4640 HW4 Q1

who

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1. *Chapter 5 Exercise 14, BDA3, Gelman et al.*

(a)

$$\begin{aligned} p(\theta,\alpha,\beta|y) &\propto p(\alpha,\beta)p(\theta|\alpha,\beta)p(y|\theta,\alpha,\beta) \\ &\propto p(\alpha,\beta)\times\prod_{j=1}^{J}\frac{\beta^{\alpha}e^{-\beta\theta_j}\theta_j^{\alpha-1}}{\Gamma(\alpha)}\times\prod_{j=1}^{J}e^{-\theta_j}\frac{\theta_j^{y_j}}{y_j!} \end{aligned} \tag{1}$$

(b)

$$\begin{aligned} p(\theta_j|\alpha,\beta,y_j) &\propto p(y_i|\theta)p(\theta|\alpha,\beta) \\ &\propto e^{-(\beta+1)\theta_j}\theta_j^{\alpha+y_j-1} \end{aligned} \tag{2}$$

Thus, we can see that

$$\theta_j|\alpha,\beta,y_j \sim Gamma(\alpha+y_j,\beta+1)$$

i.e.

$$p(\theta_j|\alpha,\beta,y) = \frac{(\beta+1)^{\alpha+y_j}e^{-(\beta+1)\theta_j}\theta_j^{\alpha+y_j-1}}{\Gamma(\alpha+y_j)y_j!}$$

Then we have the joint density of θ is:

$$p(\theta|\alpha,\beta,y) = \prod_{j=1}^{J}\frac{(\beta+1)^{\alpha+y_j}e^{-(\beta+1)\theta_j}\theta_j^{\alpha+y_j-1}}{\Gamma(\alpha+y_j)y_j!} \tag{3}$$

so the marginal posterior distribution of the hyperparameters is:

$$\begin{aligned} p(\alpha,\beta|y) &= \frac{p(\theta,\alpha,\beta|y)}{p(\theta|\alpha,\beta,y)} \\ &\propto p(\alpha,\beta)\prod_{j=1}^{J}\frac{\Gamma(\alpha+y_j)\beta^{\alpha}}{\Gamma(\alpha)(\beta+1)^{\alpha+y_j}} \end{aligned} \tag{4}$$

here we set the prior distribution for the hyperparameters is:

$$p(\alpha,\beta) \propto (\alpha\beta)^{-(1-\phi)}e^{-\phi(\alpha+\beta))}$$

here the hyperparameters follow the Gamma distribution:

$$\alpha, \beta \propto Gamma(\phi, \phi), i.i.d.$$

let $\phi = 0.001$,

$$p(\alpha, \beta) \propto (\alpha\beta)^{-0.999} e^{-0.001(\alpha+\beta))}$$

In ordet to do the sampling in a more convenient manner, we would like the know the distribution of $p(log(\frac{\alpha}{\beta}), log\beta)$, thus we firstly and get the corresponding Jacobian Matrix, which equals to $\alpha\beta$ and we can also calculate the following result:

$$\alpha = e^{log(\frac{\alpha}{\beta})+log\beta}$$

$$\beta = e^{log\beta}$$

In case of computational overflow, we also take the logarithm of the posterior distribution during the computation:

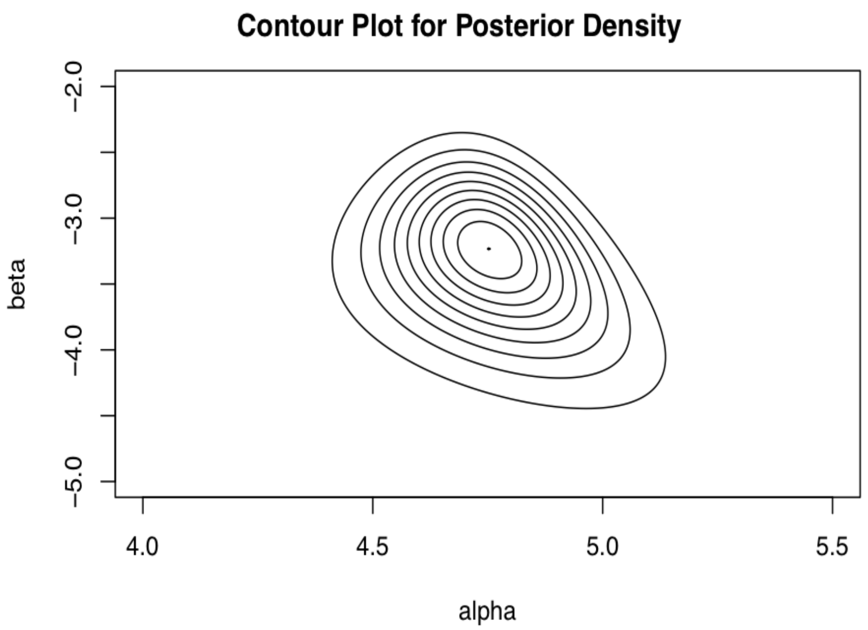
$$log(p(log(\frac{\alpha}{\beta}), log\beta|y)) \sim 0.001log(e^{log(\frac{\alpha}{\beta})+2log\beta}) - 0.001(e^{log(\frac{\alpha}{\beta})+log\beta} + e^{log\beta})$$
$$+ \sum_{j=1}^{J}\{log\Gamma(\alpha + y_j) + \alpha log\beta - log\Gamma(\alpha) - (\alpha + y_j)log(\beta + 1)\}$$

```
#y is the number of the bicycles
y<-c(16,9,10,13,19,20,18,17,35,55)
#x is the number of the other vehicles
x<-c(58,90,48,57,103,57,86,112,273,64)
#n=y+x is the total number of vehicles
n<-y+x


library(stats)
library(graphics)
a <- seq(3,7,4/1000)
b <- seq(-8,-1,7/1000)
z <- matrix(0,length(a),length(b))

for (i in 1:length(a)){
  for (j in 1:length(b)){
    t1<-exp(a[i]+b[j])
    t2<-exp(b[j])
z[i,j]<- log(t1*t2)+log((t1*t2)^(-0.999)*exp(-0.001*(t1+t2)))+
  sum(lgamma(t1+n)+log(t2^t1)-lgamma(t1)-log((t2+1)^(t1+n)))
  }
}
z<-z-max(z)
z<-exp(z)

plot2<-contour(a, b, z,xlim=c(4,5.5),ylim=c(-5,-2), drawlabels = FALSE,xlab = "alpha", ylab="beta")
title("Contour Plot for Posterior Density", outer = TRUE, line = -3)
```



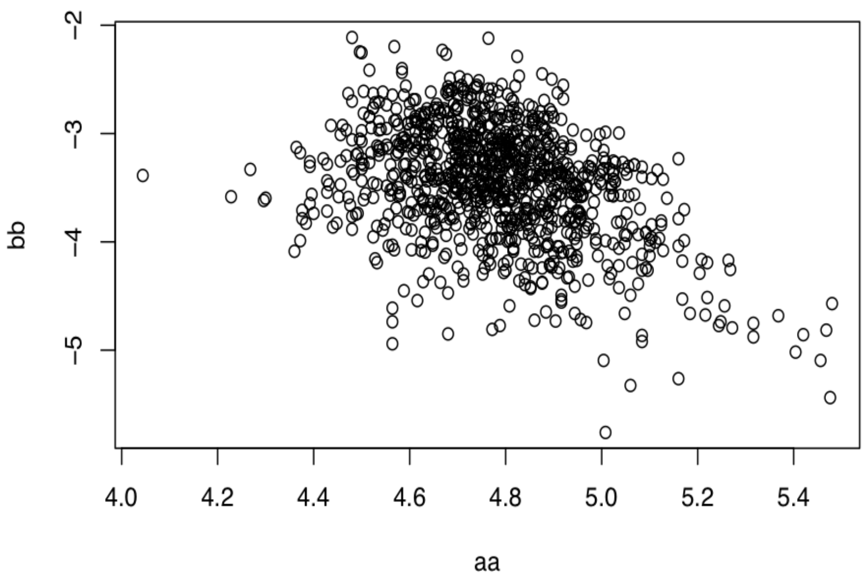
Then we draw the sampling scatter plot:

```
prob<-apply(z,1,sum)
prob<-prob/sum(prob)
#create two vectors to store the simulation result.
aa <- rep(0,1000)
bb <- rep(0,1000)

for (i in 1:1000) {

  t <- sample(length(a),1,prob=prob)
  aa[i]<- a[t]
  bb[i]<- sample(b,1,prob=z[t,])

}
plot(bb~aa)
```



(c)

When β remains unchanged, let α goes to infinity:

```
t1<-exp(100-3.5)
    t2<-exp(-3.5)
exp(log(t1*t2)+log((t1*t2)^(-0.999)*exp(-0.001*(t1+t2)))+
        sum(lgamma(t1+n)+log(t2^t1)-lgamma(t1)-log((t2+1)^(t1+n))))
```

```
## [1] 0
```

the posterior density of the hyperpriors goes to 0, which means that tthe posterior density is integrable.