2024/1/17

CSCI 570 HW1

1. State whether the following statement is True or False: "It is possible to have an instance of the Stable Matching problem in which two women have the same best valid partner." (5pts)

False.

In Stable Matching problem, for valid partner, we know that:

> A man *m* is a valid partner of a woman *w* if there is a stable matching that contains the pair (m,w).

For best valid partner, we know that:

> *m* is the best valid partner of *w* if *m* is a valid partner of *w* and no man whom *w* ranks higher than *m* is a valid partner of her.

Also, for stable matching problem, we know that it is guaranteed to produce a stable marriage for all participants. And GS algorithm can find at most two stable matchings, one with men proposing *S* and one with women proposing *S'*.

Thus, for this statement, women will have unique best valid partner in stable matching *S'*, and it is not possible to have two women to have the same best valid partner.

2. In the context of a stable roommate problem involving four students (a, b, c, d), each student ranks the others in a strict order of preference. A matching involves forming two pairs of students, and it is considered stable if no two separated students would prefer each other over their current roommates. The question is whether a stable matching always exists in this scenario. If it does, provide proof; if not, present an example of roommate preferences where no stable matching is possible. (10pts)

Assume:

Ranking	**A**	**B**	**C**	**D**
1	B	C	A	A
2	C	A	B	B
3	D	D	D	C

Step 1: (A,B)

Step 2: (B,C), A is dropped, since C is preferred than A.

Step 3: (C,A), B is dropped, since A is preferred than B.

Step 4: (D,B), A is already paired up with C, thus go to B instead.

Step 5: Looks like (C,A) and (D, B) is stable, however, (A, B) will cause an instability, since A prefers B over C, and B prefers A over D.

Moreover, in other cases like (A,B), (C,D); (B,C) will cause instability. (A,D),(B,C); (A,C) will cause instability.

Thus, every possible matching is unstable in this situation.

3. Solve Kleinberg and Tardos, Chapter 1, Exercise 1. (10pts)

Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

> *True or false? In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.*

False.

Assume:

Ranking	m_1	m_2	w_1	w_2
1	w_1	w_2	m_2	m_1
2	w_2	w_1	m_1	m_2

For man proposing:

Step1: (m_1, w_1), since w_1 is ranked first on the preference list of m_1. In observation, for w_1, the top choice is m_2 not m_1.

Step2: (m_2, w_2), since w_2 is ranked first on the preference list of m_2 and w_2 is not engaged. In observation, for w_2, the top choice is m_1 not m_2.

In this case, the statement is false, since there is no pair (m, w) such that *m* is ranked first on the preference list of *w* and *w* is ranked first on the preference list of *m*.

Same idea for woman proposing:

Step1: (w_1, m_2), since m_2 is ranked first on the preference list of w_1. In observation, for m_2, the top choice is w_2 not w_1.

Step2: (w_2, m_1), since m_1 is ranked first on the preference list of w_2 and m_1 is not engaged. In observation, for m_1, the top choice is w_1 not w_2.

Thus, the statement is false for all stable matchings in this problem.

4. Solve Kleinberg and Tardos, Chapter 1, Exercise 2. (10pts)

Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

> *True or false? Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m. Then in every stable matching S for this instance, the pair (m, w) belongs to S.*

True.

Proof (by contradiction):

Suppose there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m. In stable matching S, m is paired with w', which is (m, w'), and w is paired with m', which is (w, m'). In this case, (m, w) will cause an instability.

Thus, the statement is true. Every stable matching S contains the pair (m, w).

5. Determine whether the following statement is true or false. If it is true, give an example. If it is false, give a short explanation. (10pts)

> *For some $n \geq 2$, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm when men are proposing, every woman is matched with their most preferred man, even though that man does not prefer that woman the most.*

True.

Assume:

Ranking	m_1	m_2	m_3	w_1	w_2	w_3
1	w_3	w_3	w_1	m_1	m_2	m_3
2	w_1	w_2	w_3	m_3	m_1	m_2
3	w_2	w_1	w_2	m_2	m_3	m_1

Step1: (m_1, w_3).

Step2: (m_2, w_3), m_1 is dropped, since w_3 prefer m_2 over m_1.

Step3: (m_3, w_1).

Step4: (m_1, w_1), m_3 is dropped, since w_1 prefer m_1 over m_3.

Step5: (m_3, w_3), m_2 is dropped, since w_3 prefer m_3 over m_2.

Step6: (m_2, w_2).

Now, in the end, we have (m_1, w_1), (m_2, w_2), (m_3, w_3). From this preference list, every woman is matched with their most preferred man, even though that man does not prefer that woman the most.

6. Determine whether the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample. (10pts)

> *For all $n \geq 2$, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm when men are proposing, every woman is matched with their least preferred man.*

True.

Assume that each woman's least favorite man happens to be the man who likes her the most, and in each woman's preference list, these men are all different. In this case, due to the nature of the Gale-Shapley algorithm, each man will first propose to the woman he likes the most. Since each woman's least favorite man happens to be the one who likes her the most, this means she will accept this proposal, because she will not receive a better one (as all other men are proposing to women they prefer more). Therefore, each woman will eventually end up matched with her least favorite man.

Here is an example, assume n = 2:

Ranking	m_1	m_2	w_1	w_2
1	w_1	w_2	m_2	m_1
2	w_2	w_1	m_1	m_2

When man is proposing,(m_1, w_1), (m_2, w_2) will be stable. In observation, for w_2, m_2 is the least preferred man; for w_1, m_1 is the least preferred man.

Thus, the statement is true.

7. Solve Kleinberg and Tardos, Chapter 1, Exercise 8. (10pts)

For this problem, we will explore the issue of truthfulness in the Stable Matching Problem and specifically in the Gale-Shapley algorithm. The basic question is: Can a man or a woman end up better off by lying about his or her preferences? More concretely, we suppose each participant has a true preference order. Now consider a woman w. Suppose w prefers man m to m′, but both m and m′ are low on her list of preferences. Can it be the case that by switching the order of m and m′ on her list of preferences (i.e., by falsely claiming that she prefers m′ to m) and running the algorithm with this false preference list, w will end up with a man m″ that she truly prefers to both m and m′? (We can ask the same question for men, but will focus on the case of women for purposes of this question.)

Resolve this question by doing one of the following two things:

(a) Give a proof that, for any set of preference lists, switching the order of a pair on the list cannot improve a woman's partner in the Gale-Shapley algorithm; or

(b) Give an example of a set of preference lists for which there is a switch that would improve the partner of a woman who switched preferences.

(b)

Initial preference list:

Ranking	m_1	m_2	m_3	w_1	w_2	w_3
1	w_1	w_2	w_1	m_2	m_1	m_1
2	w_2	w_1	w_2	m_1	m_2	m_2
3	w_3	w_3	w_3	m_3	m_3	m_3

Step1: (m_1, w_1)

Step2: (m_2, w_2)

Step3: (m_3, w_1), m_3 is rejected since m_1 is preferred for w_1.

Step4: (m_3, w_2), m_3 is rejected since m_2 is preferred for w_2.

Step5: (m_3, w_3)

The final result is (m_1, w_1), (m_2, w_2), (m_3, w_3)

Falsely claiming preference list:

Ranking	m_1	m_2	m_3	w_1	w_2	w_3
1	w_1	w_2	w_1	m_2	m_1	m_1
2	w_2	w_1	w_2	m_3	m_2	m_2
3	w_3	w_3	w_3	m_1	m_3	m_3

Step1: (m_1, w_1)

Step2: (m_2, w_2)

Step3: (m_3, w_1), m_1 is dropped since m_3 is preferred for w_1.

Step4: (m_1,w_2), m_2 is dropped since m_1 is preferred for w_2.

Step5: (m_2,w_1), m_3 is dropped since m_2 is preferred for w_1.

Step6: (m_3,w_3), m_3 is rejected by w_2.

The final result is (m_1, w_2), (m_2, w_1), (m_3, w_3). The partner for w_1 is improved from m_1 to m_2.

Thus, the statement is true, there is a switch that would improve the partner of a woman who switched preferences.

8. There are six students, Harry, Ron, Hermione, Ginny, Draco, and Cho. This class requires them to pair up and work on pair programming. Each has preferences over who they want to be paired with. The preferences are:

Harry: Cho > Ron > Hermione > Ginny > Draco

Ron: Ginny > Harry > Hermione > Cho > Draco

Hermione: Ron > Harry > Ginny > Cho > Draco

Ginny: Harry > Cho > Hermione > Ron > Draco

Draco : Cho > Ron > Ginny > Hermione > Harry

Cho: Hermione > Harry > Ron > Ginny > Draco

Show that there is no stable matching. That means showing that no matter who you put together, there will always be two potential partners who are not matched but prefer each other to the current partner. (10pts)

Assume a stable matching does exist, we need to check if there is a way to match them such that no two students would prefer to be with each other over their current partners.

Firstly, note that Draco is at the Bottom of everyone’s preference list. This means that if Draco is paired with any student, that student would always prefer to be with someone else from their list over Draco, unless they are already with their top choice.

Consider any pair within ABCDE (Harry, Ron, Hermione, Ginny, Cho). Since each person has at least one other person in ABCDE who ranks higher than Draco on their preference list, if Draco is paired with anyone from ABCDE, their partner (within ABCDE) would always prefer them over Draco. This creates an instability.

For example, if Draco is paired with Harry, then according to Harry's preference list, he would prefer to be paired with Cho, Ron, Hermione, or Ginny rather than Draco. Similarly, whether it's Cho, Ron, Hermione, or Ginny, their preference for Harry is always higher than for Draco. Thus, a pairing of Harry with any of these more preferred individuals would disrupt the original stability.

Since there will always be at least one person who prefers someone else over their current partner, no matter how the pairs are formed, a stable matching does not exist. This proves the initial assumption in the problem statement to be false, i.e., there is no stable matching.