Probabilistically Optimized Airline Overbooking Strategies

or

ANYONE WILLING TO TAKE A LATER FLIGHT?!

Kevin Leder

Saverio Spagnolie

Stefan Wild

Abstract

As the airlines industry struggles to maintain economic stability in the face of a post-September 11 (terrorism-minded) market, airline survival becomes increasingly dependent upon effective company policies and sound decision making. One significant aspect of an airline's revenue management involves the optimization of the company's booking strategy, or more appropriately, *over*booking strategy. We herein develop a series of mathematical models, developed with the intent of investigating relationships between overbooking strategies and revenue. Our first set of models are static, in the sense that passenger behavior is predominantly time-independent; a binomial random variable structure is developed in order to consider variable consumer behavior patterns. We consider numerous alternatives for handling bumped passengers, and construct an auction-style model for passenger compensation. Our second set of models is more dynamic, employing Poisson processes to capture the effects of a continuous time-dependence on ticket purchasing/cancelling information. This dynamic simulation is also applied to the initial static models to test them in a more realistic setting. Finally, the effects of the post-September market on the industry's financial interests are considered. In order to model this problem in a real-world setting, we chose to consider the concerns of a particular company and flight: Frontier Airlines Flight 502. Applying the models to the problem of revenue optimization leads to somewhat intuitive results: an optimal booking limit of 15% over flight capacity theoretically nets Frontier Airlines an additional $\$2.7\,million/year$ on Flight 502, given sufficient ticket demand. We also find that regardless of post-September effects, implementing an overbooking strategy should be of great financial import to any airline.

Contents

1 Introduction

The September 2001 attacks on the World Trade Center in New York delivered a severe financial blow to the American airlines industry. Consumer safety concerns translated directly to a sizeable loss of the companies' primary income source: passenger ticket sales. Several companies have since folded under the intense monetary pressures of the surrounding economy, and the remaining companies have had to walk a fine line between somehow 'buying' customers back and having to file for Chapter 11 bankruptcy protection. By late September a 15 billion dollar aid package for the airlines industry was signed into law by the US government, with the intention of providing "urgently needed tools to assure the safety and immediate stability of (the) nation's commercial airline system."[3]

For the airline companies in this time of economic uncertainty, survival depends both sensitively and crucially upon informed decision making. For example, educated decisions must be made in light of what is entitled in the literature, "The Airline Revenue Management Problem." The objective of this problem is to maximize profits, primarily through seeking the most mathematically supported optimal booking policies. Of significant interest in the problem, and thus to the airline companies, is achieving an understanding of how overbooking a flight affects revenue. Since the September attacks, the number of paying passengers has greatly decreased, magnifying the importance of maximizing profits on any given flight when given the chance to overbook. We will build and investigate, through the course of this paper, mathematical models for various overbooking strategies. Both static, and dynamic models will be developed to obtain the optimal booking limit given a variety of parameters.

In order to most adequately model this problem in a real-world setting, we chose to consider the concerns of a particular company: Frontier Airlines. By doing so, we gain valuable real-world data and policy information, and we expect the results of such a consideration to be generalizable to the interests of a significant portion of the American airlines industry.

Clearly, overbooking strategies are inapplicable if there are very few tickets being purchased by consumers. Before overbooking schemes may even be considered, an airline must adapt its routes, fares, and capacities to ensure a sufficient ticket demand.

2 Frontier Airlines: Company Overview

Frontier Airlines is the second largest airline operating out of Denver International Airport (DIA). The company services twenty-five cities in eighteen states, employing a fleet of 29 aircraft: seventeen Boeing 737-300 jets, seven smaller 737-200s, and five Airbus 319 jets. Frontier is a discount airline[1] In September 2001, Fortune magazine ranked Frontier 41st on its 100 Fastest Growing Companies list.

Frontier offers two flights per day from DIA to LaGuardia Airport in New York. Morning flight 502 departs at 10:40am, and afternoon flight 516 departs at 3:15pm. We will focus primarily upon Flight 502 in our comparisons and analyses of overbooking strategies.

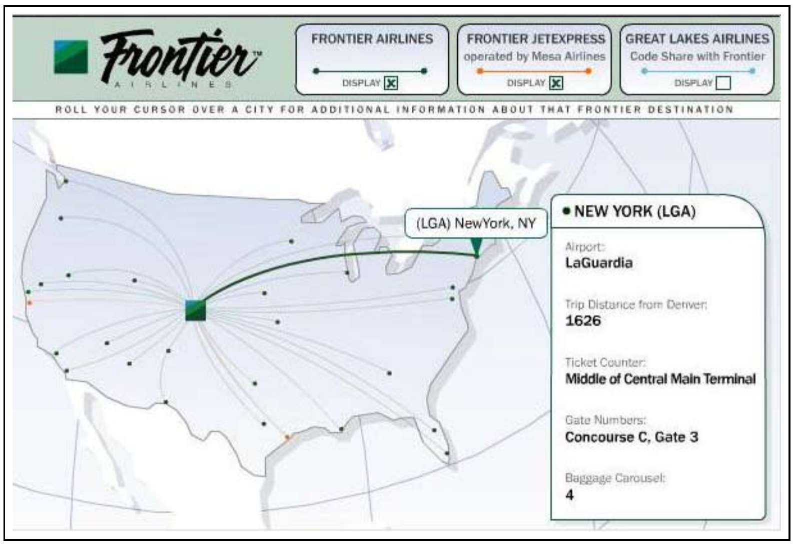

Figure 1: Frontier Airlines Route Map

3 Frontier Airlines: Technical Considerations and Details

In order to understand what is important in the attempt to model realistic overbooking strategies, a technical exploration of Frontier Airlines is necessary. We now discuss regulations for handling bumped passengers, airplane specifications, and financial interests, as they will translate into valuable mathematical tools in constructing overbooking strategies.

[1]Discount airlines draw less than one billion dollars in annual revenue.

3.1 Overbooking Regulations

The Department of Transportation (DOT) is very clear in its requirements for how airlines handle bumped customers. When overbooking results in overflow, the DOT requires airlines to ask for volunteers willing to be bumped in exchange for compensation.[1] However, the DOT has *not* specified how much in the way of compensation the airlines must give to the subsequent volunteers; in other words, negotiations and auctions may be held at the gate until the flight's departure. Each passenger who is bumped involuntarily is entitled to the following compensation:

- If the airline arranges substitute transportation such that the passenger will reach his/her destination within one hour of the original flight's arrival time, there is no obligatory compensation.
- If the airline arranges substitute transportation such that the passenger will reach his/her destination between one and two hours after the original flight's arrival time, the airline must pay the passenger an amount equal to the one-way fare for flight to the final destination.
- If the substitute transportation is scheduled to arrive any later than two hours after the original flight's arrival time, or if the airline does not make any substitute travel arrangements, the airline must pay an amount equal to twice the cost of the fare to the final destination.

3.2 Aircraft Information

Frontier currently employs the fleet of aircraft listed in the introduction. However, the company plans to transition its entire fleet from Boeing to Airbus jets. To keep Frontier's fares affordable, the company offers only one class of service to all passengers (e.g. no first-class seating). The Airbus A318 and A319 (134 seats in the single-class model) will compose the entire Frontier fleet by the year 2005.[11] Thus, we will base our models on overbooking single-class aircraft models.

3.3 Financial Interests

Airline booking considerations are frequently based on the *break-even load-factor*, a percentage of airplane seat capacity that must be filled to acquire neither loss or profit on a particular flight. The break-even load-factor for Flight 502 from Denver to New York in 2001 was 57.8%. Of course, considering variable market conditions over time, future company growth, dividends, etc., Frontier

Airlines must maximize its profitability whenever possible. Thus, if overbooking is at all profitable, it must be an integral aspect of the company's booking policy.

To sum up, the following parameters will be used in the development of the subsequent models and strategies:

- The process of compensating passengers bumped voluntarily is inexact and is generally dependent upon negotiation or auction at the gate.
- Involuntarily bumped passengers are entitled to no compensation, one-way fare, or twice the original fare, dependent upon the delay before the airline arranges substitute transportation.
- Frontier will fly only single-class Airbus A319 jets.
- The break-even load-factor for Flight 502 is 57.8%.

4 Assumptions

1. We need only concern ourselves with the sale of restricted tickets[2].
2. Ticket-holders who don't show up to the gate will spend the $60 to transfer the ticket to another flight.
3. The number of potential customers is negligible, so we may assume an infinite pool of purchasing customers while developing overbooking strategies.
4. Bumped passengers from Frontier's morning flight 502 can be placed, at the *latest*, 4 hrs. and 35 min. later on Frontier's afternoon flight 513 to the same destination.
5. The annual effects/costs associated with bumping involuntary passengers is negligible in comparison to the annual effects/costs of bumping voluntary passengers.

5 Justification of Assumptions

1. The sale of restricted tickets (which are much more affordable than unrestricted tickets) represent all but a very small percentage of all ticket purchases. Also, many high-profile ticket brokers,

[2]Frontier restricted tickets are non-refundable, save for the ability to transfer to another flight for $60.[11]

such as Priceline.com, only sell restricted tickets.

2. An overwhelming proportion of the paying ticket-holders, after having shown an interest in air travel, will spend the relatively inexpensive $60 for another flight.

3. A decrease in the number of potential customers, whether from post-September insecurities or otherwise, will clearly decrease the profitability of the airline. We make the simplifying assumption that the overbooking strategies need not concern themselves with such dynamics. If there are too few interested customers to overbook a given flight, then overbooking is a physical impossibility.

4. Frontier Airlines will first attempt to place bumped passengers on other airlines' flights to the same destination. Barring the ability to do so, Frontier will bump other passengers off of the following Frontier flight to make room for the originally bumped passengers. This is an efficient policy from the standpoint of customer service.

5. According to statistics provided by the Department of Transportation, 4% of all airline passengers are bumped voluntarily, while only 1.06 passengers in 10,000 are bumped involuntarily.[19] Under the previous assumption that there is a maximum delay for bumped passengers of 4 hrs. 35 min., and in consideration of the DOT compensation rules discussed previously, the average annual cost of bumping involuntary passengers is on the order of $100,000. (See Appendix A for a detailed calculation of this figure) This figure is negligible when compared to even the most underestimated annual costs for bumping voluntary passengers (>> $ 1million).

6 The Static Model

Our first, most straightforward model for optimizing revenues is a static one, in the sense that passenger behavior is predominantly time-independent. We accept that this model does not account for when passengers purchase their tickets, and that all passengers (save no-shows) arrive at the departure gate independently. This system may then be modeled by the following steps:

- Introduce a binomial random variable for the number of passengers who will show up for the flight,

- Define a total profit function dependent upon this random variable,
- Apply this function to various consumer behavior patterns,
- Compute (for each behavioral pattern) an optimal number of passengers to overbook.

6.1 A Binomial Random Variable Approach

We first introduce the concept of assigning a binomial random variable to the number of passengers who will show up for the flight. If X represents the number of successes that occur in B trials, where each trial is independent of every other trial, and each trial results in a "success" with fixed probability p ($0 \leq p \leq 1$), then X is said to be a binomial random variable.[22] In this model, we assume that ticket-holding consumers act independently of each other, and we presume that each will show up to the departure gate with the same probability p. Therefore, we can assign the binomial random variable X to the number of ticket-holding customers who actually arrive at the gate after B tickets have been sold. Numerous airlines consistently report that approximately 12% of all booked passengers do not actually show up to the gate (due to cancellations and no-shows)[3][20]; thus, in this model we take $p = .88$. The B trials correspond to the number of passengers holding tickets. Thus $X = Binomial(B, p)$.

With variable X in hand, we may consider the probability that i passengers will actually show up to the gate.

$$\text{P\{i passengers arrive at the gate\}} = \text{P\{X = i\}} = \binom{B}{i} p^i (1-p)^{B-i}$$

The expected value of a binomial random variable, given the above parameters, is $E[X] = B * p$. We also use the result that $P\{X = i\}$ first monotonically increases, then monotonically decreases, reaching its largest value when i is the largest integer less than or equal to $(B + 1) * p$. [22] It is also of interest in this model that this probability is continuous in the parameter p.[4]

6.2 Modeling Revenue

In order to optimize revenues and overbooking schemes, it is vital to understand income as a function of both the number of passengers booked and general passenger behavior. In other words,

[3]The reported value of this percentage varies from 10% to 15% depending on the airline.

[4]To see this, we recognize that the expansion of $p^i(1-p)^{B-i}$ results in a polynomial in p of order B, which is a continuous function.

the financial interests of the company must be tied to both the probability of passenger arrival and to how passengers react to being bumped. We define the following per-flight total profit function, and subsequently present a detailed explanation.

$$T_p(X) = (B - X) * R + \begin{cases} Airfare * X - Cost_{Flight}, & X \leq C_{\bar{\$}}; \\ (Airfare - Cost_{Add}) * (X - C_{\bar{\$}}), & C_{\bar{\$}} < X \leq C; \\ (Airfare - Cost_{Add}) * (X - C_{\bar{\$}}) - Bump(X - C), & X > C. \end{cases}$$

where

R = Transfer fee for no-show's and cancellations

B = Total number of passengers booked (as before)

$Airfare$ = Constant, as previously assumed

$Cost_{Flight}$ = Total operating cost of flying the plane

$Cost_{Add}$ = Total cost to place one passenger on the flight

$Bump(X - C)$ = The Bump Function (to be defined)

$C_{\bar{\$}}$ = Number of flying passengers required to break-even financially on the flight[5]

C = The full capacity of the plane (number of seats)

This total profit function encapsulates the following logic. There are a variety of costs associated with flying a plane, such as fuel, aircraft maintenance, advertising, and administrative costs. A certain number of tickets must be sold to break-even on any given flight. After that number of passengers is on the plane, flying any extra passengers will translate into profit, minus any costs associated with placing them on the plane (meals, booking fees, etc.). When the plane is overbooked, profits become dependent upon how bumped passengers are handled; this is represented by the Bump Function, which will be introduced shortly to model consumer behavior. $(B - X) * R$ is the revenue from ticket-holders who either cancel or no-show. The actual figures that will be used in the following models are derived from Frontier Airlines' policies and information specific to Flight 502.

Particular to Frontier's Flight 502 from Denver to New York, and thus to our simulations, is the following data. Although restricted tickets are non-refundable, there is a \$60 transfer fee which will be the *only* profit from a cancelling or a no-show customer on *this particular flight*. Airfare is derived using the average cost of restricted-ticket fare over a one week period in 2002. Assuming that ticket sales are evenly distributed for flights over the course of a week, we set $Airfare$ =\$316. $Cost_{Flight}$ is based on the previously introduced break-even load-factor of 57.8% For Flight 502, we take $Cost_{Flight}$ = \$24,648.[6] We make the approximation that the average costs associated with placing one passenger on the plane is $Cost_{Add}$ =\$16. The break-even occupancy is determined by the previously noted break-even load-factor, 57.8%. Since we assumed Flight 502 will be using an Airbus A319 with 134 seats, we take $C = 134$, and $C_{\bar{\$}} = 78$. Of course, the number of passengers who show to the gate (X) is still variable, and we still want to optimize the number of people to book (B).

- $R = \$60$
- $Airfare = \$316$
- $Cost_{Flight} = \$24{,}648$
- $Cost_{Add} = \$16$
- $C_{\bar{\$}} = 78$ passengers
- $C = 134$ passengers

6.3 Handling Bumped Passengers: The Bump Function

As stated before, the revenue model is dependent upon consumer behavior when passengers are bumped. Bumping passengers introduces numerous costs, some obvious, some not. One obvious cost is in direct compensation, say, coming in the form of a voucher for free travel on a later date. Less obvious are costs buried in the ability or lack thereof to place bumped passengers onto other airlines' flights, as well as food and hotel costs for holding passengers in the interim. Even more obscure are the costs associated with customer dissatisfaction; for example, the loss of future business with these customers or even worse after word-of-mouth complaints.[7] Therefore, we now consider a variety of possible overbooking strategies, the last three of which will translate directly into applying various Bump Functions.

[6] This value is taken from the most recent quarterly financial report, the total flight operating cost.

[7] The most the devastating effects, from a customer service standpoint, on future customer return is involuntary bumping.

1. No Overbooking

   In the interest of determining whether or not overbooking is at all profitable, we first consider the case in which our airline sells exactly the same number of tickets as there are seats in the plane.

2. Bump Threshold Model

   Second, we imagine that the cost of bumping customers is well understood, and can be applied to our model simply by assigning a "Bump Threshold"' (BT) to each flight: a probability of having to bump one or more customers from a flight given B and p.

$$P\{X > \text{flight capacity}\} < BT$$

We take $BT = 0.05$ of flight capacity. This is an intuitive estimation, based on an assumed desire to overbook flights without gaining a reputation for bumping customers.

To compute the optimal number of passengers to book, given a Bump Threshold, we first show the probability distribution function for the binomial random variable X. The probability that more than N ticket-holders will arrive at the gate, given B total tickets sold, is given by the following:

$$P\{X > N\} = 1 - P\{X \leq N\} = 1 - \sum_{i=1}^{N} \binom{B}{i} p^i (1-p)^{B-i}$$

This simplistic model is independent of revenue, and produces (through simple iteration) an optimal number of ticket sales (B) for expecting bumping to occur on less than 5% of flights.

3. Linear Compensation Plan

   The next advancement of the model comes in the form of a specific Bump Function, introducing revenue considerations via the total profit model: a linear compensation plan. This plan assumes that there is a fixed cost associated with bumping a passenger, *and* that this cost is the same for each passenger regardless of the number bumped. This cost, for example, might be the value of flight vouchers for future use. The related Bump Function is:

$$Bump(X - C) = B_{\$} * (X - C)$$

$$\text{where}$$
$$\text{(X-C)} = \text{number of bumped passengers}$$
$$B_\$ = \text{Cost of handling each bumped passenger}$$

In the case of handing out flight vouchers, $B_\$ = Airfare$. If we take $B_\$ > Airfare$, we are assuming that the *average* cost of handling a bumped passenger exceeds simple voucher distribution; for example, a higher average might take into account that some passengers will require hotel rooms for the evening. This Bump Function will be inserted into the total profit function.

4. Nonlinear Compensation Plan

The next step in the advancement of our model is to penalize the airline for bumping too many ticket-holders. This effect is mathematically represented by an application of nonlinear Bump Functions. Nonlinear functions are capable of appearing linear locally (say, for small numbers of bumped passengers), but placing steeper and steeper penalties upon the airline in the general picture. Steeper penalties must be considered since there is a chain reaction of expenses incurred when bumping passengers from one flight causes future bumps on later flights. In this spirit, we assume that the Bump Function is exponential. Then for a few bumped passengers, the costs of bumping will be similar to those modeled by the linear compensation plan, but will ensure a higher average cost per bumped passenger for dealing with them in greater numbers. Assuming flight vouchers are still adequate compensation approximations for a *few* bumped passengers, and assuming an average cost of $2 * Airfare + \$100 = \$732$ when there are 20 bumped passengers, we apply the following cost equation:

$$Bump_{NL}(X - C) = B_\$(X - C)e^{r\,(X-C)}$$
$$\text{where}$$
$$B_\$ = \text{Compensation constant}$$
$$r = r(B_\$) = \text{Exponential rate, chosen to fit the curve to the points (0,316) and (20,732).}$$

5. Time-Dependent Compensation Plan (Auction)

The primary short-coming of the nonlinear compensation plan is that it does not deal with the flights where there are too few voluntarily bumped passengers, and the airline must increase its compensation offering. Our final adjustment to the static model is to introduce a dependence on time; we now approximate the costs associated with implementing an auction-type compensation plan. This plan assumes that the airline knows the number of no-shows and cancellations 1/2 hour prior to departure time. Until the number of passengers that need to be bumped actually *are* bumped, the following auction system is employed. At 30 minutes until departure, the airline offers flight vouchers to volunteers who are willing to be bumped, equivalent in cost to the original airfare. This offer stands for fifteen minutes, at which time the offer increases exponentially up to the equivalent of $948 by departure time. We chose this number as twice the original airfare (which is the maximum obligatory compensation for involuntary passengers if they are forced to wait over two hours), plus one more cost of airfare in the hopes that treating the customers so favorably will result in future business from the same customers. These specifications are enough to determine the corresponding time-dependent *Compensation* function, which is plotted in Figure 2.

$$Compensation(t) = \begin{cases} 316, & 0 \leq t \leq 15\, min; \\ 105.33e^{.07324\, t}, & 15\, min < t \leq 30\, min. \end{cases}$$

Implicit in the auction-style setup for this model is a dependence on *when* passengers are going to accept this compensation offer. Common sense in the consideration of realistic passenger behavior dictates our propoposition to use a Chebyshev weighting distribution for this effort (shown in Figure 3). This function indicates that there will be a significant number of passengers who will desire flight vouchers as soon as they become available, in exchange for being bumped. Once these customers have spoken up, there should be a lull in the desire to accept such a compensation offer. Finally, once the offer increases to quite attractive levels, even the most stubborn passengers may be willing to be bumped in exchange for the weighty compensation.

Our task is now to simulate a random variable given the density function above. We first wish to find the probability distribution function from the integration of the Chebyshev density f(s):[8]

[8]The Chebyshev weighting function, $f(s) \equiv \frac{1}{\pi * \sqrt{1-s^2}}$, is used in polynomial functional approximation and is defined on the interval [-1,1].

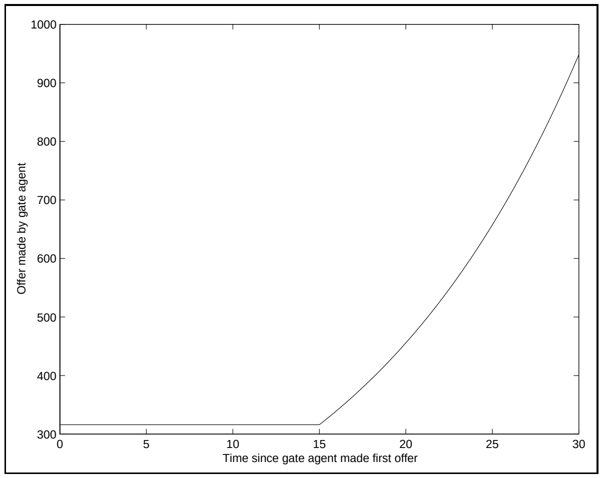

Figure 2: Auction Offering (Compensation)

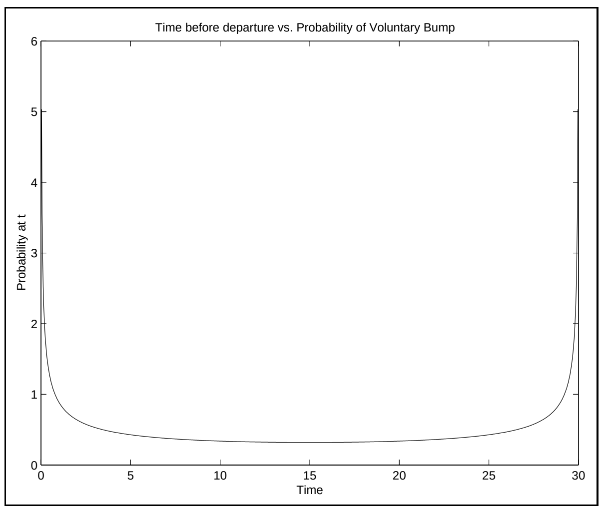

Figure 3: Chebyshev Weighting Function for Offer Acceptance

$$F(\tau) = \int_{-1}^{\tau} \frac{1}{\pi * \sqrt{1-\eta^2}}\, d\eta = \frac{1}{2} + sin^{-1}(\tau)$$

where η is a dummy variable.

Inverting this probability distribution results in a method for generating random variables with the Chebyshev distribution. [21]

$$F^{-1}(\tau) = sin[\pi * (U - \tfrac{1}{2})]$$

where U is a random uniform variable on [0,1].

With a linear transformation from the Chebyshev interval of [-1,1] to the time interval of [0,30] via $t = 15\tau + 15$, we find a random variable t that takes on values from 0 to 30 according to the density function f(s). Figure 4 shows the results of using this process to generate 100,000 time values. We can now use this random variable to assign times for compensation offer acceptance under the auction plan.

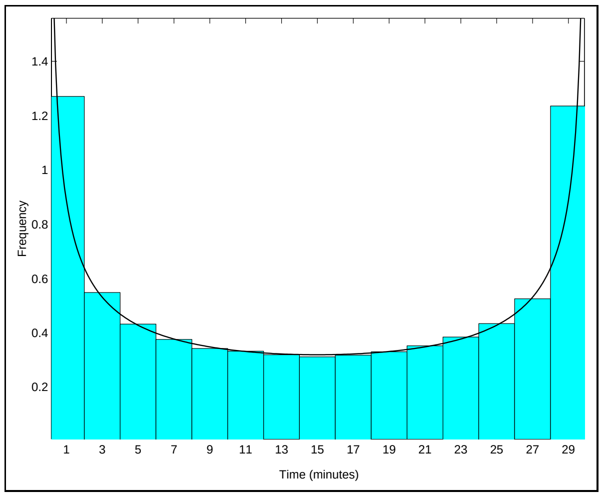

Figure 4: Histogram of 100,000 Draws from the Chebyshev Distribution

Therefore, total costs in bumping X-C passengers is $\sum_{i=1}^{X-C} Compensation(t_i)$

6.4 Optimizing Overbooking Strategies Based on the Revenue Model

Now that we have a working revenue model, we may turn to optimization. Our goal is to **maximize the expected value of the total profit function**, E[T_P(X)], given the variability of the Bump Functions and the probabilistic passenger arrival model. There are competing dynamic effects at work in the total profit function. Ticket sales are usually desired, but there is a point at which the costs of bumping passengers becomes too great. Also, the variability of the number of passengers who show up to the gate will affect the dynamics. We note that the expected value of the total profit function is found through a normalization process with the probability of passengers showing up to the gate:

$$E[T_P(X)] = \sum_{i=1}^{B} T_P(i) * \binom{B}{i} p^i (1-p)^{B-i}$$

Finally, all of the pieces for optimizing the static model are in place. We now develop a method for optimizing the revenue by finding the most appropriate booking limit (B) for any given Bump

Function. Solving such a problem analytically is unrealistic; any solution would require the inversion of a sum of factorial functions. Therefore, we turn to computation for our results. Computation here entails calculating the expected value of the total profit function over a range of B. Numerous programs to this end were written in Matlab, and are attached (with commentary) as Appendix B. The programs were tested by solving for B over a range of trivial Bump Functions. The results being to our great satisfaction, we uncovered results for choosing the most profitable booking limit (B) and the subsequent financial consequences of such overbooking strategies. The results are compiled in the following section.

7 Results of Static Model Analysis

7.1 No Overbooking

In a situation where Frontier Airlines does not overbook its flights, we find a significant loss of opportunity cost. We can obtain an extremely accurate first-order approximation to the expected profits in this scenario, where the number of people that are booked (B) is equivalent to plane capacity (C). First, we recall that the expected value of our random variable X (number of passengers who arrive at the gate) is exactly $p * B = p * C$ (in this case). Thus, E[X]=.88*134≈ 118 passengers. Assuming, (as is done in the total profit function), that each passenger beyond the 78^{th} is worth \$300 in profit, the expected value of the profit is *very* nearly the following:

$$(134 - 118) * \$60 + \$300 * (118 - 78) = \$12,960 \quad \text{per flight}$$

This is technically only an approximation, since there is a probability, negligible as is it may be, that less than 57.8% of ticket-holding passengers will arrive at the gate (probability$\approx 1 * 10^{-4}$). \$12,960 is certainly a sizeable profit on each flight, but there are still (on average) 16 empty seats when the plane departs! Aside from the possibility of re-selling even more than these 16 tickets (counting on those new ticket-holders also to arrive with probability p), we can gain an approximation of lost opportunity cost in this scenario as \$300*16 = \$4800! Thus, even with an estimation of further possibilities in ticket selling strategies, failure to accept overbooking as a possible strategy sends Frontier Airlines Flight 502 on its way with only 63% of its potential profitability. By now it should be clear that overbooking strategies can be an overwhelming factor in airline profitability!

7.2 Bump Threshold Model

Using a 0.05 Bump Threshold, we computed an optimal number of passengers to book on Flight 502. Programs included in Appendix B were written simply to optimize the probability distribution function modeling passenger arrival to the departure gate ($P\{X > i\}$), given different numbers of tickets sold (B). Given the Airbus A319 capacity of 134 passengers, and a passenger arrival probability of p=.88, we find that the optimal number of tickets to sell to essentially guarantee that bumping occurs less than five percent of the time is **B=145**, or 107% of flight capacity.

7.3 Linear Compensation Plan

The results from computing the expected value of the total profit using the following linear Bump Functions are now tabulated:

Bump Functions compared		
Bump Function	Optimal # to Book	Expected profit per flight
Bump(X-C) = 200*(X-C)	∞	∞
Bump(X-C) = 316*(X-C)	162	$17,816.64
Bump(X-C) = 400*(X-C)	156	$17,393.50
Bump(X-C) = 500*(X-C)	153	$17,121.33
Bump(X-C) = 600*(X-C)	152	$16,939.97
Bump(X-C) = 700*(X-C)	151	$16,799.34
Bump(X-C) = 800*(X-C)	151	$16,691.93
Bump(X-C) = 900*(X-C)	150	$16,601.31
Bump(X-C) = 1000*(X-C)	150	$16,525.88

Employing the linear compensation plan, if Frontier were to compensate bumped passengers less than the cost of airfare, bumping passengers would *always* cost less than revenue gained from ticket sales. Thus, Frontier would realize an unbounded profit on each flight! Obviously, the linear compensation plan is not realistic in this regime, and we must wait for subsequent models to see increased real-world applicability. If Frontier recognizes that, on average, bumping passengers will cost at least the price of airfare, the model returns some acceptable results. These results agree reasonably with the result of using a simple Bump Threshold (above), and indicates an average profit of approximately $17,000. In comparison with using no overbooking strategy at all, Frontier gains a profit of approximately \$4,000 per flight!

The actual dynamics of the problem may be seen in Figure 5, where competing effects form an optimal number of tickets to sell (B) when Frontier assumes a sizeable enough compensation average. We can also see the unbounded profit available in the unrealistic regime.

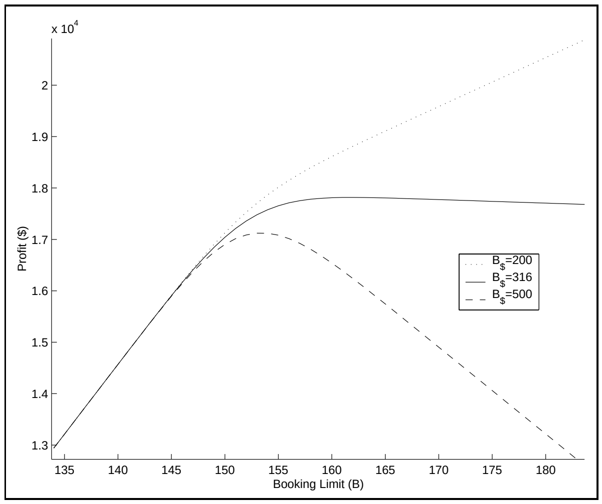

Figure 5: Per-Flight Profit vs. Booking Limit (B) for 3 Different Bump Costs (Linear Compensation Plan)

7.4 Nonlinear Compensation Plan

While Frontier Airlines may enjoy the thought of unlimited profitability, numerical results for the more realistic nonlinear model paint a more reasonable picture.

Bump Functions compared		
Bump Function	Optimal to Book	Profit per flight
$Bump_{NL}(X-C) = 50 * e^{.134(X-C)}(X-C)$	160	$\$18,699.66$
$Bump_{NL}(X-C) = 100 * e^{.100(X-C)}(X-C)$	158	$\$18,239.69$
$Bump_{NL}(X-C) = 200 * e^{.065(X-C)}(X-C)$	156	$\$17,722.26$
$Bump_{NL}(X-C) = 316 * e^{.042(X-C)}(X-C)$	154	$\$17,363.02$

This table recommends similar (though slightly higher) booking limits as previous models. These results are also more reasonable, as expected. For example, the second $Bump_{NL}$ Function in the table assumes that for a few bumped passengers, the average cost of compensation is less than the price of airfare. However, in contrast to the linear model, there is still a balancing effect in the probabilistic normalization process from the larger average costs of bumping too many passengers. Once again, the actual dynamics of the problem may be seen in the following graph (Figure XXXX), where competing effects form an optimal booking limit(B) when Frontier employs the nonlinear overbooking compensation expectations.

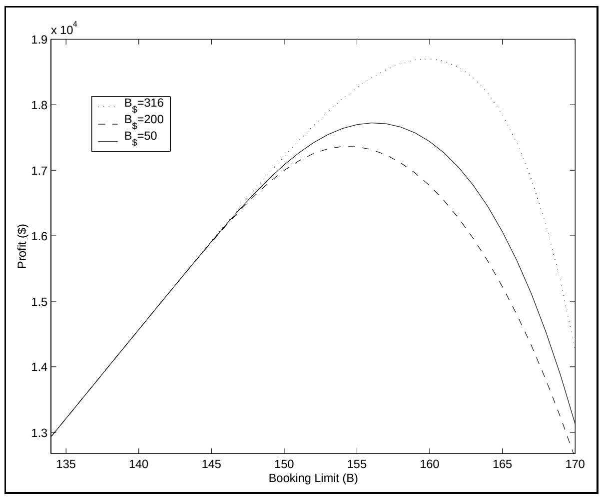

Figure 6: Per-Flight Profit vs. Booking Limit (B) for 3 Different $Bump_{NL}$ Coefficients (Nonlinear Compensation Plan)

All investigated nonlinear Bump Function coefficients result in a maximum realizable profit, as expected. Also as expected, the lower the initial compensation attempt (lower Bump Function coefficient) results in greater revenues. Of course, this model still assumes that voluntary customers might be immediately willing to take a later flight for a rather modest monetary compensation.

7.5 Time-Dependent Compensation Plan

The results of computation for this case are as follows. The histogram of 1000 runs using the time-dependent compensation plan in Figure 7 shows that the optimal booking limit is most frequently **B=154**. Figure 8 is a graph of expected total profit versus the optimal booking limit for 15 trials, displaying the randomness due to the Chebyshev draws at higher values of B. If B is too low then all models have the same profit behavior because the randomness from the overbooking scheme is not introduced until customers are being bumped. This graph also shows that regardless of random effects profitability is maximized around $B = 160$.

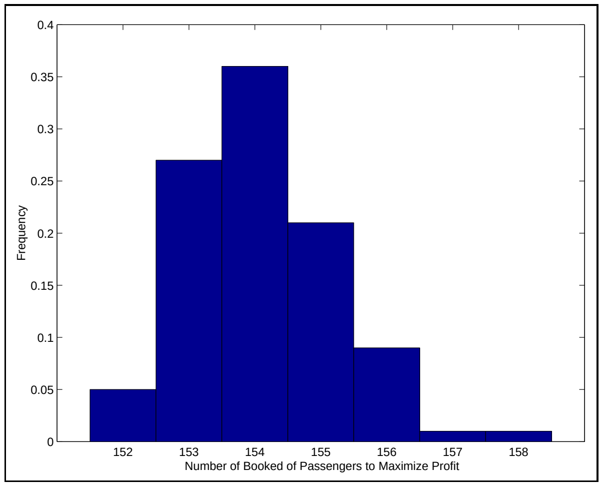

Figure 7: Time-Dependent Compensation Plan simulated 1,000 times

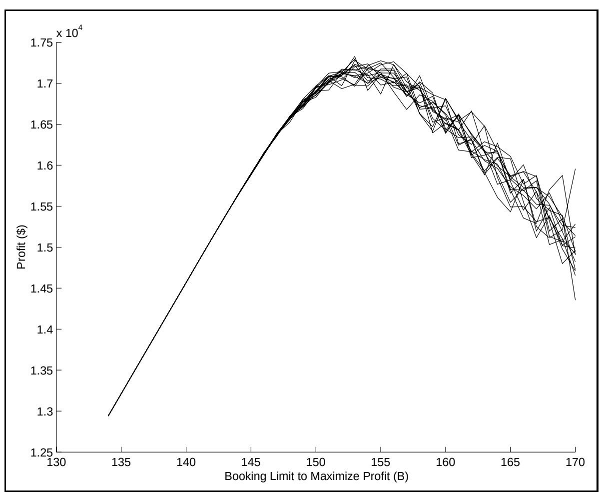

Figure 8: 15 Time-Dependent Compensation Plan Simulations

8 The Dynamic Model

The true nature of the airline overbooking problem is difficult to capture with a static booking strategy. Many of the assumptions made in the binomial-based models will be loosened in this dynamic setting. This is because continuous time allows for more detailed analysis of the order of events occurring in the airline booking problem. Keeping track of the order of different reservation requests, ticket bookings and cancellations results in a model that attempts to recommend what ticketing agents should do at a certain time. This decision-making process could be guided by various time-dependent booking limits. We developed one such scheme, and named it the Firesale Model, where we attempt to increase revenue by selling the tickets of cancellations to customers who would otherwise be denied tickets due to a fixed Booking Limit.

8.1 Reservation Process

The first step in implementing a dynamic booking scheme is to simulate the booking/reservations process that often begins weeks before the flight actually departs and usually continues right up until departure from the gate (due, for example, to other airlines booking their bumped customers in Frontiers' empty seats).

To model the stream of reservation requests we employ a Poisson Process. A Poisson Process, $\{N(t), t \geq 0\}$ is a counting process that begins at zero ($N(0) = 0$) and has independent increments, with the number of events in any interval of length t Poisson distributed with mean λt.[22] As a result of these conditions, the inter-arrival times of a Poisson Process are distributed according to an exponential distribution with rate parameter λ. This follows from the memory-less property of the exponential random variable: that an intervarrival time is independent of the previous time. If reservation requests are taken to be phoned in to a pool of ticketing agents[9] at Frontier Airlines, the independent increment assumption means that reservation requests are equally likely to occur at any time in the Reservations period (from $t = 0$ to flight departure time, $t = T$, measured in days). Since a reservation request corresponds to a call to an agent, each reservation request comes with a variable number of tickets requested for that reservation (due to single customers versus families etc.). The number of tickets requested in one of these calls is generated from some specified Batch Distribution, BatchD, that will be introduced later. Consequently, it should be noted that a reservation request does not guarantee a single ticket sale; the caller first must specify how many tickets they are attempting to reserve and then the entire reservation must be approved by the operator.

[9] By calling a pool of ticketing agents we mean submitting a reservation request by telephone, through an e-travel website, or through a travel agent.

This arrangement results in a compound[10] Poisson process because the arrival of reservations are independent but a number is also assigned to each reservation. Compound Poisson Processes are preferred to their homogeneous (constant rate λ) and non-homogeneous (varying rate λ) counterparts for two reasons. First, they better model the true state of the reservation process because they avoid the assumption made in the binomial model that each customer acts completely independent of every other customer; customers are now allowed to behave in batches of varying size. Second, it may be found in the available literature [18] that compound Poisson Processes result in variation that provides more reasonable fits to real world reservation request arrival data.

Simulating the first T time units of a Poisson Process using the method in [21] results in a vector **at** of arrival times for the $A = length(\mathbf{at})$ reservation requests received. Another vector **Bnum** of number-of-tickets-requested in each of the A reservations is also generated according to the Batch Distribution that we now define for the remainder of this paper. The density BatchD is shown in Figure 9. This particular density states that callers reserve anywhere from 1 to 4 tickets at a time[11], with varying probabilities for each number of tickets requested. The total number of tickets (potential fares) requested is then $\sum_i(\mathbf{Bnum}(i))$.

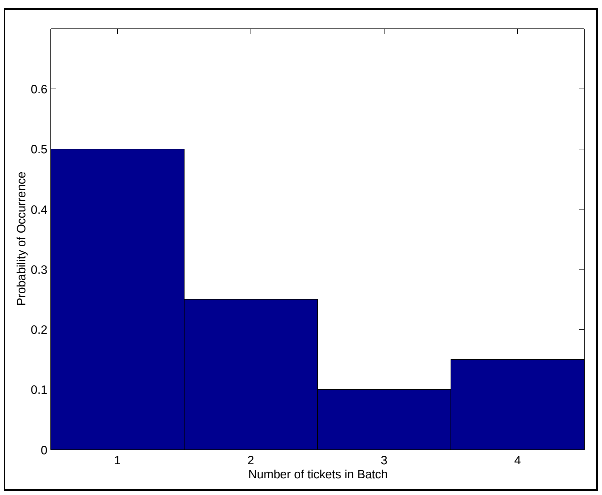

Figure 9: Density Function for Number of Tickets in a Batch of Reservations

The arrival rates for these reservation requests are derived by setting the expected value of the Poisson Process over an interval of length T equal to the average ticket demand A_D that we expect.

[10]This specific compound Poisson Process is often referred to as a "Stuttering" process (See [18]).

[11]This is a simplified distribution because most current e-travel sites limit reservations to a maximum of 8 tickets.

Then a rate of $\lambda = \frac{A_D}{E_B T}$, where E_B is the expected value of BatchD (1.9 in this case), will, on average, generate A_D number of tickets. The histogram in Figure 10 shows the results of a simulation of 10,000 Poisson Processes outputting the number of reservations requested when the average demand for tickets was 134 (A_D=C).

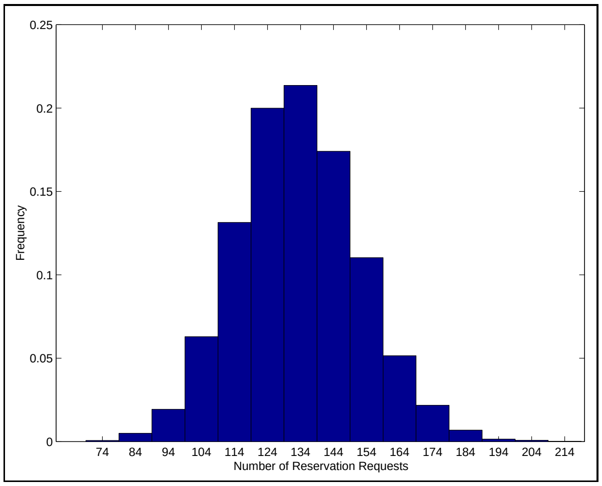

Figure 10: Histogram of Number of Reservation Requests for 10,000 Flights with an Average Demand of 134 Tickets

8.2 Booking Process

Once a reservation has been phoned in, an agent must then determine whether or not to approve the request. The number of tickets booked (approved) at any time t is stored in the state variable Tix. The operator must decide whether or not to approve a request for a reservation at time t based on several variables, including: Tix, number of tickets requested in that particular reservation, and the pre-determined booking limit, B. Various time-dependent booking schemes could be applied in this step by making the booking limit a function of time.

8.3 Treatment of Cancellations and No-Shows

The benefits of utilizing a dynamic model are maximized when the effects of cancellations and no-shows are included. The binomial-based static models did not distinguish between cancellations (tickets that are voided before the flight departs) and no-shows (tickets that are not used or voided by flight departure); however, the dynamic model is well-suited for monitoring these events. An obvious assumption that must immediately be made is the number of unused tickets that are cancellations versus the number that are no-shows. For all subsequent models and tests, we assume[12] that 75% of unused tickets are cancellations, with the remaining 25% representing no-shows. Additionally, we assume that the time of cancellation for a set of tickets reserved together is uniformly distributed from the time that the tickets are granted to the time that the flight departs. This means that some cancellations will occur almost immediately after the ticket(s) are granted (e.g. due to a typo on an online ticket service form) while some will occur just before a plane is scheduled to depart (e.g. a last minute change of plans). Lastly, we assume that multiple tickets in a single reservation will all behave equivalently (i.e. families act as unbreakable groups!).

To simulate this process for each requested reservation, a biased coin is flipped to determine, with probability p, if the group represented by the batch of tickets in question will keep their tickets. If the tickets will not be used, another biased coin is flipped to determine whether the unused tickets will result in cancellations or no-shows. Lastly, if a cancellation occurs, a time of cancellation is drawn uniformly between that batch's arrival time and the flight departure time in order to determine when the tickets will be canceled (thus updating the state variable Tix).

8.4 Modeling Revenue

At time T, the flight departure time, the total number of customers that reported to the gate is the end number of tickets, Tix at time $t = T$, less the number of no-shows. The profit made on a given flight is then computed as before (using the Time-Dependent Compensation Plan) where any number of passengers at the gate greater than the flight's capacity must be bumped. Because of the randomization used throughout this dynamic model, simulation is repeated many times to obtain a value of expected profit[13]. The objective of the dynamic model will be to maximize this expected profit through numerical simulation.

[12] The motivation for the well-informed Frontier Airlines customer to cancel his/her restricted ticket(s) before departure may be smaller than that of customers of other carriers & fares because they will receive another ticket (minus a $60 fee) in either case.

[13] Expected Profit after simulating is equal to the product of an observed profit and that profit's observed frequency, summed up over all observations.

8.5 Implementation of Dynamic Booking

As stated, the dynamic model outlined above lends itself to applying various booking schemes, two of which are now developed.

8.5.1 Dynamic Test Model

The first application of the dynamic model is as a test of results of the various binomial-based models, and is henceforth referred to as the Dynamic Test. In this case, the dynamic model is not used at its full potential because it does not need to use time-dependent conditioning (i.e. we don't care *when* the various events occur.). Instead, the model is used to make the binomial-based models more realistic by reducing some of its assumptions as well as introducing a previously absent element of randomness. The Dynamic Test allows for "group tickets" (for both reservations and cancellations) previously not taken into account in earlier models. While earlier models assumed an infinite pool of customers, the Dynamic Test requires that an average ticket demand, A_D, be specified in order to confirm the expected effects of less demand for tickets. This test will be used to simulate a real-world environment through which the static models will be evaluated (see Section 10).

8.5.2 Firesale Model

The Firesale model is the model that we have developed to take into account the possible time-dependency of airline booking. The model uses the cancellation times to essentially sell all possible tickets. If the number of tickets requested (at time t) for a particular reservation plus Tix (the number of tickets approved and still held at time t) is less than the pre-determined booking limit (B), then a reservation request is approved. Conversely, if $Tix(t)$ is equal to the booking limit or if the sale of the multiple tickets requested in a reservation batch would push $Tix(t)$ over the booking limit, the request is rejected. Thus, for a process with no cancellations, reservation requests totaling less than the booking limit[14] would be approved while subsequent requests would be rejected. Clearly, the Firesale model is highly dependent on the average demand (i.e. if demand is high enough, the airline would end up with an overwhelming majority of no-shows, as opposed to Cancellations). However, the model

[14] Making sure that a multi-fare reservation is only accepted if the addition of all tickets in the batch would fall within the booking limit.

represents a possible solution in the pursuit for short-term revenue. Matlab code for this model can be found in Appendix B. The Firesale Model is the most realistic model developed in this paper.

9 Results of Dynamic Model Analysis

The Firesale Model attempts to capture a scenario where all tickets of cancellations are sold as long as there are customers willing to buy them. If demand for tickets is high enough, we expect to sell all tickets of cancellations, resulting in a large number of bumped passengers. However, because the airline profits \$60 from each cancellation or no-show and because the number of both cancellations and no-shows will continue to increase as more tickets are sold, reasonable results are expected for reasonable ticket demand.

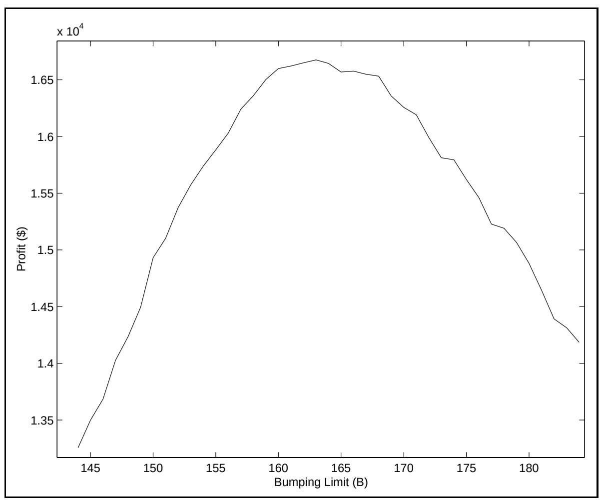

Figure 11: 1,000 Simulations of the Firesale Model

Figure 11 plots expected profit as a function of bumping limit as determined from 1000 Firesale Model simulations. An average demand twice that of capacity ($A_D = 268$) was used and a maximum profit was realized at a Booking Limit of 163. Most importantly, this figure displays how a small variation in Booking Limit could significantly alter profit. A change in either direction of 3 in the Booking Limit corresponds to a loss of more than \$1000 profit.

10 Dynamic Testing of the Static Model

The creation of a dynamic model allows us to test the results obtained from the static (binomial-based) models in a more realistic setting. The Dynamic Test (see Appendix - file poisslevel0.m) allows tickets to be reserved in batches and introduces randomness experienced in real world airline booking.

In all testing, 10,000 simulations were performed for each Booking Limit (B) and then expected profits were computed. Booking Limit versus Profit ($) was then plotted for appropriate Booking Limit values. The average demand (A_D) used in this test was kept constant at twice the capacity of the airplane (so $A_D = 268$). This was done to simulate a very large pool of customers so that the actual *over*-booking process could be tested[15]. The resulting figures for applying the dynamic test to the various compensation plans are shown and analyzed below.

10.1 Linear Compensation Plan

Two Bump constants ($B_\$ = 316$ and $B_\$ = 600$) with different behaviors (as predicted by the static model) were chosen for testing.

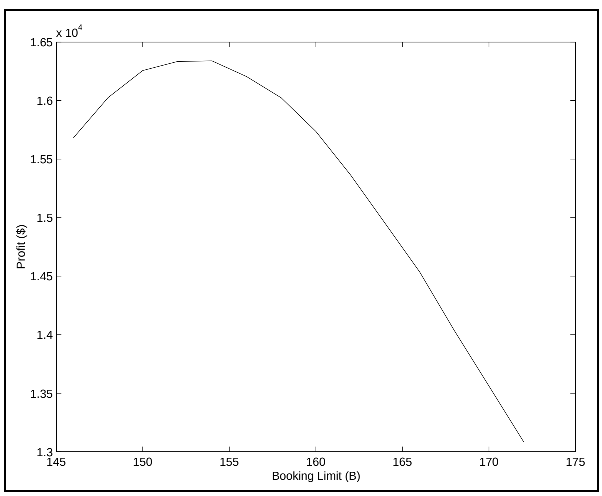

Figure 12: 10,000 Simulations of the Linear Compensation Plan with $B_\$$=600

A constant bump of 600 corresponds to giving each bumped customer a $600 compensation package. Figure 12 shows that, for this compensation plan, an optimal Booking Limit is **B=155**. This Booking Limit represents an increase of 3 from the optimal value reported by the static model. One

[15] Recall that if the pool of customers wanting to fly is too low, overbooking will never occur.

could take this to mean that, in a more realistic setting, booking limit could in fact be increased to maximize profit. However, profit drops off steeply for Booking Limits greater than 155, indicating that a more conservative strategy might be to lower the Bumping Limit to insure that this steep decline is rarely reached.

Figure 13 corresponds to a Bump constant of 316. The optimal Booking Limit is now 166, displaying once again an increase (from 162) over the initial results for the static model. However, this figure also displays some of the instability found when constant bump functions are not high enough. If the constant were lowered even ten dollars, profit would grow indefinitely because the plan does not penalize the airline enough for bumping customers.

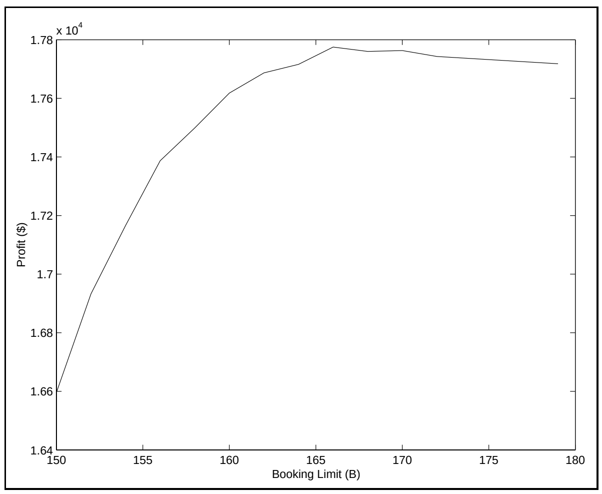

Figure 13: 10,000 Simulations of the Linear Compensation Plan with $B_{\$}$=316

10.2 Nonlinear Compensation Plan

Two nonlinear Bump coefficients ($B_{\$} = 316$ and $B_{\$} = 100$) with different behaviors (as predicted by the static model) were chosen for testing.

Both Figure 14 and Figure 15 clearly demonstrate the negative effect of a Booking Limit set too high. For nonlinear bump coefficients $B_{\$} = 316$ and $B_{\$} = 100$, optimal booking limits were predicted by the static model to be 154 and 160, respectively, with Dynamic Test result values of B=154 and B=158, respectively. Thus, a decrease of optimal booking limit occurred for the latter case, while the former remained constant. Once again, a conservative strategy would be to re-test the latter case in order to determine the true optimal booking limit. As expected, the nonlinear compensation plans are

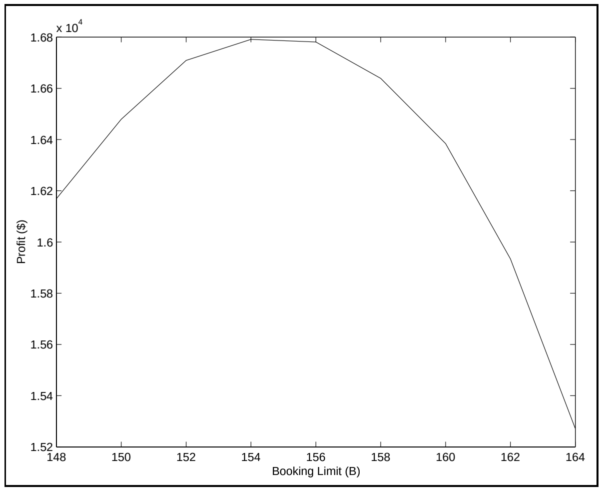

Figure 14: 10,000 Simulations of the Non-Linear Compensation Plan with $B_{\$}$=316

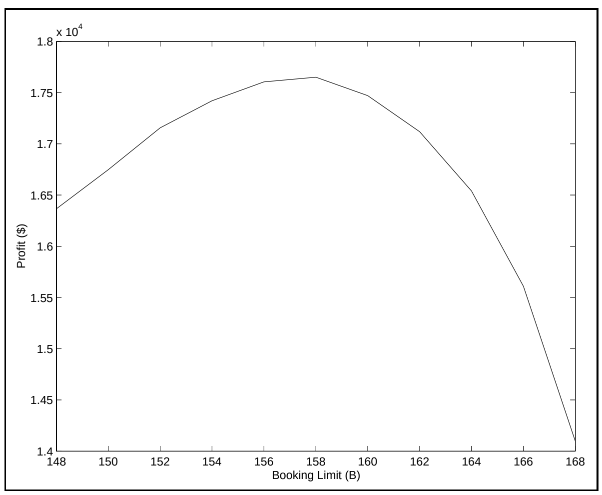

Figure 15: 10,000 Simulations of the Non-Linear Compensation Plan with $B_{\$}$=100

still more realistic than their linear counterparts when applied in a more realistic simulation.

10.3 Time-Dependent Compensation Plan

Figure 16 shows that the optimal booking limit for the time-dependent compensation plan is B=155, an increase of 1 from the static model. Profit appears to rise relatively steeply until the optimal booking limit is reached, and then falls steeply. Thus, in our most realistic static model, a careful overbooking plan matters the most! A significant gain may then be achieved by the optimal

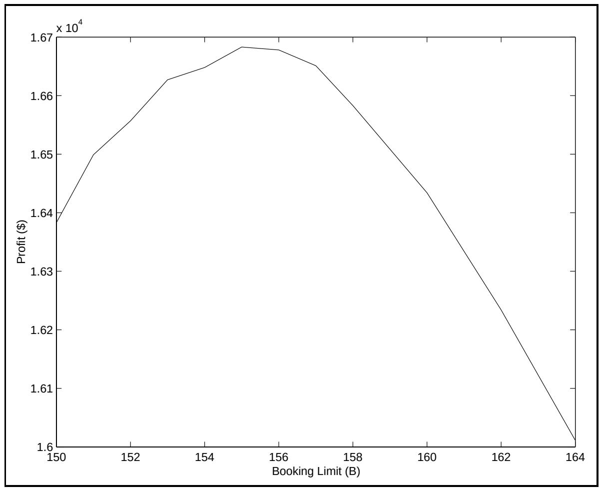

Figure 16: 10,000 Simulations of the Time-Dependent Compensation Plan

booking limit. The graph indicates that if the booking limit were altered by either three to the left or to the right, the profit would shrink by more than \$1000 per flight, similar to the result detailed in the Firesale Model.

11 Post-September Effects

As noted earlier, the attacks in September 2001 altered the business horizon for the entire American airlines industry. Thus, any applicable model should deal with post-September alterations of costs and customer behavior. One effect on the airline companies has been heightened security at and around airports. Directly following September 11, 2001, this increase in security resulted in significant delays and subsequent no-shows (while ticket-holders were waiting in quarter-mile long baggage check lines). This effect might indeed be cause for concern in calculating ticket-holder arrival percentages. However, Frontier Airlines has noticed that "the security checks (at Denver International Airport) add only 10 minutes to the check-in time." [9] Thus, the mathematical models of interest to Frontier Airlines may consider these effects to be negligible.

The most significant post-September situation that the airlines must consider is the fallout from consumer fear. Americans are less confident in the safety of flying, and have slowed in airline ticket purchases. A good mathematical model must then contain probabilities reflective of consumer behavior. It should first be noted that the individual probability of passenger arrival p, defined for use in the static model, should not vary drastically from the norm since ticket-purchasing customers after September 11 are fully aware of the risks involved. Lastly, a consequence of September 11 that is difficult to model is the decrease in average demand for flight reservations.

12 Model Strengths and Weaknesses

As with all models, ours are subject? to numerous weaknesses from over-aggressive assumptions and variable applicability. However, there are aspects of the models developed here that are quite. The major strengths and weaknesses are as follows.

12.1 Strengths

- Time-dependent auction model for pre-flight compensation
  When Frontier begins to offer compensation to voluntarily bumped passengers 1/2 hour before departure, our model allows consumer behavior to mathematically influence the financial results.

- Time-dependent decision process in the dynamic model
  The dynamic model allows ticketing agents to decide whether or not to accept reservation requests based on the number of tickets sold by that time, and based on time until flight departure.

- Multiple considerations of consumer behavior via Bump Functions
  The implementation of multiple Bump Functions allow for the testing of alternative strategies for compensating bumped customers. Profit and customer satisfaction may then be balanced depending upon the company's short-term or long-term interests.

- Varying degrees of model complexity
  Our early models are simple, making sizeable simplifying assumptions to exhibit the most basic dynamics inherent in the problem. We take small steps of increasing complexity towards a more realistic model. The intuitive relationships between the results from each step lead to increased confidence in the stability and applicability of the most involved models.

12.2 Weaknesses

- Absence of a stability analysis

  We lack an adequate mathematical understanding of the stability of our models. Varying parameters like p, the probability of independent passenger arrival in the static model, could potentially alter our results.

- Infinite customer pool in the static model

  In our static model we assume an infinite demand for airline travel. In other words, for any booking limit we set we assume that all tickets will be sold.

- Insufficient data

  The only operational data we were able to get from Frontier Airlines was from its quarterly report. This contains general information on how many people flew, operating costs, revenues, number of flights flown, and occupancy rates. However, our model lacked real information regarding cancellation rates, no-show rates, actual cost per flights, rates of reservation requests, and ratio of restricted tickets sold to unrestricted tickets sold. The lack of this information limited us because our parameters were not based on conclusive historical data and therefore we can not be confident in the accuracy of our rates.

13 Conclusion and Recommendations

The purpose of this study has been to evaluate methods of increasing airline profitability through overbooking strategy. We have succeeded in developing numerous probabilistic models, each of which resulting in mathematically supported booking limit recommendations. Our static models account for the basic underlying dynamics associated with revenue optimization through overbooking. Our dynamic models incorporate further complexities inherent in the problem by adding more probabilistic realism to the environment. Finally, our static models pass a more rigorous dynamic test, hinting at the possible stability of the corresponding compensation strategies. Every booking model, when employed, has indicated a significant increase in profitablilty over the implementation of no overbooking scheme whatsoever. We thus note that regardless of post-September 11 effects, the implementation of an overbooking strategy should be of great financial import to any airline.

The models are quite consistent, in the sense that they all result in a recommendation of similar booking limits. The most ubiquitous result in the entire study is an optimal booking limit recommendation of 154 passengers on the 134 seat Flight 502, %115 of flight capacity. This booking limit, on average in our models, results in a total profit for Frontier of $17,000 per flight. Frontier Airlines Flight 502 alone, by employing one of the overbooking strategies developed herein, nets the company an extra $2.7 million profit per year under the limiting assumption of an infinite demand pool. While the assumption of infinite demand is a significant one, we still realize a first-order approximation to the profitable potential of employing an overbooking strategy. We also find that regardless of post-September 11 effects, implementing

Finally, it cannot be overstated that the loss of billions of dollars in revenue to date following the September attacks demands that airline companies maximize profits in every possible realm. We have seen in this study that choosing a wise overbooking policy can translate into a significant recovery of financial health. Then again, overbooking strategies are irrelevant if there are no tickets being purchased by consumers. In this case, the problem of revenue optimization extends beyond the scope of this problem, into flight alterations and airfare adjustments.

14 Future Research

This problem is quite open-ended and contains numerous avenues for future research. Very generally, simulations may be made more realistic and various assumptions may be eliminated. However, this problem is in many ways disjoint from other operating areas of the airline industry, which in some sense bounds the possible problem-space.

Perhaps the first necessary advancement in creating a fully applicable model would be to incorporate variable flights routes, airplane sizes, and ticket prices. A good example of this is that the route planning has a direct effect on the demand for a particular flight. If it can be shown that it is possible to make a large profit on a given flight by a certain overbooking scheme, the routes could be adjusted to increase demand on that route. Another possible direction for further study is to consider a more dynamic cancellation scheme. It may be possible to create a cancel-stream of times that tickets for a given flight are canceled. This would allow for a greater understanding of passenger cancellation effects versus the effects of no-shows. Also, by considering multiple leg flights, a more complex system would obviously ensue. Involving the passenger arrival and cancellation rates on multiple flights as a dynamic coupling of multiple aircraft booking schemes would greatly enhance an overbooking model.

References

[1] Air Travel Consumer Report, http://www.dot.gov/airconsumer/

[2] http://www2.alaskaair.com.

[3] *BBC News*," Airlines Receive $15bn Aid Boost" http://news.bbc.co.uk/hi/english/business/newsid_1558000/15588

[4] Boudreaux, Donald *Notes from FEE*, "Julian Simon Life Saver", http://www.fee.org/freeman/98/9804/notes.html., April 1998

[5] *Business Wire* "Frontier Airlines Reports Fiscal Third-Quarter Profit," , Feb. 5, 2002.

[6] Chatwin, Richard, "Continuous-Time Airline Overbooking with Time-Dependent Fares and Refunds," *Transportation Science*, Vol. 33, No. 2, May 1999.

[7] Davis, Paul, "Airline Ties Profitability Yield to Management," *SIAM News*, Vol. 27, No. 5, May/June 1994.

[8] http://www.delta.com/care/service_plan/index.jsp.

[9] *Denver Bussiness Journal* "Frontier Operating at 80%" http://denver.bizjournals.com/denver/stories/2001/09/17 September 20, 2001

[10] Frontier Airlines Inc., "Form 10-Q, Quarterly Report Pursuant to Section 13 or 15(d) of the Securities Exchange Act of 1934," Commision file number: 0-24126, quarterly period ending September 30, 2001.

[11] http://www.frontierairlines.com.

[12] Graves, Gary, "Airline Decline," CBC News Online, http://www.cbc.ca/news/indepth/background/airlinedecline. Nov. 7, 2001.

[13] Gupta, Diwikar and Cooper, William, "When is Yield Improvement Really an Improvement?" Working Paper, University of Minnesota, Department of Mechanical Engineering, September 4, 2001.

[14] Inse, Amy. *Rocky Mountain News*, "Airport's a Lonely Place These Days". September 19, 2001. http://www.insidedenver.com/drmn/america_under_attack/article/0,1299,DRMN_664_828402,00.html.

[15] Karaesmen, Itir and Ryzin, Garrett van, "Overbooking with Substitutable Inventory Classes," Working Paper, Columbia University, Graduate Scool of Business, September 20, 1998.

[16] Kesmodel, David.*Rocky Mountain News*, "Frontier Expects Profits for Quarter". Feb.7 2002.

[17] Masselli, Jennifer *Information Week* "IT's Critical Role in Arlines" http://www.informationweek.com/story/IWK20011112S0001. Novermber 12, 2001

[18] McGill, Jeffery and Ryzin, Garrett van, "Revenue Management: Research Overview and Prospects," *Transportation Science*, Vol. 33, No. 2, May 1999.

[19] http://www.webcenter.independenttraveler.netscape.com/travel/resources/article.cfm?AID=15category=1.

[20] "Overbooking," http://cms.lufthansa.com/de/dlh/en/nws/0,1774,0-0-77892,00.html.

[21] Ross, Sheldon M.,Simulation, 2nd edition, Harcourt Academic Press, Ssan Diego, 1990.

[22] Ross, Sheldon M.,Probability Models, 7th edition, Harcourt Academic Press, San Diego, 2000.

[23] Subramanian, J., Stidham, S., Lautenbacher, C. J., "Airline Yield Management with Overbooking, Cancellations, and No-Shows," *Transportation Science*, Vol. 33, No.2, May 1999.

A Calculation of average annual cost of bumping involuntary passengers

A first-order approximation to the problem of putting multiple bumped passengers onto other planes is to assume that there is no difference in the probability of putting the first person onto another plane than putting the n^{th} person onto another plane. Since Frontier flies an average of 3,076,000 passengers per year.[10], the average annual cost of involuntarily bumped passengers then becomes \$140,300, a figure negligible when compared to even the most underestimated annual costs of bumping voluntary passengers (¿¿\$1 million).

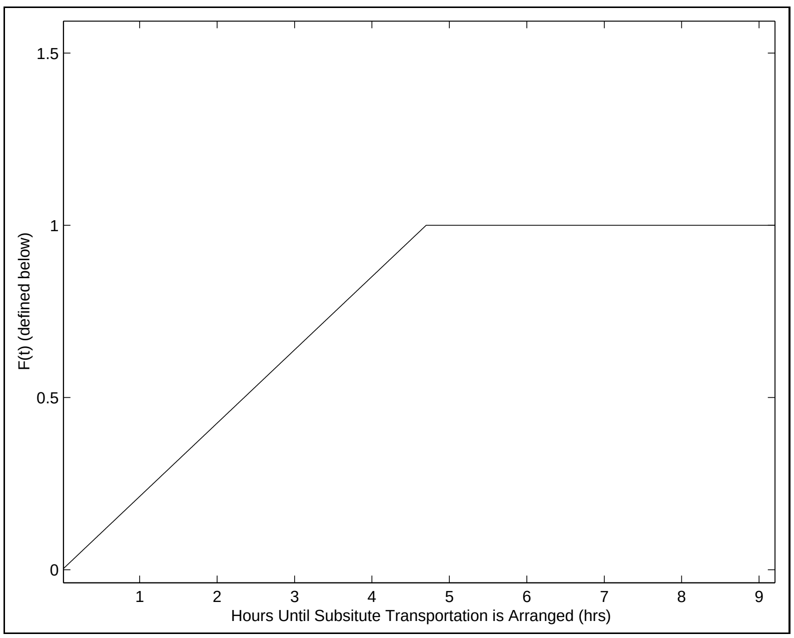

Figure 17: Probability Distribution for Time Until Substitute Travel Arrangements are Made

B Matlab Program Files

See attached.

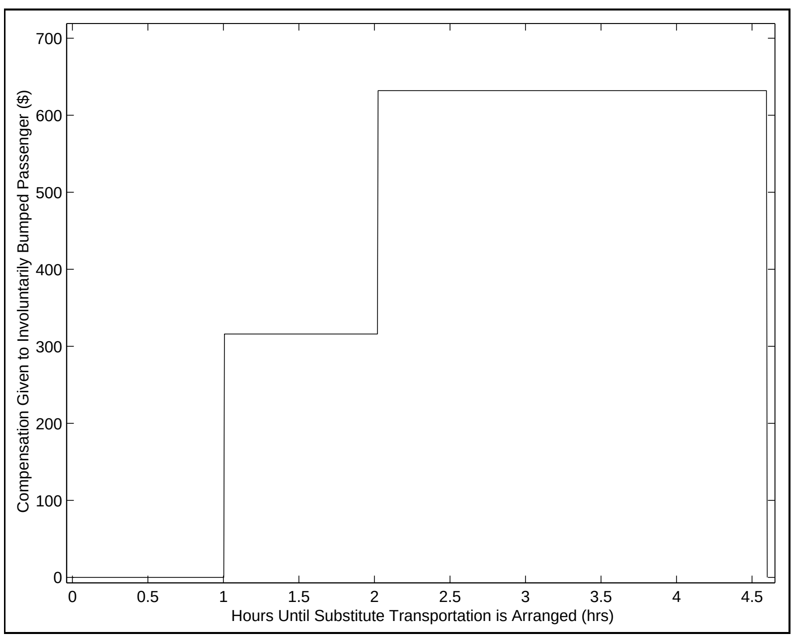

Figure 18: Compensation Schedule for Involuntarily Bumped Passengers