Mechanical Behavior of Materials
Lec.16-20

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This recitation class will cover lecture 16$\sim$20, the topics include:

- Plasticity in cracked member (Lec.16)
- Statistics of brittle fracture (Lec.17)
- Fatigue (Lec.18$\sim$20)

Agenda

1. Plasticity in cracked member
2. Statistics of brittle fracture
3. Fatigue

Plastic Zone Size

- As materials cannot withstand the theoretically infinite stress at crack tip, a plastic zone is formed by yielding which redistributes the stress field and changes the shape of the crack.
- If the size of the plastic zone is too large, then LEFM does not apply. Therefore, we need to estimate the size of the plastic zone.
- For plane stress condition $\sigma_z = 0$, the plastic zone size is estimated as:

$$2r_{o\sigma} = \frac{1}{\pi}\left(\frac{K}{\sigma_o}\right)^2$$

- For plane strain condition $\varepsilon_z = 0$, the plastic zone size is estimated as:

$$2r_{o\varepsilon} = \frac{1}{3\pi}\left(\frac{K}{\sigma_o}\right)^2$$

- As K increases, the plastic zone size increases; as σ_o increases, the plastic zone size decreases.

Plasticity Limitations on LEFM

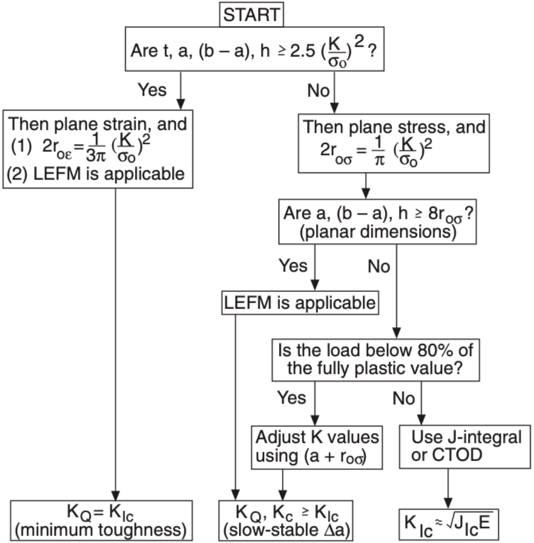
- If the plastic zone size is small, there is still a region outside of it where LEFM applies.
- For plane stress condition, the overall limit is:

$$a, (b-a), h \geqslant \frac{4}{\pi}\left(\frac{K}{\sigma_o}\right)^2 = 8r_{o\sigma}$$

- From empirical observation, plane strain does not occur unless the thickness satisfies

$$t, a, (b-a), h \geqslant 2.5\left(\frac{K}{\sigma_o}\right)^2 \approx 47r_{o\varepsilon}$$

Summary


START
Are t, a, (b – a), h ≥ 2.5 $(\frac{K}{\sigma_o})^2$?
Yes
No
Then plane strain, and
(1) $2r_{o\varepsilon} = \frac{1}{3\pi}(\frac{K}{\sigma_o})^2$
(2) LEFM is applicable
Then plane stress, and
$2r_{o\sigma} = \frac{1}{\pi}(\frac{K}{\sigma_o})^2$
Are a, (b – a), h ≥ $8r_{o\sigma}$?
(planar dimensions)
Yes
No
LEFM is applicable
Is the load below 80% of
the fully plastic value?
Yes
No
Adjust K values
using $(a + r_{o\sigma})$
Use J-integral
or CTOD
$K_Q = K_{Ic}$
(minimum toughness)
$K_Q, K_c \geq K_{Ic}$
(slow-stable Δa)
$K_{Ic} \approx \sqrt{J_{Ic}E}$

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Statistics of Strength

- Brittle materials are defect-sensitive. No single tensile strength, but a certain probability that a given sample will have a given strength.
- The average strength of the small samples is greater than that of the large sample, because larger samples are more likely to have larger cracks.
- Brittle materials tend to be stronger under bending than in tension. In bending, only the presence of large surface cracks is dangerous because max tension occurs on surface.

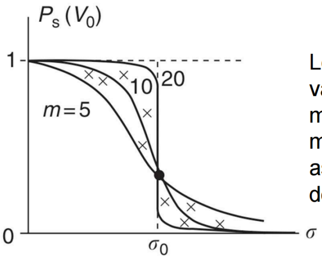
Weibull Survival Probability

The survival probability $P_s(V_0)$ under stress σ is

$$P_s(V_0) = \exp\left\{-\left(\frac{\sigma}{\sigma_0}\right)^m\right\}$$

where σ_0 and m are constants.

- m represents how rapidly the strength falls as approaching σ_0, which is called the Weibull modulus.
- $\sigma = 0,\ P_s = 1;\ \sigma \to \infty,\ P_s \to 0;\ \sigma = \sigma_0,\ P_s = \frac{1}{e}$.



Lower m, the greater variation of strength. For m greater than 20, a material can be treated as having a single well defined failure stress.

Volume Dependence of Weibull Probability

For $V = nV_0$, it can be considered as n independent samples of volume V_0. The survival probability is:

$$P_s(V) = \{P_s(V_0)\}^{V/V_0}$$

Therefore,

$$P_s(V) = \exp\left\{-\frac{V}{V_0}\left(\frac{\sigma}{\sigma_0}\right)^2\right\}$$

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Cyclic Loading

- Stress amplitude:

$$\sigma_a = \frac{\Delta\sigma}{2} = \frac{\sigma_{max} - \sigma_{min}}{2}$$

- Mean stress:

$$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2}$$

- Stress ratio:

$$R = \frac{\sigma_{min}}{\sigma_{max}}$$

- Stress amplitude ratio:

$$A = \frac{\sigma_a}{\sigma_m}$$

Cyclic Loading

- Two independent parameters are needed to determine a cyclic loading. Commonly use σ_a and σ_m, or σ_{min} and σ_{max}.
- The relationship between these two groups of parameters is:

$$\sigma_a = \frac{\sigma_{max}}{2}(1 - R) \quad \text{and} \quad \sigma_m = \frac{\sigma_{max}}{2}(1 + R)$$

- Hence the relationship between R and A is:

$$R = \frac{1 - A}{1 + A} \quad \text{and} \quad A = \frac{1 - R}{1 + R}$$

Typical Cyclic Loadings

- **Tension-to-tension:** $\sigma_{min} > 0$ or $0 < R < 1$.
- **Zero-to-tension:** $\sigma_{min} = 0$ or $R = 0$.
- **Tensile-mean-stress:** $\sigma_m > 0$ or $-1 < R < 0$.
- **Completely reversed:** $\sigma_{min} = -\sigma_{max}$ or $\sigma_m = 0$ or $R = -1$.
- **Compressive-mean-stress:** $\sigma_m < 0$ or $R < -1$.

$S - N$ Curves

- Fatigue life (N_f): the number of cycles to failure under specified stress condition S_a
- Fatigue strength: maximum stress amplitude for a specified life time.
- Fatigue limit (σ_e): stress amplitude below which fatigue does not occur.
- Low cycle fatigue: $S_a > \sigma_o$; $S - N$ curve is a straight line; significant amount of plastic strain, only strain-based approach is applicable.
- High cycle fatigue: $S_a < \sigma_o$; $\log S - \log N$ is a straight line; stresses and strains are mostly elastic, stress-based approach is applicable.

Basquin's Law

- Used for fully-reversed, high-cycle fatigue, i.e.

$$\sigma_m = 0 \quad \text{and} \quad \sigma_a < \sigma_o$$

- The general form is as follows:

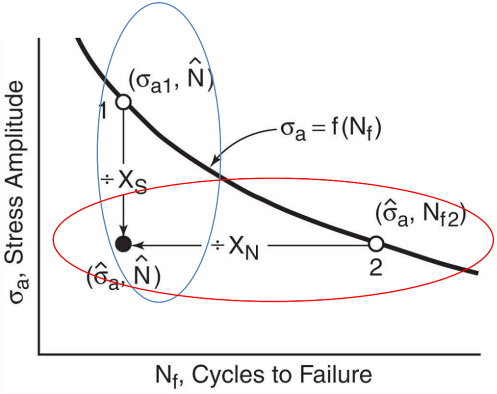
$$\sigma_{ar} = AN_f^B$$

- Equivalently, it can be expressed as:

$$\sigma_{ar} = \sigma_f'(2N_f)^b$$

where σ_f' is a constant and $\sigma_f' \approx \sigma_u$.

Safety Factor for $S - N$ Curves



For the stress amplitude and cycles $(\hat{\sigma}_a, \hat{N})$, the safety factor is defined as:

- In stress:

$$X_S = \frac{\sigma_{a1}}{\hat{\sigma}_a}, \left(N_f = \hat{N}\right)$$

- In life:

$$X_N = \frac{N_{f2}}{\hat{N}}, (\sigma_a = \hat{\sigma}_a)$$

- According to Basquin's law,

$$X_S = \frac{A\hat{N}^B}{AN_{f2}^B} = X_N^{-B} \quad \text{or} \quad X_N = X_S^{-\frac{1}{B}}$$

Factors Affecting Fatigue Behaviors

- Mean stress effect:
  - Tensile mean stress $\Rightarrow$ shorter fatigue life, lower fatigue strength
  - Compressive mean stress $\Rightarrow$ longer fatigue life, higher fatigue strength
- Notch effect:
  - Stress concentration factor: $k_t = \frac{\sigma_y}{S} = 1 + \sqrt{\frac{c}{\rho}}$
  - $\rho \downarrow \Rightarrow k_t \uparrow \Rightarrow N_f \downarrow, \sigma_a \downarrow$
- Temperature and frequency effect:
  - Usually, lower T and higher cycling frequency leads to higher fatigue strength
- Microstructure effect:
  - Increased dislocation density, smaller grain size $\Rightarrow$ higher fatigue strength

Normalized Amplitude-mean Diagram

- Plot the ratio $\frac{\sigma_a}{\sigma_{ar}}$ versus σ_m; pass $(0, 1)$ and $(\sigma_u, 0)$
- Goodman's equation:

$$\frac{\sigma_a}{\sigma_{ar}} + \frac{\sigma_m}{\sigma_u} = 1$$

- Morrow's equation:

$$\frac{\sigma_a}{\sigma_{ar}} + \frac{\sigma_m}{\sigma_f'} = 1$$

- SWT:

$$\sigma_{ar} = \sqrt{\sigma_{max}\sigma_a} = \sigma_{max}\sqrt{\frac{1-R}{2}} \quad (\sigma_{max} > 0)$$

Life estimate for $\sigma_m \neq 0$

- By Morrow's equation:

$$\sigma_{ar} = \frac{\sigma_a}{1 - \sigma_m / \sigma'_f}$$

- General stress-life equation:

$$\sigma_a = \left(\sigma'_f - \sigma_m\right)\left(2N_f\right)^b$$

- σ_{ar} can be thought of the equivalent completely reversed stress amplitude of (σ_a, σ_m).
- Safety factors:
  - $(\hat{\sigma}_a, \hat{\sigma}_m, \hat{N}) \Rightarrow (\hat{\sigma}_{ar}, \hat{N})$
  - In stress:

$$X_S = \frac{\sigma_{ar1}}{\hat{\sigma}_{ar}} \quad \left(N_f = \hat{N}\right)$$

  - In life:

$$X_N = \frac{N_{f2}}{\hat{N}} \quad \left(\sigma_{ar} = \hat{\sigma}_{ar}\right)$$

Palmgren-Miner Rule

- Used for variable amplitude loading
- Fatigue failure happens when sum of life fractions used equals to unity:

$$B_f \sum_i \frac{N_i}{N_{fi}} = 1$$

where B_f is the number of repetitions of the same load cycle; N_i is the number of cycles at amplitude σ_{ai}; N_{fi} is the number of cycles to failure at σ_{ai}.

- N_{fi} can be calculated using general stress-life equation:

$$N_{fi} = \frac{1}{2}\left(\frac{\sigma_{ai}}{\sigma'_f - \sigma_m}\right)^{1/b}$$

Fatigue Notch Factor

- The fatigue notch factor k_f is defined as:

$$k_f = \frac{\sigma_{ar}}{S_{ar}}$$

- Notch sensitivity of fatigue is defined as:

$$q = \frac{k_f - 1}{k_t - 1} \in [0, 1]$$

where $k_f < k_t$; for brittle materials with large notch radius $k_f \approx k_t$

- Empirical estimates of k_f:

$$k_f = 1 + q(k_t - 1) = \begin{cases} 1 + \dfrac{k_t - 1}{1 + \alpha/\rho} & (Peterson) \\ 1 + \dfrac{k_t - 1}{1 + \sqrt{\beta/\rho}} & (Neuber) \end{cases}$$

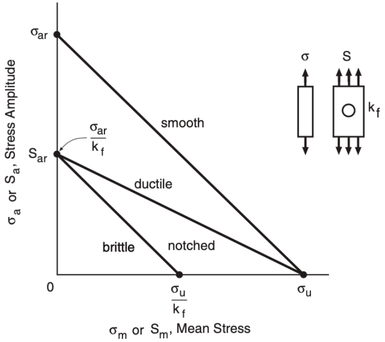
where α, β are material constants and ρ is the notch radius.

Combined Effect of Notches and Mean Stresses

- Goodman equation for notched member:

$$\frac{S_a}{S_{ar}} + \frac{S_m}{\sigma_u / k_{fm}} = 1$$

- For brittle materials, k_{fm} typically taken to be the same value as k_f.
- For ductile materials, k_{fm} is usually taken to be 1.



Exercise

Question

A cyclic uniaxial stress to an un-notched member made of steel. The material constants are $\sigma'_f = 1700$ MPa, $b = -0.15$, and the ultimate tensile strength is $\sigma_{uts} = 786$ MPa.

(a) For the cyclic uniaxial stress ($\sigma_m = 0, \sigma_{ar} = 200$ MPa), estimate the number of cycles to failure (N_f).

(b) If the cyclic uniaxial stress is not completely reversible ($\sigma_m = 200$ MPa$, \sigma_a = 200$ MPa), estimate the number of cycles to failure (N_f).

(c) If we made notches on the surface of the component, whose stress concentration factor k_t is 2.43 and the notch sensitivity q is 0.95, could you determine the fatigue limit for the notched member under fully reversible cyclic loading? Given that the fatigue limit for smooth member under fully reversible cyclic loading is half of the ultimate tensile strength of this material.

Solutions I

(a) As $\sigma_m = 0$, we can apply the Basquin's law:

$$\sigma_{ar} = \sigma'_f (2N_f)^b$$

then

$$N_f = \frac{1}{2}\left(\frac{\sigma_{ar}}{\sigma'_f}\right)^{1/b} = \frac{1}{2}\left(\frac{200 \text{ MPa}}{1700 \text{ MPa}}\right)^{1/-0.15} = 7.85 \times 10^5 \text{ cycles}$$

(b) As $\sigma_a = \left(\sigma'_f - \sigma_m\right)(2N_f)^b$, we have

$$N_f = \frac{1}{2}\left(\frac{\sigma_a}{\sigma'_f - \sigma_m}\right)^{1/b} = \frac{1}{2}\left(\frac{200 \text{ MPa}}{1700 \text{ MPa} - 200 \text{ MPa}}\right)^{1/-0.15} = 3.41 \times 10^5 \text{ cycles}$$

Solutions II

(c) As $k_t = 2.43$ and $q = 0.95$, we have

$$k_f = 1 + q(k_t - 1) = 1 + 0.95(2.43 - 1) = 2.36$$

As for smooth member,

$$\sigma_{er} = \frac{\sigma_{uts}}{2} = \frac{786 \text{ MPa}}{2} = 393 \text{ MPa}$$

for notched member we have

$$S_{er} = \frac{\sigma_{er}}{k_f} = \frac{393 \text{ MPa}}{2.36} = 166.5 \text{ MPa}$$

Thank you!