

Leaning In !

Combinatorial and Positional Games

Yves Jäckle

TU Berlin



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Pick Up Bricks

Pick Up Bricks

- ▶ 2 Players
- ▶ take turns picking 1 or 2 bricks from a stack of n bricks
- ▶ last player to pick a brick wins

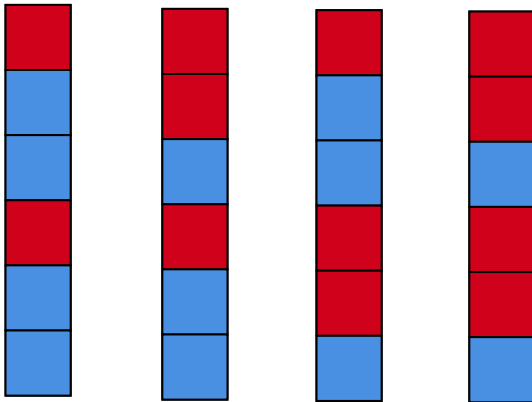
Pick Up Bricks

Let's play !



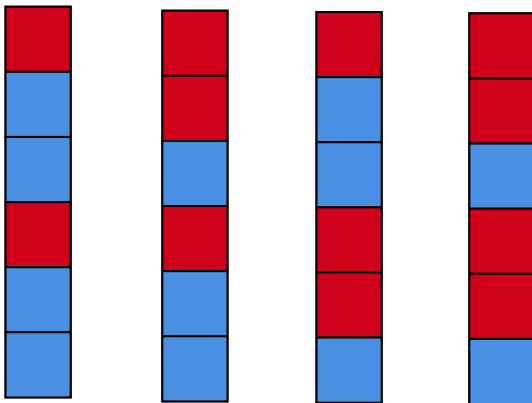
Pick Up Bricks

... didn't even have a chance !



Pick Up Bricks

Initial number of bricks is divisible by 3 \Rightarrow second player has a *winning strategy*



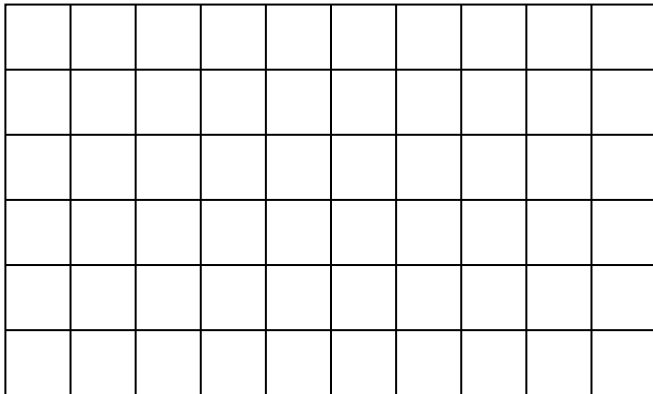
Chomp

Chomp

- ▶ 2 Players take turns selecting tiles of an $n \times m$ board
- ▶ All tiles "dominated" by the selected one get "chomped" off
 $\rightarrow (x, y)$ is dominated by $(a, b) \Leftrightarrow x \geq a$ and $y \geq b$
- ▶ Last player to select a tile *loses*

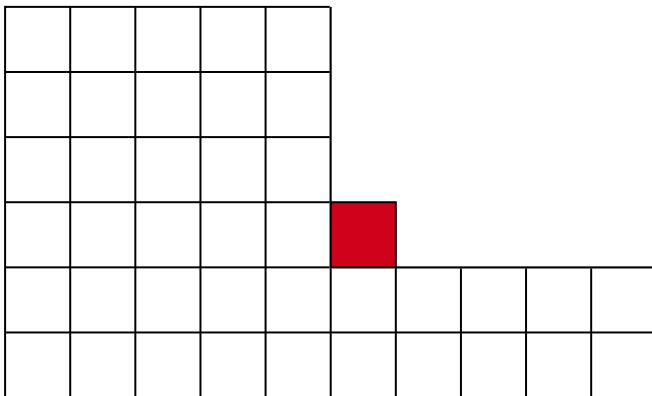
Chomp

Example of a few moves.



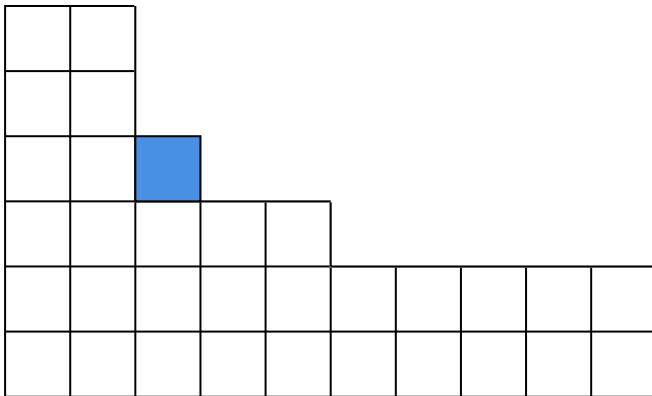
Chomp

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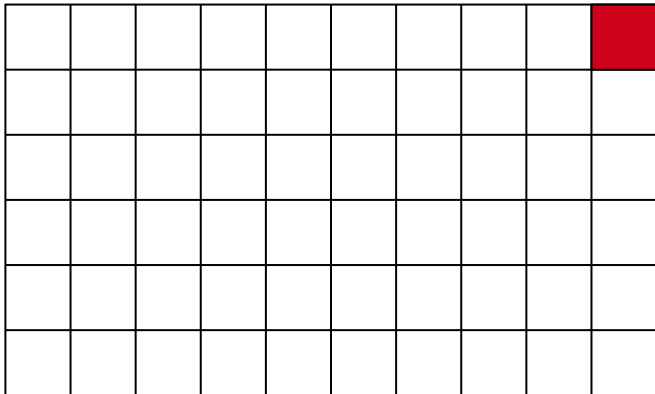
Chomp

Example of a few moves.



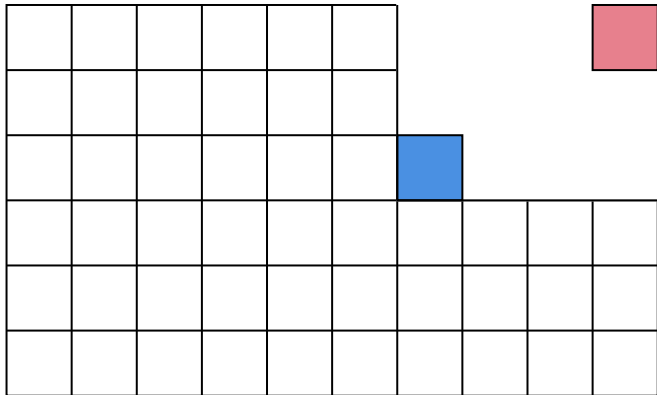
Chomp

Strategy stealing for Chomp.



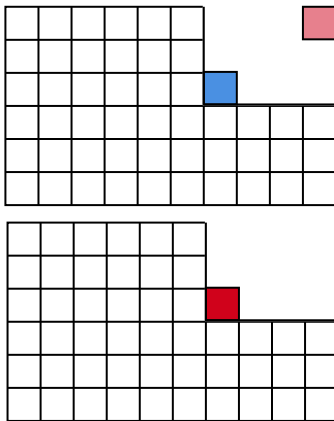
Chomp

Strategy stealing for Chomp.



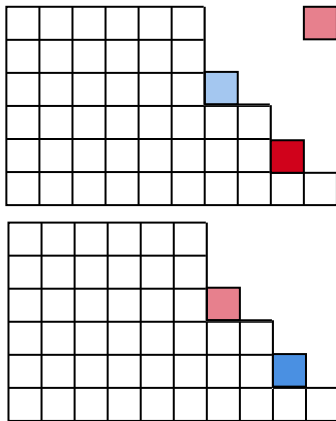
Chomp

Strategy stealing for Chomp.



Chomp

Strategy stealing for Chomp.



Tic-Tac-Toe

Tic-Tac-Toe

If you don't know this game, then you've never lived

Tic-Tac-Toe
It's a draw, right ?

Tic-Tac-Toe

- ▶ 3×3 is indeed a draw (enumerate !)
- ▶ And it stays that way in $n \times n$ from 5 onward, due to *pairing strategies*:

$$\begin{bmatrix} 11 & 1 & 8 & 1 & 12 \\ 6 & 2 & 2 & 9 & 10 \\ 3 & 7 & * & 9 & 3 \\ 6 & 7 & 4 & 4 & 10 \\ 12 & 5 & 8 & 5 & 11 \end{bmatrix}$$

Tic-Tac-Toe

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- ▶ What about ... in 3D ?

Tic-Tac-Toe

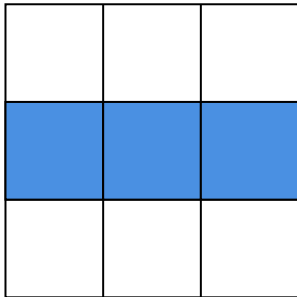
Combinatorial lines in dimension d on a cube of length n :

- ▶ image of a sequence of points
- ▶ coordinate sequences are of for $1, 2, \dots, n$ or $n, n - 1, \dots, 1$ or x, x, \dots, x for some $1 \leq x \leq n$
- ▶ not all sequences are constant (else, it's a point)

Tic-Tac-Toe

Combinatorial lines in dimension d on a cube of length n :

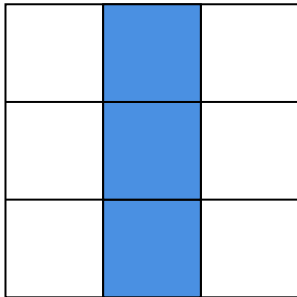
$(1, 2), (2, 2), (3, 2)$ or $(3, 2), (2, 2), (1, 2)$



Tic-Tac-Toe

Combinatorial lines in dimension d on a cube of length n :

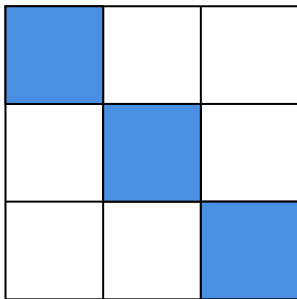
$(2, 1), (2, 2), (2, 3)$ or $(2, 3), (2, 2), (2, 1)$



Tic-Tac-Toe

Combinatorial lines in dimension d on a cube of length n :

$(1, 3), (2, 2), (3, 1)$ or $(3, 1), (2, 2), (1, 3)$



Tic-Tac-Toe

Combinatorial lines in dimension d on a cube of length n :

$(1, 3, 2), (2, 2, 2), (3, 1, 2)$

... left to the imagination

Tic-Tac-Toe

The theorem

If $n \geq 3^d - 1$, then Tic-Tac-Toe admits pairing strategies, and is therefore a draw.

Tic-Tac-Toe

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If $n \geq 3^d - 1$, then Tic-Tac-Toe admits pairing strategies, and is therefore a draw.

Sketch of the proof

- ▶ Double-counting : at most $\frac{3^d-1}{2}$ lines incident to a point
- ▶ Consider bipartite graph with one vertex per point, two per line, and an edge between them if the point is on the line.
- ▶ If $\frac{n}{3^d-1} \geq 1$, *Hall's theorem* applies and we get a matching

Thank you for your time!
Do you have any questions ?