Chapter 7

Estimating Sensitivities

Outline

- Introduction
- Finite Difference Approximation
- Pathwise Method
- Likelihood Ratio Method
- Comparisons

1. Introduction

Value of a financial derivative and Greeks

$V = E[Y(\cdot)]$, $Y(\cdot)$ --- **discounted payoff** **(dependsonpathormultipleassets)**

Sensitivities (Greeks):

$$\Delta = \frac{\partial V}{\partial S_0}, \quad \Gamma = \frac{\partial^2 V}{\partial S_0^2}$$

$$\rho = \frac{\partial V}{\partial r}, \quad \mathbf{Vega} = \frac{\partial V}{\partial \sigma}.$$

Greeks provide a way to measure the sensitivity of an option's price to quantifiable factors.
They are needed for hedging and risk management.

1. Introduction

Delta (Δ) is the rate of change of the option price w.r.t the underlying asset. It is the slope of the curve that relates the option price to the underlying asset price:

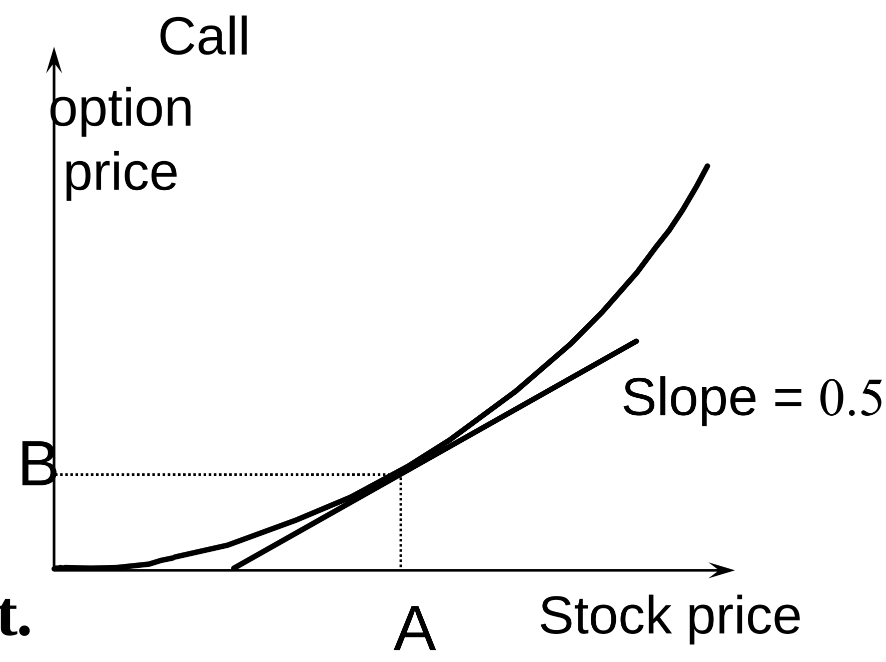
$$\Delta = \frac{\partial c}{\partial S}$$

c **is the price of call option**

S is the stock price

If $\Delta = 0.5$:

when the stock price changes by a small amount, the option price changes about 50% of that amount.



Ex: Black-Scholes delta

$$Y = e^{-rT}[S(T) - K]^+,$$

$$S(T) = S(0)\exp((r - \sigma^2/2)T + \sigma\sqrt{T}\,Z)),\ Z \sim N(0,1).$$

$$V = S(0)\,N(d_1) - K\,e^{-rT}N(d_2)$$

where $$d_1 = \frac{\ln(S(0)/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S(0)/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T},$$

$$\Rightarrow \Delta = \frac{dV}{dS(0)} = N(d_1)$$ **analytically tractable!**

Delta of European call option

$$c = S_0 \, N(d_1) - K \, e^{-rT} N(d_2)$$

$$\text{where} \quad d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}, \; d_2 = d_1 - \sigma\sqrt{T},$$

$$\Rightarrow \Delta = \frac{\partial c}{\partial S_0} = \frac{\partial}{\partial S_0}\left[S_0 \, N(d_1) - K \, e^{-rT} N(d_2)\right]$$

$$= N(d_1) + S_0 \, \phi(d_1) \frac{\partial d_1}{\partial S_0} - K \, e^{-rT} \phi(d_2) \frac{\partial d_2}{\partial S_0}$$

$$= N(d_1) \qquad (\text{note}: S_0 \, \phi(d_1) = K \, e^{-rT} \phi(d_2))$$

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$(\phi(.)$ **is the density of standard normal** $)$

From $d_1 = d_2 + \sigma\sqrt{T}$,

$$\Rightarrow \phi(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_2+\sigma\sqrt{T})^2}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \exp(-\sigma d_2\sqrt{T} - \frac{1}{2}\sigma^2 T)$$

$$= \phi(d_2)\exp(-\sigma d_2\sqrt{T} - \frac{1}{2}\sigma^2 T) \qquad (*)$$

From $d_2 = \dfrac{\ln(S_0/K) + (r-\sigma^2/2)T}{\sigma\sqrt{T}}$

$$\Rightarrow \sigma d_2\sqrt{T} = \ln(S_0/K) + (r-\sigma^2/2)T$$

$$\Rightarrow \sigma d_2\sqrt{T} + \frac{1}{2}\sigma^2 T = \ln(S_0/K) + rT$$

$$\Rightarrow \exp(-\sigma d_2\sqrt{T} - \frac{1}{2}\sigma^2 T) = e^{-rT}K/S_0$$

From (*), we have $S_0\,\phi(d_1) = K\,e^{-rT}\phi(d_2))$

Greeks for European call in BS model

Name	Symbol	Definition	Value in BS model
Delta	Δ	$\dfrac{\partial V}{\partial S(0)}$	$N(d_1)$
Gamma	Γ	$\dfrac{\partial^2 V}{\partial S^2(0)}$	$\dfrac{\phi(d_1)}{S(0)\sigma\sqrt{T}}$
Vaga	ν	$\dfrac{\partial V}{\partial \sigma}$	$S(0)\phi(d_1)\sqrt{T}$
Rho	ρ	$\dfrac{\partial V}{\partial r}$	$KT\,e^{-rT}N(d_2)$

1. Introduction

$$V(\theta) = E[Y(\theta)],$$

$Y(\theta)$ **is the discounted payoff function,**

depending on the path or multiple assets,

θ **can be any of the model or markel parameters.**

- The problem of estimating sensitivities consists of finding a way to estimate **the derivative of V with respect to** θ **.**

- **Finite difference sensitivities**
- **Pathwise sensitivities**
- **Likelihood ratio method**

2. Finite Difference Sensitivities

If $V(\theta) = E[Y(\theta)]$ is sufficiently differentiable, then

$$\frac{\partial V}{\partial \theta} \approx \frac{V(\theta + h) - V(\theta)}{h} \equiv \Delta_{\mathbf{F}} \quad \textbf{(forward difference)}$$

$$\Delta_{\mathbf{F}} \approx \frac{1}{h\,N} \sum_{i=1}^{N} \left[Y^{(i)}(\theta + h) - Y^{(i)}(\theta) \right] \equiv \hat{\Delta}_{\mathbf{F}}(N, h)$$

This approach is referred as getting Greeks by ``bumping'' the input parameters

(resimulation using CRN)

$$E\left[\hat{\Delta}_{\mathbf{F}}(N, h)\right] = \frac{V(\theta + h) - V(\theta)}{h} = \Delta_{\mathbf{F}}. \quad \textbf{(unbiased)}$$

This method is very simple, but is expensive and inaccurate. Moreover, the FD estimate is biased. (However, many banks use it because of its simplicity)

Mean Square Error

Error :

$$\frac{\partial V}{\partial \theta} - \hat{\Delta}_{\mathbf{F}}(N,h) = \frac{\partial V}{\partial \theta} - E\left[\hat{\Delta}_{\mathbf{F}}(N,h)\right] + E\left[\hat{\Delta}_{\mathbf{F}}(N,h)\right] - \hat{\Delta}_{\mathbf{F}}(N,h)$$

Mean Square Error :

$$\mathbf{MSE}(\hat{\Delta}_{\mathbf{F}}) = \mathbf{E}\left[\hat{\Delta}_{\mathbf{F}}(N,h) - \frac{\partial V}{\partial \theta}\right]^2$$

$$\Rightarrow \mathbf{MSE}(\hat{\Delta}_{\mathbf{F}}) = \mathbf{Bias}^2 + \mathbf{Var}\,(\hat{\Delta}_{\mathbf{F}})$$

Bias

If $V(\theta)$ **is twice differentiable at** θ**, then**

$$V(\theta+h)=V(\theta)+V'(\theta)h+\frac{1}{2}V''(\theta)h^2+o(h^2)$$

$$\frac{V(\theta+h)-V(\theta)}{h}=V'(\theta)+\frac{1}{2}V''(\theta)h+o(h)$$

Bias :

$$\frac{V(\theta+h)-V(\theta)}{h}-V'(\theta)=\frac{1}{2}V''(\theta)h+o(h)$$

Variance

$$\hat{\Delta}_F = \frac{1}{Nh}\sum_{i=1}^{N}\left[Y^{(i)}(\theta+h) - Y^{(i)}(\theta)\right]$$

$$\Rightarrow \mathbf{Var}(\hat{\Delta}_F) = \frac{\mathbf{1}}{\mathbf{Nh^2}}\mathbf{Var}\left[Y(\theta+h) - Y(\theta)\right]$$

If independent samples are taken for $Y(\theta+h)$ **and** $Y(\theta)$,

then

$$\mathbf{Var}\,(\hat{\Delta}_F) = \frac{\mathbf{1}}{\mathbf{Nh^2}}\left(\mathbf{Var}\left[Y(\theta+h)\right] + \mathbf{Var}[Y(\theta)]\right) \approx \frac{\mathbf{2Var(Y)}}{\mathbf{Nh^2}},$$

$\Rightarrow$ **which is very large for h << 1.**

Remark

$$\hat{\Delta}_{\mathbf{F}} = \frac{1}{Nh}\sum_{i=1}^{N}\left[Y^{(i)}(\theta+h) - Y^{(i)}(\theta)\right]$$

If the same random imputs (CRN) are used for $Y(\theta+h)$ **and** $Y(\theta)$**, then**

$$Y(\theta+h) - Y(\theta) \approx h\frac{\partial Y}{\partial \theta}$$

and hence

$$\mathbf{Var}(\hat{\Delta}_{\mathbf{F}}) \approx N^{-1}\,\mathbf{Var}\left(\frac{\partial Y}{\partial \theta}\right)$$

This indicates the importance of using CRN

which behavies well for h << 1.

Bias and Variance

$$\text{Bias}: \quad \frac{V(\theta+h)-V(\theta)}{h} - V'(\theta) = \frac{1}{2}V''(\theta)h + o(h)$$

$$\text{Variance}: \quad \frac{2\mathrm{Var}(Y)}{Nh^2}$$

- ✓ **Significant bias error**
  if h is too large
- ✓ **Large variance**
  if h is small

Decreasing h can increase variance while decreasing bias.

Optimal Mean Square Error (MSE)

$$\textbf{Bias}: \quad \frac{V(\theta+h)-V(\theta)}{h} - V'(\theta) = \frac{1}{2}V''(\theta)h + o(h)$$

$$\textbf{Variance}: \quad \frac{\mathbf{2Var(Y)}}{\mathbf{Nh^2}}$$

$$\mathbf{MSE} = \mathbf{Bias}^2 + \mathbf{Variance}$$

Minimizing MSE requires balancing the two terms.

Optimal MSE

In general, let $\textbf{Bias} = bh^{\beta}$, $\quad$ $\textbf{Variance} = \dfrac{\mathbf{c}}{\mathbf{Nh}^{\eta}}$

$$\Rightarrow \textbf{MSE} = \textbf{Bias}^2 + \textbf{Variance} = b^2 h^{2\beta} + \frac{\mathbf{c}}{\mathbf{Nh}^{\eta}}$$

Let $\mathbf{h} = h^{*} N^{-\gamma}$ $\quad$ (γ to be determined)

Let the two terms have the same order

The value of γ that maximizes the rate of convergence is $\quad \gamma = 1/(2\beta + \eta)$.

$$\Rightarrow \textbf{RMSE} = O\left(N^{-\frac{\beta}{2\beta+\eta}} \right) = \begin{cases} O(N^{-1/4}), \ \textbf{for } \beta = 1, \eta = 2; \\ O(N^{-1/2}), \ \textbf{for } \beta = 1, \eta = 0. \end{cases}$$

Central Difference

- If $V(\theta) = E[Y(\bullet)]$ is sufficiently differentiable, then one may use **central difference**

$$\frac{\partial V}{\partial \boldsymbol{\theta}} \approx \frac{V(\boldsymbol{\theta}+h)-V(\boldsymbol{\theta}-h)}{2h} \quad \textbf{(central difference)}$$

$$\approx \frac{1}{2Nh}\sum_{i=1}^{N}\left[Y^{(i)}(\boldsymbol{\theta}+h)-Y^{(i)}(\boldsymbol{\theta}-h)\right] \quad \textbf{(resimulation)}$$

$$\frac{\partial^2 V}{\partial \boldsymbol{\theta}^2} \approx \frac{V(\boldsymbol{\theta}+h)-2V(\boldsymbol{\theta})+V(\boldsymbol{\theta}-h)}{h^2} \approx \ldots$$

Bias for Central Difference

If $V(\theta)$ **is twice differentiable at** θ**, then**

$$V(\theta+h)=V(\theta)+V'(\theta)h+\frac{1}{2}V''(\theta)h^2+\frac{1}{6}V'''(\theta)h^3+o(h^3)$$

$$V(\theta-h)=V(\theta)-V'(\theta)h+\frac{1}{2}V''(\theta)h^2-\frac{1}{6}V'''(\theta)h^3+o(h^3)$$

$$\frac{V(\theta+h)-V(\theta-h)}{2h}=V'(\theta)+\frac{1}{6}V'''(\theta)h^2+o(h^2)$$

Bias :

$$\frac{V(\theta+h)-V(\theta-h)}{2h}-V'(\theta)=\frac{1}{6}V'''(\theta)h^2+o(h^2)$$

Variance

$$\hat{\Delta}_{\mathbf{F}} = \frac{1}{2Nh}\sum_{i=1}^{N}\left[Y^{(i)}(\theta+h) - Y^{(i)}(\theta-h)\right]$$

$$\Rightarrow \mathbf{Var}(\hat{\Delta}_{\mathbf{F}}) = \frac{\mathbf{1}}{\mathbf{4Nh^2}}\mathbf{Var}\left[Y(\theta+h) - Y(\theta-h)\right]$$

If independent samples are taken for $Y(\theta+h)$ **and** $Y(\theta-h)$,

then

$$\mathbf{Var}\ (\hat{\Delta}_{\mathbf{F}}) = \frac{\mathbf{1}}{\mathbf{4Nh^2}}\left(\mathbf{Var}\left[Y(\theta+h)\right] + Var[Y(\theta-h)]\right) \approx \frac{\mathbf{Var(Y)}}{\mathbf{2Nh^2}},$$

$\Rightarrow$ **which is very large for h << 1.**

Using CRN for Central Difference

If the same random imputs (CRN) are used for $Y(\theta+h)$ and $Y(\theta-h)$, then

$$Y(\boldsymbol{\theta}+h)-Y(\boldsymbol{\theta}-h)\approx 2h\frac{\partial Y}{\partial \boldsymbol{\theta}}$$

and hence

$$\mathbf{Var}(\hat{Q})\approx N^{-1}\mathbf{Var}\left(\frac{\partial Y}{\partial \boldsymbol{\theta}}\right)$$

This indicates the advantage of using CRN

which behavies well for h << 1.

Optimal MSE

Let $\quad \text{Bias} = bh^{\beta}, \qquad \text{Variance} = \dfrac{\mathbf{c}}{\mathbf{Nh}^{\eta}}$

$$\Rightarrow \text{MSE} = \text{Bias}^2 + \text{Variance} = b^2 h^{2\beta} + \frac{\mathbf{c}}{\mathbf{Nh}^{\eta}}$$

Two terms have the same order

Let $\;\mathbf{h} = h^{*} N^{-\gamma}$.

The value of γ that maximizes the rate of convergence is $\quad \gamma = 1/(2\beta + \eta)$.

$$\Rightarrow \text{RMSE} = O\left(N^{-\frac{\beta}{2\beta+\eta}} \right) = \begin{cases} O(N^{-1/3}), & \textbf{for } \beta = 2, \eta = 2; \\ O(N^{-1/2}), & \textbf{for } \beta = 2, \eta = 0. \end{cases}$$

Disadvantages of FD Method

- Expensive (two extra sets of calculations for central differences)
- If the parameter increment is large, then bias is large;
- If the parameter increment is small (and/or payoff is discontinuous), then the variance of the estimate is large;

- **How to choose the parameter increment?**

  --- Minimizing the MSE.

Remarks about FD

- Estimating Greeks is an important task, usually more important than estimating the prices.
- FD is the simplest approach, but least accurate and most expensive.
- Very important to use the CRN for the "bumped" path simulations to minimize the variance.
- In some cases, the optimum step size is a tradeoff between the variance and bias (due to the FD discretization error).

3. Pathwise Method

- FD method is simple to implement, but it is not accurate and is expensive. Below we discuss two other methods:
  - **Pathwise method (PW)**
  - **Likelihood Ratio method (LR)**

The value of a European option :

$$V(\theta) = E\left[f(S(T)) \right]$$

Where is the parameter of interest ?

--- two different point of views …

Estimating Greeks: Two Points of View

Ex: Black-Scholes delta: dV/dS(0)

Parameter of interest is S(0)

$$Y = e^{-rT}[S(T) - K]^{+} := f(S(T)),$$

$$S(T) = S(0)\exp((r - \sigma^2/2)T + \sigma\sqrt{T}\, Z)),\ Z \sim N(0,1).$$

$$V = \int f(S(T))\, P_Z(z)\, dz \qquad \textbf{(z : normal)}$$

Parameter of interest S(0) is in S(T)

$$\textbf{or } V = \int f(S)\, P_{S(0)}(S)\, dS \quad (S := S(T))$$

Parameter of interest S(0) is in density

$P_{S(0)}(S)$ **is the probability density of S(T)**

(What is the density of S(T)? --- lognormal)

3. Pathwise Method

Goal: Estimating $\frac{d}{d\theta}E[Y(\theta)]$

Idea: If the following condition is satisfied

$$\frac{d}{d\boldsymbol{\theta}}E[Y(\boldsymbol{\theta})] = E\left[\frac{d}{d\boldsymbol{\theta}}Y(\boldsymbol{\theta})\right] \quad (*) \ (\textbf{WHEN?})$$

(Interexchange of differenti ation and expectation)

then we may estimate

$$\frac{d}{d\boldsymbol{\theta}}E[Y(\boldsymbol{\theta})] = E[\frac{d}{d\boldsymbol{\theta}}Y(\boldsymbol{\theta})] \approx \frac{\mathbf{1}}{\mathbf{N}}\sum_{i=1}^{N}\frac{d}{d\boldsymbol{\theta}}Y^{(i)}(\boldsymbol{\theta}).$$

3. Pathwise Method

- **Under some conditions:**

$$\text{If } V(\theta) = E[f(S(T))] = \int f(S(T)p(z)\,dz,$$

then

Condition needed

$$\frac{\partial V}{\partial \theta} = \frac{\partial}{\partial \theta} E[f(S(T))] = E\left[\frac{\partial}{\partial \theta} f(S(T))\right]$$

$$= E\left[\frac{\partial f}{\partial S(T)} \cdot \frac{\partial S(T)}{\partial \theta}\right]$$

This factor is the same for different sensitivities (of the same option)

BS delta using PW method

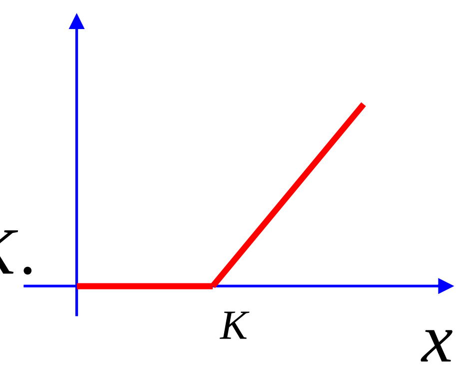
$$Y = e^{-rT}[S(T)-K]^+, \quad (Y \underset{\text{leads to}}{\overset{\text{depends on}}{\Leftrightarrow}} S(T) \underset{\text{leads to}}{\overset{\text{depends on}}{\Leftrightarrow}} S(0)=\theta)$$

$$S(T) = S(0)\exp((r-\sigma^2/2)T + \sigma\sqrt{T}\,Z)),\ Z \sim N(0,1).$$

Take θ to be S(0), applying **chain rule**

$$\frac{dY}{dS(0)} = \frac{dY}{dS(T)}\frac{dS(T)}{dS(0)} = e^{-rT} I\{S(T)>K\}\frac{S(T)}{S(0)},$$

$$\textbf{since}: \frac{d}{dx}\max(0, x-K) = \begin{cases} 0, & x<K, \\ 1, & x>K, \\ \textbf{does not exist}, & x=K. \end{cases}$$



$$\frac{dS(T)}{dS(0)} = \frac{S(T)}{S(0)}.$$

BS delta using PW method

$$\Delta = E\left[\frac{dY}{dS(0)}\right] = E\left[e^{-rT}\mathbf{I}\{S(T) > K\}\frac{S(T)}{S(0)}\right] = N(d_1).$$

Direct integration as in BS formula

If using simulation

$$\Delta \approx e^{-rT} \frac{1}{\mathrm{N}}\sum_{i=1}^{N} I\{S^{(i)}(T) > K\}\frac{S^{(i)}(T)}{S(0)}.$$

- So one can estimate delta in the process of pricing.
- Please compare the analytical value and the simulation value.

Ex 2: BS vega using PW method

$$Y = e^{-rT}[S(T) - K]^+,$$

$$S(T) = S(0)\exp((r - \sigma^2/2)T + \sigma\sqrt{T}\,Z)),\ Z \sim N(0,1).$$

Take θ to be σ, applying the chain rule

$$\frac{dY}{d\sigma} = \frac{dY}{dS(T)}\frac{dS(T)}{d\sigma} = e^{-rT} I\{S(T) > K\}\ [-\sigma T + \sqrt{T}Z]\, S(T)$$

Replace Z

$$= e^{-rT}\left[\frac{\log(S(T)/S(0)) - (r + \sigma^2/2)T}{\sigma}\right] S(T)\ I\{S(T) > K\}$$

$$\Rightarrow \mathbf{vega} = \mathbf{E}\left[\frac{dY}{d\sigma}\right]$$

simulation

$$\approx \frac{1}{N}\sum_{i=1}^{N} e^{-rT}\left[\frac{\log(S^{(i)}(T)/S(0)) - (r + \sigma^2/2)T}{\sigma}\right] S^{(i)}(T)\ I\{S^{(i)}(T) > K\}$$

Ex 3: Path-dependent delta (Asian option)

$$Y = e^{-rT}[\bar{S} - K]^{+}, \quad \bar{S} = \frac{1}{m}\sum_{i=1}^{m} S(t_i),$$

$$S(t_i) = S(0)\exp((r - \sigma^2/2)t_i + \sigma B(t_i)).$$

Take θ to be $S(0)$, applying the chain rule

$$\frac{dY}{dS(0)} = \frac{dY}{d\bar{S}} \frac{d\bar{S}}{dS(0)} = e^{-rT} I\{\bar{S} > K\} \frac{d\bar{S}}{dS(0)}$$

$$\text{Since } \frac{d\bar{S}}{dS(0)} = \frac{1}{m}\sum_{i=1}^{m} \frac{dS(t_i)}{dS(0)} = \frac{1}{m}\sum_{i=1}^{m} \frac{S(t_i)}{S(0)} = \frac{\bar{S}}{S(0)}$$

$$\Rightarrow \frac{dY}{dS(0)} = e^{-rT} I\{\bar{S} > K\} \frac{\bar{S}}{S(0)}.$$

Ex 3: Path-dependent delta (Asian option)

Thus the delta of Asian option can be estimated by

$$\Delta = \frac{\partial}{\partial \theta} E[Y(\theta)] \approx \frac{1}{N}\sum_{i=1}^{N} \frac{\partial}{\partial \theta} Y^{(i)}(\theta) \quad (\text{where } \theta = S(0))$$

$$= e^{-rT} \frac{1}{N}\sum_{i=1}^{N} \underline{I\{\overline{S}^{(i)} > K\}} \frac{\overline{S}^{(i)}}{S(0)}$$

The delta can be calculated in the process of pricing the Asian options.

This could cause inefficiency for QMC due to the discontinuity

The case of Asian options is particular interesting: there are no closed form formulas for their pricing.

Ex 4: Path-dependent vega (Asian option)

$$Y = e^{-rT}[\bar{S} - K]^+, \quad \bar{S} = \frac{1}{m}\sum_{i=1}^{m} S(t_i),$$

$$S(t_i) = S(0)\exp((r - \sigma^2/2)t_i + \sigma B(t_i)).$$

Applying the chain rule

$$\frac{dY}{d\sigma} = \frac{dY}{d\bar{S}}\frac{d\bar{S}}{d\sigma} = e^{-rT} I\{\bar{S} > K\}\frac{d\bar{S}}{d\sigma}$$

$$\frac{d\bar{S}}{d\sigma} = \frac{1}{m}\sum_{i=1}^{m}\frac{dS(t_i)}{d\sigma}$$

$$\frac{dS(t_i)}{d\sigma} = S(t_i)[-\sigma t_i + B(t_i)]$$

$$= S(t_i)[\log(S(t_i)/S(0)) - (r + \sigma^2/2)t_i]/\sigma$$

Ex 4: Path-dependent vega (Asian option)

The estimate for the vega of Asian call option

$$\mathbf{vega} = E\left[\frac{dY}{d\sigma}\right]$$

$$\approx \frac{1}{N}\sum_{j=1}^{N} e^{-rT} I\{\overline{S}^{(j)} > K\}\cdot$$

$$\cdot\frac{1}{m\sigma}\sum_{i=1}^{m} S^{(j)}(t_i)\left[\log\left(\frac{S^{(j)}(t_i)}{S(0)}\right) - (r+\sigma^2/2)t_i\right].$$

Delta of digital option using PW method?

$$Y = e^{-rT} I\{S(T) > K\} = \begin{cases} e^{-rT}, & S(T) > K \\ 0, & \text{otherwise.} \end{cases}$$

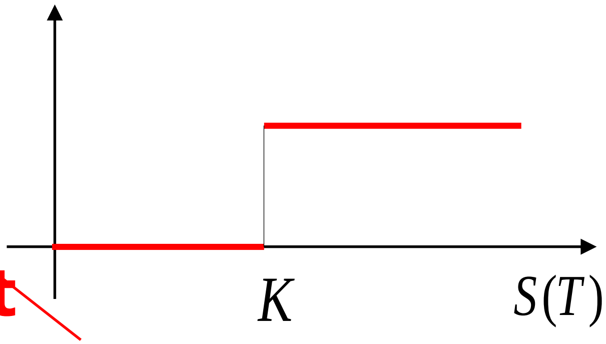
$$S(T) = S(0)\exp((r - \sigma^2/2)T + \sigma\sqrt{T}\, Z)),\ Z \sim N(0,1).$$

Applying the chain rule

$$\frac{dY}{dS(0)} = \frac{dY}{dS(T)}\frac{dS(T)}{dS(0)} = 0 \cdot \frac{S(T)}{S(0)} = 0 \Rightarrow E\left[\frac{dY}{dS(0)}\right] = 0.$$

- So the estimated value of delta is zero

On the other hand, $\dfrac{d}{dS(0)} E(Y) \neq 0.$



PW method gives incorrect result when payoff is discontinuous!

We may obtain the formula for E(Y), then get a formula for the delta

When is pathwise method applicable?

Simple rule of thumb:

- PW is OK for payoff functions which are continuous and piecewise differentiable.

- Digital and barrier options are not even continuous, thus the PW method does not apply.

- In general, PW tends to fail when the payoff has discontinuities.

When is PW method applicable?

- **A Sufficient Condition**

When an interchange of a derivative and an integral (expectation) is possible?

$$\frac{d}{d\boldsymbol{\theta}}E[Y(\boldsymbol{\theta})] \overset{???}{=} E\left[\frac{d}{d\boldsymbol{\theta}}Y(\boldsymbol{\theta})\right] \qquad (*)$$

A sufficient condition for the interchange is:

$$Y(\boldsymbol{\theta}) = f(X_1(\boldsymbol{\theta}), \cdots, X_m(\boldsymbol{\theta})),$$

$$\left| f(x) - f(y) \right| \le L \| x - y \|, \forall x, y \in R^m \text{ (\textbf{Lipschitz condition})}$$

The proof is based on

Dominated Convergence Theorem.

Dominated Convergence Theorem

If a sequence of RVs $\{Z_n\}$ converges to a RV Z with probability 1, where $|Z_n| \le M$, $\forall n$ with probability 1 and $E[M] < \infty$, then

$$\lim_{n\to\infty} E(Z_n) = E(\lim_{n\to\infty} Z_n) = E(Z).$$

$$\text{For PW : } Z_h = \frac{Y(\theta + h) - Y(\theta)}{h}.$$

$$\text{For LR : } Z_h = f(x)\,\frac{g_{\theta+h}(x) - g_\theta(x)}{h}. \qquad \text{(Latter)}$$

Proposition

Assume that $Y(\theta)$ is a.s. differentiable ~~at θ_0~~

and that $Y(\theta)$ satisfies the Lipschitz condition

$$\left|Y(\theta_1) - Y(\theta_2)\right| \le |\theta_1 - \theta_2| M,$$

for θ_1, θ_2 in a nonrandom neighborhood of θ_0,

where $EM < \infty$. Then

$$\frac{d}{d\theta} E[Y(\theta)] = E\left[\frac{d}{d\theta} Y(\theta)\right] \qquad (*)$$

holds at $\theta = \theta_0$.

Proof

Note that

$$\frac{\mathbf{d}}{\mathbf{d}\theta}\mathbf{E}[\mathbf{Y}(\theta)] = \lim_{h\to 0}\frac{\mathbf{E}[\mathbf{Y}(\theta+h)] - \mathbf{E}[\mathbf{Y}(\theta)]}{\mathbf{h}}$$

$$= \lim_{h\to 0}\mathbf{E}\left[\frac{\mathbf{Y}(\theta+h) - \mathbf{Y}(\theta)}{\mathbf{h}}\right] =: \lim_{h\to 0}\mathbf{E}[\mathbf{Z}_{\mathbf{h}}].$$

Here [...] is bounded by M and tends to $\mathbf{Y}'(\theta_0)$**.**

So by the dominated convergence theorem,

$$\lim_{h\to 0}\mathbf{E}[\mathbf{Z}_{\mathbf{h}}] = \mathbf{E}\left[\lim_{h\to 0}\mathbf{Z}_{\mathbf{h}}\right] = \mathbf{E}[\mathbf{Y}'(\theta_0)].$$

When is PW method applicable?

- **Please verify:**

Whether the condition for interchange is satisfied for the examples of European option and Asian option.

3. Pathwise Method

- **Pros:**
  - Less expensive
    (1 cheap calculation for each sensitivity);
  - No bias

- **Cons:**
  - Can't handle discontinuous payoff
  - (Smoothing techniques are required)

Remark

- The PW differentiates the path evolution and reduces the payoff's smoothness. This leads to inapplicability of PW method to non-Lipschitz payoffs.
- These challenges can be addressed in three different ways:

  - payoff smoothing using conditional expectations of the payoff;
  - approximating the previous technique with path splitting for the final timestep;
  - using of a hybrid combination of PW and LR.

4. Likelihood Ratio Method

Why LR method?

- ✓ It provides an alternative approach to derivative estimation;
- ✓ It requires no smoothness at all in the payoff and thus complements the pathwise method.

Let's recall …

Estimating Greeks: Two Points of View

Ex: BS delta: dV/dS(0)

Parameter of interest: S(0)

$$Y = e^{-rT}[S(T)-K]^+ := f(S(T)),$$

$$S(T) = S(0)\exp((r-\sigma^2/2)T + \sigma\sqrt{T}\,Z)),\ Z \sim N(0,1).$$

$$V = \int f(S(T))\,P_Z(z)\,dz \qquad \textbf{(z : normal)}$$

Parameter of interest S(0) is in S(T)

$$\textbf{or } V = \int f(S)\,P_{S(0)}(S)\,dS \quad (S := S(T))$$

Parameter of interest S(0) is in density

$P_{S(0)}(S)$ **is the probability density of S(T)**

(What is the density of S(T)? - - - lognormal)

Estimating Greeks by LR method --- Special case

Let $f(S(T))$ **be the discounted payoff** ,

$S(T) = S(0)\exp((r - \sigma^2/2)T + \sigma\sqrt{T}\,Z)),\ Z \sim N(0,1).$

$V = \int f(S)\, P_{S(0)}(S)\, dS \quad (S := S(T)$ **--- lognormal** $)$

where $P_{S(0)}(S)$ **is the p.d.f of S(T).**

Dependence on input parameters comes in through p.d.f. $P_{S(0)}(S)$ **and so**

$$\frac{\partial V}{\partial \theta} = \int f(S) \frac{\partial P_{S(0)}(S)}{\partial \theta}\, dS.$$ If exchangeable

4. Likelihood Ratio Method

- Suppose that the discounted payoff Y depends on a RV X=(X_1, ..., X_m), say, the values of a path or the values of multiple underlying assets:

$$Y = f(X) = f(X_1, \dots, X_m)$$

- Suppose that X has a probability density g and that θ **is a parameter of this density**.
- The value of the financial derivative is

$$V = E[Y] = E[f(X_1, \dots, X_m)]$$
$$= \int_{R^m} f(x)\, g_\theta(x)\, dx.$$

Suppose that the order of differentiation and integration is interchangeable

$$\frac{\partial V}{\partial \boldsymbol{\theta}} = \frac{d}{d\boldsymbol{\theta}} \int_{R^m} f(x)\, g_{\boldsymbol{\theta}}(x)\, dx$$

$$= \int_{R^m} f(x)\, \frac{d}{d\boldsymbol{\theta}} g_{\boldsymbol{\theta}}(x)\, dx \quad (\textbf{When?})$$

$$= \int_{R^m} f(x)\, \frac{dg_{\boldsymbol{\theta}}(x)/d\boldsymbol{\theta}}{g_{\boldsymbol{\theta}}(x)}\, g_{\boldsymbol{\theta}}(x)\, dx$$

$$= E\left[\underbrace{f(X) \cdot \frac{dg_{\boldsymbol{\theta}}(X)/d\boldsymbol{\theta}}{g_{\boldsymbol{\theta}}(X)}}_{} \right]$$

an unbiased estimate of $\frac{d}{d\boldsymbol{\theta}} E[Y]$

where $\frac{dg_{\boldsymbol{\theta}}(X)/d\boldsymbol{\theta}}{g_{\boldsymbol{\theta}}(X)} = \frac{d \log g_{\boldsymbol{\theta}}(X)}{d\boldsymbol{\theta}}$ **is called "score function".**

4. Likelihood Ratio Method

$$\frac{\partial V}{\partial \theta} = E\left[f(X) \cdot \frac{dg_{\theta}(X)/d\theta}{g_{\theta}(X)} \right]$$

$$\approx \frac{1}{N} \sum_{i=1}^{N} f(X^{(i)}) \frac{dg_{\theta}(X^{(i)})/d\theta}{g_{\theta}(X^{(i)})}$$

$\uparrow$

$\boxed{\text{Simulation}}$

Recall: Dominated Convergence Theorem

If a sequence of RVs $\{Z_n\}$ converges to a RV Z with probability 1, where $|Z_n| \le M$, $\forall n$ with probability 1 and $E[M] < \infty$, then

$$\lim_{n\to\infty} E(Z_n) = E(\lim_{n\to\infty} Z_n) = E(Z).$$

For PW : $Z_h = \dfrac{Y(\theta+h) - Y(\theta)}{h}$.

For LR : $Z_h = f(x)\,\dfrac{g_{\theta+h}(x) - g_{\theta}(x)}{h}$. **(Latter)**

$$\frac{\partial V}{\partial \theta} = E\left[f(X) \cdot \frac{dg_\theta(X) / d\theta}{g_\theta(X)} \right]$$

4. Likelihood Ratio Method

CV in Likelihhood Ratio Method

Note that when f = 1, we get

$$\frac{\partial}{\partial \theta} E[1] = 0,$$

and therefore

$$E\left[\frac{dg_\theta(X) / d\theta}{g_\theta(X)} \right] = 0.$$

This means we can use the score function as a CV

--- can be useful to reduce variance.

Ex 5: BS delta using LR method

$$Y = e^{-rT}[S(T)-K]^+,$$

$$S(T) = S(0)\exp((r-\sigma^2/2)T + \sigma\sqrt{T}\,Z)),\ Z \sim N(0,1).$$

View S(0) as a parameter of density of X=S(T).

S(T) is lognormal

Since $\log S(T) \sim N(\log(S(0)) + (r-\boldsymbol{\sigma}^2/2)T, \boldsymbol{\sigma}^2 T),$

the density of $X = S(T)$ **is**: (**ϕ is the density of N(0,1)**)

$$g(x) = \frac{1}{x\boldsymbol{\sigma}\sqrt{T}}\phi(\xi(x)),\ \ \xi(x) = \frac{\log(x/S(0)) - (r-\boldsymbol{\sigma}^2/2)T}{\boldsymbol{\sigma}\sqrt{T}},$$

,page Next

$$\Rightarrow \frac{dg(x)/d(S(0))}{g(x)} = -\xi(x)\frac{d\xi(x)}{dS(0)} = \frac{\log(x/S(0)) - (r-\boldsymbol{\sigma}^2/2)T}{S(0)\boldsymbol{\sigma}^2 T}.$$

Score function

Remark

If $Y \sim$ **LN** (μ, σ^2)**, i.e.,** $\log Y \sim$ **N**(μ, σ^2),

then the density of Y **is**

Y = exp(σX + μ)

then use transformation.

$$g(x) = \frac{1}{x\sigma}\phi\left(\frac{\log(x) - \mu}{\sigma}\right),$$

where ϕ **is the density of standard normal.**

Note:

Since $\log Y \sim N(\mu, \sigma^2) \Rightarrow [\log Y - \mu]/\sigma := Z \sim N(0,1)$

$\Rightarrow P(Y \le x) = P(Z \le [\log x - \mu]/\sigma) = \Phi([\log x - \mu]/\sigma)$

from here we can obtain the density of Y**.**

BS delta using LR method

$$Y = e^{-rT}[S(T) - K]^+,$$

$$S(T) = S(0)\exp((r - \sigma^2/2)T + \sigma\sqrt{T}\, Z)),\ Z \sim N(0,1).$$

The delta is

$$\Delta = E\left[f(X)\ \frac{dg(X)/dS(0)}{g(X)} \right] \qquad (\textbf{where } X = S(T))$$

$$= E\left[e^{-rT}(S(T) - K)^+\ \frac{\log(S(T)/S(0)) - (r - \sigma^2/2)T}{S(0)\sigma^2 T} \right]$$

$$= E\left[e^{-rT}(S(T) - K)^+\ \frac{Z}{S(0)\sigma\sqrt{T}} \right] \qquad \left(S(T)\ \textbf{is generated from S(0) using Z}\right)$$

$$\approx \frac{1}{N}\sum_{i=1}^{N} e^{-rT}(S^{(i)}(T) - K)^+\ \frac{Z^{(i)}}{S(0)\sigma\sqrt{T}}.$$

Simulation

Remark

- **The form of the payoff is irrelevant**, any other function of S(T) would result in an estimate of the similar form (i.e., the form of the estimator does not depend on the details of the payoff).
- For example the **delta of a digital option** is

$$\Delta = E\left[\underline{e^{-rT} I\{S(T) > K\}} \frac{Z}{S(0)\sigma\sqrt{T}} \right]$$

$$\approx \frac{1}{N} \sum_{i=1}^{N} \underline{e^{-rT} I\{S^{(i)}(T) > K\}} \frac{Z^{(i)}}{S(0)\sigma\sqrt{T}}.$$

- **LR method can handle discontinuous, non-differential payoff function**.

Remark

Once the score function is calculated, it can be multiplied by many different payoffs (of different options) to estimate their corresponding sensitivities:

$$\frac{\partial V}{\partial \boldsymbol{\theta}} = E\left[\underline{f(X)} \cdot \frac{dg_{\boldsymbol{\theta}}(X)/d\boldsymbol{\theta}}{g_{\boldsymbol{\theta}}(X)} \right]$$

$$\approx \frac{1}{N}\sum_{i=1}^{N} f(X^{(i)}) \cdot \frac{dg_{\boldsymbol{\theta}}(X^{(i)})/d\boldsymbol{\theta}}{g_{\boldsymbol{\theta}}(X^{(i)})}.$$

Ex 6: BS vega using LR method

$$Y = e^{-rT}[S(T)-K]^+,$$

$$S(T) = S(0)\exp((r-\sigma^2/2)T + \sigma\sqrt{T}\,Z)),\ Z \sim N(0,1).$$

The density of $X = S(T)$ **is :(**ϕ **is the density of N(0,1))**

$$g(x) = \frac{1}{x\sigma\sqrt{T}}\phi(\xi(x)),\ \ \xi(x) = \frac{\log(x/S(0)) - (r-\sigma^2/2)T}{\sigma\sqrt{T}},$$

$\Rightarrow$ **Score function** $\dfrac{dg(x)/d\sigma}{g(x)} = -\dfrac{1}{\sigma} - \xi(x)\dfrac{d\xi(x)}{d\sigma}$

with $\dfrac{d\xi(x)}{d\sigma} = \dfrac{\log(S(0)/x) + (r+\sigma^2/2)T}{\sigma^2\,T}.$

BS vega using LR method

$$Y = e^{-rT}[S(T)-K]^+,$$

$$S(T) = S(0)\exp((r-\sigma^2/2)T + \sigma\sqrt{T}\,Z)),\ Z \sim N(0,1).$$

$$\textbf{vega} = E\left[e^{-rT}(X-K)^+ \frac{dg(X)/d\sigma}{g(X)}\right] \quad (X = S(T))$$

$$= E\left[e^{-rT}(S(T)-K)^+ \left(\frac{Z^2-1}{\sigma} - Z\sqrt{T}\right)\right]$$

$$(Z = \frac{\log(S(T)/S(0)) - (r-\sigma^2/2)T}{\sigma\sqrt{T}})$$

$$\approx \frac{1}{N}\sum_{i=1}^{N} e^{-rT}(S^{(i)}(T)-K)^+ \left(\frac{Z^{(i)^2}-1}{\sigma} - Z^{(i)}\sqrt{T}\right)$$

$\boxed{\text{By Simulation}}$

Ex 7: Path-dependent delta (Asian option)

$$Y = e^{-rT}[\bar{S} - K]^+, \qquad \bar{S} = \frac{1}{m}\sum_{i=1}^{m} S(t_i),$$

$$S(t_i) = S(t_{i-1})\exp((r - \sigma^2/2)(t_i - t_{i-1}) + \sigma\sqrt{(t_i - t_{i-1})}\ Z_i),\ Z_i \sim N(0,1)$$

The payoff is a function of $X = (X_1, \ldots, X_m) = (S(t_1), \ldots, S(t_m))$.

We need density of path X, viewing S(0) as parameter.

Using the **Markov** **property of GBM**, we have

$$g(x_1, \ldots, x_m) = g_1(x_1 \mid S(0)) \cdot g_2(x_2 \mid x_1) \cdots g_2(x_m \mid x_{m-1}).$$

S(0) is a parameter of **the first factor** $g_1(x_1 \mid S(0))$,
but does not appear in any of the factors.

The score function for path-dependent delta

$$\frac{dg(x_1,\ldots,x_m)/dS(0)}{g(x_1,\ldots,x_m)} = \frac{dg_1(x_1\,|\,S(0))/dS(0)}{g_1(x_1\,|\,S(0))}$$

Other factors cancelled

$$= \frac{\log(S(t_1)/S(0)) - (r - \sigma^2/2)t_1}{S(0)\sigma^2 t_1}$$

$$= \frac{Z_1}{S(0)\sigma\sqrt{t_1}} \quad \textbf{(as in BS delta by LR)}$$

The score function for path-dependent delta

The delta of Asian option is

$$\Delta = E\left[e^{-rT}[\overline{S} - K]^{+} \frac{Z_1}{S(0)\sigma\sqrt{t_1}} \right]$$

$$\approx \frac{1}{N}\sum_{i=1}^{N} e^{-rT}[\overline{S}^{(i)} - K]^{+} \frac{Z_1^{(i)}}{S(0)\sigma\sqrt{t_1}}$$

Again, the specific form of the discounted payoff is irrelevant (one may obtain delta for other path dependent options, e.g., barrier options).

Ex. 8: Path-dependent vega (Asian option)

$$Y = e^{-rT}[\bar{S} - K]^+, \qquad \bar{S} = \frac{1}{m}\sum_{i=1}^{m} S(t_i),$$

$$S(t_i) = S(t_{i-1})\exp((r - \sigma^2/2)(t_i - t_{i-1}) + \sigma\sqrt{(t_i - t_{i-1})}\ Z_i),$$

$$Z_i \sim N(0,1).$$

$$g(x_1,\ldots,x_m) = g_1(x_1 \mid S(0))\, g_2(x_2 \mid x_1)\cdots g_2(x_m \mid x_{m-1}).$$

σ **appears in each factor.**

Ex. 8: Path-dependent vega (Asian option)

The score function is

$$\frac{dg(x_1,\ldots,x_m)/d\sigma}{g(x_1,\ldots,x_m)} = \frac{d\ln g(x_1,\ldots,x_m)}{d\sigma} = \sum_{j=1}^{m} \frac{d\ln g_j}{d\sigma}$$

$$= \sum_{j=1}^{m} \frac{dg_j(x_j \mid x_{j-1})/d\sigma}{g_j(x_j \mid x_{j-1})}$$

$$= \sum_{j=1}^{m} \left(\frac{Z_j^{\,2}-1}{\sigma} - Z_j\sqrt{t_j - t_{j-1}} \right)$$

$$(Z_j = \frac{\log(S(t_j)/S(t_{j-1})) - (r-\sigma^2/2)(t_j - t_{j-1})}{\sigma\sqrt{t_j - t_{j-1}}})$$

Ex. 8: Path-dependent vega (Asian option)

Thus the vega of Asian call option can be estimated as

$$E\left[e^{-rT}[\overline{S}-K]^{+} \; \frac{dg(X_1,\ldots,X_m)/d\sigma}{g(X_1,\ldots,X_m)} \right]$$

$$\approx \frac{1}{N}\sum_{i=1}^{N} e^{-rT}[\overline{S}^{(i)}-K]^{+} \sum_{j=1}^{m}\left(\frac{\left(Z_j^{(i)}\right)^2-1}{\sigma} - Z_j^{(i)}\sqrt{t_j-t_{j-1}} \right)$$

Ex. 9: BS Gamma using LR method

$$Y = e^{-rT}[S(T)-K]^+,$$

$$S(T) = S(0)\exp((r-\sigma^2/2)T + \sigma\sqrt{T}\,Z)),\ Z \sim N(0,1).$$

The density of $X = S(T)$ **is** $g(x)$

$$V = \int_0^\infty e^{-rT}(x-K)^+ g(x)dx$$

$$\frac{dV}{dS(0)} = \int_0^\infty e^{-rT}(x-K)^+ \frac{dg(x)}{dS(0)}dx = E\left[e^{-rT}(x-K)^+ \frac{dg(x)/dS(0)}{g(x)}\right]$$

$$\frac{d^2V}{dS^2(0)} = \int_0^\infty e^{-rT}(x-K)^+ \frac{d^2g(x)}{dS^2(0)}dx = E\left[e^{-rT}(x-K)^+ \frac{d^2g(x)/dS^2(0)}{g(x)}\right]$$

$$\frac{d^2g(x)/dS^2(0)}{g(x)} = \frac{\xi^2(x) - \xi(x)\sigma\sqrt{T} - 1}{S^2(0)\sigma^2 T}$$ **(see Ex. 6 for** $\xi(x)$**)**

Using (*) IN the previous page

With LR method, the second derivative is no more difficult to estimate.

Ex. 10: Path-dependent Gamma (Asian option)

$$Y = e^{-rT}[\bar{S} - K]^+, \qquad \bar{S} = \frac{1}{m}\sum_{i=1}^{m} S(t_i),$$

$$S(t_i) = S(t_{i-1})\exp((r - \sigma^2/2)(t_i - t_{i-1}) + \sigma\sqrt{(t_i - t_{i-1})}\, Z_i), \; Z_i \sim N(0,1)$$

The payoff is a function of $X = (X_1, \ldots, X_m) = (S(t_1), \ldots, S(t_m))$.
We have the p.d.f . of X:

$$g(x_1, \ldots, x_m) = g_1(x_1 \mid S(0))\, g_2(x_2 \mid x_1) \cdots g_2(x_m \mid x_{m-1}).$$

$$\frac{dg(x_1, \ldots, x_m)/dS(0)}{g(x_1, \ldots, x_m)} = \frac{dg_1(x_1 \mid S(0))/dS(0)}{g_1(x_1 \mid S(0))} \quad \textbf{(other factors cancelled)}$$

$$\frac{d^2 g(x_1, \ldots, x_m)/dS^2(0)}{g(x_1, \ldots, x_m)} = \frac{dg_1^2(x_1 \mid S(0))/dS^2(0)}{g_1(x_1 \mid S(0))} \quad \textbf{(similar as in 1D)}$$

4. Likelihood Ratio Method

- In practice, the applicability of the LR method is limited by

- ✓ The need for explicit knowledge of the density;

- ✓ Large variance (comparing with PW method).

Summary: 10 Examples

European Call			European Asian		
Delta	Vega	Gamma	Delta	Vega	Gamma
PW	PW	**??**	PW	PW	**??**
LR	LR	LR	LR	LR	LR

Analytically tractable for Greeks of European Call options.

Hybrid Approaches for Gamma

One can also use a **hybrid approach** to calculate Gamma: resimulation of Pathwise Delta estimate.

Example: Arithmetic Asian option

$$\Delta = E\left[\frac{dY}{dS(0)}\right] = E\left[e^{-rT} I\{\bar{S} > K\} \frac{\bar{S}}{S(0)}\right].$$

PW + FD

Gamma can be calculated by (finite difference)

$$\frac{1}{h}\left(E\left[e^{-rT} I\{\bar{S}(S_0+h) > K\} \frac{\bar{S}(S_0+h)}{S(0)+h}\right] - E\left[e^{-rT} I\{\bar{S} > K\} \frac{\bar{S}}{S(0)}\right]\right).$$

Use common random numbers

Hybrid Approaches for Gamma

BS Gamma by a combination of LR and PW

- The delta by LR is

$$\Delta = E\left[e^{-rT} (S(T) - K)^{+} \frac{Z}{S(0)\sigma\sqrt{T}} \right]$$

LR + PW

then by PW

$$\Gamma = \frac{d\Delta}{dS(0)} = E\left[e^{-rT} \frac{K\,Z}{S^{2}(0)\sigma\sqrt{T}} \mathbf{1}_{\{S(T)>K\}} \right].$$

? ?

- The above is "LR+PW".
- **How about "PW + LR" ?**

PW vs LR

- ✓ PW is based on the relationship of the security payoff and the parameter of interest. Differencing this relationship leads to an unbiased estimate for the sensitivity (under appropriate condition).

$$V = E\,[Y(S(\boldsymbol{\theta}))] \Rightarrow \frac{dV}{d\boldsymbol{\theta}} = E\left[\frac{dY}{dS}\,\frac{dS}{d\boldsymbol{\theta}}\right].$$

- ✓ LR is based on the relationship between the probability density of the price of underlying security and parameter of interest.

$$V = E[Y] = E(f\,(X_1,\ldots,X_m)) = \int_{R^m} f\,(x)\,g_\theta(x)\,dx.$$

PW vs LR

- The PW method relies on differentiating the payoff function with respect to the parameter of interest.
- The LR method relies on differentiating the density with respect to the parameter of interest.
- The former assumes that the parameter of interest belongs to the payoff function.
- The latter assumes that the parameter of interest is a part of the density.

PW vs LR

- Both PW and LR use a **interchange of integration and differentiation**, which is only justified when certain regularity conditions are satisfied.

$$\text{PW}: \frac{d}{d\theta} E[Y(\theta)] = E\left[\frac{d}{d\theta} Y(\theta)\right] \qquad (*)$$

$$\text{LR}: \frac{d}{d\theta} \int_{R^m} f(x)\, g_\theta(x)\, dx = \int_{R^m} f(x) \frac{d}{d\theta} g_\theta(x)\, dx \; (**)$$

- In general, the conditions for LR is easier to justify than those for PW, since density functions are in general smooth functions of their parameters, but payoff functions are not.

PW vs LR

- PW is OK for payoff functions which are continuous and piecewise differentiable. PW depends in an essential way on the form of the payoff. If payoff is suitable, then PW is more accurate (it can better exploit the structure of the individual problems).
- LR can handle discontinuous, non-differential payoff. It can be used for digital options, barrier options. It does not make use of any properties of the dependence of payoff on the underlying assts. Essentially the **same** estimator applies to **any** derivative security (an implementation advantage).
- FD method is the simplest, but is least accurate and most expensive.

PW vs LR

- LR can handle discontinuous payoffs, but a little complicated for multivariate case.
- PW is usually the best approach (simplest, lowest variance and least cost) when it is applicable – needs continuous payoff for first derivatives.
- Payoff smoothing can be used to make PW approach applicable to discontinuous payoffs and for second derivatives.
- Alternatively, combine PW with FD for second derivatives – e.g. use PW to compute delta, then FD to get Gamma.

Final Remarks

- Greeks are needed for hedging and risk analysis.
- Whereas prices can be obtained to some extent from market prices, simulation is the only way to determine the Greeks.
- We have explored 3 approaches:
  - Finite Differences
  - Pathwise method
  - Likelihood Ratio method

For high order Greeks, hybrid approaches can be used.

Final Remarks

Estimating Greeks is an important task, often more important than estimating the prices. Greeks are vital for hedging and risk analysis

- **FD (Finite Differences)**
  - simplest, but least accurate and most expensive
  - always use the same random numbers for both calcs
  - optimum bump size is a tradeoff between var & bias.
- **PW (Pathwise Method)**
  - simple, lowest variance and least cost
  - needs continuous payoff for first derivatives, but smoothing can be used for discontinuous payoffs.
- **LR (Likelihood Ratio Method)**
  - OK for discontinuous payoffs, but a little complicated

Summary

- **Further reading:**

  P. Glasserman. Monte Carlo Methods in Financial Engineering, Springer, 2004.

  (Chapter 7)

Projects

- Derive the formulas for various Greeks (delta, gamma, vega, rho) of European call option in BS model.

- Derive the PW and LR estimates for various Greeks (delta, gamma, vega, rho) of European call option in BS model.

- Compare the theoretical values and the simulation values (FD, PW and LR) of various Greeks.

Projects

- **Derive the PW and/or LR estimates for various Greeks (delta, gamma, vega, rho, at least two of them) of geometric Asian option in BS model.**
- **Compare the FD, PW and LR methods for these Greeks based on MC and QMC methods.**

Hint : Let $\mathrm{S_G} = \prod_{i=1}^{d} S^{1/d}(t_i)$. In BS model :

1. $\mathrm{S_G} = \exp(\log \mathrm{S_G}) = \exp\left(\frac{1}{d}\sum_{i=1}^{d}\log(S(t_i))\right)$.

2. $\log \mathrm{S_G} \sim N(\text{mean, variance})$, with

$$\text{mean} = \log(\mathrm{S}(0)) + \frac{d+1}{2d}(\mathrm{r} - \sigma^2/2)\mathrm{T},$$

$$\text{variance} = \frac{(d+1)(2d+1)}{6d^2}\sigma^2\mathrm{T}$$

Projects

- Derive the PW and/or LR estimates for various Greeks (delta, gamma, vega, rho, or at least two of them) of arithmetic Asian option in BS model.
- Compare the FD, PW and LR methods for these Greeks based on MC and QMC methods.
- Try to use BB and PCA in their QMC estimates.

Projects

- **Use variance reduction techniques in the estimation of Greeks by PW and LR in MC and QMC settings.**
- ***Derives PW and LR estimates for various Greeks for Basket options, and then use MC and QMC.**
- **Use various hybrid approaches for Gamma and compare their accuracy.**
- **Compute the Greeks of some exotic options (barrier options, lookback options …) by PW and/or LR method.**

Projects

- Develop smoothing techniques when using PW method for estimating Greeks when the payoffs are discontinuous.

End of Chapter 7