N-body theory

We will put N = 15000 stars in two loose clusters and let gravity work.

Stars live for tens of millions of years, often billions of years, so we will choose a time step of t = 10^5 years.

The force F between two stars with masses m_1 and m_2 and distance r_{12} is

$$\vec{F} = G \cdot \frac{m_1 \cdot m_2}{\vec{r}_{12}^{\,2}} \quad \text{(Newton's law of gravitation)}$$

Each star has

- a mass m_i
- a position $\vec{x}_i = (x_{i1}, x_{i2}, x_3)$
- a velocity $\vec{v}_i = (v_{i1}, v_{i2}, v_3)$
- an acceleration $\vec{a}_i = (a_{i1}, a_{i2}, a_3)$

We know that $v = a \cdot t$ and $s = v \cdot t$ so

$$s = s + v \cdot \Delta t$$
$$v = v + a \cdot \Delta t \quad \text{where}$$
$$\vec{a} = \frac{\vec{F}}{m} \quad \text{(From Newton's 2. law)}$$

The distance between star *i* and star *j* is i $\vec{r}_{ij} = \vec{x}_j - \vec{x}_i$

The gravitational constant G is G = 6,67$\cdot 10^{-11}\ N \cdot m^2/kg^2 \approx 10 \cdot 10^{-11}\ N \cdot m^2/kg^2 \approx 10^{-10}\ N \cdot m^2/kg^2$

The mass of the sun is $2 \cdot 10^{30}\ kg \approx 10^{30}\ kg$

A light year is $9.46 \cdot 10^{15}\ m \approx 10 \cdot 10^{15}\ m \approx 10^{16}\ m$

If we plug these numbers into $\vec{a} = \frac{\vec{F}}{m}$ we get

$$\vec{a} = \frac{\vec{F}}{m} = G \cdot \frac{m}{r^2} = 10^{-10} \cdot \frac{10^{30}}{(10^{16})^2} = 10^{-12}\ N/kg$$

As 1 year = $3.15 \cdot 10^7\ s$ we have $t = 10^5\ years \approx 10^{12}\ s$.

This means v = a·t = 1 m/s after 100,000 years for two stars starting out resting one light year apart.

We would get the same result if we used a time unit of 100 kilo years, a mass unit of 10^{30} kg (one solar mass), kept the distance in light years and used G = 1:

$$\vec{a} = \frac{\vec{F}}{m} = G \cdot \frac{m}{r^2} = 1 \cdot \frac{1}{1^2} = 1\ \frac{m/s}{100k\ year} \quad \text{(For } r = 1 \text{ light year)}$$

This simplifies the computations considerably.

Especially if we assume that all the stars in the clusters weigh 1 solar mass.

Then we have $a = 1/r^2$ where r is in light years.

Newton’s shell theorem

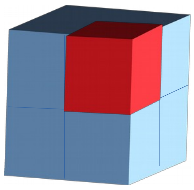
Another simplification is given by Newton’s shell theorem (https://en.wikipedia.org/wiki/Shell_theorem)

It states that all mass inside a sphere can be treated as if all the mass is in the centre of the sphere, and that a mass inside a uniform shell do not feel gravitation from that shell, no matter where the mass is placed inside the sphere.

In other words: If you could hollow out a sphere in the middle of the Earth without the sphere instantly being crushed, then you would be weightless inside the sphere.

A box is roughly spherical, so let us approximate the cluster as a box.

If we divide the box into eight smaller boxes:



then a star in the red box will be outside the other “spheres” and we can treat all stars in these boxes as if their mass was in the centre of each box. This saves a LOT of computations as we only need 7 contributions instead of 7/8th of 15000 stars!

1/8th of 15000 stars are still a lot, so let’s subdivide the red box into eight boxes and subdivide the resulting boxes again and again until we have a handful or two stars in each box.

This way our interactions drops from an impossible 30000! (the faculty of 2 times 15000 stars in each cluster interacting with each other) to only about 50000 computations each time step. This allows us, with a gaming computer screaming along on all 8 cores, to calculate the evolution of the two clusters over 10 to 100 millions of years in a few hours.

By being smart about it we can ask our program to do the subdivision and passing results up and down the chain by using recursive programming.