§2-1 换元积分法

在一般情况下：

设 $G'(u)=g(u)$，则 $\int g(u)du=G(u)+C.$

如果 $u=\varphi(x)$（可微）

$$\because \quad dG[\varphi(x)]=g[\varphi(x)]\varphi'(x)dx$$

$$\therefore \quad \int g[\varphi(x)]\varphi'(x)dx=G[\varphi(x)]+C$$

$$=[\int g(u)du]_{u=\varphi(x)}$$ 由此可得换元法定理

定理2.1

（1） 设$g(u)$具有原函数$G(u), u=\varphi(x)$可导，

记$f(x)=g[\varphi(x)]\varphi'(x)$，则有换元公式

$$\int f(x)dx=\int g[\varphi(x)]\varphi'(x)dx=[\int g(u)du]_{u=\varphi(x)}$$

$$=G(u)\Big|_{u=\varphi(x)}+C=G[\varphi(x)]+C.$$

第一类换元公式（凑微分法）

说明　**使用此公式的关键在于将**

$$\int f(x)dx \text{ 化为 } \int g[\varphi(x)]\varphi'(x)dx.$$

(2) 设$x=\psi(t)$是单调的、可导的函数，

并且$\psi'(t)\neq 0$，又设$f[\psi(t)]\psi'(t)$具有原函数，

则有换元公式

$$\int f(x)dx=\left[\int f[\psi(t)]\psi'(t)dt\right]_{t=\psi^{-1}(x)}$$

第二类换元公式

注 第一类换元公式和第二类换元公式本质上相同，只是公式使用的方向不一样.

例1 求$\int \sin 2x dx$.

解（一）$\int \sin 2x dx = \frac{1}{2}\int \sin 2x d(2x)$

$= -\frac{1}{2}\cos 2x + C;$

（二）$\int \sin 2x dx = 2\int \sin x \cos x dx$

$= 2\int \sin x d(\sin x) = (\sin x)^2 + C;$

（三）$\int \sin 2x dx = 2\int \sin x \cos x dx$

$= -2\int \cos x d(\cos x) = -(\cos x)^2 + C.$

例2 求 $\int \frac{1}{3+2x} dx.$

解 $$\int \frac{1}{3+2x} dx = \frac{1}{2}\int \frac{1}{3+2x} \cdot d(3+2x)$$

$$= \frac{1}{2}\ln|3+2x| + C.$$

一般地 $$\int f(ax+b)dx = \frac{1}{a}\left[\int f(u)du\right]_{u=ax+b}$$

例3 求$\int \frac{x}{(1+x)^3}dx.$

解 $$\int \frac{x}{(1+x)^3}dx = \int \frac{x+1-1}{(1+x)^3}dx$$

$$= \int [\frac{1}{(1+x)^2} - \frac{1}{(1+x)^3}]dx$$

$$= \int [\frac{1}{(1+x)^2} - \frac{1}{(1+x)^3}]d(1+x)$$

$$= -\frac{1}{1+x} + \frac{1}{2(1+x)^2} + C.$$

例4 求$\int \frac{1}{1+e^x}dx.$

解 $$\int \frac{1}{1+e^x}dx = \int \frac{1+e^x-e^x}{1+e^x}dx$$

$$= \int \left(1-\frac{e^x}{1+e^x}\right)dx = \int dx - \int \frac{e^x}{1+e^x}dx$$

$$= \int dx - \int \frac{1}{1+e^x}d(1+e^x)$$

$$= x - \ln(1+e^x) + C.$$

例5　求 $\int \frac{1}{\sqrt{2x+3}+\sqrt{2x-1}}dx.$

解 原式 $=\int \frac{\sqrt{2x+3}-\sqrt{2x-1}}{\left(\sqrt{2x+3}+\sqrt{2x-1}\right)\left(\sqrt{2x+3}-\sqrt{2x-1}\right)}dx$

$$=\frac{1}{4}\int \sqrt{2x+3}dx-\frac{1}{4}\int \sqrt{2x-1}dx$$

$$=\frac{1}{8}\int \sqrt{2x+3}d(2x+3)-\frac{1}{8}\int \sqrt{2x-1}d(2x-1)$$

$$=\frac{1}{12}\left(\sqrt{2x+3}\right)^3-\frac{1}{12}\left(\sqrt{2x-1}\right)^3+C.$$

例6 求 $\int \frac{1}{a^2+x^2}dx.$

解 $\int \frac{1}{a^2+x^2}dx = \frac{1}{a^2}\int \frac{1}{1+\frac{x^2}{a^2}}dx$

$$= \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}d\left(\frac{x}{a}\right) = \frac{1}{a}\arctan\frac{x}{a}+C.$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\frac{x}{a}+C.$$

例7 求 $\int \frac{1}{x^2 - 8x + 25} dx.$

解 $\int \frac{1}{x^2 - 8x + 25} dx = \int \frac{1}{(x-4)^2 + 9} dx$

$$= \int \frac{1}{(x-4)^2 + 3^2} d(x-4)$$

$$= \frac{1}{3} \arctan \frac{x-4}{3} + C.$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C$$

例8 求 $\int \frac{1}{1+\cos x} dx.$

$$= \int \frac{1}{2\cos^2 \frac{x}{2}} dx = \tan \frac{x}{2} + C$$

解 $\int \frac{1}{1+\cos x} dx = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)} dx$

$$= \int \frac{1-\cos x}{1-\cos^2 x} dx = \int \frac{1-\cos x}{\sin^2 x} dx$$

$$= \int \frac{1}{\sin^2 x} dx - \int \frac{1}{\sin^2 x} d(\sin x)$$

$$= -\cot x + \frac{1}{\sin x} + C.$$

例9 求 $\int \sin^2 x \cdot \cos^5 x dx.$

解 $\int \sin^2 x \cdot \cos^5 x dx = \int \sin^2 x \cdot \cos^4 x d(\sin x)$

$= \int \sin^2 x \cdot (1 - \sin^2 x)^2 d(\sin x)$

$= \int (\sin^2 x - 2\sin^4 x + \sin^6 x) d(\sin x)$

$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C.$

说明 当被积函数是三角函数相乘时，拆开奇次项去凑微分.

例10 求 $\int \sin^2 x \cdot \cos^4 x dx$.

解 $$\int \sin^2 x \cdot \cos^4 x dx = \int \frac{1-\cos(2x)}{2} \cdot \left(\frac{1+\cos(2x)}{2} \right)^2 dx$$

$$= \int \frac{1+\cos(2x)-\cos^2(2x)-\cos^3(2x)}{8} dx$$

$$= \int \frac{1}{8} dx + \frac{1}{16}\int \cos(2x) d(2x) - \frac{1}{16}\int (1-\cos(4x)) dx$$

$$-\frac{1}{16}\int (1-\sin^2(2x)) d(\sin(2x))$$

$$= \frac{x}{16} + \frac{\sin(4x)}{64} + \frac{\sin^3(2x)}{48} + C.$$

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说明 求形为$\int \sin^m x \cdot \cos^n x dx$($m,n$为非负整数)的积分时,

(1)若m为奇数,则$\int \sin^m x \cdot \cos^n x dx$

$$= \int \sin^{m-1} x \cdot \cos^n x \cdot \sin x dx$$

$$= -\int (1-\cos^2 x)^{\frac{m-1}{2}} \cdot \cos^n x d(\cos x);$$

令$u = \cos x$,原积分转化为求关于u的多项式的积分问题.

(2)若n为奇数,则$\int \sin^m x \cdot \cos^n x dx$

$$= \int \sin^m x \cdot \cos^{n-1} x \cdot \cos x dx$$

$$= \int \sin^m x \cdot (1-\sin^2 x)^{\frac{n-1}{2}} d(\sin x);$$

令$u = \sin x$,原积分转化为求关于u的多项式的积分问题.

(3)若m,n为偶数,则

$$\int \sin^m x\cdot\cos^n x dx=\int(\sin^2 x)^{\frac{m}{2}}\cdot(\cos^2 x)^{\frac{n}{2}}dx$$

$$=\int\left(\frac{1-\cos(2x)}{2}\right)^{\frac{m}{2}}\cdot\left(\frac{1+\cos(2x)}{2}\right)^{\frac{n}{2}}dx$$

$$=\int P_{\frac{m+n}{2}}(\cos(2x))dx,$$

其中$P_{\frac{m+n}{2}}(\cos(2x))$是关于$\cos(2x)$的$\frac{m+n}{2}$阶多项式.

令$u=2x$,原积分转化为求$\int\cos^k u du$的积分计算问题.

思考 如何求形为$\int\tan^m x\cdot\sec^n x dx$($m,n$为非负整数)的积分?

例11 求$\int \cos 3x \cos 2x dx$.

解 $\cos A \cos B = \dfrac{1}{2}[\cos(A-B)+\cos(A+B)]$,

$$\cos 3x \cos 2x = \frac{1}{2}(\cos x + \cos 5x),$$

$$\int \cos 3x \cos 2x dx = \frac{1}{2}\int (\cos x + \cos 5x) dx$$

$$= \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C.$$

例12 求 $\int \csc x dx$.

解 方法(1) $\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx$

$$= -\int \frac{1}{1-\cos^2 x} d(\cos x) \qquad u = \cos x$$

$$= -\int \frac{1}{1-u^2} du = -\frac{1}{2}\int \left(\frac{1}{1-u} + \frac{1}{1+u} \right) du$$

$$= \frac{1}{2}\int \left(\frac{1}{u-1} - \frac{1}{1+u} \right) du = \frac{1}{2}(\ln|u-1| - \ln|u+1|) + C$$

$$= \frac{1}{2}\ln\left|\frac{1-u}{1+u}\right| + C = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C.$$

方法(2) $\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx$

$$= \int \frac{1}{\tan\frac{x}{2}\left(\cos\frac{x}{2}\right)^2} d\left(\frac{x}{2}\right) = \int \frac{1}{\tan\frac{x}{2}} d\left(\tan\frac{x}{2}\right)$$

$$= \ln\left|\tan\frac{x}{2}\right| + C = \ln|\csc x - \cot x| + C.$$

$$\boxed{\int \csc x dx = \ln|\csc x - \cot x| + C.}$$

$$\int \sec x dx = \int \frac{1}{\cos x} dx$$

$$= \int \frac{1}{\sin(x+\frac{\pi}{2})} d(x+\frac{\pi}{2})$$

$$= \ln|\csc(x+\frac{\pi}{2}) - \cot(x+\frac{\pi}{2})| + C$$

$$= \ln|\sec x + \tan x| + C$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C.$$

例13 求$\int \frac{1}{x(1+2\ln x)}dx.$

解 $$\int \frac{1}{x(1+2\ln x)}dx$$

$$=\int \frac{1}{1+2\ln x}d(\ln x)$$

$$=\frac{1}{2}\int \frac{1}{1+2\ln x}d(1+2\ln x)$$

$$=\frac{1}{2}\ln|1+2\ln x|+C.$$

例14 求$\int(1-\frac{1}{x^2})e^{x+\frac{1}{x}}dx.$

解

$\because \left(x+\frac{1}{x}\right)' = 1-\frac{1}{x^2},$

$$\therefore \int(1-\frac{1}{x^2})e^{x+\frac{1}{x}}dx$$

$$=\int e^{x+\frac{1}{x}}d(x+\frac{1}{x})$$

$$=e^{x+\frac{1}{x}}+C.$$

例15 求 $\displaystyle\int \frac{1}{\sqrt{4-x^2}\arcsin\frac{x}{2}}dx.$

解 $\displaystyle\int \frac{1}{\sqrt{4-x^2}\arcsin\frac{x}{2}}dx = \int \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}\arcsin\frac{x}{2}}d\left(\frac{x}{2}\right)$

$$= \int \frac{1}{\arcsin\frac{x}{2}}d\left(\arcsin\frac{x}{2}\right) = \ln\left|\arcsin\frac{x}{2}\right| + C.$$

$$\left(\arcsin\frac{x}{2}\right)' = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \cdot \frac{1}{2}$$

(2) 设 $x=\psi(t)$ 是单调的、可导的函数，并且 $\psi'(t)\neq 0$，又设 $f[\psi(t)]\psi'(t)$ 具有原函数，则有换元公式

$$\int f(x)dx=\left[\int f[\psi(t)]\psi'(t)dt\right]_{t=\psi^{-1}(x)}$$

第二类换元公式

注 第一类换元公式和第二类换元公式本质上相同，只是公式使用的方向不一样.

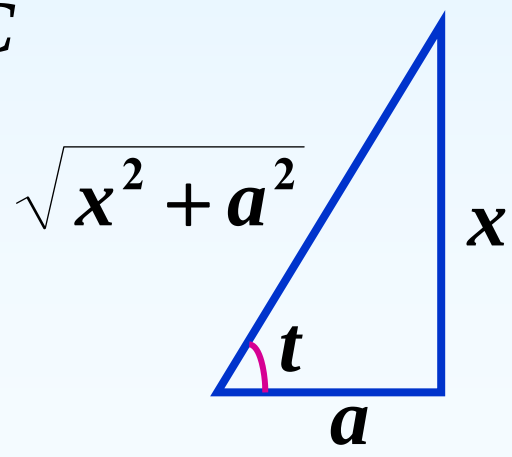
例16 求 $\int \frac{1}{\sqrt{x^2+a^2}}dx \quad (a>0).$

解 令 $x = a\tan t \Rightarrow dx = a\sec^2 t dt \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \int \frac{1}{a\sec t} \cdot a\sec^2 t dt$$

$$= \int \sec t dt = \ln|\sec t + \tan t| + C$$

$$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right) + C.$$

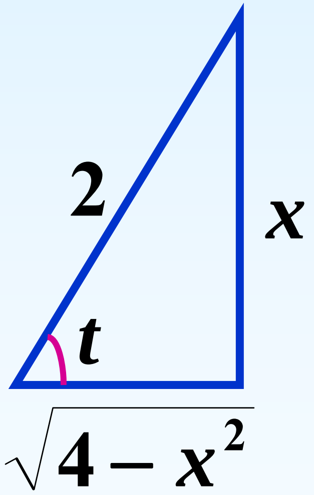


例17 求 $\int x^3\sqrt{4-x^2}\,dx.$

解 令 $x=2\sin t \quad dx=2\cos t\,dt \quad t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$$\int x^3\sqrt{4-x^2}\,dx=\int(2\sin t)^3\sqrt{4-4\sin^2 t}\cdot 2\cos t\,dt$$

$$=32\int\sin^3 t\cos^2 t\,dt$$



$$=-32\int(\cos^2 t-\cos^4 t)\,d\cos t$$

$$=-32\left(\frac{1}{3}\cos^3 t-\frac{1}{5}\cos^5 t\right)+C$$

$$=-\frac{4}{3}\left(\sqrt{4-x^2}\right)^3+\frac{1}{5}\left(\sqrt{4-x^2}\right)^5+C.$$

例18　求 $\int \frac{1}{\sqrt{x^2-a^2}}dx \quad (a>0).$

解　$1.\ x>a$

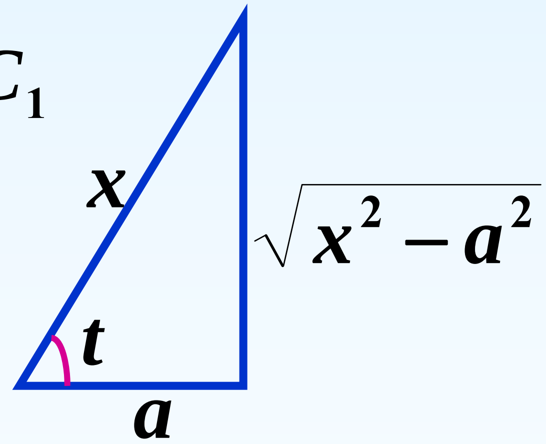
令$x = a\sec t \quad dx = a\sec t\tan t dt \qquad t \in \left(0, \frac{\pi}{2}\right)$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{a\sec t \cdot \tan t}{a\tan t}dt$$

$$= \int \sec t dt = \ln|\sec t + \tan t| + C_1$$

$$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right) + C_1$$

$$= \ln\left(x + \sqrt{x^2-a^2}\right) + C.$$



2. $x < -a$

令 $x = -u$ 那么$u > a$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = -\int \frac{1}{\sqrt{u^2 - a^2}} du$$

$$= -\ln\left(u + \sqrt{u^2 - a^2}\right) + C_2 = -\ln\left(-x + \sqrt{x^2 - a^2}\right) + C_2$$

$$= \ln\left(\frac{-x - \sqrt{x^2 - a^2}}{a^2}\right) + C_2 = \ln\left(-x - \sqrt{x^2 - a^2}\right) + C.$$

故原式 $= \ln|x + \sqrt{x^2 - a^2}| + C.$

基本积分表

(16) $\int \tan x dx = -\ln|\cos x| + C;$

(17) $\int \cot x dx = \ln|\sin x| + C;$

(18) $\int \sec x dx = \ln|\sec x + \tan x| + C;$

(19) $\int \csc x dx = \ln|\csc x - \cot x| + C;$

(20) $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + C;$

$$(21)\quad \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C;$$

$$(22)\quad \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| + C;$$

$$(23)\quad \int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + C;$$

$$(24)\quad \int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C.$$

方法(1) 三角代换

三角代换的目的是化掉根式.

一般规律如下：当被积函数中含有

(1) $\sqrt{a^2 - x^2}$ 可令 $x = a\sin t$;

(2) $\sqrt{a^2 + x^2}$ 可令 $x = a\tan t$;

(3) $\sqrt{x^2 - a^2}$ 可令 $x = a\sec t$.

化掉根式是否一定采用三角代换，需根据被积函数的情况来定.

例19 求 $\int \frac{x^5}{\sqrt{1+x^2}}dx$

解 令 $t=\sqrt{1+x^2}$ $\boxed{\Rightarrow x^2=t^2-1,}$ $\boxed{xdx=tdt,}$

$$\int \frac{x^5}{\sqrt{1+x^2}}dx=\int \frac{(t^2-1)^2}{t}tdt=\int (t^4-2t^2+1)dt$$

$$=\frac{1}{5}t^5-\frac{2}{3}t^3+t+C=\frac{1}{15}(8-4x^2+3x^4)\sqrt{1+x^2}+C.$$

例20 求 $\int \frac{1}{\sqrt{1+e^x}}dx.$

解 令 $t=\sqrt{1+e^x}$ $\boxed{\Rightarrow e^x=t^2-1,}$

$$x=\ln\left(t^2-1\right),\quad \boxed{dx=\frac{2t}{t^2-1}dt,}$$

$$\int \frac{1}{\sqrt{1+e^x}}dx=\int \frac{2}{t^2-1}dt=\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$

$$=\ln\left|\frac{t-1}{t+1}\right|+C=2\ln\left(\sqrt{1+e^x}-1\right)-x+C.$$

方法(2) 当分母的阶较高时，可采用倒代换 $x=\frac{1}{t}$.

例21 求 $\int\frac{1}{x(x^7+2)}dx$ $=\frac{1}{7}\int\frac{dx^7}{x^7(x^7+2)}$

解 令 $x=\frac{1}{t}\Rightarrow dx=-\frac{1}{t^2}dt,$

$$\int\frac{1}{x(x^7+2)}dx=\int\frac{t}{\left(\frac{1}{t}\right)^7+2}\cdot\left(-\frac{1}{t^2}\right)dt=-\int\frac{t^6}{1+2t^7}dt$$

$$=-\frac{1}{14}\ln|1+2t^7|+C=-\frac{1}{14}\ln|2+x^7|+\frac{1}{2}\ln|x|+C.$$

例22 求 $\int \frac{1}{x^4\sqrt{x^2+1}}dx.$ （分母的阶较高）

解 令 $x=\frac{1}{t} \Rightarrow dx=-\frac{1}{t^2}dt,$

$$\int \frac{1}{x^4\sqrt{x^2+1}}dx = \int \frac{1}{\left(\frac{1}{t}\right)^4\sqrt{\left(\frac{1}{t}\right)^2+1}}\left(-\frac{1}{t^2}\right)dt$$

$$=-\int \frac{t^3}{\sqrt{1+t^2}}dt = -\frac{1}{2}\int \frac{t^2}{\sqrt{1+t^2}}dt^2$$

$$=\frac{1}{2}\int\frac{1-(t^2+1)}{\sqrt{1+t^2}}d(t^2+1)$$

$$=\frac{1}{2}\int\left(\frac{1}{\sqrt{u}}-\sqrt{u}\right)du \qquad u=t^2+1$$

$$=\sqrt{u}-\frac{1}{3}\left(\sqrt{u}\right)^3+C$$

$$=\frac{\sqrt{1+x^2}}{x}-\frac{1}{3}\left(\frac{\sqrt{1+x^2}}{x}\right)^3+C.$$

方法(3) 当被积函数含有两种或两种以上的根式 $\sqrt[k]{x},\cdots,\sqrt[l]{x}$ 时，可采用令 $x=t^n$（其中 n 为各根指数的最小公倍数）

例23 求 $\int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})}dx.$

解 令 $\sqrt[6]{x}=t$, 则 $x=t^6, dx=6t^5dt$

$$\int \frac{1}{\sqrt{x}(1+\sqrt[3]{x})}dx=\int \frac{6t^5}{t^3(1+t^2)}dt=\int \frac{6t^2}{1+t^2}dt$$

$$= 6\int \frac{t^2 + 1 - 1}{1 + t^2} dt$$

$$= 6\int \left(1 - \frac{1}{1 + t^2}\right) dt$$

$$= 6[t - \arctan t] + C$$

$$= 6[\sqrt[6]{x} - \arctan \sqrt[6]{x}] + C.$$