Introduction to Electrodynamics II

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Introductions

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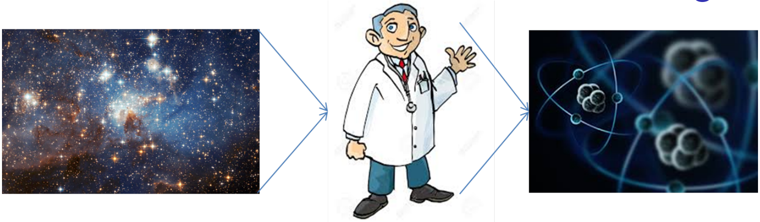
- 教材：Introduction to Electrodynamics (Griffiths)
- 参考书 1：Classical Electrodynamics (Jackson)
- 参考书 2：Electrodynamics of continuous media (Landau and Lifshitz).
- 关键概念：Lorentz invariance(spacetime), Gauge invariance (interaction, internal space)
- 数学要求：矢量运算、微分方程、多元微积分、张量分析、群论、拓扑。。。

Four fundamental forces and their ranges

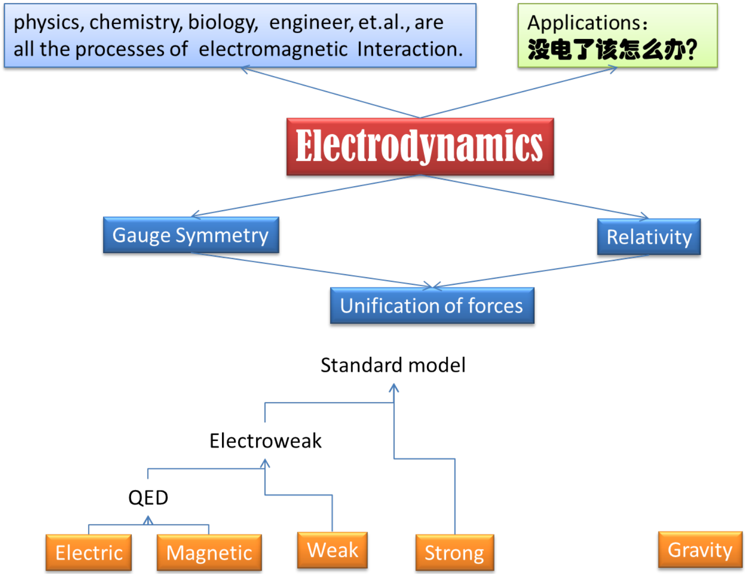


- In the universe, the dominant force is gravity, which is always attractive, and much weaker than electromagnetic force. Consider the forces between proton and electron

$$\frac{F_G}{F_{EM}} = \frac{Gm_p m_e/r^2}{e^2/(4\pi\epsilon_0 r^2)} = \frac{G}{4\pi\epsilon_0}\frac{m_p}{m_e}\left(\frac{m_e}{e}\right)^2 \simeq 4.4 \times 10^{-40}$$

  Question: why is the gravity dominant in the universe?

- **In the macroscopic world in our size, "everything" is controlled by electromagnetism, which is the basic force behind daily life.**
- In the subatomic world, the strong and weak forces take effect, which is of short range.


physics, chemistry, biology, engineer, et.al., are all the processes of electromagnetic Interaction.
Applications:
没电了该怎么办?
Electrodynamics
Gauge Symmetry
Relativity
Unification of forces
Standard model
Electroweak
QED
Electric
Magnetic
Weak
Strong
Gravity

Brief history of electromagnetism

- phenomenological description of electromagnetism
- 1784, Coulomb's Law, C. A. de Coulomb(French).

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{\hat{r}}{r^2}$$

- 1802, Gian Domenico Romagnosi(Italian), discovered "Oersted's Law". *Electric current creates a magnetic field surrounding it*.
- 1820, Oersted's Law, Hans Cristian Oersted(Danish).
- 1820, Biot-Savart Law, Jean-Baptiste Biot and Felix Savart(French)

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{s} \times (\vec{r} - \vec{s})}{|\vec{r} - \vec{s}|^3}$$

- 1826, Ampere's law, Andre-Marie Ampere(French)

$$\oint \vec{B} \cdot d\vec{r} = \mu_0 I_{enc}$$

continue......

- 1831, Faraday's Law, M. Faraday(English); 1832, M. Henry (American). *A current will be induced in a conductor which is exposed to a changing magnetic field.*
- 1835, Gauss's Law, C. F. Gauss(German)

  $$\oiint \vec{E} \cdot d\vec{S} = \frac{Q_{enc}}{\epsilon_0} \Leftrightarrow \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

  It is equivalent to the Coulomb's law.
- 1873, Maxwell equations, J. C. Maxwell(English)
- 20 century—, Dirac, Heisenberg, Jordan, et.al. Quantum Electrodynamics,
- Electroweak theory, Standard model, grand unification, all start with electrodynamics
- $\vdots$

Maxwell's seminal work establishing Electrodynamics

THE

LONDON, EDINBURGH AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[FOURTH SERIES.]

MARCH 1861.

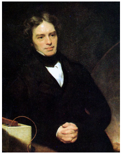
XXV. *On Physical Lines of Force. By* J. C. MAXWELL, *Professor of Natural Philosophy in King's College, London**.

VIII. *A Dynamical Theory of the Electromagnetic Field. By* J. CLERK MAXWELL, *F.R.S.*

Received October 27,—Read December 8, 1864.

Maxwell's equations

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

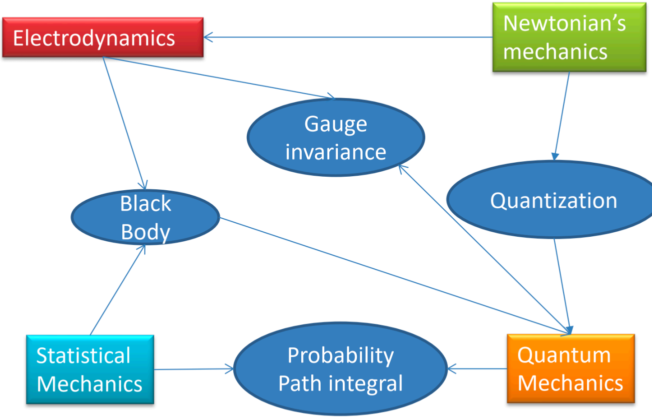


$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

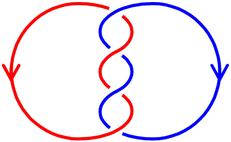
$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{j}$$

Relation with other mechanics



Physics and mathematics

1. Linking number vs. Biot-Savart and Ampere's laws



Loop 1(red) carries current I generating magnetic field $\vec{B}$ via Biot-Savart law, and we do line integral over $\vec{B}$ along loop 2 (blue) via Ampere law to get the an integer N_L.

$$\mu_0 I N_L = \oint \vec{B}(\vec{r}_2) \cdot d\vec{r}_2 = \frac{\mu_0 I}{4\pi} \oint \oint \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3} \cdot (d\vec{r}_2 \times d\vec{r}_1)$$

The integer N_L is known as the *linking number*.

2. Electrodynamics is a *gauge theory* corresponds to the fiber bundle theory in mathematics.

3. The special relativity is mathematically formulated as the SO(3,1) transformation on the spacetime. Electrodynamics is the vector representation of SO(3,1) group and its tensor extension.

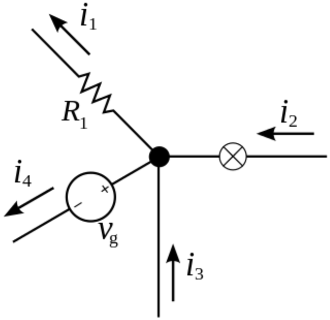
4. There are more connections between mathematics and physics.

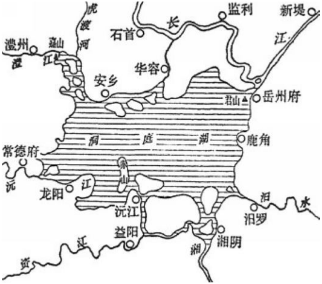
Table of Contents

Outline

The continuity equation

The Law of Mass Conservation

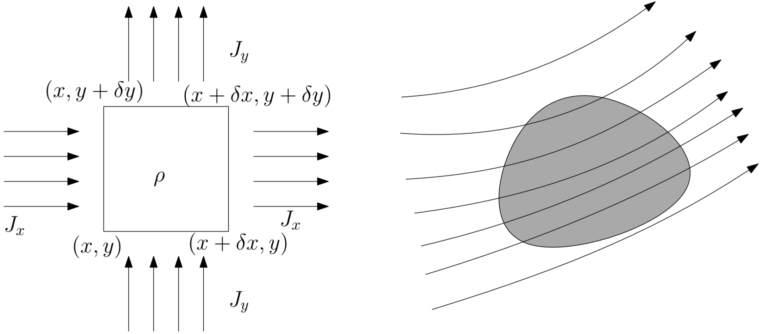




- Kirchhoff's current law $i_1 + i_4 = i_2 + i_3$.
- (water flowing in) $-$ (water flowing out) $=$ Δ(water in lake) *without* raining and evaporation(source and drain).

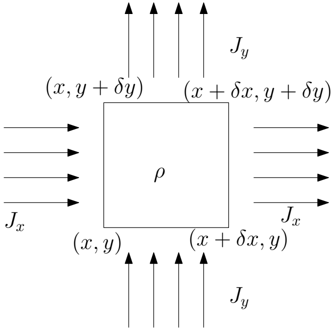
These observations reflect the *law of mass conservation*.

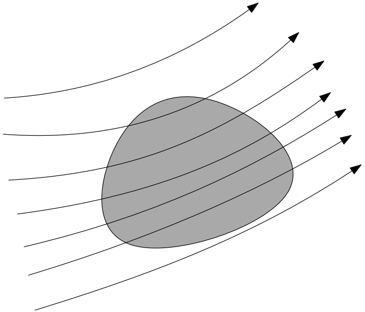
Continuity Equation (2D example)



- Suppose the space is filled with flowing charges. At the space-time point (x, y, t), the *charge density* is $\rho(x, y, t)$ and the *current density* is $\vec{J}(x, y, t)$.
- In the left pannel above, the number of charges in the small volume is given by: $Q(t) = \rho(x, y, t)\delta x \delta y$.
- When the time evolves from t to $t + \delta t$, the charge variation in the volume can be written as $Q(t + \delta t) - Q(t) = \partial_t \rho(x, y, t)\delta t \delta x \delta y$.

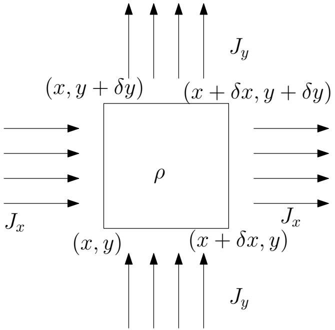
Continuity Equation (2D example)

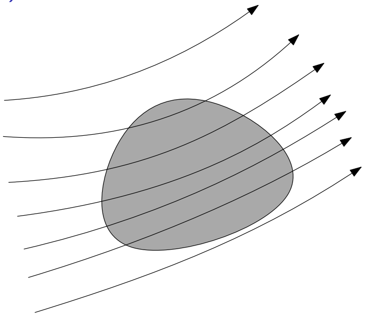




- The charges flowing in is given by: $J_x(x, y, t)\delta t\delta y + J_y(x, y, t)\delta t\delta x$.
- The charges flowing out is given by: $J_x(x + \delta x, y, t)\delta t\delta y + J_y(x, y + \delta y, t)\delta t\delta x$.
- The net charges leaving the volume is given by their difference: $(\partial_x J_x + \partial_y J_y)\delta t\delta x\delta y$.
- The mass conservation requires that: $\partial_t \rho(x, y, t)\delta t\delta x\delta y = -(\partial_x J_x + \partial_y J_y)\delta t\delta x\delta y \Rightarrow \partial_t \rho + \partial_x J_x + \partial_y J_y = 0$.

Continuity Equation (2D example)





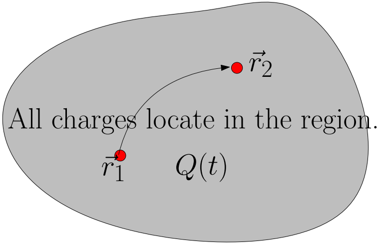
- The differential form of the continuity eq. is usually written as: $\boxed{\partial_t \rho + \vec{\nabla}\cdot\vec{J} = 0}$.
- In the right panel, the total charge in the shaded region Ω is $Q(t) = \iiint_\Omega d\tau\, \rho(\vec{r}, t)$. Then from the continuity eq., we find that $\frac{dQ}{dt} = \iiint_\Omega \frac{\partial\rho}{\partial t} d\tau = -\iiint_\Omega (\vec{\nabla}\cdot\vec{J}) d\tau = -\oiint_{\partial\Omega} d\vec{a}\cdot\vec{J}$. This is the integral form of the continuity eq.
- These Eqs. hold in all dimensions as long as charge is conserved.

Examples of continuity equations

- In Quantum Mechanics, the Schrödinger equation reads $i\hbar\partial_t\Psi(\vec{r},t) = -\frac{\hbar^2\vec{\nabla}^2}{2m}\Psi(\vec{r},t) + V(\vec{r})\Psi(\vec{r},t)$.
  $\star$ Probability density: $\rho(\vec{r},t) = \Psi^*(\vec{r},t)\Psi(\vec{r},t)$.
  $\star$ Current density: $\vec{J}(\vec{r},t) = \frac{\hbar}{2im}[\Psi^*(\vec{r},t)\vec{\nabla}\Psi(\vec{r},t) - (\vec{\nabla}\Psi^*(\vec{r},t))\Psi(\vec{r},t)]$
  Using the Schrödinger's equation, we can verify the continuity equation $\partial_t\rho + \vec{\nabla}\cdot\vec{J} = 0$.
- Before Maxwell, the fields and sources satisfy the following two laws:
  $\star$ $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$, Gauss's Law(electric field and charge),
  $\star$ $\vec{\nabla}\times\vec{B} = \mu_0\vec{J}$, Ampére's Law(static magnetic field with current)
  Obviously, $\dot{\rho} + \vec{\nabla}\cdot\vec{J} = \epsilon_0\partial_t\vec{\nabla}\cdot\vec{E} \neq 0$, i.e., the conti. eq. is violated. Maxwell provides a remedy by introducing the *displacement current* into the Ampére's law, $\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\partial_t\vec{E}}$. This is known as *modified* Ampére's law. Then, the conti. eq. is restored $\vec{\nabla}\cdot\vec{J} = \vec{\nabla}\cdot(\mu_0^{-1}\vec{\nabla}\times\vec{B} - \epsilon_0\partial_t\vec{E}) = -\epsilon_0\partial_t\vec{\nabla}\cdot\vec{E} = -\dot{\rho}$
  Interestingly, Maxwell's displacement current is also important for the *relativistic invariance* of the whole theory.

Local and Global charge conservation

- The continuity equation derived earlier is based on the *local* charge conservation, leading to *global* charge conservation; however, the reverse may not hold.



- In the right figure, all charges lie within the shaded region, which is conserved ($\partial_t Q = 0$). This conservation law is global and cannot be used to derive the continuity equation, unless some additional requirement is added, as demonstrated below.

- At time t, a point charge vanishes at $\vec{r}_1$ and reappears at $\vec{r}_2$. This does not violate the *global* conservation law. However, if one focuses on monitoring either $\vec{r}_1$ or $\vec{r}_2$ instead of the entire region, it appears as though the charge simply disappears at $\vec{r}_1$ and materializes at $\vec{r}_2$ without any apparent source. This violates *the local conservation law without breaking the global conservation law*.

- The preceding process suggests that the speed of the charge is infinite, which contradicts the principles of special relativity. If we adhere to the tenets of special relativity, such a process would not occur, and the local conservation law would be upheld.
- It may be tempting to consider the scenario where the charge is destroyed at spacetime $(\vec{r}_1, t)$ and subsequently created at $(\vec{r}_2, t')$, satisfying the condition $|\vec{r}_1 - \vec{r}_2|/(t' - t) < c$, thus ensuring that the velocity of the charge is smaller than the speed of light (c) and avoiding a violation of special relativity. However, this process would violate the *global* charge conservation, as there would be a gap in the time interval $[t, t']$ where some charges would be missing.
- Hence, the combination of global conservation principles and special relativity leads to local conservation. Subsequently, our focus will be solely on *local* charge conservation in the context of electrodynamics, which conforms to the principles of special relativity. Additionally, this concept is intertwined with the fundamental law of nature known as *local gauge invariance*.

Outline

Poynting's Theorem

Energy density of electromagnetic field

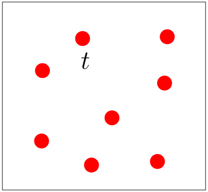
- Thanks to Einstein's mass-energy equivalence, when mass is conserved, so is energy. In order to formulate the continuity equation for energy in electrodynamics, we must introduce the concepts of energy density and energy current of the electromagnetic field.
- The work necessary to assemble a static charge distribution against the Coulomb forces can be expressed as $W_e = \frac{1}{2}\epsilon_0 \iiint d\tau \vec{E}^2$(Eq.2.45 in textbook).
- The work needed to drive currents against the back electromotive force (Faraday's law) can be calculated as $W_m = \frac{1}{2\mu_0} \iiint d\tau \vec{B}^2$. (Eq.7.34 in textbook).
- The total energy stored in the electromagnetic fields is the sum of W_e and W_m $\boxed{U_{em} = \frac{1}{2} \iiint d\tau (\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0} \vec{B}^2)}$. Here the integration is over the whole space.

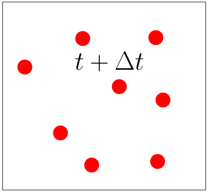
Charge-current vs. fields

In electrodynamics, it is important to recognize that

1. EM field is generated by the distribution of charges in space and time, depending not only on the positions but also on the velocities of the charges;
2. EM field, in turn, influences the movement of the charges.

Let's consider a scenario where we have a charge-current configuration at time t (left panel), generating $\vec{E}$ and $\vec{B}$. In the next instant $t + dt$, the charges undergo some movement (right panel).





Question: How much work dW is done by the EM forces on the charges during the process from t to $t + \Delta t$?

Energy conservation and transformation

Let's proceed with this exercise in the following manner:

- Since the E&M force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$, dW can be expressed as: $dW = \vec{F} \cdot d\vec{l} = q(\vec{E} + \vec{v} \times \vec{B}) \cdot (\vec{v}dt) = q\vec{E} \cdot \vec{v}dt \rightarrow \frac{dW}{dt} = q\vec{E} \cdot \vec{v}$. Clearly, $\boxed{\text{Lorentz force does no work}}$.
- In the case of a continuous charge distribution, we can assign q as the charge within an infinitesimal volume, then $\frac{dW}{dt} = \sum_q q\vec{E} \cdot \vec{v} = \iiint_{\mathcal{V}} (\rho d\tau)\vec{E} \cdot \vec{v} = \iiint_{\mathcal{V}} (\vec{E} \cdot \vec{J}) d\tau$. Here $\vec{J} = \rho\vec{v}$ is the electric current.
- $\vec{E} \cdot \vec{J}$ can be interpreted as the power delivered per unit volume.
- In order to express the energy in terms of the electromagnetic field rather than the matter field, it is necessary to replace the charge current $\vec{J}$ with the fields, utilizing the modified Ampère's law: $\vec{E} \cdot \vec{J} = \vec{E} \cdot \left(\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = \frac{1}{\mu_0} \vec{E} \cdot (\vec{\nabla} \times \vec{B}) - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t}$

- Using vector algebra(detailed derivation will be given later),
  $\vec{\nabla}\cdot(\vec{E}\times\vec{B})=\vec{B}\cdot(\vec{\nabla}\times\vec{E})-\vec{E}\cdot(\vec{\nabla}\times\vec{B})$
  $\Rightarrow\vec{E}\cdot(\vec{\nabla}\times\vec{B})=\vec{B}\cdot(\vec{\nabla}\times\vec{E})-\vec{\nabla}\cdot(\vec{E}\times\vec{B})=-\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}-\vec{\nabla}\cdot(\vec{E}\times\vec{B})$
  where we use the Faraday's law $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$. Then, we find that:
  $\vec{E}\cdot\vec{J}=\frac{1}{\mu_0}\vec{E}\cdot(\vec{\nabla}\times\vec{B})-\epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$
  $=-\frac{1}{\mu_0}\vec{B}\cdot\frac{\partial\vec{B}}{\partial t}-\frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})-\epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$
  $=-\frac{1}{2\mu_0}\frac{\partial\vec{B}^2}{\partial t}-\frac{1}{2}\epsilon_0\frac{\partial\vec{E}^2}{\partial t}-\frac{1}{\mu_0}\vec{\nabla}\cdot(\vec{E}\times\vec{B})$
- Finally, we arrive at the work-energy theorem in E&M,
  $$\boxed{\frac{dW}{dt}=-\frac{d}{dt}\iiint_{\mathcal{V}}\frac{1}{2}\left(\frac{\vec{B}^2}{\mu_0}+\epsilon_0\vec{E}^2\right)d\tau-\frac{1}{\mu_0}\oiint_{\partial\mathcal{V}}(\vec{E}\times\vec{B})\cdot d\vec{a}.}$$
  It is also known as *Poynting's theorem*.
- This theorem reflects the energy conservation and transformation: The work $\frac{dW}{dt}$ performed on the charges by the electromagnetic force is equivalent to the reduction in energy stored in the field (first term on the right-hand side), minus the energy that exited through the surface (second term on the right-hand side), which would manifest as electromagnetic radiation if it were to propagate to infinity.

- By defining the Poynting vector as $\vec{S} \equiv \frac{1}{\mu_0}(\vec{E} \times \vec{B})$, which represents the energy flow, the theorem can be expressed as $\frac{dW}{dt} = -\frac{dU_{em}}{dt} - \oiint_{\partial \mathcal{V}} \vec{\mathcal{S}} \cdot d\vec{a}$.
- The work dW is transferred to the mechanical energy of the charge. By introducing the density of mechanical energy u_{mech}, we may write $\frac{dW}{dt} = \frac{d}{dt} \iiint_{\mathcal{V}} u_{mech} d\tau$.
- The energy density of the fields can be defined as $u_{em} = \frac{1}{2}\left(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right)$.
  By using u_{em} and u_{mech}, we can rewrite the Poynting's theorem in a differential form as follows
  $\boxed{\frac{\partial}{\partial t}(u_{mech} + u_{em}) + \vec{\nabla} \cdot \vec{S} = 0.}$
  This is the continuity equation for the energy conservation.

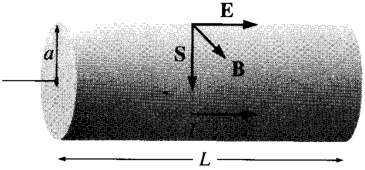
A little knowledges on the Levi-Civita antisymmetric tensor of rank 3

All nonzero elements of Levi-Civita tensor ϵ_{ijk} of rank 3 are listed as $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = -\epsilon_{213} = -\epsilon_{132} = -\epsilon_{321} = 1$. It is easy to prove that $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ and $\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl}$. In terms of the Levi-Civita tensor, we have the following expression for vectors:

$$
\begin{aligned}
\vec{A} \times \vec{B} =& \epsilon_{ijk}\vec{e}_i A_j B_k \\
\vec{C} \cdot (\vec{A} \times \vec{B}) =& \epsilon_{ijk} C_i A_j B_k \\
\vec{\nabla} \cdot (\vec{A} \times \vec{B}) =& \epsilon_{ijk}\partial_i (A_j B_k) = \epsilon_{ijk}(\partial_i A_j) B_k + \epsilon_{ijk} A_j (\partial_i B_k) \\
=& \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \\
\vec{A} \times (\vec{B} \times \vec{C}) =& \epsilon_{ijk}\vec{e}_i A_j (\epsilon_{klm} B_l C_m) = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\vec{e}_i A_j B_l C_m \\
=& \vec{e}_i A_j B_i C_j - \vec{e}_i A_j B_j C_i = A_j (\vec{B} C_j) - (\vec{A} \cdot \vec{B})\vec{C} \\
\vec{A} \times (\vec{\nabla} \times \vec{A}) =& A_i (\vec{\nabla} A_i) - (\vec{A} \cdot \vec{\nabla})\vec{A} = \frac{1}{2}\vec{\nabla}(\vec{A}^2) - (\vec{A} \cdot \vec{\nabla})\vec{A}
\end{aligned}
$$

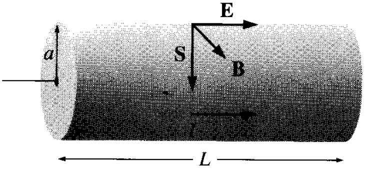
Here, the repeated indices imply summation.

Example 8.1, Joule heating



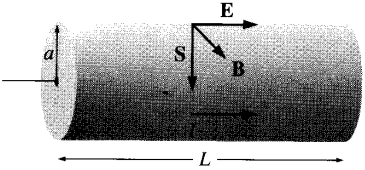
- When the current flows down a wire, the charges undergo collisions, resulting in the transfer of their mechanical energy to the lattices of the wire, known as Joule heating. This energy is eventually released into the surrounding environment in the form of radiation. To sustain a steady current, work is required to be done on the charges by an external potential(electric field) applied across the wire.
- In a time interval δt, a total of $I\delta t$ charges are transferred across the voltage V. The work supplied by the battery can be expressed as $\delta W = V(I\delta t)$, indicating that the power delivered is $P = VI$. As the kinetic energy of the charges remains unchanged, this work is done against the resistance and is converted into Joule heat.
- This is the high school story of Joule heat and resistance. Now we try to understand the Joule heating from the Poynting's theorem.

Example 8.1, Joule heating



- Let's consider a wire segment of length L over which a potential difference V is applied. The electric field within this segment is given by $E = V/L$, which is along the wire.
- The magnetic field on the surface follows from the Ampére's law, $B = \frac{\mu_0 I}{2\pi a}$ which is along the circumferential direction. Here a is the radius of the wire. Since we consider a steady state $\partial \vec{E}/\partial t$, there is no displacement current.
- The Poynting vector is then pointing inward and its value is given by $S = \frac{1}{\mu_0}\left(\frac{V}{L}\right)\left(\frac{\mu_0 I}{2\pi a}\right) = \frac{VI}{2\pi a L}$.
- Since $\vec{S}$ is antiparallel to the surface and perpendicular to the cross sections, the energy power passing in through the surface of the wire segment can be written as $\oiint \vec{S} \cdot d\vec{a} = -S(2\pi a L) = -VI$.

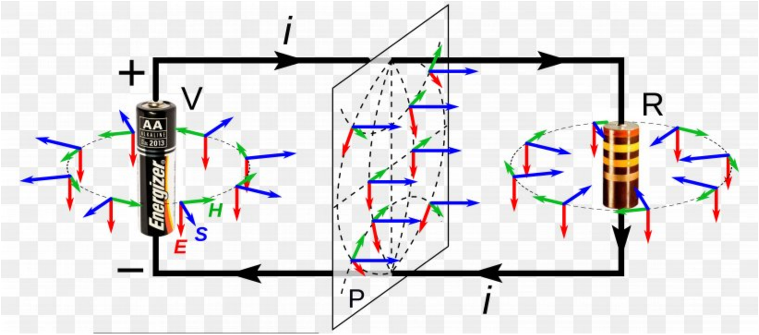
Example 8.1, Joule heating



- More analysis: the total energy of the charge carriers and the associated electromagnetic fields within the segment of a wire includes (1) the mechanical energy of charges u_{mech} and (2) the field energy u_{em}. These energies can be transferred to the lattice as (3) the heat power W_{heat} or (4) flow out through the surface $\oiint \vec{S} \cdot d\vec{a}$:
$\frac{d}{dt} \iiint d\tau (u_{mech} + u_{em}) = -\oiint \vec{S} \cdot d\vec{a} - W_{heat}$
- Since we consider a steady state, the l.h.s. vanishes. Then, we find
$W_{heat} = -\oiint \vec{S} \cdot d\vec{a} = IV$.
It implies that the Joule heat can be viewed as the energy flowing into the wire in the form of the electromagnetic fields. The corresponding electromagnetic fields are generated by the battery.

Schematic show of the energy flow

Figure taken from web.



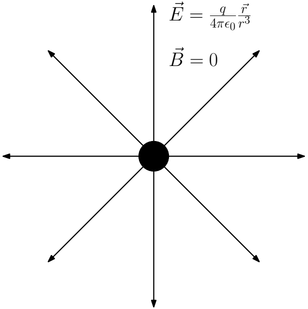
Energy of the electromagnetic field flows out of the battery and flows into the resistor.

Outline

Newton's Third Law in Electrodynamics

One point charge

- For a point charge q located at the origin, the electric field is determined by the Coulomb's law: $\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$ and the magnetic field $\vec{B} = 0$ since there is no current.



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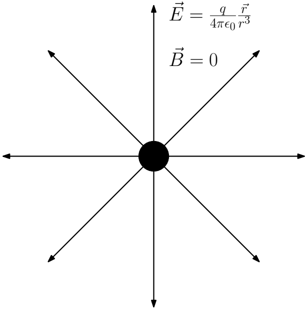


- If the point charge is now travelling along the x axis at a constant speed v. $\boxed{\text{What is the electric field and magnetic field?}}$

Newton's Third Law in Electrodynamics

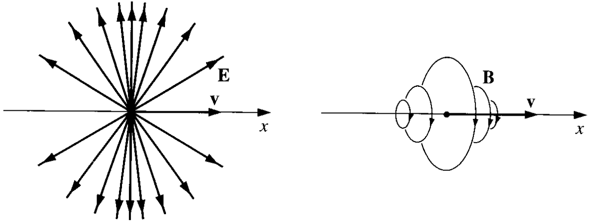
One point charge

- Clearly, the electric field $\vec{E}$ is no longer determined by Coulomb's law due to the motion of charges. For instance, if you were positioned at $x = -1$, you would observe the electric field $\vec{E}$ diminishing gradually over time as the charge moves further away.
- The magnetic field $\vec{B}$ is also non-zero, but it cannot be described using the Biot-Savart law. For example, if you were situated at $x = 1$, you would only sense the passage of charge at a specific moment, with no charge present at other times. This situation does not involve a steady current, rendering the Biot-Savart law inapplicable.

Newton's Third Law in Electrodynamics

One point charge

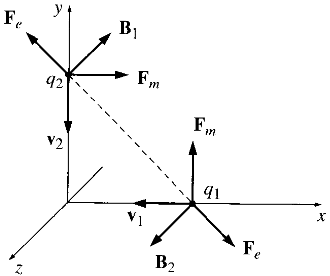
- The actual EM field of a moving charge is depicted above. $\vec{E}$ still points outward from the instantaneous position of the charge, and $\vec{B}$ circles around the axis. The detailed derivation is postponed to chap. 10.



- Next, let's check what happens to two moving point charges.

Two point charges

- Now, consider two identical point charges moving on the x-axis and y-axis with velocities $\vec{v}_1$ and $\vec{v}_2$, respectively.

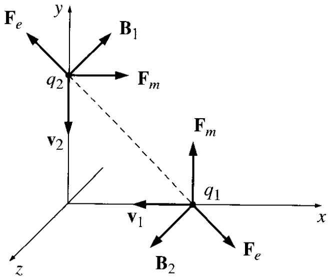


As depicted in the left figure, $\vec{B}_{1,2}$ represent the magnetic fields generated by charges 1 and 2 respectively. $\vec{F}_e$ and $\vec{F}_m$ denote the electric and magnetic forces acting on the point charges.

- Based on the EM field distribution shown in the previous slide, the electric forces $\vec{F}_e$ have equal magnitudes and are directed along the dashed line in opposite directions. However, the magnetic forces between the two charges are not opposite to each other, but rather perpendicular, despite having the same magnitude.
- **Question:** *Is there something weird?*

Two point charges

- Now, consider two identical point charges moving on the x-axis and y-axis with velocities $\vec{v}_1$ and $\vec{v}_2$, respectively.



As depicted in the left figure, $\vec{B}_{1,2}$ represent the magnetic fields generated by charges 1 and 2 respectively. $\vec{F}_e$ and $\vec{F}_m$ denote the electric and magnetic forces acting on the point charges.

- Yes, **the Newton's third law is broken!** The electromagnetic forces exerted by q_1 and q_2 on each other are not opposite, though equal in magnitude, in contrast to the static case.
- There is a more serious implication that the law of conservation of momentum may have been violated, as it is typically derived from the third law in high school physics.

Force is not instantaneous

To better understand the situation, let's examine the following cartoon:

- There is a force between Chunli and Ken that is mediated by " 气". If you cannot see " 气", the force appears to be a direct interaction.
- Since “气” requires time to propagate from Chunli to Ken(velocity can not be ∞), the force can not be instantaneous.
- If you take “气” into account, the momentum conservation is intact. It holds *at different instants* between the fighters and “气” , but not between two fighters at any instant.

Field carries momentum!

- If Chunli and Ken are the two charges, then “气” is the EM field that mediates electromagnetic forces between them.
- The electromagnetic field must possess momentum to ensure the conservation of momentum as a fundamental law in physics. However, Newton’s Third Law regarding the interaction of two charges should be disregarded, as it cannot be employed to derive the law of momentum conservation.
- We shall give the derivation of the momentum of the electromagnetic field in a moment.

Outline

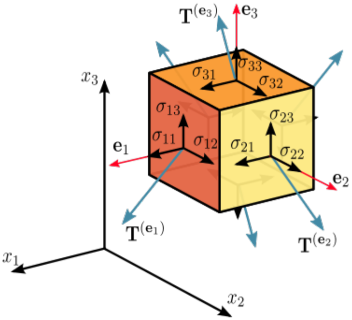
Maxwell's Stress Tensor

- Let's calculate the total E&M force on the charges in a volume $\mathcal{V}$,
  $\vec{F} = \iiint_{\mathcal{V}} d\tau(\vec{E} + \vec{v} \times \vec{B})\rho = \iiint_{\mathcal{V}} d\tau(\vec{E}\rho + \vec{J} \times \vec{B}) \equiv \iiint_{\mathcal{V}} d\tau \vec{f}$
- Since we are interested in the fields, we replace ρ and $\vec{J}$ by the fields using Maxwell equations,
  $\vec{\nabla} \times \vec{B} = \mu_0(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t})$ (Ampére law)
  $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$(Gauss law)
  $\vec{f} = \epsilon_0 \vec{E}(\vec{\nabla} \cdot \vec{E}) + (\frac{1}{\mu_0}\vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t}) \times \vec{B}$
  $= \epsilon_0 \vec{E}(\vec{\nabla} \cdot \vec{E}) - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \times \vec{B} - \frac{1}{\mu_0}\vec{B} \times (\vec{\nabla} \times \vec{B})$
- Reformulate $\vec{f}$ using the Faraday's law in the following way
  (1) $\frac{\partial \vec{E}}{\partial t} \times \vec{B} = \frac{\partial}{\partial t}(\vec{E} \times \vec{B}) - \vec{E} \times \frac{\partial \vec{B}}{\partial t} = \mu_0 \frac{\partial \vec{S}}{\partial t} + \vec{E} \times (\vec{\nabla} \times \vec{E})$
  (2) $\vec{f} = \epsilon_0[\vec{E}(\vec{\nabla} \cdot \vec{E}) - \vec{E} \times (\vec{\nabla} \times \vec{E})] - \frac{1}{\mu_0}\vec{B} \times (\vec{\nabla} \times \vec{B}) - \epsilon_0\mu_0 \frac{\partial \vec{S}}{\partial t}$
  (3) $\vec{X} \times (\vec{\nabla} \times \vec{X}) = \frac{1}{2}\vec{\nabla}(\vec{X}^2) - (\vec{X} \cdot \vec{\nabla})\vec{X}$(see previous slide)
  (4) $\vec{f} = \epsilon_0[\vec{E}(\vec{\nabla} \cdot \vec{E}) + (\vec{E} \cdot \vec{\nabla})\vec{E}] + \frac{1}{\mu_0}[\underline{\vec{B}(\vec{\nabla} \cdot \vec{B})} + (\vec{B} \cdot \vec{\nabla})\vec{B}]$
  $-\frac{1}{2}\vec{\nabla}(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2) - \epsilon_0\mu_0 \frac{\partial \vec{S}}{\partial t}$

Maxwell stress tensor

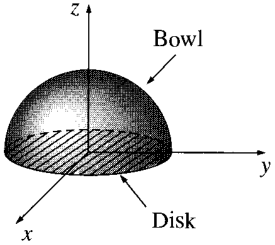
- We define the Maxwell stress tensor as follows
  $T_{ij} = \epsilon_0(E_iE_j - \frac{1}{2}\delta_{ij}\vec{E}^2) + \frac{1}{\mu_0}(B_iB_j - \frac{1}{2}\delta_{ij}\vec{B}^2)$.
- T is a tensor with two indices, and it can be represented as a "double-vector $\overleftrightarrow{T}$" with the following rules for calculations.
  $(\vec{a}\cdot\overleftrightarrow{T})_j = \sum_{i=x,y,z} a_iT_{ij}, \qquad (\overleftrightarrow{T}\cdot\vec{a})_j = \sum_{i=x,y,z} T_{ji}a_i$.
  If T is symmetric, $\vec{a}\cdot\overleftrightarrow{T} = \overleftrightarrow{T}\cdot\vec{a}$.
- $$\begin{aligned}(\vec{\nabla}\cdot\overleftrightarrow{T})_j &= \epsilon_0[\partial_i(E_iE_j) - \tfrac{1}{2}\partial_i\delta_{ij}\vec{E}^2] + \tfrac{1}{\mu_0}[\partial_i(B_iB_j) - \tfrac{1}{2}\partial_i\delta_{ij}\vec{B}^2] \\ &= \epsilon_0[(\partial_iE_i)E_j + E_i(\partial_iE_j) - \tfrac{1}{2}\partial_j\vec{E}^2] \\ &\quad + \tfrac{1}{\mu_0}[(\partial_iB_i)B_j + B_i(\partial_iB_j) - \tfrac{1}{2}\partial_j\vec{B}^2] \\ &= \epsilon_0[(\vec{\nabla}\cdot\vec{E})E_j + (\vec{E}\cdot\vec{\nabla})E_j - \tfrac{1}{2}\partial_j\vec{E}^2] \\ &\quad + \tfrac{1}{\mu_0}[(\vec{\nabla}\cdot\vec{B})B_j + (\vec{B}\cdot\vec{\nabla})B_j - \tfrac{1}{2}\partial_j\vec{B}^2]\end{aligned}$$
- Thus, $\vec{f} = \vec{\nabla}\cdot\overleftrightarrow{T} - \epsilon_0\mu_0\frac{\partial\vec{S}}{\partial t}$.
- The total force can then be written as
  $\vec{F} = \iiint_{\mathcal{V}} d\tau[\vec{\nabla}\cdot\overleftrightarrow{T} - \epsilon_0\mu_0\frac{\partial\vec{S}}{\partial t}] = \oiint_{\partial\mathcal{V}} d\vec{a}\cdot\overleftrightarrow{T} - \epsilon_0\mu_0\frac{d}{dt}\iiint_{\mathcal{V}}\vec{S}d\tau$
  where we use the divergence theorem, and $d\vec{a}$ is to the left of T.

- **Remark 1:** In a static scenario, the force can be fully expressed in terms of the stress tensor on the boundary.
- **Remark 2:** T_{ij} represents the force (per unit area) acting in the j-th direction on an element of surface oriented in the i-th direction. The "diagonal" elements (T_{xx}, T_{yy}, T_{zz}) correspond to pressures, while the "off-diagonal" elements (T_{xy}, T_{xz}, etc.) represent shearing forces.



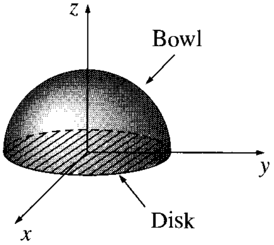
The figure is taken from web, where the cyan vectors $\vec{T}^{\vec{e}_i}$ correspond to the stress tensor.

Example 8.2, application of Maxwell's stress tensor



Calculate the total force acting on the "northern" hemisphere of a uniformly charged solid sphere with radius R and charge Q.
What will you do? if you don't know the Maxwell stress tensor(see Prob.2.43).

Example 8.2, application of Maxwell's stress tensor



By understanding the Maxwell stress tensor, we only require information about the field at the boundary of the hemispherical bowl.

Solution:

- In this electrostatic problem, the total force can be calculated using the equation $\vec{F} = \oiint d\vec{a} \cdot \overleftrightarrow{T}$, and the Poynting vector does not play a role in this context. Because of the rotational symmetry, only the z-component of $\vec{F}$ does not vanish, which is given by $F_z = \oiint_{\partial\Omega} da_i T_{iz}$.
  Here, the boundary surface $\partial\Omega$ is divided into two parts: a hemispherical "bowl" with a radius of R and a circular disk with $\theta = \pi/2$.

- The Maxwell stress tensor reads $T_{ij} = \epsilon_0(E_i E_j - \frac{1}{2}\delta_{ij}\vec{E}^2)$, so
  $F_z = \oiint da_j T_{jz} = \oiint \epsilon_0(\vec{E}\cdot d\vec{a} E_z - \frac{1}{2} da_z \vec{E}^2) = F_{bowl} + F_{disk}$.
- For the **bowl**,
  $\star$ The surface area $d\vec{a} = da\hat{r} = R^2 \sin\theta d\theta d\phi \hat{r}$,
  $\star$ The electric field $\vec{E} = E\hat{r} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}\hat{r}$
  $\star$ The radial vector $\hat{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$
  $F_{bowl} = \iint \epsilon_0(E\cos\theta \times E da - \frac{1}{2}E^2 \times da\cos\theta) = \frac{\epsilon_0}{2}\iint E^2 \cos\theta da$
  $= \frac{\epsilon_0}{2}\iint (\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2})^2 R^2 \cos\theta\sin\theta d\theta d\phi$
  $= \frac{\epsilon_0}{2}(\frac{1}{4\pi\epsilon_0}\frac{Q}{R^2})^2 R^2 \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi \cos\theta\sin\theta d\theta d\phi = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{8R^2}$
- For the **equatorial disk**,
  $\star$ The surface area $d\vec{a} = -rdrd\phi\hat{z} = da_z\hat{z}$,
  $\star$ The electric field $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\vec{r} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3} r(\cos\phi\hat{x} + \sin\phi\hat{y})$.
  Since $\vec{E}\cdot d\vec{a} = 0$, the force on the equatorial disk is given by
  $F_{disk} = \iint \epsilon_0(-\frac{1}{2}\vec{E}^2 da_z) = \frac{\epsilon_0}{2}\int_0^R rdr \int_0^{2\pi} d\phi (\frac{1}{4\pi\epsilon_0})^2 \frac{Q^2}{R^6} r^2 = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{16R^2}$
- The total force is the sum of F_{bowl} and F_{disk}:
  $F_z = F_{bowl} + F_{disk} = \frac{1}{4\pi\epsilon_0}\frac{3Q^2}{16R^2}$.

Alternative Solution:

- In this model, it is possible to extend the surface of the bowl by letting its radius R approach infinity. As a result, the equatorial disk will encompass the entire $x-y$ plane. In this scenario, the surface of the bowl does not contribute anything since the electric field becomes zero at infinity. Therefore, we only need to consider the $x-y$ plane as the sole surface enclosing the hemisphere.
- Due to the varying forms of the electric field inside ($r < R$) and outside ($r > R$) the region, we must address these two areas independently. Given that we have already evaluated the scenario for $r < R$, we will now focus solely on the region where $r > R$.
  In this case, we observe that
  $\star$ The surface area $d\vec{a} = -rdrd\phi\hat{z} = da_z\hat{z}$,
  $\star$ The electric field $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}(\cos\phi\hat{x} + \sin\phi\hat{y})$,
  $\star$ The force in this region is given by:
  $\iint(-\frac{\epsilon_0}{2}\vec{E}^2 da_z) = \frac{\epsilon_0}{2}\int_R^\infty rdr\int_0^{2\pi} d\phi(\frac{1}{4\pi\epsilon_0})^2\frac{Q^2}{r^4} = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{8R^2}$
  This is exactly equal to F_{bowl} which as given earlier.

Outline

Conservation of Momentum

- Based on Newton's second law, the force acting on the charges in the volume $\mathcal{V}$ is equivalent to the rate of change of their mechanical momentum: $\vec{F} = \frac{d\vec{p}_{mech}}{dt}$. As a consequence, we find that
$\frac{d\vec{p}_{mech}}{dt} + \epsilon_0\mu_0 \frac{d}{dt} \iiint_{\mathcal{V}} \vec{S} d\tau = \oiint_{\partial\mathcal{V}} d\vec{a} \cdot \overleftrightarrow{T}$
$\frac{d}{dt}\left(\vec{p}_{mech} + \epsilon_0\mu_0 \iiint_{\mathcal{V}} \vec{S} d\tau\right) = \oiint_{\partial\mathcal{V}} d\vec{a} \cdot \overleftrightarrow{T} \qquad (\star)$
Here, the term associated with Poynting's vector $\vec{S}$ is moved from the r.h.s. to the l.h.s. and combined the mechanical momentum of charges.
- If we interpret the r.h.s. as the force acting on the entire volume rather than just the part of force on charges within the volume, then the r.h.s. represents the total momentum of the volume. Therefore, the integral within the brackets can be interpreted as the momentum of the electromagnetic field: $\vec{p}_{em} = \epsilon_0\mu_0 \iiint_{\mathcal{V}} \vec{S} d\tau$.
- Given that Poynting's vector $\vec{S}$ is originally introduced as the energy flux carried by the electromagnetic fields, it is logical that it is associated with the momentum contained within these fields.

- As a result, we arrive at Newton's second law in electrodynamics: $\frac{d}{dt}(\vec{p}_{mech} + \vec{p}_{em}) = \oiint_{\partial\mathcal{V}} d\vec{a} \cdot \overleftrightarrow{T}$.
- This law can also be viewed as a manifestation of momentum conservation: Any change in the total momentum within a volume is balanced by the momentum transferred across the surface $\partial\mathcal{V}$, which in this context is represented by the surface integral $\oiint_{\partial\mathcal{V}} d\vec{a} \cdot \overleftrightarrow{T}$.
  Remark: If the volume $\mathcal{V}$ encompasses the entire space, there would be no momentum entering or leaving. In this case, the total momentum of the charges and electromagnetic fields, represented by $\vec{p}_{mech} + \vec{p}_{em}$, should be conserved.
- In the present context, the Maxwell's stress tensor $\overleftrightarrow{T}$ is understood as the density of momentum flux. More specifically, T_{ij} represents the momentum along the j direction flowing out across a surface oriented in the i direction, per unit area, and per unit time.

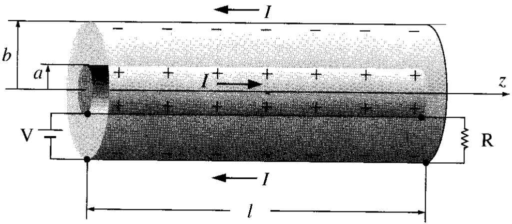
Differential form of momentum conservation

- By defining the mechanical momentum density as $\vec{\mathcal{P}}_{mech}$ and the field momentum density as $\vec{\mathcal{P}}_{em} \equiv \mu_0 \epsilon_0 \vec{S}$, the law of momentum conservation can be expressed as
  $\iiint_{\mathcal{V}} d\tau \frac{\partial \mathcal{P}_{mech}}{\partial t} + \iiint_{\mathcal{V}} d\tau \frac{\partial \mathcal{P}_{em}}{\partial t} = \iiint_{\mathcal{V}} d\tau \vec{\nabla} \cdot \overleftrightarrow{T}$
  $\iiint_{\mathcal{V}} d\tau \left(\frac{\partial \mathcal{P}_{mech}}{\partial t} + \frac{\partial \mathcal{P}_{em}}{\partial t} - \vec{\nabla} \cdot \overleftrightarrow{T} \right) = 0.$
  Here we use the divergence theorem for the surface integral of T.
- As the law of momentum conservation applies to any volume, we obtain the differential form of the momentum conservation:
  $\boxed{\frac{\partial}{\partial t}(\vec{\mathcal{P}}_{mech} + \vec{\mathcal{P}}_{em}) = \vec{\nabla} \cdot \overleftrightarrow{T}}.$

Example 8.3

A long coaxial cable with length ℓ comprises an inner conductor of radius a and an outer conductor of radius b. It is connected to a battery at one end and a resistor at the other. The inner conductor carries a uniform charge per unit length λ and a steady current I to the right, while the outer conductor holds the opposite charge and current.

Question: *What is the electromagnetic momentum stored in the fields in this setup?*



Solution: The electric and magnetic fields are determined by Gauss's law and Ampère's law, respectively. Specifically, at radius s, we have:

$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{s}, \qquad \vec{B} = \frac{\mu_0}{2\pi}\frac{I}{s}\hat{\phi}, \text{ and } \hat{s}\times\hat{\phi} = \hat{z}.$$

Thus, Poynting's vector can be expressed as $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{\lambda I}{4\pi^2\epsilon_0}\frac{1}{s^2}\hat{z}$, which aligns with the direction of the cable. The energy transferred from the battery to the resistor is given by the following surface integral: $P = \iint \vec{S}\cdot d\vec{a} = \int_a^b \frac{\lambda I}{4\pi^2\epsilon_0 s^2}(2\pi s ds) = \int_a^b \frac{\lambda I}{2\pi\epsilon_0}\frac{ds}{s} = I(\frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{a}) = IV$, where the voltage is determined by $V = \int_a^b \vec{E}\cdot d\vec{s} = \frac{\lambda}{2\pi\epsilon_0}\ln(\frac{b}{a})$.

The momentum of the electromagnetic field has the following form:

$$\boxed{\vec{p}_{em} = \iiint(\mu_0\epsilon_0\vec{S})d\tau = \mu_0\epsilon_0\frac{\lambda I}{4\pi^2\epsilon_0}\hat{z}\int_a^b\frac{ds}{s^2}(2\pi s\ell) = \frac{\mu_0\lambda I\ell}{2\pi}\ln\left(\frac{b}{a}\right)\hat{z}}$$

- **Question**: Note that the cable is stationary, the steady current is in equilibrium, and the fields are static. Therefore, the total momentum of the system must be zero. But, where can we find a source of momentum to counterbalance the momentum of the fields?
- In Chap.12, we will uncover a relativistic effect that introduces a "hidden" mechanical momentum connected to the loop current within an electrostatic field. This hidden momentum precisely offsets the momentum within the fields.

In the following let's try to reveal the essence of the momentum of electromagnetic field.

- As we slowly increase the resistance, the current decreases, resulting in a varying magnetic field depicted by $\vec{B} \approx \frac{\mu_0}{2\pi s}\frac{dI}{dt}\hat{\phi}$ under the quasi-static assumption.
- This varying magnetic field will, in turn, generate an electric field as per Faraday's law, given by $\vec{E}(s) = \left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln s + K\right]\hat{z}$ (refer to Equation 7.19), where K is an undetermined constant.
- This electric field exerts a force on the charges on the inner and outer conductors. The total force is given by: $\vec{F} = \vec{E}(a)\ell\lambda - \vec{E}(b)\ell\lambda = -\frac{\mu_0\lambda\ell}{2\pi}\frac{dI}{dt}\ln\frac{b}{a}\hat{z}$
- The overall momentum transferred to the cable as the current decreases from I to 0 is calculated as: $\vec{p}_{mech} = \int \vec{F}dt = -\frac{\mu_0\lambda\ell}{2\pi}\ln\frac{b}{a}\int_I^0 dI = \frac{\mu_0\lambda I\ell}{2\pi}\ln\frac{b}{a}$
- This process demonstrates that the momentum of the EM field can be converted into mechanical momentum.
- Although the cable acquires momentum stored in the EM field, it will not recoil due to an equal and opposite impulse provided by the simultaneous disappearance of the "hidden" momentum.

Outline

8.2.4 Angular Momentum

Fields exist as physical entities that possess energy u_{em}, momentum $\vec{\mathcal{P}}$, and consequently, angular momentum $\vec{\ell}$.

- $u_{em} = \frac{1}{2}\left(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right)$
- $\vec{\mathcal{P}}_{em} = \mu_0\epsilon_0\vec{S} = \epsilon_0(\vec{E}\times\vec{B})$,
- $\vec{\ell}_{em} = \vec{r}\times\vec{\mathcal{P}}_{em} = \epsilon_0[\vec{r}\times(\vec{E}\times\vec{B})]$.

Remark: Even static fields can carry momentum and angular momentum. Therefore, it is only when these field contributions are taken into account that the classical conservation laws are upheld.

Example 8.4

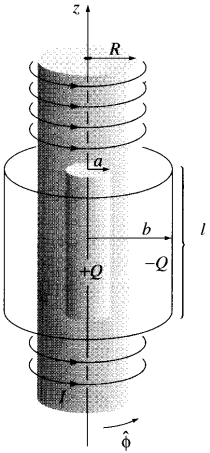


Figure 8.7

Imagine a very long solenoid with radius R, n turns per unit length, and current I. Coaxial with the solenoid are two long cylindrical shells of length l –one, inside the solenoid at radius a, carries a charge $+Q$, uniformly distributed over its surface; the other, outside the solenoid at radius b, carries charge $-Q$ (see Fig. 8.7; l is supposed to be much greater than b). When the current in the solenoid is gradually reduced, the cylinders begin to rotate, as we found in Ex. 7.8. Question: Where does the angular momentum come from?

Solution:

- There was an electric field between the charged cylindrical shells and a magnetic field inside the solenoid.
  Electric field: $\vec{E} = \frac{Q}{2\pi\epsilon_0 \ell} \frac{1}{s} \hat{s}$ for $a < s < b$, and $\vec{E} = 0$ elsewhere.
  Magnetic field: $\vec{B} = \mu n I \hat{z}$ for $s < R$, and $\vec{B} = 0$ elsewhere.
- Momentum is only stored in the EM field for the range of $a < s < R$, where the momentum density is given by:
  $\vec{\mathcal{P}}_{em} = \epsilon_0 \vec{E} \times \vec{B} = -\frac{\mu_0 n I Q}{2\pi \ell s} \hat{\phi}$.
- Accordingly, the angular momentum density can be written as
  $\vec{\ell}_{em} = \vec{r} \times \vec{\mathcal{P}}_{em} = -\frac{\mu_0 n I Q}{2\pi \ell} \hat{z} + \frac{\mu_0 n I Q}{2\pi \ell s} \hat{s}$, where $\vec{r} = s\hat{s} + z\hat{z}$.
  The total angular momentum can be easily obtained as below
  $\vec{L}_{em} = \iiint \vec{\ell}_{em} d\tau = -\frac{\mu_0 n I Q}{2} (R^2 - a^2) \hat{z}$.
  Here the radial part of $\vec{\ell}_{em}$ does not contribute post integration.
- Next, we gradually turn off the current. The resulting changing magnetic field induces a tangential electric field,
  for $s > R$, $\vec{E} = -\frac{1}{2} \mu_0 n \frac{dI}{dt} \frac{R^2}{s} \hat{\phi} \Leftrightarrow 2\pi s E = -\frac{\partial}{\partial t} (\pi R^2 B)$;
  for $s < R$, $\vec{E} = -\frac{1}{2} \mu_0 n \frac{dI}{dt} s \hat{\phi} \Leftrightarrow 2\pi s E = -\frac{\partial}{\partial t} (\pi s^2 B)$;
  This electric field applies a force, thus creating torque on the charged cylinders.

- The *total* torque on the outer cylinder (after integration)
  $\vec{N}_b = (s\hat{s}) \times (-Q\vec{E})|_{s=b} = \frac{1}{2}\mu_0 nQR^2 \frac{dI}{dt}\hat{z}$.
  Note that the contribution of $z\hat{z} \times (-Q\vec{E})$ to the torque can be disregarded as it disappears after integration.
  As the current goes to zero, the outer cylinder picks up an angular momentum,
  $\vec{L}_b = \frac{1}{2}\mu_0 nQR^2 \hat{z} \int_I^0 \frac{dI}{dt} dt = -\frac{1}{2}\mu_0 nIQR^2 \hat{z}$
- Similarly for the inner cylinder, the *total* torque reads
  $\vec{N}_a = (s\hat{s}) \times (-Q\vec{E})|_{s=a} = -\frac{1}{2}\mu_0 nQa^2 \frac{dI}{dt}\hat{z}$,
  As the current goes to zero, the inner cylinder also picks up an angular momentum,
  $\vec{L}_a = \frac{1}{2}\mu_0 nIQa^2 \hat{z}$
- The total mechanical angular momentum is given by the following summation
  $\vec{L} = \vec{L}_a + \vec{L}_b = \frac{1}{2}\mu_0 nIQ(a^2 - R^2)\hat{z}$.
  This is exactly the angular momentum stored in the field.
- Remark: (1) If a localized system is not moving, its total linear momentum has to be zero, but there is no such requirement for the angular momentum. Thus, (2) there is no need for the "hidden" angular momentum to compensate that in the fields.