§15.3(2) 中值定理、Taylor公式

凸区域

定义3.1 设$D\subset R^n$是区域.若连结D中任意两点的直线段都完全属于D，即对于任意两点$\vec{x}_0,\vec{x}_1\in D$和一切$\lambda\in[0,1]$,恒有$\vec{x}_0+\lambda(\vec{x}_1-\vec{x}_0)\in D$,则称$D$为凸区域.

例如 R^2中开圆盘

$$D=\{(x,y)\in R^2\,|\,(x-a)^2+(y-b)^2<r^2\}$$

中值定理

定理3.2 设$D \subset R^n$为凸区域，$f : D \to R$可微，则任给$\vec{a},\vec{b} \in D$，存在$\vec{\xi} \in D$，使得

$$f(\vec{b}) - f(\vec{a}) = gradf(\vec{\xi}) \cdot (\vec{b} - \vec{a}),$$

其中$\vec{\xi} = \vec{a} + \theta(\vec{b} - \vec{a}), \theta \in (0,1)$是连接$\vec{a},\vec{b}$的直线段上的一点.

推论 如果可微函数 f在区域 D上偏导数恒为零，那么f在D上必为常值函数.

当$n=2$时,此定理即为:

设$f(x,y)$在凸区域D上可微,

则对于D内任意两点(x_0,y_0)和$(x_0+\Delta x, y_0+\Delta y)$

至少存在一个$\theta(0<\theta<1)$,使得

$$f(x_0+\Delta x, y_0+\Delta y)-f(x_0,y_0)$$

$$=f_x(x_0+\theta\Delta x, y_0+\theta\Delta y)\Delta x+f_y(x_0+\theta\Delta x, y_0+\theta\Delta y)\Delta y$$

证 令$\varphi(t)=f(x_0+t\Delta x, y_0+t\Delta y)$,

则$\varphi(t)$在[0,1]上连续，在 (0,1)内可导,

由Lagrange中值定理可得

$$\varphi(1)-\varphi(0)=\varphi'(\theta)$$

而$\varphi'(t)=f_x(x_0+t\Delta x, y_0+t\Delta y)\Delta x$

$$+f_y(x_0+t\Delta x, y_0+t\Delta y)\Delta y$$

代入上式即可证明定理 结果.

Taylor公式

定理3.4 设k,n是两个正整数，那么

$$(x_1+\cdots+x_n)^k=\sum_{\alpha_1+\cdots+\alpha_n=k}\frac{k!}{\alpha_1!\cdots\alpha_n!}x_1^{\alpha_1}\cdots x_n^{\alpha_n},$$

其中$\alpha_1,\cdots,\alpha_n$是非负整数．

称$\vec{\alpha}=(\alpha_1,\cdots,\alpha_n)$为一个多重指标，记

$|\vec{\alpha}|=\alpha_1+\cdots+\alpha_n,\vec{\alpha}!=\alpha_1!\cdots\alpha_n!,$

$\vec{x}=(x_1,\cdots,x_n)$，$\vec{x}^{\vec{\alpha}}=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$，则上式记为

$$(x_1+\cdots+x_n)^k=\sum_{|\vec{\alpha}|=k}\frac{k!}{\alpha!}\vec{x}^{\vec{\alpha}}.$$

记$D^{\vec{\alpha}} f(\vec{x}) = \dfrac{\partial^{|\vec{\alpha}|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} f(\vec{x})$. 高阶偏导数

定理3.5 设$D \subset R^n$为凸区域, $f: D \to R$具有$m+1$阶连续偏导数, $\vec{a} = (a_1, \cdots, a_n) \in D$, 则任给$\vec{x} \in D$, 存在$\theta \in (0,1)$, 使得

$$f(\vec{x}) = \sum_{k=0}^{m} \sum_{|\vec{\alpha}|=k} \frac{D^{\vec{\alpha}} f(\vec{a})}{\vec{\alpha}!} (\vec{x} - \vec{a})^{\vec{\alpha}} + R_m,$$

带*Lagrange*余项的*Taylor*公式

其中$R_m = \sum_{|\vec{\alpha}|=m+1} \dfrac{D^{\vec{\alpha}} f(\vec{a} + \theta(\vec{x} - \vec{a}))}{\vec{\alpha}!} (\vec{x} - \vec{a})^{\vec{\alpha}}$为*Lagrange*余项.

余项 $R_m = o(\|\vec{x} - \vec{a}\|^m)$的 *Taylor* 公式称为带 *Peano* 余项的 *Taylor* 公式.

二元函数$f(x,y)$在点(x_0,y_0)处的*Taylor*公式可表示为

$$f(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0)+$$

$$\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)f(x_0,y_0)+\frac{1}{2!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)$$

$$+\cdots+\frac{1}{k!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^k f(x_0,y_0)+R_k$$

$$R_k=\frac{1}{(k+1)!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^{k+1} f(x_0+\theta\Delta x, y_0+\theta\Delta y)$$

$$(0<\theta<1)$$

$$\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^{p} f(x_0, y_0)$$

$$=\sum_{i=0}^{p} C_p^i \frac{\partial^p f(x_0, y_0)}{\partial x^{p-i}\partial y^i}(\Delta x)^{p-i}(\Delta y)^i .$$

也可记为

$$f(x,y)=f(x_0,y_0)+\left((x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right)f(x_0,y_0)$$

$$+\cdots+\frac{1}{k!}\left((x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right)^{k} f(x_0,y_0)+R_k$$

$$R_k=\frac{1}{(k+1)!}\left((x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right)^{k+1} f(x_0+\theta(x-x_0),y_0+\theta(y-y_0))$$

证 令 $\varphi(t) = f(x_0 + t\Delta x, y_0 + t\Delta y)$,

则 $\varphi(t)$在$t = 0$处Taylor 公式展开

$$\varphi(t) = \varphi(0) + \varphi'(0)t + \cdots + \frac{1}{m!}\varphi^{(m)}(0)t^m$$

$$+ \frac{1}{(m+1)!}\varphi^{(m+1)}(\theta t)t^{m+1}, 0 < \theta < 1.$$

$$\varphi(1) = \varphi(0) + \varphi'(0) + \cdots + \frac{1}{m!}\varphi^{(m)}(0) + \frac{1}{(m+1)!}\varphi^{(k+1)}(\theta)$$

应用链式求导法则有

$$\varphi'(t)=\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)f(x_0+t\Delta x,y_0+t\Delta y)$$

$$\varphi''(t)=\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^2 f(x_0+t\Delta x,y_0+t\Delta y)$$

……

$$\varphi^{(k)}(t)=\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^k f(x_0+t\Delta x,y_0+t\Delta y)$$

带入$\varphi(1)$表达式即可.

二元函数$f(x,y)$在点$(0,0)$处的*Taylor*公式为

$$f(x,y)=f(0,0)+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0,0)$$

$$+\frac{1}{2!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f(0,0)+\cdots+\frac{1}{k!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^k f(0,0)+R_k$$

$$R_k=\frac{1}{(k+1)!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{k+1} f(\theta x,\theta y) \qquad (0<\theta<1)$$

如果是k阶Taylor公式的Peano余项,为$o((\sqrt{x^2+y^2})^k)$

例1 求$\sqrt{1+x^2+y^2}$在$(0,0)$处的2阶带*Lagrange*余项的*Taylor*公式.

解 $f_x=\dfrac{x}{\sqrt{1+x^2+y^2}},\ f_y=\dfrac{y}{\sqrt{1+x^2+y^2}},$

$$f_{xy}=\frac{xy}{(\sqrt{1+x^2+y^2})^3},\ f_{xx}=\frac{1+y^2}{(\sqrt{1+x^2+y^2})^3},\ f_{yy}=\frac{1+x^2}{(\sqrt{1+x^2+y^2})^3},$$

$$f_{xxx}=-\frac{3x(1+y^2)}{(\sqrt{1+x^2+y^2})^{\frac{5}{2}}},\ f_{xxy}=-\frac{y^3+y-2x^2y}{(1+x^2+y^2)^{\frac{5}{2}}},$$

$$f_{xyy}=-\frac{x^3+x-2y^2x}{(1+x^2+y^2)^{\frac{5}{2}}},\ \ f_{yyy}=-\frac{3y(1+x^2)}{(1+x^2+y^2)^{\frac{5}{2}}}$$

$$f(0,0)=f_{xx}(0,0)=f_{yy}(0,0)=1,\ f_x(0,0)=f_y(0,0)=f_{xy}(0,0)=0,$$

$$\sqrt{1+x^2+y^2}=f(0,0)+f_x(0,0)x+f_y(0,0)y+$$

$$+\frac{1}{2}(f_{xx}(0,0)x^2+2f_{xy}(0,0)xy+f_{yy}(0,0)y^2)+R_2$$

$$=1+\frac{1}{2}(x^2+y^2)+R_2$$

$$R_2=-\frac{1}{3!}\frac{1}{(1+\theta^2x^2+\theta^2y^2)^{\frac{5}{2}}}[3\theta(1+\theta^2y^2)x^3+3(\theta^3y^3+\theta y-2\theta^3yx^2)yx^2$$

$$+3(\theta^3x^3+\theta x-2\theta^3xy^2)xy^2+3\theta y(1+\theta^2x^2)y^3]$$

$$=-\frac{1}{2}\frac{\theta(x^2+y^2)^2}{(1+\theta^2x^2+\theta^2y^2)^{\frac{5}{2}}}\left(0<\theta<1\right)$$

例2 求函数 $f(x,y)=\ln(1+x+y)$ 在点 $(0,0)$ 带 *Lagrange* 余项的三阶 *Taylor* 公式.

解法1

$$f_x = f_y = \frac{1}{1+x+y},$$

$$f_{xx} = f_{xy} = f_{yy} = -\frac{1}{(1+x+y)^2},$$

$$\frac{\partial^3 f}{\partial x^p \partial y^{3-p}} = \frac{2!}{(1+x+y)^3},\ p=0,1,2,3,$$

$$\frac{\partial^4 f}{\partial x^p \partial y^{4-p}} = \frac{3!}{(1+x+y)^4},\ p=0,1,2,3,4.$$

$$\ln(1+x+y)=x+y-\frac{1}{2}(x+y)^2+\frac{1}{3}(x+y)^3+R_3,$$

其中 $R_3=\dfrac{1}{4!}\left(x\dfrac{\partial}{\partial x}+y\dfrac{\partial}{\partial y}\right)^4 f(\theta x,\theta y)$

$$=-\frac{(x+y)^4}{4(1+\theta x+\theta y)^4},\ 0<\theta<1.$$

解法2 令$t=x+y$,则$f(1+x+y)=\ln(1+t)$.

一元函数 $\ln(1+t)$在$t=0$的三阶 *Taylor* 公式为

$$\ln(1+t)=t-\frac{1}{2}t^2+\frac{1}{3}t^3-\frac{1}{4}\frac{t^4}{(1+\theta t)^4},\ 0<\theta<1.$$

将$t=x+y$代入，即得$f(x,y)$在点(0,0)的三阶 *Taylor* 公式

$$\ln(1+x+y)=x+y-\frac{1}{2}(x+y)^2+\frac{1}{3}(x+y)^3-\frac{(x+y)^4}{4(1+\theta x+\theta y)^4},\ 0<\theta<1.$$

例3 求函数 $f(x,y)=\sin x\sin y$ 在点 $\left(\frac{\pi}{4},\frac{\pi}{4}\right)$ 的带 *Lagrange* 余项的二阶 *Taylor* 公式.

解法1 $f_x=\cos x\sin y,\ f_y=\sin x\cos y,$

$$f_{xx}=f_{yy}=-\sin x\sin y,\ \ f_{xy}=f_{yx}=\cos x\cos y,$$

$$\frac{\partial^3 f}{\partial x^3}=\frac{\partial^3 f}{\partial x\partial y^2}=-\cos x\sin y,$$

$$\frac{\partial^3 f}{\partial y^3}=\frac{\partial^3 f}{\partial x^2\partial y}=-\sin x\cos y,$$

所以,

$$\left[(x-\frac{\pi}{4})\frac{\partial}{\partial x}+(y-\frac{\pi}{4})\frac{\partial}{\partial y}\right]f\left(\frac{\pi}{4},\frac{\pi}{4}\right)$$

$$=(x-\frac{\pi}{4})f_x\left(\frac{\pi}{4},\frac{\pi}{4}\right)+(y-\frac{\pi}{4})f_y\left(\frac{\pi}{4},\frac{\pi}{4}\right)=\frac{1}{2}(x-\frac{\pi}{4})+\frac{1}{2}(y-\frac{\pi}{4}),$$

$$\left[(x-\frac{\pi}{4})\frac{\partial}{\partial x}+(y-\frac{\pi}{4})\frac{\partial}{\partial y}\right]^2 f\left(\frac{\pi}{4},\frac{\pi}{4}\right)$$

$$=(x-\frac{\pi}{4})^2 f_{xx}\left(\frac{\pi}{4},\frac{\pi}{4}\right)+2(x-\frac{\pi}{4})(y-\frac{\pi}{4})f_{xy}\left(\frac{\pi}{4},\frac{\pi}{4}\right)$$

$$+(y-\frac{\pi}{4})^2 f_{yy}\left(\frac{\pi}{4},\frac{\pi}{4}\right)$$

$$=-\frac{1}{2}(x-\frac{\pi}{4})^2+(x-\frac{\pi}{4})(y-\frac{\pi}{4})-\frac{1}{2}(x-\frac{\pi}{4})^2,$$

又$f\left(\frac{\pi}{4},\frac{\pi}{4}\right)=\frac{1}{2}$,

$$\sin x\sin y=\frac{1}{2}+\frac{1}{2}(x-\frac{\pi}{4})+\frac{1}{2}(y-\frac{\pi}{4})$$

$$-\frac{1}{4}(x-\frac{\pi}{4})^2+\frac{1}{2}(x-\frac{\pi}{4})(y-\frac{\pi}{4})-\frac{1}{4}(y-\frac{\pi}{4})^2+R_2,$$

其中

$$R_2=\frac{1}{3!}\left[\left((x-\frac{\pi}{4})\frac{\partial}{\partial x}+(y-\frac{\pi}{4})\frac{\partial}{\partial y}\right)^3 f(\frac{\pi}{4}+\theta(x-\frac{\pi}{4}),\frac{\pi}{4}+\theta(y-\frac{\pi}{4}))\right]$$

$$=-\frac{1}{6}\{\left[(x-\frac{\pi}{4})^3+(y-\frac{\pi}{4})^3\right]\cos\left(\frac{\pi}{4}+\theta x\right)\sin(\frac{\pi}{4}+\theta y)$$

$$+\left(3(x-\frac{\pi}{4})^2(y-\frac{\pi}{4})+3(x-\frac{\pi}{4})(y-\frac{\pi}{4})^2\right)$$

$$\sin\left(\frac{\pi}{4}+\theta(x-\frac{\pi}{4})\right)\cos\left(\frac{\pi}{4}+\theta(y-\frac{\pi}{4})\right)\},$$

$0<\theta<1.$

解法2 利用一元函数的Taylor公式，有

$$\sin x = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}\cdot(x-\frac{\pi}{4}) - \frac{1}{2!}\sin\frac{\pi}{4}\cdot(x-\frac{\pi}{4})^2 + R_2^x,$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x-\frac{\pi}{4}) - \frac{\sqrt{2}}{4}(x-\frac{\pi}{4})^2 + R_2^x,$$

$$\sin y = \sin\frac{\pi}{4} + \cos\frac{\pi}{4}\cdot(y-\frac{\pi}{4}) - \frac{1}{2!}\sin\frac{\pi}{4}\cdot(y-\frac{\pi}{4})^2 + R_2^y,$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(y-\frac{\pi}{4}) - \frac{\sqrt{2}}{4}(y-\frac{\pi}{4})^2 + R_2^y,$$

其中R_2^x, R_2^y分别是两个*Taylor*公式展开的*Lagrange*余项.

则

$$\sin x\sin y=\left[\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}(x-\frac{\pi}{4})-\frac{\sqrt{2}}{4}(x-\frac{\pi}{4})^2+R_2^x\right]$$

$$\times\left[\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}(y-\frac{\pi}{4})-\frac{\sqrt{2}}{4}(y-\frac{\pi}{4})^2+R_2^y\right]$$

$$=\frac{1}{2}+\frac{1}{2}(x-\frac{\pi}{4})+\frac{1}{2}(y-\frac{\pi}{4})-\frac{1}{4}(x-\frac{\pi}{4})^2-\frac{1}{4}(y-\frac{\pi}{4})^2$$

$$+\frac{1}{2}(x-\frac{\pi}{4})(y-\frac{\pi}{4})+R_2$$

R_2的表达式与解法一相同.

例 4 求$1.04^{2.02}$ 的近似值，要求误差不超过 0.0001.

解 设函数$f(x,y)=x^y,(x_0,y_0)=(1,2)$,

$h=0.04, k=0.02.$ $f(1,2)=1,$

$$f_x(1,2)=yx^{y-1}\Big|_{(1,2)}=2,\quad f_y(1,2)=x^y\ln x\Big|_{(1,2)}=0,$$

$$f_{xx}(1,2)=y(y-1)x^{y-2}\Big|_{(1,2)}=2,$$

$$f_{xy}(1,2)=[x^{y-1}+yx^{y-1}\ln x]\Big|_{(1,2)}=1,$$

$$f_{yy}(1,2)=[x^y\ln^2 x]\Big|_{(1,2)}=0,$$

$$1.04^{2.02}\approx 1+2\times 0.04+0\times 0.02$$

$$+\frac{1}{2}[2\times 0.04^2+2\times 0.04\times 0.02+0\times 0.02^2]=1.0824$$

$$R_2\le\frac{2^3}{3!}\sqrt{0.02^2+0.04^2}^{\,3}\sqrt{2}^{\,3}<0.0001.$$