Network Performance Analysis: Tutorial 3

1. A telephone call centre services complaints from bank customers about the quality of their service. They expect at 'busy hour' calls to arrive at a rate of 120 customers an hour. Dealing with each customer takes on average 3 minutes. When all serving assistants are occupied, customers are placed in a holding system.

   (a) The service centre has 10 serving assistants. What is the probability of a customer having to join the queue?

   (b) What is the expected queueing time for each customer?

   (c) Given that customers who do not have to queue have an average queueing time of zero, what is the average waiting time for customers who have to queue?

   (d) The bank wishes to combine two of these service centers for efficiency. Both service centers have arrival rates of 120 customers an hour in 'busy hour'. How many staff will they need at the new centre to maintain a queueing probability at least as low as in a)?

   (e) Answer parts b) and c) for the new system.

   For this question you may assume that arrivals are Poisson distributed and service times are exponentially distributed.

2. Consider a finite M/M/1 queue capable of accommodating N packets (customers). Calculate the values of N required for the following situations and comment on your answers:

(i)	$\rho = 0.5$	$P_B = 10^{-3}$
(ii)	$\rho = 0.5$	$P_B = 10^{-6}$
(iii)	$\rho = 0.8$	$P_B = 10^{-3}$
(iv)	$\rho = 0.8$	$P_B = 10^{-6}$

3. Let n be the total number of objects in two identical M/M/1 queues, each operating independently with arrival rate λ and service rate μ.

   (a) Show that the probability that the two queues are in state n is

   $$P_n = (n+1)\rho^n(1-\rho)^2, \qquad \rho = \frac{\lambda}{\mu} < 1$$

   (b) Show that $P_n < 1$.

   (c) Show that

   $$\sum_{n=0}^{\infty} P_n = 1.$$

(d) If the two separate M/M/1 systems are merged into one system to form one M/M/2 system (one system with one queue and two servers), show that the number of items k in the steady state is

$$P_k = \begin{cases} \frac{2(1-\rho)\rho^k}{1+\rho} & k \geq 1 \\ \frac{(1-\rho)}{1+\rho} & k = 0 \end{cases}$$

4. A telephone company plans to rent trunk lines between Birmingham and London for \$1,000 per year per trunk line. They expect to have a demand of 20 calls per hour, and it is expected that the average length of a call is 15 minutes. They can charge 0.01\$/minute for each customer. They will provide no holding facility, and thus a queue is not permitted.

   (a) How many trunk lines will they need to keep the blocking probability below 0.01?

   (b) How much profit will they make?

   For this question you may assume that arrivals are Poisson distributed and service times are exponentially distributed.

5. A service time distribution is defined as follows:

$$P(X) = \begin{cases} \frac{2}{b} - \frac{2}{b^2}X & 0 < X \leq b \\ 0 & X > b \end{cases} \tag{1}$$

   (a) Sketch this distribution.

   (b) The Pollaczek-Khinchin formula states the average waiting time for a queue is given by

$$\bar{W} = \frac{\lambda E\left\{X^2\right\}}{2(1-\rho)}$$

   By determining the first two moments, $E\{X\}$ and $E\left\{X^2\right\}$, of the distribution given in (1), derive an expression for the average waiting time for the queue.

6. **Queue with discouragement** Consider a single-server queuing system in which the flow of packets is controlled, that is, the state-dependent arrival rate is

$$\lambda_k = \frac{\lambda}{k+1}, \qquad k = 0, 1, \ldots$$

   where λ is a given constant. The service rate is constant $\mu_k = \mu$.

   This type of system is used to model packet transmission in which the maximum possible arrival rate is λ, but as the queue length increases, a system controller discourages packet arrivals, either by blocking or shunting packets elsewhere, so that the true arrival rate decreases as k increases.

   (a) Draw the state transition diagram for this system and derive an expression for P_k, the probability that the system is in state k.

   (b) Calculate the average number of packets in the system.

(c) Compare this result with the average number of packets in an M/M/1 queue in which there is no blocking. Comment on your answer.

(d) Calculate the average arrival rate of the packets.

(e) Calculate the average time delay in the system.

7. Consider an $M/M/1$ queue with arrival rate λ and service rate μ. If $\rho = 0.3$ and there is a probability of 97% that there are m or fewer customers in the system, calculate m.

8. Is the following table realisable for an $M/M/1$ queue?

k	P_k
0	0.40
1	0.30
2	0.20
3	0.10
≥ 4	0.00

9. For a system with a single server (such as the $M/M/1$ and $M/M/1/m$ queue) the utilisation factor, ρ, is the probability that the queue state is not zero. For a lossless system we saw, using Little's formula (which is valid for $M/G/1$ queues), that the utilisation factor is given by $\rho = \frac{\lambda}{\mu}$. Use the stationary distribution for the $M/M/1/m$ queue to compute the utilisation factor for the $M/M/1/m$ queue. Explain why it is not $\frac{\lambda}{\mu}$.

10. Mrs Cut runs a hair salon and she is the only employee. She does not make appointments but runs the salon on a first-come-first-served basis. She is extremely busy on Saturday mornings and is considering hiring an assistant or moving to larger premises. Before she does this, however, she analyses the situation carefully. She has kept careful records of the salon, including the arrival and service rates of the customers. In particular, her analysis shows that customers arrive according to a Poisson process at a rate of $\lambda = 5$ customers per hour, and it takes about 10 minutes to cut a client's hair.

    (a) Calculate the average number of people (waiting to be served and being served) in the salon.

    (b) Calculate the average number of people waiting to be served.

    (c) What is the percentage of the time that a customer can enter the salon and be served immediately?

    (d) The waiting room in Mrs Cut's salon has four seats. Calculate the probability that a customer will not find a seat.

    (e) Calculate the average waiting time in the system and the average number of people in the queue, where the average is taken over non-empty queues.

    (f) Calculate the average waiting time in the queue.