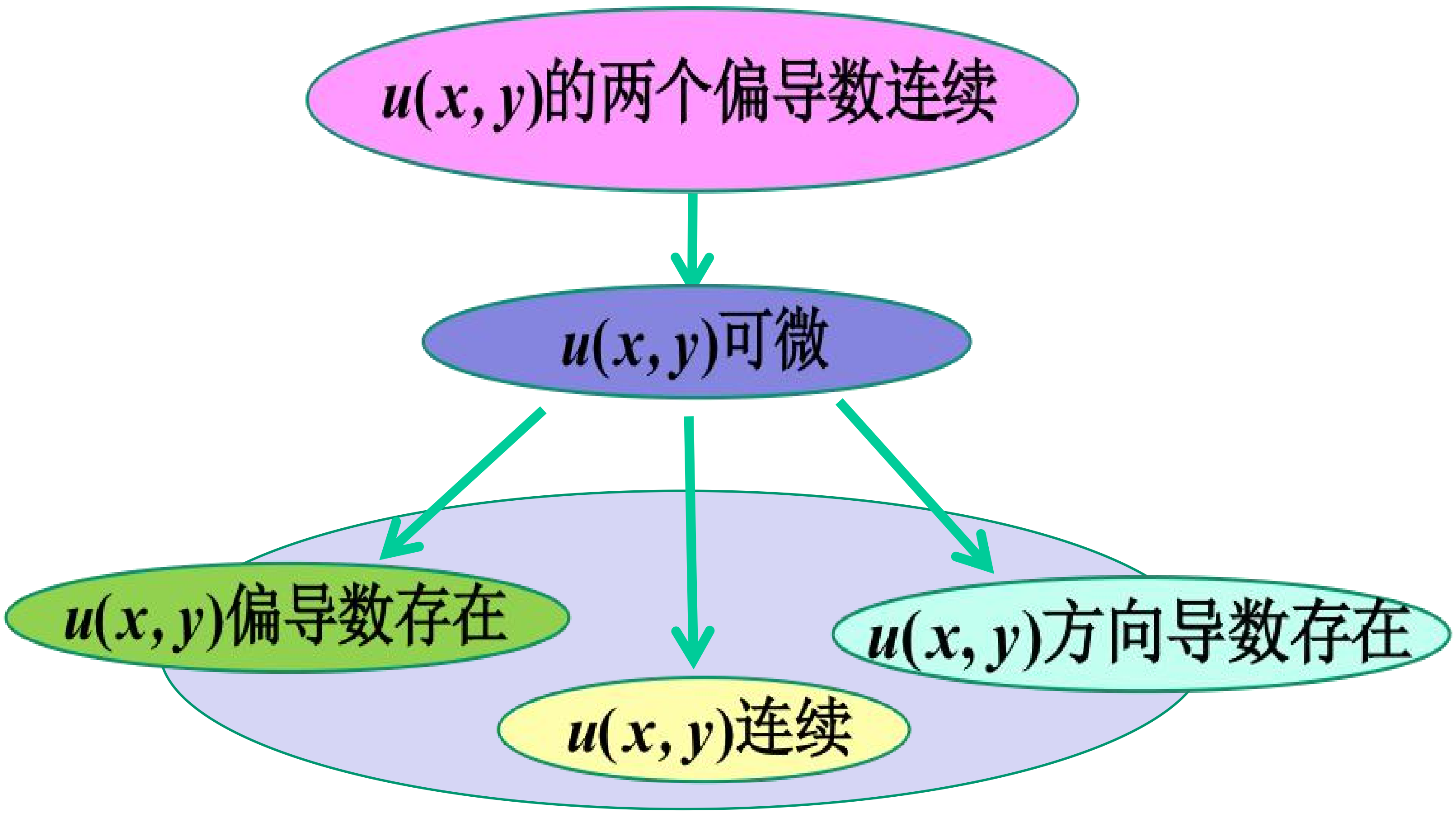
第15章 习题课

一、主要内容

- 多元函数微分学
  - 微分方法
    - 基本概念
    - 求导法则
      - 复合函数链式求导
      - 微分形式不变性
      - 隐函数求导
  - 微分应用
    - 近似计算
    - 几何应用
    - 极值求法

1. 多元函数连续、可偏导、可微、方向导数



上图“箭头”的反方向都不成立， 但逆否命题都成立.

2. 隐函数的应用

1. 曲面的切平面与法线

(1) 曲面 $F(x,y,z)=0$，**法向量**： $\boldsymbol{n}=\{F'_x, F'_y, F'_z\}$

(2) 曲面 $z=f(x,y)$ $\boldsymbol{n}=\{f'_x, f'_y, -1\}$

2. 空间曲线的切线与法平面

(1)曲线 $\begin{cases} x=x(t) \\ y=y(t) \\ z=y(t) \end{cases}$，**切向量**： $\boldsymbol{\tau}=(x'(t_0), y'(t_0), z'(t_0))$

(2)曲线 $\begin{cases} F(x,y,z)=0 \\ G(x,y,z)=0 \end{cases}$，**切向量**： $\boldsymbol{\tau}=\boldsymbol{n}_1\times\boldsymbol{n}_2$

其中，法向量: $\boldsymbol{n}_1=\{F'_x, F'_y, F'_Z\}$ $\boldsymbol{n}_2=\{G'_x, G'_y, G'_z\}$

3. 极值与最值

(1) 无条件极值

	极大	极小
• 定义	$f(x,y) \le f(x_0,y_0)$	$f(x,y) \ge f(x_0,y_0)$

• 必要条件

若偏导处存在,则$f_x(x_0,y_0) = f_y(x_0,y_0) = 0,$

极值点和偏导数不存在的点都是可能的极值点.

• 充分条件

(2) 最大最小值

(3) 条件极值与拉格朗日乘数法

二、典型例题

例1 设 $u=f(xyz)$，其中f有三阶连续导数，

$$f(1)=0,\ f'(1)=1,\quad \frac{\partial^3 u}{\partial z\partial y\partial x}=x^2y^2z^2f'''(xyz)$$

求$f(x)$.

解 令 $t=xyz$，则$u=f(t)$，

$$\frac{\partial u}{\partial x}=yzf'(t),\qquad \frac{\partial^2 u}{\partial y\partial x}=zf'(t)+xyz^2f''(t),$$

$$\frac{\partial^3 u}{\partial z\partial y\partial x}=f'(t)+3xyzf''(t)+x^2y^2z^2f'''(t).$$

由题意知 $f'(t)+3xyzf''(t)=0,$

即 $f'(t)+3tf''(t)=0.$

令 $p(t)=f'(t)$，则 $p(t)+3tp'(t)=0.$

解得 $p(t)=ct^{-\frac{1}{3}}$，从而有 $f(t)=C_1t^{\frac{2}{3}}+C_2.$

再由 $f(1)=0, f'(1)=1$ 可得 $C_1=\frac{3}{2}, C_2=-\frac{3}{2}.$

$f(t)=\frac{3}{2}t^{\frac{2}{3}}-\frac{3}{2}.$ $f(x)=\frac{3}{2}x^{\frac{2}{3}}-\frac{3}{2}.$

例2 设$z = xf(\frac{y}{x}) + 2y\varphi(\frac{x}{y})$, f, φ具有二阶连续偏导数,

求$\frac{\partial^2 z}{\partial y \partial x}$.

解 $$\frac{\partial z}{\partial x} = f(\frac{y}{x}) - f'(\frac{y}{x})\frac{y}{x} + 2\varphi'(\frac{x}{y})$$

$$\frac{\partial^2 z}{\partial y \partial x} = f'(\frac{y}{x})\frac{1}{x} - f''(\frac{y}{x})\frac{y}{x^2} - f'(\frac{y}{x})\frac{1}{x} + 2\varphi''(\frac{x}{y})\frac{-x}{y^2}$$

$$= -f''(\frac{y}{x})\frac{y}{x^2} - 2\varphi''(\frac{x}{y})\frac{x}{y^2}$$

例3 若 $f(tx,ty,tz)=t^k f(x,y,z),(t\in R)$,

证明$x\dfrac{\partial f(x,y,z)}{\partial x}+y\dfrac{\partial f(x,y,z)}{\partial y}+z\dfrac{\partial f(x,y,z)}{\partial z}=kf(x,y,z)$.

证明 设$u=tx,v=ty,w=tz$,则

$$\frac{\partial}{\partial t}\big(f(tx,ty,tz)\big)=\frac{\partial f}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial t}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial t}$$

$$=\frac{\partial f(u,v,w)}{\partial u}\cdot x+\frac{\partial f(u,v,w)}{\partial v}\cdot y+\frac{\partial f(u,v,w)}{\partial w}\cdot z,$$

则$t\dfrac{\partial f(tx,ty,tz)}{\partial t}=u\dfrac{\partial f(u,v,w)}{\partial u}+v\dfrac{\partial f(u,v,w)}{\partial v}+w\dfrac{\partial f(u,v,w)}{\partial w}$,

又 $\dfrac{\partial}{\partial t}(t^k f(x,y,z)) = k\cdot t^{k-1} f(x,y,z),$

$$f(tx,ty,tz) = t^k f(x,y,z)$$

从而 $t\cdot\dfrac{\partial}{\partial t}(t^k f(x,y,z)) = kt^k f(x,y,z)$

$$= kf(tx,ty,tz) = kf(u,v,w),$$

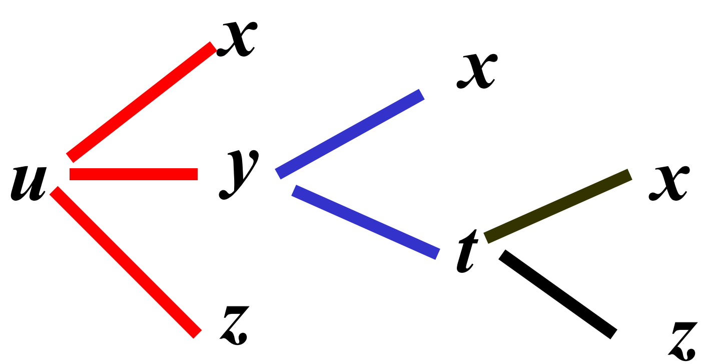
于是 $u\dfrac{\partial f(u,v,w)}{\partial u} + v\dfrac{\partial f(u,v,w)}{\partial v} + w\dfrac{\partial f(u,v,w)}{\partial w} = kf(u,v,w),$

也即是 $x\dfrac{\partial f(x,y,z)}{\partial x} + y\dfrac{\partial f(x,y,z)}{\partial y} + z\dfrac{\partial f(x,y,z)}{\partial z} = kf(x,y,z).$

例4 设 $u=f(x,y,z),y=\varphi(x,t),t=\psi(x,z)$

各函数满足求偏导条件，求 $\dfrac{\partial u}{\partial x}$

解一 变量间的关系如下图所示



$$\Rightarrow \frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{\partial \varphi}{\partial x}+\frac{\partial f}{\partial y}\cdot\frac{\partial \varphi}{\partial t}\cdot\frac{\partial \psi}{\partial x}$$

解二 用全微分形式不变性

$$du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}[\frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial t}dt] + \frac{\partial f}{\partial z}dz$$

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}[\frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial t}(\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial z}dz)] + \frac{\partial f}{\partial z}dz$$

注意到 x, z 是独立自变量

故

$$du=\left(\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial t}\frac{\partial \psi}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial t}\frac{\partial \psi}{\partial z}+\frac{\partial f}{\partial z}\right)dz$$

由全微分定义

$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial t}\frac{\partial \psi}{\partial x}$$

$$\frac{\partial u}{\partial z}=\frac{\partial f}{\partial y}\frac{\partial \varphi}{\partial t}\frac{\partial \psi}{\partial z}+\frac{\partial f}{\partial z}$$

注 解法二在实际计算中显得十分灵便且不易出错

例5 已知 $u=\dfrac{x+y}{x-y}$，求 $\dfrac{\partial^{m+n}u}{\partial y^n\partial x^m}$.

解 由 $u=\dfrac{x+y}{x-y}=1+\dfrac{2y}{x-y}=1+2y\dfrac{1}{x-y}$，可得

$$\frac{\partial^m u}{\partial x^m}=2y(-1)^m m!(x-y)^{-m-1}=2(-1)^m m!\left(\frac{x}{(x-y)^{m+1}}-\frac{1}{(x-y)^m}\right),$$

$$\frac{\partial^{m+n}u}{\partial y^n\partial x^m}=2(-1)^m m!\frac{\partial^n\left(\frac{x}{(x-y)^{m+1}}-\frac{1}{(x-y)^m}\right)}{\partial y^n}$$

$$=2(-1)^m m!\left[\frac{x(m+1)(m+2)\cdots(m+n)}{(x-y)^{m+n+1}}-\frac{m(m+1)\cdots(m+n)}{(x-y)^{m+n}}\right]$$

$$=2(-1)^m(m+n-1)!\left(\frac{nx+my}{(x-y)^{m+n+1}}\right).$$

例6 设$u(x,y)=\int_0^{xy} f(t)(xy-t)\mathrm{d}t$,其中$f(t)$连续,求$u_{xx}+u_{yy}$.

解 由$u(x,y)=\int_0^{xy} f(t)(xy-t)\mathrm{d}t = xy\int_0^{xy} f(t)\mathrm{d}t-\int_0^{xy} tf(t)\mathrm{d}t$

得 $u_x(x,y)=y\int_0^{xy} f(t)\mathrm{d}t+xy^2 f(xy)-xy^2 f(xy)$

$$=y\int_0^{xy} f(t)\mathrm{d}t,$$

故 $u_{xx}(x,y)=y^2 f(xy)$,

同理 $u_{yy}(x,y)=x^2 f(xy)$,

所以 $u_{xx}+u_{yy}=(x^2+y^2)f(xy)$.

例7 设$f(x,y)=|x-y|\varphi(x,y)$, $\varphi(x,y)$在(0,0)的某邻域内连续，证明 $f(x,y)$在(0,0)处可微的充要条件是 $\varphi(0,0)=0.$

可微 $$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(x_0+\Delta x,y_0+\Delta y)-f(x_0,y_0)-f_x(x_0,y_0)\Delta x-f_y(x_0,y_0)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}=0.$$

证明 "$\Rightarrow$"

$f(x,y)$在(0,0)处可微，所以$f_x(0,0)$, $f_y(0,0)$存在，

则 $\lim\limits_{\Delta x\to 0}\dfrac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{|\Delta x|}{\Delta x}\varphi(\Delta x,0)$极限存在

$$\lim_{\Delta x\to 0^+}\frac{|\Delta x|}{\Delta x}\varphi(\Delta x,0)=\varphi(0,0),\ \lim_{\Delta x\to 0^-}\frac{|\Delta x|}{\Delta x}\varphi(\Delta x,0)=-\varphi(0,0)$$

所以$\varphi(0,0)=-\varphi(0,0)$，即$\varphi(0,0)=0$. 必要性得证 .

"$\Leftarrow$"　已知$\varphi(0,0)=0$,则

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{|\Delta x|}{\Delta x}\varphi(\Delta x,0)=0,$$

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{|\Delta y|}{\Delta y}\varphi(0,\Delta y)=0.$$

$$\left|\frac{f(\Delta x,\Delta y)-f(0,0)-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}\right|$$

$$=\frac{|\Delta x-\Delta y|}{\sqrt{(\Delta x)^2+(\Delta y)^2}}|\varphi(\Delta x,\Delta y)|\le 2|\varphi(\Delta x,\Delta y)|$$

所以$\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\dfrac{f(\Delta x,\Delta y)-f(0,0)-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}=0$

从而$f(x,y)$在$(0,0)$处可微,充分性得证.

例8 讨论下列函数在原点的连续性、可微分性以及偏导数的存在性、连续性.

$$f(x,y)=\begin{cases}\dfrac{1-e^{x(x^2+y^2)}}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}.$$

解 (1) $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)=\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{1-e^{x(x^2+y^2)}}{x(x^2+y^2)}x=0=f(0,0)$

所以$f(x,y)$在$(0,0)$点连续.

(2) $f_x(0,0)=\lim\limits_{x\to 0}\dfrac{1-e^{x^3}}{x^3}=-1,\quad f_y(0,0)=\lim\limits_{y\to 0}\dfrac{1-e^{0}}{y^3}=0,$

$$f_x(x,y)=\begin{cases}-\dfrac{(3x^2+y^2)(x^2+y^2)e^{x(x^2+y^2)}+(1-e^{x(x^2+y^2)})2x}{(x^2+y^2)^2}, & x^2+y^2\neq 0\\ -1, & x^2+y^2=0\end{cases},$$

$$f_y(x,y)=\begin{cases}-\dfrac{2xy(x^2+y^2)e^{x(x^2+y^2)}+(1-e^{x(x^2+y^2)})2y}{(x^2+y^2)^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}.$$

可计算得到：$\lim\limits_{\substack{x\to 0\\ y\to 0}} f_x(x,y)=-1,\ \ \lim\limits_{\substack{x\to 0\\ y\to 0}} f_y(x,y)=0,$

因此 $f_x(x,y), f_y(x,y)$均在$(0,0)$处连续.

所以在二元函数 $f(x,y)$ 在原点可微.

或由以下证明可微性

$$(3)\quad \lim_{\substack{x\to 0\\ y\to 0}} \frac{f(x,y)-f(0,0)-f_x(0,0)x-f_y(0,0)y}{\sqrt{x^2+y^2}}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{\dfrac{1-e^{x(x^2+y^2)}}{x^2+y^2}+x}{\sqrt{x^2+y^2}}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{1+x(x^2+y^2)-e^{x(x^2+y^2)}}{(x^2+y^2)^{\frac{3}{2}}}$$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{1+x(x^2+y^2)-\left(1+x(x^2+y^2)+\dfrac{1}{2}x^2(x^2+y^2)^2\right)-o\left(x^2(x^2+y^2)^2\right)}{(x^2+y^2)^{\frac{3}{2}}} = 0.$$

例9 设函数 $f(x,y)=\begin{cases}\dfrac{x^3}{x^2+y^2}, & x^2+y^2\neq 0,\\ 0, & x^2+y^2=0.\end{cases}$

证明函数$f(x,y)$在$(0,0)$点不可微，但是在$(0,0)$点沿任意方向的方向导数都存在.

证明

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x}=1.$$

$$f_y(0,0)=\lim_{\Delta y\to 0}\frac{f(0,\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{0}{\Delta y}=0.$$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(\Delta x,\Delta y)-f(0,0)-f_x(0,0)\Delta x-f_y(0,0)\Delta y}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(\Delta x,\Delta y)-f_x(0,0)\Delta x}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$$

$$=\lim_{\substack{x\to 0\\ y\to 0}}\frac{-xy^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$\lim_{\substack{y=kx\\ x\to 0^+}}\frac{-xy^2}{(x^2+y^2)\sqrt{x^2+y^2}}=-\frac{k^2}{(1+k^2)^{\frac{3}{2}}}$$

因此$\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{-xy^2}{(x^2+y^2)\sqrt{x^2+y^2}}$不存在,从而不可微.

设$\vec{l}=(\cos\varphi,\sin\varphi)$为任意向量，$\varphi$为$x$轴正向到方向$\vec{l}$的转角，则

$$\left.\frac{\partial f}{\partial \vec{l}}\right|_{(0,0)}=\lim_{t\to 0^+}\frac{f(t\cos\varphi,t\sin\varphi)-f(0,0)}{t}$$

$$=\lim_{t\to 0^+}\frac{t^3\cos^3\varphi}{t^3}=\cos^3\varphi$$

所以$f(x,y)$在$(0,0)$点沿任意方向的方向导数都存在.

例10 求方程组$\begin{cases} x^2+y^2-z=0 \\ x^2+2y^2+3z^2=4a^2 \end{cases}$所确定的 y, z 关于x 的隐函数的导数$\dfrac{dy}{dx}, \dfrac{dz}{dx}$.

解 令$\begin{cases} F(x,y,z)=x^2+y^2-z=0 \\ G(x,y,z)=x^2+2y^2+3z^2-4a^2=0 \end{cases}$，则

$$F_x=2x, F_y=2y, F_z=-1; G_x=2x, G_y=4y, G_z=6z.$$

$$\frac{dy}{dx}=-\frac{\begin{vmatrix} 2x & -1 \\ 2x & 6z \end{vmatrix}}{\begin{vmatrix} 2y & -1 \\ 4y & 6z \end{vmatrix}}=-\frac{x(1+6z)}{2y(1+3z)},$$

$$\frac{dz}{dx}=-\frac{\begin{vmatrix} 2y & 2x \\ 4y & 2x \end{vmatrix}}{\begin{vmatrix} 2y & -1 \\ 4y & 6z \end{vmatrix}}=\frac{x}{1+3z}.$$

例11 试证锥面 $z=\sqrt{x^2+y^2}+3$ 的所有切平面都通过锥面的顶点.

证明 设$P_0(x_0,y_0)$是锥面上任意一点，令

$F(x,y,z)=\sqrt{x^2+y^2}-z+3$，则在点 P_0处的法向量为

$$\left(\frac{x_0}{\sqrt{x_0^2+y_0^2}},\frac{y_0}{\sqrt{x_0^2+y_0^2}},-1\right),$$

于是过点 P_0的切面方程为

$$\frac{x_0}{\sqrt{x_0^2+y_0^2}}(x-x_0)+\frac{y_0}{\sqrt{x_0^2+y_0^2}}(y-y_0)-(z-z_0)=0,$$

将锥面顶点坐标 $\boxed{(0,0,3)}$ 代入上述方程左端，有

$$-\frac{x_0^2}{\sqrt{x_0^2+y_0^2}}-\frac{y_0^2}{\sqrt{x_0^2+y_0^2}}-(3-z_0)$$

$$=-\sqrt{x_0^2+y_0^2}-3+z_0=0.$$

故结论成立.

例12 设 $z=g(x,y)=f(e^{x+y},x^2+y^2)+1$，$f(x,y)$ 具有二阶连续偏导数，且 $\lim\limits_{\substack{x\to1\\y\to0}}\dfrac{f(x,y)-x-y+1}{\sqrt{(x-1)^2+y^2}}=0$，求曲面 $z=g(x,y)$ 在点 $(0,0)$ 处的切平面和法线方程.

，

解 由 $\lim\limits_{\substack{x\to1\\y\to0}}\dfrac{f(x,y)-x-y+1}{\sqrt{(x-1)^2+y^2}}=0$，可知

$$f(1,0)=0,\ \ g(0,0)=f(1,0)+1=1.$$

曲面 $z=g(x,y)$ 在点 $(0,0,1)$ 处的法向量为

$$\vec{n}=(g_x(0,0),\ \ g_y(0,0),\ \ -1)$$

又因为

$$g_x = f_1(e^{x+y}, x^2+y^2)e^{x+y} + 2f_2(e^{x+y}, x^2+y^2)x,$$

$$g_y = f_1(e^{x+y}, x^2+y^2)e^{x+y} + 2f_2(e^{x+y}, x^2+y^2)y,$$

所以 $g_x(0,0) = f_1(1,0)$. $g_y(0,0) = f_1(1,0)$.

又由 $\lim\limits_{\substack{x\to 1\\ y\to 0}} \dfrac{f(x,y)-x-y+1}{\sqrt{(x-1)^2+y^2}} = 0$ 可知

$$f(x,y)-x-y+1 = \alpha(x,y)\sqrt{(x-1)^2+y^2},$$

其中 $\lim\limits_{\substack{x\to 1\\ y\to 0}} \alpha(x,y) = 0.$

即 $f(x,y)=x+y-1+\alpha(x,y)\sqrt{(x-1)^2+y^2}$.

$$f(1+x,0)=x+\alpha(1+x,0)|x|.$$

$$f_1(1,0)=f_x(1,0)=\lim_{\Delta x\to 0}\frac{f(1+\Delta x,0)-f(1,0)}{\Delta x}$$

$$=\lim_{x\to 0}\frac{f(1+x,0)}{x}=\lim_{x\to 0}\frac{x+\alpha(1+x,0)|x|}{x}$$

$$=\lim_{x\to 0}\left(1+\frac{\alpha(1+x,0)|x|}{x}\right)=1.$$

因此 $\vec{n}=(1,\ 1,\ -1)$, 从而切平面方程为

$\boxed{x+y-z+1=0.}$ 法线方程为 $\boxed{\dfrac{x}{1}=\dfrac{y}{1}=\dfrac{z-1}{-1}.}$

例13 求$f(x,y,z)=xyz$在条件$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{r}$

$(x,y,z,r>0)$下的极小值.

解 设拉格朗日函数为

$$L(x,y,z,\lambda)=xyz+\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{r}\right).$$

由$\begin{cases} L_x=0 \\ L_y=0 \\ L_z=0 \\ L_\lambda=0 \end{cases}$ 知L稳定点为$:x=y=z=3r,$

$$\lambda=(3r)^4$$

如何判断 $f(3r,3r,3r)=(3r)^3$ 是否为条件极值？

方法一：$H_L(3r,3r,3r)$正定

方法二：把条件 $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{r}$ 看成隐函数$z=z(x,y)$,

$$f(x,y,z)=xy\cdot z(x,y)=F(x,y).$$

计算出$z_x, z_y, F_x, F_y, F_{xx}, F_{xy}, F_{yy}$，$H_F(3r,3r)$正定,

所以 $xyz\geq(3r)^3, (x,y,z,r>0$ 且 $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{r})$

所以 $xyz \geq (3r)^3, (x, y, z, r > 0$ 且 $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{r})$

令 $x = a, y = b, z = c$，则 $r = (\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^{-1}$

代入 $xyz \geq (3r)^3$ 得

$$abc \geq [3(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^{-1}]^3$$

或 $\quad 3(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})^{-1} \leq \sqrt[3]{abc}.$

练习 证明：$\forall a, b, c$，有 $abc^3 \leq 27\left(\frac{a+b+c}{5}\right)^5$

例14 证明 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 与平面 $Ax+By+Cz=0$ 相交

所截得的椭圆面积为 $\pi\sqrt{\frac{(A^2+B^2+C^2)a^2b^2c^2}{A^2a^2+B^2b^2+C^2c^2}}$.

证明 设椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 与平面 $Ax+By+Cz=0$

相交的所截的椭圆的长短轴为 d_1,d_2

则 d_1,d_2 是椭圆上的点到原点距离的最大值和最小值

$$L(x,y,z,\lambda,\mu)=x^2+y^2+z^2-\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+\mu(Ax+By+Cz)$$

$$\begin{cases} L_x = 2x - 2\lambda \dfrac{x}{a^2} + \mu A = 0 & (1) \\ L_x = 2y - 2\lambda \dfrac{y}{b^2} + \mu B = 0 & (2) \\ L_x = 2z - 2\lambda \dfrac{z}{c^2} + \mu C = 0 & (3) \\ L_\lambda = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} - 1 = 0 \\ L_\mu = Ax + By + Cz = 0 \end{cases}$$

$$(1)\cdot x + (2)\cdot y + (3)\cdot z$$

$$\Rightarrow x^2 + y^2 + z^2 = \lambda$$

$\therefore$ 半长轴和半短轴的值即为驻点处的$\sqrt{\lambda}$

$$(1)\cdot A\left(1-\frac{\lambda}{b^2}\right)\left(1-\frac{\lambda}{c^2}\right) + (2)\cdot B\left(1-\frac{\lambda}{a^2}\right)\left(1-\frac{\lambda}{c^2}\right) + (3)\cdot C\left(1-\frac{\lambda}{a^2}\right)\left(1-\frac{\lambda}{b^2}\right)$$

$$\Rightarrow A^2\left(1-\frac{\lambda}{b^2}\right)\left(1-\frac{\lambda}{c^2}\right) + B^2\left(1-\frac{\lambda}{a^2}\right)\left(1-\frac{\lambda}{c^2}\right) + C^2\left(1-\frac{\lambda}{a^2}\right)\left(1-\frac{\lambda}{b^2}\right) = 0$$

设上述方程的解为λ_1, λ_2,

则椭圆面积为$\pi\sqrt{\lambda_1\lambda_2}$

$$=\pi\sqrt{\frac{\left(A^2+B^2+C^2\right)a^2b^2c^2}{A^2a^2+B^2b^2+C^2c^2}}$$

二元函数Taylor公式

二元函数$f(x,y)$在点(x_0,y_0)处的k阶$Taylor$公式

$$f(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0)+$$

$$\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)f(x_0,y_0)+\frac{1}{2!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^2 f(x_0,y_0)$$

$$+\cdots+\frac{1}{k!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^k f(x_0,y_0)+R_k$$

$$R_k=\frac{1}{(k+1)!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^{k+1} f(x_0+\theta\Delta x, y_0+\theta\Delta y)$$

$$(0<\theta<1)$$

---$Lagrange$ 型余项

$$\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^{p} f(x_0, y_0)$$

$$=\sum_{i=0}^{p} C_p^i \frac{\partial^p f(x_0, y_0)}{\partial x^{p-i}\partial y^i}(\Delta x)^{p-i}(\Delta y)^i .$$

余项$R_k = o(\rho^k)$称为$Peano$型余项，其中

$$\rho = \|(x_0+\Delta x, y_0+\Delta y)-(x_0, y_0)\|$$

$$= \|(\Delta x, \Delta y)\| = \sqrt{(\Delta x)^2+(\Delta y)^2}$$

二元函数$f(x,y)$在点$(0,0)$处的$Taylor$公式

$$f(x,y)=f(0,0)+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)f(0,0)$$

$$\frac{1}{2!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f(0,0)+\cdots+\frac{1}{k!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^k f(0,0)+R_k$$

$$R_k=\frac{1}{(k+1)!}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^{k+1} f(\theta x,\theta y)$$

---Lagrange型余项

$$R_k=o(\rho^k),\quad \rho=\|(x,y)\|=\sqrt{x^2+y^2}$$

---Peano 型余项

常用一元函数Taylor公式

1. $$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + o(x^n)$$

2. $$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{(-1)^{n-1}}{n} x^n + o(x^n)$$

$$\ln(1-x) = -[x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}] + o(x^n)$$

3. $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{n-1}}{(2n-1)!} x^{2n-1} + o(x^{2n})$$

4. $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n}{(2n)!} x^{2n} + o(x^{2n+1})$$

5. $f(x)=(1+x)^{\lambda},(x>-1)$

广义二项式

$$=\sum_{k=0}^{n}\frac{\lambda(\lambda-1)\cdots(\lambda-k+1)}{k!}x^{k}+o(x^{n})$$

$$=\sum_{k=0}^{n}C_{\lambda}^{k}x^{k}+o(x^{n})$$

特例

$$\frac{1}{1+x}=1-x+x^{2}-x^{3}+\cdots+(-1)^{n}x^{n}+o(x^{n})$$

$$=\sum_{k=0}^{n}(-1)^{k}x^{k}+o(x^{n})$$

例15 写出下列函数在(0,0)的带peano型余项的Taylor公式

$$(1)\ z=\frac{\cos y}{1-x},\ 2阶;\ (2)\ z=\sin x\cos y,\ 3阶;$$

$$(3)\ z=xye^{-(x^2+y^2)},\ 4阶.$$

解 (1) $$\frac{\cos y}{1-x}=(1+x+x^2+o(x^2))(1-\frac{y^2}{2}+o(y^3))$$

$$=1+x+x^2-\frac{y^2}{2}+o(\rho^2)\qquad 其中\rho=\sqrt{x^2+y^2}$$

(2) $$z=\sin x\cos y=(x-\frac{x^3}{3!}+o(x^4))(1-\frac{y^2}{2!}+o(y^3))$$

$$=x-\frac{x^3}{6}-\frac{xy^2}{2}+o(\rho^4)$$

$$(3)\ e^x = 1 + x + \frac{x^2}{2!} + o(x^2)$$

$$z = xye^{-(x^2+y^2)}$$

$$= xy(1-(x^2+y^2) + \frac{(x^2+y^2)^2}{2!} + o((x^2+y^2)^2))$$

$$= xy - xy(x^2+y^2) + o(\rho^4)$$