# 抛物线的外切三角形和内接三角形的有趣性质

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文[1]给出了抛物线内接三角形的三边所在直线的斜率的两个性质:

**性质1**　设$A$、$B$、$C$为抛物线$y^2=2px(p>0)$上的三个不同点,$\triangle ABC$的三边$AB$、$BC$、$AC$所在直线的斜率分别为$k_1$、$k_2$、$k_3$,当$\triangle ABC$的重心为抛物线的焦点$F$时,有

(1)$\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}=0$;

(2)$\frac{1}{k_1^2}+\frac{1}{k_2^2}+\frac{1}{k_3^2}=\frac{3}{4}$.

**性质2**　设$A$、$B$、$C$为抛物线$y^2=2px(p>0)$上的三个不同点,$\triangle ABC$的三边$AB$、$BC$、$AC$所在直线的斜率分别为$k_1$、$k_2$、$k_3$,点$G(m,n)$为$\triangle ABC$的重心,则有

(1)$\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}=\frac{3n}{p}$;

(2)$\frac{1}{k_1^2}+\frac{1}{k_2^2}+\frac{1}{k_3^2}=\frac{9n^2+6pm}{4p^2}$.

通过研究,笔者发现了抛物线的外切三角形和内接三角形的两个性质,为了叙述的方便,先给出一个定义.

**定义**　过抛物线内接三角形的三个顶点的三条切线围成的三角形称为抛物线的外切三角形.

**性质3**　抛物线$y^2=2px(p>0)$上不同的三点$A$、$B$、$C$处的切线两两相交于$P_1$、$P_2$、$P_3$,设外切$\triangle P_1P_2P_3$的三边$P_1P_2$、$P_2P_3$、$P_1P_3$与内接$\triangle ABC$的三边$AB$、$BC$、$AC$所在直线的斜率分别为$k_i(i=1,2,\cdots,6)$,若内接$\triangle ABC$的重心为焦点$F$,则

(1)$\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}=\frac{1}{k_4}+\frac{1}{k_5}+\frac{1}{k_6}=0$;

(2)$\frac{1}{k_1^2}+\frac{1}{k_2^2}+\frac{1}{k_3^2}=4(\frac{1}{k_4^2}+\frac{1}{k_5^2}+\frac{1}{k_6^2})=3$.

**证明**　设$A(\frac{y_1^2}{2p},y_1)$,$B(\frac{y_2^2}{2p},y_2)$,$C(\frac{y_3^2}{2p},y_3)$,则在点$A$、$B$、$C$处的切线方程分别为$yy_i=p(x+\frac{y_i^2}{2p})$ $(i=1,2,3)$.

于是可求得$P_1(\frac{y_1y_2}{2p},\frac{y_1+y_2}{2})$,$P_2(\frac{y_2y_3}{2p},\frac{y_2+y_3}{2})$,$P_3(\frac{y_1y_3}{2p},\frac{y_1+y_3}{2})$,从而

$$k_1=\frac{\frac{y_2+y_3}{2}-\frac{y_1+y_2}{2}}{\frac{y_2y_3}{2p}-\frac{y_1y_2}{2p}}=\frac{p}{y_2},$$

$$k_2=\frac{\frac{y_1+y_3}{2}-\frac{y_2+y_3}{2}}{\frac{y_1y_3}{2p}-\frac{y_2y_3}{2p}}=\frac{p}{y_3},$$

$$k_3=\frac{\frac{y_1+y_3}{2}-\frac{y_1+y_2}{2}}{\frac{y_1y_3}{2p}-\frac{y_1y_2}{2p}}=\frac{p}{y_1},$$

$$k_4=\frac{y_2-y_1}{\frac{y_2^2}{2p}-\frac{y_1^2}{2p}}=\frac{2p}{y_1+y_2},$$

$$k_5=\frac{y_3-y_2}{\frac{y_3^2}{2p}-\frac{y_2^2}{2p}}=\frac{2p}{y_2+y_3},$$

$$k_6=\frac{y_3-y_1}{\frac{y_3^2}{2p}-\frac{y_1^2}{2p}}=\frac{2p}{y_1+y_3}.$$

又$F(\frac{p}{2},0)$是$\triangle ABC$的重心,所以$\frac{y_1^2}{2p}+\frac{y_2^2}{2p}+\frac{y_3^2}{2p}=\frac{3}{2}p$,$y_1+y_2+y_3=0$,所以

$$y_1^2+y_2^2+y_3^2=3p^2,$$

$y_1y_2+y_2y_3+y_1y_3=-\frac{3}{2}p^2$.

(1) $\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}=\frac{y_2}{p}+\frac{y_3}{p}+\frac{y_1}{p}$

$=\frac{y_1+y_2+y_3}{p}=0$,

$\frac{1}{k_4}+\frac{1}{k_5}+\frac{1}{k_6}=\frac{y_1+y_2}{2p}+\frac{y_2+y_3}{2p}+\frac{y_1+y_3}{2p}$

$=\frac{y_1+y_2+y_3}{p}=0$,

所以有 $\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}=\frac{1}{k_4}+\frac{1}{k_5}+\frac{1}{k_6}=0$.

(2) $\frac{1}{k_1^2}+\frac{1}{k_2^2}+\frac{1}{k_3^2}=\frac{y_2^2}{p^2}+\frac{y_3^2}{p^2}+\frac{y_1^2}{p^2}$

$=\frac{y_1^2+y_2^2+y_3^2}{p^2}=\frac{3p^2}{p^2}=3$,

$\frac{1}{k_4^2}+\frac{1}{k_5^2}+\frac{1}{k_6^2}$

$=\frac{(y_1+y_2)^2}{4p^2}+\frac{(y_2+y_3)^2}{4p^2}+\frac{(y_1+y_3)^2}{4p^2}$

$=\frac{2}{4p^2}[(y_1^2+y_2^2+y_3^2)+(y_1y_2+y_2y_3+y_1y_3)]$

$=\frac{2}{4p^2}(3p^2-\frac{3}{2}p^2)$

$=\frac{3}{4}$,

所以 $\frac{1}{k_1^2}+\frac{1}{k_2^2}+\frac{1}{k_3^2}=4(\frac{1}{k_4^2}+\frac{1}{k_5^2}+\frac{1}{k_6^2})=3$.

我们将上述问题推广到一般情形,得到如下结论.

**性质 4** 抛物线 $y^2=2px(p>0)$ 上不同的三点 $A$、$B$、$C$ 处的切线两两相交于 $P_1$、$P_2$、$P_3$,设外切 $\triangle P_1P_2P_3$ 的三边 $P_1P_2$、$P_2P_3$、$P_1P_3$ 与内接 $\triangle ABC$ 的三边 $AB$、$BC$、$AC$ 所在直线的斜率分别为 $k_i(i=1,2,\cdots,6)$,若点 $G(m,n)$ 是内接 $\triangle ABC$ 的重心,则

(1) $\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}=\frac{1}{k_4}+\frac{1}{k_5}+\frac{1}{k_6}=\frac{3n}{p}$;

(2) $\frac{1}{k_1^2}+\frac{1}{k_2^2}+\frac{1}{k_3^2}=\frac{6m}{p}$;

(3) $\frac{1}{k_4^2}+\frac{1}{k_5^2}+\frac{1}{k_6^2}=\frac{9n^2+6pm}{4p^2}$.

**证明** 设 $A(\frac{y_1^2}{2p},y_1)$,$B(\frac{y_2^2}{2p},y_2)$,$C(\frac{y_3^2}{2p},y_3)$,同性质 3 的证明可得:$k_1=\frac{p}{y_2}$,$k_2=\frac{p}{y_3}$,$k_3=\frac{p}{y_1}$,$k_4=\frac{2p}{y_1+y_2}$,$k_5=\frac{2p}{y_2+y_3}$,$k_6=\frac{2p}{y_1+y_3}$.

若点 $G(m,n)$ 是内接 $\triangle ABC$ 的重心,则 $\frac{y_1^2}{2p}+\frac{y_2^2}{2p}+\frac{y_3^2}{2p}=3m$,$y_1+y_2+y_3=3n$,所以 $y_1^2+y_2^2+y_3^2=6pm$,$y_1y_2+y_2y_3+y_1y_3=\frac{1}{2}(9n^2-6pm)$,于是可得

$\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}=\frac{y_2}{p}+\frac{y_3}{p}+\frac{y_1}{p}$

$=\frac{y_1+y_2+y_3}{p}=\frac{3n}{p}$,

$\frac{1}{k_4}+\frac{1}{k_5}+\frac{1}{k_6}=\frac{y_1+y_2}{2p}+\frac{y_2+y_3}{2p}+\frac{y_1+y_3}{2p}$

$=\frac{y_1+y_2+y_3}{p}=\frac{3n}{p}$,

$\frac{1}{k_1^2}+\frac{1}{k_2^2}+\frac{1}{k_3^2}=\frac{y_2^2}{p^2}+\frac{y_3^2}{p^2}+\frac{y_1^2}{p^2}=\frac{y_1^2+y_2^2+y_3^2}{p^2}$

$=\frac{6pm}{p^2}=\frac{6m}{p}$,

$\frac{1}{k_4^2}+\frac{1}{k_5^2}+\frac{1}{k_6^2}$

$=\frac{(y_1+y_2)^2}{4p^2}+\frac{(y_2+y_3)^2}{4p^2}+\frac{(y_1+y_3)^2}{4p^2}$

$=\frac{2}{4p^2}[(y_1^2+y_2^2+y_3^2)+(y_1y_2+y_2y_3+y_1y_3)]$

$=\frac{2}{4p^2}[6pm+\frac{1}{2}(9n^2-6pm)]$

$=\frac{9n^2+6pm}{4p^2}$.

**参考文献:**

[1] 阮灵东,胡晓.抛物线内接三角形的有趣性质[J].数学通讯(上半月),2013(7,8).

(收稿日期:2017-06-06)