Answers to: Network Performance Analysis: 2019-2020

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1.

(a.i) The service rate is [**4**] marks each

$$\mu_k = k\mu \qquad \text{and} \qquad \lambda_k = \lambda.$$

The system is in state k when there are k people in the system. Since there is an infinite number of servers, there is never a queue and a customer can always go to a server. It follows that the service rate of the system is proportional to the number of people in the system. [**5**]

The arrival rate is independent of the service rate and thus the arrival rate is constant. [**2**]

(a.ii) The balance equations yield [**5**]

$$\lambda P(k-1) = \mu_k P(k)$$

and thus

$$\lambda P(k-1) = k\mu P(k)$$

The solution of this equation is [**6**]

$$P(k) = P(0)\left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}$$

The probability that the system is empty is calculated from the normalisation condition [**5**]

$$\sum_{k=0}^{\infty} P(k) = 1$$

This yields [**6**] marks each

$$P(0) = \left(\sum_{k=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}\right)^{-1} = \exp(-a), \qquad a = \frac{\lambda}{\mu}$$

The steady state probability is therefore [**5**] marks each

$$P(k) = \frac{a^k}{k!}\exp(-a)$$

which is a Poisson distribution with parameter a. [**3**]

(a.iii) The average number of people in the system is [**5**]

$$\bar{N} = \frac{\lambda}{\mu}$$

and the average delay is [**5**]

$$\frac{\bar{N}}{\lambda} = \frac{1}{\mu}$$

(b.i) The arrival rate is [**10**]

$$\lambda_k = \begin{cases} \lambda & k = 0, 1, \ldots, m-1 \\ 0 & k \geq m \end{cases}$$

because no more than m customers are allowed in the queue.

The service rate is [**10**]

$$\mu_k = \begin{cases} k\mu & k = 0, 1, \ldots, m \\ 0 & k > m \end{cases}$$

(b.ii) The steady state probability is [**5**]

$$P(k) = \frac{a^k}{k!} P(0), \qquad a = \frac{\lambda}{\mu}$$

where [**5**]

$$P(0) \sum_{k=0}^{m} \frac{a^k}{k!} = 1$$

and thus [**5**]

$$P(k) = \frac{\frac{a^k}{k!}}{\sum_{k=0}^{m} \frac{a^k}{k!}}$$

The probability of blocking is obtained by setting $k = m$. [**10**]

$$P_B = \frac{\frac{a^m}{m!}}{\sum_{k=0}^{m} \frac{a^k}{k!}}$$

2(a.i) λ and μ are the arrival rate and service rate respectively, and $\rho < 1$. **[5,5,5]**

(a.ii) The expression for P_0 is calculated from $\sum_{k=0}^{\infty} P_k = 1$. **[3]**

This yields **[3]**

$$P_0\left[\sum_{k=0}^{m-1}\frac{(m\rho)^k}{k!}+\sum_{k=m}^{\infty}\frac{m^m\rho^k}{m!}\right]=1$$

Since **[10]**

$$\sum_{k=m}^{\infty}\frac{m^m\rho^k}{m!}=\frac{(m\rho)^m}{m!}\sum_{k=m}^{\infty}\rho^{k-m}=\frac{(m\rho)^m}{m!(1-\rho)}$$

for $\rho < 1$, it follows that the steady state probabilities for an M/M/m queue are **[4]**

$$P_k=\begin{cases} P_0\left(\frac{(m\rho)^k}{k!}\right) & k<m \\ P_0\left(\frac{m^m\rho^k}{m!}\right) & k\geq m \end{cases}, \qquad \rho=\frac{\lambda}{m\mu}<1$$

where

$$P_0=\left[\sum_{k=0}^{m-1}\frac{(m\rho)^k}{k!}+\frac{(m\rho)^m}{m!(1-\rho)}\right]^{-1}$$

(b.i) $m = 3$ **[4,4,2]**

$$P_k=\begin{cases} P_0\left(\frac{(3\rho)^k}{k!}\right) & k<3 \\ P_0\left(\frac{9\rho^k}{2}\right) & k\geq 3 \end{cases}, \qquad \rho=\frac{\lambda}{3\mu}.$$

(b.ii) From above, with $m = 3$, **[5]**

$$P_0=\left[\sum_{k=0}^{2}\frac{(3\rho)^k}{k!}+\frac{9\rho^3}{2(1-\rho)}\right]^{-1}$$

and this is equal to **[10]**

$$P_0\left[1+3\rho+\frac{9\rho^2}{2}+\frac{9\rho^3}{2(1-\rho)}\right]^{-1}$$

(b.iii) The average number of people in the system is (**[5]** for each term)

$$\sum_{k=0}^{m-1}kP_k+\sum_{k=m}^{\infty}kP_k$$

and its simplification yields, using the formulae above

$$P_0\sum_{k=0}^{2}\frac{k(3\rho)^k}{k!}+\frac{9P_0}{2}\sum_{k=3}^{\infty}k\rho^k$$

The first term yields $P_0(3\rho + 9\rho^2)$. **[5]**

The second term yields **[5]**

$$\frac{9P_0}{2}\sum_{k=3}^{\infty} k\rho^k = \frac{9P_0}{2}\left[\sum_{k=0}^{\infty} k\rho^k - \rho - 2\rho^2\right]$$

Since **[10]**

$$\sum_{k=0}^{\infty} k\rho^k = \frac{\rho}{(1-\rho)^2}$$

it follows that the answer is **[10]**

$$P_0\left[3\rho + 9\rho^2 + \frac{9\rho}{2(1-\rho)^2} - \frac{9\rho}{2} - 9\rho^2\right] = P_0\left[\frac{9\rho}{2(1-\rho)^2} - \frac{3\rho}{2}\right].$$

(3) (a.i) The mean of a Poisson distribution is

$$\begin{aligned}
E\{k\} &= \sum_{k=0}^{\infty} kp(k|t,\lambda) \\
&= \sum_{k=0}^{\infty} \frac{k(\lambda t)^k \exp(-\lambda t)}{k!} \qquad \textbf{[3]} \\
&= \exp(-\lambda t)\sum_{k=0}^{\infty} \frac{k(\lambda t)^k}{k!} \\
&= \exp(-\lambda t)\sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!} \qquad \textbf{[3]} \\
&= \exp(-\lambda t)\left((\lambda t) + (\lambda t)^2 + \frac{(\lambda t)^3}{2!} + \cdots\right) \\
&= \exp(-\lambda t)(\lambda t)\exp(\lambda t) \\
&= \lambda t \qquad \textbf{[3]}
\end{aligned}$$

The variance of the Poisson distribution is **[3]**

$$\text{var }\{k\} = E\left\{k^2\right\} - \left(E\{k\}\right)^2$$

Consider the first term:

$$\begin{aligned}
E\left\{k^2\right\} &= \sum_{k=0}^{\infty} k^2 p(k|t,\lambda) \\
&= \sum_{k=0}^{\infty} \frac{k^2(\lambda t)^k \exp(-\lambda t)}{k!} \qquad \textbf{[3]} \\
&= \exp(-\lambda t)\sum_{k=0}^{\infty} \frac{k^2(\lambda t)^k}{k!} \\
&= \exp(-\lambda t)\sum_{k=1}^{\infty} \frac{k(\lambda t)^k}{(k-1)!} \qquad \textbf{[3]} \\
&= \exp(-\lambda t)\sum_{k=2}^{\infty} \frac{(k-1)(\lambda t)^k}{(k-1)!} + \exp(-\lambda t)\sum_{k=1}^{\infty} \frac{(\lambda t)^k}{(k-1)!} \\
&= \exp(-\lambda t)(\lambda t)^2\exp(\lambda t) + \exp(-\lambda t)(\lambda t)\exp(\lambda t) \\
&= (\lambda t)^2 + \lambda t \qquad \textbf{[3]}
\end{aligned}$$

It therefore follows that **[4]**

$$\text{var }\{k\} = E\left\{k^2\right\} - \left(E\{k\}\right)^2 = (\lambda t)^2 + \lambda t - (\lambda t)^2 = \lambda t.$$

It follows that

$$E\{k\} = \text{var }\{k\} = \lambda t$$

(a.iii) The probability that there are no arrivals in the time interval T is **[5]**

$$p(k|t,\lambda) = \frac{(\lambda t)^k \exp(-\lambda t)}{k!}$$

evaluated at $k = 0$. This yields $\exp(-\lambda T)$.

(a.iv) The probability that there is at least one arrival in the time interval T is **[5]**

$$1 - p(k|t,\lambda) = \frac{(\lambda t)^k \exp(-\lambda t)}{k!}$$

evaluated at $k = 0$. This yields $1 - \exp(-\lambda T)$.

(b.i) The balance equation is **[4]**

$$\lambda_{k-1}P_{k-1} + \mu_{k+1}P_{k+1} = (\lambda_k + \mu_k)P_k$$

which yields **[5]**

$$\lambda_{k-1}P_{k-1} = \mu_k P_k$$

The solution of this equation is **[8]**

$$P_k = \left(\frac{\lambda}{\mu}\right)\alpha^{k-1}P_{k-1}$$

and cycling through $k = 0, 1, 2, 3, \ldots,$ yields **[6]**

$$P_k = P_0\left(\frac{\lambda}{\mu}\right)^k \alpha^{\sum_{i=1}^{k-1} i} = P_0\left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)}{2}} \qquad \mathbf{[7]}$$

(b.ii) Since **[5]**

$$\sum_{k=0}^{\infty} P_k = 1$$

it follows that **[5]**

$$P_0 = \frac{1}{\sum_{k=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)}{2}}}$$

The probability that there are 2 or more people in the system is **[5]**

$$1 - P_0 - P_1 = 1 - P_0 - P_0\left(\frac{\lambda}{\mu}\right)\alpha$$

(b.iii) From above

$$P_0 = \frac{1}{\sum_{k=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^k \alpha^{\frac{k(k-1)}{2}}} > \frac{1}{\sum_{k=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^k} = 1 - \frac{\lambda}{\mu} \qquad \mathbf{[5]}, \mathbf{[5]}$$

since $\alpha < 1$

(b.iv) The condition $\frac{\lambda}{\mu} < 1$ is not necessary for a steady state solution to exist. A steady state solution can exist with $\frac{\lambda}{\mu} > 1$. **[5]**

Even if $\frac{\lambda}{\mu} > 1$, α may be sufficiently small that the infinite sum in the expression for P_0 converges. **[5]**