Black-Scholes-Merton 方程解（基于热传导方程）

假设股票价格为 $S(t)$, 服从过程

$$dS(t) = \mu S(t)dt + \sigma S(t)dz. \tag{1}$$

其中 μ 为增长率，σ 为波动率，z 为维纳过程。

考虑该股票上欧式看涨期权，执行价格为 K，到期时间为 T，无风险利率为 r, 在时刻 t $(0 \le t \le T)$ 价格为 $c(t, S(t))$。

Black-Scholes-Merton 偏微分方程为:

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} = rc, \tag{2}$$

$$c(T, S(T)) = \max(S(T) - K, 0). \tag{3}$$

方程和股票价格增值率 μ 无关。

下面通过替换变量和热传导方程基本解计算该欧式看涨期权价格。过程主要为通过替换变量先将 Black-Scholes-Merton 方程表示为标准热传导方程，然后使用热传导方程基本解，将其与边界条件进行卷积计算。通过对积分函数进行重新配分和替换变量，将积分用正态分布累计概率函数表示，即得到 Black-Scholes-Merton 方程解。

首先进行替换变量，

$$x = \ln S, \quad \tau = T - t, \tag{4}$$

$$\frac{\partial c}{\partial S} = \frac{\partial c}{\partial x}\frac{1}{S}, \quad \frac{\partial^2 c}{\partial S^2} = \frac{1}{S^2}\left(\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial x}\right). \tag{5}$$

代入微分方程后，

$$-\frac{\partial c}{\partial \tau} + r\frac{\partial c}{\partial x} + \frac{1}{2}\sigma^2\left(\frac{\partial^2 c}{\partial x^2} - \frac{\partial c}{\partial x}\right) = rc,$$

$$\frac{1}{2}\sigma^2\frac{\partial^2 c}{\partial x^2} + (r - \frac{1}{2}\sigma^2)\frac{\partial c}{\partial x} - \frac{\partial c}{\partial \tau} - rc = 0. \tag{6}$$

为了进一步简化微分方程，将 c 表示为:

$$c = e^{\alpha x + \beta\tau} f. \tag{7}$$

其中 α 和 β 为待定参数，此时微分关系有:

$$\frac{\partial c}{\partial \tau} = e^{\alpha x + \beta\tau}(\beta f + \frac{\partial f}{\partial \tau}), \quad \frac{\partial c}{\partial x} = e^{\alpha x + \beta\tau}(\alpha f + \frac{\partial f}{\partial x}),$$

$$\frac{\partial^2 c}{\partial x^2} = e^{\alpha x + \beta\tau}(\alpha^2 f + 2\alpha\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2}). \tag{8}$$

代入微分方程，再确定 α 和 β 以简化微分方程，

$$\frac{1}{2}\sigma^2(\alpha^2 f + 2\alpha\frac{\partial f}{\partial x} + \frac{\partial^2 f}{\partial x^2}) + (r - \frac{1}{2}\sigma^2)(\alpha f + \frac{\partial f}{\partial x}) - rf - \beta f - \frac{\partial f}{\partial \tau} = 0,$$

$$\frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2} + \underbrace{(\alpha\sigma^2 + r - \frac{1}{2}\sigma^2)}_{A}\frac{\partial f}{\partial x} + \underbrace{(\frac{1}{2}\sigma^2\alpha^2 + (r - \frac{1}{2}\sigma^2)\alpha - r - \beta)}_{B}f - \frac{\partial f}{\partial \tau} = 0. \tag{9}$$

让 $A = 0, \quad B = 0,$

$$\Rightarrow \quad \alpha = -\frac{r - \sigma^2/2}{\sigma^2}, \quad \beta = -\frac{(r - \sigma^2/2)^2}{2\sigma^2} - r. \tag{10}$$

此时微分方程简化为**热传导方程**，

$$\frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}=\frac{\partial f}{\partial \tau}, \tag{11}$$

$$f(0,x)=e^{-\alpha x}\max(e^x-K,0). \tag{12}$$

当把 $\tau=0$ 时的边界条件换为 $f(0,x)=\delta(x)$ 后，热传导方程有**基本解**:

$$U(\tau,x)=\frac{1}{\sigma\sqrt{2\pi\tau}}e^{-\frac{x^2}{2\tau\sigma^2}}. \tag{13}$$

所求函数 f 则为基本解和边界条件12的卷积，

$$
\begin{aligned}
f(\tau,x)&=U(\tau,x)*f(0,x) \qquad (14)\\
&=\int_{-\infty}^{+\infty}U(\tau,x-y)f(0,y)dy\\
&=\frac{1}{\sigma\sqrt{2\pi\tau}}\int_{-\infty}^{+\infty}e^{-\frac{(y-x)^2}{2\sigma^2\tau}}e^{-\alpha y}\max(e^y-K,0)dy\\
&=\frac{1}{\sigma\sqrt{2\pi\tau}}\int_{\ln K}^{+\infty}e^{-\frac{(y-x)^2}{2\sigma^2\tau}}e^{-\alpha y}(e^y-K)dy\\
&=\frac{1}{\sigma\sqrt{2\pi\tau}}\left[\int_{\ln K}^{+\infty}e^{-\frac{(y-x)^2}{2\sigma^2\tau}+(1-\alpha)y}dy-K\int_{\ln K}^{+\infty}e^{-\frac{(y-x)^2}{2\sigma^2\tau}-\alpha y}dy\right]\\
&\text{让 } z=\frac{y-x}{\sigma\sqrt{\tau}},\\
&=\frac{1}{\sqrt{2\pi}}\left[\int_{\frac{\ln K-x}{\sigma\sqrt{\tau}}}^{+\infty}e^{-\frac{1}{2}z^2+(1-\alpha)(\sigma\sqrt{\tau}z+x)}dz-K\int_{\frac{\ln K-x}{\sigma\sqrt{\tau}}}^{+\infty}e^{-\frac{1}{2}z^2-\alpha(\sigma\sqrt{\tau}z+x)}dz\right]\\
&\text{对指数部分平方项重新配分，}\\
&=\frac{1}{\sqrt{2\pi}}\left[\int_{\frac{\ln K-x}{\sigma\sqrt{\tau}}}^{+\infty}e^{-\frac{1}{2}(z+(\alpha-1)\sigma\sqrt{\tau})^2+\frac{1}{2}(\alpha-1)^2\sigma^2\tau+(1-\alpha x)}dz-K\int_{\frac{\ln K-x}{\sigma\sqrt{\tau}}}^{+\infty}e^{-\frac{1}{2}(z+\alpha\sigma\sqrt{\tau})^2+\frac{1}{2}\alpha^2\sigma^2\tau-\alpha x}dz\right]\\
&\text{让 } y_1=z+(\alpha-1)\sigma\sqrt{\tau},\ \ y_2=z+\alpha\sigma\sqrt{\tau},\\
&=\frac{1}{\sqrt{2\pi}}e^{\frac{1}{2}(\alpha-1)^2\sigma^2\tau+(1-\alpha)x}\int_{\frac{\ln K-x+\sigma^2\tau(\alpha-1)}{\sigma\sqrt{\tau}}}^{+\infty}e^{-\frac{1}{2}y_1^2}dy_1-\frac{1}{\sqrt{2\pi}}Ke^{\frac{1}{2}\alpha^2\sigma^2\tau-\alpha x}\int_{\frac{\ln K-x+\sigma^2\tau\alpha}{\sigma\sqrt{\tau}}}^{+\infty}e^{-\frac{1}{2}y_2^2}dy_2. \qquad (15)
\end{aligned}
$$

由于积分 $\frac{1}{\sqrt{2\pi}}\int_x^{+\infty}e^{-\frac{y^2}{2}}dy=(1-N(x))=N(-x)$，$N(x)$ 为正态分布累计概率函数。上式可表示为:

$$
\begin{aligned}
&e^{\frac{1}{2}(\alpha-1)^2\sigma^2\tau+(1-\alpha)x}(1-N(\frac{\ln K-x+\sigma\tau(\alpha-1)}{\sigma\sqrt{\tau}}))-Ke^{\frac{1}{2}\alpha^2\sigma\tau-\alpha x}(1-N(\frac{\ln K-x+\sigma^2\tau\alpha}{\sigma\sqrt{\tau}}))\\
=&e^{\frac{1}{2}(\alpha-1)^2\sigma^2\tau+(1-\alpha)x}N(\frac{x-\ln K+\sigma\tau(1-\alpha)}{\sigma\sqrt{\tau}})-Ke^{\frac{1}{2}\alpha^2\sigma\tau-\alpha x}N(\frac{x-\ln K-\sigma^2\tau\alpha}{\sigma\sqrt{\tau}}). \qquad (16)
\end{aligned}
$$

由于 $c(\tau,x)=e^{\alpha x+\beta\tau}f(\tau,x)$，且由10知 $\beta=-\frac{\alpha^2}{2\sigma^2}-r,\ \alpha=-\frac{r-\sigma^2/2}{\sigma^2}$，指数系数可以化简，

$$
\begin{aligned}
c(\tau,x)&=e^{\alpha x+\beta\tau}f(\tau,x)\\
&=e^{x-r\tau-\alpha\sigma^2\tau-\frac{1}{2}\sigma^2\tau}N(\frac{x-\ln K+(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}})-Ke^{-r\tau}N(\frac{x-\ln K+(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}})\\
&=e^xN(\frac{x-\ln K+(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}})-Ke^{-r\tau}N(\frac{x-\ln K+(r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}). \qquad (17)
\end{aligned}
$$

再把变量替换回 t 和 S，$\tau=T-t,\ x=\ln S$,

$$c(t,S)=SN(\frac{\ln\frac{S}{K}+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}})-Ke^{-(T-t)r}N(\frac{\ln\frac{S}{K}+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}), \tag{18}$$

$$d_1=\frac{\ln\frac{S}{K}+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},\quad d_2=\frac{\ln\frac{S}{K}+(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$c(t,S)=SN(d_1)-Ke^{-(T-t)r}N(d_2). \tag{19}$$

即 Black-Scholes-Merton 方程解。