2025(p2)

$$Given : m, n, k \in N_+ \ , \ x_1 \geq x_2 \geq \ldots\ldots \geq x_n \geq 0$$

$$\sum_{i=1}^{n} x_i \leq k$$

$$\sum_{i=1}^{n} {x_i}^m \geq 1$$

$$m \geq 2 \, , n \geq k \geq 2$$

$$prove : \sum_{i=1}^{k} x_i \geq 1$$

My personal solution:

Obviously, if $x_1 \geq 1$, then it is proved directly ,since $\sum_{i=1}^{k} x_i \geq x_1 \geq 1$

therefore,for any $x_i, 0 \leq x_i \leq 1$, we have that:

$$\sum_{i=1}^{n} {x_i}^2 \geq \sum_{i=1}^{n} {x_i}^m \geq 1$$

suppose that:

$$M = x_1 \geq x_2 \geq \ldots\ldots \geq x_k = t \geq x_{k+1} \geq \ldots\ldots \geq x_n \geq 0$$

then:

$$\sum_{i=1}^{k} (x_i - M) \cdot (x_i - \mathrm{t}) \leq 0 \Leftrightarrow \sum_{i=1}^{k} {x_i}^2 + kMt \leq (M+t) \cdot \sum_{i=1}^{k} x_i$$

$$\sum_{j=k+1}^{\mathrm{n}} x_j (x_j - \mathrm{t}) \leq 0 \Leftrightarrow \sum_{j=k+1}^{n} {x_j}^2 \leq t \cdot \sum_{j=k+1}^{n} x_j$$

combine the two sums(sum up the two equations):

$$\sum_{i=1}^{n} {x_i}^2 \le (M+t)\cdot\sum_{i=1}^{k} x_i + t\cdot\sum_{j=k+1}^{n} x_j + (-kMt) = M\cdot\sum_{i=1}^{k} x_i + t\cdot\sum_{i=1}^{n} x_i - kMt$$

$$\sum_{i=1}^{n} {x_i}^2 \ge 1, \sum_{i=1}^{n} x_i \le k$$

$$\Rightarrow 1 \le M\cdot\sum_{i=1}^{k} x_i + t\cdot k - kMt$$

$$\Leftrightarrow (1-kt)(1-M) + M \le M\cdot\sum_{i=1}^{k} x_i$$

since $M = x_1 \le 1$

if we have $t \ge \frac{1}{k}$ then: $\sum_{i=1}^{k} x_i \ge kt \ge 1$, the problem is proved;

otherwise :

$$\because (1-kt)(1-M) \ge 0$$

$$\therefore M \le (1-kt)(1-M) + M \le M\cdot\sum_{i=1}^{k} x_i$$

$$\therefore \sum_{i=1}^{k} x_i \ge 1$$

the problem is proved,totally