Network Performance Analysis: Solutions to Tutorial 3

1. (a) An example of an M/M/m queue.

$$\lambda = 2, \qquad \mu = \frac{1}{3} \qquad \text{and} \qquad m = 10$$

Thus

$$a = m\rho = \frac{\lambda}{\mu} = 6$$

and from the Erlang C curve, it follows that the probability of waiting is about 0.10.

(b) The expected waiting time for each customer is

$$\bar{W} = \frac{\bar{N}_Q}{\lambda} = \frac{C(m,a)}{m\mu(1-\rho)} = \frac{0.1}{10 \times \frac{1}{3} \times 0.4} = 0.075$$

This is the average total time spent in the queue, and includes customers who do not queue.

(c) The expected waiting time for customers who have to queue is calculated using the answer to part (b).

The total average queuing time, where the average is taken over all people in the system, can be written as

average waiting time in queue = (probability of queueing) x *(average waiting in queue for people who do queue)*

$$+$$

(probability of not queueing) x *(average waiting time in queue for people who do not need to queue)*

Since the probability of queueing is $C(m,a)$, it follows that $\bar{W}$, the total average queuing time, where the average is taken over all people in the system, can be written as

$$\bar{W} = C(m,a)\bar{W}_Q + (1 - C(m,a))\bar{V}_Q$$

where $\bar{W}_Q$ is the average queuing time for people who queue, and $\bar{V}_Q$ is the average queuing time for people who do not queue. Since $\bar{V}_Q = 0$, it follows that

$$\bar{W}_Q = \frac{\bar{W}}{C(m,a)} = \frac{1}{m\mu(1-\rho)} = 0.75$$

Note the difference between parts b) and c):

average waiting time, given that you must queue
$= 10 \times$ total average waiting time

and so if you have to queue, it is better to terminate the call and try again.

(d) $\lambda = 240$ calls per hour $= 4$ calls per minute, $\mu = \frac{1}{3}$ and $P(\text{waiting}){=}0.1$. The Erlang C curve with $a = \frac{\lambda}{\mu} = 12$ and $P(\text{waiting}){=}0.1$ yields $m = 18$.

(e) The total average waiting time for each customer is, since $\rho = \frac{2}{3}$,

$$\bar{W} = \frac{\bar{N}_Q}{\lambda} = \frac{C(m,a)}{m\mu(1-\rho)} = \frac{0.1}{18 \times \frac{1}{3} \times \frac{1}{3}} = 0.05$$

The average waiting time for customers who have to queue is

$$\bar{W}_Q = \frac{\rho}{\lambda(1-\rho)} = \frac{\frac{2}{3}}{4 \times \frac{1}{3}} = 0.50$$

Note again that if you have to queue, it is better to terminate the call and try again.

2. (i) $\rho = 0.5$, $P_B = 10^{-3}$

$$P_B = \frac{(0.5)(0.5)^N}{1 - 0.5^{N+1}} = 10^{-3}$$

This yields $N = 9$.

(ii) $\rho = 0.5$, $P_B = 10^{-6}$

$$P_B = \frac{(0.5)(0.5)^N}{1 - 0.5^{N+1}} = 10^{-6}$$

This yields $N = 19$.

(iii) $\rho = 0.8$, $P_B = 10^{-3}$

$$P_B = \frac{(0.2)(0.8)^N}{1 - 0.8^{N+1}} = 10^{-3}$$

This yields

$$(0.2)(0.8)^N = 10^{-3} - 10^{-3}(0.8^{N+1})$$

and hence

$$(0.8)^N \left(0.2 + 10^{-3}(0.8)\right) = 10^{-3}$$

which yields $N = 24$.

(iv) $\rho = 0.8$, $P_B = 10^{-6}$

$$P_B = \frac{(0.2)(0.8)^N}{1 - 0.8^{N+1}} = 10^{-6}$$

This yields $N = 55$.

A reduction in the blocking probability by three orders of magnitude causes the number of people N required to achieve blocking to increase by less than one order of magnitude, and thus N is stable with respect to changes in P_B.

3. This question is on two M/M/1 queues.

   (a) The number n of objects in the queue is given by

   $$\sum_{k=0}^{n} (k \text{ objects in first queue AND } (n-k) \text{ objects in second queue})$$

   The probabilities that there are k objects in the first queue and $(n-k)$ objects in the second queue are

   $$\rho^k(1-\rho) \qquad \text{and} \qquad \rho^{n-k}(1-\rho)$$

   respectively. The summation above is therefore equal to

   $$P_n = \sum_{k=0}^{n} \left(\rho^k(1-\rho)\right)\left(\rho^{n-k}(1-\rho)\right) = (n+1)\rho^n(1-\rho)^2$$

   (b) Differentiating P_n with respect to ρ yields

   $$(n+1)\rho^{(n-1)}(1-\rho)\left(-2\rho + n(1-\rho)\right) = 0$$

   and thus ignoring the trivial solutions $\rho = 0, 1$,

   $$\rho = \frac{n}{n+2} < 1$$

   at a stationary point.
   It follows that, for this value of ρ,

   $$P_n = (n+1)\left(\frac{n}{n+2}\right)^n \left(\frac{2}{n+2}\right)^2 = \left(\frac{n}{n+2}\right)^n \frac{4(n+1)}{(n+2)^2}$$

   and since $(n+2)^2 > 4(n+1)$, it follows that $P_n < 1$. This stationary point must be a maximum because P_n is a minimum at $\rho = 0, 1$.

   (c)

   $$\sum_{n=0}^{\infty} P_n = (1-\rho)^2 \sum_{n=0}^{\infty} (n+1)\rho^n = (1-\rho)^2 \left(\sum_{n=0}^{\infty} n\rho^n + \sum_{n=0}^{\infty} \rho^n\right)$$

   and this is equal to

   $$(1-\rho)^2\left(\frac{\rho}{(1-\rho)^2} + \frac{1}{1-\rho}\right) = 1$$

   and thus the quantities P_n are probabilities.

(d) From the lecture notes for an M/M/2 queue

$$P_k = \frac{2(1-\rho)\rho^k}{1+\rho}, \qquad k \geq 1.$$

which is the answer to the first part. Consider now the P_0. Since

$$P_0 + \sum_{k=1}^{\infty} P_k = 1$$

it follows that

$$P_0 = 1 - \sum_{k=1}^{\infty} P_k = 1 - \frac{2(1-\rho)}{1+\rho} \sum_{k=1}^{\infty} \rho^k$$

and this yields

$$P_0 = 1 - \frac{2\rho}{1+\rho} = \frac{1-\rho}{1+\rho}$$

4. This question is on the M/M/m/m queue.

   (a) The arrival rate is $\lambda = \frac{1}{3}$ call per minute, the service rate is $\mu = \frac{1}{15}$, and thus $a = \lambda/\mu = 5$. It follows from the Erlang B curve that $m = 11$ trunk lines are required.

   (b) They expect to receive 20 calls per hour, the blocking probability is 0.01, and thus there are $20 \times 0.99 = 19.8$ successful calls per hour. Each call lasts on average 15 minutes, and thus there is a total of 19.8×15 telephone minutes per hour. Since there are 365×24 hours per year, it follows that there are

$$365 \times 24 \times 19.8 \times 15$$

telephone minutes per year. The annual income is therefore

$$365 \times 24 \times 19.8 \times 15 \times 0.01 = \$26,017.20$$

The profit is therefore \$26,017.20-(\$1,000 $\times$ 11)= \$15,017.20.

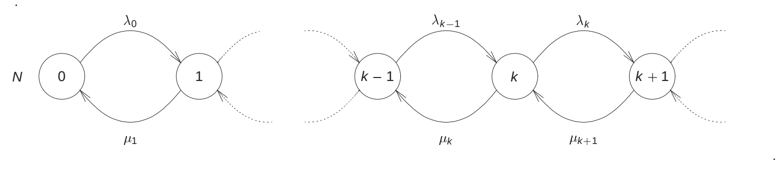
5. (a) This is a straight line for $0 < X \leq b$ that passes through the points $(0, 2/b)$ and $(b, 0)$.

   (b) The first and second moments are

$$\begin{aligned} E\{x\} &= \frac{2}{b}\int_0^b \left(x - \frac{x^2}{b}\right) \mathrm{dx} = \frac{b}{3} \\ E\{x^2\} &= \frac{2}{b}\int_0^b \left(x^2 - \frac{x^3}{b}\right) \mathrm{dx} = \frac{b^2}{6} \end{aligned}$$

Since $\rho = \lambda E\{x\} = \lambda b/3$, it follows that

$$\bar{W} = \frac{\lambda b^2/6}{2\,(1-\lambda b/3)} = \frac{\lambda b^2}{4(3-\lambda b)}$$



6. (a) The state transition diagram is
where $\lambda_k = \lambda/(k+1)$ and $\mu_k = \mu$.
The balance equations yield

$$\lambda_{k-1}P_{k-1} + \mu P_{k+1} = \lambda_k P_k + \mu P_k$$

that is

$$\frac{\lambda}{k}P_{k-1} + \mu P_{k+1} = \frac{\lambda}{k+1}P_k + \mu P k$$

The solution of this equation is

$$\frac{\lambda}{k+1}P_k = \mu P_{k+1}$$

The general solution is therefore

$$P_k = \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k P_0$$

where P_0 is calculated from the condition

$$\sum_{k=0}^{\infty} P_k = 1$$

This assumes a queue of infinite length. Thus

$$P_0 \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k = 1$$

and hence $P_0 = \exp(-\lambda/\mu)$. It follows that

$$P_k = \exp\left(-\frac{\lambda}{\mu}\right)\frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k, \qquad k = 0, 1, \ldots$$

(b) The average number of packets in the system is

$$\begin{aligned}
\sum_{k=0}^{\infty} kP_k &= \exp\left(-\frac{\lambda}{\mu}\right)\sum_{k=1}^{\infty}\frac{1}{(k-1)!}\left(\frac{\lambda}{\mu}\right)^k \\
&= \exp\left(-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)\sum_{k=1}^{\infty}\frac{1}{(k-1)!}\left(\frac{\lambda}{\mu}\right)^{k-1} \\
&= \frac{\lambda}{\mu}
\end{aligned}$$

(c) The average number of packets in an M/M/1 queue is

$$\frac{\rho}{1-\rho}, \qquad \rho = \frac{\lambda}{\mu}$$

Since $\rho < \rho/(1-\rho)$ for $\rho < 1$, the result is expected because queuing with discouragement reduces the average number of packets in the queue.

(d) The average arrival rate of the packets is

$$\begin{aligned}
\gamma &= \sum_{k=0}^{\infty} \lambda_k P_k \\
&= \lambda \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \exp\left(-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^k \\
&= \mu \exp\left(-\frac{\lambda}{\mu}\right) \sum_{k=0}^{\infty} \frac{1}{(k+1)!}\left(\frac{\lambda}{\mu}\right)^{k+1} \\
&= \mu \exp\left(-\frac{\lambda}{\mu}\right)\left(\exp\left(\frac{\lambda}{\mu}\right) - 1\right) \\
&= \mu\Big(1 - \exp(-\rho)\Big), \qquad \rho = \frac{\lambda}{\mu}
\end{aligned}$$

(e) The average time delay in the system is

$$\bar{T} = \frac{\bar{N}}{\lambda}, \qquad \bar{N} = \rho = \frac{\lambda}{\mu}$$

The average value of λ is, from above,

$$\gamma = \sum_{k=0}^{\infty} \lambda_k P_k = \mu\Big(1 - \exp(-\rho)\Big)$$

and thus

$$\begin{aligned}
\bar{T} &= \frac{\rho}{\mu\Big(1 - \exp(-\rho)\Big)} \\
&= \frac{1}{\mu} \quad \text{if } \rho \ll 1
\end{aligned}$$

7. The formula is

$$P_k = \rho^k(1-\rho) = 0.7(0.3)^k$$

k	P_k
0	0.700
1	0.210
2	0.063
sum	0.973

and thus $m = 2$.

8. Strictly speaking no because $P_k/P_{k-1} = \rho$ is not constant, which it must be. Real world problems may depart from this ideal scenario, and thus the data in the table may be realised in practice.

9. The utilisation factor is defined as the steady probability that the system is busy.

   For an M/M/1 system with a finite length queue, we have

$$P(k) = \frac{(1-\rho)\rho^k}{1-\rho^{N+1}}$$

   and thus the probability that the system is empty is

$$P(0) = \frac{1-\rho}{1-\rho^{N+1}}$$

   The utilisation factor is therefore

$$1 - P(0) = 1 - \frac{1-\rho}{1-\rho^{N+1}} = \frac{\rho - \rho^{N+1}}{1-\rho^{N+1}}$$

   Note that as $N \to \infty$, the utilisation factor approaches ρ.

10. This is an M/M/1 queue. The arrival rate is $\lambda = 5$ and the service rate is $\mu = 6$ customers per hour. This yields $\rho = \frac{5}{6}$.

    (a) The average number of people in the system is

$$E\{k\} = \frac{\lambda}{\mu - \lambda} = \frac{\rho}{1-\rho} = 5$$

    (b) The average number of people waiting to be served is given by

$$\begin{aligned}
\sum_{k=1}^{\infty}(k-1)p_k &= \sum_{k=2}^{\infty} kp_k - \sum_{k=2}^{\infty} p_k \\
&= \left(\sum_{k=1}^{\infty} kp_k - p_1\right) - (1 - p_0 - p_1) \\
&= \sum_{k=1}^{\infty} kp_k - (1 - p_0) \\
&= \sum_{k=0}^{\infty} kp_k - (1 - p_0) \\
&= \frac{\rho}{1-\rho} - (1 - p_0) \\
&= \frac{\rho}{1-\rho} - \rho \\
&= 4\frac{1}{6}.
\end{aligned}$$

(c) The probability that the salon is empty is $p_0 = 1 - \rho = \frac{1}{6}$, that is 16.67% of the time the salon is empty and a customer can be served immediately.

(d) The probability that a customer will not find a seat is

$$\sum_{k=5}^{\infty}(1-\rho)\rho^k = 1 - \left(\sum_{k=0}^{4}(1-\rho)\rho^k\right) = \rho^5 = 0.402$$

(e) From Little's theorem, the average waiting time in the system is

$$\frac{E\{k\}}{\lambda} = 1 \text{ hour}$$

A queue is formed if there are two or more people in the system. The probability that there are k people in the system is

$$P_k = \rho^k(1-\rho)$$

and the probability that there are two or more people in the system is

$$\sum_{k=2}^{\infty}\rho^k(1-\rho) = 1 - (1-\rho) - \rho(1-\rho) = \rho^2$$

It follows that given that there is a queue, that is, $k \geq 2$, the probability that the system is in state $k \geq 2$ is

$$\bar{p}_k = \frac{\rho^k(1-\rho)}{\rho^2}, \qquad k \geq 2$$

The average number of people in the queue, taken over non-empty queues, is therefore

$$L = \sum_{k=2}^{\infty}(k-1)\bar{p}_k = \frac{(1-\rho)}{\rho^2}\sum_{k=2}^{\infty}(k-1)\rho^k$$

which is equal to

$$\frac{(1-\rho)}{\rho^2}\left(\left(\sum_{k=1}^{\infty}k\rho^k - \rho\right) - \left(\sum_{k=0}^{\infty}\rho^k - 1 - \rho\right)\right)$$

and this is equal to

$$\frac{(1-\rho)}{\rho^2}\left(\frac{\rho}{(1-\rho)^2} - \frac{1}{1-\rho} + 1\right) = \frac{1}{1-\rho} = \frac{1}{1-\frac{1}{6}} = 6$$

and there are therefore, on average, six people waiting in the queue, where the average is taken over non-empty queues.

(f) The average average number of people in the queue is $4\frac{1}{6}$, and since $\lambda = 5$, it follows that the average waiting time in the queue is $\frac{5}{6}$ hour. The average waiting times are therefore unacceptable long, either one hour (in all queues), or $\frac{5}{6}$ hour for non-empty queues only.