北京航空航天大學
BEIHANG UNIVERSITY

第七章习题课

北京航空航天大学 BEIHANG UNIVERSITY

例1 已知$a=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin x\cos^4 x}{1+x^2}dx, b=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin^3 x+\cos^4 x)dx,$

$c=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(x^2\sin^3 x-\cos^4 x)dx,$ 则（ ）

(A) $b<c<a$ (B) $a<c<b$

(C) $b<a<c$ (D) $c<a<b$

解 $a=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin x\cos^4 x}{1+x^2}dx=0,$ 因为被积函数是奇函数.

$$b=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin^3 x+\cos^4 x)dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4 x dx>0,$$

$$c=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(x^2\sin^3 x-\cos^4 x)dx=-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4 x dx<0,$$

$$\therefore c<a<b.$$

例2 求极限 $\lim\limits_{n\to\infty}\sin\frac{\pi}{n}\sum\limits_{i=1}^{n}\frac{1}{2+\frac{\pi i}{n}}$.

$\sim \frac{\pi}{n}$

$[0, \frac{\pi}{n}, \frac{2\pi}{n}, \ldots, \frac{i-1}{n}\pi, \frac{i}{n}\pi, \ldots, \pi]$ $\quad \Delta x_i = \frac{\pi}{n}$

$\sum\limits_{i=1}^{n}\frac{1}{2+\frac{i}{n}\pi}\cdot\frac{\pi}{n}$ $\quad f=\frac{1}{2+x}$

解 $\sin\frac{\pi}{n}\approx\frac{\pi}{n}(n\to\infty)$

$$\lim_{n\to\infty}\sin\frac{\pi}{n}\sum_{i=1}^{n}\frac{1}{2+\frac{\pi i}{n}}=\lim_{n\to\infty}\cancel{\frac{\pi}{n}}\sum_{i=1}^{n}\frac{1}{2+\frac{\pi i}{n}} \quad =\int_0^{\pi}\frac{1}{2+\cancel{x}}dx$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\cdot\frac{\pi}{2+\frac{\pi i}{n}}=\int_0^1\frac{\pi}{2+\pi x}dx=\ln(1+\frac{\pi}{2}),$$

$$\therefore \lim_{n\to\infty}\sum_{i=1}^{n}\sin\frac{\pi}{n}\cdot\frac{1}{2+\frac{\pi i}{n}}=\ln(1+\frac{\pi}{2}).$$

例3 求 $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}dx.$

解　设 $I=\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}dx, J=\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x+\cos x}dx.$

则 $I+J=\int_0^{\frac{\pi}{2}} dx=\frac{\pi}{2},$

$(\cos x+\sin x)'$

$= -\sin x+\cos x$

$$I-J=\int_0^{\frac{\pi}{2}} \frac{\sin x-\cos x}{\sin x+\cos x}dx=-\int_0^{\frac{\pi}{2}} \frac{d(\cos x+\sin x)}{\sin x+\cos x}=0$$

$-\ln|\cos x+\sin x|\Big|_0^{\frac{\pi}{2}}$

故得 $2I=\frac{\pi}{2}$, 即 $I=\frac{\pi}{4}.$

例4 计算 $\int_0^{\pi}\sqrt{\sin^3 x-\sin^5 x}\,dx.$

解 $\because f(x)=\sqrt{\sin^3 x-\sin^5 x}=|\cos x|(\sin x)^{\frac{3}{2}}$

$$\therefore \int_0^{\pi}\sqrt{\sin^3 x-\sin^5 x}\,dx=\int_0^{\pi}|\cos x|(\sin x)^{\frac{3}{2}}dx$$

$$=\int_0^{\frac{\pi}{2}}\cos x(\sin x)^{\frac{3}{2}}dx-\int_{\frac{\pi}{2}}^{\pi}\cos x(\sin x)^{\frac{3}{2}}dx$$

$$=\int_0^{\frac{\pi}{2}}(\sin x)^{\frac{3}{2}}d\sin x-\int_{\frac{\pi}{2}}^{\pi}(\sin x)^{\frac{3}{2}}d\sin x$$

$$=\frac{2}{5}(\sin x)^{\frac{5}{2}}\Big|_0^{\frac{\pi}{2}}-\frac{2}{5}(\sin x)^{\frac{5}{2}}\Big|_{\frac{\pi}{2}}^{\pi}=\frac{4}{5}.$$

例5 求 $\int_0^{\ln 2}\sqrt{1-e^{-2x}}\,dx.$

$\sqrt{1-(e^{-x})^2}$

$\sqrt{a^2-x^2}$

解 令 $e^{-x}=\sin t,$

则 $x=-\ln\sin t,\ dx=-\dfrac{\cos t}{\sin t}dt.$

$$\text{原式}=\int_{\frac{\pi}{2}}^{\frac{\pi}{6}}\cos t\left(-\frac{\cos t}{\sin t}\right)dt=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{\cos^2 t}{\sin t}dt$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{dt}{\sin t}-\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\sin t\,dt=\ln(2+\sqrt{3})-\frac{\sqrt{3}}{2}.$$

例6 求 $\int_{-\frac{1}{2}}^{\frac{1}{2}}[\frac{\sin x}{x^8+1}+\sqrt{\ln^2(1-x)}]dx.$

解 原式 $=0+\int_{-\frac{1}{2}}^{\frac{1}{2}}|\ln(1-x)|dx$

$$=\int_{-\frac{1}{2}}^{0}\ln(1-x)dx-\int_{0}^{\frac{1}{2}}\ln(1-x)dx$$

$$=\frac{3}{2}\ln\frac{3}{2}+\ln\frac{1}{2}.$$

例7 设 $f''(x)$ 在 $[0,1]$ 上连续， 且 $f(0)=1$，

$f(2)=3$，$f'(2)=5$，求 $\int_0^1 xf''(2x)dx$.

解 $\int_0^1 xf''(2x)d\overset{2}{x} = \frac{1}{2}\int_0^1 xdf'(2x)$

$= \frac{1}{2}\left[xf'(2x)\right]_0^1 - \frac{1}{2}\int_0^1 f'(2x)d\overset{2}{x} = \frac{1}{2}f'(2) - \frac{1}{4}\left[f(2x)\right]_0^1$

$= \frac{5}{2} - \frac{1}{4}\left[f(2) - f(0)\right] = 2.$

例8 设 $f(x)=x+\sqrt{1-x^2}\int_0^1 tf(t)dt$, 求 $f(x)$.

$= C$

解 $$xf(x)=x^2+x\sqrt{1-x^2}\int_0^1 tf(t)dt$$

$= C$

$$\int_0^1 xf(x)dx=\int_0^1 x^2dx+\int_0^1 x\sqrt{1-x^2}\left(\int_0^1 tf(t)dt\right)dx$$

C

$$=\int_0^1 x^2dx+\int_0^1 tf(t)dt\cdot\int_0^1 x\sqrt{1-x^2}dx$$

$$=\frac{1}{3}+\frac{1}{3}\int_0^1 tf(t)dt$$

C

$$\int_0^1 tf(t)dt=\frac{1}{2}.$$

$f(x)=x+\frac{1}{2}\sqrt{1-x^2}$

例9 设$f(x)$满足$\int_0^1 f(tx)dt = f(x) + x\sin x$，$f(0)=0$，

且有一阶导数，求$f(x)$（$x \neq 0$）.

解 将$\int_0^1 f(tx)dt$被积函数的x体现在积分限上

$\int_0^1 f(tx)\,d(tx) =$

$tx = y$

设 $y = tx$，则

$$\int_0^x f(y)dy = xf(x) + x^2\sin x$$

两边对x求导可得

$$f(x) = f(x) + xf'(x) + 2x\sin x + x^2\cos x \quad (x \neq 0),$$

则 $f'(x) = -2\sin x - x\cos x.$

$f = \int [2\sin x - x\cos x]dx = \ ?$

例10 设 $f(x)=\int_0^x e^{-y^2+2y}dy$，求 $\int_0^1 (x-1)^2 f(x)dx$.

解 原式 $=\int_0^1 (x-1)^2[\int_0^x e^{-y^2+2y}dy]dx$ $=\frac{1}{3}\int_0^1 [\underbrace{\int_0^x e^{-y^2+2y}dy}_{u}]\, \underbrace{d(x-1)^3}_{v}$

$$=[\frac{1}{3}(x-1)^3\int_0^x e^{-y^2+2y}dy]_0^1-\int_0^1 \frac{1}{3}(x-1)^3 e^{-x^2+2x}dx$$

$f(x)$ $(x-1)^2$ $(x-1)dx$

$$=-\frac{1}{6}\int_0^1 (x-1)^2 e^{-(x-1)^2+1}d[(x-1)^2]$$

$$\underline{\underline{\text{令 }(x-1)^2=u}}\ -\frac{e}{6}\int_1^0 ue^{-u}du=-\frac{1}{6}(e-2).$$

例11 设 $f(x)$ 连续，且 $\int_0^{x^3-1} f(t)dt = x$，求 $f(7)$.

解 两边关于 x 求导可得

$$f(x^3-1)3x^2 = 1$$

$$f(7) = \frac{1}{12}$$

例12 设$F(x)=\int_0^x [\underbrace{\int_0^u \sin(u-t)^2 dt}_{g(u)}]du$，求$F''(x)$.

解 $F'(x)=\int_0^x \sin(\underbrace{x-t}_{y})^2 dt = g(x)$

令 $x-t=y$,

$$\int_0^x \sin(x-t)^2 dt = -\int_x^0 \sin y^2 dy = \int_0^x \sin y^2 dy.$$

$F''(x)=\sin x^2$.

例13 设$f(x)$在$x=1$处可导，且$f(1)=0$，$f'(1)=1$，求极限

$$\lim_{x\to 1}\frac{\int_1^x\left(t\int_t^1 f(u)du\right)dt}{(1-x)^3}.$$

解 应用洛比达法则，可得

$$\lim_{x\to 1}\frac{\int_1^x\left(t\int_t^1 f(u)du\right)dt}{(1-x)^3}=\lim_{x\to 1}\frac{x\int_x^1 f(u)du}{-3(1-x)^2}$$

$$=\lim_{x\to 1}\left[\frac{\int_x^1 f(u)du}{6(1-x)}-\frac{xf(x)}{6(1-x)}\right].$$

$$\lim_{x\to1}\frac{\int_x^1 f(u)du}{6(1-x)}=\lim_{x\to1}\frac{-f(x)}{-6}=0,$$

$$\lim_{x\to1}\frac{-xf(x)}{6(1-x)}=\lim_{x\to1}\frac{x}{6}\cdot\frac{f(x)-f(1)}{x-1}=\frac{f'(1)}{6}=\frac{1}{6},$$

$\to f'(1)$

$$\therefore \lim_{x\to1}\frac{\int_1^x\left(t\int_t^1 f(u)du\right)dt}{(1-x)^3}=\frac{1}{6}.$$

1 $\quad \to 0$

$$\text{或}\lim_{x\to1}\frac{\int_1^x\left(t\int_t^1 f(u)du\right)dt}{(1-x)^3}=\lim_{x\to1}\frac{x\int_x^1 f(u)du}{-3(1-x)^2}=\lim_{x\to1}\frac{\int_x^1 f(u)du}{-3(1-x)^2}$$

$$=\lim_{x\to1}\frac{f(x)}{6(x-1)}==\lim_{x\to1}\frac{f(x)-f(1)}{6(x-1)}=\frac{1}{6}f'(1)$$

例14 求极限 $\lim\limits_{n\to\infty}\int_0^{\frac{\pi}{2}}\sin^n x\,dx \quad (n>0).$

解一 $0\le\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\sin^n\xi_n\cdot\dfrac{\pi}{2}\to 0\ (n\to\infty)$?

$0<\sin^n\xi_n<1$

反例：$\xi_n=\dfrac{\pi}{2}-\dfrac{1}{n}\in(0,\dfrac{\pi}{2})$， 但 $\lim\limits_{n\to\infty}\sin^n\xi_n=1.$ **解法错误**

$0 \qquad \frac{\pi}{2}-\varepsilon \qquad \frac{\pi}{2}$

解二 $\forall\varepsilon\in(0,\dfrac{\pi}{2}),\exists N,$ 当 $n>N$ 时，$\left|\sin^n(\dfrac{\pi}{2}-\varepsilon)\cdot(\dfrac{\pi}{2}-\varepsilon)\right|<\varepsilon,$

$$\int_0^{\frac{\pi}{2}}\sin^n x\,dx=\int_0^{\frac{\pi}{2}-\varepsilon}\sin^n x\,dx+\int_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}}\sin^n x\,dx$$

$$=\sin^n(\frac{\pi}{2}-\varepsilon)\cdot(\frac{\pi}{2}-\varepsilon)+1\cdot\varepsilon<2\varepsilon.$$

例15 $f(x)$在$[0,1]$连续、单调减少，证明对$\forall\alpha\in[0,1]$,有

$$\int_0^{\alpha} f(x)dx \ge \alpha\int_0^1 f(x)dx$$

例14

$\alpha\int_0^{\alpha} + \alpha\int_{\alpha}^{1}$

解 问题等价于对$\forall\alpha\in[0,1]$,有

$$(1-\alpha)\int_0^{\alpha} f(x)dx \ge \alpha\int_{\alpha}^{1} f(x)dx$$

0 x_1 α x_2 1

两端分别使用积分中值定理得

$$(1-\alpha)\alpha f(x_1) \ge (1-\alpha)\alpha f(x_2)$$

$\alpha t \le t$

显然有$f(x_1)\ge f(x_2)$，所以结论成立.

$x=\alpha t$

$左 = \alpha\int_0^1 f(\alpha t)dt \ge \alpha\int_0^1 f(t)dt = 右$

例16 设$f(x)$在$[0,b]$上连续且单调减少，证明对于任何$a\in(0,b)$，有 $a\int_0^b f(x)dx < b\int_0^a f(x)dx.$

$\frac{1}{b}\int_0^b f(t)dt < \frac{1}{a}\int_0^a f(t)dt$

解1 令$F(x)=a\int_0^x f(t)dt - x\int_0^a f(t)dt,\ F(a)=0.$

$$F'(x)=af(x)-\int_0^a f(t)dt = \int_0^a f(x)dt - \int_0^a f(t)dt$$

$$=\int_0^a [f(x)-f(t)]dt. \qquad F'(x)<0,\ x\in[a,b].$$

$F(b) > F(a) = 0$

解2 令$F(x)=\dfrac{\int_0^x f(t)dt}{x}$，$F'(x)=\dfrac{\int_0^x (f(x)-f(t))dt}{x^2}<0$

解3 令 $u=\dfrac{b}{a}x$ $\quad x=\frac{b}{a}u \quad a<b$

解4

$$a\int_0^b f(x)dx = a\left(\int_0^a f(x)dx + \int_a^b f(x)dx\right)$$

$$= a\int_0^a f(x)dx + a\int_a^b f(x)dx$$

$$= a\int_0^a f(x)dx + a(b-a)f(\xi) \quad (\xi \in (a,b))$$

$$b\int_0^a f(x)dx = [a+(b-a)]\int_0^a f(x)dx$$

$$= a\int_0^a f(x)dx + (b-a)\int_0^a f(x)dx$$

$$= a\int_0^a f(x)dx + (b-a)af(\eta) \quad (\eta \in (0,a))$$

例17

$f(x)$在$[a,b]$上连续且大于零,则方程

$$\int_a^x f(t)\mathrm{d}t+\int_b^x \frac{1}{f(t)}\mathrm{d}t=0$$在(a,b)内的实根的个数为多少？

解 $$F(x)=\int_a^x f(t)\mathrm{d}t+\int_b^x \frac{1}{f(t)}\mathrm{d}t,$$

$$F'(x)=f(x)+\frac{1}{f(x)}\overset{\geq 2}{>}0;$$

$$F(a)=\int_b^a \frac{1}{f(t)}\mathrm{d}t<0, F(b)=\int_a^b f(t)\mathrm{d}t>0$$

例18 设 $f(x)$ 在区间 $[a,b]$ 上连续，且 $f(x)>0$.

证明 $\int_a^b f(x)dx\cdot\int_a^b \frac{dx}{f(x)}\geq(b-a)^2$.

证 作辅助函数

$$F(x)=\int_a^x f(t)dt\int_a^x \frac{dt}{f(t)}-(x-a)^2,$$

$$\because F'(x)=f(x)\int_a^x \frac{1}{f(t)}dt+\int_a^x f(t)dt\cdot\frac{1}{f(x)}-2(x-a)$$

$$=\int_a^x \frac{f(x)}{f(t)}dt+\int_a^x \frac{f(t)}{f(x)}dt-\int_a^x 2dt,$$

$$\because\ f(x)>0,\qquad \therefore \frac{f(x)}{f(t)}+\frac{f(t)}{f(x)}\geq 2$$

即 $F'(x)=\int_a^x\left(\frac{f(x)}{f(t)}+\frac{f(t)}{f(x)}-2\right)dt\geq 0$

$F(x)$ 单调增加.

又 $\because F(a)=0,\qquad \therefore F(b)\geq F(a)=0,$

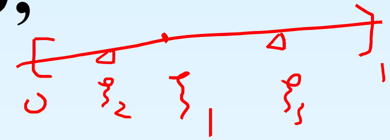
$\therefore$即 $\int_a^b f(x)dx\cdot\int_a^b\frac{dx}{f(x)}\geq(b-a)^2.$

例19 设 $f(x)$在$[0,1]$上连续可导，且$\int_0^1 f(x)dx=0,$ $= f(\xi_1)=0$

记 $F(x)=\int_0^x f(t)dt$，证明：若 $\int_0^1 F(x)dx=0$，

则存在$\xi\in(0,1)$，使 $f'(\xi)=0.$

证 记 $G(x)=\int_0^x F(t)dt$，则 $G(1)=G(0)=0,$



$\therefore \exists\xi_1\in(0,1)$使 $G'(\xi_1)=0$，即$F(\xi_1)=0.$

由$F(0)=F(\xi_1)=F(1)=0$知

$\exists\xi_2\in(0,\xi_1)$，$\xi_3\in(\xi_1,1)$使

$F'(\xi_2)=f(\xi_2)=0, F'(\xi_3)=f(\xi_3)=0.$

$\therefore$ 存在$\xi\in(0,1)$，使 $f'(\xi)=0.$

例20 设 $f(x)$ 在 $[0,1]$ 上连续，且 $f(1)=\int_0^1 e^{1-x} f(x)\,\mathrm{d}x.$

证明 $\exists \xi \in (0,1)$ 使得 $f'(\xi) = f(\xi).$

$f'(\xi) - f(\xi) = 0$

$e^{-\xi}[f'(\xi) - f(\xi)] = 0$

证 设 $F(x)=e^{-x} f(x)$，则 $F(x)$ 在 $[0,1]$ 上可导.

$[e^{-\xi} f(\xi)]' = 0$

$[e^{-x} f(x)]'_{x=\xi} = 0$

而 $F(1)=e^{-1} f(1) = e^{-1}\cdot\int_0^1 e^{1-x} f(x)\,\mathrm{d}x$

$F = e^{-x} f(x)$

$$= \int_0^1 F(x)\mathrm{d}x = F(\xi_1)$$

$\xi_1 \in (0,1)$

根据罗尔定理知 $\exists \xi \in (\xi_1, 1)$ 使 $F'(\xi) = 0$，即

$$e^{-\xi}[f'(\xi) - f(\xi)] = 0 \Rightarrow f'(\xi) = f(\xi)$$

例21 $f(x)$在$[0,1]$上连续，且

$$\int_0^1 f(t)\,\mathrm{d}t = 3\int_0^{\frac{1}{3}} e^{1-x^2}\left(\int_0^x f(t)\,\mathrm{d}t\right)\mathrm{d}x$$

证明：至少存在一个$\xi\in(0,1)$,

$$s.t.\ f(\xi) = 2\xi\int_0^{\xi} f(x)\,\mathrm{d}x.$$

$$\left(\int_0^\xi f(x)dx\right)' - 2\xi\int_0^\xi f(x)dx = 0$$

$$e^{-\xi^2}\left[\left(\int_0^\xi f(x)dx\right)' - 2\xi\int_0^\xi f(x)dx\right] = 0$$

$$\left[e^{-\xi^2}\int_0^\xi f(x)dx\right]' = 0$$

$$\int_0^1 f(t)\,\mathrm{d}t = e^{1-\eta^2}\left(\int_0^{\eta} f(t)\,\mathrm{d}t\right), \eta\in(0,1)$$

$$F(x) = e^{-x^2}\int_0^x f(t)dt$$

$$e^{-1}\int_0^1 f(t)\,\mathrm{d}t = e^{-\eta^2}\left(\int_0^{\eta} f(t)\,\mathrm{d}t\right)$$

$$F'(\xi) = 0$$

$$F(1) = F(\eta)$$

例22 设$f(x)$在$[a,b]$上二阶可导, 且$f''(x)>0$, 证明

将 b 看成变量

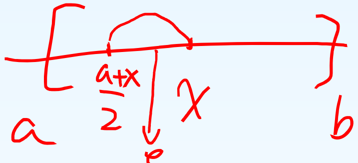
$$(b-a)f\left(\frac{a+b}{2}\right)\le\int_a^b f(x)dx.$$

$F(b)=\int_a^b-(b-a)f(\frac{a+b}{2})$

证 令$F(x)=\int_a^x f(t)dt-(x-a)f\left(\frac{a+x}{2}\right)$, $F(a)=0$

$=x-\frac{a+x}{2}$

$$F'(x)=f(x)-f\left(\frac{a+x}{2}\right)-\frac{x-a}{2}f'\left(\frac{a+x}{2}\right)$$



$$=f'(\xi_1)\frac{x-a}{2}-\frac{x-a}{2}f'\left(\frac{a+x}{2}\right)\qquad \xi_1\in\left(\frac{a+x}{2},x\right)$$

$\xi_1\in(\frac{a+x}{2},x)$

$=\frac{x-a}{2}\left[f'(\xi_1)-f'\left(\frac{a+x}{2}\right)\right]\geq 0$

$$= \frac{x-a}{2}\left(f'(\xi_1) - f'\left(\frac{a+x}{2}\right)\right)$$

$$= \frac{x-a}{2} f''(\xi_2)\left(\xi_1 - \frac{a+x}{2}\right) \qquad \xi_2 \in \left(\frac{a+x}{2}, \xi_1\right)$$

$$> 0$$

$$\therefore F(b) \geq F(a) = 0.$$

例23 设函数$f(x)$在$[a,b]$上连续可微, 证明

$$\max_{x\in[a,b]}|f(x)|\le \frac{1}{b-a}\left|\int_a^b f(x)dx\right|+\int_a^b|f'(x)|dx$$

$\int_a^b f(x)dx=f(\xi)(b-a)$

$\left|\frac{1}{b-a}\int_a^b f(x)dx\right|=|f(\xi)|$

$|f(\xi)|$

证 已知$f(x)$在$[a,b]$上连续，由积分中值定理，

存在$\xi\in[a,b]$，满足$\int_a^b f(x)dx=f(\xi)(b-a).$

又因为 $f(x)=\int_\xi^x f'(t)dt+f(\xi)$ （N-L公式）

$f(x)-f(\xi)=\int_\xi^x f'(t)dt$

$\le\left|\int_\xi^x |f'(t)|dt\right|\le\int_a^b|f'(t)|dt$

所以 $|f(x)|\le\left|\int_\xi^x f'(t)dt\right|+\frac{1}{b-a}\left|\int_a^b f(x)dx\right|$

$\le\int_a^b|f'(x)|dx+\frac{1}{b-a}\left|\int_a^b f(x)dx\right|$

$$\le\left|\int_a^b|f'(t)|dt\right|+\frac{1}{b-a}\left|\int_a^b f(x)dx\right|$$

从而 $$\max_{x\in[a,b]}|f(x)|\le\frac{1}{b-a}\left|\int_a^b f(x)dx\right|+\int_a^b|f'(x)|dx$$

例24 设 $f(x), g(x)$在 $[a,b]$上连续, 且满足

$$\int_a^x f(t)dt \ge \int_a^x g(t)dt,\ x \in [a,b],$$

$F(x) \geqslant G(x)$

$F(x) = \int_a^x f(t)dt$

$G(x) = \int_a^x g(t)dt$

$$\int_a^b f(t)dt = \int_a^b g(t)dt,$$

$= \int_a^b x\, dF(x) = xF(x)\big|_a^b - \int_a^b F(x)dx = b\int_a^b f(t)dt - \int_a^b F(x)dx$

证明 $\int_a^b xf(x)dx \le \int_a^b xg(x)dx.$ $= b\int_a^b g(t)dt - \int_a^b G(x)dx$

证 令 $F(x) = \int_a^x f(t)dt,\ G(x) = \int_a^x g(t)dt.$

$$\int_a^b xf(x)dx = \int_a^b xdF(x) = xF(x)\Big|_a^b - \int_a^b F(x)dx$$

$$= bF(b) - \int_a^b F(x)dx$$

$$= b\int_a^b f(t)dt - \int_a^b F(x)dx$$

$$\int_a^b xg(x)dx = \int_a^b xdG(x) = xG(x)\Big|_a^b - \int_a^b G(x)dx$$

$$= bG(b) - \int_a^b G(x)dx$$

$$= b\int_a^b g(t)dt - \int_a^b G(x)dx$$

例24设 $f(x)$ 在$[a,b]$上连续，且$f(x)>0$,证明存在

$\xi \in (a,b)$，使得$\int_a^{\xi} f(x)dx = \int_{\xi}^{b} f(x)dx.$

证 $F(x) = \int_a^x f(t)dt - \int_x^b f(t)dt$

$F(x) = \int_a^x f(t)dt - \int_x^b f(t)dt$

$F(x)$是区间$[a,b]$上的连续函数

$F(a) = -\int_a^b f(t)dt < 0$, $F(b) = \int_a^b f(t)dt > 0$

由连续函数的介值定理

$\exists \xi \in [a,b]$, 使得 $F(\xi) = 0.$

例25 设 $f(x), g(x)$在$[a,b]$上连续，证明

$$(1)\left(\int_a^b f(x)g(x)dx\right)^2 \le \int_a^b f^2(x)dx\int_a^b g^2(x)dx$$

$$(2)\sqrt{\int_a^b (f(x)+g(x))^2dx} \le \sqrt{\int_a^b f^2(x)dx}+\sqrt{\int_a^b g^2(x)dx}$$

证　(1)对所有$t \in R$, $[(tf(x)+g(x)]^2 \ge 0.$

$$0 \le \int_a^b (tf(x)+g(x))^2 dx$$

$$= t^2\int_a^b f^2(x)dx + 2t\int_a^b f(x)g(x)dx + \int_a^b g^2(x)dx$$

$$\therefore \Delta = \left(2\int_a^b f(x)g(x)dx\right)^2 - 4\int_a^b f^2(x)dx \cdot \int_a^b g^2(x)dx \le 0$$

$$(2)\int_a^b (f(x)+g(x))^2 dx$$

$$=\int_a^b f^2(x)dx+2\int_a^b f(x)g(x)dx+\int_a^b g^2(x)dx$$

$$\left(\sqrt{\int_a^b f^2(x)dx}+\sqrt{\int_a^b g^2(x)dx}\right)^2$$

$$=\int_a^b f^2(x)dx+2\sqrt{\int_a^b f^2(x)dx\cdot\int_a^b g^2(x)dx}+\int_a^b g^2(x)dx$$

利用(1)的结论

北京航空航天大学
BEIHANG UNIVERSITY

例26 设$f(x)$在$[a,b]$上有连续的二阶导函数，$f(a)=f(b)=0$

$= \int_a^b f(x)\,d(x-a)$

证明：(1) $\int_a^b f(x)dx = \frac{1}{2}\int_a^b (x-a)(x-b)f''(x)dx$

(2) $\left|\int_a^b f(x)dx\right| \le \frac{1}{12}(b-a)^3 \max_{a\le x\le b}|f''(x)|$

证明(1) $\int_a^b f(x)dx = \int_a^b f(x)d(x-a)$

$$= f(x)(x-a)\Big|_a^b - \int_a^b (x-a)f'(x)dx$$

$$= -\int_a^b (x-a)f'(x)dx \quad = -\int_a^b (x-a)f'(x)\,d(x-b)$$

$$= -\int_a^b (x-a)f'(x)d(x-b)$$

$$= -(x-a)f'(x)(x-b)\Big|_a^b + \int_a^b \{(x-a)f''(x)+f'(x)\}(x-b)dx$$

$$= \int_a^b \{(x-a)f''(x)+f'(x)\}(x-b)dx$$

$$= \int_a^b (x-a)(x-b)f''(x)dx + \int_a^b f'(x)(x-b)dx$$

$$= \int_a^b (x-a)(x-b)f''(x)dx + \underbrace{f(x)(x-b)\Big|_a^b}_{=0} - \int_a^b f(x)dx$$

所以 $$\int_a^b f(x)dx = \frac{1}{2}\int_a^b (x-a)(x-b)f''(x)dx$$

$$2)\left|\int_a^b f(x)dx\right| \le \frac{1}{2}\int_a^b |x-a||x-b||f''(x)|dx$$

$$\le \frac{1}{2}\max_{a\le x\le b}|f''(x)|\int_a^b |x-a||x-b|dx$$

$$= \frac{1}{2}\max_{a\le x\le b}|f''(x)|\int_a^b (x-a)(b-x)dx$$

$$= \frac{1}{12}(b-a)^3\max_{a\le x\le b}|f''(x)|$$

例27 设 $f(x)$ 在 $[a,b]$ 上有连续二阶导数，证明 $\exists\xi\in[a,b]$，

$$\text{满足}\int_a^b f(x)\mathrm{d}x=(b-a)f\left(\frac{a+b}{2}\right)+\frac{(b-a)^3}{24}f''(\xi).$$

$= F(b)$ $\quad$ $= F'(x_0)$

证 设 $F(x)=\int_a^x f(t)\mathrm{d}t$,

则 $F'(x)=f(x), F''(x)=f'(x), F'''(x)=f''(x)$.

将 $F(x)$ 在 $x_0=\dfrac{a+b}{2}$ 展开为二阶Taylor公式，代入 a,b 点的值，得

$a \quad x_0 \quad \xi_1 \quad b$

$$F(b)=F(x_0)+F'(x_0)(b-x_0)+\frac{F''(x_0)}{2!}(b-x_0)^2+\frac{F'''(\xi_1)}{3!}(b-x_0)^3$$

$$F(a)=F(x_0)+F'(x_0)(a-x_0)+\frac{F''(x_0)}{2!}(a-x_0)^2+\frac{F'''(\xi_2)}{3!}(a-x_0)^3$$

$a \quad \xi_2 \quad x_0$

其中 $x_0<\xi_1<b, a<\xi_2<x_0$.

上面两式相减得，

$F(b)$

$$F(b)-F(a)=F'(x_0)(b-a)+\frac{(b-a)^3}{48}[F'''(\xi_1)+F'''(\xi_2)].$$

由介质定理得，$\exists\xi\in[\xi_1,\xi_2]$，使得

$$F'''(\xi_1)+F'''(\xi_2)=f''(\xi_1)+f''(\xi_2)=2f''(\xi),$$

$$\frac{f''(\xi_1)+f''(\xi_2)}{2}=f''(\xi)$$

又$F'(x_0)=f(x_0)=f(\frac{a+b}{2})$，

所以$\int_a^b f(x)\mathrm{d}x=F(b)-F(a)=(b-a)f(\frac{a+b}{2})+\frac{(b-a)^3}{24}f''(\xi).$