偏导数的注意事项

性质1：链式求导法则

$$\frac{\partial f(u)}{\partial x}=\frac{d(f(u))}{du}\cdot\frac{\partial u}{\partial x}=f'(u)\cdot\frac{\partial u}{\partial x}$$

类似的有：

$$\frac{dz}{dx}=\frac{df(u,v)}{dx}=\frac{\partial f}{\partial u}\cdot\frac{du}{dx}+\frac{\partial f}{\partial v}\cdot\frac{dv}{dx}$$

其中

$$u=u(x),v=v(x)$$

problem1

> 设函数$f(u)$具有二阶连续导数，而$z=f(e^x\sin y)$满足$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=e^{2x}z$，求$f(u)$（1997）

我们的目标是通过链式求导法则，将原方程中关于 x 和 y 的偏导数，全部用关于 u 的导数（即 f′(u) 和 f′′(u)）来表示，从而得到一个只含有 f(u) 及其导数的常微分方程，然后解出 f(u)。

解答：

$$dz=df(e^x\sin y)=df(u)=\frac{df}{du}\cdot(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy)=f'(u)\cdot(e^x\sin y+e^x\cos y)=f'(u)(udx+e^x\cos ydy)$$
$$\frac{\partial z}{\partial x}=f'(u)\cdot u$$
$$\frac{\partial z}{\partial y}=f'(u)e^x\cos y$$
$$\frac{\partial\frac{\partial z}{\partial x}}{\partial x}=\frac{\partial^2 z}{\partial x^2}=\frac{\partial(f'(u)\cdot u)}{\partial x}=\frac{\partial u}{\partial x}\cdot f'(u)+\frac{\partial(f'(u))}{\partial x}\cdot u=uf'(u)+\frac{df'(u)}{du}\cdot\frac{\partial u}{\partial x}=uf'(u)+f''(u)\cdot u$$
$$\frac{\partial\frac{\partial z}{\partial y}}{\partial y}=\frac{\partial^2 z}{\partial y^2}=\frac{\partial(f'(u)\cdot e^x\cdot\cos y)}{\partial y}=-uf'(u)+e^{2x}\cdot\cos^2 y\cdot f''(u)$$
$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=e^{2x}\cdot f''(u)=e^{2x}\cdot f(u)$$
$$f''(u)=f(u)$$
$$f(u)=c1\cdot e^x+c2\cdot e^{-x}$$
$$c1,c2\in R$$

problem 2

> 设函数$f(u)$在$(0,+\infty)$内具有二阶导数，且$z=f\left(\sqrt{x^2+y^2}\right)$满足等式$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$，
>
> 验证$f''(u)+\frac{f'(u)}{u}=0$（2006）

解答：

part 1:

$$假设：x = r\cos\theta, y = r\sin\theta$$
$$z = f\cdot(\sqrt{x^2+y^2}) = f(r)$$
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$
$$因为z与\theta无关系$$
$$所以可以得到：\frac{\partial z}{\partial x} = f'(r)\cdot\frac{\partial r}{\partial x}$$
$$其中\frac{\partial r}{\partial x} = cos\theta$$
$$同理：$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}$$
$$得到：\begin{matrix}\frac{\partial z}{\partial x} = f'(r)\cdot\cos\theta \\ \frac{\partial z}{\partial y} = f'(r)\cdot\sin\theta\end{matrix}$$

part2:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial(\frac{\partial z}{\partial x})}{\partial x} = \frac{\partial(f'(r)\cos\theta)}{\partial x} = \frac{\partial(f'(r)\cos\theta)}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial(f'(r)\cos\theta)}{\partial\theta}\cdot\frac{\partial\theta}{\partial x} = \frac{f'(r)}{r}$$
$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial(\frac{\partial z}{\partial x})}{\partial y} = \frac{\partial(f'(r)\sin\theta)}{\partial y} = \frac{\partial(f'(r)\sin\theta)}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial(f'(r)\sin\theta)}{\partial\theta}\cdot\frac{\partial\theta}{\partial y} = f''(r)$$

part 3：得到答案

性质2

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

其中有:

极坐标与直角坐标偏导数全解

在进行二维坐标系变换时，我们经常需要在直角坐标 (x, y) 和极坐标 (r, θ) 之间转换。理解它们之间相互的偏导数关系是至关重要的。

我们有两组基础的换元关系式:

1. 用极坐标表示直角坐标:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

2. 用直角坐标表示极坐标:

$$\begin{cases} r = \sqrt{x^2+y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \end{cases}$$

我们将利用这两组关系式来求出全部八个偏导数。

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1. $\frac{\partial x}{\partial r}$

将 θ 视为常数，对 $x = r\cos\theta$ 求 r 的偏导:

$$\frac{\partial x}{\partial r} = \cos\theta$$

2. $\frac{\partial x}{\partial \theta}$

将 r 视为常数，对 $x = r\cos\theta$ 求 θ 的偏导:

$$\frac{\partial x}{\partial \theta} = -r\sin\theta$$

3. $\frac{\partial y}{\partial r}$

将 θ 视为常数，对 $y = r\sin\theta$ 求 r 的偏导：

$$\frac{\partial y}{\partial r} = \sin\theta$$

4. $\frac{\partial y}{\partial \theta}$

将 r 视为常数，对 $y = r\sin\theta$ 求 θ 的偏导：

$$\frac{\partial y}{\partial \theta} = r\cos\theta$$

前面四个内容相对来说比较基础，不必记忆，当x作为变量的时候才会相对难处理

5. $\frac{\partial r}{\partial x}$

$\frac{\partial r}{\partial x} = \frac{r\cos\theta}{r} = \cos\theta \frac{\partial r}{\partial x} = \frac{r\cos\theta}{r} = \cos\theta$

6. $\frac{\partial r}{\partial y}$

用极坐标表示结果：

$$\frac{\partial r}{\partial y} = \frac{r\sin\theta}{r} = \sin\theta$$

7. $\frac{\partial \theta}{\partial x}$

用极坐标表示结果：

$$\frac{\partial \theta}{\partial x} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$

8. $\frac{\partial \theta}{\partial y}$

用极坐标表示结果：

$$\frac{\partial \theta}{\partial y} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r}$$

problem 3

设函数$u = u(x,y,z)$具有二阶连续偏导数，且满足关系式$\frac{u_x}{x} = \frac{u_y}{y} = \frac{u_z}{z}$

（1）证明：在球面坐标下，u仅依赖于ρ，而与θ和φ无关

（2）若u还满足关系式$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$，试求出$u(x,y,z)$的表达式

解答：

第一问

$\frac{\partial u}{\partial x} = u_x \quad \frac{\partial u}{\partial y} = u_y \quad \frac{\partial u}{\partial z} = u_z$

要说明与其无关，就是说明$\frac{\partial u}{\partial \rho} \neq 0$, $\frac{\partial u}{\partial \theta} = 0$, $\frac{\partial u}{\partial \varphi} = 0$

$$\begin{cases} x=\rho\sin\varphi\cos\theta \\ y=\rho\sin\varphi\sin\theta \\ z=\rho\cos\varphi \end{cases}$$

$\nabla = (u_x, u_y, u_z)$ 梯度，又因为：$\frac{u_x}{x} = \frac{u_y}{y} = \frac{u_Z}{z}$ 说明梯度改变的方向与（x,y,z）的位置直接相关与（x,y,z)的方向无关

从代数上理解

先计算$\frac{\partial u}{\partial \theta}$

$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial \theta} = u_x\cdot(-\sin\theta)\cdot\sin\varphi\cdot\rho + u_y\cdot\cos\theta\cdot\sin\varphi\cdot\rho =$

$k\cdot x\cdot(-\sin\theta)\cdot\sin\varphi\cdot\rho + k\cdot y\cdot\cos\theta\cdot\sin\varphi\cdot\rho = k\rho(-x\sin\theta\sin\varphi + y\cos\theta\sin\varphi - z\sin\varphi) = k\rho\cdot(\rho\sin^2\varphi\sin\theta\cos\theta\cdot(-1) + \rho$

再计算$\frac{\partial u}{\partial \varphi}$:

$$\frac{\partial u}{\partial \varphi}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \varphi}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \varphi}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial \varphi}=u_x\cdot\rho\cos\varphi\cos\theta+u_y\cdot\rho\cos\varphi\sin\theta+u_z\cdot\rho(-\sin\varphi)$$
$$=k\rho^2\cdot(\sin^2\varphi\cos\theta\sin\theta+\cos^2\varphi\sin\theta\cos\theta-\sin\theta\cos\theta)=0$$

第二问:

先假设:

$$\frac{\partial u}{\partial x}=\frac{dF(\rho)}{d\rho}\cdot\frac{\partial \rho}{\partial x}=\frac{x}{\sqrt{x^2+y^2+z^2}}\cdot F'(\rho)$$
$$\frac{\partial u}{\partial y}=\frac{dF(\rho)}{d\rho}\cdot\frac{\partial \rho}{\partial y}=\frac{y}{\sqrt{x^2+y^2+z^2}}\cdot F'(\rho)$$
$$\frac{\partial u}{\partial z}=\frac{dF(\rho)}{d\rho}\cdot\frac{\partial \rho}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}\cdot F'(\rho)$$

第二步：计算二阶偏导数

使用乘法法则和链式法则：

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=\frac{\partial}{\partial x}\left(F'(\rho)\frac{x}{\rho}\right)\\&=\left(\frac{\partial}{\partial x}F'(\rho)\right)\frac{x}{\rho}+F'(\rho)\left(\frac{\partial}{\partial x}\frac{x}{\rho}\right)\\&=\left(F''(\rho)\frac{\partial\rho}{\partial x}\right)\frac{x}{\rho}+F'(\rho)\frac{1\cdot\rho-x\cdot\frac{\partial\rho}{\partial x}}{\rho^2}\\&=F''(\rho)\frac{x}{\rho}\frac{x}{\rho}+F'(\rho)\frac{\rho-x(x/\rho)}{\rho^2}\\&=F''(\rho)\frac{x^2}{\rho^2}+F'(\rho)\frac{\rho^2-x^2}{\rho^3}\end{aligned}$$

代入得: $F''(\rho)+2\cdot\frac{F'(\rho)}{\rho}=0$

最后解得答案:

$$u(x,y,z)=\frac{c_1}{\rho}+c_2$$