§15.4 隐函数定理

显函数 $y = 2x, \qquad z = x^2 + y^2.$

隐函数 $F: X \times Y \to R, \quad F(x, y) = 0$

如对于 $\forall x \in I \subset X$, 恒有唯一确定的 $y \in J \subset Y$, 它与 x 一起满足 $F(x, y) = 0$, 就称 $F(x, y) = 0$ 确定了一个定义在 I 上, 值域含于 J 的隐函数 .

单个方程的隐函数定理

定理4. 1 设$D \subset R^2$为开集,若函数F: $D \to R$满足：

(i) F连续且具有连续偏导数 ；

(ii) $F(x_0, y_0) = 0;$

(iii) $F_y(x_0, y_0) \neq 0.$

则在D中存在一个包含(x_0, y_0)的开矩形$I \times J$, 使得

$a)$任给$x \in I = (x_0 - \alpha, x_0 + \alpha)$, 存在唯一的$y = f(x) \in J$,

满足$F(x, f(x)) = 0$和 $y_0 = f(x_0)$;

$b)$ $f(x)$在I内连续;

$c)$ $y = f(x)$在I上有连续导数,且$\dfrac{dy}{dx} = -\dfrac{F_x(x, y)}{F_y(x, y)}.$

证明 不妨设$F_y(x_0,y_0)>0$.

1. 先证隐函数$y=f(x)$的存在性和唯一性

由条件(i), F_y在D内连续, 由连续函数的局部保号 性

存在$(x_0-\beta,x_0+\beta)\times(y_0-\beta,y_0+\beta)\subset D$,使得此矩形

内$F_y(x,y)>0$.

$F(x,y)$作为y的一元函数,在$[y_0-\beta,y_0+\beta]$上严格单调增

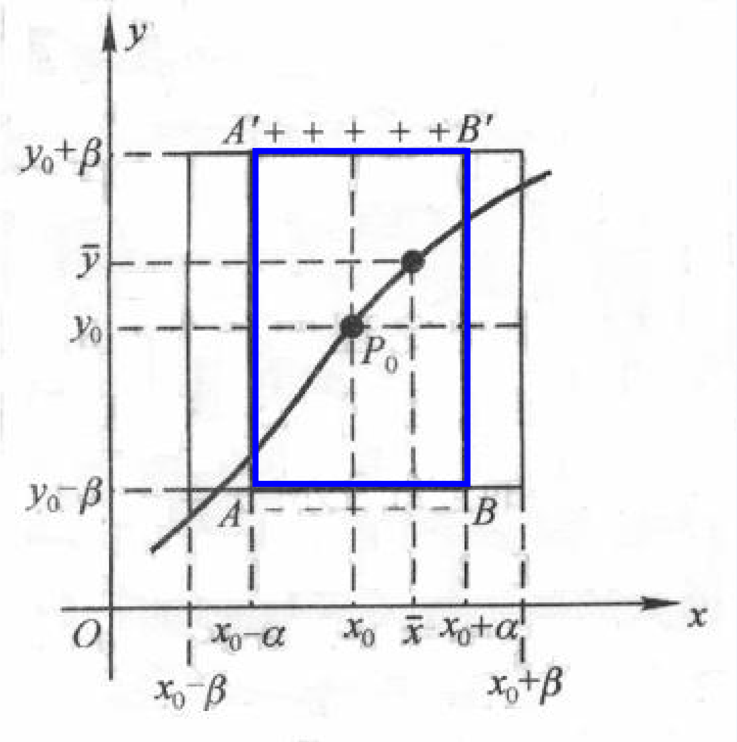
$\therefore$由$F(x_0,y_0)=0\Rightarrow F(x_0,y_0-\beta)<0,\ \ F(x_0,y_0+\beta)>0$

$\because F(x, y_0 - \beta)$与$F(x, y_0 + \beta)$在$[x_0 - \beta, x_0 + \beta]$上连续

由保号性, 存在 $\alpha > 0(\alpha \le \beta)$, 当 $x \in (x_0 - \alpha, x_0 + \alpha)$时,
恒有$F(x, y_0 - \beta) < 0$, $F(x, y_0 + \beta) > 0$.

即在矩形$ABB'A'$的边AB上
F取负值,边$A'B'$上F取正值.

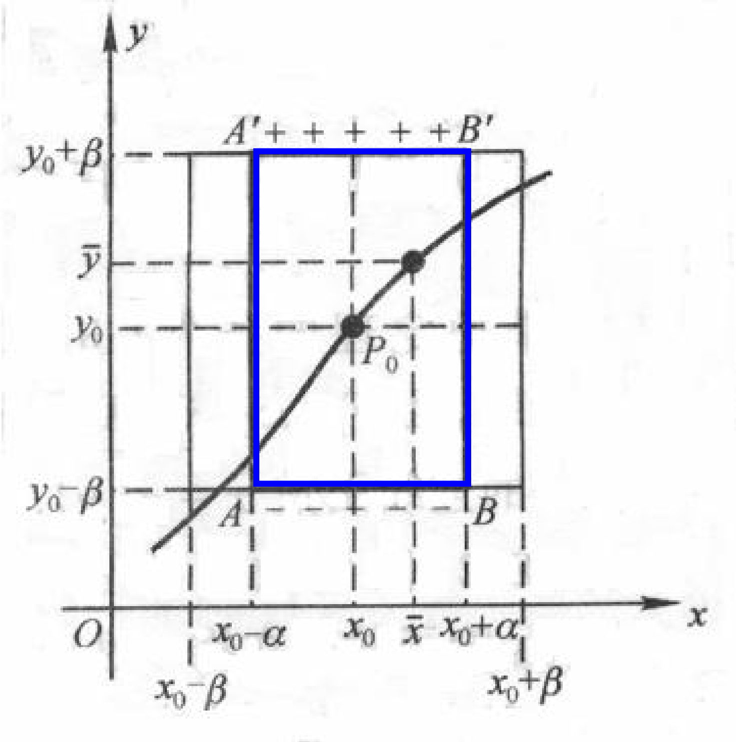
$\therefore$ 对$\forall\ \bar{x} \in (x_0 - \alpha, x_0 + \alpha)$,
$F(\bar{x}, y)$在$[y_0 - \beta, y_0 + \beta]$上
严格增且连续



$$F(\bar{x}, y_0 - \beta) < 0, F(\bar{x}, y_0 + \beta) > 0,$$

$\therefore$ 介值定理 $\Rightarrow$ 存在唯一的 $\bar{y} \in (y_0 - \beta, y_0 + \beta)$,

使得 $F(\bar{x}, \bar{y}) = 0$.



由 $\bar{x}$ 的任意性, 确定了一个定义域为 $(x_0 - \alpha, x_0 + \alpha)$, 值域含于 $(y_0 - \beta, y_0 + \beta)$ 的隐函数 $y = f(x)$.

2. 再证 $y = f(x)$ 的连续性

对 $\forall\, \bar{x} \in (x_0 - \alpha, x_0 + \alpha), \bar{y} = f(\bar{x})$,

$$y_0 - \beta < \bar{y} < y_0 + \beta.$$

$\forall\, \varepsilon > 0$, 且 $\varepsilon < \min\{y_0 + \beta - \bar{y}, \bar{y} - y_0 + \beta\}$,

使得 $y_0 - \beta \le \bar{y} - \varepsilon < \bar{y} + \varepsilon \le y_0 + \beta$.

从而 $F(\bar{x}, \bar{y} - \varepsilon) < 0, F(\bar{x}, \bar{y} + \varepsilon) > 0$.

由保号性, 存在 $\bar{x}$ 的某邻域 $(\bar{x} - \delta, \bar{x} + \delta)$

$\subset (x_0 - \alpha, x_0 + \alpha)$, 使得$x$属于该邻域时,

$F(x, \bar{y} - \varepsilon) < 0, F(x, \bar{y} + \varepsilon) > 0$.

因此存在唯一的y,使得$F(x, y) = 0, |y - \bar{y}| < \varepsilon$,

由 y 的唯一性, $y = f(x)$.

即：$\forall \varepsilon > 0, \exists \delta > 0$，当 $|x - \bar{x}| < \delta$ 时，

$$\left| f(x) - f(\bar{x}) \right| < \varepsilon.$$

进而 $y = f(x)$ 在 $(x_0 - \alpha, x_0 + \alpha)$ 上连续.

3. 证明 $y = f(x)$ 的可导性

设 $x, x + \Delta x \in (x_0 - \alpha, x_0 + \alpha)$，则

$$y = f(x),\ \ y + \Delta y = f(x + \Delta x) \in (y_0 - \beta, y_0 + \beta).$$

$$F(x, y) = 0, F(x + \Delta x, y + \Delta y) = 0.$$

由F_x和F_y的连续性及二元函数的 中值定理知：

$$0 = F(x+\Delta x, y+\Delta y) - F(x,y)$$

$$= F_x(x+\theta\Delta x, y+\theta\Delta y)\Delta x + F_y(x+\theta\Delta x, y+\theta\Delta y)\Delta y$$

其中 $0<\theta<1.$

$$\frac{\Delta y}{\Delta x} = -\frac{F_x(x+\theta\Delta x, y+\theta\Delta y)}{F_y(x+\theta\Delta x, y+\theta\Delta y)}$$

所以

$$f'(x) = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = -\frac{F_x(x,y)}{F_y(x,y)}$$

且 $f'(x)$在 $(x_0-\alpha, x_0+\alpha)$内连续.

注 1. 定理中的条件仅仅是充分的.

例如: $y^3 - x^3 = 0$, 在点(0,0)不满足(iii),

但仍能确定惟一的连续函数 $y = x$.

2. 若将条件(iii)改为 $F_x(x_0, y_0) \neq 0$, 定理

结论变成存在惟一的连续函数 $x = g(y)$.

3. 若方程 $F(x, y) = 0$ 存在连续可微隐函数, 则

也可利用复合函数求导求隐函数的导数:

对方程两边关于 x 求导得 $F_x(x, y) + F_y(x, y)y' = 0$

解方程得到 $y' = -\dfrac{F_x(x, y)}{F_y(x, y)}$

隐函数求导
常用方法

例 1 验证方程$x^2+y^2-1=0$在点$(0,1)$的某邻域内能唯一确定一个具有连续导数、且$f(0)=1$的隐函数$y=f(x)$，并求这函数的一阶和二阶导数在$x=0$的值.

解 令 $F(x,y)=x^2+y^2-1$

则 $F_x=2x,\quad F_y=2y,$

$$F(0,1)=0,\qquad F_y(0,1)=2\neq 0,$$

依定理知方程$x^2+y^2-1=0$在点$(0,1)$的某邻域内能唯一确定一个具有连续导数的函数$y=f(x)$,且$f(0)=1$.

函数的一阶和二阶导数为

$$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{x}{y},\qquad \left.\frac{dy}{dx}\right|_{x=0}=0,$$

$$\frac{d^2y}{dx^2}=-\frac{y-xy'}{y^2}=-\frac{y-x\left(-\frac{x}{y}\right)}{y^2}=-\frac{1}{y^3},$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0}=-1.$$

例 2 已知$\ln\sqrt{x^2+y^2}=\arctan\frac{y}{x}$，求$\frac{dy}{dx}$.

解 令 $F(x,y)=\ln\sqrt{x^2+y^2}-\arctan\frac{y}{x}$,

则 $F_x(x,y)=\frac{x+y}{x^2+y^2}$, $F_y(x,y)=\frac{y-x}{x^2+y^2}$,

当$y-x\neq 0$时, $\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{x+y}{y-x}$.

定理4. 2 设$D\subset R^{n+1}$为开集,若函数F：$D\to R$满足：

(i) $F(x_1,\cdots,x_n,y)$连续且具有连续偏导数；

(ii) 在点$(x_1^0,\cdots,x_n^0,y^0)$处有$F(x_1^0,\cdots,x_n^0,y^0)=0$;

(iii) $F_y(x_1^0,\cdots,x_n^0,y^0)\neq 0$,

则在D中存在一个包含$(x_1^0,\cdots,x_n^0,y^0)$的邻域$G\times J$,其中$G\subset R^n$是包含$(x_1^0,\cdots,x_n^0)$的邻域，使得

$a)$任给$(x_1,\cdots,x_n)\in G$，$F(x_1,\cdots,x_n,y)=0$在J中唯一确定隐函数$y=f(x_1,\cdots,x_n)$,满足

$F(x_1,\cdots,x_n,f(x_1,\cdots,x_n))=0$和 $y^0=f(x_1^0,\cdots,x_n^0)$;

$b)\ y = f(x_1, \cdots, x_n)$在$G$上连续;

$c)\ y = f(x_1, \cdots, x_n)$在$G$上有连续偏导数,且

$$\frac{\partial y}{\partial x_i} = -\frac{F_{x_i}(x_1, \cdots, x_n, y)}{F_y(x_1, \cdots, x_n, y)}, \quad i = 1, \cdots, n.$$

例 3 设$x^2+y^2+z^2-4z=0$，求$\dfrac{\partial^2 z}{\partial x^2}$.

解 将方程两端对x求偏导:$2x+(2z-4)\dfrac{\partial z}{\partial x}=0.\quad(1)$

$$\Rightarrow \frac{\partial z}{\partial x}=\frac{x}{2-z},z\neq 2.$$

二阶偏导求解方法1

将等式(1)两端再对x求偏导:$1+\left(\dfrac{\partial z}{\partial x}\right)^2+(z-2)\dfrac{\partial^2 z}{\partial x^2}=0.$

将$\dfrac{\partial z}{\partial x}$代入此式，可得$\dfrac{\partial^2 z}{\partial x^2}=\dfrac{x^2+(2-z)^2}{(2-z)^3}.$

二阶偏导求解方法2

在$\dfrac{\partial z}{\partial x}=\dfrac{x}{2-z}$两边对$x$求偏导，

$$\frac{\partial^2 z}{\partial x^2}=\frac{(2-z)+x\dfrac{\partial z}{\partial x}}{(2-z)^2}=\frac{(2-z)+x\cdot\dfrac{x}{2-z}}{(2-z)^2}$$

$$=\frac{(2-z)^2+x^2}{(2-z)^3}.$$

例 4 设$z = f(x+y+z,\ xyz)$，求$\dfrac{\partial z}{\partial x}, \dfrac{\partial x}{\partial y}, \dfrac{\partial y}{\partial z}$.

解 把z看成是x,y的函数，两端对x求偏导数得

$$\frac{\partial z}{\partial x} = f_1 \cdot \left(1+\frac{\partial z}{\partial x}\right) + f_2 \cdot \left(yz + xy\frac{\partial z}{\partial x}\right),$$

整理得 $$\frac{\partial z}{\partial x} = \frac{f_1 + yzf_2}{1 - f_1 - xyf_2},$$

把x看成y,z的函数对y求偏导数得

$$0 = f_1 \cdot \left(\frac{\partial x}{\partial y} + 1\right) + f_2 \cdot \left(xz + yz\frac{\partial x}{\partial y}\right),$$

整理得 $$\frac{\partial x}{\partial y}=-\frac{f_1+xzf_2}{f_1+yzf_2},$$

把y看成x,z的函数对z求偏导数得

$$1=f_1\cdot(\frac{\partial y}{\partial z}+1)+f_2\cdot(xy+xz\frac{\partial y}{\partial z}),$$

整理得 $$\frac{\partial y}{\partial z}=\frac{1-f_1-xyf_2}{f_1+xzf_2}.$$

也可以用全微分方法做此题

隐函数组

$$\begin{cases} F(x,y,u,v)=0 \\ G(x,y,u,v)=0 \end{cases} \Rightarrow u=f(x,y), v=g(x,y).$$

唯一存在，连续，可微的条件？

分析：设F,G,u,v可微，对方程组分别对x,y求偏导

$$\begin{cases} F_x+F_u u_x+F_v v_x=0 \\ G_x+G_u u_x+G_v v_x=0 \end{cases}$$

$$\begin{cases} F_y+F_u u_y+F_v v_y=0 \\ G_y+G_u u_y+G_v v_y=0 \end{cases}$$

$$\frac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}\neq 0$$

定理4. 3

设$D\subset R^4$为开集,若函数F,G：$D\to R$满足：

(i) $F(x,y,u,v)$、$G(x,y,u,v)$连续且具有连续偏导数；

(ii) $F(x_0,y_0,u_0,v_0)=0, G(x_0,y_0,u_0,v_0)=0.$

(iii) 在点(x_0,y_0,u_0,v_0)处, $\dfrac{\partial(F,G)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial F}{\partial u} & \dfrac{\partial F}{\partial v}\\ \dfrac{\partial G}{\partial u} & \dfrac{\partial G}{\partial v}\end{vmatrix}\neq 0.$

则在D中存在包含(x_0,y_0,u_0,v_0)的邻域$G\times J$, 使得

(a)对每一个$(x,y)\in G$,方程组唯一确定两个隐 函数

$u=f(x,y),v=g(x,y)$，满足

$F(x,y,f(x,y),g(x,y))=0, G(x,y,f(x,y),g(x,y))=0$

和$u_0=f(x_0,y_0),v_0=g(x_0,y_0)$;

(b) $u=f(x,y),v=g(x,y)$在G上连续;

(c) $u=f(x,y),v=g(x,y)$在G上有连续偏导数，且

$$\begin{cases}\dfrac{\partial u}{\partial x}=-\dfrac{1}{J}\dfrac{\partial(F,G)}{\partial(x,v)}=-\begin{vmatrix}F_x & F_v\\ G_x & G_v\end{vmatrix}\Big/\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}\\[2ex] \dfrac{\partial v}{\partial x}=-\dfrac{1}{J}\dfrac{\partial(F,G)}{\partial(u,x)}=-\begin{vmatrix}F_u & F_x\\ G_u & G_x\end{vmatrix}\Big/\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}\end{cases}. \quad \begin{cases}\dfrac{\partial u}{\partial y}=-\dfrac{1}{J}\dfrac{\partial(F,G)}{\partial(y,v)}=-\begin{vmatrix}F_y & F_v\\ G_y & G_v\end{vmatrix}\Big/\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}\\[2ex] \dfrac{\partial v}{\partial y}=-\dfrac{1}{J}\dfrac{\partial(F,G)}{\partial(u,y)}=-\begin{vmatrix}F_u & F_y\\ G_u & G_y\end{vmatrix}\Big/\begin{vmatrix}F_u & F_v\\ G_u & G_v\end{vmatrix}\end{cases}.$$

证明 把u, v看成x, y的函数，恒等式

$$\begin{cases} F[x, y, u(x, y), v(x, y)] \equiv 0 \\ G[x, y, u(x, y), v(x, y)] \equiv 0 \end{cases}$$

两边对x求导（应用复合函数求导法则）得：

$$\begin{cases} F_x + F_u \dfrac{\partial u}{\partial x} + F_v \dfrac{\partial v}{\partial x} = 0, \\ G_x + G_u \dfrac{\partial u}{\partial x} + G_v \dfrac{\partial v}{\partial x} = 0, \end{cases}$$

当$J = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} \neq 0$时，得：

$$\begin{cases} \dfrac{\partial u}{\partial x} = -\dfrac{1}{J}\dfrac{\partial(F,G)}{\partial(x,v)} = -\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix} \Big/ \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} \\ \dfrac{\partial v}{\partial x} = -\dfrac{1}{J}\dfrac{\partial(F,G)}{\partial(u,x)} = -\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix} \Big/ \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} \end{cases}.$$

同理可得:

$$\begin{cases} \dfrac{\partial u}{\partial y} = -\dfrac{1}{J}\dfrac{\partial(F,G)}{\partial(y,v)} = -\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix} \Big/ \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} \\ \dfrac{\partial v}{\partial y} = -\dfrac{1}{J}\dfrac{\partial(F,G)}{\partial(u,y)} = -\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix} \Big/ \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix} \end{cases}.$$

例5 设 $xu - yv = 0$， $yu + xv = 1$，

求 $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial v}{\partial x}$和$\dfrac{\partial v}{\partial y}$.

解 运用公式推导的方法,

将所给方程的两边对x 求导

$$\begin{cases} x\dfrac{\partial u}{\partial x} - y\dfrac{\partial v}{\partial x} = -u \\ y\dfrac{\partial u}{\partial x} + x\dfrac{\partial v}{\partial x} = -v \end{cases}, \qquad J = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2 + y^2,$$

在$J \neq 0$的条件下，

$$\frac{\partial u}{\partial x} = \frac{\begin{vmatrix} -u & -y \\ -v & x \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} = -\frac{xu + yv}{x^2 + y^2}, \quad \frac{\partial v}{\partial x} = \frac{\begin{vmatrix} x & -u \\ y & -v \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} = \frac{yu - xv}{x^2 + y^2},$$

将所给方程的两边对y求导，用同样方法得

$$\frac{\partial u}{\partial y} = \frac{xv - yu}{x^2 + y^2}, \qquad \frac{\partial v}{\partial y} = -\frac{xu + yv}{x^2 + y^2}.$$

例6 已知 $\begin{cases} z = x^2 + y^2 \\ x^2 + 2y^2 + 3z^2 = 20 \end{cases}$ 求 $\dfrac{dy}{dx}$.

解 方程组两端对x求导，得

$$\begin{cases} \dfrac{dz}{dx} = 2x + 2y\dfrac{dy}{dx}, \\ 2x + 4y\dfrac{dy}{dx} + 6z\dfrac{dz}{dx} = 0. \end{cases}$$

若$y + 3yz \neq 0$,解得 $\dfrac{dy}{dx} = -\dfrac{x + 6xz}{2y + 6yz}$.

说明：计算方程组决定的隐函数偏导数的一般步骤：

1.确定决定隐函数的独立的方程个数m;

2.确定所有变量的个数n;

3.确定方程组决定的隐函数个数和自变量个数：

m为隐函数个数，n-m为自变量个数；

4.根据需要计算的偏导数确定n个变量中的自变量和因变量；

5.各个方程两边求偏导，解方程得到所求偏导数.

例7 设函数 $z = f(x, y), y = y(x, z)$ 由 $\varphi(x^2, e^y, z) = 0$ 所确定，且 f, φ 均有一阶连续偏导数，求 $\dfrac{dz}{dx}$.

解 题设给出两个独立方程 $\begin{cases} z = f(x, y) \\ \varphi(x^2, e^y, z) = 0 \end{cases}$

将 y,z 看作是关于 x 的两个一元函数，在方程两端对 x 求导，

$$\begin{cases} \dfrac{dz}{dx} = f_x + f_y \dfrac{dy}{dx}, \\ \varphi_1 \cdot 2x + \varphi_2 \cdot e^y \dfrac{dy}{dx} + \varphi_3 \cdot \dfrac{dz}{dx} = 0. \end{cases}$$

当 $e^y\varphi_2 + f_y\varphi_3 \neq 0$ 时，得

$$\frac{dz}{dx} = \frac{e^y f_x\varphi_2 - 2xf_y\varphi_1}{e^y\varphi_2 + f_y\varphi_3}.$$

例8 设 $\begin{cases} x+y^2=u \\ y+z^2=v \\ z+x^2=w \end{cases}$，求 x_{uu}, x_{uv}.

解

$$\begin{cases} x_u+2yy_u=1 \\ y_u+2zz_u=0 \\ z_u+2xx_u=0 \end{cases} \Rightarrow x_u=\frac{\begin{vmatrix} 1 & 2y & 0 \\ 0 & 1 & 2z \\ 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2y & 0 \\ 0 & 1 & 2z \\ 2x & 0 & 1 \end{vmatrix}}=\frac{1}{1+8xyz},$$

$$y_u=\frac{4xz}{1+8xyz}, z_u=\frac{2x}{1+8xyz}$$

$$x_{uu}=\frac{-8\left[x_u yz+xy_u z+xyz_u\right]}{\left(1+8xyz\right)^2}=-\frac{16x^2y-8yz-32x^2z^2}{\left(1+8xyz\right)^2}$$

$$x_v=\frac{-2y}{1+8xyz},\ y_v=\frac{1}{1+8xyz},\ z_v=\frac{4xy}{1+8xyz}$$

$$x_{uv}=-\frac{16y^2z-8xz-32x^2y^2}{\left(1+8xyz\right)^2}$$

例9 已知$x^2 = vw, y^2 = uw, z^2 = uv, f(x,y,z) = F(u,v,w)$.

证明：$xf_x + yf_y + zf_z = uF_u + vF_v + wF_w$.

证明

方程组$\begin{cases} x^2 = vw \\ y^2 = uw \\ z^2 = uv \end{cases}$ 确定一组隐函数$\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$

$$\therefore F(u,v,w) = f(x(u,v,w), y(u,v,w), z(u,v,w))$$

$$F_u = f_x\frac{\partial x}{\partial u} + f_y\frac{\partial y}{\partial u} + f_z\frac{\partial z}{\partial u}, \quad F_v = f_x\frac{\partial x}{\partial v} + f_y\frac{\partial y}{\partial v} + f_z\frac{\partial z}{\partial v},$$

$$F_w = f_x\frac{\partial x}{\partial w} + f_y\frac{\partial y}{\partial w} + f_z\frac{\partial z}{\partial w}$$

北京航空航天大學 BEIHANG UNIVERSITY

对方程组$\begin{cases} x^2 = vw \\ y^2 = uw \\ z^2 = uv \end{cases}$两边关于$u$求导得$\begin{cases} 2x\dfrac{\partial x}{\partial u} = 0 \\ 2y\dfrac{\partial y}{\partial u} = w \\ 2z\dfrac{\partial z}{\partial u} = v \end{cases}$

$$\Rightarrow \begin{cases} \dfrac{\partial x}{\partial u} = 0 \\ \dfrac{\partial y}{\partial u} = \dfrac{w}{2y} \\ \dfrac{\partial z}{\partial u} = \dfrac{v}{2z} \end{cases}$$

同理可得$\begin{cases} \dfrac{\partial x}{\partial v} = \dfrac{w}{2x} \\ \dfrac{\partial y}{\partial v} = 0 \\ \dfrac{\partial z}{\partial v} = \dfrac{u}{2z} \end{cases}$，$\begin{cases} \dfrac{\partial x}{\partial w} = \dfrac{v}{2x} \\ \dfrac{\partial y}{\partial w} = \dfrac{u}{2y} \\ \dfrac{\partial z}{\partial w} = 0 \end{cases}$

$$uF_u + vF_v + wF_w = f_y\frac{uw}{y} + f_z\frac{uv}{z} + f_x\frac{vw}{x} = yf_y + zf_z + xf_x$$