§15.2 链式法则

链式法则

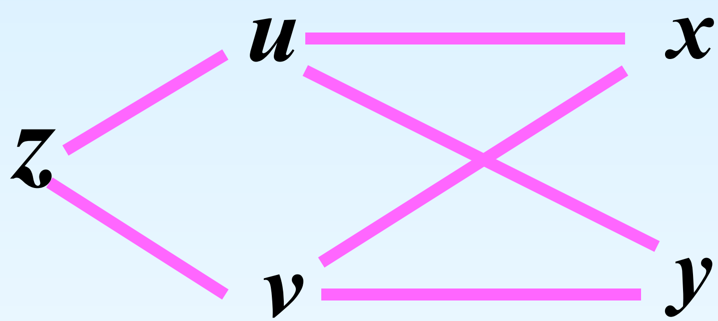
定理 如果 $u=\varphi(x,y)$ 及 $v=\psi(x,y)$ 都在点 (x,y) 具有对 x 和 y 的偏导数，且函数 $z=f(u,v)$ 在对应点 (u,v) 可微，则复合函数 $z=f[\varphi(x,y),\psi(x,y)]$ 在对应点 (x,y) 的两个偏导数存在，且可用下列公式计算

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x},\qquad \frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}.$$

链式法则

连线相乘
分线相加

链式法则如图示



$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x},\quad \frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

公式中的 $\frac{\partial f}{\partial u},\frac{\partial f}{\partial v}$ 也可以记为 f_1, f_2.

证 固定y,设x有改变量Δx,

则 $\Delta u = u(x+\Delta x, y) - u(x, y)$,

$\Delta v = v(x+\Delta x, y) - v(x, y)$,

从而$\Delta z = f(u+\Delta u, v+\Delta v) - f(u, v)$

$$= \frac{\partial f}{\partial u}\cdot\Delta u + \frac{\partial f}{\partial v}\cdot\Delta v + o(\rho), \quad \rho = \sqrt{(\Delta u)^2 + (\Delta v)^2}.$$

$$\frac{\Delta z}{\Delta x} = \frac{\partial f}{\partial u}\cdot\frac{\Delta u}{\Delta x} + \frac{\partial f}{\partial v}\cdot\frac{\Delta v}{\Delta x} + \frac{o(\rho)}{\Delta x}.$$

由于u,v关于x,y的偏导数存在,且

$$\lim_{\Delta x\to 0}\left|\frac{o(\rho)}{\Delta x}\right| = \lim_{\Delta x\to 0}\left|\frac{o(\rho)}{\rho}\right|\left|\frac{\rho}{\Delta x}\right|$$

$$= \lim_{\Delta x\to 0}\left|\frac{o(\rho)}{\rho}\right| \lim_{\Delta x\to 0}\sqrt{\left(\frac{\Delta u}{\Delta x}\right)^2 + \left(\frac{\Delta v}{\Delta x}\right)^2} = 0.$$

所以

$$\frac{\partial z}{\partial x} = \lim_{\Delta x\to 0}\frac{\Delta z}{\Delta x} = \frac{\partial f}{\partial u}\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} + \frac{\partial f}{\partial v}\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x} = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

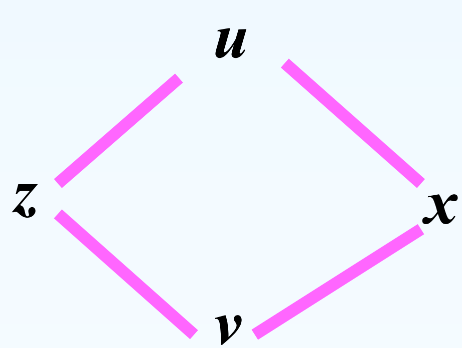
同理可证 $\dfrac{\partial z}{\partial y} = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y} + \dfrac{\partial f}{\partial v}\dfrac{\partial v}{\partial y}.$

除定理中的复合函数情形外，我们还会遇到其他的复合关系，例如，

1、设函数$u=\varphi(x),v=\psi(x)$在点x可导，且函数$f(u,v)$在对应点(u,v)可微，则复合函数$z=f(\varphi(x),\psi(x))$在x可导，且有公式

$$\frac{dz}{dx}=\frac{\partial f}{\partial u}\cdot\frac{du}{dx}+\frac{\partial f}{\partial v}\cdot\frac{dv}{dx},$$

$\frac{dz}{dx}$称为全导数.



说明 定理中函数$z=f(u,v)$的可微性不能省略.

例如 $z=f(x,y)=\begin{cases}\dfrac{x^2y}{x^2+y^2}, & x^2+y^2\neq 0\\ 0\quad, & x^2+y^2=0\end{cases}$

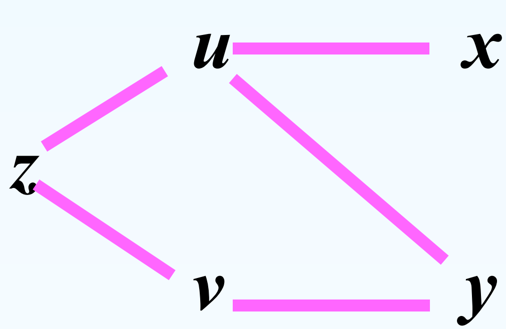
$f_x(0,0)=f_y(0,0)=0$, $f(x,y)$ 在 $(0,0)$不可微.

如 $x=t, y=t$ 有 $z=F(t)=f(t,t)=\dfrac{t}{2}$, $\dfrac{dz}{dt}=\dfrac{1}{2}$,

$$\left.\frac{dz}{dt}\right|_{t=0}=\left.\frac{\partial z}{\partial x}\right|_{(0,0)}\left.\frac{dx}{dt}\right|_{t=0}+\left.\frac{\partial z}{\partial y}\right|_{(0,0)}\left.\frac{dy}{dt}\right|_{t=0}=0.$$

2、若函数$u=\phi(x,y)$在点(x,y)关于x和y的偏导数存在，$v=\psi(y)$在点y可导，且函数$z=f(u,v)$在对应点(u,v)可微，则复合函数$z=f(\varphi(x,y),\psi(y))$在点$(x,y)$的两个偏导数存在，且有公式

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x},\quad \frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\psi'(y).$$



特别地，若 $z = f(\varphi(x, y), y)$,则

$$\frac{\partial z}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial y}.$$

此处 $\boxed{\frac{\partial z}{\partial y} \neq \frac{\partial f}{\partial y}}.$

$\frac{\partial z}{\partial y}$是复合函数 $z = f(u(x, y), y)$中把x看作是常数时关于y的偏导数, $\frac{\partial f}{\partial y}$是在函数 $z = f(u, y)$中把u看作常数时关于y的偏导数.

3. 如果 $u=\varphi(x,y)$ 及 $v=\psi(x,y)$ 都在点 (x,y) 可微

且函数 $z=f(u,v)$ 在对应点 (u,v) 可微，则

$z=f[\varphi(x,y),\psi(x,y)]$ 在对应点 (x,y) 可微,且有

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x},\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

4.此定理可推广到 n 元函数.

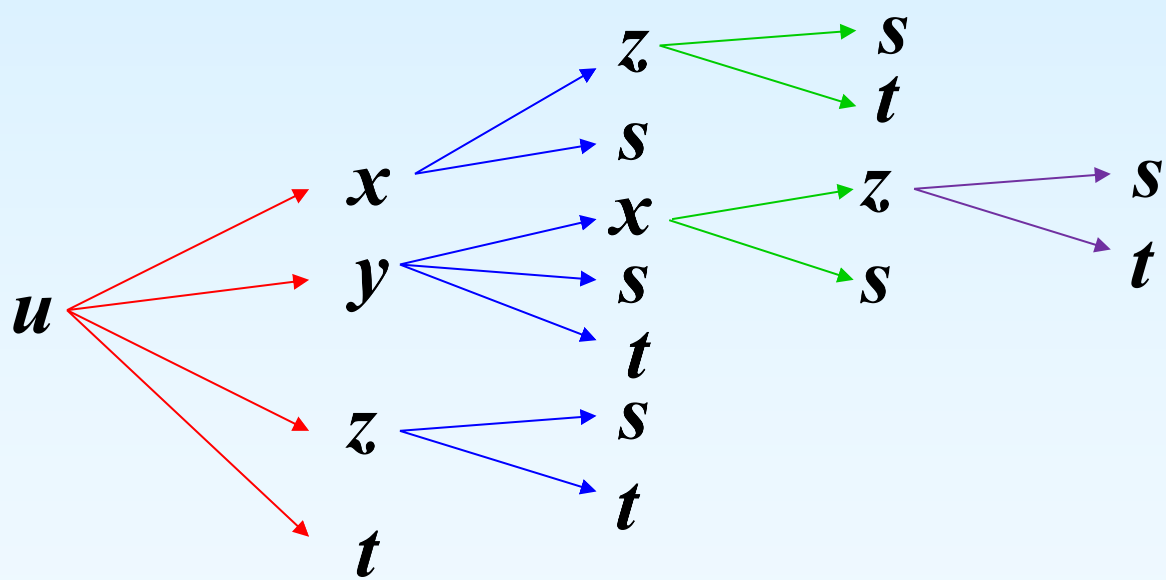
$f(u_1,u_2,\ldots\ldots u_m)$在$(u_1,u_2,\ldots\ldots u_m)$可微,

$u_k(x_1,x_2,\ldots\ldots x_n),k=1,2,3\ldots\ldots,m$在

$(x_1,x_2,\ldots\ldots x_n)$可微

$$\Rightarrow \frac{\partial f}{\partial x_i}=\sum_{k=1}^{m}\frac{\partial f}{\partial u_k}\frac{\partial u_k}{\partial x_i},i=1,2,3,\ldots\ldots n.$$

$$u = f(x, y, z, t), x = \varphi(z, s), y = \psi(x, s, t), z = \omega(s, t)$$



$$u_s = \frac{\partial f}{\partial x}\left[\frac{\partial \varphi}{\partial z}\frac{\partial \omega}{\partial s} + \frac{\partial \varphi}{\partial s}\right] + \frac{\partial f}{\partial y}\left[\frac{\partial \psi}{\partial x}\frac{\partial \varphi}{\partial z}\frac{\partial \omega}{\partial s} + \frac{\partial \psi}{\partial x}\frac{\partial \varphi}{\partial s} + \frac{\partial \psi}{\partial s}\right] + \frac{\partial f}{\partial z}\frac{\partial \omega}{\partial s}$$

$$u_t = \frac{\partial f}{\partial x}\frac{\partial \varphi}{\partial z}\frac{\partial \omega}{\partial t} + \frac{\partial f}{\partial y}\left[\frac{\partial \psi}{\partial x}\frac{\partial \varphi}{\partial z}\frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial t}\right] + \frac{\partial f}{\partial z}\frac{\partial \omega}{\partial t} + \frac{\partial f}{\partial t}$$

例 1 设 $z = e^u \sin v$，而 $u = xy$，$v = x + y$，

求 $\dfrac{\partial z}{\partial x}$ 和 $\dfrac{\partial z}{\partial y}$.

解

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = e^u \sin v \cdot y + e^u \cos v \cdot 1$$

$$= e^{xy}(y\sin(x+y) + \cos(x+y)),$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y} = e^u \sin v \cdot x + e^u \cos v \cdot 1$$

$$= e^{xy}(x\sin(x+y) + \cos(x+y)),$$

例 2 设 $z = uv + \sin t$，而 $u = e^t$， $v = \cos t$，

求 $\dfrac{dz}{dt}$.

解

$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\cdot\frac{du}{dt} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dt} + \frac{\partial z}{\partial t}$$

$$= ve^t - u\sin t + \cos t$$

$$= e^t\cos t - e^t\sin t + \cos t$$

$$= e^t(\cos t - \sin t) + \cos t.$$

例 3 设 $z = f(xy, x^2 + y^2), y = \varphi(x), f, \varphi$ 具有连续偏导数，求 $\dfrac{dz}{dx}$.

解 $u = xy = x\varphi(x),\ \ v = x^2 + y^2 = x^2 + \varphi^2(x),$

$$\frac{dz}{dx} = f_1 \cdot \frac{\partial u}{\partial x} + f_2 \cdot \frac{\partial v}{\partial x}$$

$$= f_1 \cdot (\varphi(x) + x\varphi'(x)) + f_2 \cdot (2x + 2\varphi(x)\varphi'(x))$$

全微分形式不变性

设函数$z=f(u,v)$具有连续偏导数，则有全微分

$$dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv;$$

当 $u=\varphi(x,y), v=\psi(x,y)$ 时，有

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv.$$

全微分形式不变性的实质：

无论z是自变量u、v的函数或中间变量u、v的函数，它的全微分形式是一样的.

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= \left(\frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} \right) dx + \left(\frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} \right) dy$$

$$= \frac{\partial z}{\partial u} \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right) + \frac{\partial z}{\partial v} \left(\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy \right)$$

$$= \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv.$$

多元函数全微分运算法则（设u,v为多元可微函数，f为一元可微函数）：

$$(1)\quad d(u \pm v) = du \pm dv;$$

$$(2)\quad d(ku) = kdu \quad (k\text{为常数});$$

$$(3)\quad d(uv) = udv + vdu;$$

$$(4)\quad d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2};$$

$$(5)\quad df(u) = f'(u)du.$$

例 4 已知$e^{-xy}-2z+e^{z}=0$，求$\dfrac{\partial z}{\partial x}$和$\dfrac{\partial z}{\partial y}$.

解 $\because d(e^{-xy}-2z+e^{z})=0,$

$\therefore e^{-xy}d(-xy)-2dz+e^{z}dz=0,$

$(e^{z}-2)dz=e^{-xy}(xdy+ydx)$

$$dz=\frac{ye^{-xy}}{(e^{z}-2)}dx+\frac{xe^{-xy}}{(e^{z}-2)}dy$$

$$\frac{\partial z}{\partial x}=\frac{ye^{-xy}}{e^{z}-2},\qquad \frac{\partial z}{\partial y}=\frac{xe^{-xy}}{e^{z}-2}.$$