STA 135
Discussion 2

Def: Let A by symmetric matrix. A is positive definite if $x^TAx > 0$ for every non-zero vector x.

$x^TAx :=$ a quadratic form

Exercise

Show that $A = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}$ is positive definite (p.d.)

$$(x_1 \; x_2) \begin{pmatrix} 1 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1 \; x_2) \begin{pmatrix} x_1 - 2x_2 \\ -2x_1 + 6x_2 \end{pmatrix}$$

$$= x_1^2 - 2x_1x_2 - 2x_1x_2 + 6x_2^2$$

$$= x_1^2 - 4x_1x_2 + 6x_2^2$$

Key is to rewrite RHS as a sum of squares

(i) $(x_1 - 2x_2)^2 = x_1^2 - 4x_1x_2 + 4x_2^2 > 0$

(ii) add $2x_2^2$ and we have

$$x^TAx = (x_1 - 2x_2)^2 + 2x_2^2 > 0$$

Also $A' = \begin{pmatrix} 1 & -2 \\ -2 & 6 \end{pmatrix}$ is symmetric

so A is p.d.

The Spectral Decomposition

Def: Every symmetric matrix has the factorization $A = P \Lambda P'$ with real eigenvalues in Λ and orthonormal eigenvectors in P.

Note: $A = \sum_{j=1}^{k} \lambda_j e_j e_j'$ where

e_j is an eigenvector of unit length with corresponding eigenvalue λ_j.

Exercise

Derive the spectral decomposition for

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Step 1 Find eigenvalues

$|A - \lambda I| = 0$ the characteristic equation

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 1)(\lambda - 3) = 0 \Rightarrow \lambda_1 = 1, \quad \lambda_2 = 3$$

Note: Since $\lambda_1, \lambda_3 > 0$ A is also p.d.
You can verify this by definition application.

Step 2 Find eigenvectors

$\lambda_1 = 1$: $(A - \lambda I)x = 0$ solve for x

Plug in for $\lambda_1 = 1$

$$\begin{pmatrix} 2-1 & -1 \\ -1 & 2-1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1 - x_2 = 0$ take $x_1 = 1 \Rightarrow x_2 = 1$

$x_2 - x_1 = 0$

$\underset{\sim}{x}' = (1 \;\; 1) \longrightarrow$ unit length : $\sqrt{x'x} = \sqrt{2}$

$e_1' = \left(\frac{1}{\sqrt{2}} \;\; \frac{1}{\sqrt{2}} \right)$

$\lambda_2 = 3$:

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$-x_1 - x_2 = 0$ take $x_1 = 1 \Rightarrow x_2 = -1$

$\underset{\sim}{x}' = (1 \; -1) \longrightarrow$ unit length

$e_2' = \left(\frac{1}{\sqrt{2}} \;\; -\frac{1}{\sqrt{2}} \right)$

Step 3 Express the decomposition

$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= 1 \times \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} + 3 \times \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

What would be the square root matrix in this case?

$$A = P \Lambda^{1/2} P' = \sum_{j=1}^{2} \sqrt{\lambda_j}\, e_j e_j'$$

Application of Positive Definite Matrices

For positive definite matrix A,
the graph of $x^T A x = c^2$
is an elipse (hyperellipsoid for $p > 2$).

Exercise 2.17

Claim: Every eigenvalue of a $k \times k$ p.d. matrix A is positive.

Proof

$$Ae = \lambda e$$

$$e'Ae = e'(\lambda e) = \lambda e'e$$

$$= \lambda (e_1^2 + e_2^2 + \cdots + e_p^2)$$

By p.d. of A $e'Ae > 0$ for non-zero e,
and $e'e > 0 \Rightarrow \lambda > 0$.

Remark: There is a strong geometric reason why the eigenvalues are positive, so we turn our attention back to the ellipse ~~to~~ investigate.

Ellipses and Positive Definite Matrices

Let's start with the p.d. matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

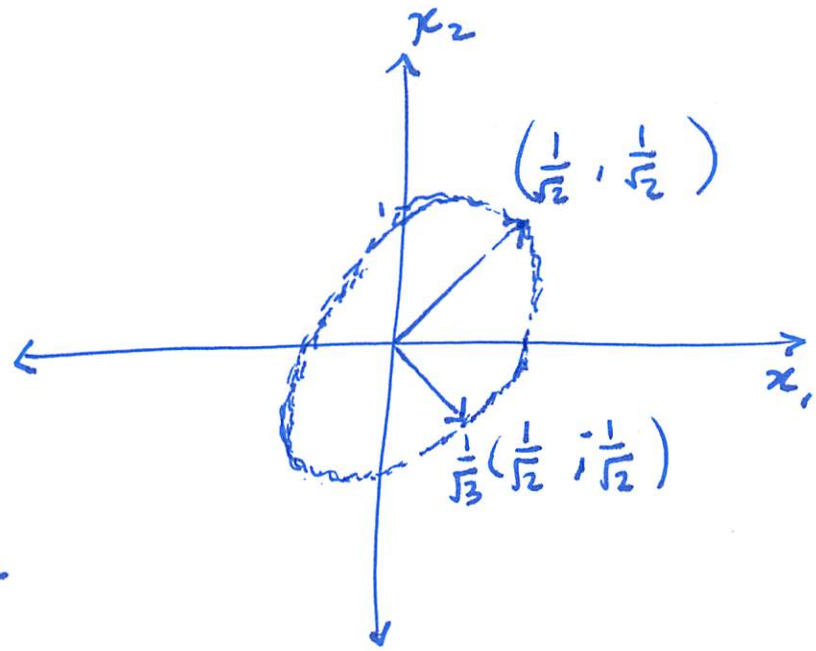
Equation for Ellipse

$$x^T A x = c^2$$

$$(x_1 \ x_2) \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c^2$$

$$(x_1 x_2) \begin{pmatrix} 2x_1 - x_2 \\ 2x_2 - x_1 \end{pmatrix} = c^2$$

$$2x_1^2 - 2x_1 x_2 + 2x_2^2 = c^2$$



when $c = 1$, the ellipse looks like this and expands and shrinks according to the distance c.

How ought one to know the geometry of the ellipse?

- know the major and minor axis direction and magnitude (a vector).

The major and minor axis of the ellipse is described by the eigenvectors of A.

Since A was ~~[illegible]~~ p.d. with

$\lambda_1 = 1$, $\lambda_2 = 3$ and $e_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ $e_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

$x^T A x = c^2$ can be re-expressed by the spectral decomposition

$x^T P \Lambda P' x = c^2$

$$x^T \left(\sum_{j=1}^{p} \lambda_j e_j e_j' \right) x = \sum_{j=1}^{p} \lambda_j x' e_j e_j' x = \sum_{j=1}^{p} \lambda_j (x' e_j)^2$$

Now in the $p = 2$ case, the ellipse is seen to be

$$c^2 = \lambda_1 \tilde{X}_1^2 + \lambda_2 \tilde{X}_2^2$$

notice no cross-product term

where $\tilde{X}_j = x' e_j$ and $\tilde{X}_1$ and $\tilde{X}_2$ are the ~~major and minor~~ principle axes of the ellipse. In the example, we have

$$c^2 = 1 * \left(\frac{x_1 + x_2}{\sqrt{2}} \right)^2 + 3 * \left(\frac{x_1 - x_2}{\sqrt{2}} \right)^2$$

In summary,

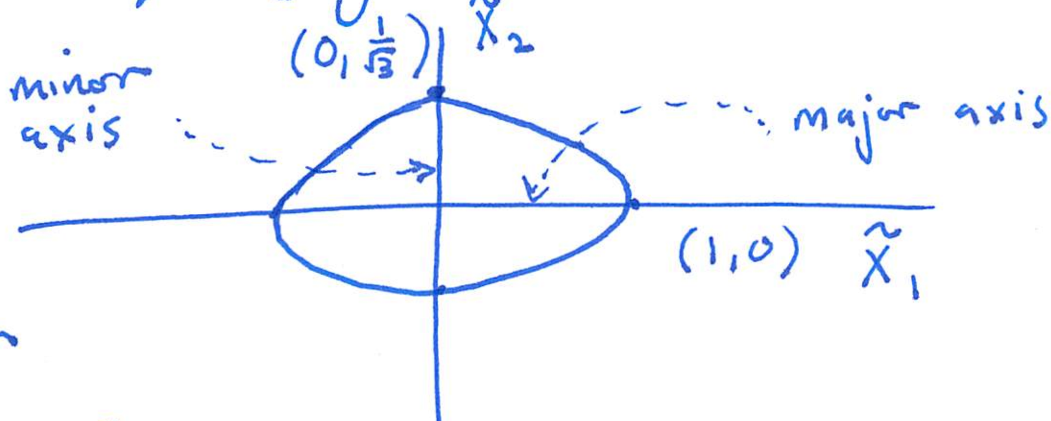
The eigenvectors dictate the major and minor axes of the ellipse direction and the eigenvalues scale the magnitude.

If $\lambda_1 < \lambda_2$, then the

minor ~~major~~ axis has length $\dfrac{c}{\sqrt{\lambda_2}}$

major ~~minor~~ axis has length $\dfrac{c}{\sqrt{\lambda_1}}$

In our example, then, we see the ellipse ~~about~~ rotated about on ~~to~~ to the principal (major and minor) axes as



Expression for Rotation onto $\tilde{X}_1, \tilde{X}_2$ axes.

$$\text{for } c^2 = 1 = [x'e_1 \;\; x'e_2]\begin{pmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{pmatrix}\begin{bmatrix}x'e_1 \\ x'e_2\end{bmatrix}$$

$$= (\tilde{X}_1, \tilde{X}_2)\,\Lambda\begin{pmatrix}\tilde{X}_1 \\ \tilde{X}_2\end{pmatrix} = \underset{\sim}{X}^T \Lambda \underset{\sim}{X}$$