第17章 习题课

第一型曲线积分的计算

$$(1)\ L:\begin{cases} x=\varphi(t), \\ y=\psi(t), \end{cases} t\in[\alpha,\beta],$$

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$$\int_L f(x,y)ds=\int_\alpha^\beta f[\varphi(t),\psi(t)]\boxed{\sqrt{\varphi'^2(t)+\psi'^2(t)}dt}$$

$x=x$

$$(2)\quad L: y=y(x) \qquad a\le x\le b.$$

$$\int_L f(x,y)ds=\int_a^b f[x,y(x)]\boxed{\sqrt{1+y'^2(x)}dx}$$

$$(3)\quad L: x=x(y) \qquad c\le y\le d.$$

$y=y$

$$\int_L f(x,y)ds=\int_c^d f[x(y),y]\boxed{\sqrt{1+x'^2(y)}dy}$$

积分下限小于积分上限

$$(4)\Gamma:\begin{cases} x=\varphi(t), \\ y=\psi(t), \\ z=\omega(t). \end{cases} \quad (\alpha \le t \le \beta)$$

$$\int_L f(x,y,z)ds$$

$$=\int_\alpha^\beta f[\varphi(t),\psi(t),\omega(t)]\boxed{\sqrt{\varphi'^2(t)+\psi'^2(t)+\omega'^2(t)}dt}$$

第二型曲线积分的计算

$$(1)\ L:\begin{cases} x=\varphi(t), \\ y=\psi(t), \end{cases} \quad t:\alpha \mapsto \beta,$$

$$\int_L Pdx+Qdy$$

$$=\int_a^b \{P[\varphi(t),\psi(t)]\varphi'(t)+Q[\varphi(t),\psi(t)]\psi'(t)\}dt.$$

$x=x$

$$(2)\quad L: y=y(x) \quad x: a \mapsto b$$

$$\int_L Pdx+Qdy=\int_a^b \{P[x,y(x)]+Q[x,y(x)]y'(x)\}dx.$$

$$(3)\quad L: x=x(y) \quad y: c \mapsto d$$

$y=y$

$$\int_L Pdx+Qdy=\int_c^d \{P[x(y),y]x'(y)+Q[x(y),y]\}dy.$$

积分下限不一定小于积分下限

$$(4)\Gamma:\begin{cases} x=\varphi(t), \\ y=\psi(t), \\ z=\omega(t). \end{cases} \quad t:\alpha \mapsto \beta$$

$$\begin{aligned}\int_\Gamma P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz \\ =\int_\alpha^\beta \{P[\varphi(t),\psi(t),\omega(t)]\varphi'(t) \\ +Q[\varphi(t),\psi(t),\omega(t)]\psi'(t) \\ +R[\varphi(t),\psi(t),\omega(t)]\omega'(t)\}dt\end{aligned}$$

两类曲线积分之间的联系

$$\int_L Pdx + Qdy = \int_L (P\cos\alpha + Q\cos\beta)ds$$

$\cos\alpha, \cos\beta$为有向曲线L的切方向余弦

$$\int_\Gamma Pdx + Qdy + Rdz = \int_\Gamma (P\cos\alpha + Q\cos\beta + R\cos\gamma)ds$$

$\cos\alpha, \cos\beta, \cos\gamma$为有向曲线$\Gamma$的切方向余弦

第二型平面曲线积分计算

$I=\int_L Pdx+Qdy$

(1) → 直接计算

(2) ↓ $\frac{\partial P}{\partial y},\frac{\partial Q}{\partial x}$ 连续(无奇点)

- 单连通 (a) ↓ $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$
  - 非闭 → $I=\int_{(x_0,y_0)}^{(x,y)} Pdx+Qdy$
  - 闭合 → $I=\int_L Pdx+Qdy=0$
- (b) ↓ $\frac{\partial P}{\partial y}\neq\frac{\partial Q}{\partial x}$
  - 闭合 → $I=\iint_D(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy$
  - 非闭 → 补充曲线 → $I=\iint_D(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y})dxdy$

(3) ↑ $\frac{\partial P}{\partial y}$ 或 $\frac{\partial Q}{\partial x}$ 不连续(有奇点) ↑ 根据被积函数补充曲线去奇点

第二型空间曲线积分:直接计算、路径无关,Stokes公式(18章)

例1 设L为$\frac{x^2}{4}+\frac{y^2}{3}=1$,其周长为$a$,求$\oint_L(2xy+3x^2+4y^2)ds$.

解 因为曲线L关于x轴对称,$2xy$关于y为奇函数,

由对称性可知$\oint_l 2xyds=0$,则

$$\oint_L(2xy+3x^2+4y^2)ds=\oint_L(3x^2+4y^2)ds$$

$$=12\oint_L(\frac{x^2}{4}+\frac{y^2}{3})ds=12\oint_L ds=12a$$

例2 求$I=\int_L (e^x \sin y - my)dx + (e^x \cos y - m)dy$，

其中为由点$(a,0)$到点$(0,0)$的上半圆周

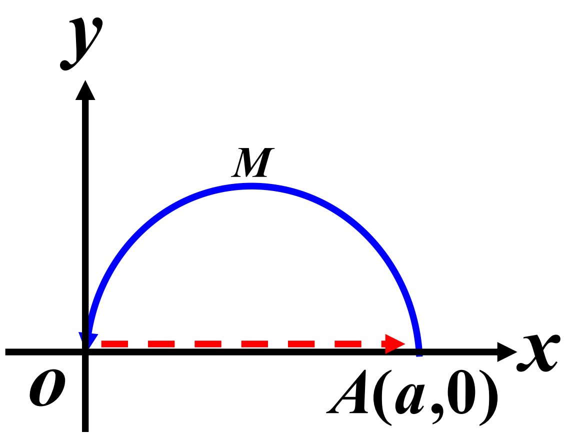
$x^2 + y^2 = ax$， $y > 0$.

解 $\because \quad \dfrac{\partial P}{\partial y} = \dfrac{\partial}{\partial y}(e^x \sin y - my) = e^x \cos y - m$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(e^x \cos y - m) = e^x \cos y$$

即 $\dfrac{\partial P}{\partial y} \neq \dfrac{\partial Q}{\partial x}$ (如下图)

$$I=\int_{L+\overline{OA}}-\int_{\overline{OA}}=\oint_{AMOA}-\int_{\overline{OA}}$$



$$\oint_{AMOA}=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

$$=m\iint_D dxdy=\frac{m}{8}\pi a^2,$$

$$\int_{\overline{OA}}=\int_0^a 0\cdot dx+(e^x-m)\cdot 0=0,$$

$$\therefore\quad I=\oint_{AMOA}-\int_{\overline{OA}}=\frac{m}{8}\pi a^2-0=\frac{m}{8}\pi a^2.$$

例3 设平面有向曲线 L 由连接点 $A(1,0)$ 与点 $B(0,1)$ 的直线段和上半圆周 $y=\sqrt{1-x^2}$ 上从$B(0,1)$ 到 $C(-1,0)$ 的弧段构成, 求$\int_L (x^2-y)dx+(x+e^y)dy$.

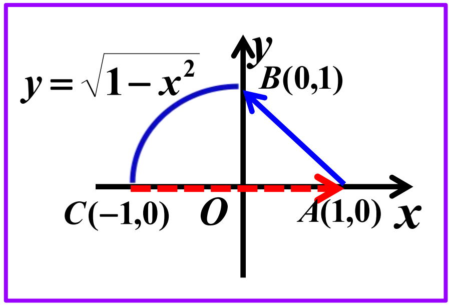
解 由格林公式可知,

$$\int_L (x^2-y)dx+(x+e^y)dy$$

$$=\int_{L+CA}(x^2-y)dx+(x+e^y)dy-\int_{CA}(x^2-y)dx+(x+e^y)dy$$

$$=\iint_D 2dxdy+\int_{AC}x^2dx$$

$$=2(\frac{\pi}{4}+\frac{1}{2})+\int_{-1}^{1}x^2dx=\frac{\pi}{2}+\frac{1}{3}.$$



例4 计算 $\oint_L \frac{xdy - ydx}{4x^2 + y^2}$ 其中 L 是以点 $(1,0)$ 为中心，R 为半径的圆周 $(R > 1)$ 取逆时针方向.

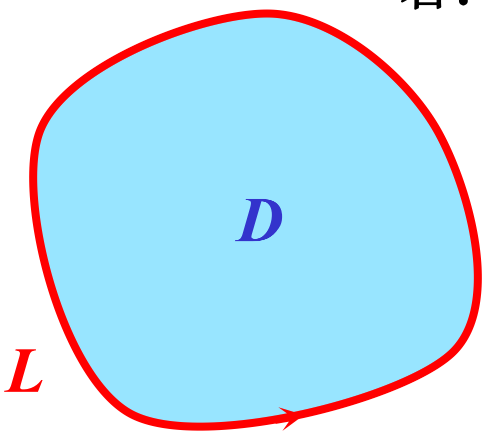
解 $\frac{\partial P}{\partial y} = \frac{y^2 - 4x^2}{(4x^2 + y^2)^2} = \frac{\partial Q}{\partial x} \quad (x, y) \neq (0,0)$

取一足够小的椭圆 $l: 4x^2 + y^2 = a^2$ 使 l 位于 L 的内区域，且取逆时针方向，记 L 和 l 所围区域为 D，则

$$I = \int_{L-l} \frac{xdy - ydx}{4x^2 + y^2} + \int_l \frac{xdy - ydx}{4x^2 + y^2}$$

$$= \iint_D 0dxdy + \frac{1}{a^2}\oint_l xdy - ydx = \frac{1}{a^2}\iint_{4x^2+y^2\leq a^2} 2dxdy = \pi$$

Green常用情形

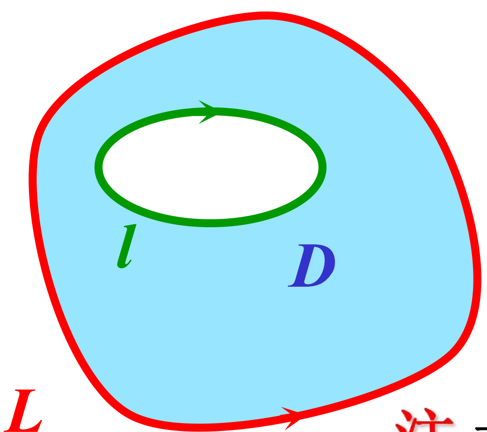


若：1. xOy平面上闭区域 D由分段光滑的曲线 L围成

2. 在D上函数$P(x,y),Q(x,y)\in C^1$

则有 $$\oint_L P\mathrm{d}x+Q\mathrm{d}y=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

其中 L 是 D 的正向边界曲线.



D是复连通区域时，格林公式为：

$$\underset{(逆)}{\oint_L} P\mathrm{d}x+Q\mathrm{d}y+\underset{(顺)}{\oint_l} P\mathrm{d}x+Q\mathrm{d}y=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y$$

注 若在D内又有 $\dfrac{\partial Q}{\partial x}\equiv\dfrac{\partial P}{\partial y}$,则 $\underset{(逆)}{\oint_L} P\mathrm{d}x+Q\mathrm{d}y=\underset{(逆)}{\oint_l} P\mathrm{d}x+Q\mathrm{d}y$

	平面曲线积分与路径无关的四个等价命题
条件	在平面单连通区域 D 上 $P(x,y),Q(x,y)$ 具有连续的一阶偏导数，则下列四个命题等价．
等价命题	(1) 在 D 内 $\int_L P dx + Q dy$ 与路径无关 (2) $\oint_C P\mathrm{d}x + Q\mathrm{d}y = 0$，闭曲线 $C \subset D$ (3) 在 D 内存在 $u(x,y)$ 使 $du = P\mathrm{d}x + Q\mathrm{d}y$ (4) 在 D 内，$\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

若区域及函数满足定理条件，并满足 $\dfrac{\partial Q}{\partial x}=\dfrac{\partial P}{\partial y}$

(1)在D内曲线积分与路径无关；

(2)在D内任意封闭曲线上的积 分为0;

(3)则在 D 内存在 $u(x,y)$, 使 $du = Pdx + Qdy$;

(4)此时方程$Pdx+Qdy=0$为全微分方程,其通解为$u(x,y)=C$.

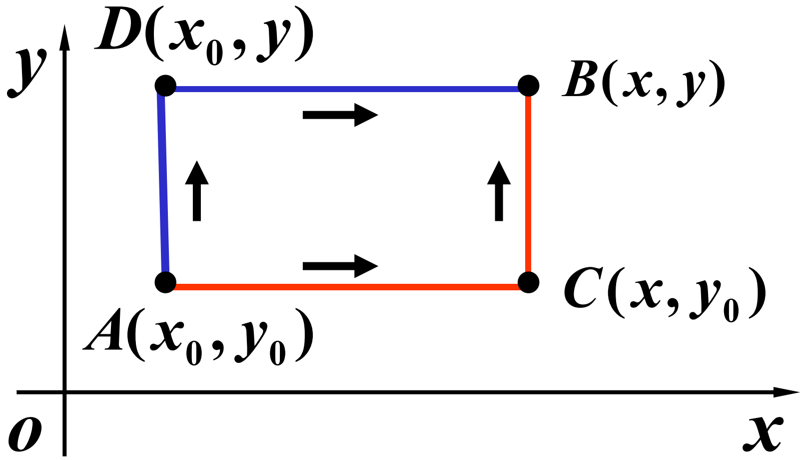
可以通过曲线积分求 $u(x,y)$,

$$u(x,y)=\int_{(x_0,y_0)}^{(x,y)} P(x,y)dx+Q(x,y)dy.$$

也可以通过不定积分法 或凑微分法求 $u(x,y)$

可通过平行于坐标轴的 折线作为积分曲线求 $u(x,y)$

ACB



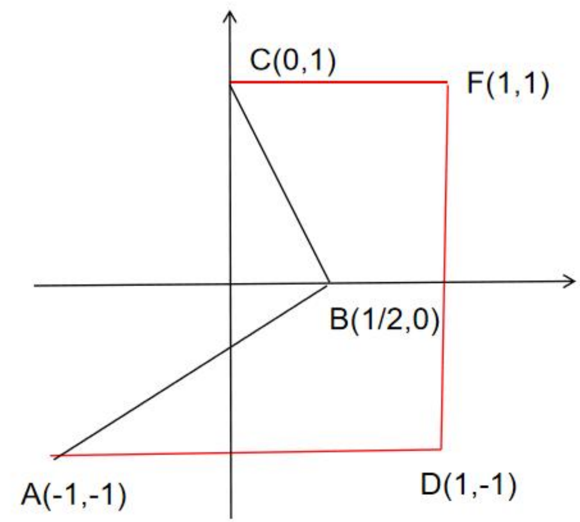
$$u(x,y)=\int_{(x_0,y_0)}^{(x,y)} P(x,y)dx+Q(x,y)dy$$

$$=\int_{x_0}^{x} P(x,y_0)dx+\int_{y_0}^{y} Q(x,y)dy$$

ADB

$$u(x,y)=\int_{y_0}^{y} Q(x_0,y)dy+\int_{x_0}^{x} P(x,y)dx$$

例5 计算 $\int_L \frac{xdy - ydx}{x^2 + y^2}$ 其中 L 是从点 $A(-1,-1)$ 到点 $B(\frac{1}{2},0)$ 再到点 $C(0,1)$ 的折线



解 $\frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial Q}{\partial x} \qquad (x, y) \neq (0,0)$

$$I = \int_L \frac{xdy - ydx}{x^2 + y^2} = \int_{L_1} \frac{xdy - ydx}{x^2 + y^2}$$

$L_1 : A(-1,-1) \to D(1,-1) \to F(1,1) \to C(0,1)$

$$I = \int_{-1}^{1} \frac{-(-1)dx}{x^2 + (-1)^2} + \int_{-1}^{1} \frac{dy}{1^2 + y^2} + \int_{-1}^{1} \frac{-dx}{x^2 + 1^2} = \frac{5}{4}\pi$$

例6 确定常数 λ，使在右半平面 $x>0$ 上向量

$$\vec{A}(x,y)=2xy(x^4+y^2)^{\lambda}\vec{i}-x^2(x^4+y^2)^{\lambda}\vec{j}$$

为某个二阶偏导连续的二 元函数 $u(x,y)$ 的梯度，并求 $u(x,y)$.

解 由 $grad\,u\,(x,y)=\dfrac{\partial u}{\partial x}\vec{i}+\dfrac{\partial u}{\partial y}\vec{j}=\vec{A}(x,y)$, 有

$$\frac{\partial u}{\partial x}=2xy(x^4+y^2)^{\lambda},\quad \frac{\partial u}{\partial y}=-x^2(x^4+y^2)^{\lambda}$$

又由 $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)=\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$，得 $4x(x^4+y^2)^{\lambda}(\lambda+1)=0.$

所以 $\lambda=-1$. 此时

$$P=2xy(x^4+y^2)^{-1},\quad Q=-x^2(x^4+y^2)^{-1}$$

$$\boxed{\text{满足}\quad \frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}\qquad (x,y)\neq(0,0),}$$

从而在右半平面这个单连通区域上, $Pdx+Qdy$是某个函数 $u(x, y)$ 的全微分.

$$u(x,y)=\int_{(1,0)}^{(x,y)}\frac{2xydx-x^2dy}{x^4+y^2}+C$$

$$=\int_0^y\frac{-x^2}{x^4+y^2}dy+C=-\arctan\frac{y}{x^2}+C.$$

例7 设$Q(x,y)$在xoy平面上具有一阶连续偏导数，曲线积分$\int_L 2xydx+Q(x,y)dy$与路径无关，且对任意 t恒有

$$\int_{(0,0)}^{(t,1)} 2xydx+Q(x,y)dy=\int_{(0,0)}^{(1,t)} 2xydx+Q(x,y)dy,$$

求$Q(x,y)$.

解 由积分与路径无关的充要条件

$$\frac{\partial Q}{\partial x}=\frac{\partial}{\partial y}(2xy)=2x,$$

因此$Q(x,y)=x^2+C(y)$，$C(y)$为待定函数，

于是

$$\int_{(0,0)}^{(t,1)} 2xydx + (x^2 + C(y))dy$$

$$= \int_0^1 (t^2 + C(y))dy = t^2 + \int_0^1 C(y)dy,$$

$$u(x,y) = \int_{x_0}^{x} P(x,y_0)dx + \int_{y_0}^{y} Q(x,y)dy$$

$$u(x,y) = \int_{y_0}^{y} Q(x_0,y)dy + \int_{x_0}^{x} P(x,y)dx$$

$$\int_{(0,0)}^{(1,t)} 2xydx + Q(x,y)dy$$

$$= \int_0^t (1 + C(y))dy = t + \int_0^t C(y)dy,$$

又$t^2 + \int_0^1 C(y)dy = t + \int_0^t C(y)dy$

两边 关于t 求导得

$$2t = 1 + C(t), \text{即} C(t) = 2t - 1,$$

$$\varphi(x,y) = x^2 + 2y - 1.$$

例8 选择常数a,b使得曲线积分

$$I=\int_L \frac{(ax^2+2xy+y^2)dx-(x^2+2xy+by^2)dy}{(x^2+y^2)^2}$$与路径无关，

并计算 $$\int_{(1,1)}^{(5,5)} \frac{(ax^2+2xy+y^2)dx-(x^2+2xy+by^2)dy}{(x^2+y^2)^2}.$$

解 $$P=\frac{ax^2+2xy+y^2}{(x^2+y^2)^2},\ Q=-\frac{x^2+2xy+by^2}{(x^2+y^2)^2},$$

则当$(x,y)\neq(0,0)$,

$$\frac{\partial P}{\partial y}=\frac{2[x^3+(1-2a)x^2y-3xy^2-y^3]}{(x^2+y^2)^2},$$

$$\frac{\partial Q}{\partial x}=\frac{2[x^3+3x^2y+(2b-1)xy^2-y^3]}{(x^2+y^2)^2},$$

所以当$1-2a=3, 2b-1=-3$时，$\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$.

即$a=-1, b=-1$时,此时曲线积分与路径无关.

$$故 I=\int_{(1,1)}^{(5,5)}\frac{(-x^2+2xy+y^2)dx-(x^2+2xy-y^2)dy}{(x^2+y^2)^2}$$

$$=\int_1^5\frac{2x^2dx-2x^2dx}{(x^2+x^2)^2}=0.$$

例9 已知平面区域 $D=\{(x,y)\big|0\le x\le\pi,0\le y\le\pi\}$，$L$为$D$的正向边界，试证：

(1) $\oint_L xe^{\sin y}dy-ye^{-\sin x}dx=\oint_L xe^{-\sin y}dy-ye^{\sin x}dx$;

(2) $\oint_L xe^{\sin y}dy-ye^{-\sin x}dx\ge 2\pi^2$.（2003年考研题）

解 (1) 由格林公式，得

$$\text{左边}=\oint_L xe^{\sin y}dy-ye^{-\sin x}dx=\iint_D(e^{\sin y}+e^{-\sin x})dxdy$$

$$\text{右边}=\oint_L xe^{-\sin y}dy-ye^{\sin x}dx=\iint_D(e^{-\sin y}+e^{\sin x})dxdy$$

因为区域D关于$y=x$对称(轮换对称性)，所以

$$\iint_D (e^{\sin y} + e^{-\sin x})dxdy = \iint_D (e^{-\sin y} + e^{\sin x})dxdy$$

因此

$$\oint_L xe^{\sin y}dy - ye^{-\sin x}dx = \oint_L xe^{-\sin y}dy - ye^{\sin x}dx.$$

(2)　由(1)知

$$\oint_L xe^{\sin y}dy - ye^{-\sin x}dx = \iint_D (e^{\sin y} + e^{-\sin x})dxdy$$

$$= \iint_D (e^{\sin x} + e^{-\sin x})dxdy \geq \iint_D 2dxdy = 2\pi^2.$$

例10 设$f(x,y)\in C^2(D)$, $D: x^2+y^2\le 1$,且$\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=e^{-(x^2+y^2)}$

求$\displaystyle\iint_D (x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y})\mathrm{d}x\mathrm{d}y$.

解

$$\iint_D (x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y})\mathrm{d}x\mathrm{d}y$$

$$=\int_0^{2\pi}\mathrm{d}\theta\int_0^1(r\cos\theta\cdot\frac{\partial f}{\partial x}+r\sin\theta\cdot\frac{\partial f}{\partial y})r\mathrm{d}r$$

$$=\int_0^1[\int_0^{2\pi}(r\cos\theta\cdot\frac{\partial f}{\partial x}+r\sin\theta\cdot\frac{\partial f}{\partial y})\mathrm{d}\theta]\,r\mathrm{d}r$$

$$\int_0^{2\pi}(r\cos\theta\cdot\frac{\partial f}{\partial x}+r\sin\theta\cdot\frac{\partial f}{\partial y})\mathrm{d}\theta=\oint_{x^2+y^2=r^2}(-\frac{\partial f}{\partial y}dx+\frac{\partial f}{\partial x}\mathrm{d}y)$$

$$\iint\limits_D \left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)\mathrm{d}x\mathrm{d}y=\int_0^1\Big[\oint\limits_{x^2+y^2=r^2}\left(-\frac{\partial f}{\partial y}dx+\frac{\partial f}{\partial x}\mathrm{d}y\right)\Big]r\mathrm{d}r$$

由格林公式

$$\iint\limits_D \left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)\mathrm{d}x\mathrm{d}y=\int_0^1 r\Big[\iint\limits_{x^2+y^2\le r^2}\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)\mathrm{d}x\mathrm{d}y\Big]\mathrm{d}r$$

$$=\int_0^1 r\Big[\iint\limits_{x^2+y^2\le r^2}\mathrm{e}^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y\Big]\mathrm{d}r$$

$$=\int_0^1 r\left(\int_0^{2\pi}\mathrm{d}\theta\int_0^r\mathrm{e}^{-\rho^2}\cdot\rho\mathrm{d}\rho\right)\mathrm{d}r$$

$$=\frac{\pi}{2\mathrm{e}}.$$

例11 假设L为逆时针方向的封闭光滑曲线, D为L所围区域, u具有连续的二阶偏导数, $\vec{n}$为L外法线的单位向量, 证明

$$\iint\limits_D [(\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2]\,dxdy = -\iint\limits_D u\,\Delta u\,dxdy + \oint\limits_L u\frac{\partial u}{\partial \vec{n}}ds.$$ Green第一公式

解 记$\vec{n} = (\cos(\vec{n}, x), \cos(\vec{n}, y))$,其中$(\vec{n}, x), (\vec{n}, y)$分别表示$\vec{n}$与$x, y$轴的夹角,

又因为u具有一阶连续偏导数,所以

$$\frac{\partial u}{\partial \vec{n}} = \frac{\partial u}{\partial x}\cos(\vec{n}, x) + \frac{\partial u}{\partial y}\cos(\vec{n}, y)$$

设逆时针方向曲线的单位切向量为$\vec{\tau} = (\cos(\vec{\tau}, x), \cos(\vec{\tau}, y))$,则

$$(\vec{n}, x) = (\vec{\tau}, y), (\vec{n}, y) = \pi - (\vec{\tau}, x)$$

则$\cos(\vec{n}, x) = \cos(\vec{\tau}, y), \cos(\vec{n}, y) = -\cos(\vec{\tau}, x)$

因此$\oint\limits_L u\frac{\partial u}{\partial \vec{n}}ds = \oint\limits_L u[\frac{\partial u}{\partial x}\cos(\vec{n}, x) + \frac{\partial u}{\partial y}\cos(\vec{n}, y)]ds$

$$\oint_L u\frac{\partial u}{\partial \vec{n}}ds=\oint_L [u\frac{\partial u}{\partial x}\cos(\vec{\tau},y)-u\frac{\partial u}{\partial y}\cos(\vec{\tau},x)]ds$$

$$=\oint_L u\frac{\partial u}{\partial x}dy-u\frac{\partial u}{\partial y}dx$$

$$=\iint_D [\frac{\partial}{\partial x}(u\frac{\partial u}{\partial x})+\frac{\partial}{\partial y}(u\frac{\partial u}{\partial y})]dxdy \quad (Green\text{公式})$$

$$=\iint_D [(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2+u(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2})]\,dxdy$$

$$=\iint_D [(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2]\,dxdy+\iint_D u\,\Delta u\,dxdy$$

问题得证.

假设L为逆时针方向的封闭光滑曲线, D为L所围区域, u,v具有连续的二阶偏导数, $\vec{n}$为L外法线的单位向量, 则

$$(1)\iint_D [(\frac{\partial u}{\partial x})^2+(\frac{\partial u}{\partial y})^2]\,dxdy=-\iint_D u\,\Delta u\,dxdy+\oint_L u\frac{\partial u}{\partial \vec{n}}ds.$$ Green第一公式

$$(2)\iint_D \begin{vmatrix}\Delta u & \Delta v\\ u & v\end{vmatrix}dxdy=\oint_L \begin{vmatrix}\dfrac{\partial u}{\partial \vec{n}} & \dfrac{\partial v}{\partial \vec{n}}\\ u & v\end{vmatrix}ds$$ Green第二公式

(3)若u为区域D上的调和函数, $r=\sqrt{(\xi-x)^2+(\eta-y)^2}$为$(x,y)$与$L$上动点$(\xi,\eta)$之间的距离, 则

$$u(x,y)=\frac{1}{2\pi}\oint_L (u\frac{\partial \ln r}{\partial \vec{n}}-\ln r\frac{\partial u}{\partial \vec{n}})ds$$

Green第三公式

例12

计算曲线积分$\oint_C (x-y)dx+(x-z)dy+(x-y)dz$其中$C$是曲线

$\begin{cases} x^2+y^2=1 \\ x-y+z=2 \end{cases}$从$z$轴正向看去，$C$的方向是顺时针方向.

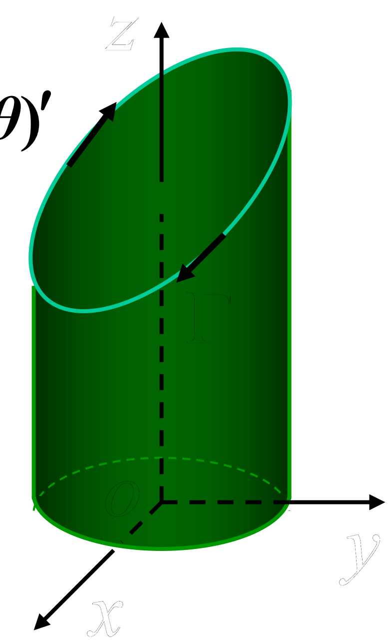
解一 曲线参数方程为 $\begin{cases} x=\cos\theta \\ y=\sin\theta \\ z=2-\cos\theta+\sin\theta \end{cases}$， $\theta:2\pi\to 0$

原式$=\int_{2\pi}^{0}[(\cos\theta-\sin\theta)(\cos\theta)'+(\cos\theta-2+\cos\theta-\sin\theta)(\sin\theta)'$

$+(\cos\theta-\sin\theta)(2-\cos\theta+\sin\theta)']d\theta$

$=\int_{2\pi}^{0}(3\cos^2\theta-2\sin\theta\cos\theta-2\cos\theta)d\theta$

$=-3\pi$



解二

取平面$x-y+z=2$上被Γ所围部分为Σ,取下侧, 由Stokes公式

$$\Sigma: z=2-x+y, (-z_x,-z_y,1)=(1,-1,1)$$

$$I=\iint_{\Sigma}\begin{vmatrix} \mathrm{d}y\mathrm{d}z & \mathrm{d}z\mathrm{d}x & \mathrm{d}x\mathrm{d}y \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x-y & x-z & x-y \end{vmatrix}=\iint_{\Sigma}-\mathrm{d}z\mathrm{d}x+2\mathrm{d}x\mathrm{d}y$$

$$=-\iint_{x^2+y^2\le 1}[(-1)\cdot(-1)+2]dxdy=-3\pi$$