Mechanical Behavior of Materials
Lec.13-15

Haoxuan Zeng

UM-SJTU Joint Institute
zenghaoxuan@sjtu.edu.cn

March 31, 2025

This recitation class will cover lecture 13∼15, the topics include:

- Failure Criterion (Lec.13)
- Linear Elastic Fracture Mechanics (Lec.14&15)

Agenda

1 Failure Criterion

2 Linear Elastic Fracture Mechanics

General Form

Effective Stress

For a given loading condition with three principal stresses of σ_1, σ_2, and σ_3, the effective stress is defined as:

$$\bar{\sigma} = f(\sigma_1, \sigma_2, \sigma_3)$$

which is a single numerical value that characterizes the state of the applied stress.

- σ_c is the critical stress for failure, which is a material property.
- If $\bar{\sigma} = \sigma_c$, failure occurs; if $\bar{\sigma} < \sigma_c$, no failure.
- $X = \frac{\sigma_c}{\bar{\sigma}}$ is the safety factor.

Types of Failure Criterion

- Fracture criterion for brittle materials:
  - Maximum normal stress criterion
  - Coulomb-Mohr fracture criterion
- Yield criterion for ductile materials:
  - Maximum shear stress criterion (Tresca criterion)
  - Octahedral shear stress criterion (Von Mises criterion)
- Different criterion correspond to different function f and critical stress σ_c.

Maximum Normal Stress Criterion

Definition

This criterion applies to fracture of brittle materials under tension-dominant loading. The effective stress is defined as the maximum principal stress:

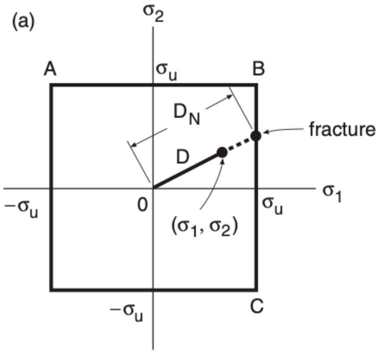
$$\bar{\sigma}_N = \max\left(|\sigma_1|, |\sigma_2|, |\sigma_3|\right)$$

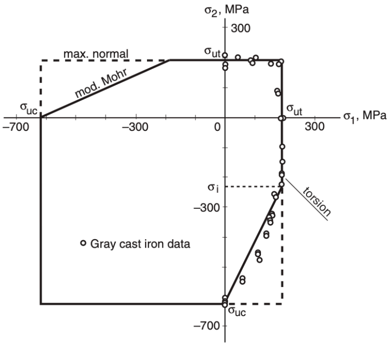
At fracture, the effective stress equals to the ultimate tensile strength:

$$\bar{\sigma}_N = \sigma_u$$

Maximum Normal Stress Criterion

The safe region is bounded by the 6 planes $|\sigma_i| = \sigma_u$ for $i = 1, 2, 3$.





- For general brittle materials, the ultimate tensile strength and ultimate compressive strength are different, $|\sigma_{ut}| < |\sigma_{ct}|$
- For plane stress, if only one principal normal stress is tensile, then the safe region is defined by Coulomb-Mohr criterion.

Coulomb-Mohr Fracture Criterion

Definition

For Coulomb-Mohr criterion, fracture happens when a critical combination of σ and τ is reached on a plane:

$$|\tau| + \mu\sigma = \tau_i \text{ (at fracture)}$$

where $\mu = \tan\phi$ and τ_i is a material intrinsic property.

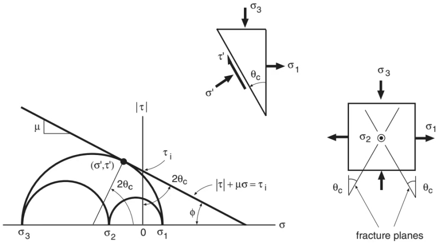


Figure 7.14 Coulomb–Mohr fracture criterion as related to Mohr's circle, and predicted fracture planes.

Failure condition: the largest circle touches failure envelope.

Coulomb-Mohr Fracture Criterion

Based on C-M criterion, the theoretical σ'_{ut} and σ'_{uc} are

$$\sigma'_{ut} = 2\tau_i\sqrt{\frac{1-m}{1+m}}$$

$$\sigma'_{uc} = -2\tau_i\sqrt{\frac{1+m}{1-m}}$$

where $m = \sin\phi$.

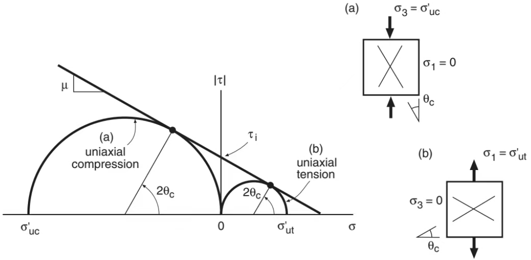


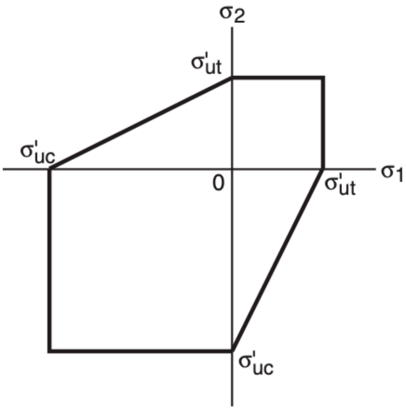
Figure 7.15 Fracture planes predicted by the Coulomb–Mohr criterion for uniaxial tests in tension and compression.

Coulomb-Mohr Fracture Criterion

The safe region is bounded by

$$|\sigma_i - \sigma_j| + m(\sigma_i + \sigma_j) = 2\tau_i\sqrt{1 - m^2} = |\sigma'_{uc}|(1 - m)$$

where $i, j = 1, 2, 3$ and $i \neq j$.



Tresca Criterion

Definition

The Tresca criterion applies to yield of ductile materials under shear stress. The effective stress is defined as the maximum shear stress:

$$\bar{\sigma}_S = \max\left(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|\right)$$

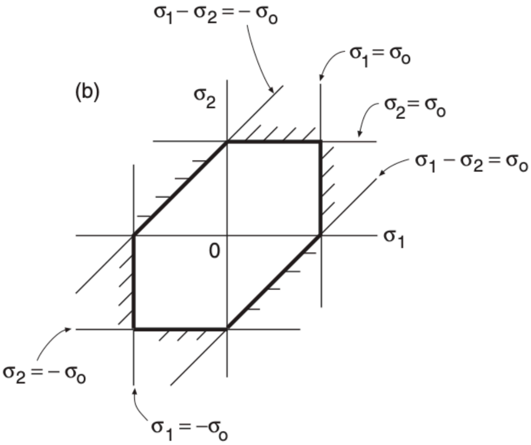
At yield, the effective stress equals to the yield strength:

$$\bar{\sigma}_S = \sigma_o$$

Note that the shear yield strength is $\tau_o = \frac{\sigma_o}{2}$.

Tresca Criterion

The safe region is bounded by $|\sigma_i - \sigma_j| = \sigma_o$ for $i \neq j$.



Von Mises criterion

Definition

This criterion applies to yield of ductile materials under shear stress. The effective stress is defined as follows:

$$\bar{\sigma}_H = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$$

For a specific coordinate system, the effective stress can be expressed as:

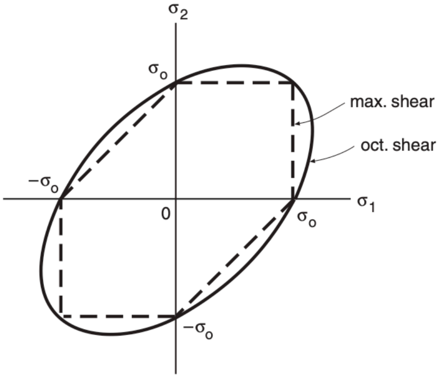
$$\bar{\sigma}_H = \frac{1}{\sqrt{2}}\sqrt{(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2\right)}$$

At yielding, the effective stress equals to the yield strength:

$$\bar{\sigma}_H = \sigma_o$$

Von Mises Criterion

The safe region is bounded by an ellipsoid.



Agenda

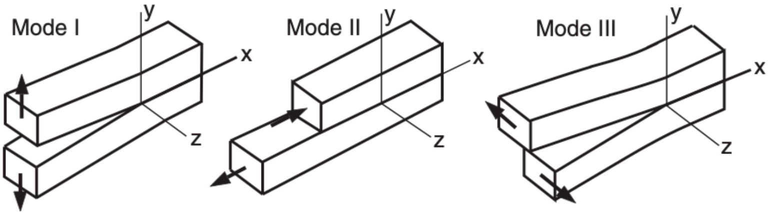
1 Failure Criterion

2 Linear Elastic Fracture Mechanics

Crack Modes

According to the movement of the crack faces, LEFM defines the following three independent crack modes:

- Mode I: opening, stress normal to crack plane and leading edge of crack.
- Mode II: sliding, stress parallel to crack plane and normal to leading edge.
- Mode III: tearing, stress parallel to crack plane and leading edge.

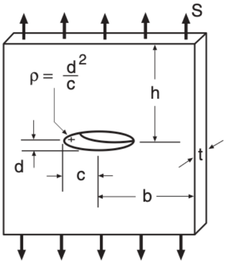


Stress Concentration of Crack

Crack can serve as stress concentration points. Consider an elliptical hole in a plate with $c \ll b$, the value of σ_y rises sharply near the hole and has a maximum value at the edge.

$$\sigma_y = S\left(1 + 2\frac{c}{d}\right) = S\left(1 + 2\sqrt{\frac{c}{\rho}}\right)$$

where $\rho = \frac{d^2}{c}$ is the radius of curvature at the edge.



The stress concentration factor is defined as:

$$k_t = \frac{\sigma_y}{S} = 1 + 2\frac{c}{d} = 1 + 2\sqrt{\frac{c}{\rho}}$$

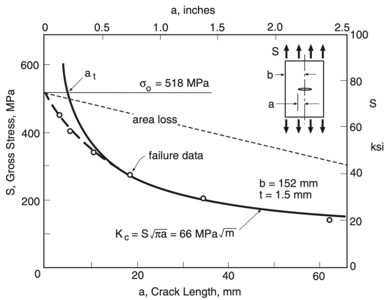
As $d \to 0$, $k_t \to \infty$. Therefore, the sharper the crack tip, the higher the stress concentration.

Stress Intensity Factor

- If the material fractures in a brittle manner (with litte plastic deformation), then we can assume the material behave in a linear-elastic manner, which LEFM applies.
- LEFM uses a single parameter, the stress intensity factor K, to characterize the severity of crack situation:

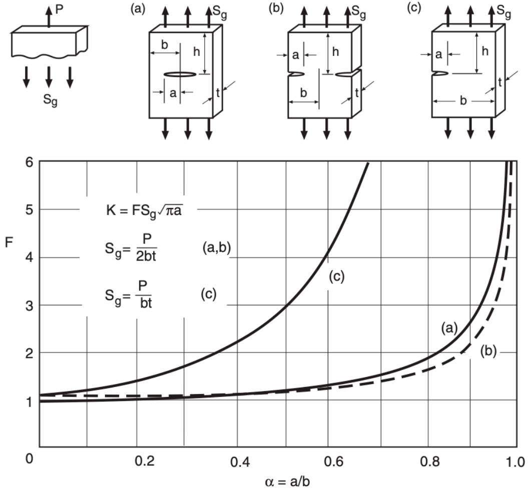
$$K = FS_g\sqrt{\pi a}$$

where F is a dimensionless function and S_g is the gross section area nominal stress.



- Safe if $K < K_C$
- Fracture if $K \geq K_C$
- K_C is the fracture toughness

Stress Intensity Factor



Values for small a/b and limits for 10% accuracy:

(a) $K = S_g\sqrt{\pi a}$ $(a/b \leq 0.4)$

(b) $K = 1.12 S_g\sqrt{\pi a}$ $(a/b \leq 0.6)$

(c) $K = 1.12 S_g\sqrt{\pi a}$ $(a/b \leq 0.13)$

Expressions for any $\alpha = a/b$:

(a) $$F = \frac{1 - 0.5\alpha + 0.326\alpha^2}{\sqrt{1-\alpha}} \qquad (h/b \geq 1.5)$$

(b) $$F = \left(1 + 0.122\cos^4\frac{\pi\alpha}{2}\right)\sqrt{\frac{2}{\pi\alpha}\tan\frac{\pi\alpha}{2}} \qquad (h/b \geq 2)$$

(c) $$F = 0.265\,(1-\alpha)^4 + \frac{0.857 + 0.265\alpha}{(1-\alpha)^{3/2}} \qquad (h/b \geq 1)$$

Figure 8.12 Stress intensity factors for three cases of cracked plates under tension. Geometries, curves, and equations labeled (a) all correspond to the same case, and similarly for (b) and (c). (Equations as collected by [Tada 85] pp. 2.2, 2.7, and 2.11.)

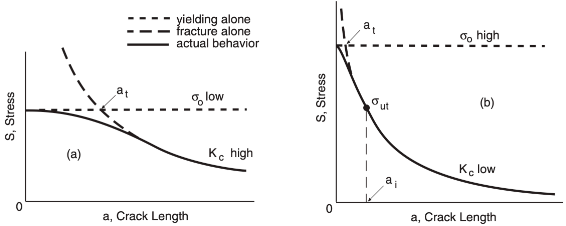
Transition Crack Length

The length of crack when critical remote stress S_C predicted by LEFM equals to yield strength is the transition crack length:

$$a_t = \frac{1}{\pi}\left(\frac{K_C}{\sigma_o}\right)^2$$

- If $a < a_t$, crack has little impact on strength reduction, strength of material close to yield strength and the material will have yielding failure.
- If $a > a_t$, LEFM shall be used for design and the material will have brittle fracture.

Trend: high strength, low fracture toughness.



Safety Factors for Crack Fracture

- On stress intensity factor K, or stress:

$$X_K = \frac{K_{IC}}{K} = \frac{K_{IC}}{FS_g\sqrt{\pi a}}$$

- On crack length:

$$X_a = \frac{a_c}{a} = \left(\frac{F}{F_c} X_K\right)^2$$

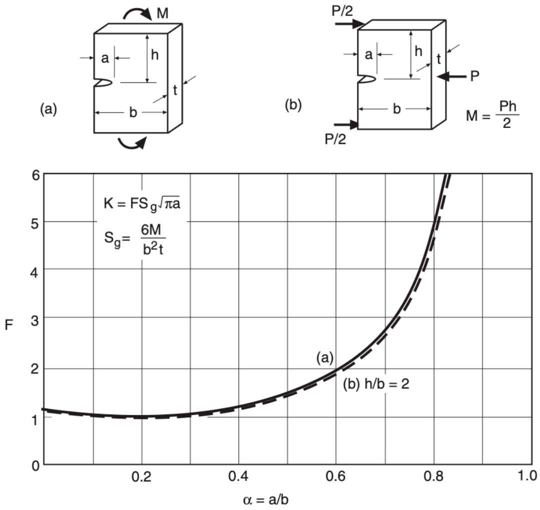
  As $X_a \propto X_K^2$, the value of X_a is usually much larger.

- On yield strength:

$$X_o = \frac{\sigma_o}{\bar{\sigma}}, \quad \bar{\sigma} = \bar{\sigma}_S \text{ or } \bar{\sigma}_H$$

- Need to make sure that there is no yielding and no crack-induced fracture through determining X_K or X_a and X_o.

Calculation of K Under Bending



Values for small a/b and limits for 10% accuracy:

$$\text{(a, b)}\quad K = 1.12 S_g \sqrt{\pi a} \qquad (a/b \le 0.4)$$

Expressions for any $\alpha = a/b$:

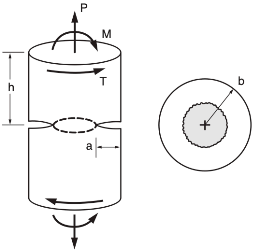
$$\text{(a)}\quad F = \sqrt{\frac{2}{\pi\alpha}\tan\frac{\pi\alpha}{2}}\left[\frac{0.923 + 0.199\left(1 - \sin\frac{\pi\alpha}{2}\right)^4}{\cos\frac{\pi\alpha}{2}}\right] \qquad (\text{large } h/b)$$

(b) F is within 3% of (a) for $h/b = 4$, and within 6% for $h/b = 2$, at any a/b:

$$F = \frac{1.99 - \alpha(1-\alpha)\left(2.15 - 3.93\alpha + 2.7\alpha^2\right)}{\sqrt{\pi}\,(1+2\alpha)(1-\alpha)^{3/2}} \qquad (h/b = 2)$$

Figure 8.13 Stress intensity factors for two cases of bending. Geometries, curves, and equations labeled (a) all correspond to the same case, and similarly for (b). Case (b) with $h/b = 2$ is the ASTM standard bend specimen. (Equations from [Tada 85] p. 2.14, and [ASTM 97] Std. E399.)

Calculation of K for Round Shaft



$K = FS_g\sqrt{\pi a}$

$\alpha = a/b$

$\beta = 1 - \alpha$

(a) Axial load P: $S_g = \dfrac{P}{\pi b^2}$, $\quad F = 1.12 \quad (10\%,\ a/b \leq 0.21)$

$$F = \frac{1}{2\beta^{1.5}}\left[1 + \frac{1}{2}\beta + \frac{3}{8}\beta^2 - 0.363\beta^3 + 0.731\beta^4\right]$$

(b) Bending moment M: $S_g = \dfrac{4M}{\pi b^3}$, $\quad F = 1.12 \quad (10\%,\ a/b \leq 0.12)$

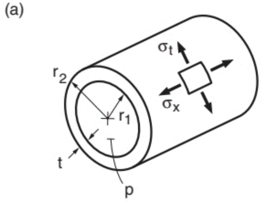
$$F = \frac{3}{8\beta^{2.5}}\left[1 + \frac{1}{2}\beta + \frac{3}{8}\beta^2 + \frac{5}{16}\beta^3 + \frac{35}{128}\beta^4 + 0.537\beta^5\right]$$

(c) Torsion T, $K = K_{\mathrm{III}}$: $S_g = \dfrac{2T}{\pi b^3}$, $\quad F = 1.00 \quad (10\%,\ a/b \leq 0.09)$

$$F = \frac{3}{8\beta^{2.5}}\left[1 + \frac{1}{2}\beta + \frac{3}{8}\beta^2 + \frac{5}{16}\beta^3 + \frac{35}{128}\beta^4 + 0.208\beta^5\right]$$

Leak-Before-Break Design

The stresss of cylindrical and spherical pressure vessels are given in figure below:

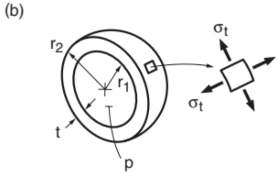


$$\sigma_t = \frac{pr_1}{t}, \qquad \sigma_x = \frac{pr_1}{2t}$$

$$(\sigma_x = 0\,,\ \text{open ends})$$

$$\sigma_r = -p \qquad \text{(inside)}$$

$$\sigma_r = 0 \qquad \text{(outside)}$$



$$\sigma_t = \frac{pr_1}{2t}$$

$$\sigma_r = -p \qquad \text{(inside)}$$

$$\sigma_r = 0 \qquad \text{(outside)}$$

Figure A.7 Approximate stresses in thin-walled pressure vessels, (a) tubular and (b) spherical. For (a), the approximations are within 5% for $t/r_1 < 0.1$, and 10% for $t/r_1 < 0.2$. For (b), they are within 5% for $t/r_1 < 0.3$, and 10% for $t/r_1 < 0.45$.

Leak-Before-Break Design I

We hope leak (crack length greater than wall thickness) occurs before brittle fracture, i.e.

$$a_c = \frac{1}{\pi}\left(\frac{K_{IC}}{FS_g}\right)^2 \geq t$$

For a cylindrical ressure vessel,

- Consider yield, as the effective stress according to Von Mises criterion is:

$$\bar{\sigma}_H \approx \frac{1}{\sqrt{2}}\sqrt{\left(\frac{PR}{t}\right)^2 + \left(\frac{PR}{2t}\right)^2 + \left(\frac{PR}{2t}\right)^2} = \frac{\sqrt{3}}{2}\frac{PR}{t}(R \gg t)$$

then $\bar{\sigma}_H < \sigma_y \Rightarrow \frac{PR}{t} < \frac{2}{\sqrt{3}}\sigma_y$

Leak-Before-Break Design II

- Consider fracture, as the maximum stress intensity factor is:

$$K_I = \frac{PR}{t}\sqrt{\pi a}$$

then $K_I < K_{IC} \Rightarrow \frac{PR}{t} < \frac{K_{IC}}{\sqrt{\pi a}}$

- The transition crack length is:

$$a_t = \frac{3}{4\pi}\left(\frac{K_{IC}}{F\sigma_y}\right)^2$$

Thank you!