2 / (a) $P(B) = P(\Omega \cap B) = P[(A \cup A^c) \cap B] = P[(A\cap B) \cup (A^c \cap B)] = P(A\cap B) + P(A^c \cap B) - P(A \cap A^c \cap B)$
$= P(A \cap B) + P(A^c \cap B)$

Then $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) - P(A)\cdot P(B) = (1-P(A))\cdot P(B) = P(A^c)\cdot P(B)$

Therefore, A^c and B are independent if A and B are independent

(b) False. Let $\Omega = \{1,2,3,4,5,6\}$; $A = \{1,2,5\}$; $B = \{2,6,3\}$; $C = \{1,2,3,4\}$. Then

$P(A,B|C) = \frac{P(A,B,C)}{P(C)} = \frac{\frac{1}{6}}{\frac{4}{6}} = \frac{1}{4}$; $P(A|C) = \frac{P(A,C)}{P(C)} = \frac{\frac{2}{6}}{\frac{4}{6}} = \frac{1}{2}$; $P(B|C) = \frac{P(B,C)}{P(C)} = \frac{\frac{2}{6}}{\frac{4}{6}} = \frac{1}{2}$

Then $P(A,B|C) = P(A|C)\cdot P(B|C) = \frac{1}{4}$ so that A, B are conditional independent given that C.

Then $P(A,B|C^c) = \frac{P(A,B,C^c)}{P(C^c)} = \frac{0}{\frac{2}{6}} = 0$ since $A \cap B \cap C^c = \phi$. $P(A|C^c) = \frac{P(A,C^c)}{P(C^c)} = \frac{\frac{1}{6}}{\frac{2}{6}} = \frac{1}{2}$

$P(B|C^c) = \frac{P(B,C^c)}{P(C^c)} = \frac{\frac{1}{6}}{\frac{2}{6}} = \frac{1}{2}$. $P(A,B|C^c) \neq P(A|C^c)\cdot P(B|C^c)$

(c) True. Let E_i and E_j be two arbitrary events in a partition such that $E_i \cap E_j = \phi$. However,

Independence implies $P(E_i \cap E_j) = P(E_i)\cdot P(E_j)$. Let $E_i = \{1,2\}$, $E_j = \{3\}$, $\Omega = \{1,2,3\}$

Then $P(E_i \cap E_j) = 0 \neq \frac{2}{3}\cdot\frac{1}{3}$

(d) False. Let $\Omega = A = \{1,2,3,4,5,6\}$ and $B = \{1,2,3,4\}$. Then $A^c = \phi$, $B^c = \{5,6\}$. Then

$P(A|B) = \frac{P(A,B)}{P(B)} = \frac{P(B)}{P(B)} = 1$; $P(B^c|A^c) = \frac{P(B^c,A^c)}{P(A^c)}$ undefined because of $\frac{0}{0}$

2. (a) ① Let $F=\Omega$, then $A\cap F = A = \Omega_A$, ~~Ω_A is in~~ $\Omega_A \in C_A$

② Let $E_1 = A\cap F_1$ for $\forall F_1 \in C$. Then it is known that $(A\cap F_1)\cup(A\cap F_1^c) = A\cap(F_1\cup F_1^c) = A$
$(A\cap F_1)\cap(A\cap F_1^c) = A\cap(F_1\cap F_1^c) = \phi$. $A\cap F_1^c$ is the complement of E_1 in terms of Ω_A. Since $F_1^c \in C$, then $(A\cap F_1^c)\in C_A$

③ Let $E_1 = A\cap F_1$, $E_2 = A\cap F_2$ for $\forall F_1, F_2 \in C$. Then $E_1\cup E_2 = (A\cap F_1)\cup(A\cap F_2) = A\cap(F_1\cup F_2)$.
Since $F_1\cup F_2 \in C$, then $A\cap(F_1\cup F_2)\in C_A$. $E_1\cup E_2 \in C_A$.

According to the definition of collection, C_A is a valid collection

① $P_A(\Omega_A) = P_A(\Omega\cap A) = \dfrac{P(\Omega\cap A)}{P(A)} = \dfrac{P(A)}{P(A)} = 1$

② $P_A(S\cap A) = \dfrac{P(S\cap A)}{P(A)} \geq 0$ as $P(A)>0$ and $P(S\cap A)\geq 0$ for $\forall S\in C$

③ for $\forall S_1, S_2 \in C$ if S_1, S_2 are disjoint, then $P_A[(S_1\cap A)\cup(S_2\cap A)] = P_A[(S_1\cup S_2)\cap A] = \dfrac{P[(S_1\cup S_2)\cap A]}{P(A)}$
$= \dfrac{P[(S_1\cap A)\cup(S_2\cap A)]}{P(A)} = \dfrac{P(S_1\cap A)+P(S_2\cap A)}{P(A)}$ [$(S_1\cap A)$ and $(S_2\cap A)$ are disjoint] $= \dfrac{P(S_1\cap A)}{P(A)} + \dfrac{P(S_2\cap A)}{P(A)}$
$= P_A(S_1\cap A) + P_A(S_2\cap A)$. According to the definition of probability measure, this measure is valid

(b) Yes. ① $P(\Omega) = \dfrac{\#\text{ of } i\text{-values such that } X_i\in\Omega}{N} = \dfrac{N}{N} = 1$ since $X_i\in\Omega$ for $\forall i$

② ~~$P(\ldots)$~~ $P(S)\geq 0$ as $N>0$ and the numerator is always non-negative.

③ for $\forall S_1, S_2 \in \Omega$ if S_1, S_2 are disjoint, then
$P(S_1\cup S_2) = \dfrac{\text{number of } i\text{-values such that } X_i\in S_1\cup S_2}{N} = \dfrac{\#\text{ of } i\text{-values for } X_i\in S_1 + \#\text{ of } i\text{-values for } X_i\in S_2}{N}$
$= \dfrac{\#\text{ of } i\text{-values for } X_i\in S_1}{N} + \dfrac{\#\text{ of } i\text{-values for } X_i\in S_2}{N} = P(S_1)+P(S_2)$

According to the definition, it's a valid probability measure.

3. Let $f = \begin{cases} 1 & \text{if food is good} \\ 0 & \text{if food is bad} \end{cases}$; $w = \begin{cases} 1 & \text{if wake up} \\ 0 & \text{otherwise} \end{cases}$; $s = \begin{cases} 1 & \text{if good sleeper} \\ 0 & \text{if bad sleeper} \end{cases}$

The quality of food and the type of the baby are independent in this case.

Then we get: $P(f=0)=0.1$, $P(w=1|f=0)=1$, $P(s=1)=0.6$, $P(f=1)=0.9$, $P(s=0)=0.4$

$P(w=1|s=1,f=1)=0.1$, $P(w=1|s=0,f=1)=0.8$

(a) $$P(w=1)=P(w=1|f=0)P(f=0)+P(w=1|f=1)\cdot P(f=1)$$

$$=P(w=1|f=0)P(f=0)+[P(w=1|s=1,f=1)\cdot P(s=1|f=1)+P(w=1|s=0,f=1)\cdot P(s=0|f=1)]\cdot P(f=1)$$

$$=P(w=1|f=0)\cdot P(f=0)+[P(w=1|s=1,f=1)\cdot P(s=1)+P(w=1|s=0,f=1)\cdot P(s=0)]\cdot P(f=1)$$

$$=1\cdot 0.1+(0.1\cdot 0.6+0.8\cdot 0.4)\cdot 0.9$$

$$=0.442$$

(b) $$P(f=0|w=1)=\frac{P(w=1|f=0)\cdot P(f=0)}{P(w=1)}=\frac{1\cdot 0.1}{0.442}=\frac{50}{221}$$

(c) $$P(f=0|s=1,w=1)=\frac{P(w=1|f=0,s=1)\cdot P(f=0)\cdot \cancel{P(s=1)}}{P(w=1|s=1)\cdot \cancel{P(s=1)}}=\frac{1\cdot 0.1}{P(w=1|f=0,s=1)\cdot P(f=0)+P(w=1|f=1,s=1)\cdot P(f=1)}$$

$$=\frac{0.1}{1\cdot 0.1+0.1\cdot 0.9}=\frac{10}{19}$$

(d) Since P(f=0|w=1) ≠ P(f=0|s=1,w=1), these two events are not conditionally independent given that w = 1 according to the definition. In the context, given the information that the baby wakes up, it is intuitive to have more certainties on the quality of the food if it is known that the baby is a good sleeper.

4. (a) $P(X=5) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$; $P(X=4) = \left(\frac{1}{2}\right)^5 \cdot 2 = \frac{1}{16}$; $P(X=3) = 2\cdot\left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 = \frac{5}{32}$

$P(X=2) = \left(\frac{1}{2}\right)^3 \times 2 + \left(\frac{1}{2}\right)^4 \times 2 - \left(\frac{1}{2}\right)^5 = \frac{11}{32}$;

$P(X=1) = 1 - P(X=5) - P(X=4) - P(X=3) - P(X=2) - P(X=0)$

$= 1 - \frac{1}{32} - \frac{1}{16} - \frac{5}{32} - \frac{11}{32} - \frac{1}{32} = \frac{3}{8}$

(b)

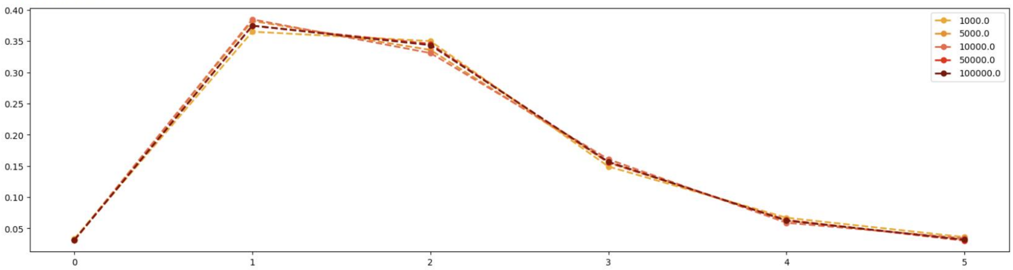
Code:

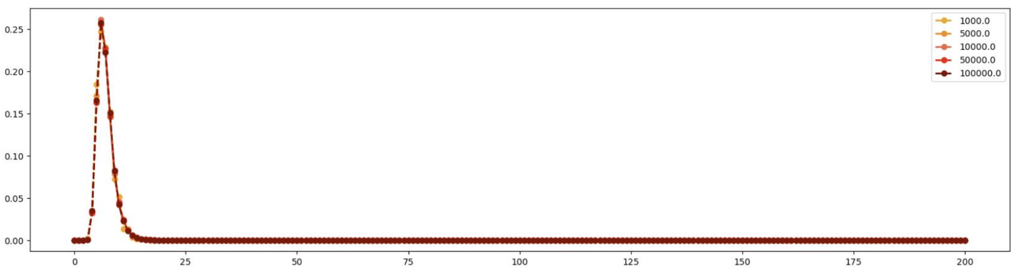
```python
def p_longest_streak(n, tries):
    count = np.zeros(n+1)
    for i in range(int(tries)):
        sequence = np.random.randint(0,2,size = n)
        currentStreak = 0
        longestStreak = 0
        for j in range(n):
            if sequence[j] == 1:
                currentStreak += 1
            else:
                if(currentStreak > longestStreak):
                    longestStreak = currentStreak
                currentStreak = 0
        if(currentStreak > longestStreak):
            longestStreak = currentStreak
        count[longestStreak] += 1
    prob = count/tries
    return prob

    # Write your Monte Carlo code here, n is the length of the sequence and tries is the number
    # of sampled sequences used to produce the estimate of the probability
```

Plot:

```
tries: 1000.0
tries: 5000.0
tries: 10000.0
tries: 50000.0
tries: 100000.0
tries: 1000.0
tries: 5000.0
tries: 10000.0
tries: 50000.0
tries: 100000.0
```





We can see that the estimated probabilities for each longest streak in a sequence of length 5 have not much difference with the exact probabilities for that

(c)

```
print("The probability that the longest streak of ones in a Bernoulli iid sequence of length 200 has length 8 or more is ")
print(sum(p_longest_streak(200,1e5)[8:])) # Compute the probability and print it here
```

```
The probability that the longest streak of ones in a Bernoulli iid sequence of length 200 has length 8 or more is
0.3181899999999999
```

The estimated probability that the longest streak of heads has length 8 or more for 200 flips is about 0.32. Therefore, the sequence of 8 ones is not an evidence that the program may not be generating truly random sequences since a randomly generated sequence can have a sequence of 8 or more ones for 200 flips with a high probability(32%).