# APPENDIX

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## I. APPENDIX

Here are the proof details of some theorems in the paper.

### *A. Proof of Lemma 1*

***Lemma** 1:* The allocation rule of ONTA-SWM is monotonic.

*Proof:* We assume $\theta_i'^j \succeq \theta_i^j$ where $\theta_i^j = \{t_{ij}^-, \eta_{ij}, D_{ij}, b_i^j\}$ and $\theta_i^j = \{t_{ij}'^-, \eta_{ij}', D_{ij}', b_i'^j\}$. Then we have to show that the fact that the bid $\theta_i^j$ is accepted implies that the bid $\theta_i'^j$ is too accepted. Recall the definition, there are four cases : (1) the smaller execution time, i.e, $\eta_{ij} \geq \eta_{ij}'$. Recall the payment rule is non-decreasing with the task exciton time. Therefore, $\pi_{ij} \geq \pi_{ij}'$. According to the allocation rule, bid $\theta_i'^j$ must wins if $\theta_i^j$ wins. (2) the less resource demand, i.e, $D_{ij} \succeq D_{ij}'$. Likewise, the payment rule is non-decreasing with less resource demand and a user will pay more if it occupies more resource. Hence, $\pi_{ij} \geq \pi_{ij}'$ and bid $\theta_i'^j$ must wins if $\theta_i^j$ wins. (3) the less bid value, i.e, $b_i^j \geq b_i'^j$. Recall the allocation rule is increasing with the bid value and bid $\theta_i'^j$ must wins if $\theta_i^j$ wins. In conclusion, the allocation rule of ONTA-SWM is monotonic

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### *B. Proof of Lemma 2*

***Lemma** 2:* The ONTA-SWM charges each winning bid by its critical value.

*Proof:* According allocation rule and payment rule, the bid $\theta_i^j$ is accepted if bid value $b_i^j \geq \pi_{ij}$ and is rejectd otherwise. By the definition , $\pi_{ij}$ is exactly the critical value $c_i^j$ for bid $\theta_i^j$. ■

### *C. Proof of Theorem 4*

*Proof:*

To prove theorem 4, we first introduce some lemmas.

***Lemma** 3:* If we can find out a constant $\sigma \geq 1$ such that the inequality $(S^{[\gamma]} - S^{[\gamma-1]}) \geq \frac{1}{\sigma}(D^{[\gamma]} - D^{[\gamma-1]})$ holds true for each iteration $\gamma$, the competitive ratio of our online algorithm is $\sigma$.

*Proof:* By summing up the inequalities for each iteration $\gamma$, we have:

$$\begin{aligned} S^{[\gamma]} - S^{[0]} &= \sum_{\gamma \in \Gamma} (S^{[\gamma]} - S^{[\gamma-1]}) \\ &\geq \sum_{\gamma \in \Gamma} \frac{1}{\sigma} (D^{[\gamma]} - D^{[\gamma-1]}). \end{aligned} \tag{1}$$

Namely, $(S^{[\Gamma]} - S^{[0]}) \geq \frac{1}{\sigma}(D^{[\Gamma]} - D(0))$. Specifically, the proposed algorithm guarantees that $S^{[0]} = 0$ and $D(0) = 0$ through initialization. Hence, it suffices to show that $S^{[\gamma]} \geq \frac{1}{\sigma} D^{[\Gamma]}$. By the weak duality, $D^{[\Gamma]} \geq OPT$ holds true. Then, we have the inequality $S^{[\gamma]} \geq \frac{1}{\sigma} OPT$, which means that our auction mechanism is $\sigma$-competitive. ■

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***Lemma** 4:* If the inequality

$$\begin{aligned} \frac{1}{\sigma_r}(\bar{O}_m^r(\rho_m^{r[\gamma]}(t)) - \bar{O}_m^r(\rho_m^{r[\gamma-1]}(t))) \leq \\ \rho_m^{r[\gamma]}(t)(z_m^{r[\gamma]}(t) - z_m^{r[\gamma-1]}(t)) \\ - (O_m^r(d_m^{r[\gamma]}(t)) - O_m^r(d_m^{r[\gamma-1]}(t)) \end{aligned} \tag{2}$$

holds, then $(S^{[\gamma]} - S^{[\gamma-1]}) \geq \frac{1}{\sigma}(D^{[\gamma]} - D^{[\gamma-1]})$.

*Proof:* Assume that the bid $\theta_i^j$ is processed in the $\gamma$-th iteration. Consider two kinds of cases, i.e., the bid $\theta_i^j$ is rejected, and the opposite where the bid $\theta_i^j$ is accepted. In the former case, we easily obtain $S^{[\gamma]} - S^{[\gamma-1]} = D^{[\gamma]} - D^{[\gamma-1]} = 0$. In the following, we elaborate the latter case. If the bid $\theta_i^j$ is accepted and scheduled on MD $m$,

$$\begin{aligned} &S^{[\gamma]} - S^{[\gamma-1]} \\ &= b_i^j - \sum_{t \in [t_{ij}^-, t_{ij}^+]} \sum_{r \in \mathcal{R}} (O_m^r(z_m^{r[\gamma]}(t)) - O_m^r(z_m^{r[\gamma-1]}(t))) \\ &\overset{(a)}{=} \mu_{ij} + \sum_{t \in [t_{ij}^-, t_{ij}^+]} \sum_{r \in \mathcal{R}} \rho_m^{r[\gamma-1]}(t)(z_m^{r[\gamma]}(t) - z_m^{r[\gamma-1]}(t)) \\ &\quad - \sum_{t \in [t_{ij}^-, t_{ij}^+]} \sum_{r \in \mathcal{R}} (O_m^r(z_m^{r[\gamma]}(t)) - O_m^r(z_m^{r[\gamma-1]}(t))) \end{aligned} \tag{3}$$

The equality (a) follows from the fact in the Algorithm. Additionally, we can obtain the difference of the dual obejective function.

$$\begin{aligned} &D^{[\gamma]} - D^{[\gamma-1]} \\ &= \mu_{ij} + \sum_{t \in [t_{ij}^-, t_{ij}^+]} \sum_{r \in \mathcal{R}} (\bar{O}_m^r(\rho_m^{r[\gamma]}(t)) - \bar{O}_m^r(\rho_m^{r[\gamma-1]}(t))) \end{aligned}$$

By summing up inequality (2) over $r \in \mathcal{R}, t \in [t_{ij}^-, t_{ij}^+]$, we can have

$$\begin{aligned} &S^{[\gamma]} - S^{[\gamma-1]} \\ &\geq \mu_{ij} + \frac{1}{\sigma} \sum_{t \in [t_{ij}^-, t_{ij}^+]} \sum_{r \in \mathcal{R}} (\bar{g}_m^r(\rho_m^{r[\gamma]}(t)) - \bar{g}_m^r(\rho_m^{r[\gamma-1]}(t))) \\ &= \mu_{ij} + \frac{1}{\sigma}(D^{[\gamma]} - D^{[\gamma-1]} - \mu_{ij}) \\ &= \frac{1}{\sigma}(D^{[\gamma]} - D^{[\gamma-1]}) + (1 - \frac{1}{\sigma})\mu_{ij} \end{aligned} \tag{4}$$

Since $\mu_{ij} \geq 0$ and $\sigma > 1$, it is obvious that $S^{[\gamma]} - S^{[\gamma-1]} \geq \frac{1}{\sigma}(D^{[\gamma]} - D^{[\gamma-1]})$. ■

Now, we can prove that our auction is $\sigma$-competitive, where $\sigma = \max_{r \in \mathcal{R}, m \in \mathcal{M}} \ln(\dfrac{2R(H_r - h_m^r)}{L_r - h_m^r})$.

Note that $\mathrm{d}P_m^r(z_m^r(t)) = \frac{1}{C_m^r}(P_m^r - h_m^r)\sigma_r \mathrm{d}z_m^r(t)$, where $\sigma_r = \ln(\dfrac{2R(H_r - h_m^r)}{L_r - h_m^r})$. If we take $\sigma = \max_{r \in R} \sigma_r$, we obtain the following inequality for $\forall m \in \mathcal{M}$, $\forall t \in \mathcal{T}$, $\forall r \in R$:

$$\frac{1}{\sigma} C_m^r \mathrm{d}\rho_m^r(t) \leq (\rho_m^r(t) - h_m^r)\mathrm{d}z_m^r(t), \tag{5}$$

In this case, $\mathrm{d}O_m^r(z_m^r(t)) = h_m^r \mathrm{d}z_m^r(t)$ and $\mathrm{d}\bar{O}_m^r(\rho_m^r(t)) = C_m^r \mathrm{d}\rho_m^r(t)$. By putting them into (5), we have the inequality:

$$\frac{1}{\sigma}\mathrm{d}\bar{O}_m^r(\rho_m^r(t)) \leq \rho_m^r(t)\mathrm{d}z_m^r(t) - \mathrm{d}O_m^r(z_m^r(t)), \tag{6}$$

which has proven inequality (2).

In conclusion, we have $S^{[\gamma]} - S^{[\gamma-1]} \geq \frac{1}{\sigma}(D^{[\gamma]} - D^{[\gamma-1]})$ for every iteration $\gamma$. By Lemma 3, we have proven that our auction is $\sigma$-competitive.

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