Chapter 1

Introduction and Warm-Up Examples

0. Introduction

- **数量金融：**
  - 利用数学工具（概率论、数理统计、偏微分方程和随机控制等）定量研究金融市场运行规律的一门新兴学科，是金融学的数学化。
  - 通过对金融问题进行数学建模、理论分析和数值计算，以求找到金融活动的内在规律，并用以指导实践。
- **核心问题：**
  1. 不确定环境下的最优投资策略的选择；
  2. 风险资产的定价与风险管理。

0. Introduction

- **数量金融（金融数学）发展简史**

  金融学是研究不确定环境下，经济资源在地域上和跨时间上的有效配置。

  在20世纪50年代之前，金融学主要是描述性学科。

  50年代之后，发生了革命性变化。两次“华尔街革命”产生了一门新兴的学科，即数量金融或“金融数学”。数量金融是金融学继定性描述阶段以后的一个更高层次的数量化的分析性交叉学科。

0. Introduction

- 1900年，Louis Bachelier（法）在其博士论文“投机的理论”中，已经开始利用布朗运动描述股票价格的变化和研究期权定价问题（那时尚无“期权”）。当时最伟大的数学家 Poincare 并不满意Bachelier 的论文。直到1965年该论文才由著名经济学家 Samuelson 推荐给金融学界知晓。

0. Introduction

- **第一次华尔街革命：**

1952年Markowitz发表“投资组合的选择”；提出用于投资分析的“均值-方差”理论。该理论为风险和回报的权衡提供了可行的量化手段。提出如何在未来结果不确定的环境下合理分配资源。一般，投资者的目标是：最大化预期收益，最小化风险（在给定风险水平下最大化预期收益；在给定预期收益水平下最小化风险）。归结为二次规划模型。该文引发了大量的对现代证券组合的分析工作，开创了金融数学的先河。

Sharp 在1964年提出了著名的资本资产定价模型（Capital Asset Pricing Model, CAPM）.

1990年，Nobel经济学奖授予了：Markowitz, Sharp和Miller(公司财务).

0. Introduction

- **第二次华尔街革命：**

上世纪70年代（1973年），Black和Scholes发表了“期权和公司债务的定价”，提出了第一个期权定价数学公式。同年，Merton对该公式进行了发展和深化，完善了期权定价理论。期权定价理论由此建立，成为现代数量金融的核心内容。

Scholes 和 Merton 获1997年Nobel经济学奖（Black英年早逝，未能分享此项殊荣）。瑞典皇家科学院在1997年度诺贝尔经济学奖的嘉奖辞中说：

“期权定价理论和公式可以说是最近25年以来经济学领域中最为重大的突破和最卓越的贡献。它不仅为金融衍生市场近10年的迅猛发展奠定了可靠的理论基础，而且它在经济生活多个领域中的广泛应用将为金融业的未来发展带来一场革命性的变化。”

0. Introduction

- 上世纪80年代以来：
  - **期权定价与套期保值：** 深化Black-Scholes理论与实用化；发展了适用于不同环境、更精确的模型。
  - **计量金融经济学：** 检验有效市场的假设的统计方法，扩散过程模型的各种估计方法，时间序列模型。
  - **最优消费投资组合：** 连续时间金融模型。
  - **利率的期限结构：** Vasicek模型，CIR模型,…
  - **风险的度量：** Value-at-Risk (VaR), Sensitivities,
  - **计算金融学：** PDE计算与Monte Carlo模拟。
  - **金融工程：** 运用工程技术的方法（数学建模、数值计算、网络图解、仿真模拟等），设计、开发和实施新型金融产品，创造性地解决金融问题。

0. Introduction

- **数量金融的重要性：**
  - **金融产品的设计，风险分析与管理：**数学方法可降低交易风险与成本，推动金融市场的发展。大量的数学家、物理学家和计算机专家涌入金融界。他们在充分认识金融市场的基础上，利用金融数学工具，设计出非常精妙的套利和投机策略。
  - **超级高科技武器：**是金融投机家手中的超级高科技武器。在和平时期，这种武器是比任何飞机、大炮、航母、核弹更具杀伤力的武器。可在举手投足之间沉重打击一个国家或地区的经济。金融安全是国家安全的重要方面。

History of Computational Finance

- The birth of **CF** as a discipline can be traced to Markowitz in the early 1950s. Markowitz conceived of the portfolio selection problem as an exercise in mean-variance optimization. This required more computer power than was available at the time, so he worked on **useful algorithms** for approximate solutions.
- **Mathematical finance** began with the same insight, but diverged by making **simplifying assumptions** to express relations in simple closed forms that did not require sophisticated computer science to evaluate.

History of Computational Finance

- In the 1960s, hedge fund (working with Markowitz, Samuelson and Merton) pioneered the **use of computers** in arbitrage trading.
- In academics, sophisticated computer processing was needed by researchers such as Fama in order to analyze large amounts of financial data in support of the **Efficient Market Hypothesis**.

History of Computational Finance

- During the 1970s, the main focus of **CF** shifted to option pricing and analyzing mortgage securitizations.
- In the late 1970s and early 1980s, a group of young quantitative practitioners who became known as **rocket scientists** arrived on Wall Street and brought along personal computers. This led to an explosion of both the amount and variety of **CF** applications.

History of Computational Finance

- By the end of 1980s, the winding down of the Cold War brought a large group of displaced physicists and applied mathematicians into finance. These people become known as **Financial Engineer** ("quant" is a term that includes both rocket scientists and financial engineers).
- This led to a second major extension of the range of computational methods used in finance. Around this time **CF** became recognized as a distinct academic subfield.
- The first degree program in CF was offered by **Carnegie Mellon University** in 1994.

History of Computational Finance

- Over the last 20 years, the field of **CF** has expanded into virtually every area of finance, and the demand for practitioners has grown dramatically.
- Moreover, many specialized companies have grown up to supply **CF** software and services.

Applications of CF

- **CF** applies engineering methodologies to problems in finance, and employs financial theory, mathematics and computation.
- **CF** is used to create new financial instruments and strategies. It is used to make pricing, hedging, trading and portfolio management decisions.
- Utilizing various derivatives, **CF** aims to precisely control the financial risk, or completely eliminate risks by utilizing combinations of derivatives and other securities.

Applications of CF

- **Areas where CF are used include:**

--- Investment banking and management

--- Forecasting

--- Securities and derivatives trading and risk manegemnt

--- Pension, Insurance

--- Mortgate agreement

--- Credit default swaps

0. Introduction

Main subjects in computational methods:

- **Monte Carlo Simulation**
- **Finite Difference Methods**
- **Optimization Methods**

1. Monte Carlo Simulation: Warm-Up Examples

- **Monte Carlo Methods:**

  --- Statistical simulation method

- **The name of the method:**

- The method is called after the city in the Monaco principality, because of a roulette, a simple random number generator.
- The name and the systematic development of Monte Carlo methods dates from about 1944.
- There are however a number of isolated and undeveloped instances on much earlier occasions.

What Is Monte Carlo?

- Take a complicated chance based system.
- Simulate the outcome multiple times on the computer.
- Keep track of what happens.

- If the system is not chance based
- Express it as chance based.
- Go through steps above.

1. Monte Carlo Simulation: Warm-Up Examples

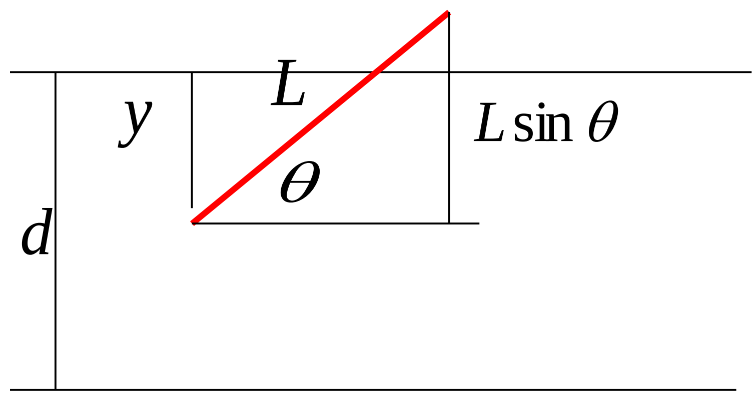
(1) Buffon's Needle Problem:

(Stated in 1733, solution published in1777)

If a needle of length *L* is dropped at random on the middle of a horizontal surface ruled with parallel lines a distance apart (d> L), what is the probability that the needle will cross one of the lines?

(1) Buffon's Needle Problem:

y : **the distance from the lowest point of the needles to the nest line above it**



θ : **the angle between the needle and the positive x-axis**

Sample space:

$$\{(\theta, y) \mid 0 \le \theta < \pi,\ 0 \le y < d\}.$$

The needle will intersect one of the lines if and only if

$$y < L\sin(\theta).$$

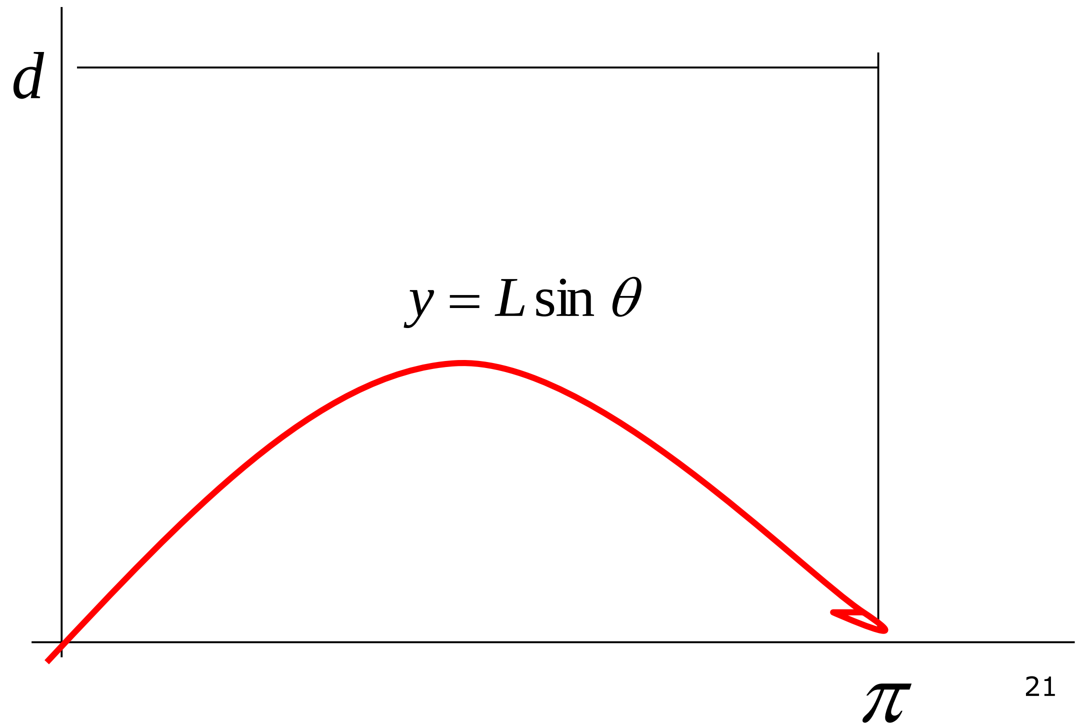
(1) Buffon's Needle Problem:

The probability:

$$P = \frac{\textbf{Area between the curves } y = L\sin\theta \textbf{ and } y = 0}{\textbf{Area of rectangle}}$$

$$= \frac{\int_0^{\pi} L\sin\theta \; d\theta}{d\pi}$$

$$= \frac{2L}{d\pi}$$



(1) Buffon's Needle Problem:

- **The probability is**

$$P(\text{crossing}) = \frac{2L}{\pi d}$$

$$\Rightarrow \pi = \frac{2L}{Pd} \approx \frac{2L}{d}\frac{n}{k} \Leftarrow \text{Simulation}$$

probability

frequency

Note: $P \approx \frac{k}{n} = \frac{\text{N.actual crossings}}{\text{N.total throws}}$

n: number of experiments;
k: number of intersection

Using "frequency" to replace the "probability".

(1) Buffon's Needle Problem

Year	L/d	Number	Crossing	Pi
1850	0.8	5000	2532	3.1596
1884	0.75	1030	489	3.1595
1902	0.83	3408	1808	3.1415929
1925	0.54	2520	859	3.1795

(1) Buffon's Needle Problem:

- **Importance:**

  In the first time to change a deterministic problem to a stochastic one.

- **Simple questions:**

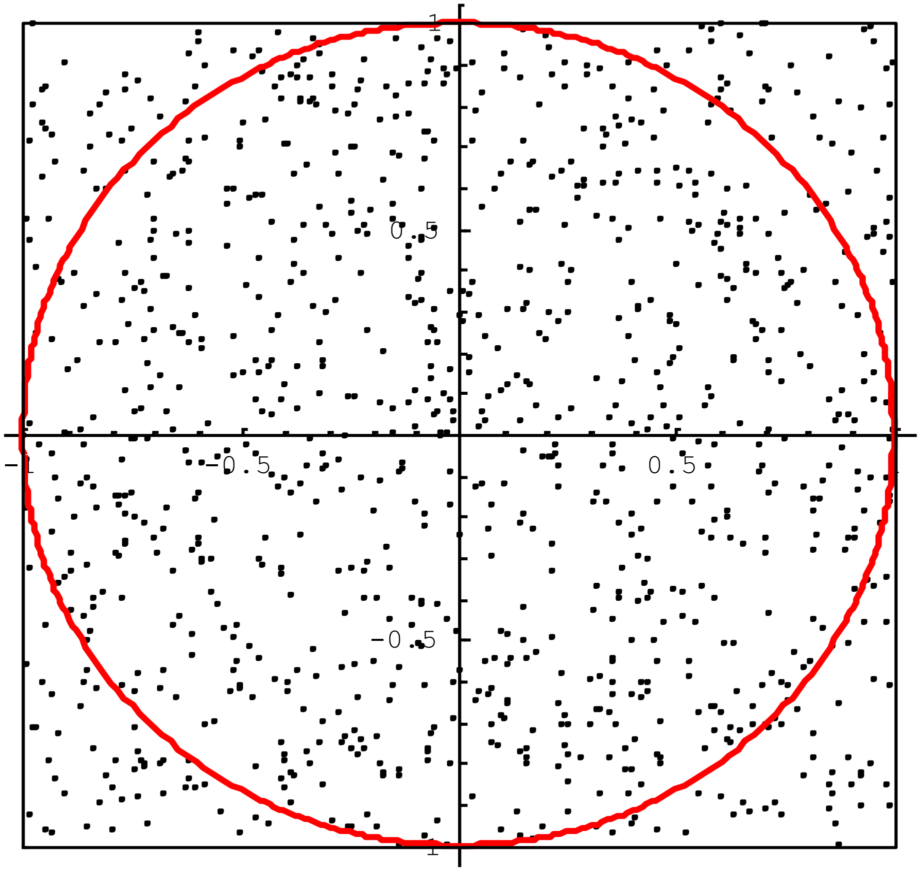
  - How to realize the Buffon's experiment on a computer?
  - Other methods to estimate the pi value?

The calculation of Pi

$$\frac{\text{N.Tosses Inside Circle}}{\text{Total N. Tosses}} =: \frac{k}{n}$$

$$\approx P = \frac{\pi R^2}{4R^2} = \frac{\pi}{4}$$

$$\Rightarrow \quad \pi \approx 4\frac{k}{n}$$



(2) Manhattan Project

This project is related to atomic weapons.

- **MC in Manhattan Project:**

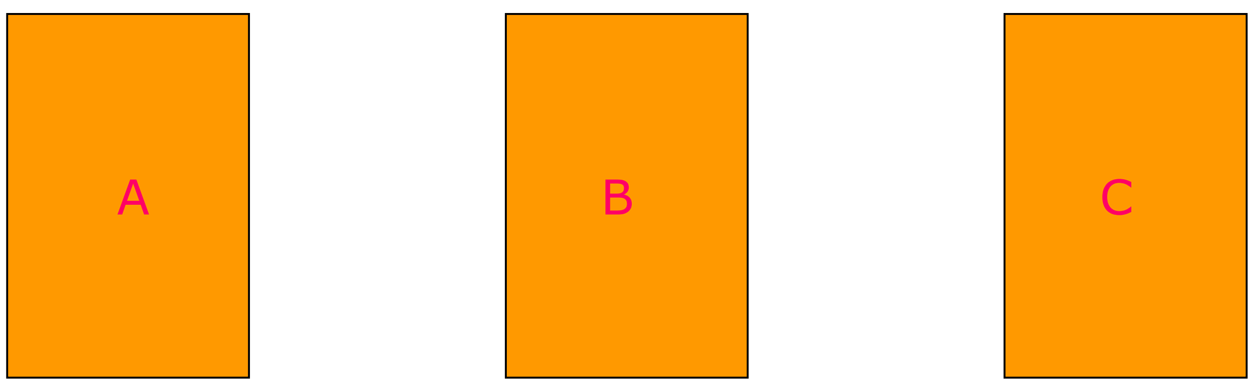
  A direct simulation of the probabilistic problems concerned with random neutron diffusion in fissile material.

- von Neumann is the founder of MC method.

(3). 电视游戏：三门问题

A，B，C 三扇门，其中只有一扇门后有奖品（汽车）。你选择某门。主持人打开另两扇门中无奖品的一扇空门，展示门后没有奖品。此时，主持人给你改变决定的机会。

问：你是坚持原来的选择，还是改选另一扇门?



(3). 电视游戏：三门问题

直观1： 无所谓。在剩下未开启的两扇门后有汽车的概率都是1/2，因此不需要改变。

直观2：

（1）若坚持，得奖概率为 1/3.

（2）若改选:

- 如果奖品在原选定的门（概率1/3），你失去获奖机会；
- 如果奖品不在原选定的门（概率2/3），因主持人已打开另两扇门中无奖品的门，你改选后，必定获奖，所以获奖概率为2/3.

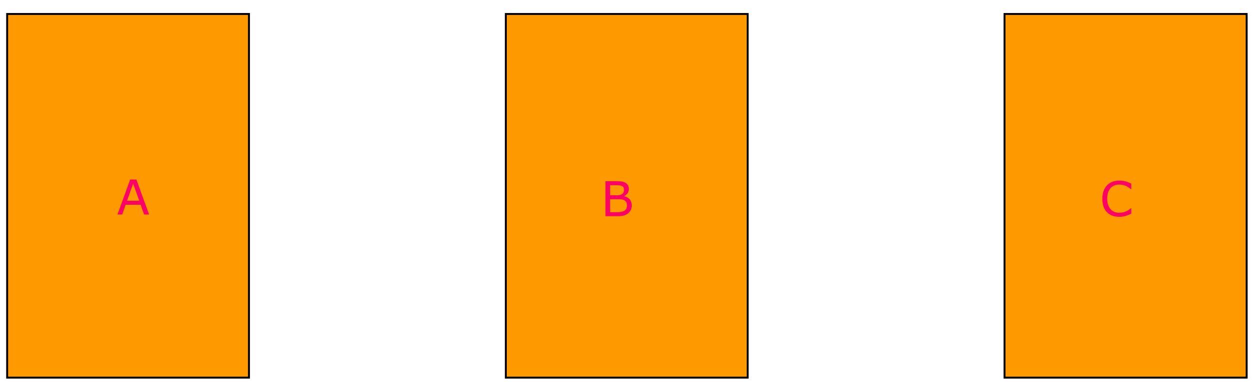
(3). 电视游戏：三门问题

设A =“汽车在A门后”， B=“汽车在B门后”， …

假设原来的选择为A门。

设B* =“主持人打开B门，没有汽车”。

比较 $P(A|B^*)$ 和 $P(C|B^*)$.



(3). 电视游戏：三门问题

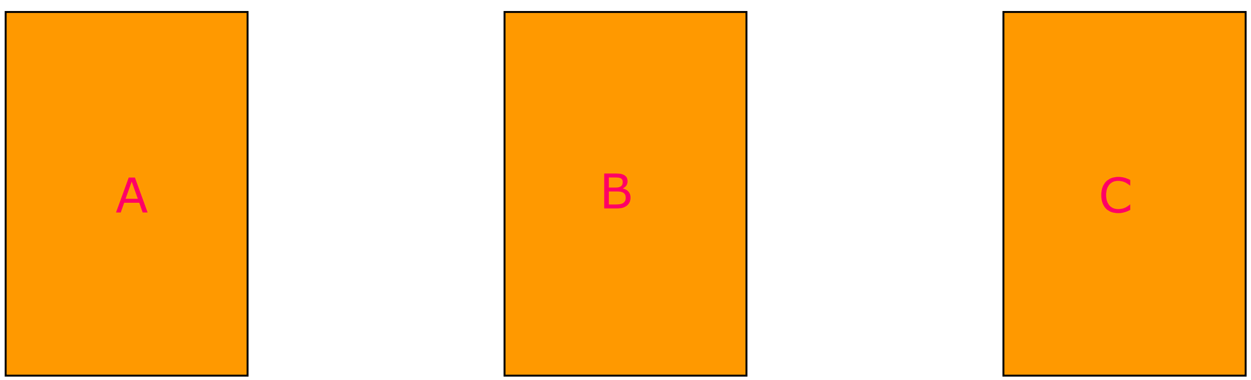
注意到：P(A)=P(B) =P(C)=1/3 ,

P(B*|A)=1/2, P(B*|B)=0, P(B*|C)=1

汽车在A后面，随意打开B，C中一个；
汽车在B后面，主持人不会打开B；
汽车在C后面，主持人必打开B；

$$P(C\mid B^*)=\frac{P(B^*\mid C)P(C)}{P(B^*\mid A)P(A)+P(B^*\mid B)P(B)+P(B^*\mid C)P(C)}$$

$$=\frac{1\times 1/3}{1/2\times 1/3+0\times 1/3+1\times 1/3}=2/3.$$



(3). 电视游戏：三门问题

- Simulation Study

如何用电脑掷骰子?

抛掷骰子 （不妨设汽车在C门）

➢ 出现1,2点 ⟶ 选A门 $\xrightarrow{\text{主持人开B门}}$ 变换选择至C门，得奖

➢ 出现3,4点 ⟶ 选B门 $\xrightarrow{\text{主持人开A门}}$ 变换选择至C门，得奖

➢ 出现5,6点 ⟶ 选C门 $\xrightarrow{\text{主持人开A或 B门}}$ 变换选择至B或A门，无奖

- Simulation Result:

Number	10	100	1000	10000	100000	1000000
Frequency	0.7000	0.6600	0.6650	0.6600	0.6663	0.6666

(4). Monte Carlo in Finance

Options:

Today is: 2020. 2. 18. Microsoft share price is 100 $/share. You have the right to buy a Microsoft share at 2020. 5. 18 at the price of 100 $/share.

(4). Monte Carlo in Finance

- For a standard European call option, the payoff is:

$$\max(S_T - K, 0)$$

- What is the fair price of an option?
- Why is option pricing important?
  - If an option is not priced correctly, someone is guaranteed to make money! This is called arbitrage.
  - If someone is guaranteed to make money, someone else is guaranteed to lose money.

(4). Monte Carlo in Finance

In real world (measure P),

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dW_t$$

We can write it as

Under P, it is a Brownian motion with drift

$$\frac{dS_t}{S_t} = r\, dt + \sigma\, (\underline{dW_t + \lambda dt}), \qquad \lambda = \frac{\mu - r}{\sigma}$$

According to Girsanov Theorem, we may construct a new measure Q, under which

$$dW_t^* \equiv dW_t + \lambda dt$$

is a standard Brownian motion. Thus in Q-measure

$$\frac{dS_t}{S_t} = r\, dt + \sigma\, dW_t^*$$

(4). Monte Carlo in Finance

- The Stock Price Assumption (under the risk-neutral measure)

$$dS_t = rS_t dt + \sigma S_t dB_t,$$

where r --- risk-free interest rate

σ --- volatility

B_t --- standard Brownian motion.

(4). Monte Carlo in Finance

- The solution to the SDE:

$$S_t = S_0 \exp((r - \sigma^2 / 2)t + \sigma B_t)$$

$$\ln S_T - \ln S_0 \sim N\left[\left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

$$\ln S_T \sim N\left[\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

--- S_T is lognormall y distribute d.

(4). Monte Carlo in Finance

- **Risk-neutral valuation principle**:

  Assume the payoff function of a option is (path-dependent)

$$g(S_{t_1}, \cdots, S_{t_d})$$

  Then the value of the option at t=0 is:

$$\textbf{Price} = e^{-rT} E_Q[g(s_{t_1}, \cdots, s_{t_d})]$$

  Option price is a **mathematical expectation**.

(4). Monte Carlo in Finance

- **The Black-Scholes Formula:**

$$c = S_0\, N(d_1) - K\, e^{-rT} N(d_2)$$

$$p = K\, e^{-rT}\, N(-d_2) - S_0\, N(-d_1)$$

where $$d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / K) + (r - \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$= d_1 - \sigma\sqrt{T}$$

(4). Monte Carlo in Finance

- However, only in rare cases we can obtain analytical solutions;
- In most cases we have to use numerical algorithms;
- Monte Carlo simulation is the method of choice.

- **Remark:**

  Two sources of dimensionality:

- The number of risk factors (underlying) involved;
- The number of time steps required.
- When both factors are present, the dimension is the product of the two numbers.

(4). Monte Carlo in Finance

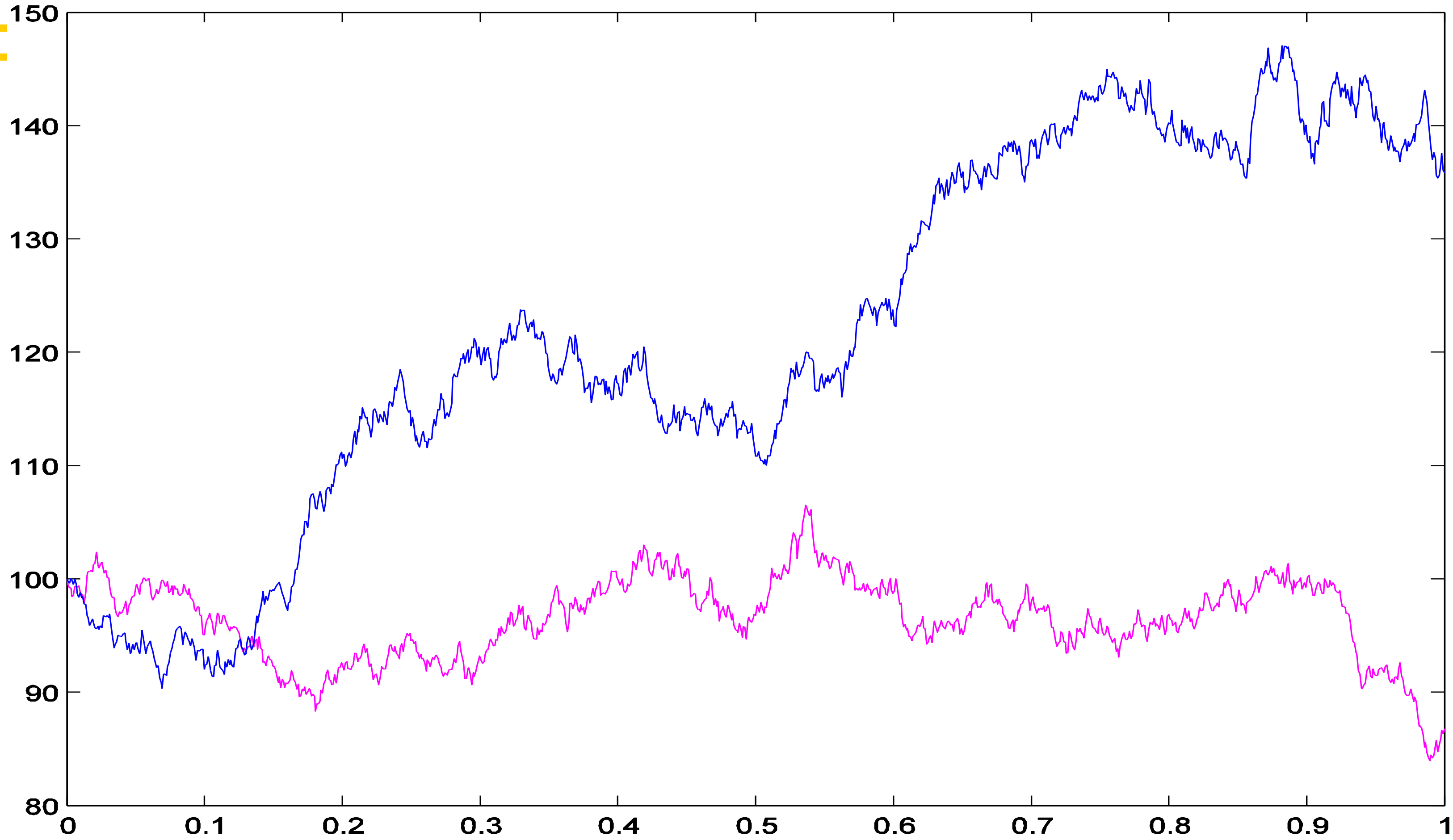
Steps of using MC to value European stock options:

1. **Simulate 1 path for the stock price in a risk neutral world (we will study how to do this)**
2. Calculate the payoff from the stock option;
3. Repeat steps 1 and 2 many times to get many sample payoff;
4. Calculate mean payoff;
5. Discount mean payoff at risk free rate to get an estimate of the value of the option.

Note:

Prof. Boyle is the first to use MC to price options.

Sample Paths



(4). Monte Carlo in Finance

- **The MC approximation is:**

$$\textbf{Price} \approx e^{-rT}\frac{1}{N}\sum_{k=1}^{N} g(S_{t_1}^k,\ldots,S_{t_d}^k),$$

where $S_{t_1}^k,\ldots,S_{t_d}^k$ **are the prices in the k - th path.**

2. Mathematical Foundation

- **(Weak) Law of Large Numbers**

Let $X_1, \ldots, X_n$ be independent, with finite expectation *a* and variance, then for any $\varepsilon > 0$ we have

$$P\left(\left|\frac{1}{n}\sum_{k=1}^{n} X_k - a\right| \geq \varepsilon\right) \to 0, \text{ as } n \to \infty$$

Or equivalently,

$$P\left(\left|\frac{1}{n}\sum_{k=1}^{n} X_k - a\right| < \varepsilon\right) \to 1, \text{ as } n \to \infty$$

2. Mathematical Foundation

Strong Law of Large Numbers

Let $X_1, \ldots, X_n$ be i.i.d, with finite expectation a, then

$$P\left(\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k = a \right) = 1.$$

2. Mathematical Foundation

Central Limit Theorem

Let $X_1, \ldots, X_n$ be independent, with finite expectation a and variance σ^2, then we have

$$P\left(\frac{\frac{1}{n}\sum_{k=1}^{n} X_k - a}{\sigma / \sqrt{n}} < x \right) \to \Phi(x), \text{ as } n \to \infty.$$

3. MC May Fail

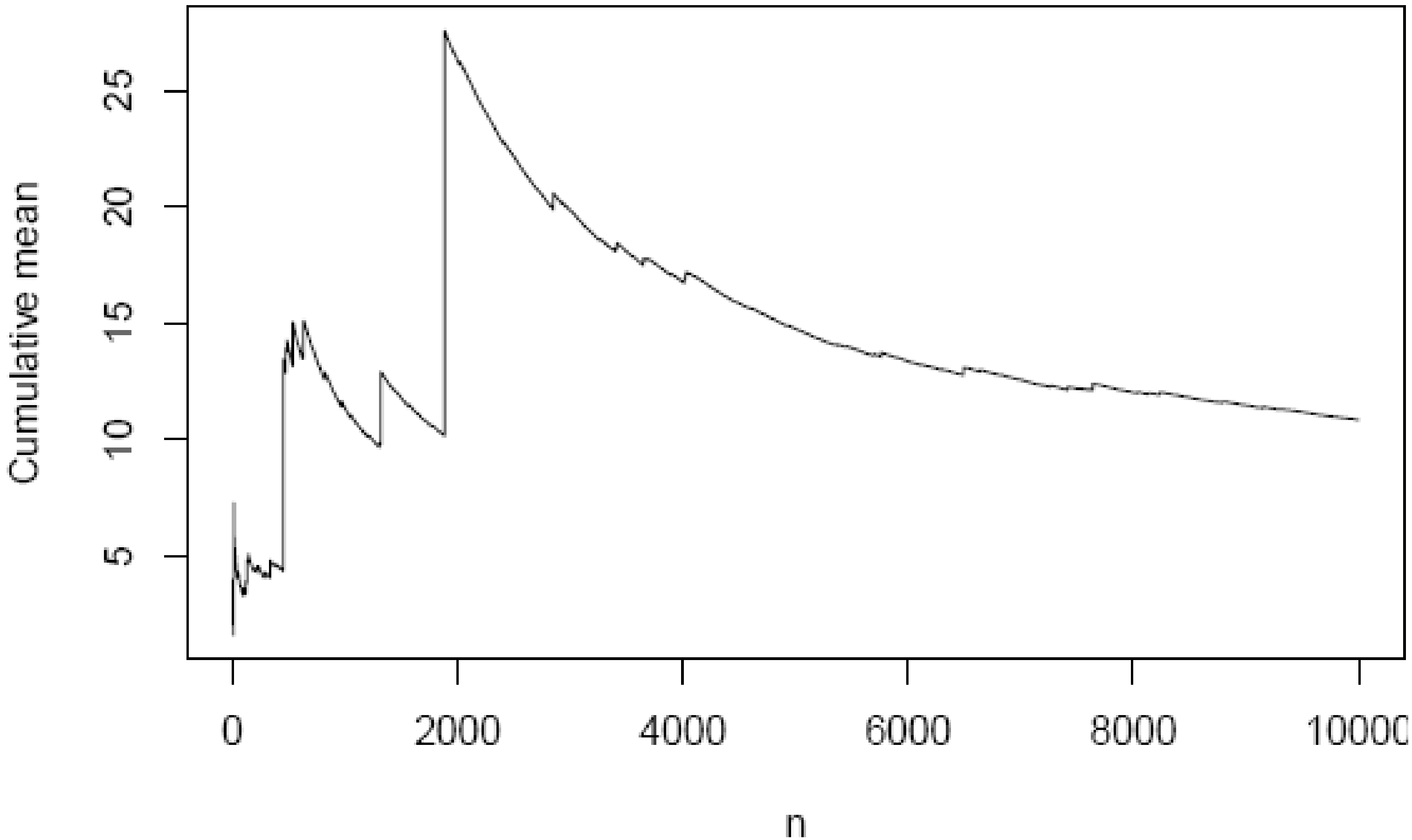
- Monte Carlo methods are extremely robust compared to alternatives, but there are still ways in which they can go wrong.

St. Petersburg Paradox*

- A casino offers a game for a player in which a fair coin is tossed. The pot starts at 2 dollar and is doubled every time a Tail appears.
- The first time a Head appears, the game ends and the player wins whatever is in the pot.
- The player wins
  - --- 2 dollar if a Head appears on the first toss,
  - --- 4 dollars if a Tail appears on the first toss, and a Head on the second, …
- In short, the player wins 2^N dollars, where N-1 Tails are tossed before the first Head appears.

What would be a fair price to pay the casino for entering the game?

St. Petersburg Paradox*--- Simulation



Every once in a while there is a sharp upward jump

St. Petersburg Paradox* --- Expectation?

- Let X be the money you win, then

  $E(X) = \frac{1}{2} * 2 + \frac{1}{4} * 4 + 1/8 * 8 + \ldots =$ **infinity**.

  The mathematical expectation does not exist!
- This problem is considered a **paradox** because although the expected payoff is infinite, most people would not be willing to pay very much to play the game (even if they could afford to lose the money).
- **The paradox** is that there is **no good answer** to the question of how much a gambler should be willing pay for a chance to play the game **based on the expected value.**
- Several approaches have been proposed for solving the paradox.

4. Practical Problems

- How a stock market evolves.
- How a flu epidemic spreads.
- How a protein folds.
- How long to wait for your espresso.
- How a scene will look when illuminated.
- How traffic jams appear.

Why Simulation?

- Many important technologies used to accomplish practical problems are based on **drawing samples** from some probability distributions and using these samples to form a **Monte Carlo estimate** of some desired quantity.

- We may wish to draw samples from a probability distribution for many reasons.

Why Simulation?

- Sampling provides a flexible way to approximate many sums and integrals a reduced cost.
- Sometimes we use this to provide a significant speedup to a costly but tractable sum.
- In other cases, algorithm requires us to approximate an intractable sum or integral.
- In many other cases, sampling is actually our goal, in the sense that we want to train a model that can sample from the training distribution.

5. MC: The Past, Present and Future

- MC began with probability and games of chance;
- The modern era of MC as a numerical technique coincided with digital computers;
- MC has become a standard method in many areas;
- Many areas of computation are intractable without MC;
- MC continues to be highly suitable for the most advanced computers due to its inherent “natural parallelism”;

5. MC: The Past, Present and Future

- Mathematical modelling likely to become more complex, leading to higher dimensional problems to be solved;
- Consequently, MC methods likely to become more important, rather than less;

- New techniques:
  - Quasi-Monte Carlo;
  - Markov Chain MC;
  - Applications in finance;
  - Applications to biology, biochemistry, biophysics, …

Exercises (warm up)

1. Calculate π using the methods described in the class. Produce a 95% confidence interval of for π based on CLT.

2. Calculate the value of a simple European Call option using Monte Carlo method, and compare it with the value obtained by the Black-Scholes formula.

The End of Chapter 1