期权二叉树定价分叉参数的计算

1 说明

在期权二叉树定价过程中，我们需要使得选择的分叉概率 p, 上升幅度 u, 下降幅度 d 满足：

$$p = \frac{e^{r\Delta t} - d}{u + d}, \tag{1}$$

$$p(u-1)^2 + (1-p)(d-1)^2 - [p(u-1) + (1-p)(d-1)]^2 = \sigma^2 \Delta t. \tag{2}$$

下面具体计算出 u 和 d 的表达式。

2 计算过程

由(2)，

$$\begin{aligned}
&p(u-1)^2 + (1-p)(d-1)^2 - [p(u-1) + (1-p)(d-1)]^2 = \sigma^2 \Delta t,\\
&p[(u-1)^2 - (d-1)^2 - 2(d-1)(u-d)] + (d-1)^2 - p^2(u-d)^2 - (d-1)^2 = \sigma^2 \Delta t,\\
&p(u^2 + d^2 - 2du) - p^2(u-d)^2 = \sigma^2 \Delta t,\\
&(u-d)^2(p - p^2) = \sigma^2 \Delta t.
\end{aligned} \tag{3}$$

代入 p 的表达式，

$$\begin{aligned}
(u-d)(e^{r\Delta t} - d) - (e^{r\Delta t} - d)^2 &= \sigma^2 \Delta t,\\
e^{r\Delta t}(u+d) - e^{2r\Delta t} - du &= \sigma^2 \Delta t.
\end{aligned} \tag{4}$$

让 $d = \frac{1}{u}$,

$$\begin{aligned}
e^{r\Delta t}(u + \frac{1}{u}) - e^{2r\Delta t} - 1 &= \sigma^2 \Delta t,\\
(u + \frac{1}{u}) &= e^{-r\Delta t}(1 + e^{2r\Delta t} + \sigma^2 \Delta t).
\end{aligned} \tag{5}$$

把右侧表达式记为 A，然后考虑 A 和 A^2 的近似表示，

$$\begin{aligned}
A &= e^{-r\Delta t} + e^{r\Delta t} + \sigma^2 \Delta t e^{-r\Delta t}\\
&\approx 2 + r^2 \Delta t^2 + \sigma^2 \Delta t(1 - r\Delta t)\\
&= 2 + \sigma^2 \Delta t + (r^2 - \sigma^2 r)\Delta t^2,
\end{aligned} \tag{6}$$

$$\begin{aligned}
A^2 &\approx 4 + 4\sigma \Delta t + (\sigma^4 + 4r^2 - 4\sigma r)\Delta t^2 + 2\sigma^2(r^2 - \sigma^2 r)\Delta t^3 + (r^2 - \sigma^2 r)^2 \Delta t^4\\
&\approx 4 + 4\sigma^2 \Delta t + (\sigma^2 - 2r)^2 \Delta t^2.
\end{aligned} \tag{7}$$

再代回计算 u，近似保留到 $O(\Delta t)$ 阶小量，

$$\begin{aligned}
u &= \frac{1}{2}(A+\sqrt{A^2-4}) \\
&\approx 1+\frac{1}{2}\sigma^2\Delta t+\frac{1}{2}(A-2)^{\frac{1}{2}}(A+2)^{\frac{1}{2}} \\
&= 1+\frac{1}{2}\sigma^2\Delta t+\frac{1}{2}(\sigma^2\Delta t+(r^2-\sigma r)\Delta t^2)^{\frac{1}{2}}(4+\sigma^2\Delta t+(r^2-\sigma^2 r)\Delta t^2)^{\frac{1}{2}} \\
&= 1+\frac{1}{2}\sigma^2\Delta t+\frac{1}{2}\sqrt{\sigma^2\Delta t}\sqrt{4}(1+\frac{(r^2-\sigma^2 r)\Delta t}{\sigma^2})^{\frac{1}{2}}(1+\frac{\sigma^2\Delta t+(r^2-\sigma^2 r)\Delta t^2}{4})^{\frac{1}{2}} \\
&\approx 1+\sigma\sqrt{\Delta t}+\frac{1}{2}\sigma^2\Delta t \\
&\approx e^{\sigma\sqrt{\Delta t}}\,.
\end{aligned} \tag{8}$$

所以，

$$u=e^{\sigma\sqrt{\Delta t}},\ \ d=e^{-\sigma\sqrt{\Delta t}}, \tag{9}$$

$$p=\frac{e^{r\Delta t}-d}{u-d}. \tag{10}$$