§15.3(1) 高阶偏导数

高阶偏导数

函数$z = f(x, y)$的二阶偏导数为

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y), \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y)$$

纯偏导数

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{yx}(x, y), \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{xy}(x, y)$$

混合偏导数

定义 二阶及二阶以上的偏导数统称为高阶偏导数.

例 1 设 $z = x^3y^2 - 3xy^3 - xy + 1$，

求 $\dfrac{\partial^2 z}{\partial x^2}$、$\dfrac{\partial^2 z}{\partial y \partial x}$、$\dfrac{\partial^2 z}{\partial x \partial y}$、$\dfrac{\partial^2 z}{\partial y^2}$ 及 $\dfrac{\partial^3 z}{\partial x^3}$.

解 $\dfrac{\partial z}{\partial x} = 3x^2y^2 - 3y^3 - y$, $\dfrac{\partial z}{\partial y} = 2x^3y - 9xy^2 - x$;

$$\frac{\partial^2 z}{\partial x^2} = 6xy^2, \quad \frac{\partial^3 z}{\partial x^3} = 6y^2, \quad \frac{\partial^2 z}{\partial y^2} = 2x^3 - 18xy;$$

$$\frac{\partial^2 z}{\partial x \partial y} = 6x^2y - 9y^2 - 1, \quad \frac{\partial^2 z}{\partial y \partial x} = 6x^2y - 9y^2 - 1.$$

例 2　设 $u = e^{ax}\cos by$，求二阶偏导数.

解 $\dfrac{\partial u}{\partial x} = ae^{ax}\cos by$,　　$\dfrac{\partial u}{\partial y} = -be^{ax}\sin by$;

$$\frac{\partial^2 u}{\partial x^2} = a^2 e^{ax}\cos by, \qquad \frac{\partial^2 u}{\partial y^2} = -b^2 e^{ax}\cos by,$$

$$\frac{\partial^2 u}{\partial x \partial y} = -abe^{ax}\sin by, \qquad \frac{\partial^2 u}{\partial y \partial x} = -abe^{ax}\sin by.$$

例 3 验证函数 $u(x,y)=\ln\sqrt{x^2+y^2}$ 满足 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$

解 $\because \ln\sqrt{x^2+y^2}=\frac{1}{2}\ln(x^2+y^2),$

$$\therefore \frac{\partial u}{\partial x}=\frac{x}{x^2+y^2}, \quad \frac{\partial u}{\partial y}=\frac{y}{x^2+y^2},$$

$$\therefore \frac{\partial^2 u}{\partial x^2}=\frac{(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2},$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{(x^2+y^2)-y\cdot 2y}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}.$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{y^2-x^2}{(x^2+y^2)^2}+\frac{x^2-y^2}{(x^2+y^2)^2}=0.$$

问题 混合偏导数都相等吗？具备怎样的条件才相等？

定理 3. 1 如果函数$z=f(x,y)$的两个二阶混合偏导数$\dfrac{\partial^2 z}{\partial y\partial x}$及$\dfrac{\partial^2 z}{\partial x\partial y}$在点$(x_0,y_0)$连续，那么在该点这两个二阶混合偏导数必相等.

证明 设 $\varphi(x)=f(x,y_0+\Delta y)-f(x,y_0)$, $\psi(y)=f(x_0+\Delta x,y)-f(x_0,y)$

则$\varphi'(x)=f_x(x,y_0+\Delta y)-f_x(x,y_0)$, $\psi'(y)=f_y(x_0+\Delta x,y)-f_y(x_0,y)$

根据微分中值定理

$$\mathrm{I}=\varphi(x_0+\Delta x)-\varphi(x_0)=\varphi'(x_0+\alpha_1\Delta x)\Delta x$$

$$=[f_x(x_0+\alpha_1\Delta x,y_0+\Delta y)-f_x(x_0+\alpha_1\Delta x,y_0)]\Delta x$$

$$=[f_{xy}(x_0+\alpha_1\Delta x,y_0+\alpha_2\Delta y)\Delta x\Delta y,$$

同理可证

$$\mathrm{I}=[f(x_0+\Delta x,y_0+\Delta y)-f(x_0+\Delta x,y_0)]-[f(x_0,y_0+\Delta y)-f(x_0,y_0)]$$

$$=\psi(y_0+\Delta y)-\psi(y_0)$$

$$=[f_{yx}(x_0+\beta_1\Delta x,y_0+\beta_2\Delta y)\Delta x\Delta y,$$

因此可得

$$[f_{xy}(x_0+\alpha_1\Delta x,y_0+\alpha_2\Delta y)\Delta x\Delta y=[f_{yx}(x_0+\beta_1\Delta x,y_0+\beta_2\Delta y)\Delta x\Delta y,$$

令$\Delta x,\Delta y\to 0$,

由于两个混合偏导数f_{xy},f_{yx}在(x_0,y_0)连续，即得

$$f_{xy}(x_0,y_0)=f_{yx}(x_0,y_0)$$

例 4 求函数 $f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0, \\ 0, & x^2+y^2=0, \end{cases}$

在(0,0)点的所有二阶偏导数.

解 $$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=0,$$

$$f_x(x,0)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,0)-f(x,0)}{\Delta x}=0,$$

$$f_{xx}(0,0)=\lim_{\Delta x\to 0}\frac{f_x(0+\Delta x,0)-f_x(0,0)}{\Delta x}=0,$$

$$f_y(0,0) = \lim_{\Delta y \to 0} \frac{f(0,0+\Delta y) - f(0,0)}{\Delta y} = 0,$$

$$f_y(0,y) = \lim_{\Delta y \to 0} \frac{f(0,y+\Delta y) - f(0,y)}{\Delta y} = 0,$$

$$f_{yy}(0,0) = \lim_{\Delta y \to 0} \frac{f_y(0,0+\Delta y) - f_y(0,0)}{\Delta y} = 0.$$

$$f_x(0,y) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x, y) - f(0,y)}{\Delta x} = -y$$

$$f_{xy}(0,0) = \lim_{\Delta y \to 0} \frac{f_x(0,0+\Delta y) - f_x(0,0)}{\Delta y} = -1,$$

$$f_y(x,0)=\lim_{\Delta x\to 0}\frac{f(x,0+\Delta y)-f(x,0)}{\Delta y}=x,$$

$$f_{yx}(0,0)=\lim_{\Delta x\to 0}\frac{f_y(0+\Delta x,0)-f_y(0,0)}{\Delta x}=1.$$

说明

$f_{xy}(0,0)\neq f_{yx}(0,0)\Rightarrow f_{xy}(x,y), f_{yx}(x,y)$至少有一个在$(0,0)$点不连续；

复合函数的高阶偏导数

标准求法 — 链式法则

例 5 设 $w = f(x+y+z, xyz)$， f 具有二阶连续偏导数，求 $\dfrac{\partial^2 w}{\partial z \partial x}$.

解 令 $u = x+y+z,\ v = xyz;$

记 $f_{12} = \dfrac{\partial^2 f(u,v)}{\partial v \partial u}$， 同理有 f_{11}, f_{22}

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = f_1 + yzf_2$$

$$\frac{\partial^2 w}{\partial z \partial x} = \frac{\partial}{\partial z}(f_1 + yzf_2) = \frac{\partial f_1}{\partial z} + yf_2 + yz\frac{\partial f_2}{\partial z};$$

$$\frac{\partial f_1}{\partial z} = \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f_1}{\partial v}\cdot\frac{\partial v}{\partial z} = f_{11} + xyf_{12};$$

$$\frac{\partial f_2}{\partial z} = \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial f_2}{\partial v}\cdot\frac{\partial v}{\partial z} = f_{21} + xyf_{22};$$

于是 $$\frac{\partial^2 w}{\partial z \partial x} = f_{11} + xyf_{12} + yf_2 + yz(f_{21} + xyf_{22})$$

$$= f_{11} + (xy + yz)f_{12} + yf_2 + xy^2 zf_{22}$$

例6 $z=f\left(xy,\dfrac{x}{y},x\right)$，求$z_{xx},z_{xy},z_{yy}$.

解 $z_x=yf_1+\dfrac{1}{y}f_2+f_3$，$z_y=xf_1-\dfrac{x}{y^2}f_2$

$$z_{xx}=y\left[yf_{11}+\frac{1}{y}f_{12}+f_{13}\right]+\frac{1}{y}\left[yf_{21}+\frac{1}{y}f_{22}+f_{23}\right]+yf_{31}+\frac{1}{y}f_{32}+f_{33}$$

$$z_{yy}=x\left[xf_{11}+\left(\frac{-x}{y^2}\right)f_{12}\right]+\frac{2x}{y^3}f_2-\frac{x}{y^2}\left[xf_{21}+\left(\frac{-x}{y^2}\right)f_{22}\right]$$

$$z_{xy}=f_1+y\left[xf_{11}+\left(\frac{-x}{y^2}\right)f_{12}\right]-\frac{1}{y^2}f_2+\frac{1}{y}\left[xf_{21}+\left(\frac{-x}{y^2}\right)f_{22}\right]+xf_{31}+\left(\frac{-x}{y^2}\right)f_{32}$$

例 7 设 $u = f(x, y)$, f 具有二阶连续偏导数，将下列表达式转换成极坐标 $x = r\cos\theta, y = r\sin\theta$ 下的形式：(1) $\left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2$； (2) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$.

解 (1) $x = r\cos\theta, y = r\sin\theta \Rightarrow r = \sqrt{x^2 + y^2}, \theta = \arctan\dfrac{y}{x}$.

设 $u = f(x, y) = F(r, \theta)$, 则

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot\frac{x}{r} - \frac{\partial u}{\partial \theta}\cdot\frac{y}{r^2} = \frac{\partial u}{\partial r}\cos\theta - \frac{\partial u}{\partial \theta}\frac{\sin\theta}{r}$$

同理可得 $\dfrac{\partial u}{\partial y}=\dfrac{\partial u}{\partial r}\sin\theta+\dfrac{\partial u}{\partial\theta}\dfrac{\cos\theta}{r}.$

于是，$\left(\dfrac{\partial u}{\partial x}\right)^2+\left(\dfrac{\partial u}{\partial y}\right)^2=\left(\dfrac{\partial u}{\partial r}\right)^2+\dfrac{1}{r^2}\left(\dfrac{\partial u}{\partial\theta}\right)^2.$

(2) $$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial r}{\partial x}+\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial x}\right)\cdot\frac{\partial\theta}{\partial x}$$

$$=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right)\cos\theta$$

$$-\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right)\cdot\frac{\sin\theta}{r}$$

$$= \frac{\partial^2 u}{\partial r^2}\cos^2\theta - 2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial^2 u}{\partial\theta^2}\frac{\sin^2\theta}{r^2}$$

$$+ 2\frac{\partial u}{\partial r}\frac{\sin\theta\cos\theta}{r^2} + \frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}.$$

同理可得

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta\cos\theta}{r} + \frac{\partial^2 u}{\partial\theta^2}\frac{\cos^2\theta}{r^2}$$

$$- 2\frac{\partial u}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2} + \frac{\partial u}{\partial r}\frac{\cos^2\theta}{r}.$$

两式相加，得

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = \frac{1}{r^2}\left[r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial\theta^2}\right].$$

例8 设$u=f(x,y)$满足$u_{xx}+u_{yy}=0$,证明:$v=f(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2})$

也满足此方程.

证 设$s=\frac{x}{x^2+y^2}, t=\frac{y}{x^2+y^2}$,则$v=f(s,t)$

$$v_x=\frac{\partial f}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial f}{\partial t}\frac{\partial t}{\partial x}=\frac{\partial f}{\partial s}s_x+\frac{\partial f}{\partial t}t_x, v_y=\frac{\partial f}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial f}{\partial t}\frac{\partial t}{\partial y}=\frac{\partial f}{\partial s}s_y+\frac{\partial f}{\partial t}t_y,$$

$$v_{xx}=\left(\frac{\partial^2 f}{\partial s^2}s_x+\frac{\partial^2 f}{\partial s\partial t}t_x\right)s_x+\frac{\partial f}{\partial s}s_{xx}+\frac{\partial f}{\partial t}t_{xx}+\left(\frac{\partial^2 f}{\partial t\partial s}s_x+\frac{\partial^2 f}{\partial t^2}t_x\right)t_x$$

$$v_{yy}=\left(\frac{\partial^2 f}{\partial s^2}s_y+\frac{\partial^2 f}{\partial s\partial t}t_y\right)s_y+\frac{\partial f}{\partial s}s_{yy}+\frac{\partial f}{\partial t}t_{yy}+\left(\frac{\partial^2 f}{\partial t\partial s}s_y+\frac{\partial^2 f}{\partial t^2}t_y\right)t_y$$

$$v_{xx}+v_{yy}=\frac{\partial^2 f}{\partial s^2}(s_x^2+s_y^2)+2\frac{\partial^2 f}{\partial s\partial t}(t_x s_x+t_y s_y)+\frac{\partial f}{\partial s}(s_{xx}+s_{yy})+\frac{\partial^2 f}{\partial t^2}(t_x^2+t_y^2)$$

$$s_x=\frac{y^2-x^2}{(x^2+y^2)^2},t_x=\frac{-2xy}{(x^2+y^2)^2},s_y=\frac{-2xy}{(x^2+y^2)^2},t_y=\frac{x^2-y^2}{(x^2+y^2)^2}.$$

$$s_{xx}=-s_{yy}=\frac{2x(x^2-3y^2)}{(x^2+y^2)^3},\quad s_x{}^2+s_y{}^2=t_x{}^2+t_y{}^2$$

$$\Rightarrow v_{xx}+v_{yy}=\left(\frac{\partial^2 f}{\partial s^2}+\frac{\partial^2 f}{\partial t^2}\right)(t_x{}^2+t_y{}^2)=0$$

结论得证

例9 设$z=(x^2+y^2)e^{x+y}$，求$\dfrac{\partial^{p+q}z}{\partial x^p\partial y^q}$，$p,q$为正整数.

证 由$\dfrac{\partial}{\partial x^k}(e^{x+y})=\dfrac{\partial}{\partial y^k}(e^{x+y})=e^{x+y},k=1,2\cdots$

关于y用$Leibniz$ 公式得

$$\frac{\partial^q z}{\partial y^q}=(x^2+y^2)e^{x+y}+C_q^1(2y)e^{x+y}+C_q^2 2e^{x+y}$$

$$=[x^2+y^2+2qy+q(q-1)]e^{x+y}$$

再对x使用*Leibniz*公式

$$\frac{\partial^{p+q} z}{\partial x^p \partial y^q} = \frac{\partial^p}{\partial x^p}\left(\frac{\partial^q z}{\partial y^q}\right) = [x^2 + y^2 + 2qy + q(q-1)]e^{x+y}$$

$$+ C_p^1 (2x) e^{x+y} + C_p^2 2 e^{x+y}$$

$$= [x^2 + y^2 + 2(px + qy) + p(p-1) + q(q-1)]e^{x+y}$$