§3 微积分基本定理

问题的提出

变速直线运动中路程的两种表示

设某物体作直线运动,$v(t)$是时间段$[T_1,T_2]$上t时刻的速度,是连续函数,且$v(t)\geq 0$,则在这段时间内物体所走过的路程为 $\int_{T_1}^{T_2} v(t)dt$

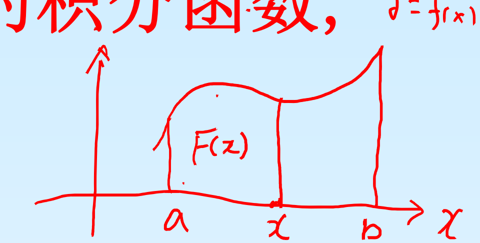
设$s(t)$是t时刻物体所走过的路程,则在这段时间内物体所走过的路程为 $s(T_2)-s(T_1)$

$$\therefore \int_{T_1}^{T_2} v(t)dt = s(T_2)-s(T_1),\text{ 其中 } s'(t)=v(t).$$

$$\boxed{\int_a^b f(x)dx \overset{?}{=} F(b)-F(a).}$$

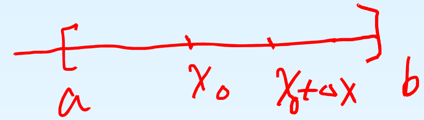
定义3.1 设函数$f(x)$在区间$[a,b]$上可积，x为$[a,b]$上任意一点，定义$F(x)=\int_a^x f(t)dt$为**变上限的积分函数**，简称为**变上限函数**.

$= \int_a^{\boxed{x}} f(x)dx$



定理3.1 设$f(x)$在$[a,b]$上可积，则$F(x)=\int_a^x f(t)dt$在$[a,b]$连续.

证 对于$\forall x_0\in(a,b)$, $x_0+\Delta x\in[a,b]$，由于



$$\left|F(x_0+\Delta x)-F(x_0)\right| = \left|\int_a^{x_0+\Delta x} f(t)dt-\int_a^{x_0} f(t)dt\right|$$

M |Δx|

$$=\left|\int_{x_0}^{x_0+\Delta x} f(t)dt\right| \le \sup_{a\le x\le b}\left|f(t)\right|\cdot\left|\Delta x\right|,$$

所以$\lim\limits_{\Delta x\to 0}\left(F(x_0+\Delta x)-F(x_0)\right)=0$，因此$F(x)$在$x_0$连续.

同理可以证明$F(x)$在$x=a$, $x=b$分别左连续，右连续.

定理3.2 设$f(x)$在$[a,b]$上连续,$F(x)=\int_a^x f(t)dt$, 则

$$\frac{dF(x)}{dx}=f(x),\qquad x\in[a,b]$$

证明 对于$\forall x_0\in(a,b), x_0+\Delta x\in[a,b]$,

$$F(x_0+\Delta x)-F(x_0)=\int_a^{x_0+\Delta x} f(t)dt-\int_a^{x_0} f(t)dt=\int_{x_0}^{x_0+\Delta x} f(t)dt,$$

则 $$\frac{F(x_0+\Delta x)-F(x_0)}{\Delta x}=\frac{\int_{x_0}^{x_0+\Delta x} f(t)dt}{\Delta x}=\frac{f(\xi(x_0,\Delta x))}{\Delta x}\Delta x.$$

$=f(\xi)\cdot\Delta x$

由于$\xi(x_0,\Delta x)$介于$x_0, x_0+\Delta x$之间,因此$\lim\limits_{\Delta x\to 0}\xi(x_0,\Delta x)=x_0$.

由上面得到$F'(x_0)=f(x_0)$.

同理可以证明$F(x)$在$x=a, x=b$结论仍然成立.

定理3.3 设$f(x)$在$[a,b]$上连续,则$F(x)=\int_a^x f(t)dt$是$f(x)$在$[a,b]$上的一个原函数.

$F'(x)=f(x)$

定理3.4（Newton-Leibniz公式）

$F(b)=\int_a^b f\,dx$

$F(a)=0$

假设$f(x)$在$[a,b]$可积,且在$[a,b]$存在原函数$F(x)$.

$\Rightarrow F(b)-F(a)$

$=\int_a^b f\,dx$

则有

$$\int_a^b f(x)dx=F(b)-F(a)\overset{\triangle}{=}F(x)\Big|_a^b=[F(x)]_a^b$$

注 (1)求定积分问题转化为求原函数的增量.

(2)当$a>b$时，$\int_a^b f(x)dx=F(b)-F(a)$仍成立.

$=-\int_b^a f\,dx=-F(x)\Big|_b^a=F(x)\Big|_a^b$

$t \to x$ $\qquad (\int_a^x f(t)dt)' = f(x)$

例1 $(\int_a^x \sin t dt)' = \sin x$

$(\int_a^{x^2} f(t)dt)' = f(x^2)\cdot 2x$ $\quad x^2 = u$

$(\int_{e^x}^{x^2} \sin t^2 dt)' = (\int_{e^x}^0 \sin t^2 dt + \int_0^{x^2} \sin t^2 dt)'$

$= (-\int_0^{e^x} \sin t^2 dt + \int_0^{x^2} \sin t^2 dt)'$

$= 2x \sin x^4 - e^x \sin e^{2x}$

特别地, 设$f(x)$在区间$[a,b]$上连续,且$\varphi_1(x),\varphi_2(x)$在$[a,b]$上可导,令$F(x)=\int_{\varphi_1(x)}^{\varphi_2(x)} f(t)dt, x\in[a,b]$, 则

$$\frac{dF}{dx}=f(\varphi_2(x))\varphi_2'(x)-f(\varphi_1(x))\varphi_1'(x),\quad x\in[a,b].$$

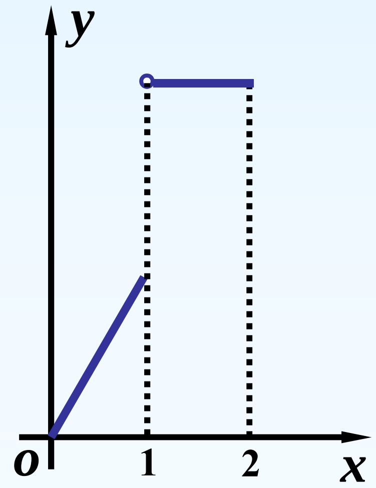
注：定理3.2，3.3证明了连续函数必有原函数, 因此被称为原函数存在定理, 并且建立了微分和积分的内在联系, 实际上是两个互逆的过程. 此定理被誉为微积分的基本定理.

例2 求 $\int_0^{\frac{\pi}{2}}(2\cos x+\sin x-1)dx.$

解 原式 $=\left[2\sin x-\cos x-x\right]_0^{\frac{\pi}{2}}=3-\dfrac{\pi}{2}.$

例3 设 $f(x)=\begin{cases}2x & 0\le x\le 1\\ 5 & 1<x\le 2\end{cases}$, 求 $\int_0^2 f(x)dx.$

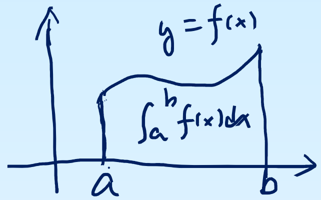
解 $\int_0^2 f(x)dx=\int_0^1 f(x)dx+\int_1^2 f(x)dx$



在$[1,2]$上规定当$x=1$时，$f(x)=5$,

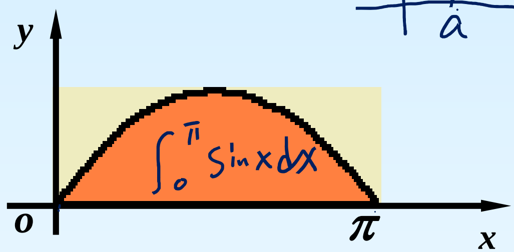
原式 $=\int_0^1 2xdx+\int_1^2 5dx=6.$

例 4 计算曲线 $y = \sin x$ 在 $[0, \pi]$ 上与 x 轴所围成的平面图形的面积.



解 面积 $A = \int_0^{\pi} \sin x dx$

$$= \left[-\cos x\right]_0^{\pi} = 2.$$



例5 求极限 $\lim\limits_{n\to\infty}\int_0^1 \frac{x^n}{1+x}dx \quad (n>0).$

$\int_a^b fg\,dx$

$= f(\xi)\int_a^b g\,dx$

解一 $\because 0 \le \frac{x^n}{1+x} \le x^n \quad x\in[0,1], n>0$

$$\therefore 0 \le \int_0^1 \frac{x^n}{1+x}dx \le \int_0^1 x^n dx = \frac{1}{n+1} \to 0 \quad n\to\infty$$

由夹逼定理，得 $\lim\limits_{n\to\infty}\int_0^1 \frac{x^n}{1+x}dx = 0.$

解二 $0\le\int_0^1 \frac{x^n}{1+x}dx = \frac{1}{1+\xi_n}\int_0^1 x^n dx = \frac{1}{1+\xi_n}\cdot\frac{1}{n+1} \to 0 (n\to\infty)$

例6 求 $\lim\limits_{x\to 0}\dfrac{\int_{\cos x}^{1} e^{-t^2}dt}{x^2}$.

分析：这是 $\dfrac{0}{0}$ 型不定式，应用洛必达法则.

解 $\dfrac{d}{dx}\int_{\cos x}^{1} e^{-t^2}dt = -e^{-\cos^2 x}\cdot(\cos x)'$

$= \sin x \cdot e^{-\cos^2 x}$,

$$\lim_{x\to 0}\frac{\int_{\cos x}^{1} e^{-t^2}dt}{x^2} = \lim_{x\to 0}\frac{\sin x\cdot e^{-\cos^2 x}}{2x} = \frac{1}{2e}.$$

例7 设 $f(x)$在$[0,1]$上连续，且 $f(x)<1$.

证明：$2x-\int_0^x f(t)dt=1$在$[0,1]$上只有一个解.

证 令 $F(x)=2x-\int_0^x f(t)dt-1$,

$\boxed{F(0)=-1<0,}$

$\boxed{F(1)=1-\int_0^1 f(t)dt}=\int_0^1 [\underbrace{1-f(t)}_{>0}]dt\boxed{>0,}$

$\because f(x)<1,\quad \therefore F'(x)=2-f(x)>0,$

$\boxed{F(x)\text{在}[0,1]\text{上为单调增加函数.}}$

所以$F(x)=0$即原方程在$[0,1]$上只有一个解.

例8 设 $f(x)$在$(-\infty,+\infty)$内连续，且 $f(x)>0$.证明函数$F(x)=\dfrac{\int_0^x tf(t)dt}{\int_0^x f(t)dt}$在$(0,+\infty)$内为单调增加函数.

证 $$\frac{d}{dx}\int_0^x tf(t)dt = xf(x),\quad \frac{d}{dx}\int_0^x f(t)dt = f(x),$$

$$x\int_0^x f(t)dt = \int_0^x x f(t)dt$$

$$F'(x)=\frac{xf(x)\cdot\int_0^x f(t)dt - f(x)\cdot\int_0^x tf(t)dt}{\left(\int_0^x f(t)dt\right)^2}$$

$$F'(x)=\frac{f(x)\int_0^x (x-t)f(t)dt}{\left(\int_0^x f(t)dt\right)^2},$$

$\boxed{\because\ f(x)>0,\quad (x>0)}$ $\boxed{\therefore \int_0^x f(t)dt>0,}$

$\because (x-t)f(t)\geq 0$且不恒为0,

$\boxed{\therefore \int_0^x (x-t)f(t)dt>0,}$

$\therefore F'(x)>0\quad (x>0).$

故$F(x)$在$(0,+\infty)$内为单调增加函数.

例9 已知$f(x), g(x)$在$[a,b]$上连续,求证

$\exists \xi \in (a,b)$,使得$f(\xi)\int_\xi^b g(x)dx = g(\xi)\int_a^\xi f(x)dx.$

$$\left[\int_a^\xi f(x)dx \int_\xi^b g(x)dx\right]'_\xi = 0$$

$$\left(\int_a^\xi f(x)dx\right)'_\xi = f(\xi)$$

$$\left(\int_\xi^b g(x)dx\right)'_\xi = -g(\xi)$$

证明 令$F(x) = \int_a^x f(t)dt \int_x^b g(t)dt,$

则$F(x)$在$[a,b]$上连续,在(a,b)内可导,

且$F(a) = F(b) = 0$, 由Rolle定理,$\exists \xi \in (a,b)$使得

$$0 = F'(\xi) = f(\xi)\int_\xi^b g(x)dx - g(\xi)\int_a^\xi f(x)dx,$$

结论成立.

例10 求 $\lim\limits_{n\to\infty}\left(\dfrac{n}{n^2+1^2}+\dfrac{n}{n^2+2^2}+\cdots+\dfrac{n}{n^2+n^2}\right)$

$\dfrac{n\cdot n}{n^2+n^2} \le \,?\, \le \dfrac{n\cdot n}{n^2+1^2}$

$\downarrow \frac{1}{2} \qquad \downarrow 1$

解

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots+\frac{n}{n^2+n^2}\right)$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{n^2}{n^2+1^2}+\frac{n^2}{n^2+2^2}+\cdots+\frac{n^2}{n^2+n^2}\right)$$

$0 \quad \frac{1}{n} \quad \frac{2}{n} \quad 1=\frac{n}{n}$

$\frac{1}{1+x^2}$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+\left(\frac{1}{n}\right)^2}+\frac{1}{1+\left(\frac{2}{n}\right)^2}+\cdots+\frac{1}{1+\left(\frac{n}{n}\right)^2}\right)$$

$$=\int_0^1\frac{1}{1+x^2}dx=\left[\arctan x\right]_0^1=\frac{\pi}{4}$$

二、定积分的计算

定理3.5(换元公式)假设(1) $f(x)$在$[a,b]$上连续；

(2) 函数$x=\varphi(t)$在$[\alpha,\beta]$上有连续导数；

(3) 当t在区间$[\alpha,\beta]$上变化时，$x=\varphi(t)$的值在$[a,b]$上变化，且$\varphi(\alpha)=a$、$\varphi(\beta)=b$，

则有

$$\int_a^b f(\underset{\varphi(t)}{\underline{x}})d\underset{\varphi(t)}{\underline{x}}=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt$$

证明 设$F(x)$是$f(x)$的一个原函数，

定义$\Phi(t)=F[\varphi(t)]$,

$$\Big[F(\varphi(t))\Big]' = F'(\varphi(t))\cdot\varphi'(t)$$
$$= f(\varphi(t))\cdot\varphi'(t)$$

易证其是$f[\varphi(t)]\varphi'(t)$的一个原函数.

$$\int_{\alpha}^{\beta} f[\varphi(t)]\varphi'(t)dt=\Phi(\beta)-\Phi(\alpha)$$

$$=F[\varphi(\beta)]-F[\varphi(\alpha)]=F(b)-F(a)$$

$$=\int_{a}^{b} f(x)dx$$

应用换元公式时

（1）由左到右时 $\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt$

相当于第二类换元法

由右到左时 $\int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt = \int_a^b f(x)dx$

相当于第一类换元法

把原变量换成新变量时，积分限也相应改变.

(2)当$\alpha > \beta$时，换元公式仍成立.

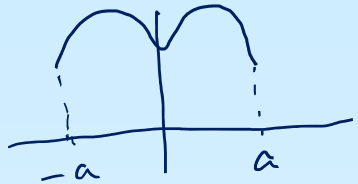
(3) $\int_a^b f(x)dx = \int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt$

求出$f[\varphi(t)]\varphi'(t)$的一个原函数$\Phi(t)$后，

不必再把$\Phi(t)$变换成原变量x的函数，

而只需将新变量t的上下限代入$\Phi(t)$然后相减，

也就是$\Phi(\beta)-\Phi(\alpha)$.

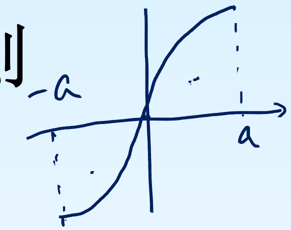


例 11

(1)若 $f(x)$为偶函数，且在$[-a, a]$上可积，则

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx\,;$$

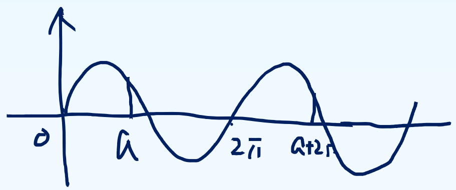
(2)若 $f(x)$为奇函数，且在$[-a, a]$上可积，则



$$\int_{-a}^{a} f(x)dx = 0.$$

(3)若$f(x)$是R上的周期为T的连续函数，则对任意实数a,成立

$[a, a+T]$ $[0, T]$



$$\int_{a}^{a+T} f(x)dx = \int_{0}^{T} f(x)dx$$

证 $\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx,$

在 $\int_{-a}^{0} f(x)dx$ 中令 $x = -t$,

$$\boxed{\int_{-a}^{0} f(x)dx =} -\int_{a}^{0} f(-t)dt = \boxed{\int_{0}^{a} f(-t)dt,}$$

$$\therefore \int_{-a}^{a} f(x)dx = \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)dx,$$

(1) $f(x)$为偶函数，则$f(-x) = f(x)$,

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx;$$

(2) $f(x)$为奇函数，则 $f(-x)=-f(x)$,

$$\int_{-a}^{a} f(x)dx = -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx = 0$$

$$(3)\int_{a}^{a+T} f(x)dx = \int_{a}^{0} f(x)dx + \int_{0}^{T} f(x)dx + \int_{T}^{a+T} f(x)dx$$

① ③

$$(u = x - T)$$ $x=u+T$

$$\boxed{\int_{T}^{a+T} f(x)dx = \int_{0}^{a} f(u)du}$$ 结论得证

$\parallel$

$\int_0^a f(x)dx$

例12 计算 $\int_{\sqrt{e}}^{e^{\frac{3}{4}}} \frac{dx}{x\sqrt{\ln x(1-\ln x)}}$.

解 原式 $= \int_{\sqrt{e}}^{e^{\frac{3}{4}}} \frac{d(\ln x)}{\sqrt{\ln x(1-\ln x)}}$

$\frac{du}{\sqrt{u}} = 2d\sqrt{u}$

$$= \int_{\sqrt{e}}^{e^{\frac{3}{4}}} \frac{d(\ln x)}{\sqrt{\ln x}\sqrt{(1-\ln x)}} = 2\int_{\sqrt{e}}^{e^{\frac{3}{4}}} \frac{d\sqrt{\ln x}}{\sqrt{1-(\sqrt{\ln x})^2}}$$

$$= 2\left[\arcsin(\sqrt{\ln x})\right]_{\sqrt{e}}^{e^{\frac{3}{4}}} = \frac{\pi}{6}.$$

例13 设$f(x)$在$[0,1]$上连续，且单调减少，证明：对任何$q\in(0,1)$，有$\int_0^q f(x)dx \ge q\int_0^1 f(x)dx$.

$\int_0^1 \ne \int_0^q$

解1 令$u=\dfrac{x}{q}$，有$\int_0^q f(x)dx = q\int_0^1 f(qu)du$.

$\because \int_0^1 f(qu)du \ge \int_0^1 f(u)du,$

$qu \le u \quad f(qu) \ge f(u)$

$\therefore$ 对任何$q\in(0,1)$，有$\int_0^q f(x)dx \ge q\int_0^1 f(u)du.$

$= q\int_0^1 f(x)dx$

解2 令$F(q)=\dfrac{\int_0^q f(x)dx}{q}$,

$$F'(q)=\frac{qf(q)-\int_0^q f(x)dx}{q^2}=\frac{qf(q)-qf(\xi)}{q^2}\le 0$$

$\therefore F(q)$为单调减函数，

$\therefore$ 对任何$q\in(0,1)$，有$F(q)\ge F(1).$ $=\int_0^1 f(x)dx$

例14 计算 $\displaystyle\int_0^a \frac{1}{x+\sqrt{a^2-x^2}}dx. \quad (a>0)$

解 令 $x = a\sin t, \qquad dx = a\cos t dt,$

$$x = a \Rightarrow t = \frac{\pi}{2}, \qquad x = 0 \Rightarrow t = 0,$$

$$\text{原式} = \int_0^{\frac{\pi}{2}} \frac{a\cos t}{a\sin t + \sqrt{a^2(1-\sin^2 t)}}dt$$

$\frac{1}{2}[(\cos t+\sin t)+(\cos t-\sin t)]$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos t}{\sin t+\cos t}dt = \frac{1}{2}\int_0^{\frac{\pi}{2}}\left(1+\frac{\cos t-\sin t}{\sin t+\cos t}\right)dt$$

$$= \frac{1}{2}\cdot\frac{\pi}{2} + \frac{1}{2}\Big[\ln|\sin t+\cos t|\Big]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

例15 计算$\displaystyle\int_{-1}^{1}\frac{2x^2+x\cos x}{1+\sqrt{1-x^2}}dx.$

解 原式$\displaystyle=\int_{-1}^{1}\boxed{\frac{2x^2}{1+\sqrt{1-x^2}}}dx+\int_{-1}^{1}\boxed{\frac{x\cos x}{1+\sqrt{1-x^2}}}dx$

偶函数　　奇函数

$$=4\int_{0}^{1}\frac{x^2}{1+\sqrt{1-x^2}}dx$$

$$=4\int_{0}^{1}\frac{x^2(1-\sqrt{1-x^2})}{1-(1-x^2)}dx=4\int_{0}^{1}(1-\sqrt{1-x^2})dx$$

$$=4-\boxed{4\int_{0}^{1}\sqrt{1-x^2}dx}=4-\pi.$$

单位圆的面积

$$\int\sqrt{a^2-x^2}dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a}+C$$

例6 若 $f(x)$在[0,1]上连续，证明

（1）$\int_0^{\frac{\pi}{2}} f(\sin x)dx = \int_0^{\frac{\pi}{2}} f(\cos x)dx$；

（2）$\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$.

并由此计算$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$.

证（1）设$x=\frac{\pi}{2}-t$，则$x=0 \Rightarrow t=\frac{\pi}{2}$； $x=\frac{\pi}{2} \Rightarrow t=0$，

$$\int_0^{\frac{\pi}{2}} f(\sin x)dx = -\int_{\frac{\pi}{2}}^{0} f\left[\sin\left(\frac{\pi}{2}-t\right)\right]dt$$

$$= \int_0^{\frac{\pi}{2}} f(\cos t)dt = \int_0^{\frac{\pi}{2}} f(\cos x)dx;$$

（2）$\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$

设 $x = \pi - t$, 则 $x = 0 \Rightarrow t = \pi, x = \pi \Rightarrow t = 0$,

$$\boxed{\int_0^{\pi} xf(\sin x)dx} = -\int_{\pi}^{0}(\pi - t)f[\sin(\pi - t)]dt$$

$$= \int_0^{\pi}(\pi - t)f(\sin t)dt,$$

$$= \pi\int_0^{\pi} f(\sin t)dt - \int_0^{\pi} tf(\sin t)dt$$

$$= \pi\int_0^{\pi} f(\sin x)dx - \boxed{\int_0^{\pi} xf(\sin x)dx},$$

$$\therefore \int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx.$$

$$\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$$

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

$$= -\frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\cos^2 x}d(\cos x)$$

$$= -\frac{\pi}{2}\left[\arctan(\cos x)\right]_0^{\pi}$$

$$= -\frac{\pi}{2}\left(-\frac{\pi}{4}-\frac{\pi}{4}\right) = \frac{\pi^2}{4}.$$

例17 设 $f(x)$ 在区间R上连续，则

$$\int_0^{2\pi} f(a\cos\theta + b\sin\theta)d\theta = \int_0^{2\pi} f(\sqrt{a^2+b^2}\cos\lambda)d\lambda$$

证明

$$\int_0^{2\pi} f\left(a\cos\theta+b\sin\theta\right)d\theta=\int_0^{2\pi} f\left(\sqrt{a^2+b^2}\left(\frac{a\cos\theta}{\sqrt{a^2+b^2}}+\frac{b\sin\theta}{\sqrt{a^2+b^2}}\right)\right)d\theta$$

$$= \int_0^{2\pi} f(\sqrt{a^2+b^2}\cos(\theta-\alpha))d\theta,$$

令 $\theta-\alpha=\lambda,$

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \sin\alpha = \frac{b}{\sqrt{a^2+b^2}}$$

$$\int_0^{2\pi} f(\sqrt{a^2+b^2}\cos(\theta-\alpha))d\theta = \int_{-\alpha}^{2\pi-\alpha} f(\sqrt{a^2+b^2}\cos\lambda)d\lambda$$

$$= \int_0^{2\pi} f(\sqrt{a^2+b^2}\cos\lambda)d\lambda$$

例18 求 $\int_0^{\frac{\pi}{4}} \ln\sin 2x\,dx,\ \int_0^{\frac{\pi}{2}} \ln\sin x\,dx.$

解 $I=\int_0^{\frac{\pi}{4}} \ln\sin 2x\,dx \ =\int_0^{\frac{\pi}{4}} \ln(2\sin x\cos x)dx$

$$=\int_0^{\frac{\pi}{4}} (\ln 2+\ln\sin x+\ln\cos x)dx$$

$$\int_0^{\frac{\pi}{4}} \ln\cos x\,dx=\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \ln\cos(\frac{\pi}{2}-t)d(\frac{\pi}{2}-t)=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\sin x\,dx$$

$$=\frac{\pi}{4}\ln 2+\int_0^{\frac{\pi}{4}} \ln\sin x\,dx+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\ln\sin x\,dx$$

$$=\frac{\pi}{4}\ln 2+\int_0^{\frac{\pi}{2}} \ln\sin \underset{2t}{x}\,d\underset{2t}{x} \ =\frac{\pi}{4}\ln 2+2I$$

$$\therefore I=-\frac{\pi}{4}\ln 2.$$

令 $x=2t,\ \int_0^{\frac{\pi}{2}} \ln\sin x\,dx=2\int_0^{\frac{\pi}{4}} \ln\sin 2t\,dt=-\frac{\pi}{2}\ln 2.$

定理3.6(分部积分公式) 设函数$u(x)$、$v(x)$

在$[a,b]$上有连续导数，则有

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

或

$$\int_a^b uv'dx = [uv]_a^b - \int_a^b u'v\,dx$$

$$\int_a^b u\,dv = \int_a^b [d(uv) - v\,du]$$

$$= \int_a^b \underbrace{d(uv)}_{(uv)'dx} - \int_a^b v\,du$$

$$= uv\Big|_a^b - \int_a^b v\,du$$

定积分的分部积分公式

例19 计算 $\int_0^{\frac{1}{2}} \arcsin x dx.$

解

$-\frac{1}{2}d(1-x^2)$

$$\int_0^{\frac{1}{2}} \arcsin x dx = \left[x \arcsin x\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{xdx}{\sqrt{1-x^2}}$$

$$= \frac{1}{2}\cdot\frac{\pi}{6} + \frac{1}{2}\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}} d(1-x^2)$$

$$= \frac{\pi}{12} + \left[\sqrt{1-x^2}\right]_0^{\frac{1}{2}} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

例20　计算 $\int_0^{\frac{\pi}{4}} \frac{xdx}{1+\cos 2x}$.

解　$\because 1+\cos 2x = 2\cos^2 x,$

$\sim d\tan x$

$$\therefore \int_0^{\frac{\pi}{4}} \frac{xdx}{1+\cos 2x} = \int_0^{\frac{\pi}{4}} \frac{xdx}{2\cos^2 x} = \frac{1}{2}\int_0^{\frac{\pi}{4}} xd(\tan x)$$

$$= \frac{1}{2}[x\tan x]_0^{\frac{\pi}{4}} - \frac{1}{2}\int_0^{\frac{\pi}{4}} \tan xdx$$

$$= \frac{\pi}{8} + \frac{1}{2}[\ln\cos x]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{\ln 2}{4}.$$

例21 计算 $\int_0^1 \frac{\ln(1+x)}{(2+x)^2}dx.$

解 $\int_0^1 \frac{\ln(1+x)}{(2+x)^2}dx = -\int_0^1 \ln(1+x)d\left(\frac{1}{2+x}\right)$

$$= -\left[\frac{\ln(1+x)}{2+x}\right]_0^1 + \int_0^1 \frac{1}{2+x}d\ln(1+x)$$

$$= -\frac{\ln 2}{3} + \int_0^1 \boxed{\frac{1}{2+x}\cdot\frac{1}{1+x}}dx \qquad \frac{1}{1+x} - \frac{1}{2+x}$$

$$= -\frac{\ln 2}{3} + [\ln(1+x) - \ln(2+x)]_0^1 = \frac{5}{3}\ln 2 - \ln 3.$$

例22 设 $f(x)=\int_1^{x^2}\frac{\sin t}{t}dt$, 求 $\int_0^1 xf(x)dx$.

$f'(x)=\frac{\sin x^2}{x^2}\,2x=2\frac{\sin x^2}{x}$

解 $\frac{\sin t}{t}$ 的原函数无法直接求出,所以用分部积分法

$$\int_0^1 xf(x)dx=\frac{1}{2}\int_0^1 f(x)d(x^2)$$

$$=\frac{1}{2}\left[x^2f(x)\right]_0^1-\frac{1}{2}\int_0^1 x^2df(x)$$

$$=-\frac{1}{2}\int_0^1 x^2f'(x)dx \quad f'(x)=\frac{\sin x^2}{x^2}\cdot 2x=\frac{2\sin x^2}{x},$$

$$=-\frac{1}{2}\int_0^1 2x\sin x^2dx=-\frac{1}{2}\int_0^1\sin x^2d(x^2)$$

$$=\frac{1}{2}\left[\cos x^2\right]_0^1=\frac{1}{2}(\cos 1-1).$$

例23 证明定积分公式

$x=\frac{\pi}{2}-t$

$$I_n=\int_0^{\frac{\pi}{2}}\sin^n x dx=\int_0^{\frac{\pi}{2}}\cos^n x dx$$

$$=\begin{cases}\dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\cdots\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2}, & n\text{为正偶数}\\[2ex] \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\cdots\dfrac{4}{5}\cdot\dfrac{2}{3}, & n\text{为大于1的正奇数}\end{cases}$$

$I_n = \int_0^{\frac{\pi}{2}} \sin^{n-1} x \ \sin x dx$

$= -\int_0^{\frac{\pi}{2}} \sin^{n-1} x \ d\cos x$

证 设 $u = \sin^{n-1} x$, $dv = \sin x dx$,

$$I_n = \underbrace{\left[-\sin^{n-1} x \cos x\right]_0^{\frac{\pi}{2}}}_{0} + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x \underbrace{\cos^2 x}_{1-\sin^2 x} dx$$

$$I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx - (n-1)\int_0^{\frac{\pi}{2}} \sin^{n} x dx$$

$$= (n-1)I_{n-2} - (n-1)I_n$$

$$I_n = \frac{n-1}{n} I_{n-2}$$ 积分 I_n 关于下标的递推公式

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$ ……，直到下标减到0或1为止

$$I_{2m}=\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdots\cdots\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}I_0,$$

$$(m=1,2,\cdots)$$

$$I_{2m+1}=\frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdots\cdots\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}I_1,$$

$$I_0=\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{2},\qquad I_1=\int_0^{\frac{\pi}{2}}\sin x dx=1,$$

于是 $$I_{2m}=\frac{2m-1}{2m}\cdot\frac{2m-3}{2m-2}\cdots\cdots\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},$$

$$I_{2m+1}=\frac{2m}{2m+1}\cdot\frac{2m-2}{2m-1}\cdots\cdots\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}.$$

小结

1.积分上限函数 $F(x)=\int_a^x f(t)dt$

2.积分上限函数的导数 $F'(x)=f(x)$

3.微积分基本公式 $\int_a^b f(x)dx=F(b)-F(a)$

牛顿—莱布尼茨公式沟通了微分学与积分学之间的关系.

4.定积分的换元法

$$\int_a^b f(x)dx=\int_\alpha^\beta f[\varphi(t)]\varphi'(t)dt$$

5.定积分的分部积分公式

$$\int_a^b udv=[uv]_a^b-\int_a^b vdu.$$

作业:

习题7.3

1(1)(3)，2，3(双数),4(单数)，5，6，7，9