Network Performance Analysis: Solutions to Tutorial 2

1. To calculate the variance

$$
\begin{aligned}
E\left\{k^2\right\} &= \sum_{k=0}^{\infty} k^2 p(k) \\
&= \sum_{k=0}^{\infty} k^2 \rho^k (1-\rho) \\
&= \sum_{k=0}^{\infty} k(k-1)\rho^k(1-\rho) + \sum_{k=0}^{\infty} k\rho^k(1-\rho) \\
&= E\{k\} + (1-\rho)\rho^2 \sum_{k=2}^{\infty} k(k-1)\rho^{k-2} \\
&= E\{k\} + (1-\rho)\rho^2 \sum_{k=2}^{\infty} \frac{d^2\rho^k}{d\rho^2} \\
&= E\{k\} + (1-\rho)\rho^2 \frac{d^2}{d\rho^2}\left(\sum_{k=2}^{\infty} \rho^k\right) \\
&= E\{k\} + (1-\rho)\rho^2 \frac{d^2}{d\rho^2}\left(\frac{\rho^2}{1-\rho}\right) \\
&= E\{k\} + (1-\rho)\rho^2 \left(\frac{2}{(1-\rho)^3}\right) \\
&= E\{k\} + \frac{2\rho^2}{(1-\rho)^2}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathrm{var}\{k\} &= E\left\{k^2\right\} - (E\{k\})^2 \\
&= E\{k\} + \frac{2\rho^2}{(1-\rho)^2} - \left(\frac{\rho}{1-\rho}\right)^2 \\
&= \frac{\rho}{1-\rho} + \frac{2\rho^2}{(1-\rho)^2} - \left(\frac{\rho}{1-\rho}\right)^2 \\
&= \frac{\rho}{(1-\rho)^2}
\end{aligned}
$$

2. Since

$$P(k|t,\lambda) = \frac{(\lambda t)^k}{k!}\exp(-\lambda t), \qquad \lambda t = \frac{1000}{3600} = \frac{1}{3.6}$$

it follows that the probability that one or more calls arrive in a one second interval is

$$1 - P(0|t,\lambda) = 1 - \exp\left(-\frac{1}{3.6}\right) = 0.243$$

3. The waiting time is equal to the sum of the queuing time and the waiting time

   (a) The probability that the machine is busy is given by

   $$1 - \text{probability that the machine is not occupied} = 1 - P(0)$$

   where $P(0)$ is the probability that the system is in state zero, that is, there are no people in the the system. Thus the probability that the machine is busy is the utilisation factor ρ,

   $$\rho = \frac{\lambda}{\mu} = \frac{0.2}{0.5} = 0.4$$

   (b) Apply Little's formula to the entire system.

   $$\bar{N} = \lambda\bar{T}$$

   where $\bar{N}$ is the average number of people in the system

   $$\bar{N} = E\{k\} = \frac{\rho}{1-\rho}, \qquad \rho = \frac{\lambda}{\mu}$$

   Thus

   $$\bar{T} = \frac{\bar{N}}{\lambda} = \left(\frac{\rho}{1-\rho}\right)\frac{1}{\lambda} = \frac{1}{\mu-\lambda} = \frac{1}{0.5-0.2} = \frac{10}{3}\text{ minutes}$$

   (c) We require

   $$\frac{1}{\mu-\lambda} \geq 5, \qquad \mu = 0.5$$

   and thus the arrival rate satisfies $\lambda \geq 0.3$ if the average waiting time is 5 minutes or longer.
   The total average delay is confirmed from Little's theorem,

   $$\bar{T} = \frac{\bar{N}}{\lambda} = \left(\frac{\rho}{1-\rho}\right)\frac{1}{\lambda} = \frac{0.6}{(1-0.6)}\frac{1}{0.3} = 5\text{ minutes}$$

   as required. This is the total waiting time, and thus the average time spent in the queue is

   $$5 - \text{serving time} = 5 - 2 = 3\text{ minutes}$$

   (d) Use Little's formula again:

   $$\bar{N} = \lambda\bar{T} = 0.3 \times 3 = 0.9\text{ people}$$