Traveling Salesman Problem (TSP)
&
Hamiltonian Cycle

TSP Problem Statement

Given a set of distances, order <u>n</u> cities in a tour $v_{i_1}, v_{i_2}, \ldots v_{i_n}$ with $i_1 = 1$, so it minimizes

$$\sum d(v_{i_j}, v_{i_{j+1}}) + d(v_{i_n}, v_{i_1})$$

TSP has applications in

- Vehicle routing
- Logistics planning
- Cutting / drilling tasks
- ...

Decision version of TSP:

Given a set of distances on $\underline{n}$ cities and a bound D, is there a tour of length/cost at most D?

Def.: A cycle $\underline{C}$ in G is a Hamiltonian Cycle if it visits each vertex exactly once.

Problem Statement

Given an undirected graph G, is there a Hamiltonian Cycle in G?

Show that the Hamiltonian Cycle (HC) problem is NP-Complete.

1- We show that the problem is in NP

a) Certificate:

b) Certifier:

2 - Choose Vertex Cover as the problem known to be NP Complete

3 - We show that Vertex Cover $\leq_p$ HC

Plan: Given an undirected graph $G = (V, E)$ and an integer $\underline{k}$, we construct $G' = (V', E')$ that has a Hamiltonian Cycle iff G has a vertex cover of size at most $\underline{k}$.

Construction of G'

For each edge (VU) in G, G' will have one gadget W_{VU} with following node labeling:

Some intuition behind the construction of the gadget:

Other vertices in G'

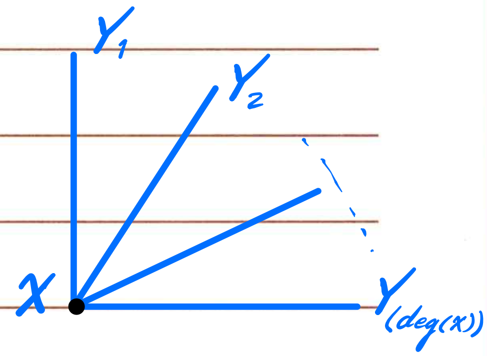
- Selector vertices: There are $\underline{k}$ selector vertices in G', $s_1 \ldots s_k$

Other edges in G'

1 - For each vertex $u \in V$, we add edges to join pairs of gadgets in order to form a path going through all the gadgets corresponding to edges incident on node u in G.

Other edges in G'

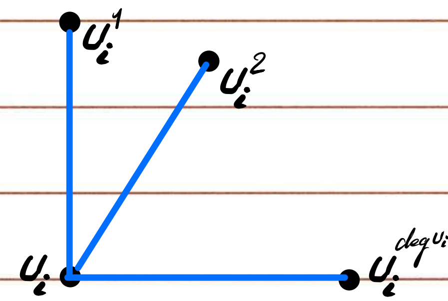
2 – Final set of edges in G' join the first vertex $[X, Y_1, 1]$ and the last vertex $[X, Y_{(deg(X))}, 6]$ of each of these paths to each of the selector vertices.



Proof: A) Suppose that $G=(V,E)$ has a vertex cover of size $\underline{k}$. Let the vertex cover set be

$$S=\{u_1, u_2, \quad u_k\}$$

We will identify the neighbors of u_i as shown here:



We can form a Ham. Cycle in G' by following the nodes in G' in this order:

start at s, and go to

$[u_1, u_1^1, 1] \quad \ldots \quad [u_1, u_1^1, 6]$

$[u_1, u_1^2, 1] \quad \ldots \quad [u_1, u_1^2, 6]$

$[u_1, u_1^{\deg u_1}, 1] \quad \ldots \quad [u_1, u_1^{\deg u_1}, 6]$

Then go to s_2 and follow the nodes

$[u_2, u_2^1, 1] \quad \cdots \quad [u_2, u_2^1, 6]$

$[u_2, u_2^2, 1] \quad \cdots \quad [u_2, u_2^2, 6]$

$[u_2, u_2^{\deg u_1}, 1] \quad \cdots \quad [u_2, u_2^{\deg u_1}, 6]$

Then go to s_3

$\vdots$

$\vdots$

$[u_k, u_k^1, 1] \quad \cdots \quad [u_k, u_k^1, 6]$

$[u_k, u_k^2, 1] \quad \cdots \quad [u_k, u_k^2, 6]$

$[u_k, u_k^{\deg u_1}, 1] \quad \cdots \quad [u_k, u_k^{\deg u_1}, 6]$

Finally return back to s_1 to complete the Ham. Cycle.

B) Suppose G' has a Ham. Cycle, then the set

$$S = \{ v_i \in V : (s_j, [v_i, v_i^1, 1]) \in C$$

$$\text{for some } 1 \leq j \leq k \}$$

will be a vertex cover set in G.
Since segments of the HC between s_j and s_{j+1} go through all gadgets corresponding to edges that are

incident on v_i in G (indicating that node v_i covers all edges incident on it in G)
And because the Ham. Cycle goes through all gadgets in G', then all corresponding edges will be covered by the nodes in the set S.

Prove that TSP is NP-Complete

1- Show that $TSP \in NP$

a. Certificate:

b. Certifier:

2- Choose an NP-Complete problem:
Hamiltonian Cycle

3- Prove that $HC \leq_p TSP$

Plan: Given an instance of the HC problem on graph $G = (V, E)$, we will construct G' such that G has a HC iff G' has a tour of cost $\leq |V|$.

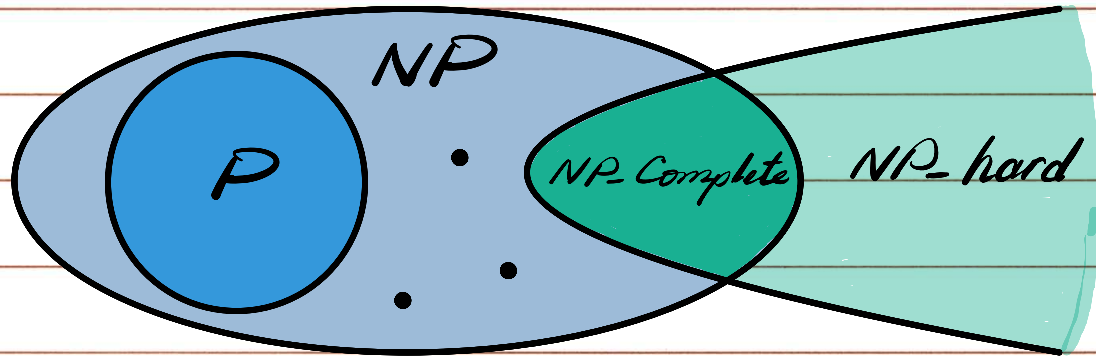
Construction of G':

The set of known NP-Complete problems we can choose from in our proof of NP-Completeness:

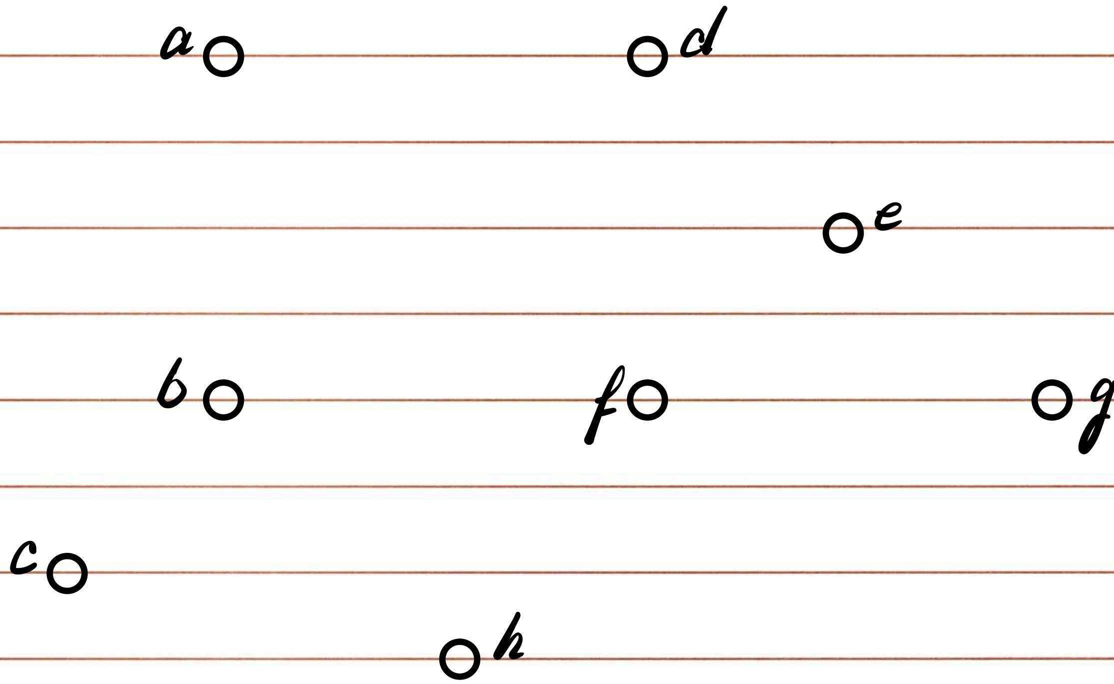
- 3-SAT
- Independent Set
- Vertex cover
- Set cover
- Set packing
- Hamiltonian Cycle
- TSP

- We can also use the decision versions of
  - 0-1 Knapsack
  - Subset sum

since we are already familiar with these problems, although a proof of their NP-Completeness has not been presented in lecture.


NP
P
NP-Complete
NP-hard

TSP with Triangle Inequalities



Claim:

The cost of our approximate slolution to TSP is within a factor of 2 of the cost of the optimal tour.

Proof:

General TSP

Theorem: If $P \neq NP$, then for any constant $\rho \geq 1$, there is no polynomial time approximation algorithm with approximation ratio ρ for the general TSP.

Plan for the proof of the theorem:

2. We saw in lecture that finding a Hamiltonian Cycle in a graph is **NP**-complete. Show that finding a Hamiltonian Path -- a path that visits each vertex exactly once, and isn't required to return to its starting point -- is also **NP**-complete.

1. In the *Min-Cost Fast Path* problem, we are given a directed graph $G=(V,E)$ along with positive integer times t_e and positive costs c_e on each edge. The goal is to determine if there is a path P from s to t such that the total time on the path is at most T and the total cost is at most C (both T and C are parameters to the problem). Prove that this problem is **NP**-complete.

3. Some **NP**-complete problems are polynomial-time solvable on special types of graphs, such as bipartite graphs. Others are still **NP**-complete.

Show that the problem of finding a Hamiltonian Cycle in a bipartite graph is still **NP**-complete.