and \ k = 1, \cdots, n$ are basis for S. Similarly evidence for $\sum_i^s \xi_{i,j} u_i) = 0$. So all coeffients $\xi_{i,j} = 0$(Contradiction).
So $u_j v_l^T \quad j = 1, \cdots, m \ and \ l = 1, \cdots, s$ be a basis for T and $u_j v_l^T \quad j = 1, \cdots, s \ and \ l = 1, \cdots, s$ be a basis for $S \cap T$.
So we can get these, $dim(S) = ns$ and $dim(T) = ms$ and $dim(S \cap T) = s^2$.
Finally, we can get the dimension of $\mathcal{W}$.

$$dim(\mathcal{W}) = dimS + T = dimS + dimT - dim(S \cap T) = ms + ns - s^2$$

ii) The projection is that

$$\begin{aligned}
P_{\mathcal{W}}(\mathcal{M}) &= (I_m - P_U)\mathcal{M} P_V + P_U \mathcal{M}(I_n - P_V) + P_U \mathcal{M} P_V \\
&= P_U \mathcal{M} + \mathcal{M} P_V - P_U \mathcal{M} P_V
\end{aligned}$$

Pick a $\mathcal{M} \in \mathbb{R}^{m \times n}$, we can write as $\mathcal{M} = \sum_i^m \sum_j^n \alpha_{i,j} u_i v_k^T$. We do projection $P_{mathcalW}$ on $\mathcal{M}$. We get that

$$\begin{aligned}
P_{\mathcal{W}}(\mathcal{M}) &= P_U \sum_i^m \sum_j^n \alpha_{i,j} u_i v_j^T + \sum_i^m \sum_j^n \alpha_{i,j} u_i v_j^T P_V - P_U \sum_i^m \sum_j^n \alpha_{i,j} u_i v_j^T P_V \\
&= \sum_i^s \sum_j^n \alpha_{i,j} u_i v_j^T + \sum_i^m \sum_j^s \alpha_{i,j} u_i v_j^T - \sum_i^s \sum_j^s \alpha_{i,j} u_i v_j^T \\
&= \sum_i^s \sum_j^n \alpha_{i,j} u_i v_j^T + \sum_{i=s+1}^m \sum_j^s \alpha_{i,j} u_i v_j^T
\end{aligned}$$

iii) And the projection onto $\mathcal{W}^\perp$ is that

$$\begin{aligned}
P_{\mathcal{W}^\perp}(\mathcal{M}) &= (I_m - P_U)\mathcal{M}(I_n - P_V) \\
&= \mathcal{M} - P_{\mathcal{W}}(\mathcal{M})
\end{aligned}$$

So we also have that,

$$\begin{aligned}
P_{\mathcal{W}^\perp}(\mathcal{M}) &= \mathcal{M} - \sum_i^s \sum_j^n \alpha_{i,j} u_i v_k^T + \sum_{i=s+1}^m \sum_j^s \alpha_{i,j} u_i v_j^T \\
&= \sum_i^m \sum_j^n \alpha_{i,j} u_i v_j^T - \sum_i^s \sum_j^n \alpha_{i,j} u_i v_j^T - \sum_{i=s+1}^m \sum_j^s \alpha_{i,j} u_i v_j^T \\
&= \sum_{i=s+1}^m \sum_j^n \alpha_{i,j} u_i v_j^T - \sum_{i=s+1}^m \sum_j^s \alpha_{i,j} u_i v_j^T \\
&= \sum_{i=s+1}^m \sum_{j=s+1}^n \alpha_{i,j} u_i v_j^T
\end{aligned}$$

So without loss of generality, we can put all dimension of $ms+ns-s^2$ first $ms+ns-s^2$ rows into a $m \times n$vector. So the $P_{\mathcal{W}^\perp}(\mathcal{M})$ can the last parts, so we have that

$$vec(P_{\mathcal{W}}(\mathcal{M}))vec(P_{\mathcal{W}^\perp}(\mathcal{M}))^T = [w_1, \cdots, w_{ms+ns-s^2}, 0, \cdots, 0]\,[0, \cdots, 0, w_{ms+ns-s^2}, \cdots, w_{m\times n}]^T = 0$$

So we have that

$$P_{\mathcal{W}}(\mathcal{M}) \perp P_{\mathcal{W}^\perp}(\mathcal{M})$$

Done. ■

Problem 2 :

i) Let U be an $n \times n$ unitary matrix. What can you say about $\sigma(U)$?

ii) Let P be an $n \times n$ real matrix representing the orthogonal projection of $\mathbb{R}^n$ onto a subspace S of dimension k. Find $\sigma(P)$. What is the algebraic and geometric multiplicity of each eigenvalue?

Solution.

i) From the theorem in the midterm we have that $trace(A)^2 \leq n\ trace(A^TA)$. So we can get that

$$(\sum \sigma(U))^2 \leq n^2$$

Suppose that (x, λ) be an eigenpair for U. We have that

$$\begin{aligned} Ux &= \lambda x \\ \|Ux\| &= \|\lambda x\| \\ x^*U^*Ux &= \|\lambda\|x^*x \\ x^*x &= \|\lambda\|x^*x \\ \|\lambda\| &= 1 \end{aligned}$$

ii) Based on the defination of eigenpair, we have $Px = \lambda x$. And based on the defination of projection, we have that $\forall s \in S \rightarrow Ps = s$. And we know that $P^2 = P$. So we can get that,

$$\begin{aligned} Px &= \lambda x \\ PPx &= \lambda Px \\ Px &= \lambda^2 x \\ \lambda x &= \lambda^2 x \end{aligned}$$

So we can get $\lambda = 0$ *or* 1. And let B_S be the orthogonal basis of s, for all $s_i \in B_s$ we can get $Ps_i = s_i$. So all s_i will be the eigenvector for eigenvalue of 1. So for $\lambda = 1$ the geometric multiplicity both are k.

And let B_{S^c} be the basis for the complement of S. For all $s_i^c \in B_{S^c}$ we have that $Ps_i^c = 0 \times s_i^c$. So the geometric multiplicity for $\lambda = 0$ are $n-k$.

And we know that $\tau_g(\lambda) \leq \tau_a(\lambda)$ and $\tau_a(\lambda = 0) + \tau_a(\lambda = 1) = n$. So the algebraic multiplicity for $\lambda = 0$ is $n-k$. And the algebraic multiplicity for $\lambda = 1$ is k.

Done. ■