Homework 1

Due: January 17, 11.59 PM PST

1. State whether the following statement is True or False: “It is possible to have an instance of the Stable Matching problem in which two women have the same best valid partner.” (5pts)

   False:
   When women propose in GS, they end up with their best valid partner. Thus, there needs to be a unique best valid partner for each woman, since, if two women had the same best valid partner, one wouldn’t end up with that partner in the matching obtained by GS with women proposing, which is a contradiction.

   Rubrics:
   3 pts for mentioning it is False, 2 for correct explanation.

2. In the context of a stable roommate problem involving four students (a, b, c, d), each student ranks the others in a strict order of preference. A matching involves forming two pairs of students, and it is considered stable if no two separated students would prefer each other over their current roommates. The question is whether a stable matching always exists in this scenario. If it does, provide proof; if not, present an example of roommate preferences where no stable matching is possible. (10pts)

   A stable matching need not exist.
   Consider the following list of preferences. Note a, b, and c all prefer d the least.
   • a : b > c > d
   • b : c > a > d
   • c : a > b > d
   Now, there can only be 3 sets of disjoint roommate pairs.
   • If the students are divided as (a, b) and (c, d), then (b, c) cause instability, since c prefers b over d and b prefers c over a.
   • If the students are divided as (a, c) and (b, d), then (a, b) cause instability, since b prefers a over d and a prefers b over c.
   • If the students are divided as (a, d) and (b, c), then (a, c) cause an instability, since a prefers c over d and c prefers a over b.
   Thus every matching is unstable, and no stable matching exists with this list of preferences.

   Rubrics:
   5 pts for mentioning A stable matching need not exist and 5 pts for the correct explanation.

3. Solve Kleinberg and Tardos, Chapter 1, Exercise 1. (10pts)

This is false. Consider, in fact, one of the examples from the text, with two men and two Women m prefers w to w'. m' prefers w' to w. w prefers m' to m. w' prefers m to m'. Note that there are no pairs at all where each ranks the other first, so clearly no such pair can show up in a stable matching.

Rubrics:
5 pts for mentioning it is False, 5 for explanation.

4. Solve Kleinberg and Tardos, Chapter 1, Exercise 2. (10pts)

True. Suppose S is a stable matching that contains the pairs (m, w_0) and (m_0, w), for some $m_0 \neq m$, $w_0 \neq w$. Clearly, the pairing (m, w) is preferred by both m and w over their current pairings, contradicting the stability of S.

Rubric:

5pt: Correctly identifies the statement as true.

5pt: Provides a correct explanation.

5. Determine whether the following statement is true or false. If it is true, give an example. If it is false, give a short explanation. (10pts)
For some $n \geq 2$, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm when men are proposing, every woman is matched with their most preferred man, even though that man does not prefer that woman the most.

True:
W1: M1>M3>M2
W2: M2>M1>M3
W3: M3>M1>M2

M1: W2>W1>W3
M2: W1>W2>W3
M3: W1>W3>W2

M1 proposes to W2, she accepts
M2 proposes to W1, she accepts
M3 proposes to W1, she accepts, now M2 is free
M2 proposes to W2, she accepts, now M1 is free
M1 proposes to W1, she accepts, now M3 is free
M3 proposes to W3, she accepts.
As you can see, all women end up with their most preferred partner, and men don’t.

Rubrics:
- 3 points for stating it is True
- 7 points for showing the example:
    - deduct 2 points if there is a minor mistake in the example

- 2 points if the student states it is False but gives some valid attempt at an argument

6. Determine whether the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample. (10pts)
   For all $n \geq 2$, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm when men are proposing, every woman is matched with their least preferred man.

   Solution: True:
   If every man's first preference in women is unique and for that particular woman, the corresponding man is their least preferred man, it would serve as the required example. More generally, we must ensure that every woman is only proposed by her least preferred partner and no one else otherwise she would get a better partner.

   For e.g.:

   M1: W1>W2
   M2: W2>W1

   W1: M2>M1
   W2: M1>M2

   M1 proposes to W1, she accepts.
   M2 proposes to W2, she accepts.

   Rubrics:
   - 3 points for stating it is True
   - 7 points for correct explanation:
     - deduct 2 points if there is a minor mistake in the explanation
   - 2 points if the student states it is False but gives some valid attempt for a counterexample

7. Solve Kleinberg and Tardos, Chapter 1, Exercise 8. (10pts)

   Assume we have three men m1, m2, and m3 and three women w1, w2, and w3 with preferences as given in the table below.
   Column w3 shows the true preferences of woman w3, while in column w′ she pretends she prefers man m3 to m1.

m_1	m_2	m_3	w_1	w_2	w_3	(w'_3)
w_3	w_1	w_3	m_1	m_1	m_2	m_2
w_1	w_3	w_1	m_2	m_2	m_1	m_3
w_2	w_2	w_2	m_3	m_3	m_3	m_1

First let us consider one possible execution of the G-S algorithm with the true preference list of w3.

m_1	w_3			w_3
m_2		w_1		w_1
m_3			$[w_3][w_1]w_2$	w_2

First m1 proposes to w3, then m2 proposes to w1. Then m3 proposes to w2 and w1 and gets rejected, finally proposes to w2 and is accepted. This execution forms pairs (m1, w3), (m2, w1) and (m3, w2), thus pairing w3 with m1, who is her second choice. Now consider execution of the G-S algorithm when w3 pretends she prefers m3 to m1 (see column w′).
Then the execution might look as follows:

m_1	w_3		$-$	w_1			w_1
m_2		w_1		$-$	w_3		w_3
m_3			w_3		$-$	$[w_1]w_2$	w_2

Man m1 proposes to w3, m2 to w1, then m3 to w3. She accepts the proposal, leaving m1 alone. Then m1 proposes to w1 which causes w1 to leave her current partner m2, who consequently proposes to w3 (and that is exactly what w3 prefers). Finally, the algorithm pairs up m3 (recently left by w3) and w2. As we see, w3 ends up with the man m2, who is her true favorite. Thus we conclude that by falsely switching order of her preferences, a woman may be able to get a more desirable partner in the G-S algorithm.

Rubrics:

- 3 points for choosing (b) as the right approach
- 7 points for showing the correct example
    - Deduct 2 points if there is a minor mistake
- 2 points if the student selects (a) as their approach and gives a valid attempt at it.

8. There are six students, Harry, Ron, Hermione, Ginny, Draco, and Cho. This class requires them to pair up and work on pair programming. Each has preferences over who they want to be paired with. The preferences are:
Harry: Cho > Ron > Hermione > Ginny > Draco
Ron: Ginny > Harry > Hermione > Cho > Draco

Hermione: Ron > Harry > Ginny > Cho > Draco
Ginny: Harry > Cho > Hermione > Ron > Draco
Draco : Cho > Ron > Ginny > Hermione > Harry
Cho: Hermione > Harry > Ron > Ginny > Draco
Show that there is no stable matching. That means showing that no matter who you put together, there will always be two potential partners who are not matched but prefer each other to the current partner. (10pts)

Notice that Draco is least preferred by everyone else. Now, someone must get matched with Draco, say A.
Among the remaining students, there must be a student, say B, who prefers A the most - this is because everyone other than Draco (Harry, Ron, Hermione, Ginny, and Cho) appear as the first preference for someone other than Draco. Then, suppose this B is paired with C (someone other than A as A is paired with Draco). Now, A prefers B over his current partner Draco, and B prefers A over C, thus there will always be an instability.

Rubrics:

- 5 points for mentioning Draco is last on everybody's list.
- 5 points for mentioning every other student has a unique, most preferred partner.