Introduction to Electrodynamics II

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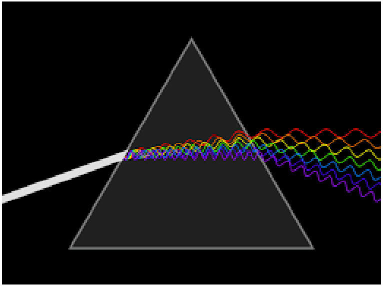
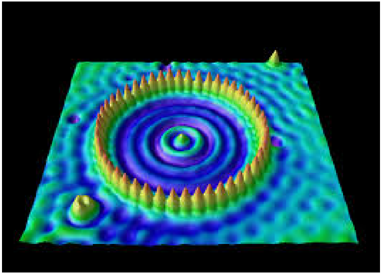
9.1.1 The Wave function

What is wave?

9.1.1 The Wave function





9.1.1 The Wave function

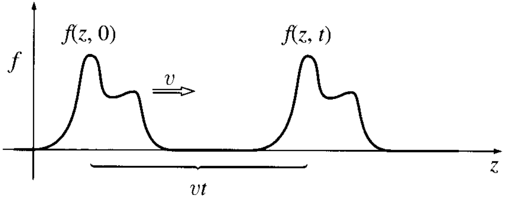
Several concepts about waves

–the most fundamental phenomenon in Nature.

- **Ideal wave**: A wave is a disturbance of a *continuous medium* that propagates with a fixed shape at constant velocity.
- **Absorption**: The wave will diminish in size as it propagates in the presence of absorption.
- **Dispersive**: In a dispersive medium, waves with different frequencies travel at different speeds.
- **Spherical wave**: In two or three dimensions, a spherical wave emanates symmetrically from a single point source, with its amplitude decreasing as it propagates.
- **Standing wave**: Standing waves don't propagate at all.

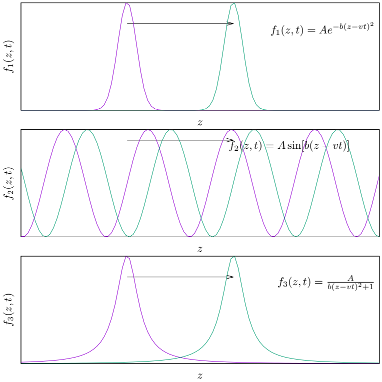
Mathematical description of Wave

Consider a physical quantity f that varies in spacetime, such as **the displacement of a string**, **the probability amplitude of a particle in quantum mechanics**, **the displacement of atoms in a crystal**, and other similar quantities.



□ Starting with the initial configuration $f(z, 0)$ at $t = 0$, each point on the wave form shifts to the right by a distance of vt at time t, where v denotes the velocity. Alternatively, the function $f(z, t)$ solely depends on $u \equiv z - vt$, indicating a wave with a fixed shape propagating along the z direction at a speed of v.

Examples



- Ideal waves,

$$f_1(z,t) = Ae^{-b(z-vt)^2}, \quad f_2(z,t) = A\sin[b(z-vt)], \quad f_3(z,t) = \frac{A}{b(z-vt)^2+1}$$

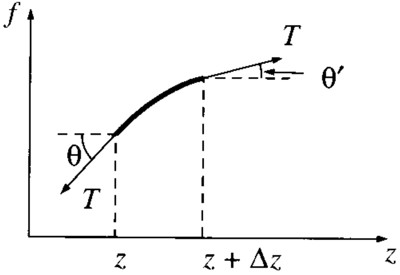
- Disturbance not being waves,

$$f_4(z,t) = Ae^{-b(bz^2+vt)}, \quad \text{and} \quad f_5(z,t) = A\sin(bz)\cos(bvt)^3,$$

Revisit: Oscillation of a stretched string.

When a stretched string with mass per unit length μ under tension T is pulled, it undergoes a slight deformation from equilibrium and begins to oscillate. The force acting on a small segment of this string with length Δz can be analyzed as follows:

☐ Both θ' and θ are very small for small displacements.



☐ The **net force along** z**-axis (horizontal)** is given by: $T \cos\theta' - T\cos\theta \sim O(\theta^2)$, which can be ignored.

☐ The **net force perpendicular to** z**-axis (vertical)** can be calculated in the following way:

$$\Delta F = T \sin\theta' - T\sin\theta \approx T(\tan\theta' - \tan\theta)$$

$$\Rightarrow \Delta F = T\left(\frac{\partial f}{\partial z}\Big|_{z+\Delta z} - \frac{\partial f}{\partial z}\Big|_{z}\right) \approx T\frac{\partial^2 f}{\partial z^2}\Delta z$$

Revisit: Oscillation of a stretched string.

□ **Wave equation.** According to Newton's second law ($F = m\frac{d^2x}{dt^2}$), we find that $T\frac{\partial^2 f}{\partial z^2}\Delta z = (\mu\Delta z)\frac{\partial^2 f}{\partial t^2}$. It leads to the wave equation for the stretched string: $\boxed{\frac{\mu}{T}\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial z^2}}$.

□ **Solutions of the wave equation.**

★ First, we recast this wave equation in the following form:

$$\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial z^2} = \left(\frac{1}{v}\frac{\partial}{\partial t} - \frac{\partial}{\partial z}\right)\left(\frac{1}{v}\frac{\partial}{\partial t} + \frac{\partial}{\partial z}\right) f = 0.$$

★ Next, we introduce the new coordinates $z_+ = z + vt$ and $z_- = z - vt$ to substitute for z and t, respectively, such that:

$$\frac{1}{v}\frac{\partial}{\partial t} = \frac{\partial}{\partial z_+} - \frac{\partial}{\partial z_-} \text{ and } \frac{\partial}{\partial z} = \frac{\partial}{\partial z_+} + \frac{\partial}{\partial z_-}.$$

★ In terms of z_+ and z_-, the wave equation can be written as:

$$-4\frac{\partial^2 f}{\partial z_+ \partial z_-} = 0.$$

★ Obviously, any function $g(z_-)$ is a solution, since $\frac{\partial g(z_-)}{\partial z_+} = 0$. Similarly, $h(z_+)$ is also a solution. Therefore, **the general solutions consist of all linear combinations of $h(z_+)$ and $g(z_-)$**, which propagate along opposite directions of the z axis, respectively.

Matter wave: Schrödinger's equation

In addition to the classical wave equation for stretched strings, there exist various other types of wave equations. One notable example is the Schrödinger equation in quantum mechanics, which serves as a wave equation for particles and is expressed in the following form:

$$i\hbar \frac{\partial}{\partial t}\psi(z,t) = -\frac{\hbar^2 \partial_z^2}{2m}\psi(z,t)$$

It possesses a plane wave solution $\psi(z,t) = e^{i[kz-\omega(k)t]}$ with a dispersion relation given by $\omega(k) = \hbar k^2/(2m)$.

□ In fact, **the plane wave** represents a general form of the wave function. For the earlier example of the stretched string, the plane wave solution takes the form $e^{ik(z\pm vt)}$ with $\omega = vk$. In the subsequent section, we will introduce the concepts of plane waves, aka, **sinusoidal waves**.

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9.1.2 Sinusoidal Waves

Sinusoidal wavefunction

The most familiar and important wave form is the sinusoidal wave given below. All the parameters are real.

Wave vector

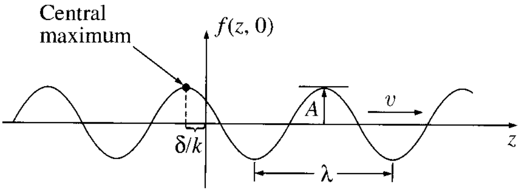
amplitude speed Phase constant

$$f(z,t) = A\cos\underbrace{[k(z - vt) + \delta]}_{\text{phase}}$$

Parameters characterizing the wavefunction:

- $\lambda = \frac{2\pi}{k}$, wavelength
- $T = \frac{\lambda}{v} = \frac{2\pi}{kv}$, period
- $\nu = \frac{1}{T} = \frac{kv}{2\pi} = \frac{v}{\lambda}$, frequency
- $\omega = 2\pi\nu = kv$, angular frequency
- In term of ω, $f(z,t) = A\cos(kz - \omega t + \delta)$

Wave is moving.



- When the phase is zero, $z = \frac{\omega t}{k} - \frac{\delta}{k}$, this position is referred to as the **central maximum**. It is evident that with the passage of time, the central maximum as well as the wave form moves towards the right with a velocity $v = \omega/k$.
- Typically, the frequency remains positive, but the sign of k can be inverted. In this scenario, the wave function $g(z, t) = A\cos(-kz - \omega t + \delta)$ represents a wave form propagating to the left with a velocity $v = -\omega/k$.

Complex wave functions

By using Euler's formula $e^{i\delta} = \cos\delta + i\sin\delta$, the sinusoidal wave can be expressed as the real part of a **complex wave function**:

$$\tilde{A}e^{i(kz-\omega t)} = Ae^{i(kz-\omega t+\delta)}.$$

Here, $\tilde{A} \equiv Ae^{i\delta}$, and a complex number is distinguished by a tilde symbol from a real number.

Example 9.1: Summation of two sinusoidal waves

Given two real sinusoidal waves, f_1 and f_2, their sum f_3 can be expressed as: $f_3 = f_1 + f_2 = Re(\tilde{f}_1) + Re(\tilde{f}_2) = Re(\tilde{f}_3)$, where $\tilde{f}_3 = \tilde{f}_1 + \tilde{f}_2$.

For sinusoidal waves, $\tilde{f}_3 = \tilde{A}_1 e^{i(kz-\omega t)} + \tilde{A}_2 e^{i(kz-\omega t)} = \tilde{A}_3 e^{i(kz-\omega t)}$, where $\tilde{A}_3 = \tilde{A}_1 + \tilde{A}_2$, or $A_3 e^{i\delta_3} = A_1 e^{i\delta_1} + A_2 e^{i\delta_2}$.

□ We observe that, in terms of complex wave functions, the interference of two sinusoidal waves can be simply expressed as a sum of two complex amplitudes. If you attempt to perform this calculation *without* utilizing the complex notation, you will need to reference trigonometric identities and engage in tedious algebraic calculations.

Linear combinations of sinusoidal waves

In fact, any real wave function can be expressed as a linear combination of sinusoidal waves:

$$f(z,t) = \int_{-\infty}^{\infty} dk A(k) \cos[kz - \omega t + \delta(k)],$$

so it can also be represented as the real part of a complex wave function defined as:

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} dk \tilde{A}(k) e^{i(kz - \omega t)},$$

where $\tilde{A}(k) = A(k) e^{i\delta(k)}$ is a complex function.

★ This is nothing but a Fourier transformation of a complex function.

Outline

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   - 9.1 Waves in One Dimension
     - 9.1.1 The Wave Equation
     - 9.1.2 Sinusoidal Waves
     - 9.1.3 Boundary Conditions: Reflection and Transmission
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   - 9.2 Electromagnetic Waves in Vacuum
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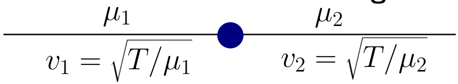
9.1.3 Boundary Conditions: Reflection and Transmission

In reality, there are no infinitely long strings; waves always encounter boundaries. The behavior at these boundaries is determined by how the string is connected to them, which is known as the **boundary conditions**.

□ **A simple model of two connected strings**



As depicted in the diagram, one string is connected to another at $z = 0$ smoothly. Both strings have the same tension T, but different mass per unit length, resulting in different velocities.

□ The wave function $\tilde{f}(z, t)$ should consist of two connected pieces, one for the first string, and the other for the second string. They are connected by the boundary conditions(B.C.) at $z = 0$.

□ Boundary conditions

- An obvious B.C. is the continuity of $\tilde{f}$ at $z = 0$ for all times, expressed as:

  $$\lim_{z\to 0^-} \tilde{f}_{z<0}(z, t) = \lim_{z\to 0^+} \tilde{f}_{z>0}(z, t) \text{ for all } t.$$

  In a simpler form, it can be written as:

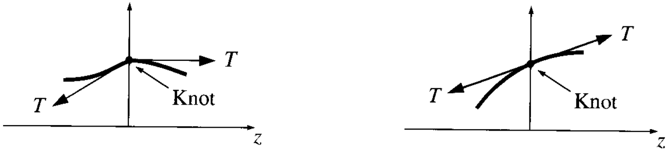
  $$\tilde{f}(0^+, t) = \tilde{f}(0^-, t).$$

  This condition ensures that the two strings remain connected.

- If the knot is massless, another B.C. arises: the derivative of $\tilde{f}$ must be continuous as well, given by:

  $$\partial_z \tilde{f}(0^+, t) = \partial_z \tilde{f}(0^-, t).$$

  This condition ensures that there is no finite force on the massless knot, avoiding an unacceptable infinite acceleration.



(a) Discontinuous slope; force on knot

(a) Continuous slope; no force on knot

Trial wave function

When only one vibration source is placed at $z = -\infty$ with frequency ω, the following observations can be made:

- In the 1st with $z < 0$, there are two waves: One is the incident wave from the left, $\tilde{f}_I(z,t) = \tilde{A}_I e^{i(k_1 z - \omega t)}$, and the other is the reflected wave to the left, $\tilde{f}_R(z,t) = \tilde{A}_R e^{i(-k_1 z - \omega t)}$.
- In the 2nd string with $z > 0$, there exists only the transmitted wave propagating to the right, $\tilde{f}_T(z,t) = \tilde{A}_T e^{i(k_2 z - \omega t)}$.

The <u>total wave function</u> can be expressed piecewise as follows:

$$\tilde{f}(z,t) = \begin{cases} \tilde{A}_I e^{i(k_1 z - \omega t)} + \tilde{A}_R e^{i(-k_1 z - \omega t)}, & \text{for } z < 0. \\ \tilde{A}_T e^{i(k_2 z - \omega t)}, & \text{for } z > 0. \end{cases}$$

$\tilde{A}_I$, $\tilde{A}_R$ and $\tilde{A}_T$ can be determined by the boundary conditions.

□ Remark: all three waves share the same frequency ω; otherwise, the boundary condition $\tilde{f}(0^+,t) = \tilde{f}(0^-,t)$ can never be satisfied. But the wave vectors can be different due to different velocities.

Determine the oscillating amplitudes with B.C.

The 1st B.C., $\tilde{f}(0^+, t) = \tilde{f}(0^-, t)$, leads to the equation:

$$\tilde{A}_I + \tilde{A}_R = \tilde{A}_T.$$

Meanwhile, the 2nd B.C., $\partial_z \tilde{f}(0^+, t) = \partial_z \tilde{f}(0^-, t)$, gives rise to

$$k_1(\tilde{A}_I - \tilde{A}_R) = k_2 \tilde{A}_T.$$

□ By solving these two linear equations, we can express the complex amplitudes $\tilde{A}_R$ and $\tilde{A}_T$ in terms of $\tilde{A}_I$:

$$\tilde{A}_R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right) \tilde{A}_I = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) \tilde{A}_I,$$

$$\tilde{A}_T = \left(\frac{2k_1}{k_1 + k_2}\right) \tilde{A}_I = \left(\frac{2v_2}{v_2 + v_1}\right) \tilde{A}_I.$$

Thus, the real amplitudes and phase satisfy the following equations:

$$A_R e^{i\delta_R} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right) A_I e^{i\delta_I}, \quad A_T e^{i\delta_T} = \left(\frac{2v_2}{v_2 + v_1}\right) A_I e^{i\delta_I}.$$

□ **Real amplitudes.** It can be easily observed that:

$$A_R = \left|\frac{v_2 - v_1}{v_2 + v_1}\right| A_I, \;\; A_T = \left|\frac{2v_2}{v_2 + v_1}\right| A_I.$$

□ **Phases.** The phase of the transmitted wave is equal to that of the incident wave, $\delta_T = \delta_I$. Regarding the relationship between δ_R and δ_I, there exist two cases as indicated below:

Case 1: If $\mu_2 < \mu_1$, resulting in $v_2 > v_1$, all three waves have the same phase angle $\delta_R = \delta_T = \delta_I$.

Case 2: If $\mu_2 > \mu_1$, leading to $v_2 < v_1$, the reflected wave is phase-shifted by π, specifically $\delta_R + \pi = \delta_T = \delta_I$.

★ In this case, the reflected wave can be expressed as

$$A_R \cos(-k_1 z - \omega t + \delta_R) = -A_R \cos(-k_1 z - \omega t + \delta_I).$$

Comparing it with the incident wave $A_I \cos(k_1 z - \omega t + \delta_I)$, the reflected wave appears to be "upside down".

★ If the second string is infinitely massive $(\mu_2 = \infty)$ and the first string is nailed down at the end, then $v_2 = 0$. Consequently, $A_R = A_I$ and $A_T = 0$. Obviously, in such a scenario, there is no transmitted wave, resulting in complete reflection.

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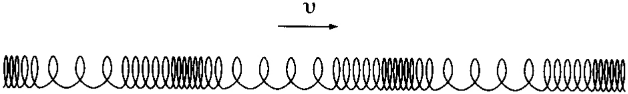
9.1.4 Polarization

If the quantity undergoing oscillation is a vector, the wave can be classified based on the relative angle between the oscillating vector and the direction of propagation into two categories:

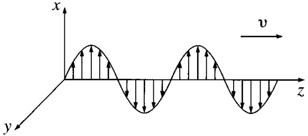
- *Transverse wave:* The oscillating vector is perpendicular to the direction of propagation. Examples include the oscillation of a stretched string and electromagnetic waves.
  Question: Can transverse waves exist in liquids or gases?
- *Longitudinal wave:* The oscillating vector is parallel to the direction of propagation. Examples include sound waves and the compressing spring in the following figure.



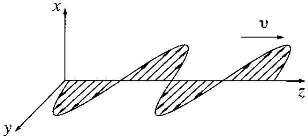
☐ **Polarization:** The direction of the oscillating vector is defined as the polarization of the wave.

Transverse waves exhibit two independent modes or polarizations. For $\vec{k} = k\hat{z}$, these polarizations are illustrated below:

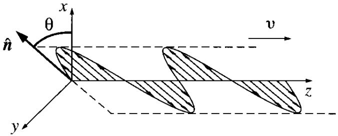
- **Vertical polarization:** $\tilde{\vec{f}}_v(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{x}$;
- **Horizontal polarization:** $\tilde{\vec{f}}_h(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{y}$;
- **Mixed polarization:** $\tilde{\vec{f}}(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{n}$.
  Here, $\hat{n}$ serves as the *polarization vector*, satisfying $\hat{n}\cdot\hat{z} = 0$.



(a) Vertical polarization



(b) Horizontal polarization



(c) Polarization vector

Outline

9.2.1 The Wave Equation for $\vec{E}$ and $\vec{B}$

Maxwell equations in a vacuum with $\rho = 0$ and $\vec{j} = 0$ are given by:

$$\text{(i)}\vec{\nabla} \cdot \vec{E} = 0, \quad \text{(ii)}\vec{\nabla} \cdot \vec{B} = 0, \quad \text{(iii)}\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \text{(iv)}\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

By taking the curl to both sides of Eqs. (iii) and (iv), we find that

$$\begin{aligned}
\text{(iii): } & \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = -\vec{\nabla}^2 \vec{E} \\
& = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}. \\
\text{(iv): } & \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \vec{\nabla}^2 \vec{B} = -\vec{\nabla}^2 \vec{B} \\
& = \vec{\nabla} \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}.
\end{aligned}$$

Electromagnetic wave equation in vacuum

□ **Wave equation.** To summarize, the wave equations for electromagnetic fields in vacuum can be written as follows:

$$\boxed{\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}, \quad \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}.}$$

□ **Wave velocity**. When comparing it with the wave equation of a one-dimensional stretched string, it becomes apparent that the wave velocity c is determined by the constants ϵ_0 and μ_0 as follows:

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = \frac{1}{\sqrt{8.854187817 \times 10^{-12} \frac{C^2}{N \cdot m^2} \cdot 4\pi \times 10^{-7} \frac{N}{A^2}}} = 299792458 m/s$$

□ **Light is EM wave**. It is worth noting that ϵ_0 and μ_0 can be obtained by measuring the EM force between charges. By utilizing these measured values, the velocity of an electromagnetic wave can be calculated, resulting in a value very close to the measured speed of light. Therefore, for physicists in the 19th century, the idea that "Light is an electromagnetic wave" was unavoidable.

Outline

9.2.2 Monochromatic Plane Waves

Color and frequency

☐ **Color** is directly related to the frequency of light. In the visible light spectrum, different colors correspond to different wavelengths and frequencies.

The Visible Range		
Frequency (Hz)	Color	Wavelength (m)
1.0×10^{15}	near ultraviolet	3.0×10^{-7}
7.5×10^{14}	shortest visible blue	4.0×10^{-7}
6.5×10^{14}	blue	4.6×10^{-7}
5.6×10^{14}	green	5.4×10^{-7}
5.1×10^{14}	yellow	5.9×10^{-7}
4.9×10^{14}	orange	6.1×10^{-7}
3.9×10^{14}	longest visible red	7.6×10^{-7}
3.0×10^{14}	near infrared	1.0×10^{-6}

Generally, light with shorter wavelengths and higher frequencies appears **bluer**, while light with longer wavelengths and lower frequencies appears **redder**.

☐ Full electromagnetic spectrum

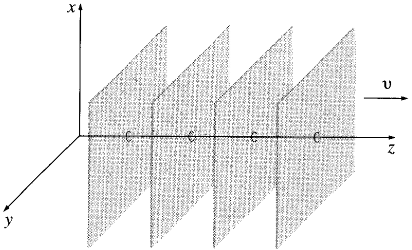
The Electromagnetic Spectrum		
Frequency (Hz)	Type	Wavelength (m)
10^{22}		10^{-13}
10^{21}	gamma rays	10^{-12}
10^{20}		10^{-11}
10^{19}		10^{-10}
10^{18}	x rays	10^{-9}
10^{17}		10^{-8}
10^{16}	ultraviolet	10^{-7}
10^{15}	visible	10^{-6}
10^{14}	infrared	10^{-5}
10^{13}		10^{-4}
10^{12}		10^{-3}
10^{11}		10^{-2}
10^{10}	microwave	10^{-1}
10^{9}		1
10^{8}	TV, FM	10
10^{7}		10^{2}
10^{6}	AM	10^{3}
10^{5}		10^{4}
10^{4}	RF	10^{5}
10^{3}		10^{6}

Monochromatic waves

☐ **Monochromatic waves** are sinusoidal waves with a specific frequency. When the wave propagates along the z-axis, the corresponding monochromatic wave functions can be expressed as:

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{i(kz-\omega t)},\ \tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_0 e^{i(kz-\omega t)}.$$

The phase of these monochromatic waves is the same on every plane perpendicular to the direction of propagation. Therefore, we refer to such waves as **plane waves**.



★ The monochromatic waves must satisfy the Maxwell equations, which impose constraints on k, ω, $\tilde{\vec{E}}_0$, and $\tilde{\vec{B}}_0$.

Constraints on monochromatic EM waves

For a plane wave propagating along the z-axis, the following insights can be derived from Maxwell's equations:

☐ In a vacuum, there are no electric or magnetic charges, hence $\vec{\nabla} \cdot \tilde{\vec{E}}(\vec{r}, t) = \vec{\nabla} \cdot \tilde{\vec{B}}(\vec{r}, t) = 0$. This implies that $(\tilde{E}_0)_z = (\tilde{B}_0)_z = 0$, indicating that **the EM wave is transverse.**

☐ From Faraday's law $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, we can derive that $k(\hat{z} \times \tilde{\vec{E}}_0) = \omega \tilde{\vec{B}}_0$ indicating $\tilde{\vec{B}}_0 \cdot \tilde{\vec{E}}_0 = 0$. Hence, **the vectors $\vec{k}$, $\tilde{\vec{E}}_0$, and $\tilde{\vec{B}}_0$ are mutually perpendicular, and form a right-handed system.**

☐ Ampere's law also yields an identity $k(\hat{z} \times \tilde{\vec{B}}_0) = -\frac{\omega}{c^2} \tilde{\vec{E}}_0$. When combined with Faraday's law, we can derive that:

$$\frac{\omega}{c^2} \tilde{\vec{E}}_0 = -k(\hat{z} \times \tilde{\vec{B}}_0) = -\frac{k^2}{\omega} [\hat{z} \times (\hat{z} \times \tilde{\vec{E}}_0)] = \frac{k^2}{\omega} \tilde{\vec{E}}_0.$$

For this equation to hold true, **a linear dispersion for EM waves** must be satisfied: $\boxed{\omega = ck}$. Therefore, $\tilde{\vec{B}}_0 = \frac{1}{c} \hat{z} \times \tilde{\vec{E}}_0$.

☐ In addition, the **polarization** of an EM wave is the defined as the direction in which the **electric** field oscillates.

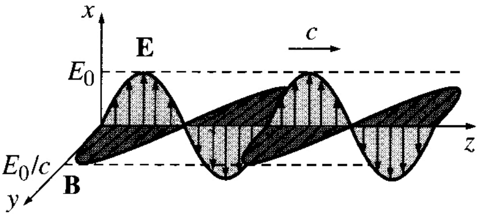
Example of Linear Polarization

For **linear polarization**, if $\vec{E}$ points in the x direction, then $\vec{B}$ points in the y direction. Therefore, we find that

$$\tilde{\vec{E}}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)}\hat{x}, \quad \tilde{\vec{B}}(z,t) = \frac{1}{c}\tilde{E}_0 e^{i(kz-\omega t)}\hat{y}.$$

By taking the real part of the wave function, the field configuration can be written as:

$$\vec{E}(z,t) = E_0 \cos(kz - \omega t + \delta)\hat{x}, \quad \vec{B}(z,t) = \frac{1}{c}E_0 \cos(kz - \omega + \delta)\hat{y}.$$



☐ This is a typical example of **linear polarization** where $\vec{E}$ oscillates along a fixed direction. In the present model, it is $\hat{x}$. However, it can be any vector within the $x-y$ plane given $\vec{k}$ along z axis.

Example of Circular Polarization

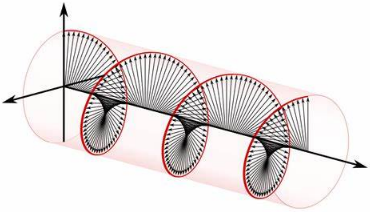
Consider a strange complex polarization vector $\hat{n} = \frac{\hat{x}+i\hat{y}}{\sqrt{2}}$, the corresponding wave functions can be written as:

$$\tilde{\vec{E}}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)} \frac{\hat{x}+i\hat{y}}{\sqrt{2}}, \quad \tilde{\vec{B}}(z,t) = -\frac{i}{c}\tilde{E}_0 e^{i(kz-\omega t)} \frac{\hat{x}+i\hat{y}}{\sqrt{2}}.$$

The real parts of these wave functions are given by:

$$\vec{E}(z,t) = \frac{E_0}{\sqrt{2}} \cos(kz-\omega t+\delta)\hat{x} - \frac{E_0}{\sqrt{2}} \sin(kz-\omega t+\delta)\hat{y},$$

$$\vec{B}(z,t) = \frac{E_0}{\sqrt{2}c} \sin(kz-\omega t+\delta)\hat{x} + \frac{E_0}{\sqrt{2}c} \cos(kz-\omega t+\delta)\hat{y}.$$



□ This is an example of **circular polarization** where $\vec{E}$ rotates clockwise around the wavevector(here, it is z axis). If the polarization is $\frac{\hat{x}-i\hat{y}}{\sqrt{2}}$, $\vec{E}$ will rotate anticlockwise around the wavevector.

Example of Elliptical Polarization

In the earlier example of circular polarization, we used a specific complex polarization vector $\hat{n} = \frac{\hat{x}+i\hat{y}}{\sqrt{2}}$. In fact, $\hat{n}$ can be any complex vector that satisfies the normalization condition $\hat{n} \cdot \hat{n}^* = 1$. Without loss of generality, we can choose $\hat{n} = \cos\alpha\hat{x} + i\sin\alpha\hat{y}$ with $0 \leq \alpha < \pi$. In this case, the field equations are given by:

$$\tilde{\vec{E}}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)}(\cos\alpha\hat{x} + i\sin\alpha\hat{y}),$$
$$\tilde{\vec{B}}(z,t) = -\frac{i}{c}\tilde{E}_0 e^{i(kz-\omega t)}(\sin\alpha\hat{x} + i\cos\alpha\hat{y}).$$

By taking the real part, the wave function can be written as:

$$\vec{E}(z,t) = E_0\cos\alpha\cos(kz-\omega t+\delta)\hat{x} - E_0\sin\alpha\sin(kz-\omega t+\delta)\hat{y},$$
$$\vec{B}(z,t) = \frac{E_0}{c}\sin\alpha\sin(kz-\omega t+\delta)\hat{x} + \frac{E_0}{c}\cos\alpha\cos(kz-\omega t+\delta)\hat{y}.$$

□ It is evident that the polarization reduces to a <u>linear one</u> when $\alpha = 0$ and to a <u>circular</u> one when $\alpha = \pi/4$ or $3\pi/4$. For other values of α, it represents **elliptical polarization** since the vector $\vec{E}(z,t)$ traces an ellipse described by $\left[\frac{E_x(z,t)}{E_0\cos\alpha}\right]^2 + \left[\frac{E_y(z,t)}{E_0\sin\alpha}\right]^2 = 1$.

EM wave propagating along arbitrary direction

If the EM wave propagates along an arbitary direction given by the wave vector $\vec{k} = k\hat{k}$, the oscillating fields are given by:

$$\tilde{\vec{E}}(\vec{r}, t) = \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\hat{n}$$
$$\tilde{\vec{B}}(\vec{r}, t) = \frac{1}{c}\tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}(\hat{k}\times\hat{n}) = \frac{1}{c}\hat{k}\times\tilde{\vec{E}}(\vec{r}, t).$$

☐ Here, the polarization vector $\hat{n}$ is a **complex vector** that satisfies the <u>transverse</u> condition $\boxed{\hat{n}\cdot\hat{k} = 0}$ and the <u>normalization</u> condition $\boxed{\hat{n}\cdot\hat{n}^* = 1}$.

Outline

9.2.3 Energy and Momentum in Electromagnetic Waves

For a monochromatic electromagnetic wave with linear polarization propagating along the z axis, the energy and momentum densities are expressed as follows:

- **Electric and Magnetic fields**:
  $\vec{E}(z,t) = E_0 \cos(kz - \omega t + \delta)\hat{x}$,
  $\vec{B}(z,t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta)\hat{y}$,
- **Energy density**:
  $u = \frac{1}{2}\left(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right) = \frac{1}{2}\left(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\frac{\vec{E}^2}{c^2}\right) = \epsilon_0 \vec{E}^2 =$
  $\epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta)$.
- **Energy flux**:
  $\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = \frac{E_0^2}{\mu_0 c}\cos^2(kz - \omega t + \delta)\hat{z} = \frac{1}{\mu_0 \epsilon_0 c} u\hat{z} = cu\hat{z}$
- **Momentum density**: $\vec{\mathcal{P}} = \frac{1}{c^2}\vec{S} = \frac{1}{c^2} cu\hat{z} = \frac{1}{c} u\hat{z}$.

Time Average of measurements on EM wave

For visible light, the wavelength is in the range of several hundred nanometers, and the frequency is about 10^{15} Hz. Consequently, macroscopic measurements will span huge amount of oscillation cycles. As a result, what we actually measure is the **time average** of the relevant quantities.

$$\langle u \rangle = \frac{1}{T}\int_0^T \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) dt = \frac{1}{2}\epsilon_0 E_0^2,$$
$$\left\langle \vec{S} \right\rangle = c\langle u \rangle \hat{z} = \frac{1}{2} c\epsilon_0 E_0^2 \hat{z},$$
$$\vec{\mathcal{P}} = \frac{1}{c}\langle u \rangle \hat{z} = \frac{1}{2c}\epsilon_0 E_0^2 \hat{z}.$$

□ The **intensity** of an EM wave is defined as the average power per unit area that it transports. This value represents a time-averaged quantity. In the case of normal incidence, the intensity is given by:

$$I \equiv \langle S \rangle = \frac{1}{2} c\epsilon_0 E_0^2.$$

□ In the case of <u>oblique incidence</u>, the intensity is adjusted by a factor of $\cos\theta$, where θ represents the angle between the energy flux and the surface's normal direction.

Radiation Pressure

Electromagnetic waves carry momentum, leading to the generation of a force known as **radiation pressure.**

☐ When light is incident normally upon a perfect absorber, it transfers its momentum to a surface area A. Within a small time interval Δt (much larger than ω^{-1}), the momentum transfer is given by $\Delta p = \langle \mathcal{P} \rangle A c \Delta t$, thereby resulting in the radiation pressure (average force per unit area):

$$P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{A}\frac{\langle \mathcal{P} \rangle A c \Delta t}{\Delta t} = c\langle \mathcal{P} \rangle = \frac{1}{2}\epsilon_0 E_0^2 = \frac{I}{c}.$$

☐ For a perfect reflector the pressure is twice as great, because the momentum is completely reflected.

☐ **An intriguing explanation** for radiation pressure is as follows: The electric field polarized in the x-direction impels charges to move in the x direction, subsequently resulting in the magnetic field in the y-direction exerting a force ($q\vec{v} \times \vec{B}$) on them in the z direction. The cumulative force acting on all charges within the surface gives rise to the pressure.

Outline

9.3.1 Propagation in Linear Media

Macroscopic Maxwell's equations in dielectrics

In practical applications, we are more concerned with the propagation of electromagnetic waves in macroscopic medium.

☐ If the medium is a dielectric with no free charges and currents, the macroscopic Maxwell's equations can be expressed as:

$$\text{(i)} \quad \nabla \cdot \vec{D} = 0, \qquad \text{(iii)} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$
$$\text{(ii)} \quad \nabla \cdot \vec{B} = 0, \qquad \text{(iv)} \quad \nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t},$$

☐ In a **linear and homogeneous dielectric**, $\vec{D} = \epsilon \vec{E}$ and $\vec{H} = \frac{1}{\mu}\vec{B}$, with ϵ and μ being constant throughout space. Subsequently,

$$\text{(i)} \quad \nabla \cdot \vec{E} = 0, \qquad \text{(iii)} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$
$$\text{(ii)} \quad \nabla \cdot \vec{B} = 0, \qquad \text{(iv)} \quad \nabla \times \vec{B} = \mu\epsilon \frac{\partial \vec{E}}{\partial t},$$

☐ Analogous to the vacuum, the speed of an EM wave propagating in dielectrics is determined by $v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n}$, where n represents the **index of refraction** defined as $n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$.

□ **On properties of μ and ϵ.** For most nonmagnetic materials, μ is very close to μ_0, so $n \approx \sqrt{\epsilon_r}$, where ϵ_r is the dielectric constant. Given that ϵ_r is typically greater than 1, the refractive index $n > 1$ and the speed of the electromagnetic wave is $v = c/n < c$.

□ **How does light propagate in dielectrics?** This observation of light propagation in dielectrics is mathematically pretty trivial, but the physical implications are astonishing: As the wave passes through, the fields busily polarize and magnetize all the molecules, and the resulting (oscillating) dipoles create their own electric and magnetic fields. These combine with the original fields in such a way as to **create a single wave with the same frequency but a different speed**. This extraordinary conspiracy is responsible for the phenomenon of transparency.

Inhomogeneous medium

Strictly speaking, the medium can never be homogeneous; therefore, we must consider the propagation of light in an **inhomogeneous medium**.

☐ For example, we need to know what happens when a wave passes from one transparent medium into another, e.g., air to water, or glass to plastic. In this case we need **boundary conditions**, which has already been derived before:

$$\begin{array}{llll} \text{(i)} & \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}, & \text{(iii)} & \vec{E}_1^{||} = \vec{E}_2^{||}, \\ \text{(ii)} & B_1^{\perp} = B_2^{\perp}, & \text{(iv)} & \frac{1}{\mu_1}\vec{B}_1^{||} = \frac{1}{\mu_2}\vec{B}_2^{||} \end{array}$$

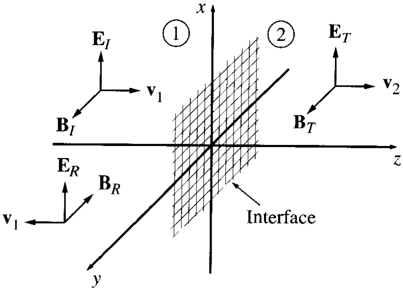
These boundary conditions establish a connection between the electromagnetic fields in the two media near the boundary.

☐ By applying the macroscopic Maxwell's equations and boundary conditions, we can quantitatively interpret optical phenomena like reflection and refraction as shown later.

Outline

9.3.2 Reflection and Transmission at Normal Incidence

In subsequent sections, we shall give the **reflection**, **refraction** and **transmission** of light a quantitative description using Maxwell equations with appropriate boundary conditions. We first consider normal incidence.



□ **Normal incidence.** The $x - y$ plane serves as the boundary separating two linear media. A plane wave with frequency ω, propagating in the z direction and polarized in the x direction, approaches the interface from the left.

Just like the stretched string, both the incident and reflected waves are located in medium ①, while the transmitted wave only exists in medium ②. Their wave functions are provided below:

☐ **Incident wave** in medium ①,

$$\tilde{\vec{E}}_I(z,t) = \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{x},$$

$$\tilde{\vec{B}}_I(z,t) = \frac{1}{v_1} \tilde{E}_{0I} e^{i(k_1 z - \omega t)} \hat{y}.$$

☐ **Reflected wave** in medium ①

$$\tilde{\vec{E}}_R(z,t) = \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{x},$$

$$\tilde{\vec{B}}_R(z,t) = -\frac{1}{v_1} \tilde{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{y}.$$

☐ **Transmitted wave** in medium ②,

$$\tilde{\vec{E}}_T(z,t) = \tilde{E}_{0T} e^{i(k_2 z - \omega t)} \hat{x},$$

$$\tilde{\vec{B}}_T(z,t) = \frac{1}{v_2} \tilde{E}_{0T} e^{i(k_2 z - \omega t)} \hat{y}.$$

The electric and magnetic fields all adhere to the relationship: $\vec{B} = \frac{\hat{k}}{v} \times \vec{E}$, possessing the identical oscillating frequency ω.

Boundary conditions

We now apply the boundary conditions, reiterated here, to determine the parameters in the wave functions.

$$\begin{array}{llll} \text{(i)} & \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp}, & \text{(iii)} & \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel} \\ \text{(ii)} & B_1^{\perp} = B_2^{\perp}, & \text{(iv)} & \dfrac{1}{\mu_1}\vec{B}_1^{\parallel} = \dfrac{1}{\mu_2}\vec{B}_2^{\parallel} \end{array} .$$

□ Conditions (i) and (ii) are automatically satisfied as the electromagnetic waves we construct are transverse, with vanishing perpendicular components: $E^{\perp} = E_z = 0$ and $B^{\perp} = B_z = 0$.

□ Conditions (iii) and (iv) lead to the following two equations:
(iii)$\Rightarrow \tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$
(iv)$\Rightarrow \frac{1}{\mu_1}\left(\frac{1}{v_1}\tilde{E}_{0I} - \frac{1}{v_1}\tilde{E}_{0R}\right) = \frac{1}{\mu_2}\left(\frac{1}{v_2}\tilde{E}_{0T}\right) \Leftrightarrow \tilde{E}_{0I} - \tilde{E}_{0R} = \beta\tilde{E}_{0T}$
Here, the parameter β is given by: $\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}$.
□ Solving these two equations allows us to express both $\tilde{E}_{0R}$ and $\tilde{E}_{0T}$ in terms of $\tilde{E}_{0I}$:

$$\tilde{E}_{0R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0I}, \quad \tilde{E}_{0T} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{0I}.$$

Nonmagnetic materials

For nonmagnetic materials, μ_1 and μ_2 are usually close to μ_0, so that $\beta \approx v_1/v_2$. In this case, we have the following expression:

$$\tilde{E}_{0R} \approx \left(\frac{v_2 - v_1}{v_2 + v_1}\right) \tilde{E}_{0I}, \quad \tilde{E}_{0T} \approx \left(\frac{2v_2}{v_1 + v_2}\right) \tilde{E}_{0I}.$$

This expression is identical to that of the stretched string. Similar conclusions regarding the wave amplitudes and phases could also be derived as follows:

□ The reflected wave is **in phase** with the incident wave when $v_2 > v_1$ and **out of phase** with the incident wave when $v_2 < v_1$.

□ The real amplitudes of the reflected and transmitted waves are proportional to the amplitude of the incident wave:

$$E_{0R} = \left|\frac{v_2 - v_1}{v_2 + v_1}\right| E_{0I}, E_{0T} = \left(\frac{2v_2}{v_2 + v_1}\right) E_{0I}.$$

Alternatively, using the indices of refraction:

$$E_{0R} = \left|\frac{n_1 - n_2}{n_1 + n_1}\right| E_{0I}, E_{0T} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{0I}.$$

Reflection and Transmission coefficients

Note that in the dielectrics, the intensity is given by $I = \frac{1}{2}\epsilon v E_0^2$.

□ The **reflection coefficient** is defined as the ratio of the reflected intensity to the incident intensity:

$$R \equiv \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{1-\beta}{1+\beta}\right)^2$$

□ Similarly, the **transmission coefficient** is defined as the ratio of the transmitted intensity to the incident intensity:

$$T \equiv \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \beta\left(\frac{2}{1+\beta}\right)^2$$

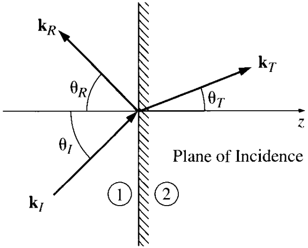
□ Clearly, $R + T = 1$, as a result of **energy conservation**. Furthermore, if $\mu_1, \mu_2 \approx \mu_0$, then $\beta \approx \frac{v_1}{v_2} = \frac{n_2}{n_1}$. In this approximation for nonmagnetic materials, we have $R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$ and $T = \frac{4n_1 n_2}{(n_1+n_2)^2}$, with $R + T = 1$ still being satisfied rigorously.

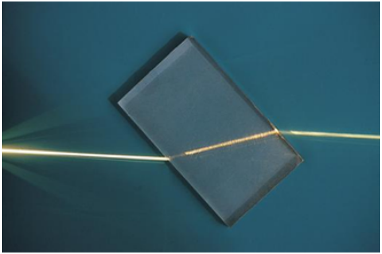
★ As an example, when light passes from air ($n_1 = 1$) into glass ($n_2 = 1.5$), we have $R = 0.04$ and $T = 0.96$. It is not surprising that most of the light is transmitted.

Outline

9.3.3 Reflection and Transmission at Oblique Incidence

Now let's consider the **oblique incidence** of a monochromatic plane wave. Unlike **normal incidence**, the transmitted light does not travel in the same direction of incident light through the medium, but is refracted, leading to many interesting optical phenomena.





☐ In this case, the reflected and incident waves are still in medium ①, while the transmitted wave is in medium ②. Furthermore, they continue to have the same oscillating frequency ω.

Wave functions and Boundary conditions

□ **Incident wave** in medium ①:

$$\tilde{\vec{E}}_I(\vec{r},t) = \tilde{\vec{E}}_{0I} e^{i(\vec{k}_I\cdot\vec{r}-\omega t)}, \qquad \tilde{\vec{B}}_I(\vec{r},t) = \frac{1}{v_1}(\hat{k}_I \times \tilde{\vec{E}}_I)$$

□ **Reflected wave** in medium ①:

$$\tilde{\vec{E}}_R(\vec{r},t) = \tilde{\vec{E}}_{0R} e^{i(\vec{k}_R\cdot\vec{r}-\omega t)}, \qquad \tilde{\vec{B}}_R(\vec{r},t) = \frac{1}{v_1}(\hat{k}_R \times \tilde{\vec{E}}_R)$$

□ **Transmitted wave** in medium ②:

$$\tilde{\vec{E}}_T(\vec{r},t) = \tilde{\vec{E}}_{0T} e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}, \qquad \tilde{\vec{B}}_T(\vec{r},t) = \frac{1}{v_2}(\hat{k}_T \times \tilde{\vec{E}}_T)$$

★ These waves have identical frequency, but different wave numbers

$$\omega = k_I v_1 = k_R v_1 = k_T v_2, \text{ or } k_I = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T.$$

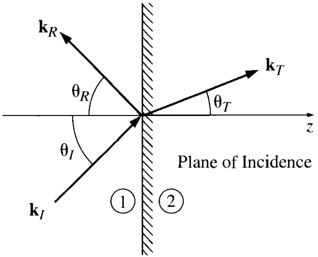
□ The components of the wavevectors in the $x-y$ plane can be determined by the continuity conditions of the EM fields at the interface located at "$z=0$":

$$(\cdots)_I e^{i(k_{Ix}x+k_{Iy}y-\omega t)} + (\cdots)_R e^{i(k_{Rx}x+k_{Ry}y-\omega t)} = (\cdots)_T e^{i(k_{Tx}x+k_{Ty}y-\omega t)}$$

★ It is evident that $(k_I)_y = (k_R)_y = (k_T)_y$ and $(k_I)_x = (k_R)_x = (k_T)_x$ for the same reason that the frequency must remain identical for the incident, reflected, and transmitted waves.

☐ Denote the wave vector in the xy-plane as $\vec{K}_{\parallel}$. Then, the wavevectors of the incident, reflected, and transmitted waves can be expressed as $\vec{k}_I = \vec{K}_{\parallel} + k_{1z}\hat{z}$, $\vec{k}_R = \vec{K}_{\parallel} - k_{1z}\hat{z}$, and $\vec{k}_T = \vec{K}_{\parallel} + k_{2z}\hat{z}$.

These three wavevectors clearly lie in the same plane spanned by $\vec{K}_{\parallel}$ and $\hat{z}$, which is referred to as **the plane of incidence**. As this plane includes the z-axis, it is perpendicular to the interface. This principle is known as **the First Law of Optics**.



☐ When aligning the x-axis with the vector $\vec{K}_{\parallel}$, the x-components of the three wavevectors are all equal to $K_{\parallel}$: $(k_I)_x = (k_R)_x = (k_T)_x = K_{\parallel}$. Furthermore, their y-components becomes zero: $(k_I)_y = (k_R)_y = (k_T)_y = 0$.

★ Introducing the **angles of incidence, reflection, and refraction** as θ_I, θ_R, and θ_T respectively, the equality of the x-components of k_I, k_R and k_T can be restated as: $k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$.

★ In addition, as previously derived, we have $k_I = k_R = \frac{n_1}{n_2} k_T$, leading to the formulation of the following two laws:

- ⋆ **Second Law (The reflection law):** $\boxed{\theta_I = \theta_R}$
- ⋆ **Third Law (Law of refraction, Snell's law):** $\boxed{\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}}$.

□ We have determined the wavevectors and frequencies of the incident, reflected, and transmitted waves. It is clear that the exponential factor $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ in the boundary conditions can be eliminated:

$$(\cdots)_I e^{i(\vec{k}_I\cdot\vec{r}-\omega t)}|_{z=0} + (\cdots)_R e^{i(\vec{k}_R\cdot\vec{r}-\omega t)}|_{z=0} = (\cdots)_T e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}|_{z=0}.$$

It results in the following relationship for the amplitudes of the EM waves:

(i) $\epsilon_1(\tilde{\vec{E}}_{0I} + \tilde{\vec{E}}_{0R})_z = \epsilon_2(\tilde{\vec{E}}_{0T})_z$, (iii) $(\tilde{\vec{E}}_{0I} + \tilde{\vec{E}}_{0R})_{x,y} = (\tilde{\vec{E}}_{0T})_{x,y}$

(ii) $(\tilde{\vec{B}}_{0I} + \tilde{\vec{B}}_{0R})_z = (\tilde{\vec{B}}_{0T})_z$, (iv) $\frac{1}{\mu_1}(\tilde{\vec{B}}_{0I} + \tilde{\vec{B}}_{0R})_{x,y} = \frac{1}{\mu_2}(\tilde{\vec{B}}_{0T})_{x,y}$

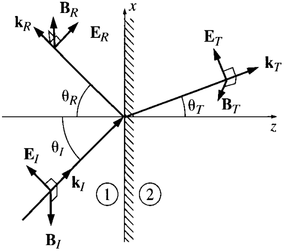
★ Before moving forward, we must distinguish between two potential polarizations: where $\vec{E}$ is either **parallel** to the plane of incidence (p-polarization) or **perpendicular** to it(s-polarization).

Parallel polarization (p-polarization)

We are presenting the results only for p-polarization, while leaving the s-polarization as an exercise for you. In this scenario, it is evident that

$\boxed{(\tilde{E}_{0I,R,T})_y = 0}$ and $\boxed{(\tilde{B}_{0I,R,T})_{x,z} = 0}$.

Since $\vec{B} = \frac{\hat{k}}{c} \times \vec{E}$, we find that $\tilde{B}_{0Iy} = \frac{\tilde{E}_{0I}}{v_1}$, $\tilde{B}_{0Ry} = -\frac{\tilde{E}_{0R}}{v_1}$ and $\tilde{B}_{0Ty} = \frac{\tilde{E}_{0T}}{v_2}$.



★ The B.C. for the **perpendicular** components (i) and (ii) can be written as:

(i) $\epsilon_1(-\tilde{E}_{0I}\sin\theta_I + \tilde{E}_{0R}\sin\theta_R) = \epsilon_2(-\tilde{E}_{0T}\sin\theta_T)$.

(ii) is satisfied automatically, since $\vec{B}$ has no z-components.

★ The B.C. for the **parallel** components (iii) and (iv) are given by:

(iii) $\tilde{E}_{0I}\cos\theta_I + \tilde{E}_{0R}\cos\theta_R = \tilde{E}_{0T}\cos\theta_T$.

(iv) $\frac{1}{\mu_1}(\tilde{B}_{0Iy} + \tilde{B}_{0Ry}) = \frac{1}{\mu_2}\tilde{B}_{0Ty} \Rightarrow \frac{1}{v_1\mu_1}(\tilde{E}_{0I} - \tilde{E}_{0R}) = \frac{1}{\mu_2 v_2}\tilde{E}_{0T}$.

Fresnel's equations

□ By applying the second law $\theta_I = \theta_R$ and Snell's Law $\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$, the condition (i) can be expressed as:

$$\epsilon_1(-\tilde{E}_{0I} + \tilde{E}_{0R}) = -\epsilon_2 \tilde{E}_{0T} \frac{\sin\theta_T}{\sin\theta_I} = -\epsilon_2 \tilde{E}_{0T} \frac{n_1}{n_2},$$

$$\Rightarrow \tilde{E}_{0I} - \tilde{E}_{0R} = \frac{\epsilon_2 n_1}{\epsilon_1 n_2} \tilde{E}_{0T} = \frac{v_1 \mu_1}{v_2 \mu_2} \tilde{E}_{0T} = \beta \tilde{E}_{0T},$$

This is exactly equivalent to the condition (iv).

□ Since the condition (ii) is already satisfied, we are left with only the condition (iii),

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \alpha \tilde{E}_{0T},$$

where $\alpha \equiv \frac{\cos\theta_T}{\cos\theta_I}$.

□ Upon solving these two equations, we derive **the Fresnel equations** for p-polarization.

$$\tilde{E}_{0R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right) \tilde{E}_{0I}, \qquad \tilde{E}_{0T} = \left(\frac{2}{\alpha+\beta}\right) \tilde{E}_{0I}.$$

Additionally, there exist two other Fresnel equations for s-polarization, which we leave as an exercise for you.

Properties of the refraction of p-polarized light

□ **Phase.** The transmitted wave is always in phase with the incident wave; the reflected wave is in phase if $\alpha > \beta$, or 180 degrees out of phase if $\alpha < \beta$.

□ **Amplitude.** Clearly, the amplitudes of the transmitted and reflected waves depend on the angle of incidence θ_I, as α is a function of it:

$$\alpha = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-[(n_1/n_2)\sin\theta_I]^2}}{\cos\theta_I}.$$

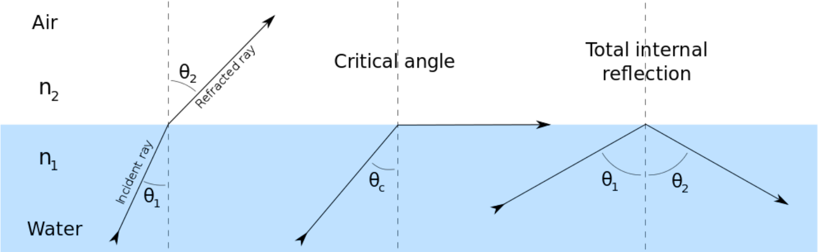
Two interesting cases are identified:

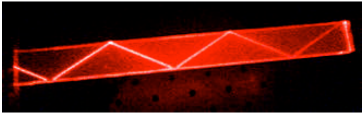
- When $\theta_I = 0$, $\alpha = 1$ and we obtain the case of normal incidence.
- When $\theta_I = \pi/2$, $\alpha = \infty$ and $\tilde{E}_{0T} = 0$, implying that the wave is completely reflected with no transmission.

□ Note that when $n_1 < n_2$, there is always a solution for θ_T as θ_I varies from 0 to $\pi/2$. However, when $n_1 > n_2$, there exists a critical value $\theta_c \equiv \arcsin\frac{n_2}{n_1}$ for the incident angle θ_I. If $\theta_I > \theta_c$, there is no solution for θ_T anymore, implying no transmitted light.

Total internal reflection

In fact, when light is incident from a medium with a larger refractive index to a medium with a smaller one, a phenomenon known as **total reflection** occurs, as depicted in the following figure.





Total internal reflection in a block of acrylic(丙烯酸塑料).

Brewster's angle

There is another interesting optical phenomenon revealed in Fresnel's equations for p-polarization. Let's examine the amplitude of the reflected wave:

$$\tilde{E}_{0R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0I}, \quad \alpha = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-[(n_1/n_2)\sin\theta_I]^2}}{\cos\theta_I}.$$

It is evident that when $\alpha = \beta$, $\tilde{E}_{0R} = 0$, implying that there is no reflection! In this case, the light is completely transmitted.

□ **Brewster's angle.** This phenomenon occurs at a specific incident angle θ_B, which is derived as $\boxed{\sin^2\theta_B = \frac{1-\beta^2}{(n_1/n_2)^2-\beta^2}}$ from $\alpha = \beta$. We call θ_B the Brewster's angle after the Scottish physicist Sir David Brewster (1781–1868).

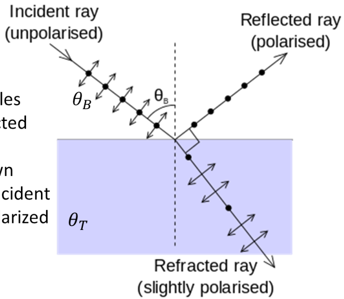
□ For nonmagnetic matter, both μ_1 and μ_2 are close to μ_0. In this case, $\beta \approx n_2/n_1$, $\sin^2\theta_B \approx \beta^2/(1+\beta^2)$. Hence, we find that $\boxed{\tan\theta_B \approx \frac{n_2}{n_1}}$.

Question: Why is there no reflection at the Brewster's angle?

This is related to the following facts:

☐ The reflected wave propagating along the direction of $\vec{k}_R$ is a superposition of the radiation of all the dipoles within the medium, stimulated by the electric fields $\vec{E}_T$ of the transmitted wave.

Incident and Reflected angles are both θ_B, and the refracted angle is θ_T.
Two polarizations are shown in the picture. When the incident angle is θ_B, the parallel polarized wave is NOT reflected.



☐ A dipole doesn't radiate along its oscillation direction, which in this case is the direction of the electric field $\vec{E}_T$. This condition is fulfilled when $\theta_B + \theta_T = \pi/2$ or $\tan\theta_B = \frac{\sin\theta_B}{\cos\theta_B} = \frac{\sin\theta_B}{\sin\theta_T} = \frac{n_2}{n_1}$ as shown in the figure.

☐ The Brewster's angle exists only for the p-polarized light, not for the s-polarized light.

Applications of Brewster's angle

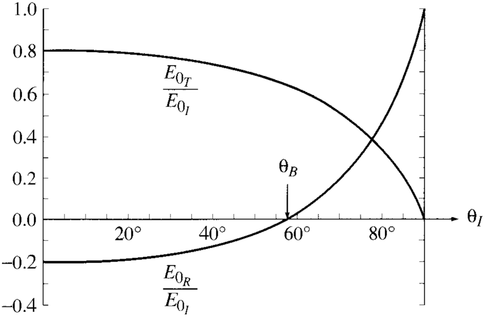
☐ When the incident angle equals to the Brewster's angle, there is no reflected light with p-polarization, but only with s-polarization. Such phenomenon can be used to produce linearly polarized light.

☐ When capturing clear images from inside a room or photographing underwater scenes from outside, the presence of reflected sunlight can pose a challenge. However, if we choose to take pictures at the Brewster's angle, the reflected light exhibits strong s-polarization. Therefore, a polarizer can effectively reduce the intensity of the reflected sunlight.

–picture taken from web.

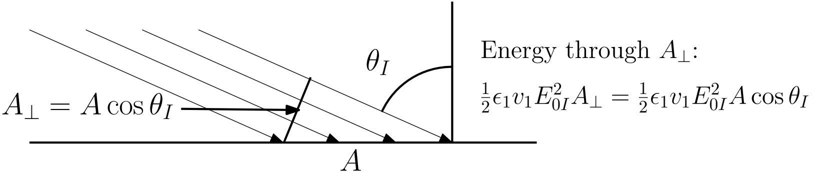
Reflection and Transmission coefficients

When light is incident on glass ($n_2 = 1.5$) from air ($n_1 = 1$), we depict the **transmitted** and **reflected** amplitudes as functions of θ_I in the following figure.



□ Here, it is noted that $n_1 < n_2$. However, if $n_1 > n_2$, the total reflection would occur at the critical angle which is greater than Brewster's angle due to the inequality: $\theta_c > \sin\theta_c = \frac{n_2}{n_1} = \tan\theta_B > \theta_B$.

□ **Energy flux.** For oblique incidence, the intensity of the incident light is given by: $I_I = \frac{1}{2}\epsilon_1 v_1 E_{0I}^2 \cos\theta_I$ as depicted in the figure.



The intensities of the reflected and transmitted intensities have the similar forms: $I_R = \frac{1}{2}\epsilon_1 v_1 E_{0R}^2 \cos\theta_R$, and $I_T = \frac{1}{2}\epsilon_2 v_2 E_{0T}^2 \cos\theta_T$.

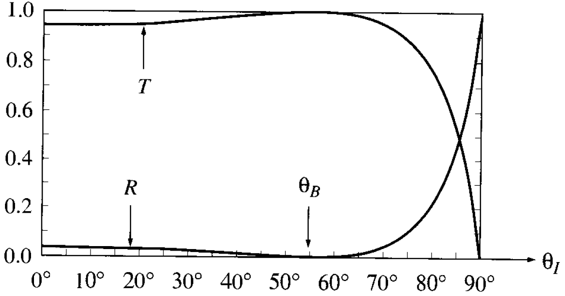
□ The **reflection** and **transmission coefficients** are defined as $R = \frac{I_R}{I_I}$ and $T = \frac{I_T}{I_I}$, respectively. These coefficients can be calculated for p-polarized light as follows:

$R \equiv \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2,$

$T \equiv \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2$

□ Clearly, $R + T = 1$ manifesting the law of energy conservation.

□ For $n_1 = 1$(air) and $n_2 = 1.5$(glass), the transmission and reflection coefficents are plotted as functions of θ_I in the following figure:



The reflection coefficient R represents the portion of incident energy that is reflected, and it approaches zero at Brewster's angle θ_B as expected. The transmission coefficient T represents the fraction that is transmitted, and it approaches 1 at θ_B.

□ This subsection describes the propagation of light through dielectrics without free charges and currents. In the following section, we will explore the propagation of light in conductors.

Outline

9.4.1 Electromagnetic Waves in Conductors

Electromagnetic response in conductors

When light strikes the surface of a conductor, the scenario is entirely different from that of dielectrics. In conductors, there are free currents that follow **Ohm's law**: $\vec{J}_f = \sigma \vec{E}$[1].

☐ **Maxwell's equations** in conductors can be expressed as:

$$\text{(i)}\vec{\nabla} \cdot \vec{E} = \frac{\rho_f}{\epsilon}, \quad \text{(iii)}\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$$

$$\text{(ii)}\vec{\nabla} \cdot \vec{B} = 0, \quad \text{(iv)}\vec{\nabla} \times \vec{B} = \mu\sigma\vec{E} + \mu\epsilon\partial_t \vec{E}.$$

☐ **Continuity Equation.** $\frac{\partial \rho_f}{\partial t} = -\vec{\nabla} \cdot \vec{J}_f = -\sigma \vec{\nabla} \cdot \vec{E} = -\frac{\sigma}{\epsilon}\rho_f$. Therefore, $\rho_f(t) = e^{-(\sigma/\epsilon)t}\rho_f(0)$ indicates that any initial free charge accumulation $\rho_f(0)$ decays over a characteristic time $\tau \equiv \epsilon/\sigma$.

[1]Here, we assume that σ is a constant independent of ω. The realistic electromagnetic response of conductors is quite complex, which is beyond the scope of this course.

☐ The characteristic time[2] τ charaterizes how good a conductor is. Three cases are listed below:

1. For a "perfect" conductor, $\sigma = \infty$ and $\tau = 0$;
2. For a "good" conductor, τ is much less than the other relevant times in the problem;
3. For a "poor" conductor, τ is greater than other characteristic times in the problem.

☐ Suppose we wait long enough for any accumulated free charge to disappear. In that case, $\rho_f = 0$, and Maxwell's equations simplify as:

$$\text{(i)}\vec{\nabla}\cdot\vec{E} = 0, \quad \text{(iii)}\vec{\nabla}\times\vec{E} = -\partial_t\vec{B},$$

$$\text{(ii)}\vec{\nabla}\cdot\vec{B} = 0, \quad \text{(iv)}\vec{\nabla}\times\vec{B} = \mu\sigma\vec{E} + \mu\epsilon\partial_t\vec{E}.$$

Here, only the transverse components of the electromagnetic fields are present, as Eqs. (i) and (ii) remove the longitudinal components.

[2] Please distinguish between τ and the **mean free time** in Drude's theory of metals. In Drude's theory, a longer mean free time indicates a higher conductivity, or a better conductor.

Electromagnetic waves in conductors

☐ **Wave equations.**

As the previous derivation for EM waves, we apply the curl to equations (iii) and (iv) to derive the wave equations for the transverse electric and magnetic fields:

$$\vec{\nabla}^2 \vec{E} = \mu\epsilon\partial_t^2 \vec{E} + \mu\sigma\partial_t \vec{E}, \quad \vec{\nabla}^2 \vec{B} = \mu\epsilon\partial_t^2 \vec{B} + \mu\sigma\partial_t \vec{B}.$$

☐ **Plane wave solutions.** Just as the case for dielectrics, these equations also allow for plane-wave solutions with normal incidence,

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{i(\tilde{k}z - \omega t)}, \quad \tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_0 e^{i(\tilde{k}z - \omega t)}.$$

☐ However, the **wave number** $\tilde{k}$ becomes **complex**,

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega.$$

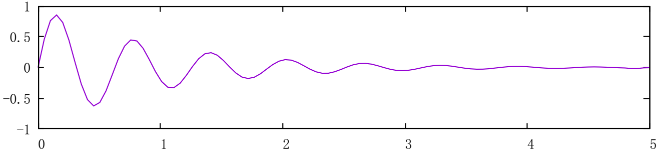
Let $\tilde{k} = k + i\kappa$, we can determine that:

$$k \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} + 1\right]^{\frac{1}{2}}, \; \kappa \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega}\right)^2} - 1\right]^{\frac{1}{2}}.$$

☐ The **real part** k still gives rise to the wavelength, propagation speed, and index of refraction in the conventional manner: $\lambda = \frac{2\pi}{k}$, $v = \frac{\omega}{k}$, and $n = \frac{ck}{\omega}$.

□ The **imaginary part** κ causes the wave amplitude to decrease exponentially with increasing z, resulting in **attenuation** as clearly shown in the following wave equations and pictures:

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}, \quad \tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_0 e^{-\kappa z} e^{i(kz-\omega t)}.$$



□ It is worth noting that $-k-i\kappa$ also satisfies the wave equation, but the amplitude increases with increasing z. However, this situation is unrealistic without energy input.

□ The **skin depth** is defined as $\boxed{d \equiv \frac{1}{\kappa}}$, representing the distance the field travels to decrease its amplitude by a factor of $e^{-1} \sim 1/3$. It also indicates the depth to which the wave penetrates into the conductor.

Skin depths of some typical metals

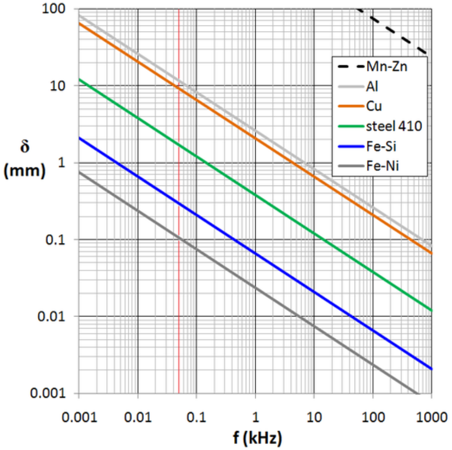
☐ The skin depth *d* decreases with increasing σ, μ, and ω, while increases with ϵ as shown in the following equation:

$$\kappa = \omega\sqrt{\frac{\epsilon\mu}{2}}[\sqrt{1+(\frac{\sigma}{\epsilon\omega})^2}-1]^{\frac{1}{2}} = \sigma\sqrt{\frac{\mu}{2}}[\sqrt{\epsilon^2+(\frac{\sigma}{\omega})^2}+\epsilon]^{-\frac{1}{2}}$$

The skin depths of some familar metals at a frequency of 10 GHz (microwave region) are less than a micrometer,

σ : Ag > Cu > Au > Al

Conductor	Skin depth
Aluminum	0.80 μm
Gold	0.79 μm
Copper	0.65 μm
Silver	0.64 μm



Polarization and Phases of EM waves

Assuming the polarization is along the x direction, the wave functions take the following form:

$$\tilde{\vec{E}}(z,t) = \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)} \hat{x}, \qquad \tilde{\vec{B}}(z,t) = \frac{\tilde{k}}{\omega} \tilde{E}_0 e^{-\kappa z} e^{i(kz-\omega t)} \hat{y}.$$

The complex amplitude of the magnetic field is determined by Ampere's law.

□ The complex wavevector can be expressed in polar form as $\tilde{k} = Ke^{i\phi}$, where the argument and the modulus have the following form:

$$\phi = \tan^{-1}\left(\frac{\kappa}{k}\right) = \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{\epsilon\omega}{\sigma}\right),$$

$$K = \sqrt{k^2+\kappa^2} = \omega\sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}.$$

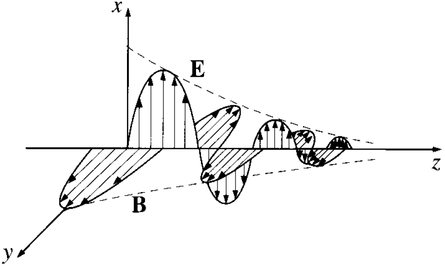
Let $\tilde{E}_0 = E_0 e^{i\delta_E}$, then $\tilde{B}_0 = B_0 e^{i\delta_B} = \frac{\tilde{k}}{\omega}\tilde{E}_0 = \frac{K}{\omega} E_0 e^{i(\phi+\delta_E)}$. Hence, the magnitudes and phases of the electric and magnetic fields are related as follows:

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\epsilon\mu\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}}, \qquad \delta_B - \delta_E = \phi.$$

☐ **Real wavefunctions.** The actual electric and magnetic fields are the real parts of $\tilde{\vec{E}}$ and $\tilde{\vec{B}}$ respectively, given by

$$\vec{E}(z,t) = E_0 e^{-\kappa z}\cos(kz - \omega t + \delta_E)\hat{x},$$
$$\vec{B}(z,t) = B_0 e^{-\kappa z}\cos(kz - \omega t + \delta_B)\hat{y}.$$

The field configurations are depicted in the following figure,



☐ Clearly, the magnetic field **lags behind** the electric field, i.e., the magnetic and electric fields are no longer in phase in a metal.

Outline

9.4.2 Reflection at a Conducting Surface

As electromagnetic waves cannot propagate through the conductor, the majority of the electromagnetic energy will be reflected by the surface of the conductor. In this subsection, we will investigate the reflection with normal incidence at a conducting surface.

☐ **Boundary conditions.** We first show the boundary conditions:

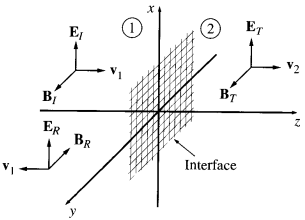
(i) $\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f$, (iii) $\vec{E}_1^{||} - \vec{E}_2^{||} = 0$.

(ii) $B_1^{\perp} - B_2^{\perp} = 0$, (iv) $\frac{1}{\mu_1}\vec{B}_1^{||} - \frac{1}{\mu_2}\vec{B}_2^{||} = \vec{K}_f \times \hat{n}$.

☐ **Requirement on σ_f and $\vec{K}_f$** which are the **possible** free surface charge and current densities at the conducting surface, respectively.

- For transverse electromagnetic waves with normal incidence, $E_1^{\perp} = E_2^{\perp} = 0$, implying $\sigma_f = 0$ based on condition (i).
- For ohmic conductors, the volume current density is given by $\vec{J}_f = \sigma \vec{E}$. Since the conductivity σ and $\vec{E}$ are both finite, there is no 2D surface current $\vec{K}_f$(otherwise $\vec{J}_f$ is infinite on the surface).

Wavefunction of electromagnetic wave



Let's say the xy plane forms the boundary between a dielectric material and a conductor. A monochromatic plane wave, traveling in the z direction and polarized in the x direction, is approaching from the dielectric toward the conductor.

- Again the wavefunction consists of three parts, and $\tilde{\vec{B}} = \frac{1}{\omega}k\hat{z}\times\tilde{\vec{E}}$,
  - Incident wave in dielectric:
    $\tilde{\vec{E}}_I(z,t) = \tilde{E}_{0I}e^{i(k_1 z-\omega t)}\hat{x}$, $\tilde{\vec{B}}_I(z,t) = \frac{1}{v_1}\tilde{E}_{0I}e^{i(k_1 z-\omega t)}\hat{y}$
  - Reflected wave in dielectric:
    $\tilde{\vec{E}}_R(z,t) = \tilde{E}_{0R}e^{i(-k_1 z-\omega t)}\hat{x}$, $\tilde{\vec{B}}_R(z,t) = -\frac{1}{v_1}\tilde{E}_{0R}e^{i(-k_1 z-\omega t)}\hat{y}$
  - Transmitted wave in conductor:
    $\tilde{\vec{E}}_T(z,t) = \tilde{E}_{0T}e^{i(\tilde{k}_2 z-\omega t)}\hat{x}$, $\tilde{\vec{B}}_T(z,t) = \frac{\tilde{k}_2}{\omega}\tilde{E}_{0T}e^{i(\tilde{k}_2 z-\omega t)}\hat{y}$

☐ Let's check the boundary conditions for the transverse waves with normal incidence. In this case, $E^{\perp} = 0$ and $B^{\perp} = 0$.

- The conditions (i) and (ii) are automatically satisfied.
- The condition (iii) requires that: $\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$.
- The condition (iv) with $\vec{K}_f = 0$ results in the following relation: $\frac{1}{\mu_1 v_1}(\tilde{E}_{0I} - \tilde{E}_{0R}) - \frac{\tilde{k}_2}{\mu_2 \omega}\tilde{E}_{0T} = 0 \Rightarrow \tilde{E}_{0I} - \tilde{E}_{0R} = \tilde{\beta}\tilde{E}_{0T}$.
  Here, we define $\boxed{\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2 = \frac{\mu_1 v_1}{\mu_2 \omega}(k_2 + i\kappa_2)}$.

It follows that the complex amplitudes of the reflected and transmitted wave can be expressed as follows:

$$\tilde{E}_{0R} = \left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{0I}, \quad \tilde{E}_{0T} = \left(\frac{2}{1+\tilde{\beta}}\right)\tilde{E}_{0I}.$$

☐ These results are formally identical to the previous results for the light propagating from one dielectric to the other, but the resemblance is deceptive since $\tilde{\beta}$ is now a complex number.

Energy conservation for light reflected at a conducting surface

When light is reflected at the interface between two dielectrics, the reflection and transmission coefficients satisfy $R + T = 1$ as a result of energy conservation. However, such a relation does not apply in the case of reflection at a conducting surface, as electromagnetic waves cannot propagate inside a conductor, making it impossible to define the transmission coefficient in the usual manner.

□ **Energy missing in the reflection.** The reflection coefficient is still defined as: $R = \frac{I_R}{I_I} = \left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2$. The difference of the intensity between the incident and reflected waves is given by:

$$\begin{aligned}\Delta I =& I_I(1-R) = \frac{1}{2}\epsilon_1 v_1 |E_{0I}|^2 (1-R) = \epsilon_1 v_1 |E_{0I}|^2 \frac{\tilde{\beta}+\tilde{\beta}^*}{|1+\tilde{\beta}|^2} \\ =& \epsilon_1 v_1 |E_{0I}|^2 \frac{\mu_1 v_1}{\mu_2 \omega} \frac{2k_2}{|1+\tilde{\beta}|^2} = \frac{2k_2}{\mu_2 \omega} \frac{|E_{0I}|^2}{|1+\tilde{\beta}|^2}\end{aligned}$$

The missing energy in the dielectric after reflection is transferred into the conductor as Joule's heat through the interface, as depicted below.

□ **Calculating Joule's heat in the conductor.**

★ The power density of Joule's heat in the conductor is given by: $\mathcal{P}_Q(\vec{r},t) = \vec{E}_T(\vec{r},t)\cdot\vec{J}(\vec{r},t) = \sigma|\vec{E}_T(\vec{r},t)|^2$. For a plane wave with normal incidence, we observe that P_Q is a function of z and t: $\mathcal{P}_Q(z,t) = \sigma E_{0T}^2 e^{-2\kappa_2 z}\cos^2(k_2 z - \omega t + \delta_E)$. Furthermore, for a rapidly oscillating field, we can conduct a time average resulting in $\mathcal{P}_Q(z) = \frac{1}{2}\sigma E_{0T}^2 e^{-2\kappa_2 z}$.

★ Then, the total power of Joule's heat can be obtained as follows: $\Delta P_Q = \int_0^\infty (\Delta A dz)\mathcal{P}_Q(z) = \frac{1}{4\kappa_2}\Delta A\sigma E_{0T}^2 = \frac{\Delta A\sigma}{\kappa_2|1+\tilde{\beta}|^2}|E_{0I}|^2$. Here, ΔA represents the area of the conducting surface, and we utilize the relationship between $\tilde{E}_{0T}$ and $\tilde{E}_{0I}$.

★ Since $\tilde{k}_2^2 = \mu_2\epsilon_2\omega^2 + i\mu_2\sigma\omega$, we deduce that $2k_2\kappa_2 = \mu_2\sigma\omega$, so that $\dfrac{\Delta P_Q}{\Delta A} = \dfrac{2k_2}{\mu_2\omega}\dfrac{|E_{0I}|^2}{|1+\tilde{\beta}|^2}$, which represents the exact missing energy intensity when the light is reflected in the dielectric.

□ **The law of energy conservation holds true.**

青铜镜

□ In the case of a *perfect* conductor with $\sigma = \infty$, we would have infinite values for $\tilde{k}_2$ and $\tilde{\beta}$, resulting in $\tilde{E}_{0R} = -\tilde{E}_{0I}$ and $\tilde{E}_{0T} = 0$.

In this scenario, the wave experiences complete reflection with a 180° phase shift. This is the reason why excellent conductors are effective as mirrors. In practical applications, a thin coating of silver is applied to the back of a glass pane. Given that the skin depth in silver at optical frequencies is around 100Å, only a thin layer is required to create a high-quality mirror.



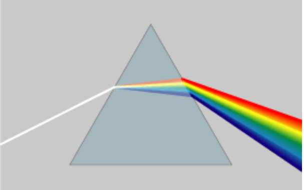
(从左到右：战国喜鹊猴子古铜镜、青铜镜正面、西汉“透光”镜)

Outline

9.4.3 The Frequency Dependence of Permittivity

☐ **Dispersion.** In a vacuum, the speed of light is always $c = 1/\sqrt{\epsilon_0\mu_0}$, independent of the frequency ω. However, in dielectrics, the speed v of an electromagnetic wave depends on its frequency ω, causing the refractive index $n = c/v$ to also depend on ω. This phenomenon is known as **dispersion**, and the supporting medium is referred to as **dispersive**.

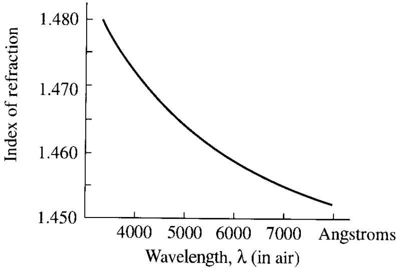
Because the color of visible light is determined by its frequency, dispersion leads to many interesting phenomena, such as:



☐ Dispersion occurs because of the collective response of various atoms with <u>different natural frequencies in dielectrics to EM waves</u>, causing the permittivity ϵ, permeability μ, and conductivity σ to all depend on the frequency ω of the EM wave.

★ As demonstrated earlier, the d'Alembert's wave equation suggests $k^2 - \epsilon\mu\omega^2 = 0$, which is valid even if ϵ and μ depend on ω. This equation defines the relationship between ω and k, known as **the dispersion relation**.

In nonmagnetic materials, the permeability μ approximates the vacuum permeability μ_0, resulting in the refractive index n being approximately equal to the relative permittivity ϵ_r. In the figure, we illustrate the relationship between n and λ for a typical glass.



Phase velocity and Group velocity

In a dispersive medium, we need to distinguish between two different types of velocities.

□ **Phase velocity.** The previously mentioned velocity is referred to as the "phase velocity", defined by $v_p = \omega/k$, representing the velocity of the equal phase plane of a sinusoidal wave $\cos(kz - \omega t + \delta)$.

The phase velocity is determined by the permittivity and permeability of the medium $v_p = 1/\sqrt{\epsilon(\omega)\mu(\omega)}$. While it is a constant c in a vacuum, it is dependent on ω in a dispersive medium.

□ **Group velocity.** In the case of a wave packet that encompasses a spectrum of frequencies, its envelope propagates as a cohesive unit at the group velocity, denoted as $v_g = \frac{d\omega}{dk}$.

As v_p is dependent on ω, each sinusoidal component of a wavepacket will propagate at different speeds, leading to the collapse of the wave form eventually. <u>To gain a deeper understanding of these velocities, we will dedicate a few pages to discussing them</u>.

More on v_g and v_p

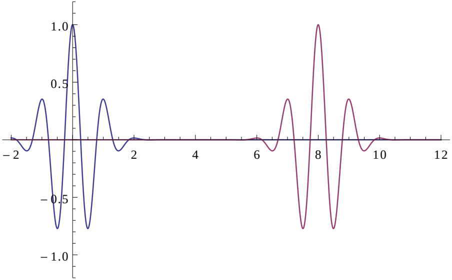
Let's examine the Gaussian wave packet as shown below:

$$\psi(x,t) = \int dk A(k) e^{ikx - i\omega(k)t} = \int dk \sqrt{\frac{a}{\pi}} e^{-a(k-k_0)^2} e^{ikx - i\omega(k)t}.$$

Here $A(k)$ represents the normalized momentum distribution satisfying $\int dk A(k) = 1$. As time progresses, the wavepacket will evolve in a manner determined by the dispersion relation $\omega(k)$.

☐ **Linear dispersion** $\omega = vk$**.** The wavefunction is given by:

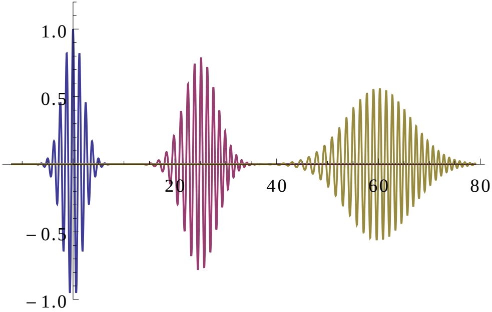
$$\psi(x,t) = \int dk \sqrt{\frac{a}{\pi}} e^{-a(k-k_0)^2} e^{ikx - ivkt} = e^{-\frac{(x-vt)^2}{4a} + ik_0(x-vt)}.$$



In this case, all the sinusoidal components have the same phase velocity $v_p = v$. Hence, the wavepacket moves also at this uniform velocity ($v_g = v$) with its shape unchanged as demonstrated in the figure.

□ **Parabolic dispersion $\omega = bk^2$.** The wavefunction of the wavepacket can be calculated explicitly as follows:

$$\psi(x,t) = \int dk \sqrt{\frac{a}{\pi}} e^{-a(k-k_0)^2} e^{ikx - ibk^2 t} = \frac{1}{\sqrt{1+\frac{ibt}{a}}} e^{-\frac{x^2 + 4iak_0(bk_0 t - x)}{4(a+ibt)}}.$$



In this instance, the phase velocity $v_p = bk$ varies for each sinusoidal component. Hence, the evolving wavepacket, as depicted in the figure, showcases its collapse due to the distinct phase velocities of each sinusoidal component.

★ Indeed, in all instances of nonlinear dispersion, the phase velocity becomes dependent on the wave number or frequency, inevitably leading to the collapse of a spatially localized wavepacket.

□ **Group velocity** is defined as $v_g = d\omega/dk$. To understand the essence of v_g, let's examine a wavepacket with a narrow momentum distribution $A(k)$. In this case, only the momenta around k_0 contribute, allowing a Taylor expansion of $\omega(k)$: $\omega(k) = \omega_0 + \frac{d\omega}{dk}\Big|_{k_0}(k - k_0) + \cdots$. To the linear order of k, we observe that

$$\begin{aligned}\psi(x,t) &= \int dk A(k) e^{ikx - i\omega t} \approx \int dk A(k) e^{ik_0 x + i(k-k_0)x - i(\omega_0 + v_g(k-k_0))t} \\ &\approx e^{ik_0 x - i\omega_0 t} \int dk \sqrt{\tfrac{a}{\pi}} e^{-a(k-k_0)^2} e^{i(k-k_0)(x - v_g t)} \\ &\approx e^{ik_0(x - v_p t)} e^{-(x - v_g t)^2/(4a)}\end{aligned}$$

This approximate wave function comprises a sinusoidal wave with a phase velocity of $v_p = \omega_0/k_0$ and a Gaussian wavepacket whose center moves at the group velocity $v_g = \frac{d\omega}{dk}$.

★ It should be noted that the decomposition of a general wave packet as described above holds only for a finite time interval; beyond that, the linear expansion will no longer be valid.

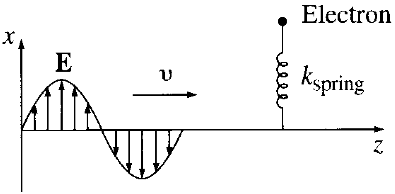
□ **Examples:** For EM wave, $\omega(k) = ck$ and $v_g = v_p = c$. For the free particle in nonrelativistic QM, $\omega(k) = \frac{\hbar k^2}{2m}$, $v_g = 2v_p = \frac{\hbar k}{m}$. For a surface wave in deep water, $\omega(k) = \sqrt{gk}$, $v_g = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}\frac{\omega}{k} = \frac{1}{2}v_p$.

Why is there dispersion?

Now let's return to the frequency dependence of ϵ in nonconductors.

A simple spring model for electrons in dielectrics

☐ **Electrons in a nonconductor** are bound to specific molecules. We can expand the binding potential in a Taylor series with respect to the displacement x from their equilibrium position as follows:



$U(x) = U(0) + xU'(0) + \frac{1}{2}x^2 U''(0) + \cdots$.

★ Since at the equilibrium position $U'(0) = 0$, in the lowest order approximation, we can visualize each electron as being connected to the end of a hypothetical spring, characterized by a force constant $k_{\text{spring}} = U''(0)$, as illustrated in the figure.

□ Then, the **binding force** on electrons is given by:

$$F_{\text{binding}} = -k_{\text{spring}} x = -m\omega_0^2 x,$$

where m represents the electron's mass, and $\omega_0 \equiv \sqrt{k_{\text{spring}}/m}$ denotes the natural oscillation frequency.

□ A **damping force** opposing the velocity is introduced as:

$$F_{\text{damping}} = -m\gamma\dot{x}.$$

Here γ is the damping parameter, and we don't worry about its origin at present.

□ In the presence of an EM wave polarized in the x direction with frequency ω, the electron experiences a **driving force** given by:

$$F_{\text{driving}} = qE = qE_0 \cos(\omega t),$$

where q represents the electron's charge and E_0 is the amplitude of the wave at the electron's location.

□ The **E.O.M.** of the electron is given by Newton's second law:

$$m\ddot{x} = F_{tot} = F_{binding} + F_{damping} + F_{driving},$$

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = qE_0 \cos(\omega t).$$

□ Solution of x and the polarization.

★ Clearly, this E.O.M. is the real part of the following one:

$$\ddot{\tilde{x}} + \gamma\dot{\tilde{x}} + \omega_0^2\tilde{x} = \frac{q}{m}E_0 e^{-i\omega t}.$$

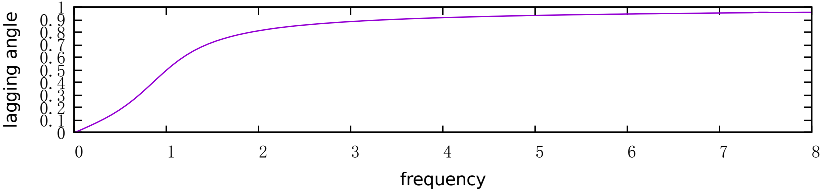
★ Assuming $\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t}$ and substituting it into the complex differential equation above, we find:

$$\tilde{x}_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0.$$

★ The dipole moment is the real part of the following quantity:

$$\tilde{p}(t) = q\tilde{x}(t) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega}E_0 e^{-i\omega t}.$$

The imaginary term in the denominator indicates that the polarization lags behind the E-field by an angle of $\tan^{-1}\left[\frac{\gamma\omega}{\omega_0^2-\omega^2}\right]$. This angle is plotted as a function of ω in the figure below.



Susceptibility, permittivity of dielectrics

☐ The natural frequency ω_0 and damping coefficient γ of an electron in a given molecule are dependent on its surroundings. Assuming there are f_j electrons sharing identical surrounding conditions, they also share the same γ and ω_0, denoted as γ_j and ω_j respectively.

★ If the number of molecules per unit volume is N, the polarization $\vec{P}$ is determined by the real part of the **complex polarization** $\tilde{\vec{P}}$:

$$\tilde{\vec{P}} = \frac{Nq^2}{m}\left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right)\tilde{\vec{E}}_0 = \epsilon_0 \tilde{\chi}_e \tilde{\vec{E}}_0.$$

Here, we introduce **the complex susceptibility** $\tilde{\chi}_e$.

★ The **actual** polarization and electric field may not exhibit a linear relationship due to the phase difference caused by the damping term.

☐ The complex permittivity is defined as $\tilde{\epsilon} = \epsilon_0(1 + \tilde{\chi}_e)$, thus the **complex dielectric constant** is given by:

$$\tilde{\epsilon}_r = \frac{\tilde{\epsilon}}{\epsilon_0} = 1 + \frac{Nq^2}{m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}.$$

Electromagnetic waves in dispersive medium

In a dispersive nonmagnetic medium, the wave equation allows for a plane wave solution with a given frequency, described as:

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{i(\tilde{k}z-\omega t)}.$$

☐ **Complex wave number.** Here, the wave number must be complex to satisfy $\tilde{k}^2 = \epsilon\mu\omega^2$, leading to:

$$\tilde{k} = \sqrt{\tilde{\epsilon}\mu}\omega \approx \sqrt{\tilde{\epsilon}\mu_0}\omega = \frac{\omega}{c}\sqrt{\tilde{\epsilon}_r} = \frac{\omega}{c}\sqrt{1+\tilde{\chi}_e}.$$

Expressing $\tilde{k}$ in terms of its real and imaginary components as $\tilde{k} = k + i\kappa$, the plane wave transforms to:

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{-\kappa z} e^{i(kz-\omega t)}.$$

★ Clearly, the wave undergoes attenuation as energy is absorbed due to damping.

☐ Since the intensity is proportional to E^2 (and thus to $e^{-2\kappa z}$), we can define the **absorption coefficient** as $\alpha \equiv 2\kappa$.

The **refractive index** is defined as $n = ck/\omega$.

□ **Weak polarization.** Generally, polarization is weak for gases, implying that $|\tilde{\chi}_e| \ll 1$. Consequently, $\tilde{k}$ can be approximated as:

$$\tilde{k} = \frac{\omega}{c}(1 + \frac{1}{2}\tilde{\chi}_e + \cdots) \approx \frac{\omega}{c}\left[1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega}\right].$$

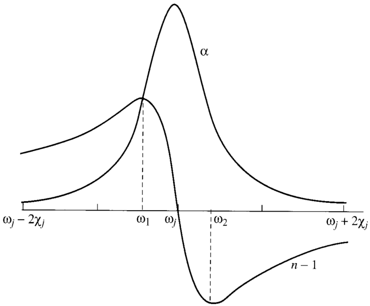
Then, the refractive index and the absorption coefficient read:

$$n = \frac{ck}{\omega} = \frac{c}{\omega}\mathrm{Re}(\tilde{k}) \approx 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j(\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2},$$

$$\alpha = 2\kappa = 2\mathrm{Im}(\tilde{k}) \approx \frac{Nq^2\omega^2}{m\epsilon_0 c}\sum_j \frac{f_j\gamma_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2\omega^2}.$$

★ We observe that at resonance frequencies ω_j's, $n = 1$, and α reaches its maximum. The behaviors of α and $n - 1$ near resonance frequencies are quite peculiar, as depicted below.

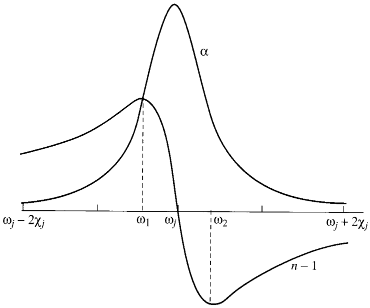
Anomalous dispersion near resonance



☐ The figure illustrates that in most cases, the index of refraction increases gradually with frequency, aligning with optical expectations. However, in the vicinity of a resonance ($\omega_1 < \omega < \omega_2$), the index of refraction sharply decreases, a phenomenon termed **anomalous dispersion**.

☐ **Large Dissipation.** The region of anomalous dispersion overlaps with maximum absorption, rendering the material practically opaque within this frequency range. This occurs because we're exciting the electrons at their preferred frequency, resulting in significant oscillation amplitudes and consequent energy dissipation through damping.

Anomalous dispersion near resonance



☐ **Faster than speed of light?** In this graph, the refractive index *n* runs below 1 above the resonance, suggesting that the **phase** speed exceeds *c*. This is no cause for alarm, since energy does not travel at the phase velocity but rather at the **group velocity** (see Prob. 9.25). Moreover, the graph does not include the contributions of other terms in the sum, which add a relatively constant "background" that, in some cases, keeps $n > 1$ on both sides of the resonance.

Cauchy's formula, far from resonance

☐ If ω is far from resonance, the condition $(\omega_j^2 - \omega^2)^2 \gg \gamma_j^2\omega^2$ holds, rendering damping negligible. Hence $n \approx 1 + \frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2 - \omega^2}$.

★ For most substances, the frequencies ω_j's are scattered throughout the spectrum in a disorganized manner. But, for transparent medium, the nearest significant resonances typically occur in the ultraviolet range. Hence, we can assume $\omega < \omega_j$, leading to:

$$n \approx 1 + \left(\frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^2}\right) + \omega^2\left(\frac{Nq^2}{2m\epsilon_0}\sum_j \frac{f_j}{\omega_j^4}\right).$$

☐ To derive **Cauchy's formula**, one can retain terms up to order ω^2 and substitute ω with the wavelength $\lambda = 2\pi c/\omega$. This yields:

$$n = 1 + A\left(1 + \frac{B}{\lambda^2}\right).$$

Here, A and B are referred to as the **coefficient of refraction** and the **coefficient of dispersion**, respectively. This formula is particularly applicable to most gases within the optical spectrum.

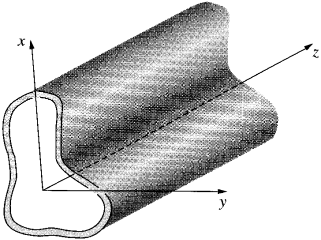
Outline

9.5.1 Wave Guides

In this section, we study electromagnetic waves propagating within a specialized confined space known as a "waveguide."

★ Waveguide supplying power for the Advanced Photon Source.

☐ The **waveguide** is a hollow pipe through which electromagnetic waves are confined. We assume that the waveguide is a perfect conductor with a conductivity value of $\sigma = \infty$, and that its cross-section remains constant along its length.



Boundary conditions of a waveguide

★ Inside a perfect conductor with a conductivity $\sigma = \infty$, the electric field is zero ($\vec{E} = 0$), since otherwise, the current density would be infinite ($\vec{J} = \sigma \vec{E} = \infty$). There is also no magnetic field present ($\vec{B} = 0$) because, according to Faraday's law, the temporal change of magnetic field is zero ($\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E} = 0$). If the magnetic field started out as zero, it will remain so within the perfect conductor.

★ In addition, the boundary conditions at the metallic surface take the following form:

(i) $\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f$, (iii) $\vec{E}_1^{\parallel} - \vec{E}_2^{\parallel} = 0$.

(ii) $B_1^{\perp} - B_2^{\perp} = 0$, (iv) $\frac{1}{\mu_1}\vec{B}_1^{\parallel} - \frac{1}{\mu_2}\vec{B}_2^{\parallel} = \vec{K}_f \times \hat{n}$.

□ Therefore, the **boundary conditions** of the electromagnetic fields at the inner surface of the waveguide are given as follows:

(1) $\vec{E}^{\parallel} = 0$, derived from condition (iii).

(2) $B^{\perp} = 0$, derived from condition (ii).

The remaining two conditions (i) and (iv) are associated with the boundary charges and currents.

Electromagnetic waves in the waveguide

We are investigating the propagation of monochromatic waves within a pipe aligned along the z-axis. In this context, $\vec{E}$ and $\vec{B}$ can be represented by the following generic forms:

(i) $\tilde{\vec{E}}(x, y, z, t) = \tilde{\vec{E}}_0(x, y)e^{i(kz-\omega t)}$

(ii) $\tilde{\vec{B}}(x, y, z, t) = \tilde{\vec{B}}_0(x, y)e^{i(kz-\omega t)}$.

The amplitudes of electric and magnetic fields can be expressed as:

$$\tilde{\vec{E}}_0(x, y) = \tilde{E}_x(x, y)\hat{x} + \tilde{E}_y(x, y)\hat{y} + \tilde{E}_z(x, y)\hat{z},$$
$$\tilde{\vec{B}}_0(x, y) = \tilde{B}_x(x, y)\hat{x} + \tilde{B}_y(x, y)\hat{y} + \tilde{B}_z(x, y)\hat{z}.$$

★ In general, both $\tilde{\vec{E}}_0(x, y)$ and $\tilde{\vec{B}}_0(x, y)$ are functions of x and y, suggesting that <u>the electromagnetic wave in the waveguide cannot be a plane wave.</u>

□ By substituting the wavefunctions (i) and (ii) into Maxwell's equations, one can determine the specific forms of $\tilde{E}_{x,y,z}(x, y)$, $\tilde{B}_{x,y,z}(x, y)$ and the dispersion relation $\omega(k)$, as demonstrated below.

In the subsequent derivation, we may replace ∂_z with ik and ∂_t with $-i\omega$ whenever these differential operators are applied to the exponential phase factor.

★ (i) Gauss's law $\vec{\nabla} \cdot \tilde{\vec{E}} = 0$:

$$\partial_x(\tilde{E}_x e^{i(kz-\omega t)}) + \partial_y(\tilde{E}_y e^{i(kz-\omega t)}) + \partial_z(\tilde{E}_z e^{i(kz-\omega t)}) = 0$$

$$\Rightarrow (1)\ \partial_x \tilde{E}_x + \partial_y \tilde{E}_y + ik\tilde{E}_z = 0$$

Using a similar derivation, we arrive at the following identities.

★ (ii) No magnetic monopole $\vec{\nabla} \cdot \tilde{\vec{E}} = 0$:

$$(2)\ \partial_x \tilde{B}_x + \partial_y \tilde{B}_y + ik\tilde{B}_z = 0$$

★ (iii) Faraday's law $\vec{\nabla} \times \tilde{\vec{E}} = -\partial_t \tilde{\vec{B}}$:

$$(3)\ \partial_x \tilde{E}_y - \partial_y \tilde{E}_x = i\omega \tilde{B}_z$$

$$(4)\ \partial_y \tilde{E}_z - ik\tilde{E}_y = i\omega \tilde{B}_x$$

$$(5)\ ik\tilde{E}_x - \partial_x \tilde{E}_z = i\omega \tilde{B}_y$$

★ (iv) Modified Ampére's law $\vec{\nabla} \times \tilde{\vec{B}} = \mu_0 \epsilon_0 \partial_t \tilde{\vec{E}}$:

$$(6)\ \partial_x \tilde{B}_y - \partial_y \tilde{B}_x = -\frac{i\omega}{c^2} \tilde{E}_z$$

$$(7)\ \partial_y \tilde{B}_z - ik\tilde{B}_y = -\frac{i\omega}{c^2} \tilde{E}_x$$

$$(8)\ ik\tilde{B}_x - \partial_x \tilde{B}_z = -\frac{i\omega}{c^2} \tilde{E}_y$$

Let's first examine Eqs (3-8) from Faraday's and Ampére's laws.

★ The components $\tilde{E}_x$ and $\tilde{B}_y$ appear in Eqs.(5) and (7), which can be rewritten in the following form

$$\frac{i\omega}{c^2}\tilde{E}_x - ik\tilde{B}_y = -\partial_y\tilde{B}_z, \qquad ik\tilde{E}_x - i\omega\tilde{B}_y = \partial_x\tilde{E}_z.$$

Then, $\tilde{E}_x$ and $\tilde{B}_y$ can be readily solved as:

$$\tilde{E}_x = i\frac{\omega\partial_y\tilde{B}_z + k\partial_x\tilde{E}_z}{(\omega/c)^2 - k^2}, \qquad \tilde{B}_y = i\frac{k\partial_y\tilde{B}_z + \frac{\omega}{c^2}\partial_x\tilde{E}_z}{\omega^2/c^2 - k^2}.$$

★ Similarly, the components $\tilde{E}_y$ and $\tilde{B}_x$ appear in Eqs. (4) and (8):

$$ik\tilde{E}_y + i\omega\tilde{B}_x = \partial_y\tilde{E}_z, \qquad \frac{i\omega}{c^2}\tilde{E}_y + ik\tilde{B}_x = \partial_x\tilde{B}_z.$$

It leads to the solution of $\tilde{E}_y$ and $\tilde{B}_x$ as follows:

$$\tilde{E}_y = i\frac{k\partial_y\tilde{E}_z - \omega\partial_x\tilde{B}_z}{\omega^2/c^2 - k^2}, \qquad \tilde{B}_x = i\frac{k\partial_x\tilde{B}_z - \frac{\omega}{c^2}\partial_y\tilde{E}_z}{\omega^2/c^2 - k^2}$$

□ Therefore, we have derived the expressions for $\tilde{E}_{x,y}$ and $\tilde{B}_{x,y}$ in relation to $\tilde{E}_z$ and $\tilde{B}_z$.

Wave equations can be derived by substituting $\tilde{E}_{x,y}$ and $\tilde{B}_{x,y}$ into the remaining equations (1), (2), (3) and (6).

★ By substituting $\tilde{E}_x$ and $\tilde{E}_y$ into Eq.(1), we obtain the wave equation for $\tilde{E}_z$: (9) $[\partial_x^2 + \partial_y^2 + (\omega/c)^2 - k^2]\tilde{E}_z = 0.$

★ Similarly, by substituting $\tilde{B}_x$ and $\tilde{B}_y$ into Eq.(2), we derive the wave equation for $\tilde{B}_z$: (10) $[\partial_x^2 + \partial_y^2 + (\omega/c)^2 - k^2]\tilde{B}_z = 0.$

★ Equations (3) and (6) are redundant and do not provide more information. In fact, substituting $\tilde{E}_{x,y}$ into Eq. (3) yields Eq. (10), and substituting $\tilde{B}_{x,y}$ into Eq. (6) yields Eq. (9).

□ By solving Eqs. (9) and (10), we can determine $\tilde{E}_z$ and $\tilde{B}_z$, which subsequently determine $\tilde{E}_{x,y}$ and $\tilde{B}_{x,y}$. We then obtain the complete wave functions in the waveguide which can be classified into the following three categories: **TE, TM** and **TEM** waves:

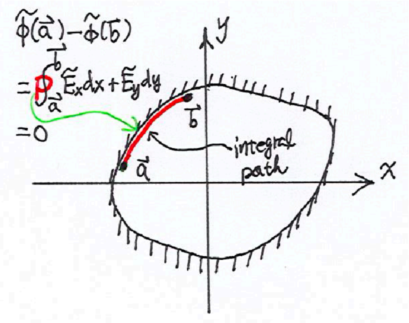
(a) If $\tilde{E}_z = 0$, they are called **TE** ("transverse electric") waves;
(b) If $\tilde{B}_z = 0$, they are called **TM** ("transverse magnetic") waves;
(c) If both $\tilde{E}_z = 0$ and $\tilde{B}_z = 0$, we call them **TEM** waves.

☐ TEM waves cannot occur in a hollow wave guide.

Proof: This statement can be proved as follows.

★ For TEM waves, as described by Eq.(3), we find $\partial_x \tilde{E}_y - \partial_y \tilde{E}_x = -i\omega \tilde{B}_z = 0$, allowing us to express $\tilde{E}_{x,y}$ as the 2D gradient of a scalar function $\tilde{\phi}$: $\tilde{E}_{x,y} = -\partial_{x,y}\tilde{\phi}$.

★ Moreover, $\tilde{E}_{x,y}$ also satisfies Eq.(1), which is $\partial_x \tilde{E}_x + \partial_y \tilde{E}_y = ik\tilde{E}_z = 0$. Consequently, we have $(\partial_x^2 + \partial_y^2)\tilde{\phi} = 0$, indicating that $\tilde{\phi}$ satisfies the 2D Laplace's equation. Since $\tilde{\phi}$ is complex, we effectively deal with two instances of Laplace's equations.



★ According to boundary condition, the parallel component of the electric field, $\vec{E}^{\parallel} = 0$, at the metallic surface implies that the scalar function $\tilde{\phi}$ must maintain a constant value on the boundary as demonstrated in the figure.

★ The solution of Laplace's equation does not allow for local maxima or minima within empty space. Any extrema occur exclusively on the boundary. As the potential remains constant at the conductor boundary, it implies a constant potential throughout the waveguide, resulting in $\vec{E} = 0$, indicating **the absence of TEM waves. Q.E.D.**

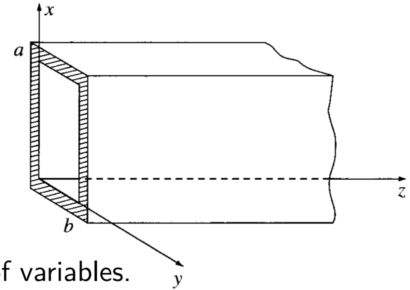
□ **Coaxial cable:** If a separate conductor is introduced along the center of the waveguide, the potential at its surface may differ from that of the outer wall, establishing a nontrivial potential within the waveguide. Here, the key difference from the previous scenario is that the boundaries of a coaxial cable are composed of separate, disconnected components, i.e., inner and outer walls. Consequently, the TEM waves may propagate in the coaxial cable. An example of TEM waves propagating in the waveguide will be presented in Sec. 9.5.3.

Outline

9.5.2 TE Waves in a Rectangular Wave Guide

In the TE mode, $\tilde{E}_z$ is zero, and $\tilde{B}_z$ obeys the wave equation $[\partial_x^2 + \partial_y^2 + (\omega/c)^2 - k^2]\tilde{B}_z = 0.$

★ For a waveguide with a rectangular cross-section as shown in the right picture, the wave equation for $\tilde{B}_z$ can be solved using the method of separation of variables.



Finding the solution of $\tilde{B}_z$

Following the method of variable separation, we express $\tilde{B}_z$ as the product of two functions: $X(x)$ and $Y(y)$. By substituting $\tilde{B}_z(x, y) = X(x)Y(y)$ into the wave equation, we obtain:

$$YX'' + XY'' + [(\omega/c)^2 - k^2]XY = 0 \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} + (\frac{\omega}{c})^2 - k^2 = 0.$$

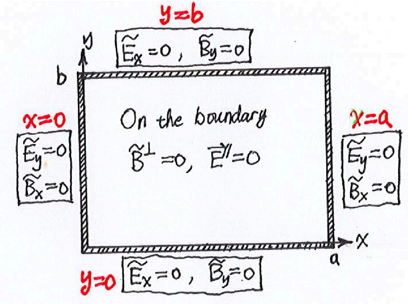
Since the x and y dependent terms must be constant, we can assume:

$$(1)\ \frac{X''}{X} = -k_x^2 \text{ and } (2)\ \frac{Y''}{Y} = -k_y^2,$$

where k_x and k_y must satisfy:

$$-k_x^2 - k_y^2 + (\omega/c)^2 - k^2 = 0.$$

☐ **Boundary conditions.** Before proceeding, we will establish the boundary conditions for the waveguide. As depicted in the right image, $B^{\perp}$ and $\vec{E}^{\parallel}$ vanish at the boundary. Hence, $\tilde{B}_x$ and $\tilde{E}_y$ are zero at the vertical boundaries.



★ As demonstrated in earlier slides, Ampere's laws yields the following expressions:

$$ik\tilde{B}_x - \partial_x\tilde{B}_z = -\frac{i\omega}{c^2}\tilde{E}_y \quad \text{and} \quad \partial_y\tilde{B}_z - ik\tilde{B}_y = -\frac{i\omega}{c^2}\tilde{E}_x.$$

Therefore, we deduce that $\partial_x\tilde{B}_z|_{x=0 \text{ or } a} = 0$ and $\partial_y\tilde{B}_z|_{y=0 \text{ or } b} = 0$.

This results in the following **boundary conditions**:

$$\boxed{X'(0) = X'(a) = 0, \text{ and } Y'(0) = Y'(b) = 0.}$$

□ **Solutions of X and Y.**

★ The general solution for X is expressed as $X(x) = A\sin(k_x x) + B\cos(k_x x)$, leading to $X'(x) = k_x A\cos(k_x x) - Bk_x\sin(k_x x)$. Upon applying the boundary conditions, we determine that $A = 0$, and $k_x = m\pi/a$ where m is an integer. Hence $X(x) = B\cos(m\pi x/a)$.

★ Similarly, we find that $Y(y) = B\cos\left(\frac{n\pi y}{b}\right)$ with n being integer.

□ **Solution of** $\tilde{B}_z(x,y) = X(x)Y(y)$. Finally, we obtain the solution of $\tilde{B}_z(x,y)$ as follows:

$$\tilde{B}_z(x,y) = \tilde{B}_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right).$$

This transverse electric (TE) wave is characterized by two integers, m and n, hence it is designated as the **TE$_{mn}$ mode**.

★ The **wave vector** of the TE$_{mn}$ mode is given by:

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]} \equiv \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}.$$

Here, we define the **cutoff frequency** $\omega_{mn} \equiv c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$.

□ **Cutoff frequency.** If the frequency of the incident wave ω is less than the mode cutoff frequency ω_{mn}, the wave number k becomes imaginary. Consequently, the fields are exponentially attenuated. This characteristic leads us to designate ω_{mn} as the **cutoff frequency** for the TE_{mn} mode.

★ Typically, it is assumed that $a \geq b$. Under this condition, the **lowest** cutoff frequency for a given waveguide corresponds to the TE_{10} mode, expressed as $\omega_{10} = \frac{c\pi}{a}$.

□ **Expression for $\tilde{E}_{x,y}$ and $\tilde{B}_{x,y}$.** Knowing the expression of $\tilde{B}_z$ for TE wave, it is straightforward to calculate $\tilde{E}_{x,y}$ and $\tilde{B}_{x,y}$ from Faraday's and Ampere's laws. For convenience, we define $k_m = \frac{m\pi}{a}$ and $k_n = \frac{n\pi}{b}$. Then, we find that

$$\tilde{E}_x = \frac{-i\omega k_n \tilde{B}_0 \cos(k_m x)\sin(k_n y)}{(\omega/c)^2 - k^2}, \qquad \tilde{B}_y = \frac{-ikk_n \tilde{B}_0 \cos(k_m x)\sin(k_n y)}{\omega^2/c^2 - k^2}.$$

$$\tilde{E}_y = \frac{i\omega k_m \tilde{B}_0 \sin(k_m x)\cos(k_n y)}{\omega^2/c^2 - k^2}, \qquad \tilde{B}_x = \frac{-ikk_m \tilde{B}_0 \sin(k_m x)\cos(k_n y)}{\omega^2/c^2 - k^2}$$

Wave velocity

According to the definition of wave velocities, we have the following observations.

□ The **phase velocity** is given by: $v = \frac{\omega}{k} = \frac{c}{\sqrt{1-(\omega_{mn}/\omega)^2}}$.

Obviously, $v > c$, however this is not an issue.

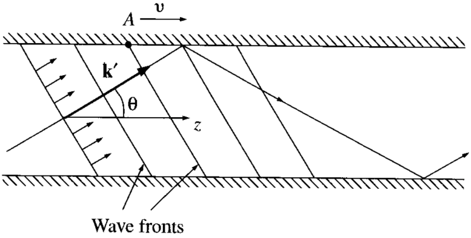
□ The **group velocity**, at which, the wavepacket propagate, has the following form: $v_g = \frac{1}{dk/d\omega} = c\sqrt{1-(\omega_{mn}/\omega)^2} < c$.

Confined electromagnetic waves

The electromagnetic waves in a waveguide can also be regarded as confined electromagnetic waves.

□ **Reflection of waves at the surfaces of a waveguide.** Consider a free plane wave characterized by a wave vector $\vec{k}' = (k_x, k_y, k_z)$. When confined within a rectangular waveguide, reflections at the metallic surfaces along the x and y directions generate a total of four plane waves with wave vectors $(\pm k_x, \pm k_y, k_z)$.

□ **Standing wave.** The interference of the four waves results in the formation of standing waves. Consequently, only integer multiples of half-wavelengths can exist, specifically, $a = m(\lambda_x/2)$ and $b = n(\lambda_y/2)$. This implies that the x and y components of the wave vector are restricted to $k_x = \frac{m\pi}{a}$ and $k_y = \frac{n\pi}{b}$. Simultaneously, in the z direction, a traveling wave persists with a wave number of $k_z = k$.



★ Hence, the allowed **wave vector** $\vec{k}'$ only take the following form:

$$\vec{k}' = \frac{\pi m}{a}\hat{x} + \frac{\pi n}{b}\hat{y} + k\hat{z}.$$

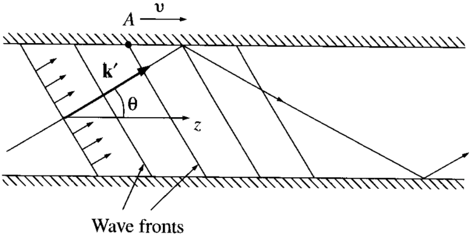
★ The dispersion can be calculated similar to that in the free space,

$$\omega = c|\vec{k}'| = c\sqrt{k^2 + \pi^2[(\frac{m}{a})^2 + (\frac{n}{b})^2]} = \sqrt{(ck)^2 + (\omega_{mn})^2}.$$

From this, we can recover the previous result: $k = \frac{1}{c}\sqrt{\omega^2 - (\omega_{mn})^2}$.

□ Next, we reproduce the wave velocities in the present framework.

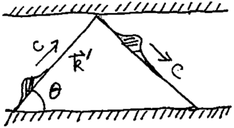
★ The wave fronts propagate along the direction $\vec{k}'$ at the speed of light c, as in free space. However, when considering propagation along the z-axis, as depicted in the figure, the **phase velocity** exceeds c:



$$v_p = \frac{\omega}{k} = \frac{c|\vec{k}'|}{k} = \frac{c}{\cos\theta} = \frac{c}{\sqrt{1-(\omega_{mn}/\omega)^2}},$$

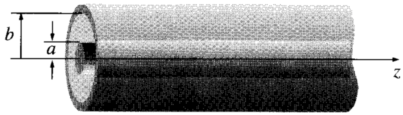
where $\cos\theta = \dfrac{k}{|\vec{k}'|} = \dfrac{c^{-1}\sqrt{\omega^2-(\omega_{mn})^2}}{\omega/c} = \sqrt{1-(\omega_{mn}/\omega)^2}$.

★ The **group velocity** represents the speed of the wave packet encompassing a spectrum centered around $\vec{k}'$. Generally, the size of the wave packet is much smaller than that of the waveguide, so it can be likened to a particle bouncing along the pipe at the speed of light. Therefore, $v_g = c\cos\theta = c\sqrt{1-(\omega_{mn}/\omega)^2}$.



Outline

9.5.3 The Coaxial Transmission Line



In this section, we examine the coaxial transmission line comprising a lengthy straight wire with a radius of a, enclosed by a cylindrical conducting sheath with a radius of b. This type of waveguide allows for the **TEM** mode, characterized by $\tilde{E}_z = \tilde{B}_z = 0$.

☐ **Linear dispersion for TEM mode.**

The expressions for $\tilde{E}_{x,y}$ and $\tilde{B}_{x,y}$ in terms of $\tilde{E}_z$ and $\tilde{B}_z$, such as,

$$\tilde{E}_x = i\frac{\omega\partial_y\tilde{B}_z + k\partial_x\tilde{E}_z}{(\omega/c)^2 - k^2}, \qquad \tilde{E}_y = i\frac{k\partial_y\tilde{E}_z - \omega\partial_x\tilde{B}_z}{\omega^2/c^2 - k^2},$$

$$\tilde{B}_x = i\frac{k\partial_x\tilde{B}_z - \frac{\omega}{c^2}\partial_y\tilde{E}_z}{\omega^2/c^2 - k^2}, \qquad \tilde{B}_y = i\frac{k\partial_y\tilde{B}_z + \frac{\omega}{c^2}\partial_x\tilde{E}_z}{\omega^2/c^2 - k^2},$$

become useless in this scenario, as the numerator vanishes. If there are still nontrivial electromagnetic waves present, the denominators of these expressions must also vanish, indicating a linear dispersion relationship $\omega = ck$.

□ Finding solutions of TEM mode.

In order to solve the TEM mode, we start directly from the equations derived from the Faraday's and Ampere's laws (repeated as follows)

$$(3)\partial_x\tilde{E}_y - \partial_y\tilde{E}_x = i\omega\tilde{B}_z, \quad (6)\partial_x\tilde{B}_y - \partial_y\tilde{B}_x = -\frac{i\omega}{c^2}\tilde{E}_z$$

$$(4)\partial_y\tilde{E}_z - ik\tilde{E}_y = i\omega\tilde{B}_x, \quad (7)\partial_y\tilde{B}_z - ik\tilde{B}_y = -\frac{i\omega}{c^2}\tilde{E}_x$$

$$(5)ik\tilde{E}_x - \partial_x\tilde{E}_z = i\omega\tilde{B}_y, \quad (8)ik\tilde{B}_x - \partial_x\tilde{B}_z = -\frac{i\omega}{c^2}\tilde{E}_y.$$

All the red terms vanish for the TEM mode.

★ By substituting the linear dispersion $\omega = ck$ into Eqs. (4,5,7,8), we find that: $\tilde{B}_x = -\frac{1}{c}\tilde{E}_y$, and $\tilde{B}_y = \frac{1}{c}\tilde{E}_x$.

Therefore, we only need to solve the electric field.

★ The x and y components of the electric field satisfy Eq.(3) and Gauss's law, namely,

$$\partial_x\tilde{E}_x + \partial_y\tilde{E}_y = 0, \text{ and } \partial_x\tilde{E}_y - \partial_y\tilde{E}_x = 0.$$

Evidently, $\tilde{E}_{x,y}$ can be expressed as the gradient of a complex function $\tilde{\phi}$, given by $\boxed{\tilde{E}_{x,y} = -\partial_{x,y}\tilde{\phi}}$.

★ The scalar function $\tilde{\phi}$ satisfies Laplace's equation, $\vec{\nabla}^2\tilde{\phi} = 0$, which in the polar coordinate system (s, φ) can be expressed as

$$\frac{1}{s}\partial_s(s\partial_s\tilde{\phi}) + \frac{1}{s^2}\partial_\varphi^2\tilde{\phi} = 0.$$

Since the coaxial cable exhibits cylindrical symmetry, we can express $\tilde{\phi}$ as a function of s. Then, it is readily to find the solution for $\tilde{\phi}$:

$$\tilde{\phi}(s, \varphi) = -\frac{\tilde{\lambda}_0}{2\pi\epsilon_0}\ln\frac{s}{s_0}$$

Here, $\tilde{\lambda}_0$ and s_0 are constants.

★ The corresponding amplitudes of the electric field and magnetic field are given by:

$$\tilde{E}_x = -\partial_x\tilde{\phi} = \tilde{\lambda}_0\frac{x}{2\pi\epsilon_0 s^2}, \qquad \tilde{E}_y = -\partial_y\tilde{\phi} = \tilde{\lambda}_0\frac{y}{2\pi\epsilon_0 s^2}, \quad \tilde{E}_z = 0$$

$$\tilde{B}_x = -\frac{1}{c}\tilde{E}_y = -c\mu_0\tilde{\lambda}_0\frac{y}{2\pi s^2}, \quad \tilde{B}_y = \frac{1}{c}\tilde{E}_x = c\mu_0\tilde{\lambda}_0\frac{x}{2\pi s^2}, \quad \tilde{B}_z = 0.$$

★ Finally, we obtain the rotationally invariant solution of the TEM wave as follows:

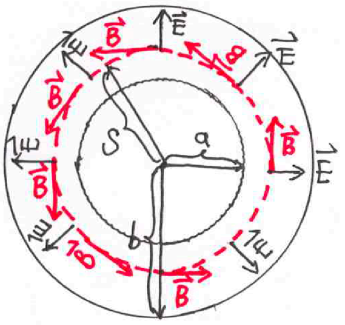
$$\tilde{\vec{E}}(\vec{r}, t) = \frac{\tilde{\lambda}_0\hat{s}}{2\pi\epsilon_0 s}e^{i(kz-\omega t)}, \qquad \tilde{\vec{B}}(\vec{r}, t) = \frac{c\mu_0\tilde{\lambda}_0\hat{\varphi}}{2\pi s}e^{i(kz-\omega t)}.$$

□ Charge and current distributions.

The real wave functions of the TEM mode are given by:

$$\vec{E}(s, \varphi, z, t) = \frac{\lambda \cos(kz - \omega t + \delta)}{2\pi\epsilon_0 s}\hat{s}, \ \vec{B}(s, \varphi, z, t) = \frac{c\mu_0\lambda \cos(kz - \omega t + \delta)}{2\pi s}\hat{\varphi},$$

Here, we take $\tilde{\lambda}_0 = \lambda e^{i\delta}$. To establish such an electromagnetic field distribution within the waveguide, the two metallic shells need to carry charges and currents that vary with z and t.



★ As depicted in the figure, the electric field exhibits radial orientation while the magnetic field is circumferential.

Applying Gauss's law and Ampere's law around the red circle with radius s, we derive the line charge density and current distribution as follows:

$$\lambda(z, t) = \lambda \cos(kz - \omega t + \delta), \quad I(z, t) = c\lambda \cos(kz - \omega t + \delta).$$