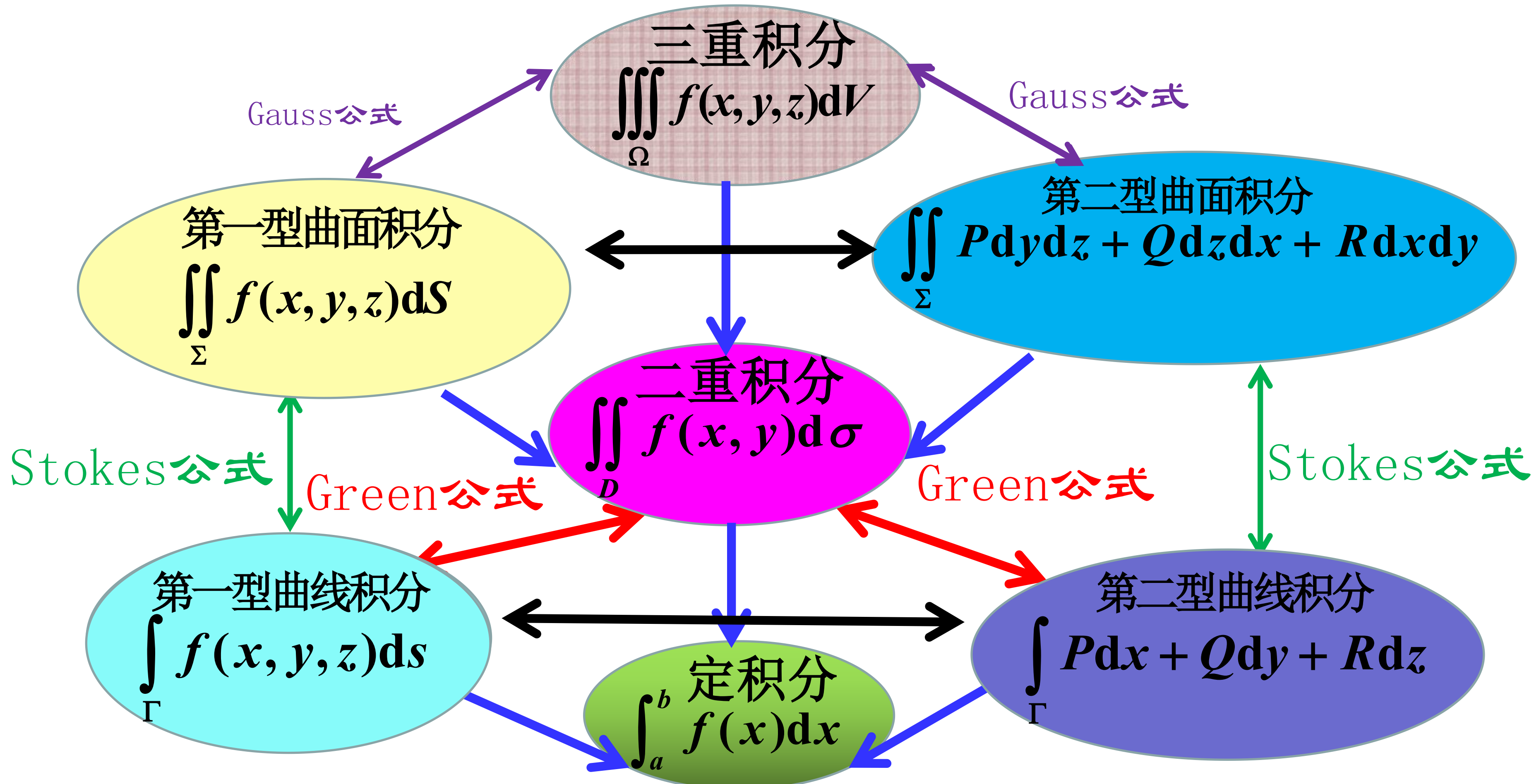
曲线积分与曲面积分
内容总结

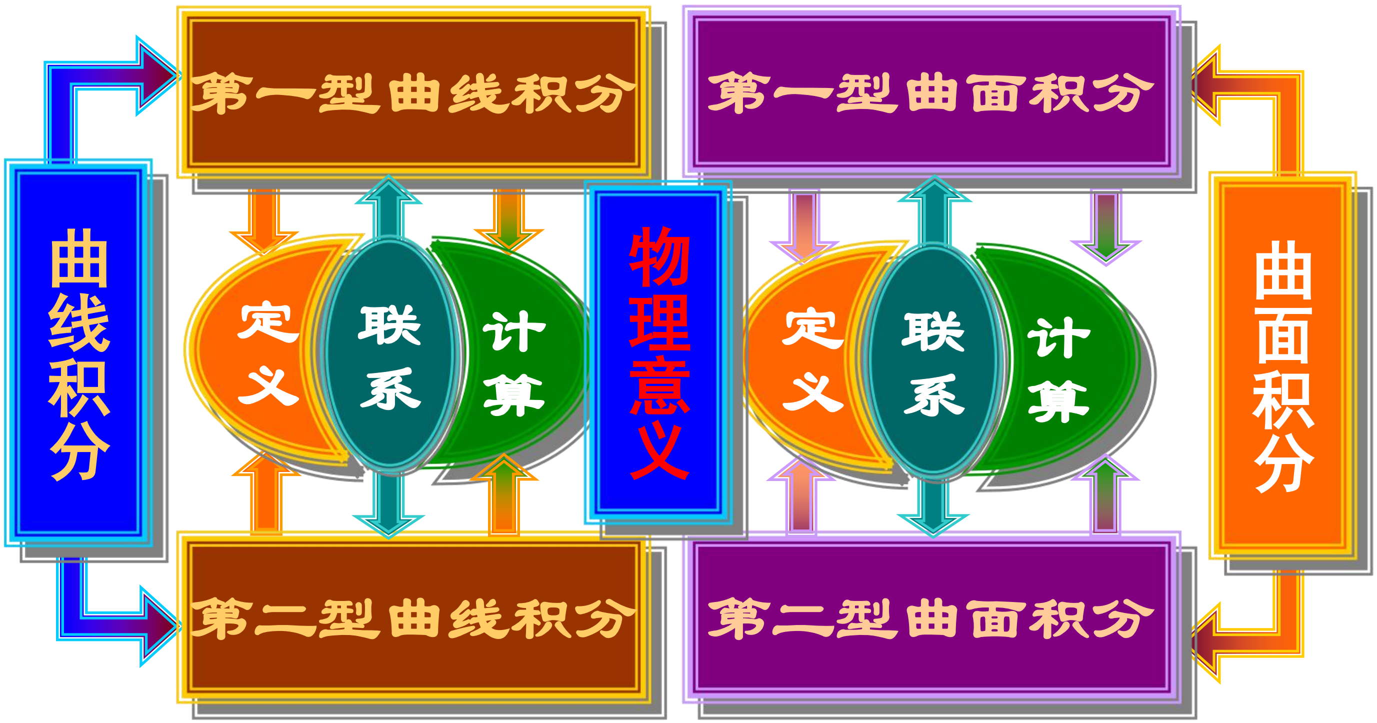
一、基本内容
二、典型例题

2025.05.26 苑佳



$$\oiint\limits_{\Sigma} Pdydz+Qdzdx+Rdxdy=\iiint\limits_{\Omega}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dV$$

曲线积分与曲面积分


曲线积分
第一型曲线积分
第二型曲线积分
定义
联系
计算
物理意义
第一型曲面积分
第二型曲面积分
定义
联系
计算
曲面积分

第一型曲线积分的直接计算

平面曲线

(1) $L:\begin{cases} x=\varphi(t), \\ y=\psi(t), \end{cases} t\in[\alpha,\beta],$

$$\int_L f(x,y)ds=\int_\alpha^\beta f[\varphi(t),\psi(t)]\sqrt{\varphi'^2(t)+\psi'^2(t)}dt$$

(2) $L: y=y(x) \qquad a\le x\le b.$

$$\int_L f(x,y)ds=\int_a^b f[x,y(x)]\sqrt{1+y'^2(x)}dx$$

(3) $L: x=x(y) \qquad c\le y\le d.$

$$\int_L f(x,y)ds=\int_c^d f[x(y),y]\sqrt{1+x'^2(y)}dy$$

空间曲线

$\Gamma:\begin{cases} x=\varphi(t), \\ y=\psi(t), \quad (\alpha\le t\le\beta) \\ z=\omega(t). \end{cases}$

积分下限
小于
积分上限

$$\int_\Gamma f(x,y,z)ds=\int_\alpha^\beta f[\varphi(t),\psi(t),\omega(t)]\sqrt{\varphi'^2(t)+\psi'^2(t)+\omega'^2(t)}dt$$

第二型曲线积分的直接计算

平面曲线

(1) $L:\begin{cases} x=\varphi(t), \\ y=\psi(t), \end{cases} \quad t:\alpha\mapsto\beta,$

$$\int_L Pdx+Qdy=\int_a^b\{P[\varphi(t),\psi(t)]\varphi'(t)+Q[\varphi(t),\psi(t)]\psi'(t)\}dt$$

(2) $L: y=y(x) \quad x:a\mapsto b$

$$\int_L Pdx+Qdy=\int_a^b\{P[x,y(x)]+Q[x,y(x)]y'(x)\}dx$$

(3) $L: x=x(y) \quad y:c\mapsto d$

$$\int_L Pdx+Qdy=\int_c^d\{P[x(y),y]x'(y)+Q[x(y),y]\}dy$$

空间曲线

$$\Gamma:\begin{cases} x=\varphi(t), \\ y=\psi(t), \quad t:\alpha\mapsto\beta \\ z=\omega(t). \end{cases}$$

积分下限未必小于积分上限

$$\int_\Gamma Pdx+Qdy+Rdz=\int_\alpha^\beta\{P[\varphi(t),\psi(t),\omega(t)]\varphi'(t)$$
$$+Q[\varphi(t),\psi(t),\omega(t)]\psi'(t)+R[\varphi(t),\psi(t),\omega(t)]\omega'(t)\}dt$$

两类曲线积分之间的联系

$$\int_L Pdx + Qdy = \int_L (P\cos\alpha + Q\cos\beta)ds$$

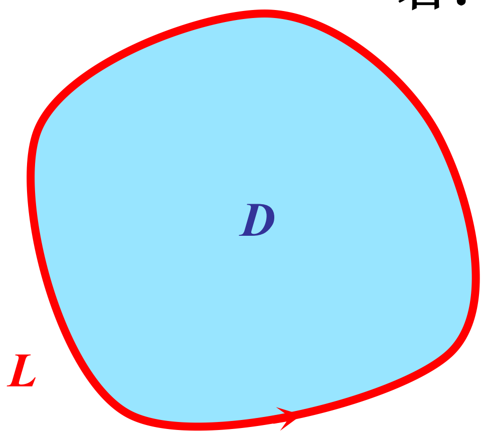
$\cos\alpha, \cos\beta$为有向曲线 L的切方向余弦

$$\int_\Gamma Pdx + Qdy + Rdz = \int_\Gamma (P\cos\alpha + Q\cos\beta + R\cos\gamma)ds$$

$\cos\alpha, \cos\beta, \cos\gamma$为有向曲线 Γ的切方向余弦

格林公式

平面曲线积分与二重积分



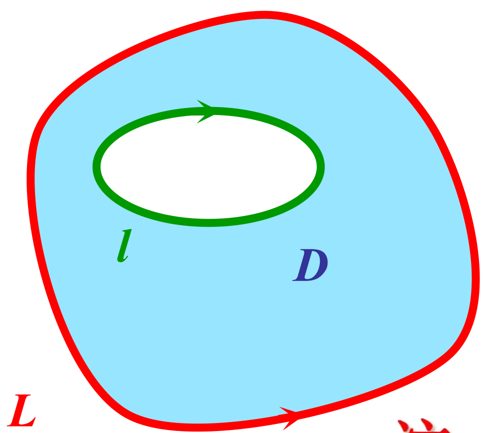
若：1. xOy平面上闭区域 D由分段光滑的曲线 L围成

2. 在D上函数$P(x,y),Q(x,y)\in C^1$

则有

$$\oint_L P\mathrm{d}x+Q\mathrm{d}y=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

其中 L 是 D 的正向边界曲线.



D是复连通区域时, 格林公式为:

$$\underset{(\text{逆})}{\oint_L} P\mathrm{d}x+Q\mathrm{d}y+\underset{(\text{顺})}{\oint_l} P\mathrm{d}x+Q\mathrm{d}y=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

注 若在D内又有 $\dfrac{\partial Q}{\partial x}\equiv\dfrac{\partial P}{\partial y}$, 则 $\underset{(\text{逆})}{\oint_L} P\mathrm{d}x+Q\mathrm{d}y=\underset{(\text{逆})}{\oint_l} P\mathrm{d}x+Q\mathrm{d}y$

平面曲线积分与路径无关

条件	在单连通区域 D 上 $P(x,y),Q(x,y)$ 具有连续的一阶偏导数, 则以下四个命题等价.
等价命题	(1) 在 D 内 $\int_L Pdx+Qdy$ 与路径无关 (2) $\oint_C Pdx+Qdy=0$, 封闭曲线 $C\subset D$ (3) 在 D 内存在 $u(x,y)$ 使 $du=Pdx+Qdy$ (4) 在 D 内, $\dfrac{\partial Q}{\partial x}=\dfrac{\partial P}{\partial y}$.

$$u(x,y)=\int_{(x_0,y_0)}^{(x,y)}P(x,y)dx+Q(x,y)dy$$

$$=\int_{x_0}^{x}P(x,y_0)dx+\int_{y_0}^{y}Q(x,y)dy$$

$$=\int_{y_0}^{y}Q(x_0,y)dy+\int_{x_0}^{x}P(x,y)dx$$

第二型平面曲线积分计算

根据被积函数补充曲线挖去奇点

$\frac{\partial P}{\partial y}$ 或 $\frac{\partial Q}{\partial x}$ 不连续 (有奇点)

(3)

$I = \int_L Pdx + Qdy$

(1) 直接计算

(2)

$\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ 连续(无奇点)

单连通 (a)

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

非闭 $I = \int_{(x_0,y_0)}^{(x,y)} Pdx + Qdy$

闭合 $I = \int_L Pdx + Qdy = 0$

(b)

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$

闭合 $I = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dxdy$

非闭 补充曲线

第一型曲面积分的计算

(1) $\Sigma: z = z(x, y) \quad (x, y) \in D_{xy}$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{xy}} f(x, y, z(x, y)) \sqrt{1 + z_x^2 + z_y^2} dxdy$$

(2) $\Sigma: y = y(z, x) \quad (z, x) \in D_{zx}$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{zx}} f(x, y(z, x), z) \sqrt{1 + y_z^2 + y_x^2} dzdx$$

(3) $\Sigma: x = x(y, z) \quad (y, z) \in D_{yz}$

$$\iint_{\Sigma} f(x, y, z) dS = \iint_{D_{yz}} f(x(y, z), y, z) \sqrt{1 + x_y^2 + x_z^2} dydz$$

(4) $\Sigma:\begin{cases} x = x(u,v), \\ y = y(u,v), \\ z = z(u,v), \end{cases} (u,v) \in D$,

$$\iint\limits_{\Sigma} f(x,y,z)\mathrm{d}S = \iint\limits_{D} f(x(u,v),y(u,v),z(u,v))\sqrt{EG-F^2}\,\mathrm{d}u\mathrm{d}v\ ,$$

其中 $E = x_u^2 + y_u^2 + z_u^2$,

$F = x_u x_v + y_u y_v + z_u z_v$,

$G = x_v^2 + y_v^2 + z_v^2$.

第二型曲面积分的计算

“一投，二代，三定号”

(1) $\Sigma: z = z(x,y) \quad (x,y)\in D$

$$\iint_{\Sigma} Pdydz + Qdzdx + Rdxdy$$

$$=\pm\iint_{D}\left[(P(x,y,z(x,y))\cdot(-z_x)+Q\cdot(-z_y)+R\cdot 1\right]dxdy$$

上侧取正，下侧取负

(2) $\Sigma: y = y(z,x) \quad (z,x)\in D$

$$\iint_{\Sigma} Pdydz + Qdzdx + Rdxdy$$

$$=\pm\iint_{D}\left[(P(x,y(z,x),z)\cdot(-y_x)+Q\cdot 1+R\cdot(-y_z)\right]dzdx$$

右侧取正，左侧取负

(3) $\Sigma: x = x(y,z) \quad (y,z)\in D$

$$\iint_{\Sigma} Pdydz + Qdzdx + Rdxdy$$

$$=\pm\iint_{D}\left[(P(x(y,z),y,z)\cdot 1+Q\cdot(-x_y)+R\cdot(-x_z)\right]dydz$$

前侧取正，后侧取负

$$(4)\ \Sigma:\begin{cases} x=x(u,v), \\ y=y(u,v), \quad\quad (u,v)\in D \\ z=z(u,v), \end{cases}$$

$$\iint\limits_{\Sigma} Pdydz+Qdzdx+Rdxdy$$

$$=\pm\iint\limits_{\Delta}[P(x(u,v),y(u,v),z(u,v))\cdot\frac{\partial(y,z)}{\partial(u,v)}+Q\cdot\frac{\partial(z,x)}{\partial(u,v)}+R\cdot\frac{\partial(x,y)}{\partial(u,v)}]dudv$$

$\left(\frac{\partial(y,z)}{\partial(u,v)},\frac{\partial(z,x)}{\partial(u,v)},\frac{\partial(x,y)}{\partial(u,v)}\right)$与Σ指定侧的法向量方向一致时取+，否则取−.

两类曲面积分之间的关系

$$\iint_{\Sigma} Pdydz + Qdzdx + Rdxdy = \iint_{\Sigma} (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS$$

$\cos\alpha, \cos\beta, \cos\gamma$为给定侧曲面的法方向余弦

第二型曲面积分也可以把三项化为一项来计算

$$\iint_{\Sigma} Pdydz + Qdzdx + Rdxdy = \iint_{\Sigma} (P\cos\alpha + Q\cos\beta + R\cos\gamma)dS$$

$$= \iint_{\Sigma} (P\frac{\cos\alpha}{\cos\gamma} + Q\frac{\cos\beta}{\cos\gamma} + R)\cos\gamma dS$$

$$= \iint_{\Sigma} (P\frac{\cos\alpha}{\cos\gamma} + Q\frac{\cos\beta}{\cos\gamma} + R)dxdy$$

高斯公式　曲面积分与三重积分

若：1. 空间闭区域 Ω 由分片光滑的闭曲面 Σ 围成；

2. 在 Ω 上函数 $P(x,y,z), Q(x,y,z), R(x,y,z) \in C^1$.

则有
$$\oint\!\!\!\oint_{\Sigma} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}V$$

$$= \oint\!\!\!\oint_{\Sigma} (P\cos\alpha + Q\cos\beta + R\cos\gamma)\mathrm{d}S$$

其中 Σ 是 Ω 的整个边界曲面的外侧.

若区域 Ω 为二维复连通区域, 外面的边界 Σ 取外侧, 内部的边界 S 取内侧. 相对于区域来说, 边界曲面整体取外侧.

$$\oint\!\!\!\oint_{\Sigma(\text{外侧})} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y + \oint\!\!\!\oint_{S(\text{内侧})} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y = \iiint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}V$$

注 若在 Ω 内又有 $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = 0$, 则 $\displaystyle\oint\!\!\!\oint_{\Sigma(\text{外侧})} = \oint\!\!\!\oint_{S(\text{外侧})}$

第二型曲面积分计算

根据被积函数补充曲面去奇点

$\frac{\partial P}{\partial x}$ 或 $\frac{\partial Q}{\partial y}$ 或 $\frac{\partial R}{\partial z}$ 不连续(有奇点)

$$\iint_{\Sigma} P\mathrm{d}y\mathrm{d}z + Q\mathrm{d}z\mathrm{d}x + R\mathrm{d}x\mathrm{d}y$$

直接计算

两类联系

$\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial R}{\partial z}$ 连续(无奇点)

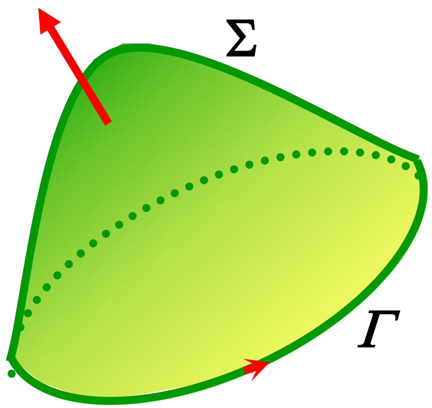
闭 合

非 闭

高斯公式

补充曲面

Stokes公式　　　　空间曲线积分与曲面积分



若：1. Γ 为分段光滑的空间有向闭曲线，

Σ 是以 Γ 为边界的分片光滑的有向曲面，

Γ 的正向与 Σ 的侧符合右手法则.

2. 在曲面 Σ (包括Γ)上，$P, Q, R \in C^1$. 则有

$$\oint_\Gamma P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = \iint_\Sigma \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathrm{d}y\mathrm{d}z + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathrm{d}z\mathrm{d}x + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

或记为

$$\oint_\Gamma P\mathrm{d}x + Q\mathrm{d}y + R\mathrm{d}z = \iint_\Sigma \begin{vmatrix} \mathrm{d}y\mathrm{d}z & \mathrm{d}z\mathrm{d}x & \mathrm{d}x\mathrm{d}y \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \iint_\Sigma \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

空间曲线积分与路径无关

条件	在单连通区域 Ω 上 $P(x,y,z),Q,R$ 具有连续的一阶偏导数，则以下四个命题等价.
等价命题	(1) 在Ω内$\int_\Gamma Pdx+Qdy+Rdz$与路径无关 (2) $\oint_\Gamma Pdx+Qdy+Rdz=0$，任意封闭曲线$\Gamma\subset\Omega$ (3) 在Ω内存在$u(x,y,z)$，使$du=Pdx+Qdy+Rdz$； (4) 在Ω内，$\frac{\partial R}{\partial y}\equiv\frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z}\equiv\frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x}\equiv\frac{\partial P}{\partial y}$.

$$u(x,y,z)=\int_{(x_0,y_0,z_0)}^{(x,y,z)}P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz$$

$$=\int_{x_0}^{x}P(x,y_0,z_0)dx+\int_{y_0}^{y}Q(x,y,z_0)dy+\int_{z_0}^{z}R(x,y,z)dz$$

第二型空间曲线积分计算

$$I=\int_L Pdx+Qdy+Rdz$$

(1) $\Rightarrow$ 直接计算

(2) $\Downarrow$

$$\frac{\partial P}{\partial y},\frac{\partial Q}{\partial x},\frac{\partial R}{\partial z}\text{连续(无奇点)}$$

单连通 (a) $\Downarrow$

(*) 成立

非闭 $\Leftarrow$

$$I=\int_{(x_0,y_0,z_0)}^{(x,y,z)} Pdx+Qdy+Rdz$$

闭合 $\Leftarrow$

$$I=\oint_\Gamma Pdx+Qdy+Rdz=0$$

(b) $\Downarrow$

(*) 不成立

闭合 $\Rightarrow$

$$I=\iint_\Sigma \begin{vmatrix} dydz & dzdx & dxdy \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

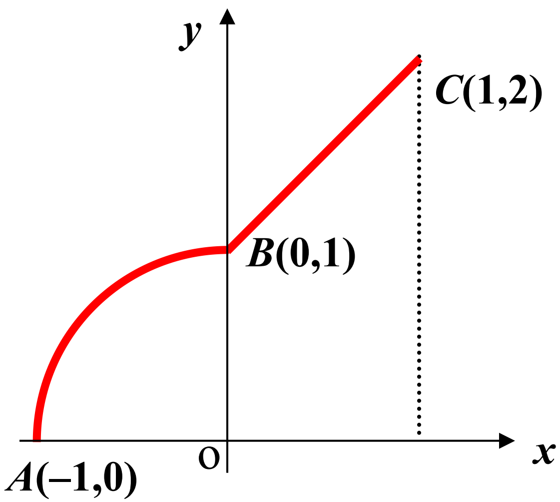
$$\frac{\partial R}{\partial y}\equiv\frac{\partial Q}{\partial z},\frac{\partial P}{\partial z}\equiv\frac{\partial R}{\partial x},\frac{\partial Q}{\partial x}\equiv\frac{\partial P}{\partial y}\quad(*)$$

典型例题---计算题

例1 $\int_L (x-y)\mathrm{d}s$，L是以原点为圆心的单位 圆介于点$A(-1,0), B(0,1)$之间

的劣弧$\overset{\frown}{AB}$与连接点$B, C(1,2)$的直线段$\overline{BC}$组成的分段光滑曲线．

解



$$\int_L (x-y)\mathrm{d}s = \int_{\overset{\frown}{AB}} (x-y)\mathrm{d}s + \int_{\overline{BC}} (x-y)\mathrm{d}s$$

其中，$\overset{\frown}{AB}: \begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad t \in [\frac{\pi}{2}, \pi]$

$$\mathrm{d}s = \sqrt{x'^2(t) + y'^2(t)}\,\mathrm{d}t = \mathrm{d}t$$

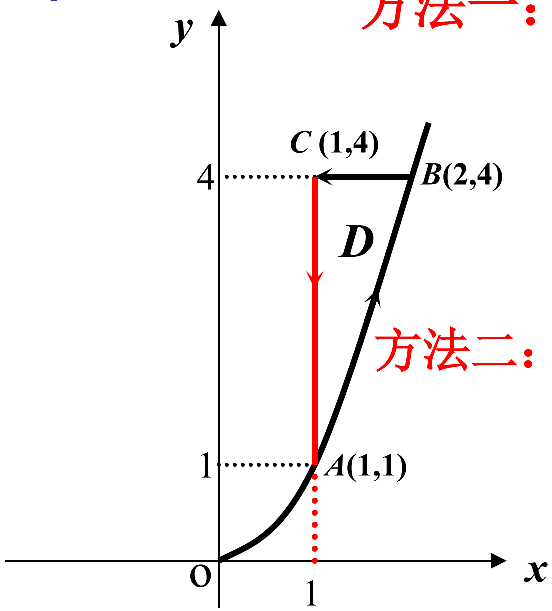
$$\overline{BC}: \quad y = x+1, \quad x \in [0,1]$$

$$\mathrm{d}s = \sqrt{1+y'^2(x)}\,\mathrm{d}x = \sqrt{2}\,\mathrm{d}x$$

$$\therefore \int_L (x-y)\mathrm{d}s = \int_{\frac{\pi}{2}}^{\pi} (\cos t - \sin t)\mathrm{d}t + \int_0^1 (-\sqrt{2})\mathrm{d}x = -2-\sqrt{2}.$$

例2 $\int_L \frac{1}{y}\mathrm{d}x+\frac{1}{x}\mathrm{d}y$，$L$是曲线 $y=x^2$上从点$A(1,1)$到$B(2,4)$的有向弧段$\widehat{AB}$与直线 $y=4$上从点B到$C(1,4)$的线段$\overline{BC}$组成的有向分段光滑曲线.

解



方法一: $\int_L = \int_{\widehat{AB}} + \int_{\overline{BC}}$

$$= \int_1^2 \left(\frac{1}{x^2}+\frac{1}{x}\cdot 2x\right)\mathrm{d}x + \int_2^1 \frac{1}{4}\mathrm{d}x = \frac{9}{4}$$

方法二: $\int_L + \int_{\overline{CA}} = \iint_D \left(\frac{1}{y^2}-\frac{1}{x^2}\right)\mathrm{d}x\mathrm{d}y$

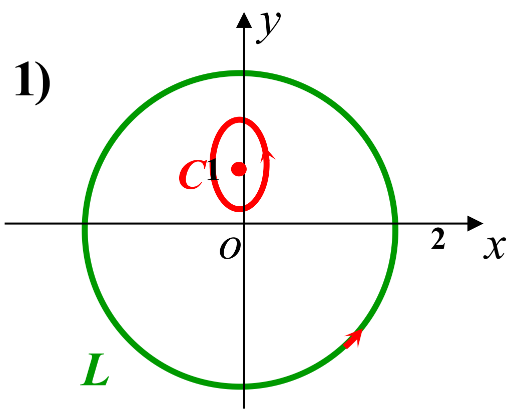
$$= \int_1^4 \mathrm{d}y\int_1^{\sqrt{y}}\left(\frac{1}{y^2}-\frac{1}{x^2}\right)\mathrm{d}x = -\frac{3}{4}$$

$$\therefore \int_L = -\frac{3}{4} - \int_{\overline{CA}} \frac{1}{y}\mathrm{d}x+\frac{1}{x}\mathrm{d}y = -\frac{3}{4}-\int_4^1 1\mathrm{d}y = \frac{9}{4}$$

例3 设有平面力场 $\vec{F}=\dfrac{1-y}{4x^2+(y-1)^2}\vec{i}+\dfrac{x}{4x^2+(y-1)^2}\vec{j}$，力场中的点$M$ 在场力 $\vec{F}$ 的作用下，沿着圆周 $x^2+y^2=4$ 的逆时针方向运动一周，试求场 力$\vec{F}$ 所作的功.

解 $$W=\oint_L \frac{1-y}{4x^2+(y-1)^2}\mathrm{d}x+\frac{x}{4x^2+(y-1)^2}\mathrm{d}y$$

$P=\dfrac{1-y}{4x^2+(y-1)^2}$ $Q=\dfrac{x}{4x^2+(y-1)^2}$ 则 $\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$ $(x,y)\neq(0,\ 1)$



取曲线 $C:4x^2+(y-1)^2=\varepsilon^2$ 逆时针

参数方程为 $C:\begin{cases}x=\dfrac{\varepsilon}{2}\cos t\\ y=1+\varepsilon\sin t\end{cases}$ $t:0\to 2\pi$

$$\therefore W=\int_{L-C}+\oint_C=\iint_D 0dxdy+\oint_C=\frac{1}{\varepsilon^2}\int_0^{2\pi}[(-\varepsilon\sin t)(-\frac{\varepsilon}{2}\sin t)+\frac{\varepsilon}{2}\cos t\cdot\varepsilon\cos t]\mathrm{d}t=\pi\ .$$

也可用Green公式:$\oint_C=\dfrac{1}{\varepsilon^2}\oint_C(1-y)dx+xdy=\dfrac{1}{\varepsilon^2}\iint\limits_{4x^2+(y-1)^2\leq\varepsilon^2}2dxdy=\pi$

例4 选择常数a,b使得曲线积分

$$I=\int_L \frac{(ax^2+2xy+y^2)dx-(x^2+2xy+by^2)dy}{(x^2+y^2)^2}$$与路径无关,

并计算$\displaystyle\int_{(1,1)}^{(5,5)} \frac{(ax^2+2xy+y^2)dx-(x^2+2xy+by^2)dy}{(x^2+y^2)^2}$.

解 令$P=\dfrac{ax^2+2xy+y^2}{(x^2+y^2)^2}, Q=-\dfrac{x^2+2xy+by^2}{(x^2+y^2)^2}$,则当$(x,y)\neq(0,0)$时,

$$\frac{\partial P}{\partial y}=\frac{2[x^3+(1-2a)x^2y-3xy^2-y^3]}{(x^2+y^2)^2}, \frac{\partial Q}{\partial x}=\frac{2[x^3+3x^2y+(2b-1)xy^2-y^3]}{(x^2+y^2)^2},$$

所以当$1-2a=3, 2b-1=-3$时,$\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$.

即$a=-1, b=-1$时曲线积分与路径无关,沿直线$y=x$可得

$$I=\int_{(1,1)}^{(5,5)} \frac{(-x^2+2xy+y^2)dx-(x^2+2xy-y^2)dy}{(x^2+y^2)^2}=\int_1^5 \frac{2x^2dx-2x^2dx}{(x^2+x^2)^2}=0.$$

例5 求$I=\int_\Gamma[(x+y)^2+xyz]ds$， 其中$\Gamma$为圆周$\begin{cases}x^2+y^2+z^2=1,\\ x+y+z=0.\end{cases}$

解 方法一：由于曲线关于原点对称，知$\int_\Gamma xyzds=0$

由轮换对称性 $\int_\Gamma xyds=\int_\Gamma yzds=\int_\Gamma zxds,\int_\Gamma x^2ds=\int_\Gamma y^2ds=\int_\Gamma z^2ds$

$$\int_\Gamma(x^2+y^2)ds=\frac{2}{3}\int_\Gamma(x^2+y^2+z^2)ds=\frac{2}{3}\int_\Gamma ds=\frac{4\pi}{3}$$

$$\int_\Gamma 2xyds=\frac{1}{3}\int_\Gamma(2xy+2yz+2zx)ds=\frac{1}{3}\int_\Gamma[(x+y+z)^2-(x^2+y^2+z^2)]ds$$

$$=-\frac{1}{3}\int_\Gamma ds=-\frac{2\pi}{3}$$

所以$I=\int_\Gamma[(x+y)^2+xyz]ds=\int_\Gamma(x^2+y^2+2xy)ds=\frac{4\pi}{3}-\frac{2\pi}{3}=\frac{2\pi}{3}$

方法二：

$$\int_\Gamma(x+y)^2ds=\int_\Gamma(-z)^2ds=\frac{1}{3}\int_\Gamma(x^2+y^2+z^2)ds=\frac{1}{3}\int_\Gamma ds=\frac{2\pi}{3}$$

方法三：也可写出曲线的参数方程来计算, 但过程比较麻烦

$$\Gamma:\begin{cases} x^2+y^2+z^2=1, \\ x+y+z=0. \end{cases}$$

联立方程得$(x^2+y^2)+(-x-y)^2=1$

即$x^2+y^2+xy=\dfrac{1}{2}$, 令$\begin{cases} x=u+v \\ y=u-v \end{cases}$, 上面方程变为$3u^2+v^2=\dfrac{1}{2}$,

可得曲线Γ的参数方程 $\begin{cases} x=\dfrac{1}{\sqrt{6}}\sin\theta+\dfrac{1}{\sqrt{2}}\cos\theta \\ y=\dfrac{1}{\sqrt{6}}\sin\theta-\dfrac{1}{\sqrt{2}}\cos\theta \\ z=-\dfrac{2}{\sqrt{6}}\sin\theta \end{cases}$

(或者将$x^2+y^2+xy=\dfrac{1}{2}$配方得$(x+\dfrac{y}{2})^2-\dfrac{3y^2}{4}=\dfrac{1}{2}$, 进而写出参数方程)

例6 计算$I=\int_\Gamma (z-y)\,\mathrm{d}x+(x-z)\,\mathrm{d}y+(x-y)\,\mathrm{d}z$,其中$\Gamma\begin{cases}x^2+y^2=1\\ x-y+z=2\end{cases}$,

从z轴正向看为顺时针方向.

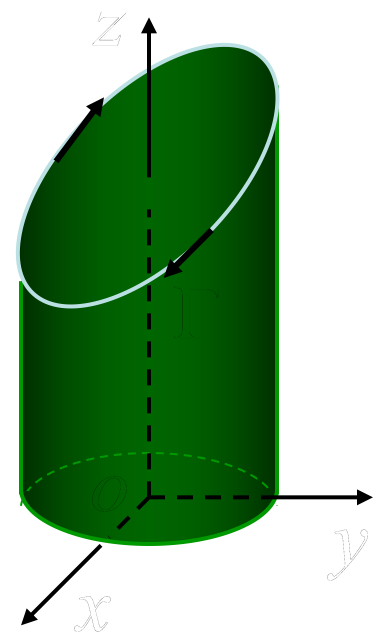
解 方法一: Γ 的参数方程 $x=\cos t,\ y=\sin t,\ z=2-\cos t+\sin t\quad (t:2\pi\to 0)$

所以 $I=\int_{2\pi}^{0}[(2-\cos t)(-\sin t)+(-2+2\cos t-\sin t)\cos t+(\cos t-\sin t)(\cos t+\sin t)]\mathrm{d}t$

$$=\int_0^{2\pi}(1-4\cos^2 t)\,\mathrm{d}t=-2\pi$$

方法二: 取平面$x-y+z=2$上被Γ所围部分为Σ,取下侧, 由Stokes公式

$$I=\iint_\Sigma\begin{vmatrix}\mathrm{d}y\mathrm{d}z & \mathrm{d}z\mathrm{d}x & \mathrm{d}x\mathrm{d}y\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ z-y & x-z & x-y\end{vmatrix}=\iint_\Sigma 2\mathrm{d}x\mathrm{d}y=-2\iint_{x^2+y^2\le 1}dxdy=-2\pi$$



例7 计算$\int_L (yz - e^{x^2})dx + (xy+1)dz + xzdy$，其中$L$是从点$A(2,0,0)$沿螺旋线$x = 2\cos t,\ y = \sin t,\ z = t$到$B(2,0,2\pi)$的一段.

解 记 $P = yz - e^{x^2}$， $Q = xz$， $R = xy + 1$，

$$\text{则}\ \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} = x, \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = y, \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = z$$

所以积分与路径无关.

取L_1为A到B的直线段，即$L_1 : \begin{cases} x = 2 \\ y = 0 \end{cases} \quad (z : 0 \mapsto 2\pi)$

$$\int_L (yz - e^{x^2})dx + (xy+1)dz + xzdy$$
$$= \int_{L_1} (yz - e^{x^2})dx + (xy+1)dz + xzdy$$
$$= \int_0^{2\pi} dz = 2\pi$$

例8 计算$I=\oint_{\Gamma}(y+z)dx+(z-\sin y)dy+2xdz$,其中$\Gamma$为柱面$x^2+y^2=1$与平面 $x+y+z=1$的交线,从 z 轴正向看去Γ为顺时针方向.

解 方法一: 设Σ为平面$x+y+z=1$被曲线所围的部分 ,取下侧

单位法向量$\vec{n}^0=(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}})=(\cos\alpha,\cos\beta,\cos\gamma)$

由Stokes公式, $I=\iint\limits_{\Sigma}\begin{vmatrix}\frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y+z & z-\sin y & 2x\end{vmatrix}dS=\frac{3}{\sqrt{3}}\iint\limits_{\Sigma}dS$

$\Sigma: z=1-x-y,\quad (x,y)\in D=\{(x,y)\,|\,x^2+y^2\le 1\}$

$$I=\sqrt{3}\iint\limits_{\Sigma}dS=\sqrt{3}\iint\limits_{D}\sqrt{1+z_x^2+z_y^2}\,dxdy=3\iint\limits_{D}dxdy=3\pi$$

例8 计算$I=\oint_\Gamma (y+z)dx+(z-\sin y)dy+2xdz$,其中$\Gamma$为柱面$x^2+y^2=1$与平面 $x+y+z=1$的交线,从 z 轴正向看去Γ为顺时针方向.

方法二: 设Σ为平面$x+y+z=1$被曲线所围的部分 ,取下侧

$$\Sigma: z=1-x-y,\quad (x,y)\in D=\{(x,y)\mid x^2+y^2\le 1\},$$

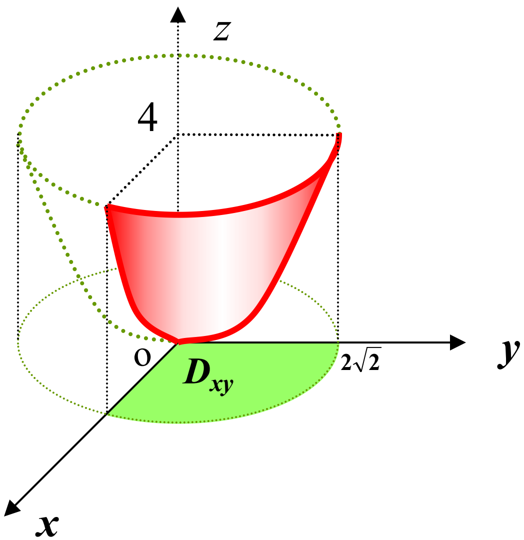
$$(-z_x,-z_y,1)=(1,1,1),$$

$$原积分=\iint_\Sigma \begin{vmatrix} dydz & dzdx & dxdy \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+z & z-\sin y & 2x \end{vmatrix} = -\iint_\Sigma dydz+dzdx+dxdy$$

$$=\iint_D [1\cdot(-z_x)+1\cdot(-z_y)+1]dxdy = \iint_D 3dxdy = 3\pi$$

例9 $\iint\limits_{\Sigma}\frac{1}{z}(z+x^2+y^2)\mathrm{d}S$，$\Sigma$是曲面 $z=\dfrac{x^2+y^2}{2}$介于$0\le z\le 4$的第一卦限部分.

解 Σ: $z=\dfrac{x^2+y^2}{2}$ $\quad D_{xy}=\{(x,y)\,|\,x^2+y^2\le 8\}$



$$\mathrm{d}S=\sqrt{1+z_x'^2+z_y'^2}\,\mathrm{d}x\mathrm{d}y=\sqrt{1+x^2+y^2}\,\mathrm{d}x\mathrm{d}y$$

$$\therefore I=\iint\limits_{\Sigma}(1+\frac{x^2+y^2}{z})\mathrm{d}S=\iint\limits_{D_{xy}}3\sqrt{1+x^2+y^2}\,\mathrm{d}x\mathrm{d}y$$

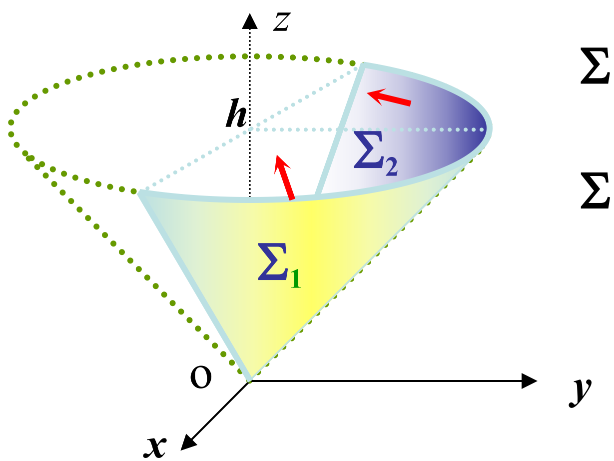
$$\therefore I=\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{2\sqrt{2}}3\sqrt{1+r^2}\cdot r\mathrm{d}r$$

$$=\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{2\sqrt{2}}\frac{3}{2}\sqrt{1+r^2}\,\mathrm{d}(1+r^2)$$

$$=\frac{\pi}{2}\left[(1+r^2)^{\frac{3}{2}}\right]_0^{2\sqrt{2}}=13\pi$$

例10 $\iint\limits_{\Sigma} xy\mathrm{d}y\mathrm{d}z$，$\Sigma$是曲面 $z=\sqrt{x^2+y^2}\ (0\le z\le h)$ 上侧的 $y\ge 0$部分.

解 方法一： Σ由第一卦限和第二卦限中的锥面Σ_1和Σ_2构成.



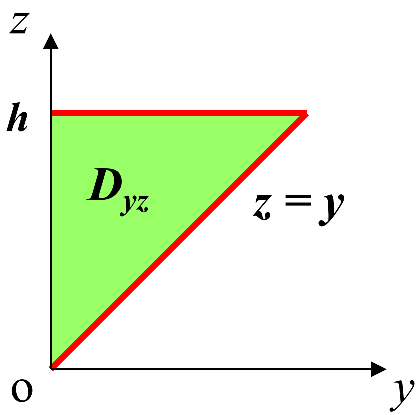
$\Sigma_1: x=\sqrt{z^2-y^2}$ 取后侧;

$\Sigma_2: x=-\sqrt{z^2-y^2}$ 取前侧.

$$\iint\limits_{\Sigma} = \iint\limits_{\Sigma_1} + \iint\limits_{\Sigma_2}$$

$$= -\iint\limits_{D_{yz}} \sqrt{z^2-y^2}\, y\mathrm{d}y\mathrm{d}z + \iint\limits_{D_{yz}} (-\sqrt{z^2-y^2})\, y\mathrm{d}y\mathrm{d}z$$

$$= -2\iint\limits_{D_{yz}} \sqrt{z^2-y^2}\, y\mathrm{d}y\mathrm{d}z$$

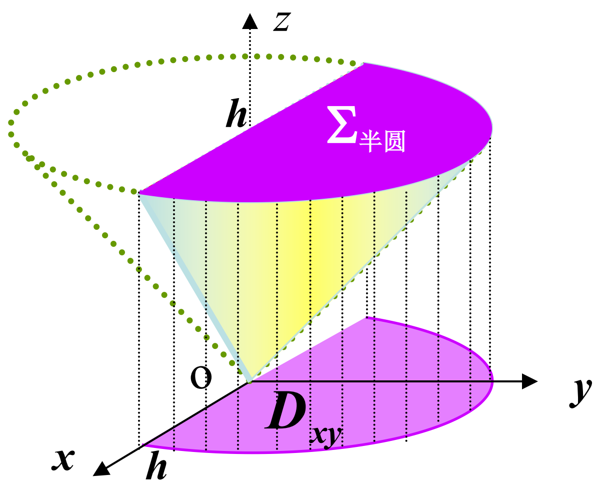


$$\underline{\underline{\text{先 } y}}\ -2\int_0^h \mathrm{d}z \int_0^z y\sqrt{z^2-y^2}\,\mathrm{d}y$$

$$= -\frac{h^4}{6}.$$

例10 $\iint\limits_{\Sigma} xy\mathrm{d}y\mathrm{d}z$，Σ是曲面 $z=\sqrt{x^2+y^2}$ $(0\le z\le h)$ 上侧的 $y\ge 0$部分.

方法二：需贴补侧面 Σ_{Δ}（右侧）和半圆顶面 $\Sigma_{半圆}$（下侧）.



$$\iint\limits_{\Sigma} + \iint\limits_{\Sigma_{\Delta}} + \iint\limits_{\Sigma_{半圆}} = -\iiint\limits_{\Omega} \frac{\partial P}{\partial x}\mathrm{d}v = -\iiint\limits_{\Omega} y\,\mathrm{d}v$$

$$= -\iint\limits_{D_{xy}} y\,\mathrm{d}x\mathrm{d}y\int_{\sqrt{x^2+y^2}}^{h}\mathrm{d}z$$

$$= -\iint\limits_{D_{xy}} y(h-\sqrt{x^2+y^2})\,\mathrm{d}x\mathrm{d}y$$

$$\underline{\underline{极坐标}}\; -\int_0^{\pi}\sin\theta\,\mathrm{d}\theta\int_0^{h} r^2(h-r)\,\mathrm{d}r = -\frac{h^4}{6}.$$

又因 $\iint\limits_{\Sigma_{\Delta}} xy\mathrm{d}y\mathrm{d}z = 0$，$\iint\limits_{\Sigma_{半圆}} xy\mathrm{d}y\mathrm{d}z = 0$，$\therefore \iint\limits_{\Sigma} xy\mathrm{d}y\mathrm{d}z = -\dfrac{h^4}{6}.$

例11 计算$I=\iint\limits_{\Sigma}[f(x,y,z)+x]dydz+[2f(x,y,z)+y]dzdx+[f(x,y,z)+z]dxdy$,

其中 $f(x,y,z)$ 为连续函数,Σ 为平面 $x-y+z=1$在第四卦限部分的上侧.

解 方法一: 利用两类曲面积分之间的关系

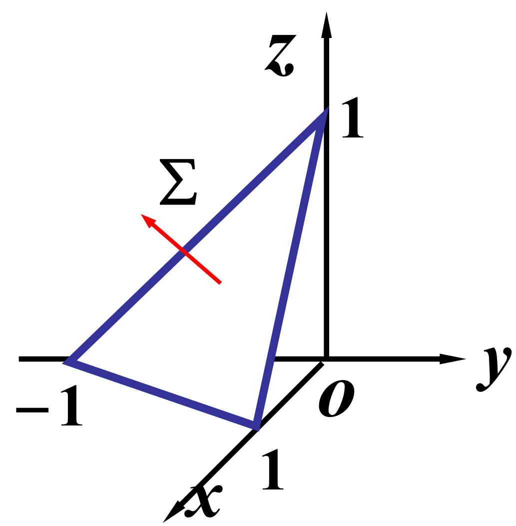
$\because \Sigma$ 的法向量为 $\vec{n}=\{1,-1,1\}$,

$\therefore$法向量的单位余弦为$\cos\alpha=\dfrac{1}{\sqrt{3}},\cos\beta=\dfrac{-1}{\sqrt{3}},\cos\gamma=\dfrac{1}{\sqrt{3}}$.

$$I=\iint\limits_{\Sigma}\{\frac{1}{\sqrt{3}}[f(x,y,z)+x]-\frac{1}{\sqrt{3}}[2f(x,y,z)+y]+\frac{1}{\sqrt{3}}[f(x,y,z)+z]\}dS$$

$$=\frac{1}{\sqrt{3}}\iint\limits_{\Sigma}(x-y+z)dS$$

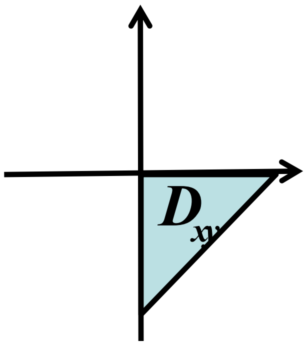
$$=\frac{1}{\sqrt{3}}\iint\limits_{\Sigma}dxdy=\frac{1}{2}.$$



例11 计算$I=\iint\limits_{\Sigma}[f(x,y,z)+x]dydz+[2f(x,y,z)+y]dzdx+[f(x,y,z)+z]dxdy$,

其中 $f(x,y,z)$ 为连续函数,Σ 为平面 $x-y+z=1$在第四卦限部分的上侧.

方法二:也可直接计算(向量点积法)



$$\Sigma: z=1-x+y,(x,y)\in D_{xy},(-z_x,-z_y,1)=(1,-1,1)$$

$$\iint\limits_{\Sigma}[f(x,y,z)+x]dydz+[2f(x,y,z)+y]dzdx+[f(x,y,z)+z]dxdy$$

$$=\iint\limits_{D_{xy}}\{[f(x,y,1-x+y)+x]\cdot 1+[2f(x,y,1-x+y)+y]\cdot(-1)$$

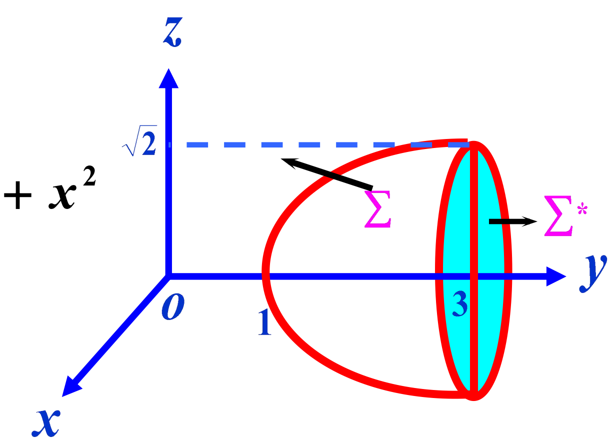
$$+[f(x,y,1-x+y)+1-x+y]\cdot 1\}dxdy$$

$$=\iint\limits_{D_{xy}}1\,dxdy=\frac{1}{2}$$

例12 计算曲面积分$I=\iint\limits_{\Sigma} x(8y+1)dydz+2(1-y^2)dzdx-4yzdxdy$

其中Σ是由$\begin{cases} z=\sqrt{y-1} \\ x=0 \end{cases}$ $(1\le y\le 3)$绕y轴旋转一周所成的曲面，

法向量与y轴正方向夹角大于$\dfrac{\pi}{2}$.



解 $\begin{cases} z=\sqrt{y-1} \\ x=0 \end{cases}$绕$y$轴旋转的曲面为$y-1=z^2+x^2$

补充曲面$\Sigma^*: y=3,\quad D_{zx}: z^2+x^2\le 2$,取右侧

记Σ和Σ^*所围区域为Ω, 则由Gauss公式

记$P=x(8y+1), Q=2(1-y^2), R=-4yz$,则$\dfrac{\partial P}{\partial x}+\dfrac{\partial Q}{\partial y}+\dfrac{\partial R}{\partial z}=1$

$$\iint\limits_{\Sigma+\Sigma^*}=\iiint\limits_{\Omega} dxdydz=\iint\limits_{z^2+x^2\le 2} dxdz\int_{1+z^2+x^2}^{3} dy=\int_0^{2\pi} d\theta\int_0^{\sqrt{2}} rdr\int_{1+r^2}^{3} dy=2\pi$$

$$\iint\limits_{\Sigma^*} x(8y+1)dydz+2(1-y^2)dzdx-4yzdxdy=\iint\limits_{z^2+x^2\le 2}(-16)dzdx=-32\pi$$

所以$I=\iint\limits_{\Sigma+\Sigma^*}-\iint\limits_{\Sigma^*}=2\pi-(-32\pi)=34\pi$

例13 计算 $\iint\limits_{\Sigma}\dfrac{x\cos\alpha+y\cos\beta+z\cos\gamma}{(x^2+y^2+z^2)^{3/2}}dS$，其中Σ是曲面 $\dfrac{(x-1)^2}{9}+\dfrac{(y-2)^2}{16}+\dfrac{(z-3)^2}{25}=1$的外侧，$\cos\alpha,\cos\beta,\cos\gamma$ 是其外法线向量的方向余弦.

解 对充分小的$\varepsilon>0$，记$\Sigma_1: x^2+y^2+z^2=\varepsilon^2$，外侧，使$\Sigma_1$位于Σ的内区域中，记Ω为Σ与$\Sigma_1$所围有界闭区域，则

$$I=\iint\limits_{\Sigma}\frac{x\cos\alpha+y\cos\beta+z\cos\gamma}{(x^2+y^2+z^2)^{3/2}}dS$$

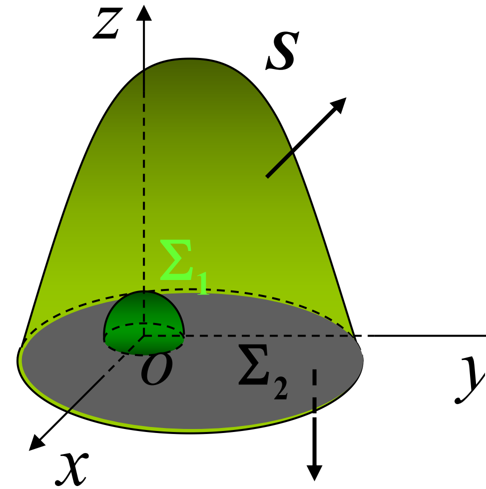
$$=\iint\limits_{\Sigma-\Sigma_1}\frac{xdydz+ydzdx+zdxdy}{(x^2+y^2+z^2)^{3/2}}+\iint\limits_{\Sigma_1}\frac{xdydz+ydzdx+zdxdy}{(x^2+y^2+z^2)^{3/2}}$$

$$=\iiint\limits_{\Omega}0dV+\frac{1}{\varepsilon^3}\iint\limits_{\Sigma_1}xdydz+ydzdx+zdxdy$$

$$=\frac{1}{\varepsilon^3}\iiint\limits_{x^2+y^2+z^2\le\varepsilon^2}3dV=4\pi$$

例14 计算曲面积分 $I=\iint\limits_{S}\dfrac{xdydz+ydzdx+zdxdy}{(x^2+y^2+z^2)^{\frac{3}{2}}}$，其中 S 是

$1-\dfrac{z}{7}=\dfrac{(x-2)^2}{25}+\dfrac{(y-1)^2}{16}\,(z\geq 0)$，取上侧.



解 取 Σ_1: $x^2+y^2+z^2=1,\quad z\geq 0$，下侧，

Σ_2: $z=0,\ \dfrac{(x-2)^2}{25}+\dfrac{(y-1)^2}{16}\leq 1,\quad x^2+y^2\geq 1$，下侧，

Σ_3: $z=0,\ x^2+y^2\leq 1$，下侧

$$I=\iint\limits_{S+\Sigma_1+\Sigma_2}\frac{xdydz+ydzdx+zdxdy}{(x^2+y^2+z^2)^{\frac{3}{2}}}-\iint\limits_{\Sigma_1}\frac{xdydz+ydzdx+zdxdy}{(x^2+y^2+z^2)^{\frac{3}{2}}}-\iint\limits_{\Sigma_2}\frac{xdydz+ydzdx+zdxdy}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

S 与 Σ_1,Σ_2 所围区域记为 Ω，边界曲面整体取外侧，则由Gauss公式

$$\iint\limits_{S+\Sigma_1+\Sigma_2}\frac{xdydz+ydzdx+zdxdy}{(x^2+y^2+z^2)^{\frac{3}{2}}}=\iiint\limits_{\Omega}\frac{3(x^2+y^2+z^2)^{\frac{3}{2}}-3(x^2+y^2+z^2)(x^2+y^2+z^2)^{\frac{1}{2}}}{(x^2+y^2+z^2)^3}dxdydz=0$$

$$\iint_{\Sigma_1} \frac{xdydz + ydzdx + zdxdy}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \iint_{\Sigma_1} xdydz + ydzdx + zdxdy$$

$$= -\iint_{-\Sigma_1+\Sigma_3} xdydz + ydzdx + zdxdy + \iint_{\Sigma_3} xdydz + ydzdx + zdxdy$$

$$= -3\iiint_{\Omega_1} dxdydz + 0 = -2\pi$$

其中Ω_1是Σ_2, Σ_3所围区域, 边界曲面整体取外侧.

$$\iint_{\Sigma_2} \frac{xdydz + ydzdx + zdxdy}{(x^2+y^2+z^2)^{\frac{3}{2}}} = 0$$

所以$I = 2\pi$

典型例题---综合题

例15 设$f(u)$为连续函数，L为平面上逐段光滑的任意闭曲线，求证

$$\oint_L f(x^2+y^2)(xdx+ydy)=0.$$

证明 $f(u)$为连续函数，则$F(u)=\int_0^u f(t)dt$可导，且$F'(u)=f(u)$

从而$\dfrac{\partial F(x^2+y^2)}{\partial x}=2xf(x^2+y^2)$，$\dfrac{\partial F(x^2+y^2)}{\partial y}=2yf(x^2+y^2)$，

即$d(\dfrac{1}{2}F(x^2+y^2))=f(x^2+y^2)(xdx+ydy)$，

由积分与路径无关的等价结论，$\oint_L f(x^2+y^2)(xdx+ydy)=0.$

例16 设$f(x,y)\in C^2(D)$，$D: x^2+y^2\le 1$，且$\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=e^{-(x^2+y^2)}$

求 $\iint\limits_D \left(x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}\right)\mathrm{d}x\mathrm{d}y$.

解

$$\iint\limits_D \left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)\mathrm{d}x\mathrm{d}y=\int_0^{2\pi}\mathrm{d}\theta\int_0^1\left(r\cos\theta\cdot\frac{\partial f}{\partial x}+r\sin\theta\cdot\frac{\partial f}{\partial y}\right)r\mathrm{d}r$$

$$=\int_0^1\left[\int_0^{2\pi}\left(r\cos\theta\cdot\frac{\partial f}{\partial x}+r\sin\theta\cdot\frac{\partial f}{\partial y}\right)\mathrm{d}\theta\right]r\mathrm{d}r$$

$$\int_0^{2\pi}\left(r\cos\theta\cdot\frac{\partial f}{\partial x}+r\sin\theta\cdot\frac{\partial f}{\partial y}\right)\mathrm{d}\theta=\oint\limits_{x^2+y^2=r^2}\left(-\frac{\partial f}{\partial y}dx+\frac{\partial f}{\partial x}\mathrm{d}y\right)$$

由格林公式

$$\iint\limits_D \left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)\mathrm{d}x\mathrm{d}y=\int_0^1 r\left[\iint\limits_{x^2+y^2\le r^2}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)\mathrm{d}x\mathrm{d}y\right]\mathrm{d}r$$

$$=\int_0^1 r\left[\iint\limits_{x^2+y^2\le r^2}e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y\right]\mathrm{d}r=\int_0^1 r\left(\int_0^{2\pi}\mathrm{d}\theta\int_0^r e^{-\rho^2}\cdot\rho\mathrm{d}\rho\right)\mathrm{d}r=\frac{\pi}{2e}.$$

例17 设$f(x,y)$及其二阶偏导在全平面上连续, 且$f(0,0)=0$,

$$\left|\frac{\partial f}{\partial x}\right| \le 2|x-y|, \left|\frac{\partial f}{\partial y}\right| \le 2|x-y|,$$ 求证 $|f(5,4)| \le 1.$

证明 因为$f(x,y)$的二阶偏导连续，所以$\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$,

可得曲线积分$\displaystyle\int_L \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$与路径无关,且$df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy.$

设$O(0,0), A(4,4), B(5,4)$,则在直线OA: $y=x$上, $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$

$$从而 f(5,4) - f(0,0) = \int_{(0,0)}^{(5,4)} \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$= \int_{OA} \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \int_{AB} \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

$$= 0 + \int_4^5 \frac{\partial f(x,4)}{\partial x}dx \le \int_4^5 2|x-4|dx = 1$$

例18 假设L为逆时针方向的封闭光滑曲线, D为L所围区域, u具有连续的二阶偏导数,$\vec{n}$为L外法线的单位向量, 证明

$$\iint_D [(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2]\, dxdy = -\iint_D u\, \Delta u\, dxdy + \oint_L u \frac{\partial u}{\partial \vec{n}} ds.$$

解 记$\vec{n} = (\cos(\vec{n}, x), \cos(\vec{n}, y))$,其中$(\vec{n}, x), (\vec{n}, y)$分别表示$\vec{n}$与$x, y$轴的夹角,

又因为u具有一阶连续偏导数,所以

$$\frac{\partial u}{\partial \vec{n}} = \frac{\partial u}{\partial x}\cos(\vec{n}, x) + \frac{\partial u}{\partial y}\cos(\vec{n}, y)$$

设逆时针方向曲线的单位切向量为$\vec{\tau} = (\cos(\vec{\tau}, x), \cos(\vec{\tau}, y))$,则

$$(\vec{n}, x) = (\vec{\tau}, y), (\vec{n}, y) = \pi - (\vec{\tau}, x)$$

则$\cos(\vec{n}, x) = \cos(\vec{\tau}, y), \cos(\vec{n}, y) = -\cos(\vec{\tau}, x)$

因此$\oint_L u \frac{\partial u}{\partial \vec{n}} ds = \oint_L u[\frac{\partial u}{\partial x}\cos(\vec{n}, x) + \frac{\partial u}{\partial y}\cos(\vec{n}, y)]ds$

$$\oint_L u\frac{\partial u}{\partial \vec{n}}ds = \oint_L [u\frac{\partial u}{\partial x}\cos(\vec{\tau}, y) - u\frac{\partial u}{\partial y}\cos(\vec{\tau}, x)]ds$$

$$= \oint_L u\frac{\partial u}{\partial x}dy - u\frac{\partial u}{\partial y}dx$$

$$= \iint_D [\frac{\partial}{\partial x}(u\frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(u\frac{\partial u}{\partial y})]dxdy \quad (Green\text{公式})$$

$$= \iint_D [(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 + u\,(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})]\,dxdy$$

$$= \iint_D [(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2]\,dxdy + \iint_D u\,\Delta u\,dxdy$$

问题得证.

假设L为逆时针方向的封闭光滑曲线, D为L所围区域, u, v具有连续的二阶偏导数, $\vec{n}$为L外法线的单位向量, 则

$$(1)\iint_D \left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dxdy=-\iint_D u\,\Delta u\,dxdy+\oint_L u\frac{\partial u}{\partial \vec{n}}ds.$$

Green第一公式

$$(2)\iint_D \begin{vmatrix} \Delta u & \Delta v \\ u & v \end{vmatrix} dxdy=\oint_L \begin{vmatrix} \dfrac{\partial u}{\partial \vec{n}} & \dfrac{\partial v}{\partial \vec{n}} \\ u & v \end{vmatrix} ds$$

Green第二公式

(3)若u为区域D上的调和函数, $r=\sqrt{(\xi-x)^2+(\eta-y)^2}$为$(x, y)$与$L$上动点$(\xi,\eta)$之间的距离, 则

$$u(x, y)=\frac{1}{2\pi}\oint_L \left(u\frac{\partial \ln r}{\partial \vec{n}}-\ln r\frac{\partial u}{\partial \vec{n}}\right)ds$$

Green第三公式

例19 设Σ是分片光滑的闭曲面,$\vec{n}$为Σ的单位外法向量,证明

$$I=\oiint_{\Sigma}\begin{vmatrix}\cos(\vec{n},x) & \cos(\vec{n},y) & \cos(\vec{n},z)\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}dS=0$$

在下面两种情况下都成立

(1)P,Q,R在$\overline{\Omega}$上二阶连续可微,Ω是Σ所围的立体;

(2)P,Q,R在Σ上一节连续可微.

证明 (1)由Gauss公式

$$I=\oiint_{\Sigma}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)dydz+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)dzdx+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

$$=\iiint_{\Omega}\left[\frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\right]dxdydz$$

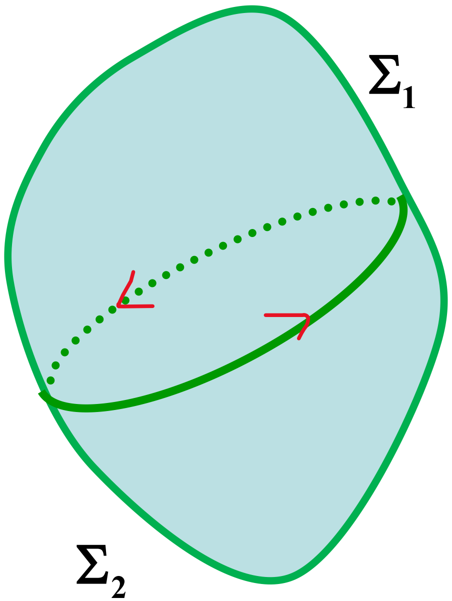
$$=0$$

(2)在Σ上任取一条分段光滑的封闭曲线Γ,Γ将Σ分成Σ_1,Σ_2,在Σ_1,Σ_2上分别应用Stokes公式,可得

$$I = \left(\iint_{\Sigma_1} + \iint_{\Sigma_2}\right) \begin{vmatrix} \cos(\vec{n}, x) & \cos(\vec{n}, y) & \cos(\vec{n}, z) \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

$$= \int_{\Gamma} Pdx + Qdy + Rdz + \int_{-\Gamma} Pdx + Qdy + Rdz$$

$$= 0$$



感谢大家