Q1

1- Prove that Max-Degree Spanning Tree (MDST) is NP.

Certificate: a subgraph T of G

Certifier: can easily check in polynomial time that

a- T is a spanning tree (connected and the number of edges is one less than the number of vertices)

b- All nodes degrees in G' are $\leq k$

a and b can be easily done in polynomial time

$\rightarrow$ MDST $\in$ NP

2- Choose Hamiltonian path for our reduction

3- Will show that Hamiltonian Path $\leq_p$ MDST

We will show that G has a Hamiltonian Path iff G has a spanning tree with maximum degree 2.

Hamiltonian Path Problem means that given a graph $G' = (V', E')$, determine if there is a simple path that visits every vertex only once.

Construction:

- Use the same graph G' for the MDST problem. i.e $G = G'$
- Set $k=2$ for the MDST problem. A spanning tree with maximum degree 2 is a path since any degree higher than 2 would create a branching point, not allowed in a simple path.

Proof:

A- If G' has a Hamiltonian path, then this path is a spanning tree of G' with maximum degree 2, since it starts and ends at nodes with degree 1, and all other nodes have degree 2.

B- If G has a spanning tree with maximum degree 2, this spanning tree must be a path that visits every vertex (since it's a spanning tree) and hence forms a Hamiltonian path in G''.

Q2.

1- Prove that k-cycle-decomposition is in NP.

Certificate: A set of k disjoint cycles in G with sizes $a_1, \dots a_k$.

Certifier: can easily check in polinomial time that

a- Each cycle has the correct size.

b- The cycles are disjoint (no shared vertices).

c- Every vertex in V is in exactly one of the cycles.

a, b and c can be easily done in polynomial time.

→ k-cycle-decomposition ∈ NP.

2- choose Hamiltoncycle for our reduction

3- will show that Hamiltonian cycle $\leq_p$ k-cycle-decomposition

Given an instance G of Hamiltonian Cycle, we will construct an instance $(G', a_1, a_2 \dots a_k)$ of k-cycle-decomposition such that G has a Hamiltonian Cycle iff G' can be decomposed into k cycles of sizes $a_1, \dots a_k$.

Construction: Given an instance $G(V, E)$ of Hamiltonian cycle, we construct an instance $(G'(V', E'), a_1, \dots a_k)$ of k-cycle-decomposition as follows.

- Let $n = |V|$. We set $a_1 = n$ and $a_2 = 1$.
- We construct G' from G by adding one new vertex v' and connecting it to an arbitrary vertex v in G

Proof:

A - If G has a Hamiltonian Cycle, then that cycle together with the new vertex v' as a 1-cycle forms a 2-cycle-decomposition of

G with size $a_1 = n$ and $a_2 = 1$.

B- If G' has a 2-cycle-decomposition with sizes $a_1 = n$ and $a_2 = 1$ then the cycle of size n must be a Hamiltonian Cycle in G, because it cannot include v' (otherwise the remaining cycle would have size 0, not 1.

The same construction can be used for any $k > 2$ by setting $a_1 = n$ and all other $a_i = 1$ (for $i > 1$) and adding $k-1$ new vertices, each connected to an arbitrary vertex in G. The argument remains the same: G has a Hamiltonian cycle iff G' can be decomposed into k cycles with the given sizes.

Q3

1- Prove that HALF-IS is in NP

Certificate: A subset of V. $|S| = |V|/2$

Certifier: can easily check in polynomial time that

a- $|S| = |V|/2$

b- No two vertices in S are adjacent.

a and b can be easily done in polynomial time

$\rightarrow$ HALF-IS $\in$ NP.

2- Choose Independent Set for our reduction

3- Will show that Independent Set $\leq_p$ HALF-IS.

Given an instance (G, k) of Independent Set, we will construct G' of HALF-IS such that G has an independent set of size k iff G' has an independent set of size $|V(G')|/2$

Construction: Given an instance $(G(V,E), k)$ of Independent Set, we construct an instance $G'(V', E')$ of HALF-IS as follows:

1) If $k = |V|/2$, let $G' = G$.

2) If $k < |V|/2$, add $m = |V| - 2k$ new vertices to G, each disconnected from all other vertices, to form G'. Note that $|V'| = |V| + m = |V| + (|V| - 2k) = 2|V| - 2k$, which is even.

3) If $k > |V|/2$, add $m = 2k - |V|$ new vertices to G, each connected to all other vertices (including each other), to form G'. Note that $|V'| = |V| + m = |V| + (2k - |V|) = 2k$, which is even.

Proof:

A - If G has an independent set S of size k:

1) If $k = |V|/2$, then S is an independent set of size $|V'|/2$ in G'.

2) If $k < |V|/2$, then S union the m new vertices is an independent set of size $k+m = \frac{|V'|}{2}$ in G'.

3) If $k > |V|/2$, then S is an independent set of size $k = \frac{|V'|}{2}$ in G', because none of the m new vertices can be in an independent set S' of size $|V'|/2$.

B- If G' has an independent set S' of size $|V'|/2$.

1) if $k = |V|/2$, then S' is an independent set of size k in G.

2) if $k < |V|/2$, then S' must include all m new vertices (since they are not connected to any other vertex), so the remaining vertices in S' form an independent set of size $|V'|/2 - m = k$ in G.

3) If $k > |V|/2$, then S' cannot include any of m new vertices (since they are all connected to each other), so S' is an independent set of size $|V'|/2 = k$ in G