The University Of Sheffield.

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NONE

DEPARTMENT OF COMPUTER SCIENCE **Spring Semester 2019-2020**

NETWORK PERFORMANCE ANALYSIS **2 hours**

ANSWER ALL QUESTIONS.

All questions carry equal weight. Figures in square brackets indicate the percentage of available marks allocated to each part of a question.

Registration number from U-Card (9 digits) — to be completed by student

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1. a) Consider a steady state queue in which the number of servers is unbounded.

    (i) Write down the equations that define the arrival rate and service rate in terms of the state k. Explain the differences between them. [15%]

    (ii) Use the steady state balance equations to derive the steady state probability that the system is in state k. [30%]

    (iii) What is the average number of people in the system and what is the average delay? [10%]

   b) The system now changes to a finite number of servers m, and a customer is not allowed to wait. Thus if a customer arrives and finds all the service tills busy, the customer leaves the system.

    (i) What are the service and arrival rates for this system? [20%]

    (ii) Calculate the blocking probability. [25%]

2. a) Consider an $M/M/m$ queue in which there is one queue and m servers. The steady state probabilities for this system are

$$P_k = \begin{cases} P_0\left(\frac{(m\rho)^k}{k!}\right) & k < m \\ P_0\left(\frac{m^m\rho^k}{m!}\right) & k \geq m \end{cases}, \qquad \rho = \frac{\lambda}{m\mu}.$$

(i) What do λ and μ represent? Also, state the restriction on the value of ρ. [15%]

(ii) Derive an expression for P_0 in terms of m and ρ. Simplify the expression as much as possible. [20%]

b) Consider a switch in a computer network that has one input port and three output ports, and is modelled as an $M/M/3$ queue.

(i) Write down the expressions for the probabilities P_k for $k < 3$ and $k \geq 3$, and the restriction on the value of ρ. [10%]

(ii) Show that [15%]

$$P_0 = \left[1 + 3\rho + \frac{9}{2}\rho^2 + \frac{9\rho^3}{2(1-\rho)}\right]^{-1}.$$

(iii) Show that the average number of packets in the system is

$$P_0 \sum_{k=0}^{2} \frac{k(3\rho)^k}{k!} + \frac{9P_0}{2} \sum_{k=3}^{\infty} k\rho^k$$

and that this expression simplifies to

$$P_0 \left[\frac{9\rho}{2(1-\rho)^2} - \frac{3\rho}{2}\right].$$

[40%]

3. a) The Poisson process is the arrival process that is most frequently used to model the behaviour of queues.

   (i) Derive expressions for the mean and variance of the Poisson distribution at a specific time in terms of the rate λ. [25%]

   (ii) What is the probability that there are no arrivals in the time interval T? [5%]

   (iii) What is the probability that there is at least one arrival in the time interval T? [5%]

b) Consider an M/M/1 queue for which the arrival and service rates at state k are

$$\begin{aligned} \lambda_k &= \lambda\alpha^k, & k \geq 0,\, 0 \leq \alpha < 1 \\ \mu_k &= \mu, & k \geq 1 \end{aligned}$$

   (i) Calculate the probability P_k that there are k customers in the system. Express your answer in terms of P_0. [30%]

   (ii) Deduce an expression for P_0 and calculate the probability that there are two or more people in the system. [15%]

   (iii) Show that if $\frac{\lambda}{\mu} < 1$, then

$$P_0 > 1 - \frac{\lambda}{\mu}$$

[10%]

   (iv) Is the condition $\frac{\lambda}{\mu} < 1$ necessary for a steady state solution to exist? Can this solution exist for $\frac{\lambda}{\mu} \geq 1$? Explain your answer. [10%]

END OF QUESTION PAPER

Answers to: Network Performance Analysis: 2019-2020

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1.

(a.i) The service rate is [**4**] marks each

$$\mu_k = k\mu \qquad \text{and} \qquad \lambda_k = \lambda.$$

The system is in state k when there are k people in the system. Since there is an infinite number of servers, there is never a queue and a customer can always go to a server. It follows that the service rate of the system is proportional to the number of people in the system. [**5**]

The arrival rate is independent of the service rate and thus the arrival rate is constant. [**2**]

(a.ii) The balance equations yield [**5**]

$$\lambda P(k-1) = \mu_k P(k)$$

and thus

$$\lambda P(k-1) = k\mu P(k)$$

The solution of this equation is [**6**]

$$P(k) = P(0)\left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}$$

The probability that the system is empty is calculated from the normalisation condition [**5**]

$$\sum_{k=0}^{\infty} P(k) = 1$$

This yields [**6**] marks each

$$P(0) = \left(\sum_{k=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}\right)^{-1} = \exp(-a), \qquad a = \frac{\lambda}{\mu}$$

The steady state probability is therefore [**5**] marks each

$$P(k) = \frac{a^k}{k!}\exp(-a)$$

which is a Poisson distribution with parameter a. [**3**]

(a.iii) The average number of people in the system is [**5**]

$$\bar{N} = \frac{\lambda}{\mu}$$

and the average delay is [**5**]

$$\frac{\bar{N}}{\lambda} = \frac{1}{\mu}$$

(b.i) The arrival rate is [**10**]

$$\lambda_k = \begin{cases} \lambda & k = 0, 1, \ldots, m-1 \\ 0 & k \geq m \end{cases}$$

because no more than m customers are allowed in the queue.

The service rate is [**10**]

$$\mu_k = \begin{cases} k\mu & k = 0, 1, \ldots, m \\ 0 & k > m \end{cases}$$

(b.ii) The steady state probability is [**5**]

$$P(k) = \frac{a^k}{k!} P(0), \qquad a = \frac{\lambda}{\mu}$$

where [**5**]

$$P(0) \sum_{k=0}^{m} \frac{a^k}{k!} = 1$$

and thus [**5**]

$$P(k) = \frac{\frac{a^k}{k!}}{\sum_{k=0}^{m} \frac{a^k}{k!}}$$

The probability of blocking is obtained by setting $k = m$. [**10**]

$$P_B = \frac{\frac{a^m}{m!}}{\sum_{k=0}^{m} \frac{a^k}{k!}}$$

2(a.i) λ and μ are the arrival rate and service rate respectively, and $\rho < 1$. **[5,5,5]**

(a.ii) The expression for P_0 is calculated from $\sum_{k=0}^{\infty} P_k = 1$. **[3]**

This yields **[3]**

$$P_0\left[\sum_{k=0}^{m-1}\frac{(m\rho)^k}{k!}+\sum_{k=m}^{\infty}\frac{m^m\rho^k}{m!}\right]=1$$

Since **[10]**

$$\sum_{k=m}^{\infty}\frac{m^m\rho^k}{m!}=\frac{(m\rho)^m}{m!}\sum_{k=m}^{\infty}\rho^{k-m}=\frac{(m\rho)^m}{m!(1-\rho)}$$

for $\rho < 1$, it follows that the steady state probabilities for an M/M/m queue are **[4]**

$$P_k=\begin{cases} P_0\left(\frac{(m\rho)^k}{k!}\right) & k<m \\ P_0\left(\frac{m^m\rho^k}{m!}\right) & k\geq m \end{cases}, \qquad \rho=\frac{\lambda}{m\mu}<1$$

where

$$P_0=\left[\sum_{k=0}^{m-1}\frac{(m\rho)^k}{k!}+\frac{(m\rho)^m}{m!(1-\rho)}\right]^{-1}$$

(b.i) $m = 3$ **[4,4,2]**

$$P_k=\begin{cases} P_0\left(\frac{(3\rho)^k}{k!}\right) & k<3 \\ P_0\left(\frac{9\rho^k}{2}\right) & k\geq 3 \end{cases}, \qquad \rho=\frac{\lambda}{3\mu}.$$

(b.ii) From above, with $m = 3$, **[5]**

$$P_0=\left[\sum_{k=0}^{2}\frac{(3\rho)^k}{k!}+\frac{9\rho^3}{2(1-\rho)}\right]^{-1}$$

and this is equal to **[10]**

$$P_0\left[1+3\rho+\frac{9\rho^2}{2}+\frac{9\rho^3}{2(1-\rho)}\right]^{-1}$$

(b.iii) The average number of people in the system is ([**5**] for each term)

$$\sum_{k=0}^{m-1}kP_k+\sum_{k=m}^{\infty}kP_k$$

and its simplification yields, using the formulae above

$$P_0\sum_{k=0}^{2}\frac{k(3\rho)^k}{k!}+\frac{9P_0}{2}\sum_{k=3}^{\infty}k\rho^k$$

The first term yields $P_0(3\rho + 9\rho^2)$. **[5]**

The second term yields **[5]**

$$\frac{9P_0}{2}\sum_{k=3}^{\infty} k\rho^k = \frac{9P_0}{2}\left[\sum_{k=0}^{\infty} k\rho^k - \rho - 2\rho^2\right]$$

Since **[10]**

$$\sum_{k=0}^{\infty} k\rho^k = \frac{\rho}{(1-\rho)^2}$$

it follows that the answer is **[10]**

$$P_0\left[3\rho + 9\rho^2 + \frac{9\rho}{2(1-\rho)^2} - \frac{9\rho}{2} - 9\rho^2\right] = P_0\left[\frac{9\rho}{2(1-\rho)^2} - \frac{3\rho}{2}\right].$$

(3) (a.i) The mean of a Poisson distribution is

$$\begin{aligned}
E\{k\} &= \sum_{k=0}^{\infty} k p(k|t,\lambda) \\
&= \sum_{k=0}^{\infty} \frac{k(\lambda t)^k \exp(-\lambda t)}{k!} \qquad \textbf{[3]} \\
&= \exp(-\lambda t) \sum_{k=0}^{\infty} \frac{k(\lambda t)^k}{k!} \\
&= \exp(-\lambda t) \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!} \qquad \textbf{[3]} \\
&= \exp(-\lambda t) \left((\lambda t) + (\lambda t)^2 + \frac{(\lambda t)^3}{2!} + \cdots \right) \\
&= \exp(-\lambda t)(\lambda t)\exp(\lambda t) \\
&= \lambda t \qquad \textbf{[3]}
\end{aligned}$$

The variance of the Poisson distribution is **[3]**

$$\text{var }\{k\} = E\left\{k^2\right\} - (E\{k\})^2$$

Consider the first term:

$$\begin{aligned}
E\left\{k^2\right\} &= \sum_{k=0}^{\infty} k^2 p(k|t,\lambda) \\
&= \sum_{k=0}^{\infty} \frac{k^2(\lambda t)^k \exp(-\lambda t)}{k!} \qquad \textbf{[3]} \\
&= \exp(-\lambda t) \sum_{k=0}^{\infty} \frac{k^2(\lambda t)^k}{k!} \\
&= \exp(-\lambda t) \sum_{k=1}^{\infty} \frac{k(\lambda t)^k}{(k-1)!} \qquad \textbf{[3]} \\
&= \exp(-\lambda t) \sum_{k=2}^{\infty} \frac{(k-1)(\lambda t)^k}{(k-1)!} + \exp(-\lambda t) \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!} \\
&= \exp(-\lambda t)(\lambda t)^2 \exp(\lambda t) + \exp(-\lambda t)(\lambda t)\exp(\lambda t) \\
&= (\lambda t)^2 + \lambda t \qquad \textbf{[3]}
\end{aligned}$$

It therefore follows that **[4]**

$$\text{var }\{k\} = E\left\{k^2\right\} - (E\{k\})^2 = (\lambda t)^2 + \lambda t - (\lambda t)^2 = \lambda t.$$

It follows that

$$E\{k\} = \text{var }\{k\} = \lambda t$$

(a.iii) The probability that there are no arrivals in the time interval T is **[5]**

$$p(k|t,\lambda) = \frac{(\lambda t)^k \exp(-\lambda t)}{k!}$$

evaluated at $k = 0$. This yields $\exp(-\lambda T)$.

(a.iv) The probability that there is at least one arrival in the time interval T is **[5]**

$$1 - p(k|t,\lambda) = \frac{(\lambda t)^k \exp(-\lambda t)}{k!}$$

evaluated at $k = 0$. This yields $1 - \exp(-\lambda T)$.

(b.i) The balance equation is **[4]**

$$\lambda_{k-1} P_{k-1} + \mu_{k+1} P_{k+1} = (\lambda_k + \mu_k) P_k$$

which yields **[5]**

$$\lambda_{k-1} P_{k-1} = \mu_k P_k$$

The solution of this equation is **[8]**

$$P_k = \left(\frac{\lambda}{\mu}\right) \alpha^{k-1} P_{k-1}$$

and cycling through $k = 0, 1, 2, 3, \ldots,$ yields **[6]**

$$P_k = P_0 \left(\frac{\lambda}{\mu}\right)^k \alpha^{\sum_{i=0}^k i} = P_0 \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k+1)}{2}} \qquad \textbf{[7]}$$

(b.ii) Since **[5]**

$$\sum_{k=0}^{\infty} P_k = 1$$

it follows that **[5]**

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k+1)}{2}}}$$

The probability that there are 2 or more people in the system is **[5]**

$$1 - P_0 - P_1 = 1 - P_0 - P_0 \left(\frac{\lambda}{\mu}\right) \alpha$$

(b.iii) From above

$$P_0 = \frac{1}{\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k+1)}{2}}} > \frac{1}{\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^k} = 1 - \frac{\lambda}{\mu} \qquad \textbf{[5]}, \textbf{[5]}$$

since $\alpha < 1$

(b.iv) The condition $\frac{\lambda}{\mu} < 1$ is not necessary for a steady state solution to exist. A steady state solution can exist with $\frac{\lambda}{\mu} > 1$. **[5]**

Even if $\frac{\lambda}{\mu} > 1$, α may be sufficiently small that the infinite sum in the expression for P_0 converges. **[5]**

Network Performance Analysis: Tutorial 1

1. Leaves fall from a tree to the ground as a Poisson process at a rate of one every minute.

   (a) Sketch the form of the Poisson distribution for several values of $\mu = \lambda t$. What happens as μ increases?

   (b) At what value of k does the curve achieve its maximum value?

   (c) You arrive at 4:00 pm. During the next ten minutes, what is the probability that you see twelve leaves fall?

2. X takes on four different values with differing probabilities, as shown in the following table.

X	1	2	3	4
$P(X)$	0.05	0.2	0.3	0.45

   (a) What is the mean of X?

   (b) What is the standard deviation of X?

   (c) What is the expectation of $\ln X$?

3. (a) Show that the standard deviation of the Poisson distribution, normalised by its mean value, tends to zero as $\mu = \lambda t$ increases.

   (b) What does this imply about the form of the distribution for large values of μ?

4. Compute the mean and variance of the exponential distribution,

   $$p\left(x\right) = \lambda \exp\left(-\lambda x\right), \quad \text{where}\, 0 < x < \infty.$$

   What is the probability that x will be between 1 and 2 if $\lambda = 0.5$?

5. Calls arrive at a telephone exchange at a rate of 1000 calls per hour. What is the probability that one or more calls arrive in an interval of one second?

6. (a) Suppose one observes a Poisson process for which a packet arrives, on average, every millisecond. No packets arrive within a 5 millisecond interval. How does this affect the probability of observing an arrival during the next millisecond?

   (b) Packets arrive according to a Poisson process at an average rate of one packet every millisecond. What is the probability that the first arrival occurs within 3 milliseconds?

Network Performance Analysis: Solutions to Tutorial 1

1. (a) The curve $P(k|\mu)$ against k has one maximum. As k increases, the value of k at which the maximum value of $P(k|\mu)$ occurs, increases, but the value of the maximum decreases.

   (b) The curve achieves its maximum value at $k \approx \mu$. Note that k is an integer and μ is a real number.

   (c) From the data, $k = 12$ and $\mu = \lambda t = 1 \times 10 = 10$. Thus

$$P(k|t, \lambda) = P(k|\mu) = \frac{10^{12} \exp(-10)}{12!} = \frac{4.54 \times 10^{7}}{12!} = 0.09478$$

2. (a) The mean value of X is

$$E\{X\} = 1(0.05) + 2(0.2) + 3(0.3) + 4(0.45) = 3.15$$

   (b) The variance of X is

$$\text{var}\{X\} = E\left\{(X - E\{X\})^2\right\} = E\left\{X^2\right\} - (E\{X\})^2$$

   Evaluate the first term.

$$E\left\{X^2\right\} = 1(0.05) + 4(0.2) + 9(0.3) + 16(0.45) = 10.75$$

   Thus

$$\text{var}\{X\} = 10.75 - (3.15)^2 = 0.8275$$

   (c) The following table is required:

X	1	2	3	4
$P(X)$	0.05	0.2	0.3	0.45
$\ln X$	0	0.693	1.099	1.386

   Thus

$$E\{\ln X\} = 0(0.05) + 0.693(0.2) + 1.099(0.3) + 1.386(0.45) = 1.092$$

3. (a)

$$\frac{\sigma}{\mu} = \frac{\sqrt{\text{var}\{k\}}}{\mu} = \frac{\sqrt{\mu}}{\mu} = \frac{1}{\sqrt{\mu}} = \frac{1}{\sqrt{\lambda t}}$$

   and the result follows.

(b) This result shows that the distribution is closely packed about the mean μ for large values of μ. Note that the mean and standard deviation increase as μ increases, but their ratio decreases, as shown in part (a).

4. (a) The mean of the exponential distribution is $\frac{1}{\lambda}$ and

$$\int_0^\infty \lambda x^2 \exp(-\lambda x)\,\mathrm{dx} = \frac{2}{\lambda^2}$$

It follows that the variance is equal to

$$\frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

(b) The probability is

$$\int_1^2 0.5 \exp\left(-\frac{x}{2}\right)\,\mathrm{dx} = -\exp\left(-\frac{x}{2}\right)\Big|_1^2 = \exp\left(-\frac{1}{2}\right) - \exp\left(-1\right) = 0.239$$

5. The arrival rate is $\lambda = \frac{1000}{3600}$ calls per second.

Since

P(1 or more calls arrive in an interval of 1 second) $=$ 1-P(there are no calls in an interval of 1 second)

the answer is

$$1 - P(0|1,\lambda) = 1 - \exp\left(-\frac{1000}{3600}\right) = 0.24$$

6. (a) Since the Poisson process has no memory, the fact that no packets arrived within a 5 millisecond interval has no effect. Thus historical information about a Poisson process does improve predictions about the future.

(b) Since

P(no arrivals in first 3 milliseconds) $+$ $P(k \geq 1$ arrivals in first 3 milliseconds)$=$1

it follows that

$P(k \geq 1$ arrivals in first 3 milliseconds)$=$1-$P(0|3,1) = 1 - \exp(-(3 \times 1)) = 0.95$

Network Performance Analysis: Tutorial 2

1. In the lecture notes we saw that the average number of customers in the system for an $M/M/1$ queue was given by

$$E\{k\} = \frac{\rho}{1-\rho}$$

   Show that the variance of k is given by

$$E\left\{k^2\right\} - (E\{k\})^2 = \frac{\rho}{(1-\rho)^2}$$

2. One thousand calls per hour arrive at a telephone exchange, and the arrivals have a Poisson distribution. What is the probability that one or more calls arrive in an interval of one second?

3. Students arrive randomly at a cash-dispensing machine with an average interarrival time of 5 minutes. The length of time that a student spends on the machine is exponentially distributed with an average of 2 minutes. The system is modelled as an M/M/1 queue.

   (a) What is the probability that a student arriving at the machine will have to wait?

   (b) What is the average total time (queuing and being served) time for a student?

   (c) The bank plans to install a second machine when the total time in the system is 5 minutes or longer. At what average arrival rate will this occur? For this arrival rate, what is the average time that a person waits in the queue at the machine?

   (d) What is the average number of people in the queue for the arrival rate in part (c)?

Network Performance Analysis: Solutions to Tutorial 2

1. To calculate the variance

$$
\begin{aligned}
E\left\{k^2\right\} &= \sum_{k=0}^{\infty} k^2 p(k) \\
&= \sum_{k=0}^{\infty} k^2 \rho^k (1-\rho) \\
&= \sum_{k=0}^{\infty} k(k-1)\rho^k(1-\rho) + \sum_{k=0}^{\infty} k\rho^k(1-\rho) \\
&= E\{k\} + (1-\rho)\rho^2 \sum_{k=2}^{\infty} k(k-1)\rho^{k-2} \\
&= E\{k\} + (1-\rho)\rho^2 \sum_{k=2}^{\infty} \frac{d^2\rho^k}{d\rho^2} \\
&= E\{k\} + (1-\rho)\rho^2 \frac{d^2}{d\rho^2}\left(\sum_{k=2}^{\infty} \rho^k\right) \\
&= E\{k\} + (1-\rho)\rho^2 \frac{d^2}{d\rho^2}\left(\frac{\rho^2}{1-\rho}\right) \\
&= E\{k\} + (1-\rho)\rho^2 \left(\frac{2}{(1-\rho)^3}\right) \\
&= E\{k\} + \frac{2\rho^2}{(1-\rho)^2}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\operatorname{var}\{k\} &= E\left\{k^2\right\} - (E\{k\})^2 \\
&= E\{k\} + \frac{2\rho^2}{(1-\rho)^2} - \left(\frac{\rho}{1-\rho}\right)^2 \\
&= \frac{\rho}{1-\rho} + \frac{2\rho^2}{(1-\rho)^2} - \left(\frac{\rho}{1-\rho}\right)^2 \\
&= \frac{\rho}{(1-\rho)^2}
\end{aligned}
$$

2. Since

$$P(k|t,\lambda) = \frac{(\lambda t)^k}{k!}\exp(-\lambda t), \qquad \lambda t = \frac{1000}{3600} = \frac{1}{3.6}$$

it follows that the probability that one or more calls arrive in a one second interval is

$$1 - P(0|t,\lambda) = 1 - \exp\left(-\frac{1}{3.6}\right) = 0.243$$

3. The waiting time is equal to the sum of the queuing time and the waiting time

   (a) The probability that the machine is busy is given by

   $$1 - \text{probability that the machine is not occupied} = 1 - P(0)$$

   where $P(0)$ is the probability that the system is in state zero, that is, there are no people in the the system. Thus the probability that the machine is busy is the utilisation factor ρ,

   $$\rho = \frac{\lambda}{\mu} = \frac{0.2}{0.5} = 0.4$$

   (b) Apply Little's formula to the entire system.

   $$\bar{N} = \lambda \bar{T}$$

   where $\bar{N}$ is the average number of people in the system

   $$\bar{N} = E\{k\} = \frac{\rho}{1-\rho}, \qquad \rho = \frac{\lambda}{\mu}$$

   Thus

   $$\bar{T} = \frac{\bar{N}}{\lambda} = \left(\frac{\rho}{1-\rho}\right)\frac{1}{\lambda} = \frac{1}{\mu-\lambda} = \frac{1}{0.5-0.2} = \frac{10}{3} \text{ minutes}$$

   (c) We require

   $$\frac{1}{\mu-\lambda} \geq 5, \qquad \mu = 0.5$$

   and thus the arrival rate satisfies $\lambda \geq 0.3$ if the average waiting time is 5 minutes or longer.
   The total average delay is confirmed from Little's theorem,

   $$\bar{T} = \frac{\bar{N}}{\lambda} = \left(\frac{\rho}{1-\rho}\right)\frac{1}{\lambda} = \frac{0.6}{(1-0.6)}\frac{1}{0.3} = 5 \text{ minutes}$$

   as required. This is the total waiting time, and thus the average time spent in the queue is

   $$5 - \text{serving time} = 5 - 2 = 3 \text{ minutes}$$

   (d) Use Little's formula again:

   $$\bar{N} = \lambda \bar{T} = 0.3 \times 3 = 0.9 \text{ people}$$

Network Performance Analysis: Tutorial 3

1. A telephone call centre services complaints from bank customers about the quality of their service. They expect at 'busy hour' calls to arrive at a rate of 120 customers an hour. Dealing with each customer takes on average 3 minutes. When all serving assistants are occupied, customers are placed in a holding system.

   (a) The service centre has 10 serving assistants. What is the probability of a customer having to join the queue?

   (b) What is the expected queueing time for each customer?

   (c) Given that customers who do not have to queue have an average queueing time of zero, what is the average waiting time for customers who have to queue?

   (d) The bank wishes to combine two of these service centers for efficiency. Both service centers have arrival rates of 120 customers an hour in 'busy hour'. How many staff will they need at the new centre to maintain a queueing probability at least as low as in a)?

   (e) Answer parts b) and c) for the new system.

   For this question you may assume that arrivals are Poisson distributed and service times are exponentially distributed.

2. Consider a finite M/M/1 queue capable of accommodating N packets (customers). Calculate the values of N required for the following situations and comment on your answers:

(i)	$\rho = 0.5$	$P_B = 10^{-3}$
(ii)	$\rho = 0.5$	$P_B = 10^{-6}$
(iii)	$\rho = 0.8$	$P_B = 10^{-3}$
(iv)	$\rho = 0.8$	$P_B = 10^{-6}$

3. Let n be the total number of objects in two identical M/M/1 queues, each operating independently with arrival rate λ and service rate μ.

   (a) Show that the probability that the two queues are in state n is

$$P_n = (n+1)\rho^n(1-\rho)^2, \qquad \rho = \frac{\lambda}{\mu} < 1$$

   (b) Show that $P_n < 1$.

   (c) Show that

$$\sum_{n=0}^{\infty} P_n = 1.$$

(d) If the two separate M/M/1 systems are merged into one system to form one M/M/2 system (one system with one queue and two servers), show that the number of items k in the steady state is

$$P_k = \begin{cases} \frac{2(1-\rho)\rho^k}{1+\rho} & k \geq 1 \\ \frac{(1-\rho)}{1+\rho} & k = 0 \end{cases}$$

4. A telephone company plans to rent trunk lines between Birmingham and London for $1,000 per year per trunk line. They expect to have a demand of 20 calls per hour, and it is expected that the average length of a call is 15 minutes. They can charge 0.01$/minute for each customer. They will provide no holding facility, and thus a queue is not permitted.

   (a) How many trunk lines will they need to keep the blocking probability below 0.01?

   (b) How much profit will they make?

   For this question you may assume that arrivals are Poisson distributed and service times are exponentially distributed.

5. A service time distribution is defined as follows:

$$P(X) = \begin{cases} \frac{2}{b} - \frac{2}{b^2}X & 0 < X \leq b \\ 0 & X > b \end{cases} \tag{1}$$

   (a) Sketch this distribution.

   (b) The Pollaczek-Khinchin formula states the average waiting time for a queue is given by

$$\bar{W} = \frac{\lambda E\left\{X^2\right\}}{2(1-\rho)}$$

   By determining the first two moments, $E\{X\}$ and $E\left\{X^2\right\}$, of the distribution given in (1), derive an expression for the average waiting time for the queue.

6. **Queue with discouragement** Consider a single-server queuing system in which the flow of packets is controlled, that is, the state-dependent arrival rate is

$$\lambda_k = \frac{\lambda}{k+1}, \qquad k = 0, 1, \ldots$$

   where λ is a given constant. The service rate is constant $\mu_k = \mu$.

   This type of system is used to model packet transmission in which the maximum possible arrival rate is λ, but as the queue length increases, a system controller discourages packet arrivals, either by blocking or shunting packets elsewhere, so that the true arrival rate decreases as k increases.

   (a) Draw the state transition diagram for this system and derive an expression for P_k, the probability that the system is in state k.

   (b) Calculate the average number of packets in the system.

(c) Compare this result with the average number of packets in an M/M/1 queue in which there is no blocking. Comment on your answer.

(d) Calculate the average arrival rate of the packets.

(e) Calculate the average time delay in the system.

7. Consider an $M/M/1$ queue with arrival rate λ and service rate μ. If $\rho = 0.3$ and there is a probability of 97% that there are m or fewer customers in the system, calculate m.

8. Is the following table realisable for an $M/M/1$ queue?

k	P_k
0	0.40
1	0.30
2	0.20
3	0.10
≥ 4	0.00

9. For a system with a single server (such as the $M/M/1$ and $M/M/1/m$ queue) the utilisation factor, ρ, is the probability that the queue state is not zero. For a lossless system we saw, using Little's formula (which is valid for $M/G/1$ queues), that the utilisation factor is given by $\rho = \frac{\lambda}{\mu}$. Use the stationary distribution for the $M/M/1/m$ queue to compute the utilisation factor for the $M/M/1/m$ queue. Explain why it is not $\frac{\lambda}{\mu}$.

10. Mrs Cut runs a hair salon and she is the only employee. She does not make appointments but runs the salon on a first-come-first-served basis. She is extremely busy on Saturday mornings and is considering hiring an assistant or moving to larger premises. Before she does this, however, she analyses the situation carefully. She has kept careful records of the salon, including the arrival and service rates of the customers. In particular, her analysis shows that customers arrive according to a Poisson process at a rate of $\lambda = 5$ customers per hour, and it takes about 10 minutes to cut a client's hair.

    (a) Calculate the average number of people (waiting to be served and being served) in the salon.

    (b) Calculate the average number of people waiting to be served.

    (c) What is the percentage of the time that a customer can enter the salon and be served immediately?

    (d) The waiting room in Mrs Cut's salon has four seats. Calculate the probability that a customer will not find a seat.

    (e) Calculate the average waiting time in the system and the average number of people in the queue, where the average is taken over non-empty queues.

    (f) Calculate the average waiting time in the queue.

Network Performance Analysis: Solutions to Tutorial 3

1. (a) An example of an M/M/m queue.

$$\lambda = 2, \qquad \mu = \frac{1}{3} \qquad \text{and} \qquad m = 10$$

Thus

$$a = m\rho = \frac{\lambda}{\mu} = 6$$

and from the Erlang C curve, it follows that the probability of waiting is about 0.10.

(b) The expected waiting time for each customer is

$$\bar{W} = \frac{\bar{N}_Q}{\lambda} = \frac{C(m,a)}{m\mu(1-\rho)} = \frac{0.1}{10 \times \frac{1}{3} \times 0.4} = 0.075$$

This is the average total time spent in the queue, and includes customers who do not queue.

(c) The expected waiting time for customers who have to queue is calculated using the answer to part (b).

The total average queuing time, where the average is taken over all people in the system, can be written as

average waiting time in queue = (probability of queueing) x *(average waiting in queue for people who do queue)*

$$+$$

(probability of not queueing) x *(average waiting time in queue for people who do not need to queue)*

Since the probability of queueing is $C(m,a)$, it follows that $\bar{W}$, the total average queuing time, where the average is taken over all people in the system, can be written as

$$\bar{W} = C(m,a)\bar{W}_Q + (1 - C(m,a))\bar{V}_Q$$

where $\bar{W}_Q$ is the average queuing time for people who queue, and $\bar{V}_Q$ is the average queuing time for people who do not queue. Since $\bar{V}_Q = 0$, it follows that

$$\bar{W}_Q = \frac{\bar{W}}{C(m,a)} = \frac{1}{m\mu(1-\rho)} = 0.75$$

Note the difference between parts b) and c):

average waiting time, given that you must queue
$= 10 \times$ total average waiting time

and so if you have to queue, it is better to terminate the call and try again.

(d) $\lambda = 240$ calls per hour $= 4$ calls per minute, $\mu = \frac{1}{3}$ and $P(\text{waiting}){=}0.1$. The Erlang C curve with $a = \frac{\lambda}{\mu} = 12$ and $P(\text{waiting}){=}0.1$ yields $m = 18$.

(e) The total average waiting time for each customer is, since $\rho = \frac{2}{3}$,

$$\bar{W} = \frac{\bar{N}_Q}{\lambda} = \frac{C(m,a)}{m\mu(1-\rho)} = \frac{0.1}{18 \times \frac{1}{3} \times \frac{1}{3}} = 0.05$$

The average waiting time for customers who have to queue is

$$\bar{W}_Q = \frac{\rho}{\lambda(1-\rho)} = \frac{\frac{2}{3}}{4 \times \frac{1}{3}} = 0.50$$

Note again that if you have to queue, it is better to terminate the call and try again.

2. (i) $\rho = 0.5$, $P_B = 10^{-3}$

$$P_B = \frac{(0.5)(0.5)^N}{1 - 0.5^{N+1}} = 10^{-3}$$

This yields $N = 9$.

(ii) $\rho = 0.5$, $P_B = 10^{-6}$

$$P_B = \frac{(0.5)(0.5)^N}{1 - 0.5^{N+1}} = 10^{-6}$$

This yields $N = 19$.

(iii) $\rho = 0.8$, $P_B = 10^{-3}$

$$P_B = \frac{(0.2)(0.8)^N}{1 - 0.8^{N+1}} = 10^{-3}$$

This yields

$$(0.2)(0.8)^N = 10^{-3} - 10^{-3}(0.8^{N+1})$$

and hence

$$(0.8)^N \left(0.2 + 10^{-3}(0.8)\right) = 10^{-3}$$

which yields $N = 24$.

(iv) $\rho = 0.8$, $P_B = 10^{-6}$

$$P_B = \frac{(0.2)(0.8)^N}{1 - 0.8^{N+1}} = 10^{-6}$$

This yields $N = 55$.

A reduction in the blocking probability by three orders of magnitude causes the number of people N required to achieve blocking to increase by less than one order of magnitude, and thus N is stable with respect to changes in P_B.

3. This question is on two M/M/1 queues.

   (a) The number n of objects in the queue is given by

   $$\sum_{k=0}^{n} (k \text{ objects in first queue AND } (n-k) \text{ objects in second queue})$$

   The probabilities that there are k objects in the first queue and $(n-k)$ objects in the second queue are

   $$\rho^k(1-\rho) \qquad \text{and} \qquad \rho^{n-k}(1-\rho)$$

   respectively. The summation above is therefore equal to

   $$P_n = \sum_{k=0}^{n} \left(\rho^k(1-\rho)\right)\left(\rho^{n-k}(1-\rho)\right) = (n+1)\rho^n(1-\rho)^2$$

   (b) Differentiating P_n with respect to ρ yields

   $$(n+1)\rho^{(n-1)}(1-\rho)\left(-2\rho + n(1-\rho)\right) = 0$$

   and thus ignoring the trivial solutions $\rho = 0, 1$,

   $$\rho = \frac{n}{n+2} < 1$$

   at a stationary point.
   It follows that, for this value of ρ,

   $$P_n = (n+1)\left(\frac{n}{n+2}\right)^n \left(\frac{2}{n+2}\right)^2 = \left(\frac{n}{n+2}\right)^n \frac{4(n+1)}{(n+2)^2}$$

   and since $(n+2)^2 > 4(n+1)$, it follows that $P_n < 1$. This stationary point must be a maximum because P_n is a minimum at $\rho = 0, 1$.

   (c)

   $$\sum_{n=0}^{\infty} P_n = (1-\rho)^2 \sum_{n=0}^{\infty}(n+1)\rho^n = (1-\rho)^2\left(\sum_{n=0}^{\infty} n\rho^n + \sum_{n=0}^{\infty}\rho^n\right)$$

   and this is equal to

   $$(1-\rho)^2\left(\frac{\rho}{(1-\rho)^2} + \frac{1}{1-\rho}\right) = 1$$

   and thus the quantities P_n are probabilities.

(d) From the lecture notes for an M/M/2 queue

$$P_k = \frac{2(1-\rho)\rho^k}{1+\rho}, \qquad k \geq 1.$$

which is the answer to the first part. Consider now the P_0. Since

$$P_0 + \sum_{k=1}^{\infty} P_k = 1$$

it follows that

$$P_0 = 1 - \sum_{k=1}^{\infty} P_k = 1 - \frac{2(1-\rho)}{1+\rho} \sum_{k=1}^{\infty} \rho^k$$

and this yields

$$P_0 = 1 - \frac{2\rho}{1+\rho} = \frac{1-\rho}{1+\rho}$$

4. This question is on the M/M/m/m queue.

   (a) The arrival rate is $\lambda = \frac{1}{3}$ call per minute, the service rate is $\mu = \frac{1}{15}$, and thus $a = \lambda/\mu = 5$. It follows from the Erlang B curve that $m = 11$ trunk lines are required.

   (b) They expect to receive 20 calls per hour, the blocking probability is 0.01, and thus there are $20 \times 0.99 = 19.8$ successful calls per hour.

   Each call lasts on average 15 minutes, and thus there is a total of 19.8×15 telephone minutes per hour. Since there are 365×24 hours per year, it follows that there are

   $$365 \times 24 \times 19.8 \times 15$$

   telephone minutes per year. The annual income is therefore

   $$365 \times 24 \times 19.8 \times 15 \times 0.01 = \$26,017.20$$

   The profit is therefore \$26,017.20-(\$1,000 $\times$ 11)= \$15,017.20.

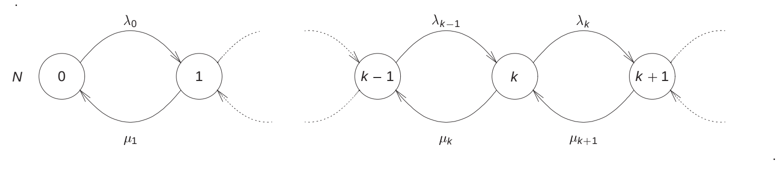
5. (a) This is a straight line for $0 < X \leq b$ that passes through the points $(0, 2/b)$ and $(b, 0)$.

   (b) The first and second moments are

   $$\begin{aligned} E\{x\} &= \frac{2}{b}\int_0^b \left(x - \frac{x^2}{b}\right) \mathrm{dx} = \frac{b}{3} \\ E\{x^2\} &= \frac{2}{b}\int_0^b \left(x^2 - \frac{x^3}{b}\right) \mathrm{dx} = \frac{b^2}{6} \end{aligned}$$

   Since $\rho = \lambda E\{x\} = \lambda b/3$, it follows that

   $$\bar{W} = \frac{\lambda b^2/6}{2\,(1-\lambda b/3)} = \frac{\lambda b^2}{4(3-\lambda b)}$$



6. (a) The state transition diagram is
where $\lambda_k = \lambda/(k+1)$ and $\mu_k = \mu$.
The balance equations yield

$$\lambda_{k-1}P_{k-1} + \mu P_{k+1} = \lambda_k P_k + \mu P_k$$

that is

$$\frac{\lambda}{k}P_{k-1} + \mu P_{k+1} = \frac{\lambda}{k+1}P_k + \mu Pk$$

The solution of this equation is

$$\frac{\lambda}{k+1}P_k = \mu P_{k+1}$$

The general solution is therefore

$$P_k = \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k P_0$$

where P_0 is calculated from the condition

$$\sum_{k=0}^{\infty} P_k = 1$$

This assumes a queue of infinite length. Thus

$$P_0 \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k = 1$$

and hence $P_0 = \exp(-\lambda/\mu)$. It follows that

$$P_k = \exp\left(-\frac{\lambda}{\mu}\right)\frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k, \qquad k = 0, 1, \ldots$$

(b) The average number of packets in the system is

$$\begin{aligned}
\sum_{k=0}^{\infty} kP_k &= \exp\left(-\frac{\lambda}{\mu}\right)\sum_{k=1}^{\infty}\frac{1}{(k-1)!}\left(\frac{\lambda}{\mu}\right)^k \\
&= \exp\left(-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\sum_{k=1}^{\infty}\frac{1}{(k-1)!}\left(\frac{\lambda}{\mu}\right)^{k-1} \\
&= \frac{\lambda}{\mu}
\end{aligned}$$

(c) The average number of packets in an M/M/1 queue is

$$\frac{\rho}{1-\rho}, \qquad \rho = \frac{\lambda}{\mu}$$

Since $\rho < \rho/(1-\rho)$ for $\rho < 1$, the result is expected because queuing with discouragement reduces the average number of packets in the queue.

(d) The average arrival rate of the packets is

$$\begin{aligned}
\gamma &= \sum_{k=0}^{\infty} \lambda_k P_k \\
&= \lambda \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \exp\left(-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^k \\
&= \mu \exp\left(-\frac{\lambda}{\mu}\right) \sum_{k=0}^{\infty} \frac{1}{(k+1)!}\left(\frac{\lambda}{\mu}\right)^{k+1} \\
&= \mu \exp\left(-\frac{\lambda}{\mu}\right)\left(\exp\left(\frac{\lambda}{\mu}\right) - 1\right) \\
&= \mu\Big(1 - \exp(-\rho)\Big), \qquad \rho = \frac{\lambda}{\mu}
\end{aligned}$$

(e) The average time delay in the system is

$$\bar{T} = \frac{\bar{N}}{\lambda}, \qquad \bar{N} = \rho = \frac{\lambda}{\mu}$$

The average value of λ is, from above,

$$\gamma = \sum_{k=0}^{\infty} \lambda_k P_k = \mu\Big(1 - \exp(-\rho)\Big)$$

and thus

$$\begin{aligned}
\bar{T} &= \frac{\rho}{\mu\Big(1 - \exp(-\rho)\Big)} \\
&= \frac{1}{\mu} \quad \text{if } \rho \ll 1
\end{aligned}$$

7. The formula is

$$P_k = \rho^k(1-\rho) = 0.7(0.3)^k$$

k	P_k
0	0.700
1	0.210
2	0.063
sum	0.973

and thus $m = 2$.

8. Strictly speaking no because $P_k/P_{k-1} = \rho$ is not constant, which it must be. Real world problems may depart from this ideal scenario, and thus the data in the table may be realised in practice.

9. The utilisation factor is defined as the steady probability that the system is busy.

   For an M/M/1 system with a finite length queue, we have

$$P(k) = \frac{(1-\rho)\rho^k}{1-\rho^{N+1}}$$

   and thus the probability that the system is empty is

$$P(0) = \frac{1-\rho}{1-\rho^{N+1}}$$

   The utilisation factor is therefore

$$1 - P(0) = 1 - \frac{1-\rho}{1-\rho^{N+1}} = \frac{\rho - \rho^{N+1}}{1-\rho^{N+1}}$$

   Note that as $N \to \infty$, the utilisation factor approaches ρ.

10. This is an M/M/1 queue. The arrival rate is $\lambda = 5$ and the service rate is $\mu = 6$ customers per hour. This yields $\rho = \frac{5}{6}$.

    (a) The average number of people in the system is

$$E\{k\} = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1-\rho} = 5$$

    (b) The average number of people waiting to be served is given by

$$\begin{aligned}
\sum_{k=1}^{\infty}(k-1)p_k &= \sum_{k=2}^{\infty} kp_k - \sum_{k=2}^{\infty} p_k \\
&= \left(\sum_{k=1}^{\infty} kp_k - p_1\right) - (1 - p_0 - p_1) \\
&= \sum_{k=1}^{\infty} kp_k - (1 - p_0) \\
&= \sum_{k=0}^{\infty} kp_k - (1 - p_0) \\
&= \frac{\rho}{1-\rho} - (1 - p_0) \\
&= \frac{\rho}{1-\rho} - \rho \\
&= 4\frac{1}{6}.
\end{aligned}$$

(c) The probability that the salon is empty is $p_0 = 1 - \rho = \frac{1}{6}$, that is 16.67% of the time the salon is empty and a customer can be served immediately.

(d) The probability that a customer will not find a seat is

$$\sum_{k=5}^{\infty}(1-\rho)\rho^k = 1 - \left(\sum_{k=0}^{4}(1-\rho)\rho^k\right) = \rho^5 = 0.402$$

(e) From Little's theorem, the average waiting time in the system is

$$\frac{E\{k\}}{\lambda} = 1 \text{ hour}$$

A queue is formed if there are two or more people in the system. The probability that there are k people in the system is

$$P_k = \rho^k(1-\rho)$$

and the probability that there are two or more people in the system is

$$\sum_{k=2}^{\infty}\rho^k(1-\rho) = 1 - (1-\rho) - \rho(1-\rho) = \rho^2$$

It follows that given that there is a queue, that is, $k \geq 2$, the probability that the system is in state $k \geq 2$ is

$$\bar{p}_k = \frac{\rho^k(1-\rho)}{\rho^2}, \qquad k \geq 2$$

The average number of people in the queue, taken over non-empty queues, is therefore

$$L = \sum_{k=2}^{\infty}(k-1)\bar{p}_k = \frac{(1-\rho)}{\rho^2}\sum_{k=2}^{\infty}(k-1)\rho^k$$

which is equal to

$$\frac{(1-\rho)}{\rho^2}\left(\left(\sum_{k=1}^{\infty}k\rho^k - \rho\right) - \left(\sum_{k=0}^{\infty}\rho^k - 1 - \rho\right)\right)$$

and this is equal to

$$\frac{(1-\rho)}{\rho^2}\left(\frac{\rho}{(1-\rho)^2} - \frac{1}{1-\rho} + 1\right) = \frac{1}{1-\rho} = \frac{1}{1-\frac{1}{6}} = 6$$

and there are therefore, on average, six people waiting in the queue, where the average is taken over non-empty queues.

(f) The average average number of people in the queue is $4\frac{1}{6}$, and since $\lambda = 5$, it follows that the average waiting time in the queue is $\frac{5}{6}$ hour. The average waiting times are therefore unacceptable long, either one hour (in all queues), or $\frac{5}{6}$ hour for non-empty queues only.