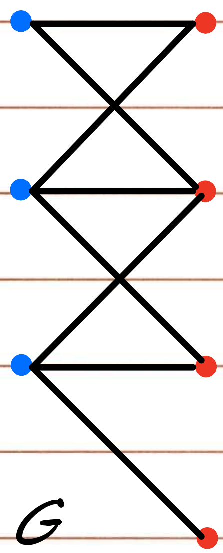
Network Flow

Bipartite Matching Problem

Recall that a bipartite graph $G=(V,E)$ is an undirected graph whose node set can be partitioned as $V = X \cup Y$ with the property that every edge $e \in E$ has one end in X and the other end in Y.

Def. A matching M in G is a subset of the edges $M \subseteq E$ such that each node appears in at most one edge in M.



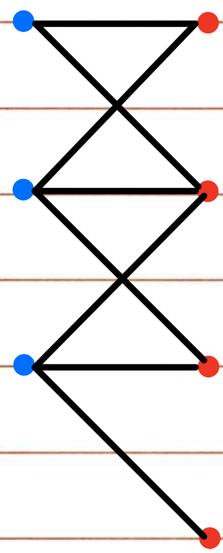
Problem Statement:

Given a bipartite graph G, find a matching of largest possible size in G.

General Plan:

Design a flow network G' that will have a flow value $v(f)=k$ iff there exists a matching of size k in G. Moreover, flow f in G' should identify the matching M in G.

Construction of G'



Solution

Find max. flow in G'. Say max. flow is f. Edges carrying flow between sets X and Y will correspond to our max. size matching in G.

Proof Strategy

To prove this, we will show that G' will have a flow of value k iff G has a matching of size k.

Proof:

A- If we have a matching of size k in G, we can find an s-t flow f of value k in G'.

B- If we have an s-t flow f of value k in G', we can find a matching of size k in G.

What is the complexity of the above solution if the basic Ford-Fulkerson alg. were used to find max. flow in G'?

Edge-Disjoint Paths Problem

Def. A set of paths is edge-disjoint if their edge sets are disjoint.

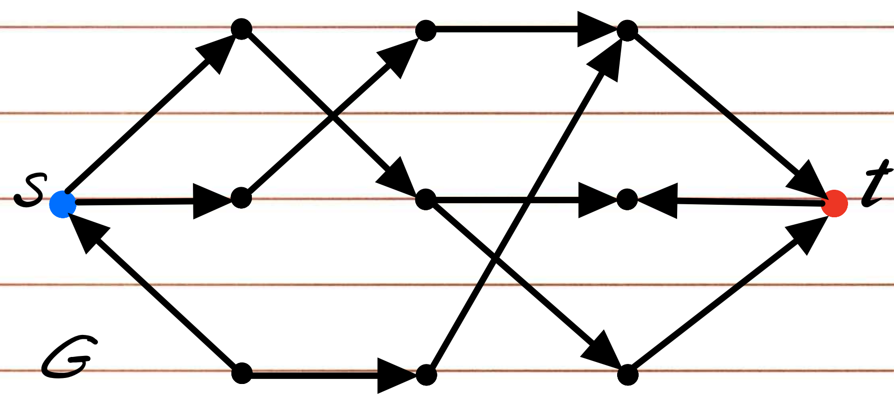
Problem Statement

Given a directed graph $G=(V,E)$ and two nodes s & $t \in V$, find the max. number of edge-disjoint s-t paths in G.

General Plan:

Design a flow network G' that will have a flow value $v(f) = k$ iff there exist k edge-disjoint s-t paths in G. Moreover, flow f in G' should identify the set of k edge-disjoint s-t paths in G.

Construction of G'



Proof:

A- If we have k edge disjoint s-t paths in G, we can find a flow of value k in G'.

B- If we have a flow of value k in G',
we can find k edge-disjoint s-t
paths in G.

How can we apply this solution to undirected graphs?

Node-Disjoint Paths Problem

<u>Def.</u> A set of paths is node-disjoint if their node sets (except for starting and ending nodes) are disjoint.

<u>Problem Statement</u>

Given a directed graph $G=(V,E)$ and two nodes s & $t \in V$, find the max. number of node-disjoint s-t paths in G.

Circulation &
Circulation with lower bounds

Def. A Circulation network is a directed graph $G=(V,E)$ with following features:

- Each edge e has a non-negative capacity c_e
- Associated with each node $v \in V$ is a demand d_v
- if $d_v > 0$, node v has demand of d_v for flow
- if $d_v < 0$, node v has a supply of $|d_v|$ for flow
- if $d_v = 0$, v is neither a sink nor a source

Notation

We will call $f(e)$ Circulation through edge e.
$f(e)$ has the following properties:

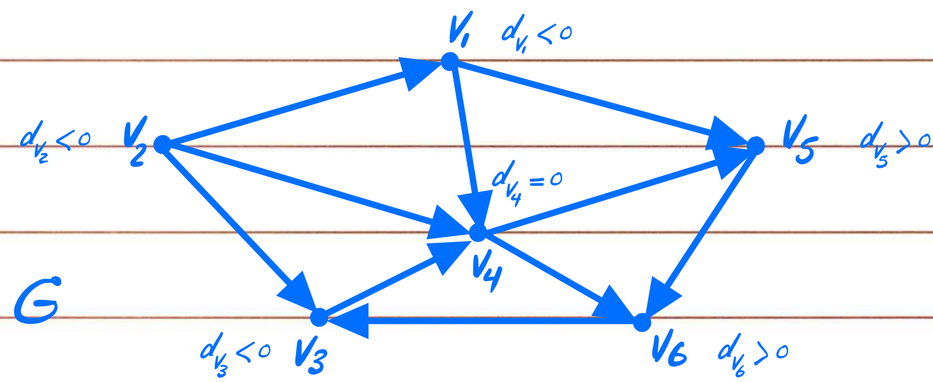
1. Capacity Constraint:
   For each edge $e \in E$, $0 \leqslant f(e) \leqslant c_e$

2. Demand Condition:
   For each node $v \in V$, $\sum_{e \text{ into } v} f(e) - \sum_{e \text{ out of } v} f(e) = d_v$

FACT: If there is a feasible circulation with demands $\{d_v\}$, then $\sum_v d_v = 0$

Problem Statement

Given a circulation network G with demands $\{d_v\}$, is there a feasible circulation in G?



Find max flow f in G'

Can we have $v(f) < D$?

Can we have $v(f) > D$?

Can we have $v(f) = D$?

Proof Template

A) If there is a feasible circulation f with demand values $\{d_v\}$ in G, we can find a max flow of value D in G'.

explain how ...

B) If there is a max flow of value D in G', we can find a feasible circulation f with demand values $\{d_v\}$ in G.

explain how ...

Def. A Circulation network with lower bounds is a directed graph $G=(V,E)$ with following features:

- Each edge e has a non-negative capacity c_e and a lower bound constraint on flow l_e.
- Associated with each node $v \in V$ is a demand d_v
  - if $d_v > 0$, node v has demand of d_v for flow
  - if $d_v < 0$, node v has a supply of $|d_v|$ for flow
  - if $d_v = 0$, v is neither a sink nor a source

Notation

We will call $f(e)$ Circulation through edge e.
$f(e)$ has the following properties:

1. Capacity Constraint:
   For each edge $e \in E$, $l_e \leqslant f(e) \leqslant c_e$

2. Demand Condition:
   For each node $v \in V$, $f^{in}(v) - f^{out}(v) = d_v$

Problem Statement

Given a circulation network G with demands $\{d_v\}$ and lower bounds $\{l_e\}$, is there a feasible circulation in G?

Solution:

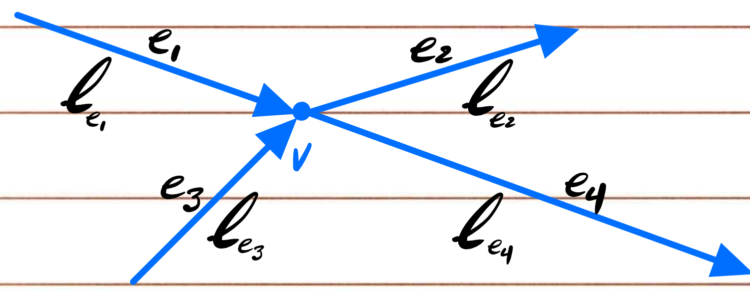
We will find the feasible circulation (if it exists) in two passes.

- Pass #1. Find f_0 to satisfy all l_e's.

- Pass #2. Use the remaining capacity of the network to find a feasible circulation f_1 (if it exists)

- Combine the two flows: $f = f_0 + f_1$

Consider node v in G and its incident edges:



Since $f_0(e_i) = \ell_{e_i}$, we will have a flow imbalance at node v due to f_0.

$$f_0^{in}(v) - f_0^{out}(v) = \sum_{e \text{ into } v} \ell_e - \sum_{e \text{ out of } v} \ell_e = L_v$$

L_v is called the flow imbalance at node v.

Solution outline

1- Push flow f_0 through G, where $f_0(e) = l_e$

2- Construct G', where $C'_e =$
and $d'_v =$

3- Find a feasible circulation in G'.
(if it exists, call this f_1)

4- If in step 3 there is no feasible circulation in G' $\Rightarrow$
There is no feasible circulation in G.

Otherwise,
feasible circulation in $G = f_0 + f_1$

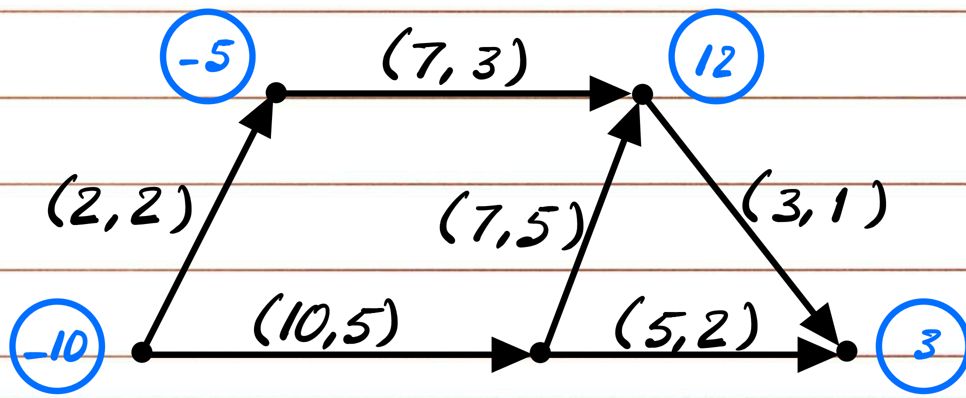
Numerical Example on next page

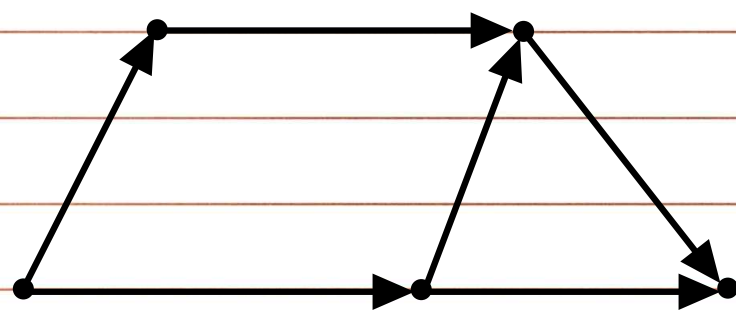
(x, y) indicates

- edge capacity of x
- lower bound flow of y

8 indicates demand value of 8

G





1. We're asked to help the captain of the USC tennis team to arrange a series of matches against UCLA's team. Both teams have n players; the tennis rating (a positive number, where a higher number can be interpreted to mean a better player) of the i-th member of USC's team is t_i and the tennis rating for the k-th member of UCLA's team is b_k. We would like to set up a competition in which each person plays one match against a player from the opposite school. Our goal is to make *as many matches as possible* in which the USC player has a higher tennis rating than his or her opponent. Use network flow to give an algorithm to decide which matches to arrange to achieve this objective.

2. CSCI 570 is a large class with n TAs. Each week TAs must hold office hours in the TA office room. There is a set of k hour-long time intervals $I_1, I_2, \ldots I_k$ in which the office room is available. The room can accommodate up to 3 TAs at any time. Each TA provides a subset of the time intervals he or she can hold office hours with the minimum requirement of l_j hours per week, and the maximum m_j hours per week. Lastly, the total number of office hours held during the week must be H. Design an algorithm to determine if there is a valid way to schedule the TA's office hours with respect to these constraints.

3. There are n students in a class. We want to choose a subset of k students as a committee. There has to be m1 number of freshmen, m2 number of sophomores, m3 number of juniors, and m4 number of seniors in the committee. Each student is from one of k departments, where k = m1 +m2 +m3 +m4. Exactly one student from each department has to be chosen for the committee. We are given a list of students, their home departments, and their class (freshman, sophomore, junior, senior). Describe an efficient algorithm based on network flow techniques to select who should be on the committee such that the above constraints are all satisfied.

4. Given a directed graph $G=(V,E)$ a source node $s \in V$, a sink node $t \in V$, and lower bound ℓ_e for flow on each edge $e \in E$, find a feasible s-t flow of minimum possible value.
Note: there are no capacity limits for flow on edges in G.