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Name: KEY

1. The following data matrix is observed for a two-dimensional random vector $\mathbf{X} = \begin{bmatrix} 3 & 4 \\ 6 & 2 \\ 3 & 3 \end{bmatrix}$ Assume that the population is multivariate normal with unknown mean vector and unknown covariance matrix .

   (a) Use the Hotelling T^2 to test $H_0 : \mu = (3, 2)^T$ against $H_a : \mu \neq (3, 2)^T$ at 0.05 level of significance.

$$\bar{x} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{3+6+3}{3} \\ \frac{4+2+3}{3} \end{pmatrix}$$

$$d_1 = x_1 - \bar{x}_1 = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$$d_2 = x_2 - \bar{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$s_{11} = \frac{1}{n-1} d_1' d_1 = 3$$

$$s_{22} = \frac{1}{n-1} d_2' d_2 = 1$$

$$s_{12} = \frac{1}{n-1} d_1' d_2 = -\frac{3}{2}$$

$$S = \begin{pmatrix} 3 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{pmatrix}$$

$$S^{-1} = \frac{1}{\frac{12}{4} - \frac{9}{4}} \begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & 3 \end{pmatrix}$$

$$S^{-1} = \begin{pmatrix} 4/3 & 2 \\ 2 & 4 \end{pmatrix}$$

$$(\bar{x} - \mu_0) = \begin{pmatrix} 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$T^2 = n(\bar{x} - \mu_0)' S^{-1} (\bar{x} - \mu_0)$$
$$= 3 \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 4/3 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 3 & 3 \end{pmatrix} \begin{pmatrix} 10/3 \\ 6 \end{pmatrix} = 10 + 18 = 28$$

$$\frac{(n-1)p}{(n-p)} F_{p,n-p}(0.05) = \frac{(3-1)2}{3-2}(199.5) = 798$$

Since $T^2 < 798$, we conclude μ_0 is in the confidence region and cannot reject H_0 at level 0.05.

   (b) (b) Construct the simultaneous 95% T^2 confidence levels for μ_1, μ_2 from the mean vector μ.

For μ_i $i = 1, 2$

$$\bar{x}_i \pm \sqrt{\frac{(n-1)p}{(n-p)} F_{p,n-p}(\alpha)} \sqrt{\frac{s_{ii}}{n}}$$

$$\mu_1: \quad 4 \pm \sqrt{798} \sqrt{\frac{3}{3}} = (-24.25, 32.25) = \star$$

$$\mu_2: \quad 3 \pm \sqrt{798} \sqrt{\frac{1}{3}} = (-13.31, 19.31) = \star\star$$

We are simultaneously confident that μ_1 and μ_2 lie between $\star$ and $\star\star$ respectively.

2. A cardiologist considered three treatments for treating patients who had recently experienced severe heart attacks. Forty patients (n = 40) enrolling in the study received all three treatments, one at a time, spaced out over sufficient time so there would not be any carry-over effect. The order of applying the treatments was also randomized. A single response variable was calculated after a week of each treatment to produce scores of efficacy. We provide the summary statistics from the study:

$$\bar{\mathbf{x}} = \begin{bmatrix} 46.1 \\ 57.3 \\ 50.4 \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} 101.3 & 63.0 & 71.0 \\ 63.0 & 80.2 & 55.6 \\ 71.0 & 55.6 & 97.4 \end{bmatrix}$$

Assume that the population is multivariate normal with unknown mean vector μ and unknown covariance matrix $\mathbf{\Sigma}$.

(a) What is the experimental design of this study?

Repeated Measures Design

(b) To test the equality of the treatments at .05 level, specify the appropriate null and alternative hypotheses through an explicit contrast matrix, C.

$H_0: C\mu = 0$ vs. $H_a: C\mu \neq 0$ where $\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$ and

$C = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$. Note for students: $c_3 = (1 \quad 0 \quad -1)$ is not included in contrast matrix because $c_3 = c_1 + c_2$ where c_i is the ith contrast row vector, All rows need to be orthogonal.

(c) Calculate the Hotelling T^2 statistic.

$C\bar{x} = \begin{pmatrix} -11.2 \\ 6.9 \end{pmatrix}$ $n = 40$, $q = 3$

$$CSC' = \begin{pmatrix} 55.5 & -32.6 \\ -32.6 & 66.4 \end{pmatrix}$$

$$T^2 = n(C\bar{x})'(CSC')^{-1}(C\bar{x}) = 90.4$$

(d) What is the distribution of T^2 under H_0?

$$\frac{(n-1)(q-1)}{n-q+1} F_{q-1,\, n-q+1}$$

(e) What can you conclude about the equality of means?

$$\frac{39(2)}{38} 3.25 = \frac{(n-1)(q-1)}{n-q+1} F_{q-1,\, n-q+1} = 6.67 < 90.4 = T^2$$

We can reject H_0 at 0.05. The treatment means are not equal in efficacy.

Cont.

3. Consider the spectral decomposition of the covariance matrix for two variables, X_1 =height of lawyer , X_2 = number of litigation cases lost throughout career:

$$\Sigma = \begin{bmatrix} 3 & -2 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

(a) What is the relationship betweeen the total variance, tr(Σ)= $\sigma_{11}+\sigma_{22}$, and the eigenvalues λ_1, λ_2?

$$tr(\Sigma) = tr(P\Lambda P') = tr(\Lambda P P') = tr(\Lambda) = \sum_{i=1}^{2} \lambda_i$$

(b) What proportion of the total variance does the first population principal component, Y_1, explain?

$$\frac{\lambda_1}{\lambda_1 + \lambda_2} = \frac{7}{9}$$

(c) Calculate ρ_{Y_1,X_1} and ρ_{Y_1,X_2}. Do these numbers shed any additional light on principal component Y_1, than say, the components of e_1?

$$\rho_{Y_1,X_1} = \frac{e_{11}\sqrt{\lambda_1}}{\sqrt{\sigma_{11}}} = \frac{\frac{1}{\sqrt{5}}\sqrt{7}}{\sqrt{3}} = 0.683$$

$$\rho_{Y_1,X_2} = \frac{e_{12}\sqrt{\lambda_2}}{\sqrt{\sigma_{22}}} = \frac{-\frac{2}{\sqrt{5}}\sqrt{2}}{\sqrt{2}} = -0.8945$$

The ranking of absolute magnitudes is consistent between the correlation coefficients and the elements of e_1. Both support the notion that X_2 contributes more to the variance of PC1, but this is not surprising given the unequal scales of X_1 and X_2.

Cont.

4. Complete Exercise 9.12(a)-(c) in your textbook.

(b) $\hat{h}_i^2 = \hat{\ell}_{i1}^2$ for an $m=1$ model $i=1,2,3$

$\hat{h}_1^2 = 0.01044$

$\hat{h}_2^2 = 0.005655$

$\hat{h}_3^2 = 0.005852$

(a) $\hat{\psi}_i = s_{ii} - \hat{h}_i^2$ for $i=1,2,3$

where s_{ii} is under S_n.

$S_n = \frac{n-1}{n} S$ and thus

$\hat{\psi}_1 = 0.0001734$ $\hat{\psi}_2 = 0.0004941$ $\hat{\psi}_3 = 0.0006342$

(c) $$\frac{\sum_{i=1}^{3} h_i^2}{tr(S_n)} = 0.9440$$

The End.