1. a)

(i) three assumption:

① P(exactly one arrival in $[t, t+\Delta t]$) $=\lambda\Delta t$

② P(exactly no arrivals in $[t, t+\Delta t]$) $=1-\lambda\Delta t$

③ P(more than one arrival in $[t, t+\Delta t]$) $=0$

where λ is the proportionality constant, Δt is the time interval.

(ii) Let $P_k(t)$ denote the probability of k arrivals in the time interval t.
Let $p_{ij}(\Delta t)$ be the probability of going from i arrivals to j arrivals in the time interval Δt.

then $P_k(t+\Delta t) = P_k(t)\, p_{k,k}(\Delta t) + P_{k-1}(t)\, p_{k-1,k}(\Delta t)$

$P_0(t+\Delta t) = P_0(t)\, p_{0,0}(\Delta t)$

because $p_{k,k}(\Delta t) = (1-\lambda\Delta t)$, $p_{k-1,k}(\Delta t) = \lambda\Delta t$,

λ is the proportionality constant

thus $P_k(t+\Delta t) = P_k(t)(1-\lambda\Delta t) + P_{k-1}(t)(\lambda\Delta t)$

$P_0(t+\Delta t) = P_0(t)(1-\lambda\Delta t)$

thus $\frac{P_k(t+\Delta t)-P_k(t)}{\Delta t} = \frac{dP_k(t)}{dt} = -\lambda P_k(t) + \lambda P_{k-1}(t)$ ①

$\frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = \frac{dP_0(t)}{dt} = -\lambda P_0(t)$ ②

the solution of ② is $P_0(t) = A\exp(-\lambda t)$

and thus $\frac{dP_2(t)}{dt} = -\lambda P_2(t) + A\lambda^2 t\exp(-\lambda t)$

where solution is $P_2(t) = \frac{A(\lambda t)^2}{2} \exp(-\lambda t)$ ②

Thus the general solution for arbitary k is

$$P_k(t) = \frac{A(\lambda t)^k}{k!} \exp(-\lambda t)$$

and because $\sum_{i=0}^{\infty} P_i(t) = 1$, so $A\exp(-\lambda t)\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} = 1$

which yields $A \cdot \exp(-\lambda t) \exp(\lambda t) = 1$ $A = 1$

thus Poisson distribution is : $\underline{\underline{P(k|t,\lambda) = \frac{(\lambda t)^k}{k!} \exp(-\lambda t)}}$

(iii) Because in Poisson distribution, it is assumed that λ is constant, so t is time interval, which is a time difference.

(iv)

$$P(\text{exactly one arrival in } [t, t+\Delta t]) = \int_t^{t+\Delta t} \lambda t \cdot \exp(-\lambda t) = -\int_t^{t+\Delta t} t\, d[\exp(-\lambda t)]$$

$$= -t\exp(-\lambda t)\Big|_t^{t+\Delta t} + \int_t^{t+\Delta t} \exp(-\lambda t) dt = \underline{\underline{\lambda \Delta t}}$$

$$P(\text{exactly no arrivals is } [t, t+\Delta t]) = \int_t^{t+\Delta t} \exp(-\lambda t) = -\frac{1}{\lambda} \exp(-\lambda t)\Big|_t^{t+\Delta t}$$

$$= \underline{\underline{1 - \lambda \Delta t}}$$

Because $P(\text{one arrival}) + P(\text{no arrivals}) = 1$

Thus $P(\text{more than one arrival is } [t, t+\Delta t]) = \underline{\underline{0}}$

1. b)

(i) $P(k \geq 2 \mid \frac{1}{2}, 3) = 1 - P(0 \mid \frac{1}{2}, 3) - P(1 \mid \frac{1}{2}, 3) = 1 - 0.223 - \frac{3}{2} \cdot 0.223$

$= 1 - 0.223 - 0.334$

$= 0.442$

(ii) $P(3 \mid 1, 3) = \frac{3^3}{3!} \exp(-3) = \frac{9}{2} \cdot \exp(-3) = \cancel{0.448}$ 0.224

2. a)

(i) The balance equation is

$$\lambda_{k-1} P_{k-1} + \mu_{k+1} P_{k+1} = (\lambda_k + \mu_k) P_k$$

which yields $\lambda_{k-1} P_{k-1} = \mu_k P_k$

The solution of this equation is

$$P_k = \left(\frac{\lambda}{\mu}\right) \alpha^{(k-1)^2} P_{k-1}$$

and cycling through $k = 0, 1, 2 \cdots$, yields

$$P_k = P_0 \left(\frac{\lambda}{\mu}\right)^k \cdot \alpha^{\sum_{i=1}^{k} i^2} = P_0 \left(\frac{\lambda}{\mu}\right)^k \cdot \alpha^{\frac{2k^3+3k^2+k}{6}}$$

(ii) Since $\sum_{k=0}^{\infty} P_k = 1$

it follows that $P_0 \left(\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{2k^3+3k^2+k}{6}} \right) = 1$

Thus $P_0 = \left(\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{2k^3+3k^2+k}{6}} \right)^{-1}$

Thus $P_k = \left(\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \cdot \alpha^{\frac{2k^3+3k^2+k}{6}} \right)^{-1} \cdot \left(\frac{\lambda}{\mu}\right)^k \cdot \alpha^{\frac{2k^3+3k^2+k}{6}}$

2 a)

(iii) If $\frac{\lambda}{\mu} < 1$, the infinite sum is guaranteed to converge. Because $0 < \alpha \leq 1$, so $\alpha^{\frac{2k^3+3k^2+k}{6}} \leq 1$

and $\left(\frac{\lambda}{\mu}\right)^k \leq 1$, so the sum ≤ 1 so the sum will converge

b)

(i) $\lambda_k = \lambda$, $k = 0, 1, 2 \cdots$

$$\mu_k = \begin{cases} k\mu & k = 0, 1, 2 \cdots m \\ m\mu & k \geq m \end{cases}$$

because $\lambda_{k-1} P_{k-1} + \mu_{k+1} P_{k+1} = (\lambda_k + \mu_k) P_k$

thus $\lambda P_{k-1} = k\mu P_k \quad k \leq m$

$\lambda P_{k-1} = m\mu P_k \quad k \geq m$

consider $k \leq m$ $\quad P_k = \frac{1}{k}(\rho m) P_{k-1}$, $\rho m = \frac{\lambda}{\mu}$

yield $P_k = \left(\frac{(m\rho)^k}{k!}\right) P_0 \quad k < m$

consider $k \geq m$ $\quad P_k = \frac{\lambda}{m\mu} P_{k-1} = \rho P_{k-1}$

yield $P_k = \rho^{k-m+1} \cdot P_{m-1} \quad k = m, m+1 \cdots$

and $P_{m-1} = \left(\frac{(m\rho)^{m-1}}{(m-1)!}\right) P_0$

Thus steady state probability distribution for the system is

$$P_k = \begin{cases} P_0 \left(\frac{(m\rho)^k}{k!}\right) & k \leq m \\ P_0 \left(\frac{m^m \rho^k}{m!}\right) & k \geq m \end{cases}$$

since $\sum_{k=0}^{\infty} P_k = 1$ $\quad P_0 \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \sum_{k=m}^{\infty} \frac{m^m \rho^k}{m!}\right] = 1$

$\therefore$ hence final results are

$$P_k = \begin{cases} P_0 \left(\dfrac{(m\rho)^k}{k!} \right) & k \le m \\ P_0 \left(\dfrac{m^m \rho^k}{m!} \right) & k \ge m \end{cases}, \quad \rho = \frac{\lambda}{m\mu} < 1$$

where
$$P_0 = \left[\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!(1-\rho)} \right]^{-1}$$

2 b)

(ii)
$$\overline{N_Q} = \sum_{k=m+1}^{\infty} (k-m) P_k = P_0 \sum_{k=m+1}^{\infty} (k-m) \frac{m^m \rho^k}{m!} = P_0 \left(\frac{m^m \rho^m}{m!} \right) \sum_{k=1}^{\infty} k \rho^k$$

since $\displaystyle\sum_{k=1}^{\infty} k\rho^k = \frac{\rho}{(1-\rho)^2}$ thus $\displaystyle\overline{N_Q} = C(m,a) \frac{\rho}{1-\rho}, \ \rho = \frac{\lambda}{m\mu}$

where $C(m,a)$ is the probability that a customer will wait.

(iii) arrival rate $\lambda = 15$

service rate $\mu = \frac{1}{5}$

Thus Erlang $a = \frac{\lambda}{\mu} = 75$

Because $C(m, 75) = 0.01$, according to the Erlang C curve,

The number of trunk lines $m = 98$

3.

a) The balance equation yield

$$\lambda_{k-1} P_{k-1} + \mu P_{k+1} = \lambda_k P_k + \mu P_k$$

that is

$$\frac{\lambda}{k+1} P_{k-1} + \mu P_{k+1} = \frac{\lambda}{k+2} P_k + \mu P_k$$

solution of above equation is

$$\frac{\lambda}{k+1} P_{k-1} = \mu P_k \text{. which yields } P_k = \left(\frac{1}{k+1}\right)\left(\frac{\lambda}{\mu}\right) P_{k-1}$$

b) According to a)

$$P_1 = \frac{1}{2} \frac{\lambda}{\mu} \cdot P_0$$

$$P_2 = \frac{1}{3} \frac{\lambda}{\mu} \cdot P_1$$

$$\vdots$$

hence the general solution is therefore

$$P_k = \frac{1}{(k+1)!} \left(\frac{\lambda}{\mu}\right)^k P_0$$

c) Because $\sum_{k=0}^{\infty} P_k = 1$, $P_0 \cdot \left[\sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left(\frac{\lambda}{\mu}\right)^k\right] = 1$

$$P_0 = \left[\sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left(\frac{\lambda}{\mu}\right)^k\right]^{-1} = \left[\frac{\mu}{\lambda} \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \left(\frac{\lambda}{\mu}\right)^{k+1}\right]^{-1}$$

$$= \frac{\lambda}{\mu} \left[\exp\left(\frac{\lambda}{\mu}\right) - 1\right]^{-1} = \frac{\lambda}{\mu} \frac{1}{\left(\exp\left(\frac{\lambda}{\mu}\right) - 1\right)}$$

Thus $P_k = \frac{1}{(k+1)!} \left(\frac{\lambda}{\mu}\right)^k P_0 = \frac{1}{(k+1)!} \left(\frac{\lambda}{\mu}\right)^{k+1} \frac{1}{\left(\exp\left(\frac{\lambda}{\mu}\right) - 1\right)}$

3. d) average number of packets in the system is

$$\bar{P} = \sum_{k=0}^{\infty} k P_k = \frac{1}{(\exp(\frac{\lambda}{\mu})-1)} \sum_{k=0}^{\infty} \frac{k}{(k+1)!} \cdot (\frac{\lambda}{\mu})^{k+1}$$

$$= \frac{1}{(\exp(\frac{\lambda}{\mu})-1)} \left[\sum_{k=0}^{\infty} \frac{k+1}{(k+1)!} \cdot (\frac{\lambda}{\mu})^{k+1} - \sum_{k=0}^{\infty} \frac{1}{(k+1)!} (\frac{\lambda}{\mu})^{k+1} \right]$$

$$= \frac{1}{(\exp(\frac{\lambda}{\mu})-1)} \left[\frac{\lambda}{\mu} \sum_{k=0}^{\infty} \frac{1}{k!} (\frac{\lambda}{\mu})^{k} - \exp(\frac{\lambda}{\mu}) + 1 \right]$$

$$= \frac{1}{(\exp(\frac{\lambda}{\mu})-1)} \left[\frac{\lambda}{\mu} \cdot \exp(\frac{\lambda}{\mu}) - \exp(\frac{\lambda}{\mu}) + 1 \right]$$

$$= \frac{\exp(\frac{\lambda}{\mu})(\frac{\lambda}{\mu}-1)+1}{\exp(\frac{\lambda}{\mu})-1}$$

e) if $\lambda << \mu$, $\frac{\lambda}{\mu} \approx 0$ $\exp(\frac{\lambda}{\mu}) \approx 1$

Then $\bar{P} \approx 0$

if $\lambda \approx \mu$ $\frac{\lambda}{\mu} \approx 1$ $\exp(\frac{\lambda}{\mu}) \approx \exp(1)$

Then $\bar{P} \approx \frac{1}{\exp(1)-1} \approx 0.58$

if $\lambda >> \mu$ $\frac{\lambda}{\mu} \to \infty$ $\exp(\frac{\lambda}{\mu}) \to \infty$

Then $\bar{P} \to \infty$