The University Of Sheffield.

Data Provided:
Erlang C curves

DEPARTMENT OF COMPUTER SCIENCE — Spring Semester 2020-2021

NETWORK PERFORMANCE ANALYSIS — 2 hours

ANSWER ALL QUESTIONS.

All questions carry equal weight. Figures in square brackets indicate the percentage of available marks allocated to each part of a question.

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1. a) The Poisson distribution is used for modelling queueing systems.

   (i) State the three assumptions that are used in the derivation of the distribution. [15%]

   (ii) Derive an expression for the Poisson distribution and define all the symbols. [35%]

   (iii) The variable t that represents time appears in the formula for the distribution. Explain clearly why this is a *time difference*, not an *absolute time*. [10%]

   (iv) Show that the three assumptions in 1.a(i) are satisfied by the formula for the Poisson distribution. [15%]

   b) Packets arrive at a switch at a rate of three packets every millisecond and follow a Poisson distribution.

   (i) What is the probability that two or more packets arrive within the first half millisecond? [15%]

   (ii) What is the probability that exactly three packets arrive between the first and second milliseconds? [10%]

2. a) Consider an M/M/1 queue for which the arrival rate λ_k and service rate μ_k at state k are

$$\begin{aligned} \lambda_k &= \lambda\alpha^{k^2}, & k = 0, 1, 2, \dots, \\ \mu_k &= \mu, & k = 1, 2, \dots, \end{aligned}$$

where λ and μ, $\lambda < \mu$, are constants, and $0 < \alpha \leq 1$.

(i) Derive an expression for P_k in terms of λ, μ and α. The following formula may be required

$$\sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6}.$$

[35%]

(ii) Show that P_0 is given by

$$P_0 = \left(\sum_{k=0}^{\infty} \left(\frac{\lambda}{\mu} \right)^k \alpha^{\frac{2k^3 - 3k^2 + k}{6}} \right)^{-1}$$

and thus deduce an expression for P_k in terms of λ, μ and α. [15%]

(iii) If $\frac{\lambda}{\mu} < 1$, is the infinite sum guaranteed to converge? Explain your answer. [10%]

b) The $M/M/m$ system consists of one queue and m servers. The arrivals are modelled as a Poisson process with rate λ and the service rate for state k is μ_k.

(i) Derive the steady state probability distribution for the system. Include restrictions on the size of any parameters. [15%]

(ii) Calculate the average number $\bar{N}_Q$ of people in the queue, that is, the number of people waiting to be served. Derive an expression for $\bar{N}_Q$ in terms of the probability that a customer will wait. [15%]

(iii) A telephone exchange is to serve $10,000$ subscribers. Assume the subscribers generate calls that follow a Poisson distribution with a rate of 15 calls per minute, and that the calls last, on average, 5 minutes. How many trunk lines are required in order that the probability of waiting is 0.01? [10%]

3. Consider a single-server queuing system in which the flow of packets is controlled, such that the arrival rate decreases as the state of the system increases according to the equation

$$\lambda_k = \frac{\lambda}{k+2}, \qquad k = 0, 1, \ldots$$

where λ is a known constant. The service rate is constant for all states, $\mu_k = \mu$.

a) Show that the steady state probability P_k satisfies the equation

$$P_k = \left(\frac{1}{k+1}\right)\left(\frac{\lambda}{\mu}\right) P_{k-1}$$

[20%]

b) Show that the steady state probability that the system is in state k is

$$P_k = \frac{1}{(k+1)!}\left(\frac{\lambda}{\mu}\right)^k P_0$$

[15%]

c) Use the results of questions 3(a) and 3(b) to show that P_k is given by

$$P_k = \frac{1}{(k+1)!}\left(\frac{\lambda}{\mu}\right)^{k+1} \frac{1}{(\exp(\lambda/\mu) - 1)}$$

[25%]

d) Show that the average number of packets in the system is

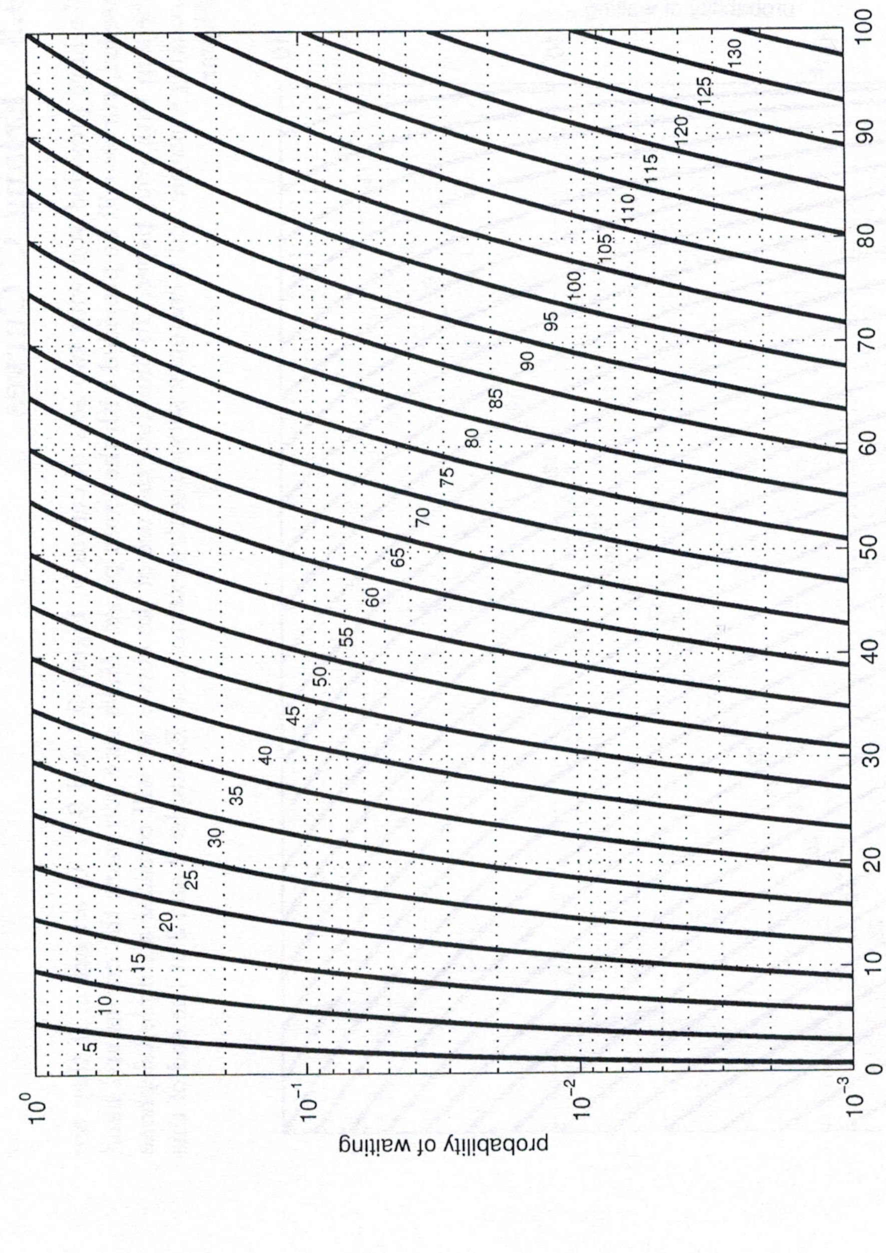
$$\bar{P} = \frac{\exp(\lambda/\mu)\left(\frac{\lambda}{\mu} - 1\right) + 1}{\exp(\lambda/\mu) - 1}$$

[25%]

e) Consider the expression for $\bar{P}$ in question 3(d). Determine the approximate values of $\bar{P}$ for $\lambda \ll \mu$, $\lambda \approx \mu$ and $\lambda \gg \mu$. [15%]

END OF QUESTION PAPER

Next page: The Erlang C Curve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