CSCI 570 - HW 12

November 29, 2022

1. [20 points] A variation of the satisfiability problem is the MIN 2-SAT problem. The goal in the MIN 2-SAT problem is to find a truth assignment that minimizes the number of satisfied clauses. Give a 2 approximation algorithm that you can find for the problem.

Answer:

We assume that no clause contains a variable as well as the complement of the variable. (Such clauses are satisfied regardless of the truth assignment used.)

We give a 2-approximation algorithm as follows:
Let I be an instance of MINSAT consisting of the clause set C_I and variable set X_I. Construct an auxiliary graph $G_I(V_I, E_I)$ corresponding to I as follows. The node set V_I is in one-to-one correspondence with the clause set C_I. For any two nodes v_i and v_j in V_I, the edge (v_i, v_j) is in E_I if and only if the corresponding clauses c_i and c_j are such that there is a variable x belongs to X_I that appears in un-complemented form in c_i and complemented form in c_j, or vice versa.

To construct a truth assignment, we construct an approximate vertex cover V' for G_I such that $|V'|$ is at most twice that of a minimum vertex cover for G_I. Then construct a truth assignment that causes all clauses in $V_I - V'$ to be false. (For a method to find an approximate vertex cover, please refer to section 11.4 in the textbook.)

2. [20 points] Write down the problem of finding a Min-s-t-Cut of a directed network with source s and sink t as an Integer Linear Program and explain your program.

$$
\begin{aligned}
\text{minimize} \sum_{(u,v)\in E} c(u,v).x_{(u,v)} \\
\text{subject to} : x_v - x_u + x_{(u,v)} \geq 0 \quad \forall (u,v) \in E \\
x_u \in \{0,1\} \quad \forall u \in V : u \neq s, u \neq t \\
x_{(u,v)} \in \{0,1\} \quad \forall (u,v) \in E \\
x_s = 1 \\
x_t = 0
\end{aligned}
$$

The variable x_u indicates if the vertex u is on the side of s in the cut. That is, $x_u = 1$ if and only if u is on the side of s. Setting $x_s = 1$ and $x_t = 0$ ensures that s and t are separated. Likewise, the variable $x_{(u,v)}$ indicates if the edge (u,v) crosses the cut. The first constraint ensures that if the edge (u,v) is in the cut, then u is on the side of s and v is on the side of t. For completeness, you should argue that with the above correspondence (that is, $x_{(u,v)}$ indicating if an edge crosses the cut), every min-s-t-cut corresponds to a feasible solution and vice versa.

Grading (20pt):

- 15 pt: Correctly write LP
- 5 pt: Correctly explain LP

3. [10 points] 720 students have pre-enrolled for the "Analysis of Algorithms" class in Fall. Each student must attend one of the 16 discussion sections, and each discussion section i has capacity for D_i students. The happiness level of a student assigned to a discussion section i is proportionaate to $\alpha_i(D_i - S_i)$, where α_i is a parameter reflecting how well the air-conditioning sysem works for the room used for section i (the higher the better), and S_i is the actual number of students assigned to that section. We want to find out how many students to assign to each section in order to maximize total student happiness. Express the problem as a integer linear program problem.

Answer:
Our variables will be the S_i. Our objective function is:

$$\begin{aligned} \text{maximize} \sum_{i=1}^{16} \alpha_i (D_i - S_i) \\ \text{subject to: } D_i - S_i \geq 0 \text{ for } 0 < i \leq 16 \\ S_i \geq 0 \text{ for } 0 < i \leq 16 \\ \sum_{i=1}^{16} S_i = 720 \end{aligned}$$

Grading:

- Objective function (4 points)
- Each constraint (2 points)

4. [16 points] A set of n space stations need your help in building a radar system to track spaceships traveling between them. The i^{th} space station is located in 3D space at coordinates (x_i, y_i, z_i). The space stations never move. Each space station i will have a radar with power r_i, where r_i is to be determined. You want to figure how powerful to make each space station's radar transmitter, so that whenever any spaceship travels in a straight line from one station to another, it will always be in radar range of either the first space station (its origin) or the second space station (its destination). A radar with power r is capable of tracking space ships anywhere in the sphere with radius r centered at itself. Thus, a space ship is within radar range through its strip from space station i to space station j if every point along the line from (x_i, y_i, z_i) to (x_j, y_j, z_j) falls within either the sphere of radius r_i centered at (x_i, y_i, z_i) or the sphere of radius r_j centered at (x_j, y_j, z_j). The cost of each radar transmitter is proportional to its power, and you want to minimize the total cost of all of the radar transmitters. You are given all of the $(x_1, y_1, z_1), ..., (x_n, y_n, z_n)$ values, and your job is to choose values for $r_1, ..., r_n$. Express this problem as a linear program.

(a) Describe your variables for the linear program (3 pts).
Answer:
r_i=the power of the i^{th} radar transmitter., i=1,2,...n (3 pts)

(b) Write out the objective function (5 pts).
Answer:
Minimize $r_1 + r_2 + ... + r_n$ or $\sum_i^n r_i$
Defining the objective function without mentioning r_i: -3 pts

(c) Describe the set of constraints for LP. You need to specify the number of constraints needed and describe what each constraint represents (8 pts).
Answer:
$r_i + r_j \geq \sqrt{((x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2)}$. Or, $r_i + r_j \geq d_{i,j}$ for each pair of stations i and j, where $d_{i,j}$ is the distance from station i to station j (6 pts).
We need $\sum_{i=1}^{n-1} i = (n^2 - n)/2$ constraints of inequality (The number of constraints is due to the number of unique paths between each pair of space stations) (2pts).

5. (Ungraded) Given a graph G and two vertex sets A and B, let $E(A, B)$ denote the set of edges with one endpoint in A and one endpoint in B. The Max Equal Cut problem is defined as follows
Instance Graph $G(V, E), V = 1, 2, ..., 2n$.
Question Find a partition of V into two n-vertex sets A and B, maximizing the size of $E(A, B)$.
Provide a factor $\frac{1}{2}$-approximation algorithm for solving the Max Equal Cut problem.

Answer:
Start with empty sets A, B, and perform n iterations:
In iteration i, pick vertices $2i - 1$ and $2i$, and place one of them in A and the other in B, according to which choice maximizes $|E(A, B)|$.
In a particular iteration, when we have cut (A, B) and we add u and v. Suppose u has N_{A_u}, N_{B_u} neighbours in A, B respectively and suppose v has N_{A_v}, N_{B_v} neighbours in A, B respectively. Then, adding u to A and v to B adds $N_{B_u} + N_{A_v}$ edges to the cut, whereas doing the other way round

adds $N_{B_v} + N_{A_u}$ edges to the cut. Since the sum of these two options is nothing but the total number of edges being added to this partial subgraph, the bigger of the two must be at least half the total number of edges being added to this partial subgraph. Since this is true for each iteration, at the end when all the nodes (and edges) are added, our algorithm is bound to add at least half of the total $||E||$ edges. Naturally since the max equal cut capacity $OPT \leq ||E||$, our solution is $\frac{1}{2}$-approximation.