§2 换元积分法和分部积分法

在一般情况下：

设 $G'(u)=g(u)$，则 $\int g(u)du = G(u)+C$.

如果 $u=\varphi(x)$（可微）

$\because \quad dG[\varphi(x)] = g[\varphi(x)]\varphi'(x)dx$

$\therefore \quad \int g[\varphi(x)]\varphi'(x)dx = G[\varphi(x)]+C$

$= [\int g(u)du]_{u=\varphi(x)}$ 由此可得换元法定理

定理2.1

（1） 设$g(u)$具有原函数$G(u), u=\varphi(x)$可导，

记$f(x)=g[\varphi(x)]\varphi'(x)$，则有换元公式

$$\int f(x)dx = \int g[\varphi(x)]\varphi'(x)dx = [\int g(u)du]_{u=\varphi(x)}$$

$$= G(u)\Big|_{u=\varphi(x)} + C = G[\varphi(x)] + C.$$

第一类换元公式（凑微分法）

说明 使用此公式的关键在于将

$$\int f(x)dx \text{ 化为 } \int g[\varphi(x)]\varphi'(x)dx.$$

(2) 设 $x=\psi(t)$ 是单调的、可导的函数，并且 $\psi'(t)\neq 0$，又设 $f[\psi(t)]\psi'(t)$ 具有原函数，则有换元公式

$$\int f(x)dx=\left[\int f[\psi(t)]\psi'(t)dt\right]_{t=\psi^{-1}(x)}$$

第二类换元公式

注 第一类换元公式和第二类换元公式本质上相同，只是公式使用的方向不一样.

例1　求$\int \sin 2x dx.$

解（一）$\int \sin 2x dx = \frac{1}{2}\int \sin 2x d(2x)$

$= -\frac{1}{2}\cos 2x + C;$

（二）$\int \sin 2x dx = 2\int \sin x \cos x dx$

$= 2\int \sin x d(\sin x) = (\sin x)^2 + C;$

（三）$\int \sin 2x dx = 2\int \sin x \cos x dx$

$= -2\int \cos x d(\cos x) = -(\cos x)^2 + C.$

例2 求 $\int \frac{1}{3+2x}dx.$

解 $$\int \frac{1}{3+2x}dx = \frac{1}{2}\int \frac{1}{3+2x}\cdot d(3+2x)$$

$$= \frac{1}{2}\ln|3+2x| + C.$$

一般地 $$\int f(ax+b)dx = \frac{1}{a}\left[\int f(u)du\right]_{u=ax+b}$$

例3 求 $\int \frac{x}{(1+x)^3}dx.$

解 $\int \frac{x}{(1+x)^3}dx = \int \frac{x+1-1}{(1+x)^3}dx$

$$= \int [\frac{1}{(1+x)^2} - \frac{1}{(1+x)^3}]dx$$

$$= \int [\frac{1}{(1+x)^2} - \frac{1}{(1+x)^3}]d(1+x)$$

$$= -\frac{1}{1+x} + \frac{1}{2(1+x)^2} + C.$$

例4 求$\int \frac{1}{1+e^x}dx.$

解 $\int \frac{1}{1+e^x}dx = \int \frac{1+e^x-e^x}{1+e^x}dx$

$= \int \left(1-\frac{e^x}{1+e^x}\right)dx = \int dx - \int \frac{e^x}{1+e^x}dx$

$= \int dx - \int \frac{1}{1+e^x}d(1+e^x)$

$= x - \ln(1+e^x) + C.$

例5 求 $\displaystyle\int \frac{1}{\sqrt{2x+3}+\sqrt{2x-1}}dx.$

解 原式$\displaystyle=\int \frac{\sqrt{2x+3}-\sqrt{2x-1}}{\left(\sqrt{2x+3}+\sqrt{2x-1}\right)\left(\sqrt{2x+3}-\sqrt{2x-1}\right)}dx$

$$=\frac{1}{4}\int\sqrt{2x+3}dx-\frac{1}{4}\int\sqrt{2x-1}dx$$

$$=\frac{1}{8}\int\sqrt{2x+3}d(2x+3)-\frac{1}{8}\int\sqrt{2x-1}d(2x-1)$$

$$=\frac{1}{12}\left(\sqrt{2x+3}\right)^3-\frac{1}{12}\left(\sqrt{2x-1}\right)^3+C.$$

例6 求 $\int \frac{1}{a^2+x^2}dx.$

解 $\int \frac{1}{a^2+x^2}dx = \frac{1}{a^2}\int \frac{1}{1+\frac{x^2}{a^2}}dx$

$= \frac{1}{a}\int \frac{1}{1+\left(\frac{x}{a}\right)^2}d\left(\frac{x}{a}\right) = \frac{1}{a}\arctan\frac{x}{a}+C.$

$$\boxed{\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\frac{x}{a}+C.}$$

例7 求 $\int \frac{1}{x^2-8x+25}dx.$

解 $\int \frac{1}{x^2-8x+25}dx = \int \frac{1}{(x-4)^2+9}dx$

$$= \int \frac{1}{(x-4)^2+3^2}d(x-4)$$

$$= \frac{1}{3}\arctan\frac{x-4}{3}+C.$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\frac{x}{a}+C$$

例8 求 $\int \frac{1}{1+\cos x}dx$. $= \int \frac{1}{2\cos^2 \frac{x}{2}}dx = \tan\frac{x}{2} + C$

解 $\int \frac{1}{1+\cos x}dx = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)}dx$

$$= \int \frac{1-\cos x}{1-\cos^2 x}dx = \int \frac{1-\cos x}{\sin^2 x}dx$$

$$= \int \frac{1}{\sin^2 x}dx - \int \frac{1}{\sin^2 x}d(\sin x)$$

$$= -\cot x + \frac{1}{\sin x} + C.$$

例9 求 $\int \sin^2 x \cdot \cos^5 x dx.$

解 $\int \sin^2 x \cdot \cos^5 x dx = \int \sin^2 x \cdot \cos^4 x d(\sin x)$

$$= \int \sin^2 x \cdot (1 - \sin^2 x)^2 d(\sin x)$$

$$= \int (\sin^2 x - 2\sin^4 x + \sin^6 x) d(\sin x)$$

$$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C.$$

说明 当被积函数是三角函数相乘时，拆开奇次项去凑微分.

例10 求 $\int \sin^2 x \cdot \cos^4 x dx.$

解 $\displaystyle\int \sin^2 x \cdot \cos^4 x dx = \int \frac{1-\cos(2x)}{2} \cdot \left(\frac{1+\cos(2x)}{2}\right)^2 dx$

$$= \int \frac{1+\cos(2x)-\cos^2(2x)-\cos^3(2x)}{8} dx$$

$$= \int \frac{1}{8} dx + \frac{1}{16}\int \cos(2x) d(2x) - \frac{1}{16}\int (1-\cos(4x)) dx$$

$$-\frac{1}{16}\int (1-\sin^2(2x)) d(\sin(2x))$$

$$= \frac{x}{16} + \frac{\sin(4x)}{64} + \frac{\sin^3(2x)}{48} + C.$$

说明 求形为$\int \sin^m x \cdot \cos^n x dx$($m,n$为非负整数)的积分时，

(1)若m为奇数，则$\int \sin^m x \cdot \cos^n x dx$

$$= \int \sin^{m-1} x \cdot \cos^n x \cdot \sin x dx$$

$$= -\int (1-\cos^2 x)^{\frac{m-1}{2}} \cdot \cos^n x d(\cos x);$$

令$u = \cos x$,原积分转化为求关于u的多项式的积分问题.

(2)若n为奇数，则$\int \sin^m x \cdot \cos^n x dx$

$$= \int \sin^m x \cdot \cos^{n-1} x \cdot \cos x dx$$

$$= \int \sin^m x \cdot (1-\sin^2 x)^{\frac{n-1}{2}} d(\sin x);$$

令$u = \sin x$,原积分转化为求关于u的多项式的积分问题.

(3)若m,n为偶数,则

$$\int \sin^m x\cdot\cos^n x dx=\int(\sin^2 x)^{\frac{m}{2}}\cdot(\cos^2 x)^{\frac{n}{2}}dx$$

$$=\int\left(\frac{1-\cos(2x)}{2}\right)^{\frac{m}{2}}\cdot\left(\frac{1+\cos(2x)}{2}\right)^{\frac{n}{2}}dx$$

$$=\int P_{\frac{m+n}{2}}(\cos(2x))dx,$$

其中$P_{\frac{m+n}{2}}(\cos(2x))$是关于$\cos(2x)$的$\dfrac{m+n}{2}$阶多项式.

令$u=2x$,原积分转化为求$\int\cos^k u du$的积分计算问题.

思考 如何求形为$\int\tan^m x\cdot\sec^n x dx$($m,n$为非负整数)的积分?

例11 求$\int \cos 3x \cos 2x dx$.

解 $\cos A \cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$,

$$\cos 3x \cos 2x = \frac{1}{2}(\cos x + \cos 5x),$$

$$\int \cos 3x \cos 2x dx = \frac{1}{2}\int (\cos x + \cos 5x) dx$$

$$= \frac{1}{2}\sin x + \frac{1}{10}\sin 5x + C.$$

例12 求 $\int \csc x dx$.

解 方法(1) $\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{\sin x}{\sin^2 x} dx$

$$= -\int \frac{1}{1-\cos^2 x} d(\cos x) \qquad u = \cos x$$

$$= -\int \frac{1}{1-u^2} du = -\frac{1}{2}\int \left(\frac{1}{1-u} + \frac{1}{1+u}\right) du$$

$$= \frac{1}{2}\int \left(\frac{1}{u-1} - \frac{1}{1+u}\right) du = \frac{1}{2}\left(\ln|u-1| - |1+u|\right) + C$$

$$= \frac{1}{2}\ln\left|\frac{1-u}{1+u}\right| + C = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right| + C.$$

方法(2) $\int \csc x dx = \int \frac{1}{\sin x} dx = \int \frac{1}{2\sin\frac{x}{2}\cos\frac{x}{2}} dx$

$$= \int \frac{1}{\tan\frac{x}{2}\left(\cos\frac{x}{2}\right)^2} d\left(\frac{x}{2}\right) = \int \frac{1}{\tan\frac{x}{2}} d\left(\tan\frac{x}{2}\right)$$

$$= \ln\left|\tan\frac{x}{2}\right| + C = \ln|\csc x - \cot x| + C.$$

$$\int \csc x dx = \ln|\csc x - \cot x| + C.$$

$$\int \sec x dx = \int \frac{1}{\cos x} dx$$

$$= \int \frac{1}{\sin(x+\frac{\pi}{2})} d(x+\frac{\pi}{2})$$

$$= \ln|\csc(x+\frac{\pi}{2}) - \cot(x+\frac{\pi}{2})| + C$$

$$= \ln|\sec x + \tan x| + C$$

$$\int \sec x dx = \ln|\sec x + \tan x| + C.$$

例13 求$\int \frac{1}{x(1+2\ln x)}dx.$

解 $\int \frac{1}{x(1+2\ln x)}dx$

$$= \int \frac{1}{1+2\ln x}d(\ln x)$$

$$= \frac{1}{2}\int \frac{1}{1+2\ln x}d(1+2\ln x)$$

$$= \frac{1}{2}\ln|1+2\ln x|+C.$$

例14 求$\int (1-\frac{1}{x^2})e^{x+\frac{1}{x}}dx.$

解

$$\therefore \quad \int (1-\frac{1}{x^2})e^{x+\frac{1}{x}}dx$$

$$=\int e^{x+\frac{1}{x}}d(x+\frac{1}{x})$$

$$=e^{x+\frac{1}{x}}+C.$$

$$\because \quad \left(x+\frac{1}{x}\right)'=1-\frac{1}{x^2},$$

例15 求 $\displaystyle\int \frac{1}{\sqrt{4-x^2}\arcsin\frac{x}{2}}dx.$

解 $\displaystyle\int \frac{1}{\sqrt{4-x^2}\arcsin\frac{x}{2}}dx = \int \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}\arcsin\frac{x}{2}}d\left(\frac{x}{2}\right)$

$$=\int \frac{1}{\arcsin\frac{x}{2}}d\left(\arcsin\frac{x}{2}\right) = \ln\left|\arcsin\frac{x}{2}\right| + C.$$

$$\left(\arcsin\frac{x}{2}\right)' = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{1}{2}$$

(2) 设$x=\psi(t)$是单调的、可导的函数，并且$\psi'(t)\neq 0$，又设$f[\psi(t)]\psi'(t)$具有原函数，则有换元公式

$$\int f(x)dx=\left[\int f[\psi(t)]\psi'(t)dt\right]_{t=\psi^{-1}(x)}$$

第二类换元公式

注 第一类换元公式和第二类换元公式本质上相同，只是公式使用的方向不一样.

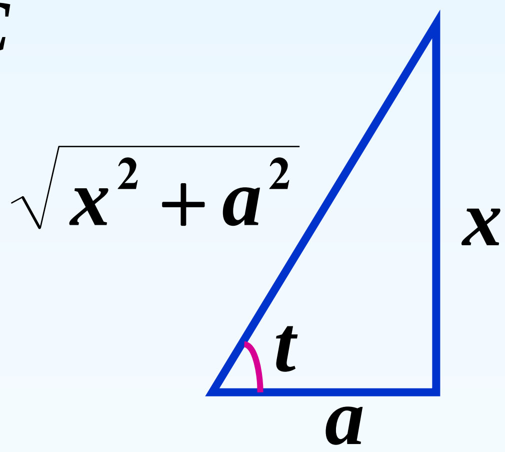
例16 求 $\int \frac{1}{\sqrt{x^2+a^2}}dx \quad (a>0).$

解 令 $x = a\tan t \Rightarrow dx = a\sec^2 t dt \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \int \frac{1}{a\sec t} \cdot a\sec^2 t dt$$

$$= \int \sec t dt = \ln|\sec t + \tan t| + C$$

$$= \ln\left(\frac{x}{a} + \frac{\sqrt{x^2+a^2}}{a}\right) + C.$$

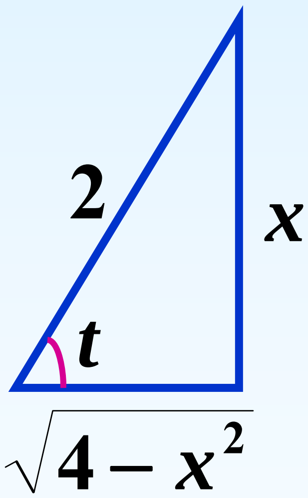


例17 求 $\int x^3\sqrt{4-x^2}dx.$

解 令 $x = 2\sin t \qquad dx = 2\cos t dt \qquad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\int x^3\sqrt{4-x^2}dx = \int (2\sin t)^3\sqrt{4-4\sin^2 t}\cdot 2\cos t dt$$

$$= 32\int \sin^3 t\cos^2 t dt$$



$$= -32\int (\cos^2 t - \cos^4 t)d\cos t$$

$$= -32\left(\frac{1}{3}\cos^3 t - \frac{1}{5}\cos^5 t\right) + C$$

$$= -\frac{4}{3}\left(\sqrt{4-x^2}\right)^3 + \frac{1}{5}\left(\sqrt{4-x^2}\right)^5 + C.$$

例18 求 $\int \frac{1}{\sqrt{x^2-a^2}}dx \quad (a>0).$

解 1. $x>a$

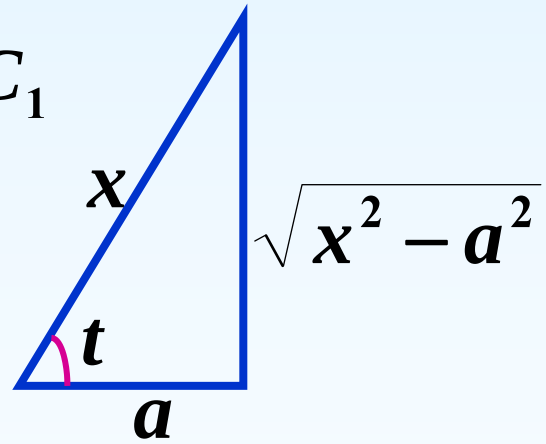
令$x = a\sec t \quad dx = a\sec t\tan t dt \qquad t\in\left(0,\frac{\pi}{2}\right)$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \int \frac{a\sec t\cdot\tan t}{a\tan t}dt$$

$$= \int \sec t dt = \ln|\sec t+\tan t| + C_1$$

$$= \ln\left(\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right)+C_1$$

$$= \ln\left(x+\sqrt{x^2-a^2}\right)+C.$$



2. $x < -a$

令 $x = -u$ 那么 $u > a$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = -\int \frac{1}{\sqrt{u^2 - a^2}} du$$

$$= -\ln\left(u + \sqrt{u^2 - a^2}\right) + C_2 = -\ln\left(-x + \sqrt{x^2 - a^2}\right) + C_2$$

$$= \ln\left(\frac{-x - \sqrt{x^2 - a^2}}{a^2}\right) + C_2 = \ln\left(-x - \sqrt{x^2 - a^2}\right) + C.$$

故原式 $= \ln|x + \sqrt{x^2 - a^2}| + C.$

基本积分表

(16) $\int \tan x dx = -\ln|\cos x| + C;$

(17) $\int \cot x dx = \ln|\sin x| + C;$

(18) $\int \sec x dx = \ln|\sec x + \tan x| + C;$

(19) $\int \csc x dx = \ln|\csc x - \cot x| + C;$

(20) $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\arctan\frac{x}{a} + C;$

$$(21)\quad \int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C;$$

$$(22)\quad \int \frac{1}{a^2-x^2}dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C;$$

$$(23)\quad \int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\frac{x}{a} + C;$$

$$(24)\quad \int \frac{1}{\sqrt{x^2\pm a^2}}dx = \ln\left|x+\sqrt{x^2\pm a^2}\right| + C.$$

方法(1) 三角代换

三角代换的目的是化掉根式.

一般规律如下：当被积函数中含有

(1) $\sqrt{a^2-x^2}$ 可令 $x=a\sin t$;

(2) $\sqrt{a^2+x^2}$ 可令 $x=a\tan t$;

(3) $\sqrt{x^2-a^2}$ 可令 $x=a\sec t$.

化掉根式是否一定采用三角代换，需根据被积函数的情况来定.

例19 求$\displaystyle\int\frac{x^5}{\sqrt{1+x^2}}dx$

解 令$t=\sqrt{1+x^2}$ $\boxed{\Rightarrow x^2=t^2-1,}$ $\boxed{xdx=tdt,}$

$$\int\frac{x^5}{\sqrt{1+x^2}}dx=\int\frac{\left(t^2-1\right)^2}{t}tdt=\int\left(t^4-2t^2+1\right)dt$$

$$=\frac{1}{5}t^5-\frac{2}{3}t^3+t+C=\frac{1}{15}(8-4x^2+3x^4)\sqrt{1+x^2}+C.$$

例20 求 $\int \frac{1}{\sqrt{1+e^x}}dx.$

解 令 $t=\sqrt{1+e^x}$ $\boxed{\Rightarrow e^x = t^2-1,}$

$x=\ln(t^2-1),$ $\boxed{dx=\frac{2t}{t^2-1}dt,}$

$$\int \frac{1}{\sqrt{1+e^x}}dx = \int \frac{2}{t^2-1}dt = \int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$

$$=\ln\left|\frac{t-1}{t+1}\right| + C = 2\ln\left(\sqrt{1+e^x}-1\right) - x + C.$$

方法(2) 当分母的阶较高时，可采用倒代换 $x=\frac{1}{t}$.

例21 求 $\int \frac{1}{x(x^7+2)}dx$

解 令 $x=\frac{1}{t} \Rightarrow dx=-\frac{1}{t^2}dt$,

$$\int \frac{1}{x(x^7+2)}dx=\int \frac{t}{\left(\frac{1}{t}\right)^7+2}\cdot\left(-\frac{1}{t^2}\right)dt=-\int \frac{t^6}{1+2t^7}dt$$

$$=-\frac{1}{14}\ln|1+2t^7|+C=-\frac{1}{14}\ln|2+x^7|+\frac{1}{2}\ln|x|+C.$$

例22 求 $\int \frac{1}{x^4\sqrt{x^2+1}}dx.$ （分母的阶较高）

解 令 $x=\frac{1}{t} \Rightarrow dx=-\frac{1}{t^2}dt,$

$$\int \frac{1}{x^4\sqrt{x^2+1}}dx = \int \frac{1}{\left(\frac{1}{t}\right)^4\sqrt{\left(\frac{1}{t}\right)^2+1}}\left(-\frac{1}{t^2}\right)dt$$

$$=-\int \frac{t^3}{\sqrt{1+t^2}}dt = -\frac{1}{2}\int \frac{t^2}{\sqrt{1+t^2}}dt^2$$

$$=\frac{1}{2}\int\frac{1-(t^2+1)}{\sqrt{1+t^2}}d(t^2+1)$$

$$=\frac{1}{2}\int\left(\frac{1}{\sqrt{u}}-\sqrt{u}\right)du \qquad u=t^2+1$$

$$=\sqrt{u}-\frac{1}{3}\left(\sqrt{u}\right)^3+C$$

$$=\frac{\sqrt{1+x^2}}{x}-\frac{1}{3}\left(\frac{\sqrt{1+x^2}}{x}\right)^3+C.$$

方法(3) 当被积函数含有两种或两种以上的根式 $\sqrt[k]{x},\cdots,\sqrt[l]{x}$ 时，可采用令 $x=t^n$（其中 n 为各根指数的最小公倍数）

例23 求 $\displaystyle\int\frac{1}{\sqrt{x}(1+\sqrt[3]{x})}dx.$

解 令 $x=t^6 \Rightarrow dx=6t^5dt,$

$$\int\frac{1}{\sqrt{x}(1+\sqrt[3]{x})}dx=\int\frac{6t^5}{t^3(1+t^2)}dt=\int\frac{6t^2}{1+t^2}dt$$

$$= 6\int \frac{t^2 + 1 - 1}{1 + t^2} dt$$

$$= 6\int \left(1 - \frac{1}{1 + t^2} \right) dt$$

$$= 6[t - \arctan t] + C$$

$$= 6[\sqrt[6]{x} - \arctan \sqrt[6]{x}] + C.$$

问题 $\int xe^x dx = ?$

解决思路 利用两个函数乘积的求导法则.

设函数$u = u(x)$和$v = v(x)$可导,

$$(uv)' = u'v + uv',$$

$$\boxed{uv' = (uv)' - u'v,}$$

$$\int uv'dx = uv - \int u'vdx, \quad \int udv = uv - \int vdu.$$

分部积分公式

定理2.2 设函数$u(x)$和$v(x)$可导,若$u'(x)v(x)$存在原函数,则$u(x)v'(x)$存在原函数,并有

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

或

$$\int u(x)dv(x) = u(x)v(x) - \int v(x)du(x)$$

选u和v的总原则:

1. v易求;
2. $\int v\,du$比$\int u\,dv$易求.

例24 求积分 $\int x\cos x dx$.

解（一） 令 $u=\cos x$, $xdx=d\left(\frac{x^2}{2}\right)=dv$

$$\int x\cos xdx=\frac{x^2}{2}\cos x+\int\frac{x^2}{2}\sin xdx$$

显然，u,v' 选择不当，积分更难进行.

（二） 令 $u=x$, $\cos xdx=d(\sin x)=dv$

$$\int x\cos xdx=x\sin x-\int\sin xdx$$

$$=x\sin x+\cos x+C.$$

例25 求积分$\int x^2 e^x dx$.

解 $u = x^2$, $e^x dx = de^x = dv$,

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx$$

（再次使用分部积分法）$u = x$, $e^x dx = dv$

$$= x^2 e^x - 2(xe^x - e^x) + C.$$

总结 若被积函数是幂函数和正(余)弦函数或幂函数和指数函数的乘积，就考虑设幂函数为u, 使其降幂一次(假定幂指数是正整数)

例26 求积分 $\int x\arctan x dx$.

解 令 $u=\arctan x$, $xdx = d\frac{x^2}{2} = dv$

$$\int x\arctan x dx = \frac{x^2}{2}\arctan x - \int \frac{x^2}{2} d(\arctan x)$$

$$= \frac{x^2}{2}\arctan x - \int \frac{x^2}{2}\cdot\frac{1}{1+x^2}dx$$

$$= \frac{x^2}{2}\arctan x - \int \frac{1}{2}\cdot\left(1-\frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2}{2}\arctan x - \frac{1}{2}(x-\arctan x)+C.$$

例27　求积分$\int x^3 \ln x dx.$

解　$u = \ln x, \quad x^3 dx = d\frac{x^4}{4} = dv,$

$$\int x^3 \ln x dx = \frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^3 dx$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C.$$

总结　若被积函数是幂函数和对数函数或幂函数和反三角函数的乘积，就考虑设对数函数或反三角函数为 u.

函数u，v的选取一般优先级：

反（三角函数)、对（数函数）、幂（函数）、

三（角函数）、指（数函数),

即排列次序在前面的函 数优先取为$u(x)$.

例28 已知 $f(x)$的一个原函数是e^{-x^2}，求$\int xf'(x)dx$.

解 $\int xf'(x)dx = \int xdf(x) = xf(x) - \int f(x)dx,$

$\boxed{\text{而} \int f(x)dx = e^{-x^2} + C,}$

两边同时对 x求导, $\because \left(\int f(x)dx\right)' = f(x),$

$\boxed{f(x) = -2xe^{-x^2},}$

$$\therefore \int xf'(x)dx = xf(x) - \int f(x)dx$$

$$= -2x^2e^{-x^2} - e^{-x^2} + C.$$

例29 求积分 $\int \frac{x\arctan x}{\sqrt{1+x^2}}dx.$

解 $\because \left(\sqrt{1+x^2}\right)' = \frac{x}{\sqrt{1+x^2}},$

$$\therefore \int \frac{x\arctan x}{\sqrt{1+x^2}}dx = \int \arctan x \, d\sqrt{1+x^2}$$

$$= \sqrt{1+x^2}\arctan x - \int \sqrt{1+x^2}\, d(\arctan x)$$

$$= \sqrt{1+x^2}\arctan x - \int \sqrt{1+x^2} \cdot \frac{1}{1+x^2}dx$$

$$= \sqrt{1+x^2}\arctan x - \boxed{\int \frac{1}{\sqrt{1+x^2}}dx}$$

$$\int \frac{1}{\sqrt{a^2+x^2}}dx = \ln(x+\sqrt{a^2+x^2})+C$$

$$\therefore \quad \int \frac{x\arctan x}{\sqrt{1+x^2}}dx$$

$$= \sqrt{1+x^2}\arctan x - \ln(x+\sqrt{1+x^2})+C.$$

例30　求积分 $\int \ln(x+\sqrt{1+x^2})dx$

解　$\int \ln(x+\sqrt{1+x^2})dx$

$$= x\ln(x+\sqrt{1+x^2}) - \int x(\ln(x+\sqrt{1+x^2}))' dx$$

$$= x\ln(x+\sqrt{1+x^2}) - \int \frac{x}{\sqrt{1+x^2}} dx$$

$$= x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C$$

造循环

例31 求积分$\int \sin(\ln x)dx$.

解 $\boxed{\int \sin(\ln x)dx}$

$$= x\sin(\ln x) - \int x\cos(\ln x)\cdot\frac{1}{x}dx$$

$$= x\sin(\ln x) - \int \cos(\ln x)dx$$

$$= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx$$

$$= x[\sin(\ln x) - \cos(\ln x)] - \boxed{\int \sin(\ln x)dx}$$

$$\therefore \int \sin(\ln x)dx = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C.$$

例32　求积分$\int e^x \sin x dx$.

解　$\boxed{\int e^x \sin x dx}$

$$= e^x \sin x - \int e^x \cos x dx$$

$$= e^x \sin x - (e^x \cos x + \int e^x \sin x dx)$$

$$= e^x (\sin x - \cos x) - \boxed{\int e^x \sin x dx}$$

$$\therefore \int e^x \sin x dx = \frac{e^x}{2}(\sin x - \cos x) + C.$$

思考 在接连几次应用分部积分公式时， 应注意什么？

注意前后几次所选的 u 应为同类型函数.

例 $\int e^x \cos x dx$

第一次时若选 $u_1 = \cos x$

$$\int e^x \cos x dx = e^x \cos x + \int e^x \sin x dx$$

第二次时仍应选 $u_2 = \sin x$

例33 求积分 $\int e^{\sqrt{x}}dx.$

解 令$u=\sqrt{x}$,则$\int e^{\sqrt{x}}dx=2\int ue^{u}du$

$$=2\int ude^{u}$$

$$=2ue^{u}-2\int e^{u}du$$

$$=2ue^{u}-2e^{u}+C$$

$$=2ue^{u}-2e^{u}+C$$

$$=2\sqrt{x}e^{\sqrt{x}}-2e^{\sqrt{x}}+C$$

例34 求积分 $I_n=\int\frac{1}{(1+x^2)^n}dx.$

解
$$I_n=\frac{x}{(1+x^2)^n}-\int xd\left(\frac{1}{(1+x^2)^n}\right)$$
$$=\frac{x}{(1+x^2)^n}+2n\int\frac{x^2}{(1+x^2)^{n+1}}dx$$
$$=\frac{x}{(1+x^2)^n}+2n\int\frac{1+x^2-1}{(1+x^2)^{n+1}}dx$$
$$=\frac{x}{(1+x^2)^n}+2nI_n-2nI_{n+1}$$

$$\therefore I_{n+1}=\frac{x}{2n(1+x^2)^n}+\frac{2n-1}{2n}I_n$$ 递推公式

由$I_1=\arctan x+C$,可求出I_n的表达式.

例35 计算$\int \cos^n xdx$, $\int \sin^n xdx$, 其中$n \in N^*$.

解 $\int \cos^n xdx = \int \cos^{n-1} x\, d(\sin x)$

$$\therefore \int \cos^n xdx = \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \sin^2 xdx$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} xdx - (n-1)\int \cos^n xdx$$

$$\therefore \boxed{\int \cos^n xdx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} xdx}$$

递推法

类似可求得:

$$\int \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx.$$

思考 建立求解$I_n = \int \frac{dx}{\sin^n x}$的递推公式

提示: $I_n = \int \frac{dx}{\sin^n x} = -\int \frac{d\cot x}{\sin^{n-2} x}$

思考　建立求解$I_n=\int\frac{dx}{\sin^n x}$的递推公式

提示: $I_n=\int\frac{dx}{\sin^n x}=-\int\frac{d\cot x}{\sin^{n-2} x}$

作业：

习题6.2

1(单数)，2(单数)，3,

4(1,3,6,10,11,13,16,18), 5