Homework 1

Due : Friday, September 19 at 11 : 59 pm

Deliverables. Submit a single PDF of your write-up to Gradescope *HW1 Write-Up*.

Start each problem on a new page.

Honor Code

Write and sign the following statement:

> "I certify that all solutions in this document are entirely my own and that I have not looked at anyone else's solution. I have given credit to all external sources I consulted."

Linear Algebra

1. **System of equations (3 points).** Consider the linear system

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 6 \\ 3 & 6 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}.$$

   How many solutions does this system have? Explain your reasoning.

2. **Asymptotic powers of 2×2 matrices (6 points).** For each matrix below, determine the behavior of

$$\lim_{n \to \infty} M^n$$

Hint: Use the eigen-decomposition $M = PDP^{-1}$, where $D = \{\lambda_1, \lambda_2, ...\}$ is a diagonal matrix of the eigenvalues. What would be the formula for M^n?

(a) $M_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) $M_2 = \begin{bmatrix} 2 & -5 \\ 1/2 & -7/6 \end{bmatrix}$.

(c) $M_3 = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$.

3. **Singular Value Decomposition (4 points).** Given the matrix

$$A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$$

Find a unit vector x ($\|x\| = 1$) for which $\|Ax\|$ is maximized.

Hint: Recall from SVD that $A = U\Sigma V^\top$, where A maps each right singular vector (a column of V) to a left singular vector (a column of U), scaled by the corresponding singular value. For which right singular vector does this scaling make $\|Ax\|$ the largest?

4. **Image Flipping (6 points).** *This question is a warmup to problem 4 in the coding section.* Consider a tiny 3×3 image represented by matrix I and the same image flipped on the Y axis I_{flip}:

$$I = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}, \quad I_{\text{flip}} = \begin{bmatrix} x_7 & x_8 & x_9 \\ x_4 & x_5 & x_6 \\ x_1 & x_2 & x_3 \end{bmatrix}.$$

   (a) What is the size of the transformation matrix T that performs this flip? *Hint: You can first convert I to a 1×9 vector, make transformation using matrix T, and then convert I_{flip} back to a 3×3 matrix.*

   (b) Construct T and verify that $I \times T$ produces I_{flip}.

   (c) Describe an algorithm for constructing the vertical-flip transformation matrix for any $N \times N$ matrix (either text explanation or pseudocode is acceptable). Show that $I \times T$ produces I_{flip}.

   (d) Describe how you would modify this algorithm to do a *horizontal* flip (along the X ais).

Calculus

5. **Partial Derivatives (8 points). (First and second derivatives)**

   (a) For $f(x_1, x_2) = x_1^3 + x_2^3 - 3x_1x_2$, find all the first and second order partial derivatives.

   (b) Let $f(x, y) = 4x^2 + y^2 - 8xy + 4x + 6y - 10$.
   Find the critical points by solving

   $$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0$$

   simultaneously and determine which point(s) yield a minimum value.

   (c) For $f(x, y) = e^{xy} + x^2y$, compute:

   (i) $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

   (ii) $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, and $\frac{\partial^2 f}{\partial x \partial y}$

   (d) For $f(x, y) = \ln(x^2 + y^2 + 1)$, compute:

   (i) $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

   (ii) $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, and $\frac{\partial^2 f}{\partial x \partial y}$

6. **Recursive expression and derivatives (6 points).** Suppose we have variables $z_1, z_2, \ldots, z_n$, $w_1, \ldots, w_{n-1}$, and $b_1, \ldots, b_{n-1} \in \mathbb{R}$, where

$$z_n = w_{n-1} z_{n-1} + b_{n-1}.$$

   (a) Compute $\frac{dz_k}{dz_{k-1}}$ for $k = 2, \ldots, n$.

   (b) Compute $\frac{dz_n}{dz_1}$.

   (c) Compute $\frac{dz_n}{db_1}$. You may express your answers in terms of $z_1, ..., z_n, w_1, ..., w_{n-1}, b_1, ..., b_{n-1}$

Probability

7. **Conditioned uniform difference (3 points).** Let $X, Y \overset{\text{iid}}{\sim} \text{Unif}(-1, 1)$. Compute

$$\mathbb{P}\big(|X - Y| \leq 0.5 \,\big|\, X\,Y > 0\big).$$

Hint: Try drawing a 2D cartesian plane where the horizontal and vertical axes represent X and Y respectively and each range from -1 to 1. For which quadrants is it true that $XY > 0$? Within these quadrants, how can we visualize the region $|X - Y| < 0.5$?

8. **Nearest–neighbor arc length (4 points).** Suppose you select 20 i.i.d. points $X_1, \ldots, X_{20}$ uniformly at random on the circumference of the unit circle.

   (a) Let D be the shortest arc distance from X_1 to the nearest of the other 19 points. Calculate $\mathbb{P}(D > t)$ where $0 \leq t \leq \frac{1}{2}$.

   Hint: What does the event $\{D > t\}$ mean in terms of where the other 19 points can be located relative to X_1?

   (b) Find the expected arc length (in degrees) between X_1 and the point nearest to it.

   Hint: The Wikipedia page on expected value of continuous variables *may be helpful here.*

9. **Cancer screening (3 points).** A medical test has sensitivity 90% and false–positive rate 3%:

$$\mathbb{P}(T = 1 \mid C = 1) = 0.9, \qquad \mathbb{P}(T = 1 \mid C = 0) = 0.03.$$

Suppose the disease prevalence is very low, $\mathbb{P}(C = 1) = 0.001$. Compute the posterior probability of disease given a positive test:

$$\mathbb{P}(C = 1 \mid T = 1).$$

10. **Follower Counts (3 points).** Suppose 420 people are sitting uniformly at random around a circle, each with a *distinct* number of TikTok followers. What is the expected number of people whose follower count is higher than both their immediate neighbors?