Mechanical Behavior of Materials
Midterm Review

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Principle Stresses

- Principle normal stresses:

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_n = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

- Principle shear stress:

$$\tau_3 = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \frac{|\sigma_1 - \sigma_2|}{2}$$

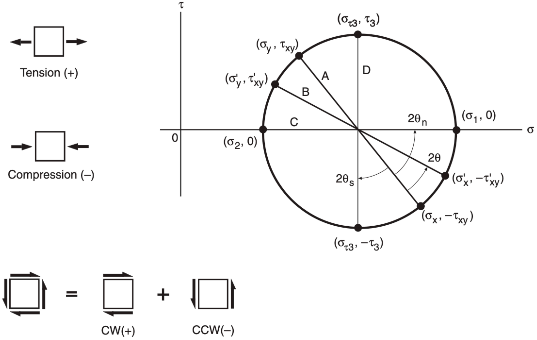
$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}, \quad \sigma_{\tau_3} = \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_1 + \sigma_2}{2}$$

- $|\theta_s - \theta_n| = \frac{\pi}{4}$

Mohr's Circle

Equation of Mohr's circle:

$$\left(\sigma - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$



- A rotation of 2θ on Mohr's circle corresponds to a rotation of θ for coordinate system in real space.
- Clockwise (CW) shear stress: positive.
- Counter Clockwise (CCW) shear stresses: negative

Generalized Plane Stress

If there is only one nonzero stress component:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{yx} & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{bmatrix}$$

then we have a state of generalized plane stress, and the stress normal to the plane of the nonzero is one of the principal stresses, i.e.

$$\sigma_3 = \sigma_z$$

- σ_z is an eigen value of the stress tensor.
- If τ_{xz} and τ_{yz} are nonzero, then need to solve $\det(\sigma - \lambda I) = 0$ to find the principal stresses.

- Maximum normal stress criterion:

$$\bar{\sigma}_N = \max\left(|\sigma_1|, |\sigma_2|, |\sigma_3|\right) = \sigma_u \text{ (at fracture)}$$

- Coulomb-Mohr fracture criterion

$$|\tau| + \mu\sigma = \tau_i \text{ (at fracture)}$$

- Maximum shear stress criterion (Tresca criterion)

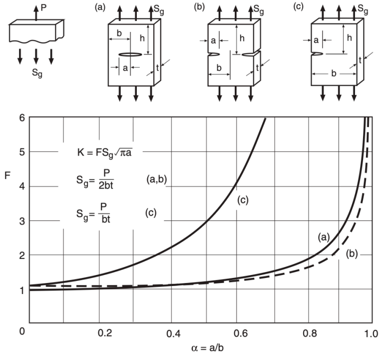
$$\bar{\sigma}_S = \max\left(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|\right) = \sigma_o \text{ (at yield)}$$

- Octahedral shear stress criterion (Von Mises criterion)

$$\begin{aligned}
\bar{\sigma}_H &= \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \\
&= \frac{1}{\sqrt{2}}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right)} \\
&= \sigma_o \text{ (at yield)}
\end{aligned}$$

General Steps for LEFM Problems

1. Check the crack configuratoin
2. Calculate $\alpha = \frac{a}{b}$
3. Calculate S_g
4. Calculate F according to α
5. Calculate stress intensity factor by $K = FS_g\sqrt{\pi a}$



Values for small a/b and limits for 10% accuracy:

(a) $K = S_g\sqrt{\pi a}$ $(a/b \leq 0.4)$

(b) $K = 1.12S_g\sqrt{\pi a}$ $(a/b \leq 0.6)$

(c) $K = 1.12S_g\sqrt{\pi a}$ $(a/b \leq 0.13)$

Expressions for any $\alpha = a/b$:

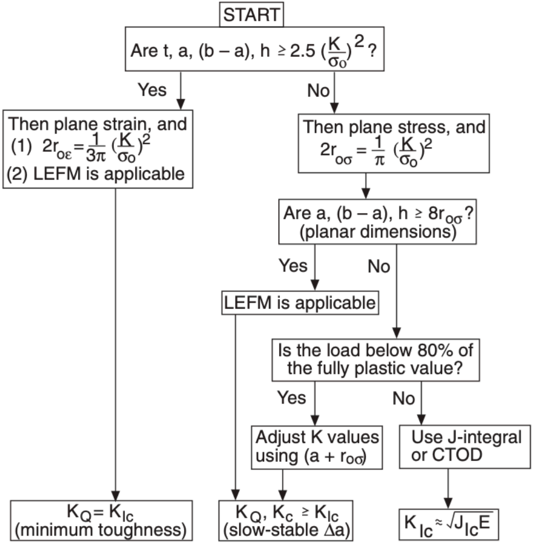
(a) $$F = \frac{1 - 0.5\alpha + 0.326\alpha^2}{\sqrt{1-\alpha}} \qquad (h/b \geq 1.5)$$

(b) $$F = \left(1 + 0.122\cos^4\frac{\pi\alpha}{2}\right)\sqrt{\frac{2}{\pi\alpha}\tan\frac{\pi\alpha}{2}} \qquad (h/b \geq 2)$$

(c) $$F = 0.265\,(1-\alpha)^4 + \frac{0.857 + 0.265\alpha}{(1-\alpha)^{3/2}} \qquad (h/b \geq 1)$$

Figure 8.12 Stress intensity factors for three cases of cracked plates under tension. Geometries, curves, and equations labeled (a) all correspond to the same case, and similarly for (b) and (c). (Equations as collected by [Tada 85] pp. 2.2, 2.7, and 2.11.)

Plasticity in Cracked Member



Exercise 1

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Which statement about the ultimate tensile strengths of ceramics is correct?

A. The ultimate tensile strength equals to the ultimate compressive strength based on the Coulomb-Mohr (CM) fracture criteria

B. The ultimate tensile strength is usually larger than the ultimate compressive strength

C. The ultimate tensile strength is dependent on the size of the largest crack present in the ceramics

D. None of the above

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D. None of the above

Answer: C

Exercise 2

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Which of the following statements concerning fracture toughness (K_C) is correct?

A. Fracture toughness is independent on temperature

B. Brittle materials usually have higher fracture toughness compared to ductile materials

C. Steels with small hardness usually have higher fracture toughness compared to those with large hardness

D. Components containing larger cracks have higher fracture toughness compared to those composed of the same material but containing small cracks

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Answer: C

Exercise 3

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An close ended vessel having inner radius $r_1 = 80$ mm and outer radius $r_2 = 85$ mm contains a pressure of 100 MPa. It is made of the AISI 4130 steel with $K_{IC} = 120\ \text{MPa}\cdot\sqrt{\text{m}}$, $\sigma_o = 1090$ MPa, $\sigma_u = 1250$ MPa. A small longitudinal crack with a half width of 1 mm along tube direction is present. $K = FS_g\sqrt{\pi a}$. For simplicity, the F factor is assumed to be 1 for this question.

(a) Whether this vessel will meet the leak-before-fracture design purpose? Please show the evidence.

(b) Please determine the safety factor for this cracked vessel against brittle fracture and against yielding?

(c) Calculate the transition crack length a_t.

(d) If the gas in the vessel will cause the diffusion of hydrogen into the steel and segregates towards the grain boundaries and defects, how the a_t in (c) and safety factors in (b) will change?

Exercise 3

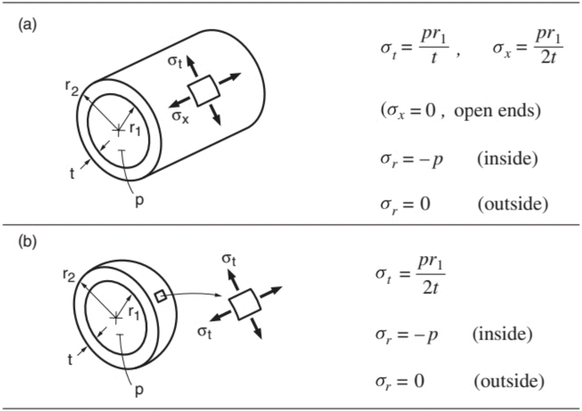


Figure A.7 Approximate stresses in thin-walled pressure vessels, (a) tubular and (b) spherical. For (a), the approximations are within 5% for $t/r_1 < 0.1$, and 10% for $t/r_1 < 0.2$. For (b), they are within 5% for $t/r_1 < 0.3$, and 10% for $t/r_1 < 0.45$.

Exercise 3 I

(a) The thickness of the vessel is

$$t = r_2 - r_1 = 85 \text{ mm} - 80 \text{ mm} = 5 \text{ mm}$$

The critical crack length for brittle fracture is

$$a_c = \frac{1}{\pi}\left(\frac{K_{IC}}{F\sigma_t}\right)^2 = \frac{1}{\pi}\left(\frac{120 \text{ MPa}\cdot\sqrt{\text{m}}}{1 \times 1600 \text{ MPa}}\right)^2 = 1.79 \text{ mm} < 5 \text{ mm}$$

Therefore, the vessel will not meet the leak-before-fracture design purpose.

Exercise 3 II

(b) The safety factor against brittle fracture is

$$X_K = \frac{K_{IC}}{F\sigma_t\sqrt{\pi a}} = \frac{120\ \text{MPa}\cdot\sqrt{\text{m}}}{1\times 1600\ \text{MPa}\sqrt{\pi\times 1\ \text{mm}}} = 1.34$$

According to the maximum shear stress criterion,

$$\bar{\sigma}_S = \max\left(|\sigma_1-\sigma_2|, |\sigma_2-\sigma_3|, |\sigma_3-\sigma_1|\right) = 1700\ \text{MPa}$$

Then the safety factor against yielding is

$$X_o = \frac{\sigma_o}{\bar{\sigma}_S} = \frac{1090\ \text{MPa}}{1700\ \text{MPa}} = 0.64$$

(c) The transition crack length is

$$a_t = \frac{1}{\pi}\left(\frac{K_{IC}}{F\sigma_o}\right) = \frac{1}{\pi}\left(\frac{120\ \text{MPa}\cdot\sqrt{\text{m}}}{1\times 1090\ \text{MPa}}\right)^2 = 3.86\ \text{mm}$$

Exercise 3 III

(d) The diffusion of hydrogen into the steel and segregation towards grain boundaries and defects will introduce impurities to the steel, resulting K_{IC} to decrease. Therefore, a_t and X_K will decrease. Also, the yield strength will increase due to lattice distortion, thus X_o will increase.

Fatigue Stress-Based Approach

- Basquin's law (used for fully-reversed, high-cycle fatigue, i.e. $\sigma_m = 0$ and $\sigma_a < \sigma_o$):

$$\sigma_{ar} = AN_f^B = \sigma_f'(2N_f)^b$$

- Life estimate with $\sigma_m \neq 0$ (mean stress effect):

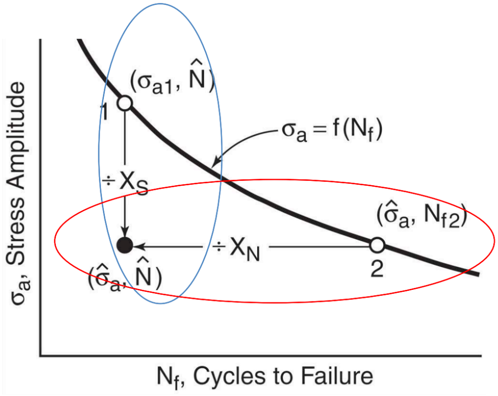
$$\sigma_a = \frac{\sigma_{max} - \sigma_{min}}{2}, \quad N_f = \frac{1}{2}\left(\frac{\sigma_a}{\sigma_f' - \sigma_m}\right)^{\frac{1}{b}} \quad \text{(Morrow)}$$

$$\sigma_{ar} = \sqrt{\sigma_{max}\sigma_a}, \quad N_f = \frac{1}{2}\left(\frac{\sigma_{ar}}{\sigma_f'}\right)^{\frac{1}{b}} \quad \text{(SWT)}$$

- Palmgren-Miner rule (for variable amplitude loading):

$$B_f \sum_i \frac{N_i}{N_{fi}} = 1$$

Safety Factor for $S - N$ Curves



- $(\hat{\sigma}_a, \hat{\sigma}_m, \hat{N}) \Rightarrow (\hat{\sigma}_{ar}, \hat{N})$
- In stress:

$$X_S = \frac{\sigma_{ar1}}{\hat{\sigma}_{ar}}, \left(N_f = \hat{N}\right)$$

- In life:

$$X_N = \frac{N_{f2}}{\hat{N}}, (\sigma_{ar} = \hat{\sigma}_{ar})$$

- According to Basquin's law,

$$X_S = \frac{A\hat{N}^B}{AN_{f2}^B} = X_N^{-B} \quad \text{or} \quad X_N = X_S^{-\frac{1}{B}}$$

Factors Affecting Fatigue Behaviors

- Mean stress effect:
  - Tensile mean stress $\Rightarrow$ shorter fatigue life, lower fatigue strength
  - Compressive mean stress $\Rightarrow$ longer fatigue life, higher fatigue strength
- Notch effect:
  - Stress concentration factor: $k_t = \frac{\sigma_y}{S} = 1 + \sqrt{\frac{c}{\rho}}$
  - $\rho \downarrow \Rightarrow k_t \uparrow \Rightarrow N_f \downarrow, \sigma_a \downarrow$
- Temperature and frequency effect:
  - Usually, lower T and higher cycling frequency leads to higher fatigue strength
- Microstructure effect:
  - Increased dislocation density, smaller grain size $\Rightarrow$ higher fatigue strength

Exercise 4

Exercise 4

What of the following approaches will decrease the fatigue limit of a steel?

A. Increasing the strength of the steel through cold working
B. Introducing sharp corner to have stress concentration
C. Increasing the frequency of cyclic loading
D. None of the above

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A. Increasing the strength of the steel through cold working

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C. Increasing the frequency of cyclic loading

D. None of the above

Answer: B

Notch Effect

- The fatigue notch factor k_f is defined as:

$$k_f = \frac{\sigma_{ar}}{S_{ar}}$$

- Notch sensitivity of fatigue is defined as:

$$q = \frac{k_f - 1}{k_t - 1} \in [0, 1]$$

where $k_f < k_t$; for brittle materials with large notch radius $k_f \approx k_t$

- Empirical estimates of k_f:

$$k_f = 1 + q(k_t - 1) = \begin{cases} 1 + \dfrac{k_t - 1}{1 + \alpha/\rho} & (Peterson) \\ 1 + \dfrac{k_t - 1}{1 + \sqrt{\beta/\rho}} & (Neuber) \end{cases}$$

where α, β are material constants and ρ is the notch radius.

Exercise 5

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Which one of the following statements about notch effect on fatigue is wrong?

A. The notch sensitivity is a material property and independent on notch radius
B. The fatigue notch factor is usually smaller than the stress concentration factor
C. The fatigue strength of brittle material is usually notch sensitive
D. None of the above

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Which one of the following statements about notch effect on fatigue is wrong?

A. The notch sensitivity is a material property and independent on notch radius

B. The fatigue notch factor is usually smaller than the stress concentration factor

C. The fatigue strength of brittle material is usually notch sensitive

D. None of the above

Answer: A

Fatigue Crack Growth I

- a_f: crack length at fatigue failure ($= \min\{a_c, a_o\}$)
- a_c: critical crack length for brittle fracture

$$a_c = \frac{1}{\pi}\left(\frac{K_c}{FS_{max}}\right)^2$$

- a_o: critical crack length for fully plastic yielding

$$a_o = b - \frac{P_{max}}{2t\sigma_o}$$

- Walker's equation:

$$C = \frac{C_0}{(1-R)^{m(1-\gamma)}}$$

Fatigue Crack Growth II

- Range of stress intensity factor:

$$\Delta K = F \Delta S \sqrt{\pi a} = \frac{F_p \Delta P}{t \sqrt{b}}$$

- Crack growth life:

$$N_{if} = \frac{a_f^{1-\frac{m}{2}} - a_i^{1-\frac{m}{2}}}{C\left(F \Delta S \sqrt{\pi}\right)^m \left(1 - \frac{m}{2}\right)} \quad (m \neq 2)$$

- Safety factor:

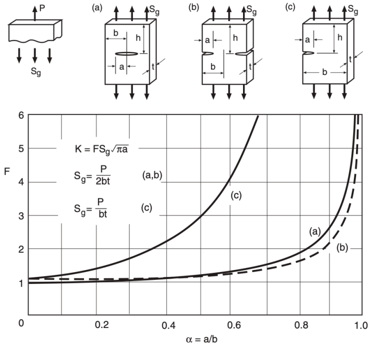
$$X_c = \frac{S_c}{S_{max}}, \quad X_N = \frac{N_{if}}{N_p}$$

Exercise 6

Example 11.4

A center-cracked plate of the AISI 4340 steel ($\sigma_u = 1296\,\text{MPa}$) of Table 11.2 has dimensions, as defined in Fig. 8.12(a), of $b = 38$ and $t = 6\,\text{mm}$, and it contains an initial crack of length $a_i = 1\,\text{mm}$. It is subjected to tension-to-tension cyclic loading between constant values of minimum and maximum force, $P_{\text{min}} = 80$ and $P_{\text{max}} = 240\,\text{kN}$.

(a) At what crack length a_f is failure expected? Is the cause of failure yielding or brittle fracture?
(b) How many cycles can be applied before failure occurs?
(c) Assume that this member is an engineering component that is expected to be subjected to 150,000 cycles in its service life, and further assume that a safety factor of three on life is required. If $a_i = 1$ mm is the minimum detectable crack length a_d for inspection, are periodic inspections required? If so, at what interval?
(d) Consider the possibility of avoiding periodic inspections by improved initial inspection, such that a smaller a_i can be justified. What new $a_i = a_d$ would be required?



	Yield	Toughness	Walker Equation				
Material	σ_o	K_{Ic}	C_0	C_0	m	γ	γ
	MPa (ksi)	MPa $\sqrt{m}$ (ksi$\sqrt{in}$)	$\frac{mm/cycle}{(MPa\sqrt{m})^m}$	$\frac{in/cycle}{(ksi\sqrt{in})^m}$		$(R \geq 0)$	$(R < 0)$
AISI 4340 steel ($\sigma_u = 1296$ MPa)	1255 (182)	130 (118)	5.11×10^{-10}	2.73×10^{-11}	3.24	0.420	0

Exercise 6 I

Solution **(a)** The crack length at fully plastic yielding can be estimated from Fig. A.16(a):

$$a_o = b\left(1 - \frac{P_{\max}}{2bt\sigma_o}\right) = (38\,\text{mm})\left(1 - \frac{240{,}000\,\text{N}}{2(38\,\text{mm})(6\,\text{mm})(1255\,\text{MPa})}\right) = 22.1\,\text{mm}$$

The yield strength (and also K_{Ic}) is obtained from Table 11.2.

The crack length a_c at brittle fracture is given by Eq. 11.33:

$$a_c = \frac{1}{\pi}\left(\frac{K_{Ic}}{F S_{\max}}\right)^2$$

With reference to Fig. 8.12(a), an initial estimate of a_c may be made by assuming that $a_c/b \le 0.4$, so that $F \approx 1$. We obtain

$$S_{\max} = \frac{P_{\max}}{2bt} = \frac{240{,}000\,\text{N}}{2(38\,\text{mm})(6\,\text{mm})} = 526\,\text{MPa}$$

$$a_c \approx \frac{1}{\pi}\left(\frac{K_{Ic}}{F S_{\max}}\right)^2 = \frac{1}{\pi}\left(\frac{130\,\text{MPa}\sqrt{\text{m}}}{1(526\,\text{MPa})}\right)^2 = 0.0194\,\text{m} = 19.4\,\text{mm}$$

Exercise 6 II

Table E11.4

Calc. No.	Trial a mm	$\alpha = a/b$	F	$K_{\max} = F S_{\max}\sqrt{\pi a}$ MPa$\sqrt{\text{m}}$
1	15	0.395	1.097	125.3
2	16	0.421	1.114	131.3
3	15.77	0.416	1.110	130.0

This corresponds to $a_c/b = 0.51$, which is beyond the region of 10% accuracy for $F \approx 1$. A trial and error solution, as in Ex. 8.1(c), is thus needed, with F taken from Fig. 8.12(a). This is shown in Table E11.4. The final K value is $K_{Ic} = 130\,\text{MPa}\sqrt{\text{m}}$ so that $a_c = 15.8\,\text{mm}$. Since this is smaller than a_o, brittle fracture determines the controlling value a_f, and

$$a_f = 15.8\,\text{mm} \qquad \textbf{Ans.}$$

Exercise 6 III

(b) If F is approximately constant, Eq. 11.32 can be employed to calculate N_{if} by substituting either the initial F or an intermediate value that is biased toward the initial one:

$$N_{if} = \frac{a_f^{1-m/2} - a_i^{1-m/2}}{C\left(F\,\Delta S\sqrt{\pi}\right)^m (1 - m/2)}$$

In this case, the value increases from $F_i = 1.00$ to $F_f = 1.11$. So the variation is small enough that constant F is a reasonable assumption, and we can use $F = 1.00$ for the N_{if} calculation. If we note that Table 11.2 gives constants for the Walker equation, we see that the nonzero R-ratio for the applied load can be handled by calculating a C value from Eq. 11.20 as follows:

$$R = \frac{S_{\min}}{S_{\max}} = \frac{P_{\min}}{P_{\max}} = \frac{80}{240} = 0.333$$

$$C = \frac{C_0}{(1-R)^{m(1-\gamma)}} = \frac{5.11 \times 10^{-10}}{(1-0.333)^{3.24(1-0.42)}} = 1.095 \times 10^{-9}\,\frac{\text{mm/cycle}}{(\text{MPa}\sqrt{\text{m}})^m}$$

However, substitution into the equation for N_{if} is most convenient if all quantities have units consistent with $\text{MPa}\sqrt{\text{m}}$ as used for ΔK, requiring a units conversion for C as follows:

$$C = 1.095 \times 10^{-9}\,\frac{\text{mm/cycle}}{(\text{MPa}\sqrt{\text{m}})^m} \times \frac{1\ \text{m}}{1000\ \text{mm}} = 1.095 \times 10^{-12}\,\frac{\text{m/cycle}}{(\text{MPa}\sqrt{\text{m}})^m}$$

Two additional calculations are useful before computing N_{if}:

$$\Delta S = S_{\max}(1-R) = 526(0.667) = 351\ \text{MPa}$$

$$\left(1 - \frac{m}{2}\right) = \left(1 - \frac{3.24}{2}\right) = -0.62$$

Exercise 6 IV

Substituting the various numerical values finally gives N_{if}:

$$N_{if} = \frac{(0.0158\ \text{m})^{-0.62} - (0.001\ \text{m})^{-0.62}}{\left(1.095 \times 10^{-12} \dfrac{\text{m/cycle}}{(\text{MPa}\sqrt{\text{m}})^m}\right)(1.00 \times 351\ \text{MPa} \times \sqrt{\pi})^{3.24}(-0.62)}$$

$$N_{if} = 77{,}600 \text{ cycles} \qquad \textbf{Ans.}$$

In the preceding substitutions, note that all units are meters, MPa, or combinations of these. Careful checking indicates that these all cancel, leaving only "cycles."

(c) With no periodic inspections, the safety factor on life from Eq. 11.2 is

$$X_N = \frac{N_{if}}{\hat{N}} = \frac{77{,}600}{150{,}000} = 0.52$$

Hence, failure is expected before the end of the service life, so inspections are clearly needed. For the required $X_N = 3$, the inspection interval can be obtained from Eq. 11.5:

$$N_p = \frac{N_{if}}{X_N} = \frac{77{,}600}{3} = 25{,}900 \text{ cycles} \qquad \textbf{Ans.}$$

Exercise 6 V

(d) To avoid periodic inspections and satisfy $X_N = 3$, we need a new, smaller $a_i = a_d$ such that N_{if} is

$$N_{if} = X_N \hat{N} = 3(150{,}000) = 450{,}000 \text{ cycles}$$

Equation 11.32 is needed again, but now with N_{if} known and a_i unknown. Noting that the same values of a_f, C, m, F, and ΔS apply as in (b), and handling units as before, we have the following substitutions:

$$450{,}000 = \frac{(0.0158)^{-0.62} - a_i^{-0.62}}{(1.095 \times 10^{-12})(1.00 \times 351\sqrt{\pi})^{3.24}(-0.62)}$$

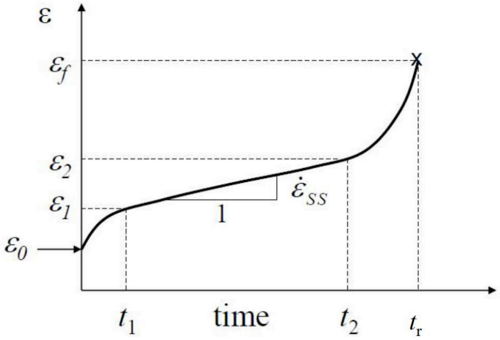
Solving for a_i gives

$$a_i = a_d = 7.63 \times 10^{-5}\text{m} = 0.0763\,\text{mm} \qquad \textbf{Ans.}$$

According to the earlier discussion in Section 11.2.1, this very small a_d is probably below the limits of any reasonable inspection. Hence, periodic inspection would be difficult to avoid in this case unless it is possible to lower the applied load through redesign or restrictions on the use of the component.

Creep

- ε_0: $t = 0$, instantaneous elongation
- Primary creep $(0 < t < t_1)$: transient
- Secondary creep $(t_1 < t < t_2)$: steady state or viscous creep
- Teritary creep $(t_2 < t < t_r)$: unstable
- Rupture $(t = t_r)$



Long term creep stress rupture:

$$P(L.M.) = T(\log t_r + C) = b_0 + b_1\sigma$$

where b_0, b_1, C are constants, T is temperature (in K), t_r is stress-rupture time (in h).

Exercise 7

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Which of the following methods would decrease the creep life time?

A. Increase the static stress in service
B. Select materials with high melting temperature
C. Use coarse-grained materials with larger crystal grain
D. Add stiffer second phase

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Answer: A

Thank you!