$$\textbf{Problem.}\quad \begin{cases} x_{n+1} = 3x_n - 6y_n - z_n, \\ y_{n+1} = -x_n + 2y_n + z_n, \quad x_1 = -2,\ y_1 = 1,\ z_1 = -1. \\ z_{n+1} = x_n + 3y_n - z_n, \end{cases}$$

$$\lim_{n\to\infty} \frac{x_n + y_n + z_n}{3^n + 5^n}.$$

Step 1. Write the recurrence in matrix form

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{bmatrix} = \begin{bmatrix} 3 & -6 & -1 \\ -1 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}.$$

Suppose

$$A = \begin{bmatrix} 3 & -6 & -1 \\ -1 & 2 & 1 \\ 1 & 3 & -1 \end{bmatrix}, \quad \mathbf{v}_{n+1} = A\mathbf{v}_n, \quad \mathbf{v}_n = A^{n-1}\mathbf{v}_1.$$

Step 2. Find eigenvalues of A

We compute the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & -6 & -1 \\ -1 & 2-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = \lambda^3 - 4\lambda^2 - 7\lambda + 10.$$

Hence

$$(\lambda + 2)(\lambda - 1)(\lambda - 5) = 0.$$

So the eigenvalues are

$$\lambda_1 = -2, \quad \lambda_2 = 1, \quad \lambda_3 = 5.$$

Step 3. Find corresponding eigenvectors

$$v_{-2} = \begin{bmatrix} -\frac{1}{7} \\ -\frac{2}{7} \\ 1 \end{bmatrix}, \qquad v_1 = \begin{bmatrix} \frac{5}{4} \\ \frac{1}{4} \\ 1 \end{bmatrix}, \qquad v_5 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}.$$

Then

$$A = PDP^{-1}, \quad P = [v_{-2}\ v_1\ v_5], \quad D = \operatorname{diag}(-2,\ 1,\ 5).$$

Step 4. Decompose the initial vector

Write

$$\mathbf{v}_1 = c_1 v_{-2} + c_2 v_1 + c_3 v_5.$$

Solving gives

$$c_1 = -1, \quad c_2 = 0, \quad c_3 = \tfrac{5}{7}.$$

Thus

$$\mathbf{v}_n = -(-2)^{n-1} v_{-2} + \tfrac{5}{7}\, 5^{\,n-1} v_5.$$

Step 5. Asymptotic behavior

For large n,

$$\mathbf{v}_n \sim \tfrac{5}{7}\, 5^{\,n-1} v_5.$$

Hence

$$x_n + y_n + z_n \sim \tfrac{5}{7}\, 5^{\,n-1}(-3+1+0) = -\tfrac{10}{7}\, 5^{\,n-1}.$$

Step 6. Compute the limit

$$\lim_{n\to\infty} \frac{x_n + y_n + z_n}{3^n + 5^n} = \lim_{n\to\infty} \frac{-\frac{10}{7} 5^{\,n-1}}{5^n(1 + (3/5)^n)} = -\frac{2}{7}.$$

$$\boxed{\lim_{n\to\infty} \frac{x_n + y_n + z_n}{3^n + 5^n} = -\frac{2}{7}.}$$