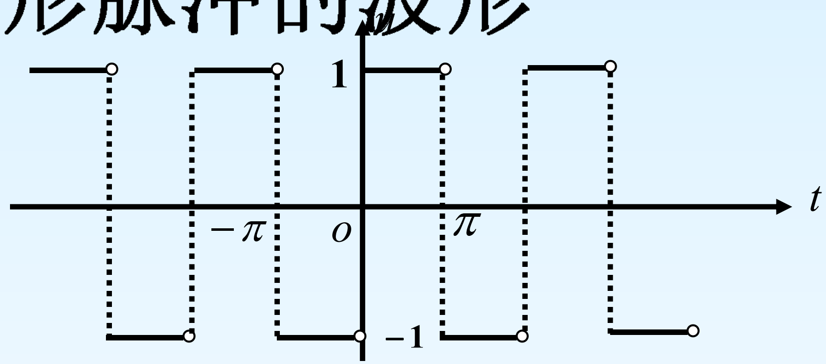
§13.2 Fourier级数的计算

一、以2π为周期的函数的*Fourier*级数展开

例 1 以2π为周期的矩形脉冲的波形

$$u(t)=\begin{cases}-1, & -\pi \le t < 0 \\ 1, & 0 \le t < \pi\end{cases}$$



将它展成Fourier级数.

解 $u(t)$相应的Fourier级数为:

$$u(t) \sim \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \sin(2n-1)t$$

在$u(t)$的不连续点$t=k\pi(k=0,\pm1,\pm2,\cdots)$处，

级数收敛于$\dfrac{-1+1}{2}=0,$

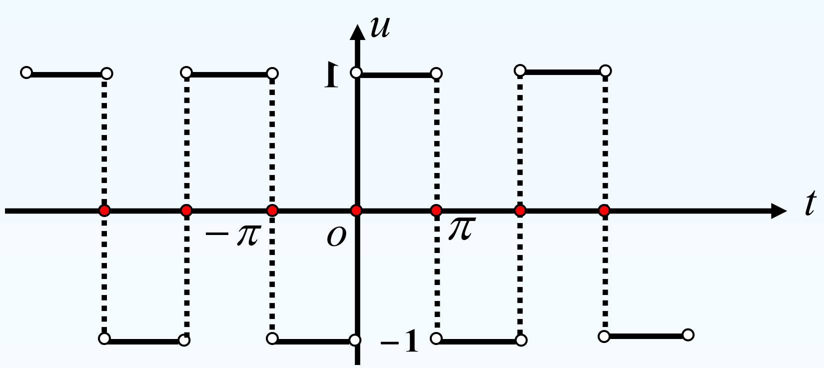
在连续点处,收敛到$u(t),$

所以函数的Fourier展开式为:

$$u(t)=\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\sin(2n-1)t$$

$$(-\infty<t<+\infty;t\neq 0,\pm\pi,\pm2\pi,\cdots)$$

和函数图象为



注 以2π为周期的函数$f(x)$，Fourier系数中的积分区间可以改成长度为2π的任意区间，不影响a_n, b_n的值，即有$\forall c$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx dx, n = 0,1,2,\cdots,$$
$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx dx, n = 1,2,\cdots$$

注意:

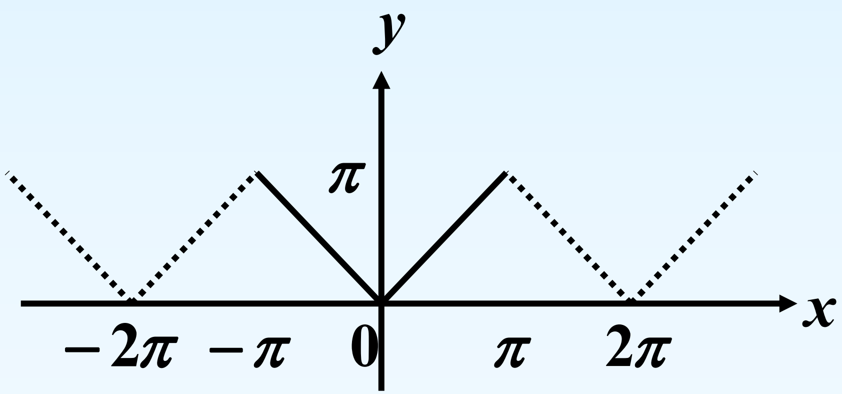
若$f(x)$为非周期函数,在$[-\pi,\pi)$上有定义,且满足收敛条件,则也可以展开成*Fourier*级数.

作法:

周期延拓$(T=2\pi)$　$F(x)=f(x), x\in[-\pi,\pi)$

例2 将$f(x)=\begin{cases}-x, & -\pi\le x<0\\ x, & 0\le x\le\pi\end{cases}$展开成$Fourier$级数.

解 将$f(x)$延拓为以2π为周期的周期函数$F(x)$



$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}F(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-x)dx+\frac{1}{\pi}\int_{0}^{\pi}xdx=\pi,$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$$= \begin{cases} -\dfrac{4}{(2k-1)^2\pi}, & n = 2k-1, k = 1,2,\cdots \\ 0, & n = 2k, k = 1,2,\cdots \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = 0$$

$$\therefore F(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x, \in (-\infty,\infty)$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x, x \in [-\pi, \pi]$$

注 可利用Fourier展开式求级数的和

$$\because \quad f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)x,$$

$$\therefore f(0)=0 \Rightarrow \frac{\pi^2}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\sigma_1$$

记$\sigma_2=\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\cdots,$

$$\sigma=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\sigma_1+\sigma_2,$$

则$\sigma_2=\frac{\sigma}{4}=\frac{\sigma_1+\sigma_2}{4} \Rightarrow \sigma_2=\frac{\pi^2}{24}, \quad \sigma=\frac{\pi^2}{6}$

$$\sigma_3=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=2\sigma_1-\sigma=\frac{\pi^2}{12}.$$

例3 设周期为 2π 的函数 $f(x)$ 在 $[-\pi,\pi)$ 内的表达式为

$$f(x)=\begin{cases} bx\,, -\pi\le x<0 \\ ax\,, 0\le x<\pi \end{cases} \quad (\text{常数}\, a>b>0)$$

求其 *Fourier* 级数的和函数 $S(x), S(6)S(5\pi)$.

解

$$S(x)=\begin{cases} f(x), & x\ne(2k+1)\pi \\ \dfrac{(a-b)\pi}{2}, & x=(2k+1)\pi \end{cases}$$

$$S(6)S(5\pi)=b(6-2\pi)\frac{(a-b)\pi}{2}$$

二、正弦级数与余弦级数

1. 奇函数和偶函数的*Fourier*级数

定理 (1)周期为2π的奇函数$f(x)$的*Fourier*系数为

$$a_n = 0 \qquad (n = 0,1,2,\cdots)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \qquad (n = 1,2,\cdots)$$

(2)周期为2π的偶函数$f(x)$的*Fourier*系数为

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \qquad (n = 0,1,2,\cdots)$$

$$b_n = 0 \qquad (n = 1,2,\cdots)$$

定义

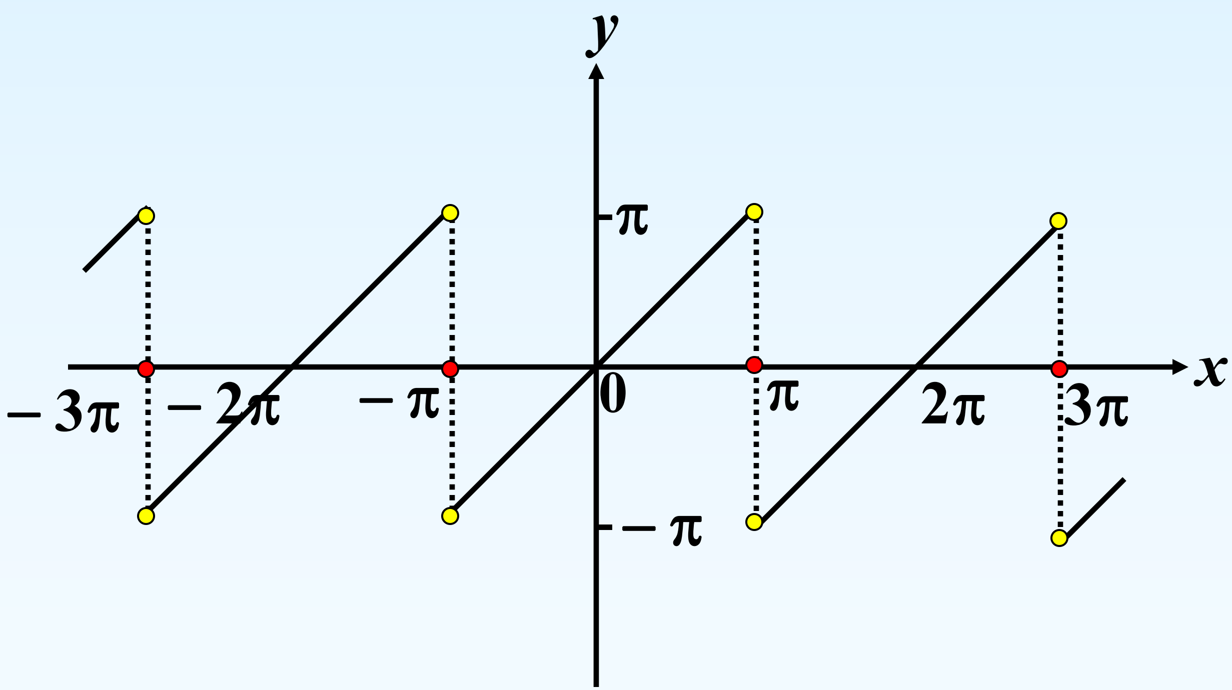
若$f(x)$为奇函数，则其$Fourier$级数$\sum_{n=1}^{\infty} b_n \sin nx$称为正弦级数;

若$f(x)$为偶函数，则其$Fourier$级数$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx$称为余弦级数.

例4 设$f(x)$是周期为2π的周期函数，它在$[-\pi,\pi)$上的表达式为$f(x)=x$，将$f(x)$展开成Fourier级数.

解

和函数图象



$$\therefore a_n = 0, \qquad (n = 0,1,2,\cdots)$$

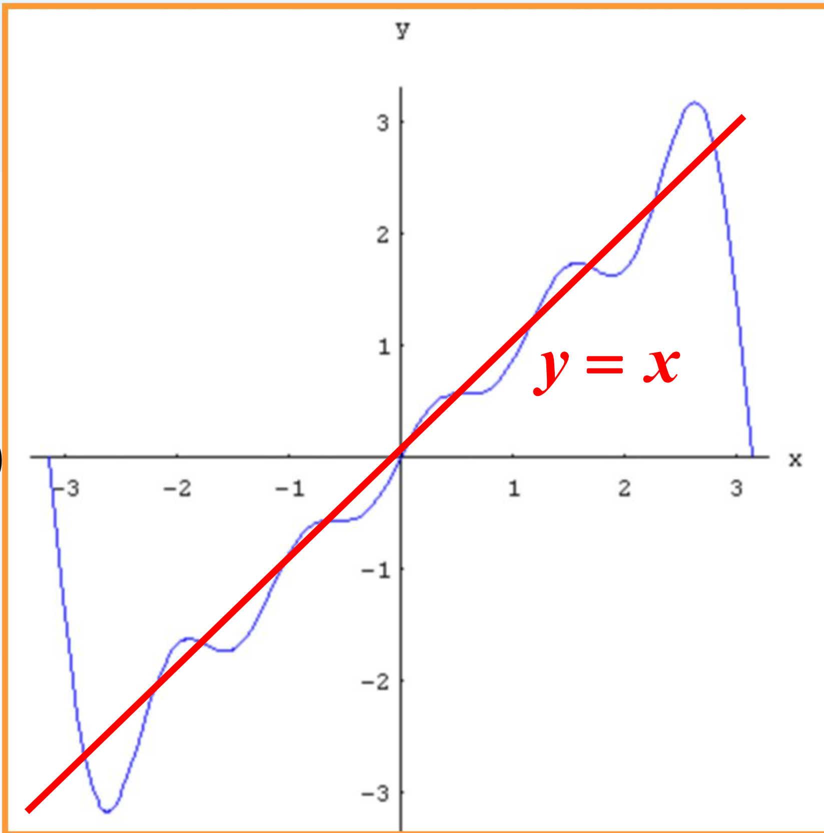
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx dx$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= -\frac{2}{n}\cos n\pi = \frac{2}{n}(-1)^{n+1},$$

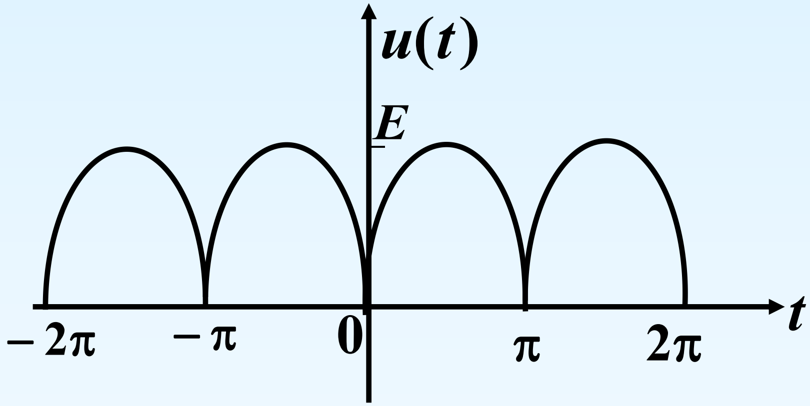
$$f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

$$(-\infty < x < +\infty; x \neq \pm\pi, \pm 3\pi, \cdots)$$



例5 将周期函数$u(t)=E\left|\sin t\right|$展开成Fourier级数,其中$E>0$为常数.

解



$$\therefore b_n=0,\ \ (n=1,2,\cdots)$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}u(t)dt=\frac{2}{\pi}\int_0^{\pi}E\sin t\,dt=\frac{4E}{\pi},$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} u(t)\cos nt dt = \frac{2}{\pi}\int_0^{\pi} E\sin t\cos nt dt$$

$$= \frac{E}{\pi}\left[-\frac{\cos(n+1)t}{n+1} + \frac{\cos(n-1)t}{n-1}\right]_0^{\pi} \quad (n \neq 1)$$

$$= \begin{cases} -\dfrac{4E}{[(2k)^2-1]\pi}, & \text{当} n = 2k \\ 0, & \text{当} n = 2k+1 \end{cases} \quad (k = 1,2,\cdots)$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} u(t)\cos t dt = \frac{2}{\pi}\int_0^{\pi} E\sin t\cos t dt = 0,$$

$$u(t) = \frac{2E}{\pi}[1 - 2\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1}] \quad (-\infty < x < +\infty)$$

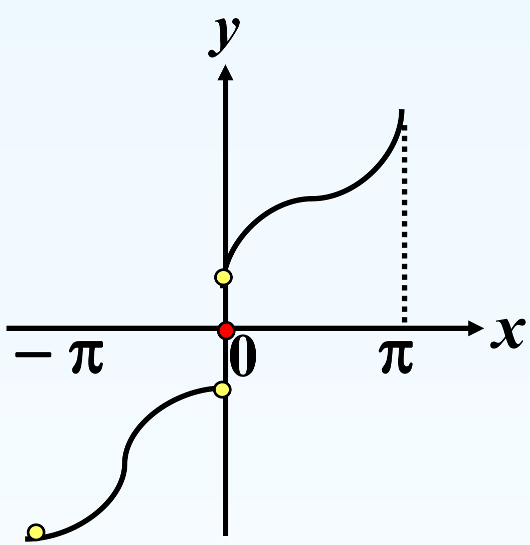
2. 非周期函数的周期性延拓

设$f(x)$定义在$[0,\pi]$上，延拓成以2π为周期的函数$F(x)$.

令 $F(x)=\begin{cases} f(x) & 0\le x\le \pi \\ g(x) & -\pi<x<0 \end{cases}$, 且$F(x+2\pi)=F(x)$,

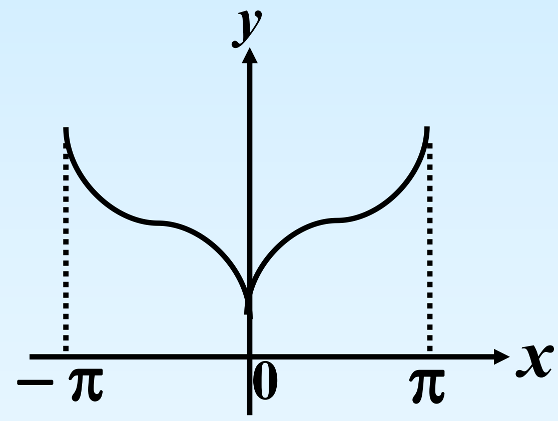
则有如下两种延拓方式：

奇延拓: $g(x)=-f(-x)$, $F(x)=\begin{cases} f(x) & 0<x\le\pi \\ 0 & x=0 \\ -f(-x) & -\pi<x<0 \end{cases}$



$$f(x)\sim\sum_{n=1}^{\infty} b_n \sin nx \quad (0\le x\le\pi)$$

偶延拓: $g(x) = f(-x)$



则$F(x) = \begin{cases} f(x) & 0 \le x \le \pi \\ f(-x) & -\pi < x < 0 \end{cases}$

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad (0 \le x \le \pi)$$

非周期函数,不同延拓下*Fourier*级数表达式不唯一

例6 将$f(x)=x+1\ (0\le x\le\pi)$分别展开为正弦级数和余项级数.

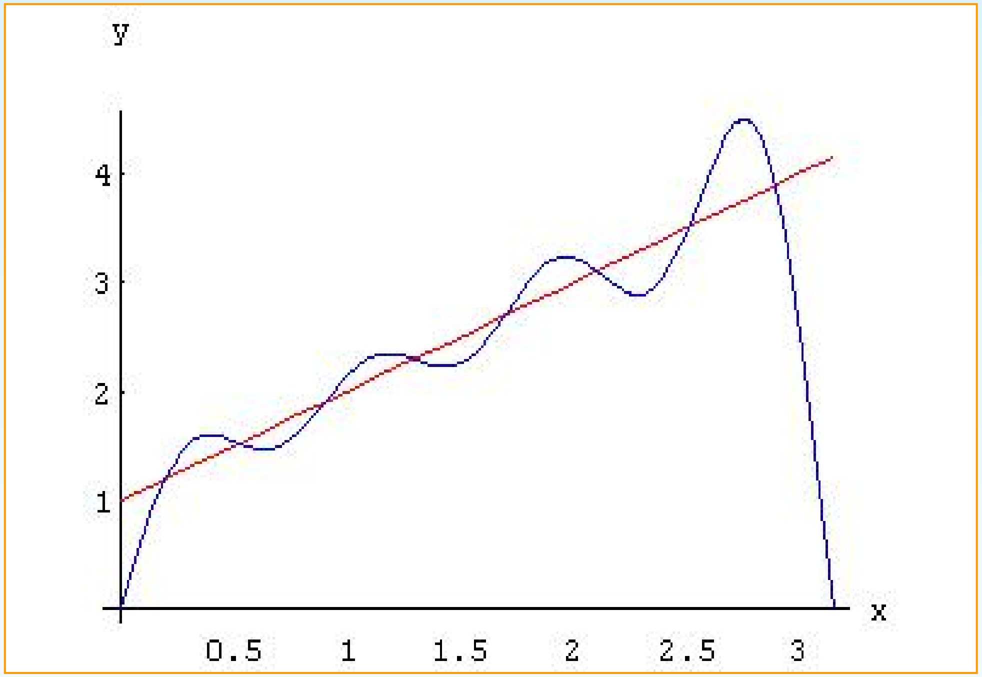
解 (1)求正弦级数. 对$f(x)$进行奇延拓:

$$b_n=\frac{2}{\pi}\int_0^{\pi}(x+1)\sin nx\,dx$$

$$=\frac{2}{n\pi}(1-\pi\cos n\pi-\cos n\pi)=\begin{cases}\dfrac{2}{\pi}\cdot\dfrac{\pi+2}{2k+1} & \text{当}n=2k+1\\[2ex] -\dfrac{1}{k} & \text{当}n=2k\end{cases}$$

$$x+1=\frac{2}{\pi}\left[(\pi+2)\sin x-\frac{\pi}{2}\sin 2x+\frac{1}{3}(\pi+2)\sin 3x-\cdots\right]$$

$$(0<x<\pi)$$


y
4
3
2
1
0.5
1
1.5
2
2.5
3
x

(2)求余弦级数. 对$f(x)$进行偶延拓：

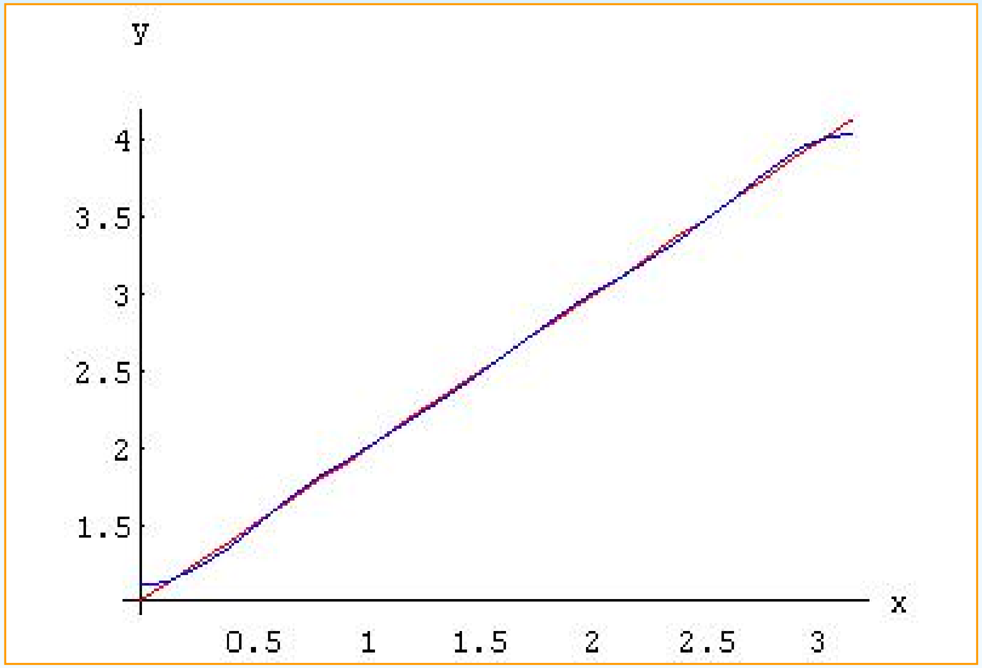
$$a_0=\frac{2}{\pi}\int_0^{\pi}(x+1)dx=\pi+2,$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}(x+1)\cos nx dx$$

$$=\frac{2}{n^2\pi}(\cos n\pi-1)=\begin{cases}0 & \text{当}n=2k\\ -\dfrac{4}{(2k+1)^2\pi} & \text{当}n=2k+1\end{cases}$$

$$x+1=\frac{\pi}{2}+1-\frac{4}{\pi}(\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\cdots]$$

$$(0\le x\le\pi)$$


y
4
3.5
3
2.5
2
1.5
x
0.5
1
1.5
2
2.5
3

3. 以2l为周期的函数的*Fourier*级数

定理 设周期为$2l$的周期函数$f(x)$满足收敛定理的条件，则它的*Fourier*级数展开式为

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right),$$

其中 $$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}dx,\qquad (n=0,1,2,\cdots)$$

$$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}dx,\qquad (n=1,2,\cdots)$$

注 如果$f(x)$为奇函数,则其$Fourier$级数展开式为

$$\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l},$$

其中$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx, (n = 1, 2, \cdots)$

如果$f(x)$为偶函数,则其$Fourier$级数展开式为

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l},$$

其中$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx, (n = 0, 1, 2, \cdots)$

证明 令$z=\dfrac{\pi x}{l}, -l \le x \le l \Rightarrow -\pi \le z \le \pi,$

设$f(x)=f(\dfrac{lz}{\pi})=F(z)$, $F(z)$以2π为周期.

$$F(z)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nz+b_n \sin nz),$$

其中$a_n=\dfrac{1}{\pi}\int_{-\pi}^{\pi}F(z)\cos nz dz, b_n=\dfrac{1}{\pi}\int_{-\pi}^{\pi}F(z)\sin nz dz.$

$$\therefore f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos\frac{n\pi}{l}x+b_n \sin\frac{n\pi}{l}x)$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}F(z)\cos nz dz=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi}{l}x dx, (z=\frac{\pi x}{l})$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}F(z)\sin nz dz=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi}{l}x dx.$$

例7 设$f(x)$是周期为4的周期函数,它在$[-2,2)$上的表达式

为$f(x)=\begin{cases}0 & -2\le x<0\\ k & 0\le x<2\end{cases}$,将它展开成Fourier级数.

解 $\because l=2 \quad \therefore a_0=\frac{1}{2}\int_{-2}^{0}0dx+\frac{1}{2}\int_{0}^{2}kdx \;=k,$

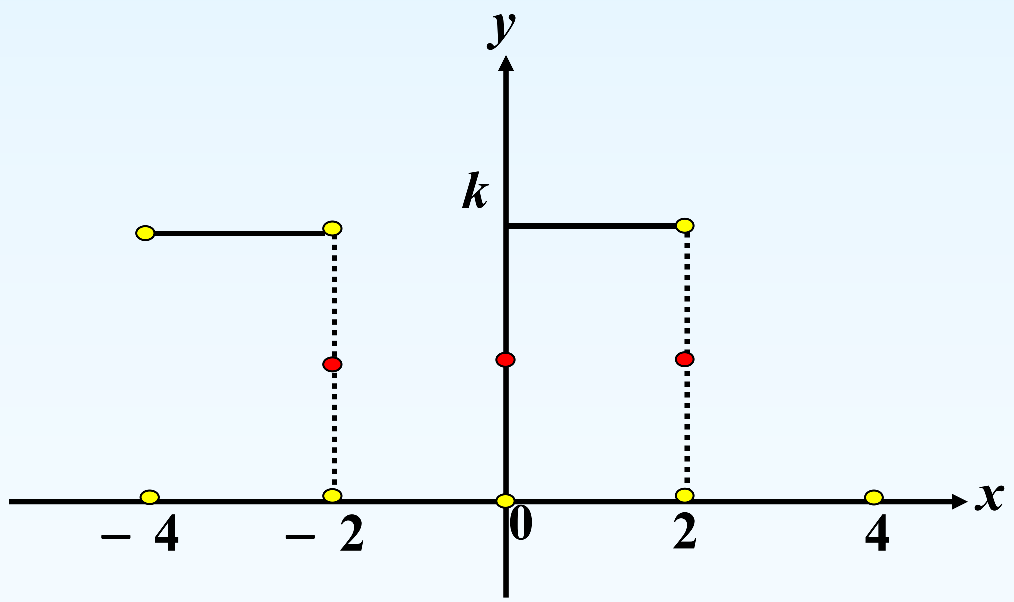
$$a_n=\frac{1}{2}\int_0^2 k\cdot\cos\frac{n\pi}{2}xdx \;=0,\quad (n=1,2,\cdots)$$

$$b_n=\frac{1}{2}\int_0^2 k\cdot\sin\frac{n\pi}{2}xdx \;=\frac{k}{n\pi}(1-\cos n\pi)$$

$$=\begin{cases}\dfrac{2k}{n\pi} & 当n=1,3,5,\cdots\\ 0 & 当n=2,4,6,\cdots\end{cases},$$

$$\therefore\ f(x)=\frac{k}{2}+\frac{2k}{\pi}\left(\sin\frac{\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}+\frac{1}{5}\sin\frac{5\pi x}{2}+\cdots\right)$$

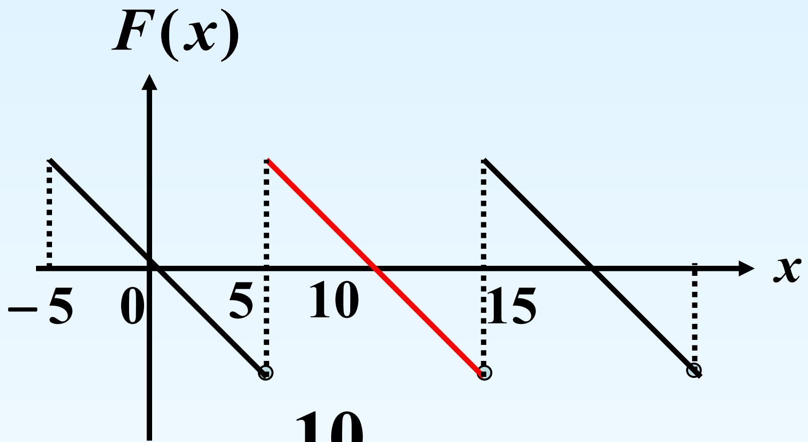
$$(-\infty<x<+\infty;x\neq 0,\pm 2,\pm 4,\cdots)$$



例8 将函数$f(x)=10-x, 5<x<15$展开为$Fourier$级数.

解1 对$f(x)$作周期为10的周期延拓, 得周期函数$F(x)$

$$F(x)=-x, -5<x<5$$



$$a_n=0, \quad (n=0,1,2,\cdots)$$

$$b_n=\frac{2}{5}\int_0^5(-x)\sin\frac{n\pi x}{5}dx=(-1)^n\frac{10}{n\pi}, \quad (n=1,2,\cdots)$$

$$\therefore F(x)=\frac{10}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{5}, \quad x\neq 5(2k+1)$$

$$\therefore 10-x=\frac{10}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{5}, 5<x<15$$

解2 $a_n = 0$

$$b_n = \frac{1}{5}\int_{-5}^{5} F(x)\sin\frac{n\pi x}{5}dx$$

令$t = x + 10$ $$= \frac{1}{5}\int_{5}^{15} F(t-10)\sin\frac{n\pi t}{5}dt$$

$$= \frac{1}{5}\int_{5}^{15} F(t)\sin\frac{n\pi t}{5}dt$$

$$= \frac{1}{5}\int_{5}^{15} (10-t)\sin\frac{n\pi t}{5}dx$$

$$= (-1)^n\frac{10}{n\pi},$$

故 $$10 - x = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi}{5}x, 5 < x < 15$$

例9 如何将定义在$[0,\frac{\pi}{2}]$上的可积函数$f(x)$延拓, 使其

*Fourier*级数为$\sum_{n=1}^{\infty} a_{2n-1}\cos(2n-1)x$.

解 $\because f(x)$的*Fourier*级数缺少$\sin nx$项

$\therefore f(x)$是以2π为周期的偶函数.

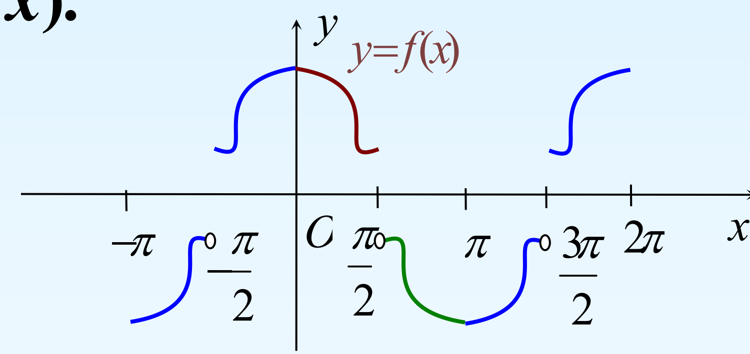
$$a_{2n}=0=\frac{2}{\pi}\int_0^{\pi} f(x)\cos 2nx dx$$

$$=\frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} f(x)\cos 2nx dx+\int_{\frac{\pi}{2}}^{\pi} f(x)\cos 2nx dx\right]$$

$$\overset{u=\pi-x}{=} -\int_{\pi}^{\frac{\pi}{2}} f(\pi-u)\cos 2n(\pi-u)du+\int_{\frac{\pi}{2}}^{\pi} f(x)\cos 2nx dx$$

$$\Rightarrow \int_{\frac{\pi}{2}}^{\pi} [f(\pi - x) + f(x)] \cos 2nx dx = 0.$$

故可令 $x \in [\dfrac{\pi}{2}, \pi]$, $f(x) = -f(\pi - x)$.



$$\Rightarrow f(x) = \begin{cases} f(x), & 0 \le x \le \dfrac{\pi}{2}, \\ -f(\pi - x), & \dfrac{\pi}{2} < x \le \pi, \\ f(-x), & -\dfrac{\pi}{2} \le x \le 0, \\ -f(\pi + x), & -\pi \le x < -\dfrac{\pi}{2}. \end{cases}$$

作业 ：

习题13.1

2(1, 3, 5)、4(3, 4)、5(2, 4)、6(2)