北京航空航天大學
BEIHANG UNIVERSITY

§18.3 Gauss公式与Stoks公式 (2)

斯托克斯(Stokes)公式

1. 双侧曲面的侧与其边界曲线的方向

设 S 为双侧曲面, Γ 为其边界曲线,

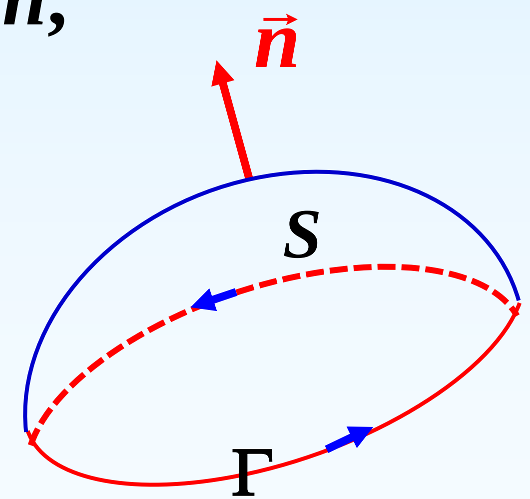
则 S 的侧和 Γ 的方向之间满足 右手法则.

在 S 取定的侧,取一代表法向量 $\vec{n}$,

若右手大拇指指向 $\vec{n}$ 的方向,

则四指所指的方向就是 Γ 的

方向.



2. 斯托克斯公式

定理3.2 设 S 为光滑的双侧曲面,其边界曲线 Γ 是按段光滑的连续曲线. 若函数 P, Q, R 在 S (连同Γ)上连续, 且有连续的一阶偏导数, 则

$$\oint_{\Gamma} Pdx + Qdy + Rdz$$

$$= \iint_{S} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)dzdx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy$$

其中 S 的侧和 Γ 的方向满足右手法则.

分析： 只证 $\iint_S \frac{\partial P}{\partial z} dzdx - \frac{\partial P}{\partial y} dxdy = \oint_\Gamma P dx$

$$\iint_S \frac{\partial P}{\partial z} \cos\beta dS$$

$$\iint_S \frac{\partial P}{\partial z} \frac{\cos\beta}{\cos\gamma} \cos\gamma dS$$

$$\oint_C P(x, y, z(x, y))dx$$

$$-\iint_{D_{xy}} \frac{\partial}{\partial y} P[x, y, z(x, y)]dxdy$$

$$-\iint_S \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y}\right)dxdy$$

证 先证 $\iint_S \frac{\partial P}{\partial z}dzdx - \frac{\partial P}{\partial y}dxdy = \oint_\Gamma Pdx$

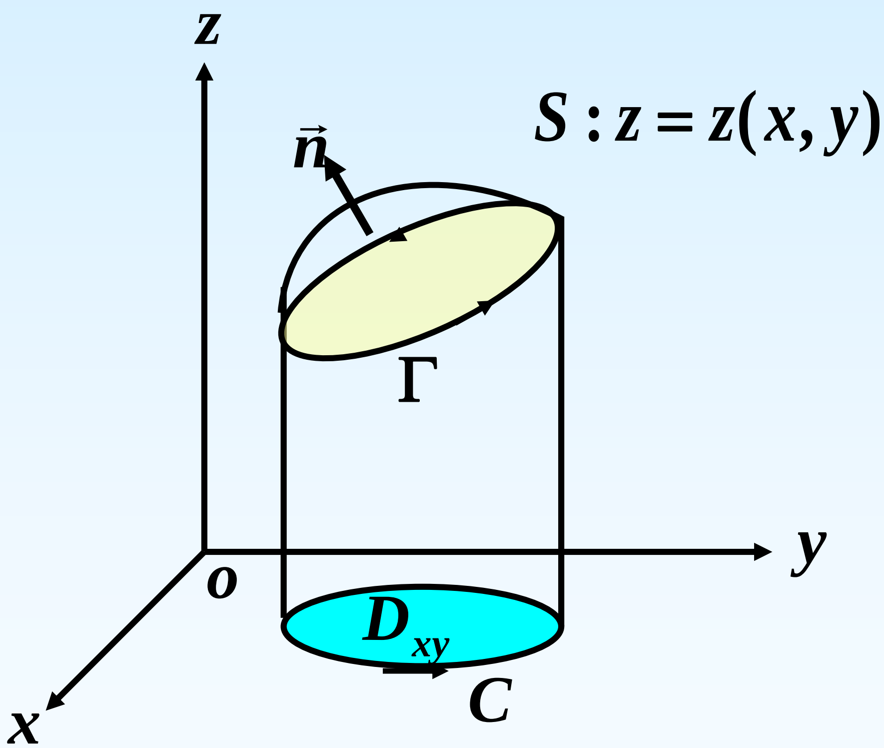
设曲面S由方程 $z = z(x, y), (x, y) \in D_{xy}$确定,

且取上侧为 S 的侧,

有向曲线C是Γ在xy平面上的投影,

易见C是D_{xy}的边界线,

方向为正向.



设 C 的参数方程为: $x = x(t), y = y(t), \alpha \le t \le \beta$.

则 $\oint_{\Gamma} P(x,y,z)dx = \oint_{\Gamma} P(x,y,z(x,y))dx$

$$= \int_{\alpha}^{\beta} P(x(t),y(t),z(x(t),y(t)))dt$$

$$= \oint_{C} P(x,y,z(x,y))dx = -\iint_{D_{xy}} \frac{\partial}{\partial y} P[x,y,z(x,y)]dxdy$$

$$= -\iint_{D_{xy}} \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y}\right)dxdy = -\iint_{S} \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y}\right)dxdy$$

而 $$\iint_S \frac{\partial P}{\partial z} dzdx - \frac{\partial P}{\partial y} dxdy = \iint_S \frac{\partial P}{\partial z} \cos\beta dS - \iint_S \frac{\partial P}{\partial y} dxdy$$

$$= \iint_S \frac{\partial P}{\partial z} \frac{\cos\beta}{\cos\gamma} \cos\gamma dS - \iint_S \frac{\partial P}{\partial y} dxdy$$

$$= \iint_S \frac{\partial P}{\partial z} \frac{\cos\beta}{\cos\gamma} dxdy - \iint_S \frac{\partial P}{\partial y} dxdy$$

$$= -\iint_S \frac{\partial P}{\partial z} \frac{\partial z}{\partial y} dxdy - \iint_S \frac{\partial P}{\partial y} dxdy$$

$$(\cos\alpha, \cos\beta, \cos\gamma) = \frac{1}{\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}} (-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1)$$

所以 $\iint_S \frac{\partial P}{\partial z}dzdx - \frac{\partial P}{\partial y}dxdy = \oint_\Gamma Pdx$

若曲面S不能由$z = z(x, y)$给出, 则可用一些光滑的曲线将之分割为若干小块, 每一小块满足要求.

类似的,

$$\iint_S \frac{\partial Q}{\partial x}dxdy - \frac{\partial Q}{\partial z}dydz = \oint_\Gamma Q(x, y, z)dy,$$

$$\iint_S \frac{\partial R}{\partial y}dydz - \frac{\partial R}{\partial x}dzdx = \oint_\Gamma R(x, y, z)dz,$$

加在一起, 可见斯托克斯公式成立.

便于记忆形式

$$\oint_{\Gamma} Pdx + Qdy + Rdz = \iint_{S} \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

第一型曲面积分的形式

$$\oint_{\Gamma} Pdx + Qdy + Rdz = \iint_{S} \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} dS$$

其中$\vec{n}^0 = (\cos\alpha, \cos\beta, \cos\gamma)$

Stokes 公式的实质

表达了有向曲面上的曲面积分与其边界曲线上的曲线积分之间的关系.

当 S 为 xy 平面内的区域时,

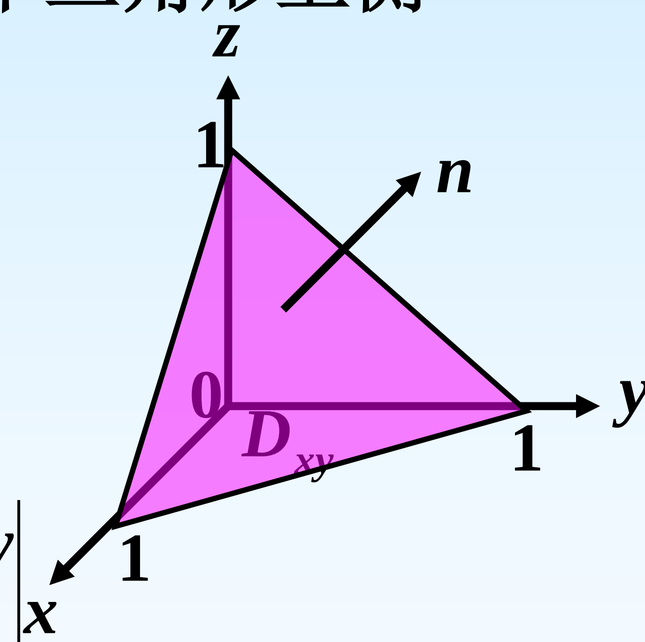
斯托克斯公式 $\xrightarrow{\text{特殊情形}}$ 格林公式

使用Stokes 公式时, 也要注意条件.

3. 简单应用

例 6 计算曲线积分$\oint_{\Gamma} zdx + xdy + ydz$,

其中Γ是平面$x + y + z = 1$被三坐标面所截成的三角形的整个边界, 它的正向与这个三角形上侧的法向量之间符合右手规则.

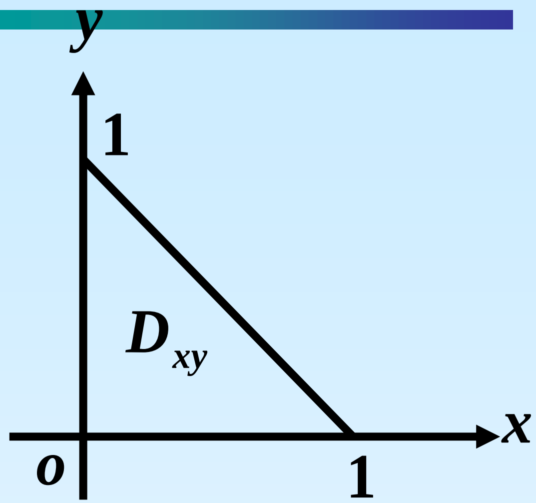


解 设S为平面$x + y + z = 1$被曲线所围的部分,取上侧

由斯托克斯公式

$$\oint_{\Gamma} zdx + xdy + ydz = \iint_S \begin{vmatrix} dydz & dzdx & dxdy \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

$$= \iint_S dydz + dzdx + dxdy$$



$$S : z = 1 - x - y, \quad (x, y) \in D_{xy},$$

$$(-z_x, -z_y, 1) = (1, 1, 1)$$

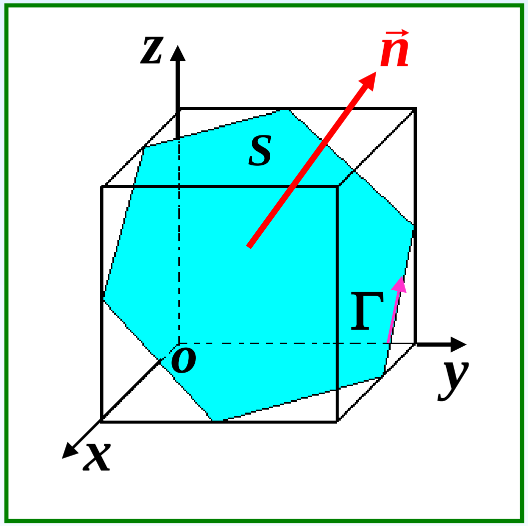
$$\oint_\Gamma zdx + xdy + ydz = \iint_S dydz + dzdx + dxdy$$

$$= 3\iint_{D_{xy}} dxdy = \frac{3}{2}$$

例 7 计算曲线积分

$$\oint_{\Gamma}(y^2-z^2)dx+(z^2-x^2)dy+(x^2-y^2)dz$$

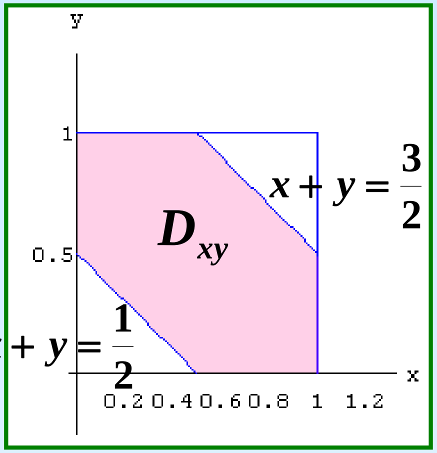
其中Γ是平面$x+y+z=\dfrac{3}{2}$截立方体:$0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$的表面所得的截痕, 若从 ox 轴的正向看去, 取逆时针方向.



解 取S为平面$x+y+z=\dfrac{3}{2}$的上侧被Γ所围成的部分.

则 $\vec{n}^0=(\dfrac{1}{\sqrt{3}},\dfrac{1}{\sqrt{3}},\dfrac{1}{\sqrt{3}})$

即 $\cos\alpha=\cos\beta=\cos\gamma=\dfrac{1}{\sqrt{3}}$,



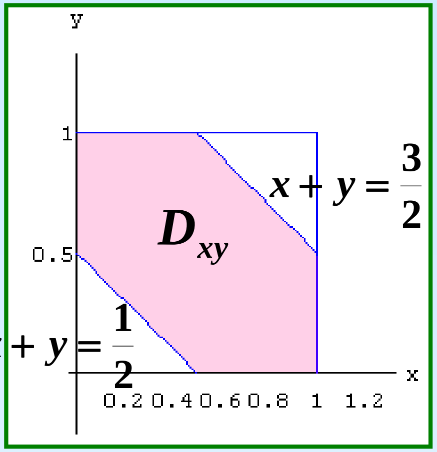
$$\therefore I=\iint_S \begin{vmatrix} \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} & \dfrac{1}{\sqrt{3}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2-z^2 & z^2-x^2 & x^2-y^2 \end{vmatrix} dS$$

$$=-\frac{4}{\sqrt{3}}\iint_S (x+y+z)dS \quad (\because \text{在}S\text{上}x+y+z=\frac{3}{2})$$

$$=-\frac{4}{\sqrt{3}}\cdot\frac{3}{2}\iint_S dS=-2\sqrt{3}\iint_{D_{xy}}\sqrt{3}dxdy=-\frac{9}{2}.$$

解二 $\Sigma: z=\dfrac{3}{2}-x-y,\quad (x,y)\in D_{xy}$(见右图),

取上侧, $-z_x=1, -z_y=1$, 由Stokes公式



$$原积分=\iint_{\Sigma}\begin{vmatrix} dydz & dzdx & dxdy \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2-z^2 & z^2-x^2 & x^2-y^2 \end{vmatrix}$$

$$=\iint_{\Sigma}(-2y-2z)dydz+(-2z-2x)dzdx+(-2x-2y)dxdy$$

$$=-\iint_{D_{xy}}\{[2y+2(\frac{3}{2}-x-y)]\cdot(-z_x)+[2(\frac{3}{2}-x-y)+2x]\cdot(-z_y)+(2x+2y)\cdot 1\}dxdy$$

$$=-\iint_{D_{xy}}(3-2x+3-2y+2x+2y)dxdy=-6S(D_{xy})=-6\times\frac{3}{4}=-\frac{9}{2}$$

例8 计算$\oint_{\Gamma}(y+z)dx+(z-\sin y)dy+2xdz$,其中$\Gamma$为柱面$x^2+y^2=1$与平面$x+y+z=1$的交线,从$z$轴正向看去$\Gamma$为顺时针方向.

解 设Σ为平面$x+y+z=1$被曲线所围的部分,取下侧

$$\vec{n}^0=\left(-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$$

$$原积分=\iint_{\Sigma}\begin{vmatrix}\dfrac{-1}{\sqrt{3}} & \dfrac{-1}{\sqrt{3}} & \dfrac{-1}{\sqrt{3}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y+z & z-\sin y & 2x\end{vmatrix}dS=\frac{3}{\sqrt{3}}\iint_{\Sigma}dS$$

$$\Sigma: z = 1 - x - y, \quad (x, y) \in D = \{(x, y) \mid x^2 + y^2 \le 1\}$$

$$\iint_{\Sigma} dS = \iint_{D} \sqrt{1 + z_x^2 + z_y^2}\, dxdy$$

$$= \sqrt{3} \iint_{D} dxdy = \sqrt{3}\pi$$

所以$\oint_{\Gamma} (y + z)dx + (z - \sin y)dy + 2xdz = \sqrt{3} \iint_{\Sigma} dS$

$$= 3\pi$$

例8 计算$\oint_{\Gamma}(y+z)dx+(z-\sin y)dy+2xdz$，其中$\Gamma$为柱面$x^2+y^2=1$与平面$x+y+z=1$的交线，从$z$轴正向看去$\Gamma$为顺时针方向.

解二 设Σ为平面$x+y+z=1$被曲线所围的部分，取下侧

$$\Sigma: z=1-x-y,\quad (x,y)\in D=\{(x,y)\,|\,x^2+y^2\le 1\},$$

$$-z_x=1,\quad -z_y=1,$$

$$\text{原积分}=\iint_{\Sigma}\begin{vmatrix} dydz & dzdx & dxdy \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y+z & z-\sin y & 2x \end{vmatrix}=-\iint_{\Sigma} dydz+dzdx+dxdy$$

$$=\iint_{D}[1\cdot(-z_x)+1\cdot(-z_y)+1]dxdy=\iint_{D}3dxdy=3\pi$$

空间曲线积分与路径无关性

1. 空间单连通区域 (一维单连通)

如果V 内任一封闭曲线皆可以不经过V外的点而连续收缩于V 内的一点, 则称V 为单连通区域. 否则,称为复连通区域.

单连通区域: 球体,

复连通区域: 环状区

2. 积分与路径无关定理

定理3. 3 设$\Omega \subset R^3$是空间单连通区域. 若P,Q,R在Ω上连续,且有一阶连续偏导数，则以下四个条件等价：

(1) 沿Ω内任一按段光滑封闭曲线L,有

$$\oint_L Pdx + Qdy + Rdz = 0;$$

(2) 在Ω内$\int_L Pdx + Qdy + Rdz$与路径无关;

(3) 在Ω内存在$u(x,y,z)$,使$du = Pdx + Qdy + Rdz$;

(4) 在Ω内,$\boxed{\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} \equiv \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} \equiv \frac{\partial P}{\partial z}}$.

证明: 略

注: (1) Ω的单连通性质很重要.

(2) 定理的两个条件都满足时,才可用.

(3) 在积分与路径无关的条件下,求原函数u时,

选用平行于坐标面 的折线作为积分曲线.

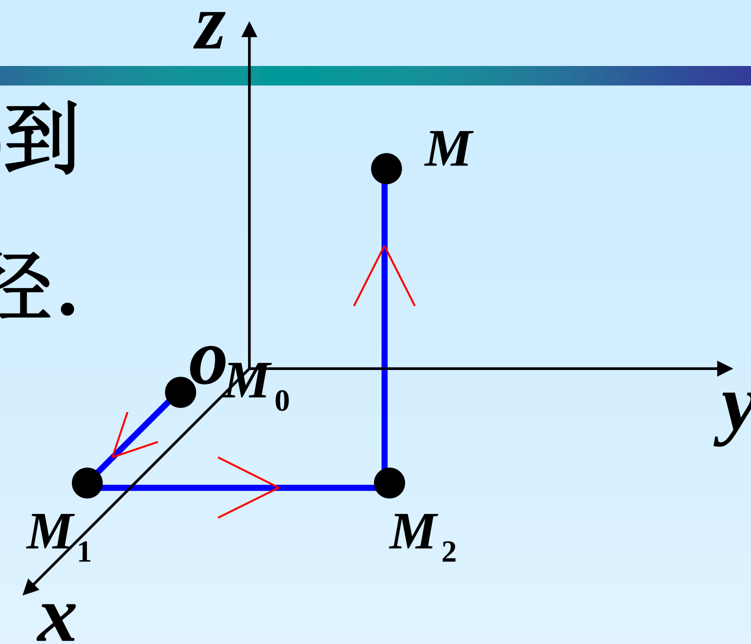
如图：取从点 $M_0(x_0,y_0,z_0)$到

$M(x,y,z)$的折线为积分路径.

即 $M_0(x_0,y_0,z_0)$

$\longrightarrow M_1(x,y_0,z_0)$

$\longrightarrow M_2(x,y,z_0) \longrightarrow M(x,y,z)$



$$u(x,y,z)=\int_{(x_0,y_0,z_0)}^{(x,y,z)} P(x,y,z)dx+Q(x,y,z)dy+R(x,y,z)dz$$

$$=\int_{x_0}^{x} P(x,y_0,z_0)dx+\int_{y_0}^{y} Q(x,y,z_0)dy+\int_{z_0}^{z} R(x,y,z)dz$$

例8 验证曲线积分

$$\int_{\Gamma}(y+z)dx+(z+x)dy+(x+y)dz$$

与路径无关,并求被积表达式的原函数$u(x,y,z)$.

解 由于

$$P=y+z, Q=z+x, R=x+y,$$

$$\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=\frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y}=\frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}=1,$$

所以曲线积分与路径无关.

于是

$$u(x,y,z)=\int_{(x_0,y_0,z_0)}^{(x,y,z)}(y+z)dx+(z+x)dy+(x+y)dz$$

$$=\int_{x_0}^{x}(y_0+z_0)dx+\int_{y_0}^{y}(z_0+x)dy+\int_{z_0}^{z}(x+y)dz$$

$$=xy+xz+yz+c$$

其中$c=-x_0y_0-x_0z_0-y_0z_0$为一常数,

若取(x_0,y_0,z_0)为原点, 则得$u(x,y,z)=xy+xz+yz.$

例9 确定函数 $f(x), \Phi(x)$ 满足 $f(0)=\Phi(0)$

使得曲线积分与路径无关.

$$\int_L [\frac{\Phi(x)}{2} y^2 + (x^2 - f(x))y]dx + \left[f(x)y + \Phi(x) \right] dy + zdz$$

解 $P = \dfrac{\Phi(x)}{2} y^2 + (x^2 - f(x))y, \quad Q = f(x)y + \Phi(x), \quad R = z$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

$$f'(x)y+\Phi'(x)=\Phi(x)y+x^2-f(x)$$

$$\begin{cases} f'(x)=\Phi(x) \\ \Phi'(x)=x^2-f(x) \end{cases}$$

$$f(x)=c_1\cos x+c_2\sin x+x^2-2,$$

$$\Phi(x)=-c_1\sin x+c_2\cos x+2x$$

由$f(0)=\Phi(0)$,可得$c_1-2=c_2=c$,所以

$$f(x)=(2+c)\cos x+c\sin x+x^2-2,$$

$$\Phi(x)=-(2+c)\sin x+c\cos x+2x.$$

例10 计算$\int_L (yz-e^{x^2})dx+(xy+1)dz+xzdy,$

其中L是从点$A(2,0,0)$沿螺旋线$x=2\cos t,$

$y=\sin t, z=t$到$B(2,0,2\pi)$的一段.

解 记 $P=yz-e^{x^2}$, $Q=xz$, $R=xy+1,$

$$\text{则}\ \frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=z,\quad \frac{\partial R}{\partial y}=\frac{\partial Q}{\partial z}=x,\quad \frac{\partial P}{\partial z}=\frac{\partial R}{\partial x}=y.$$

所以积分与路径无关.

取L_1为A到B的直线段,即$L_1:\begin{cases} x=2 \\ y=0 \\ z=z \end{cases}$ $z:0\mapsto 2\pi$

$$\int_L (yz - e^{x^2})dx + (xy+1)dz + xzdy$$

$$= \int_{L_1} (yz - e^{x^2})dx + (xy+1)dz + xzdy$$

$$= \int_0^{2\pi} dz = 2\pi$$

小结

1、Gauss公式及相关计算

常见情形：补面、挖点

2、Stokes公式及相关计算

3、空间曲线积分与路径无关性