Leaning In !

Combinatorial and Positional Games

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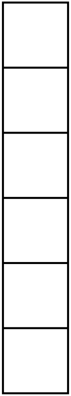
Pick Up Bricks

Pick Up Bricks

- 2 Players
- take turns picking 1 or 2 bricks from a stack of n bricks
- last player to pick a brick wins

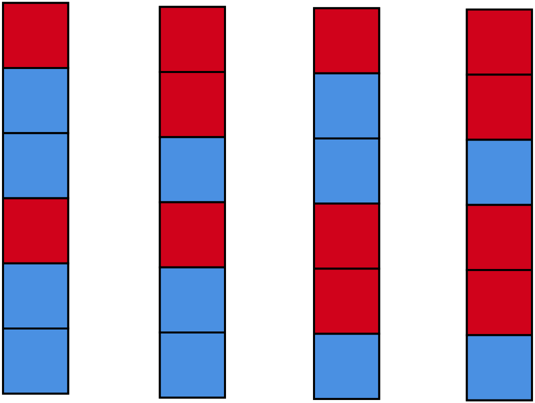
Pick Up Bricks

Let's play !



Pick Up Bricks

... didn't even have a chance !



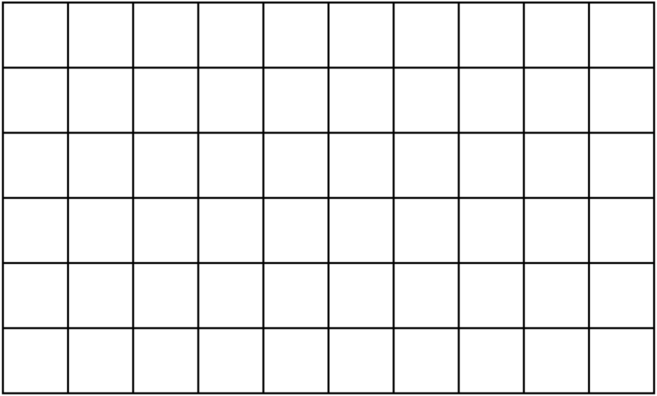
Chomp

Chomp

- 2 Players take turns selecting tiles of an $n \times m$ board
- All tiles "dominated" by the selected one get "chomped" off
  $\rightarrow (x, y)$ is dominated by $(a, b) \Leftrightarrow x \geqslant a$ and $y \geqslant b$
- Last player to select a tile *loses*

Chomp

Example of a few moves.



Chomp

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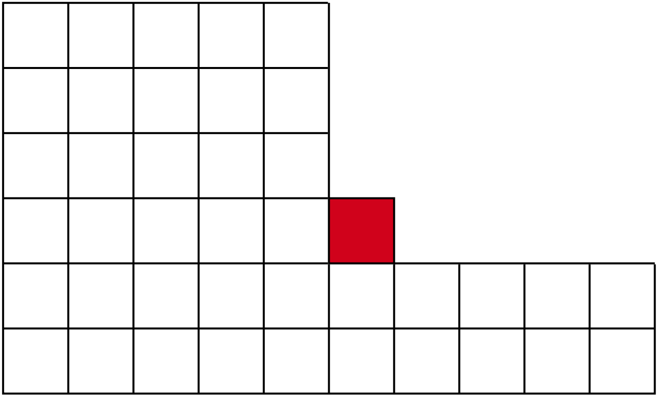

Chomp

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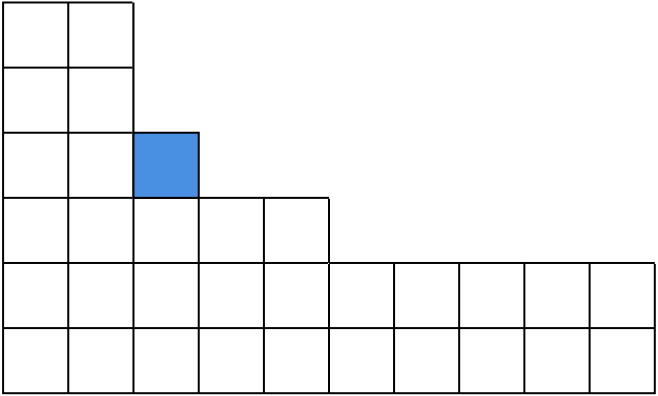
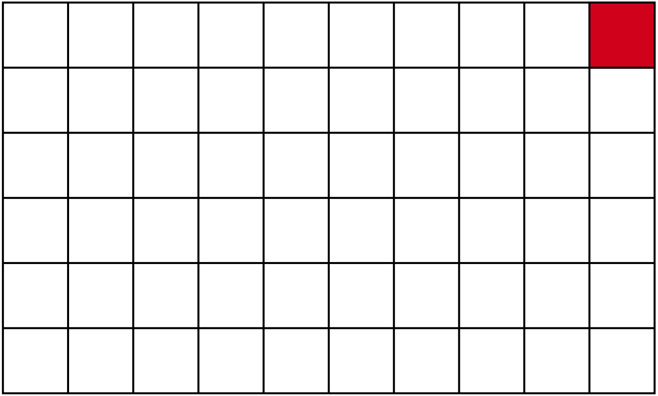


Chomp

Strategy stealing for Chomp.



Chomp

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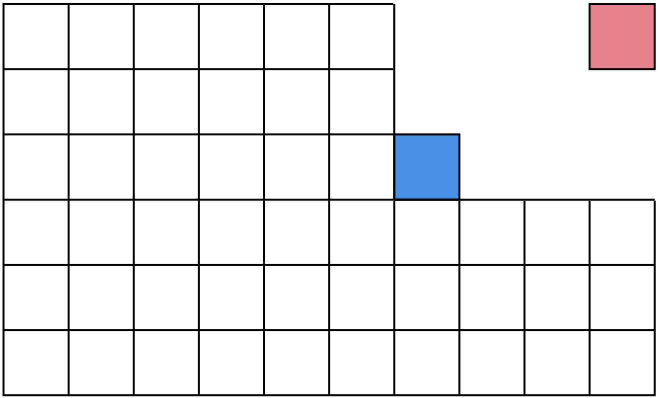

Chomp

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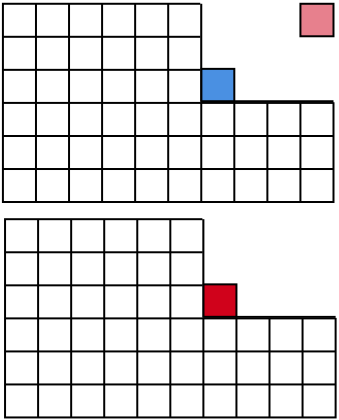
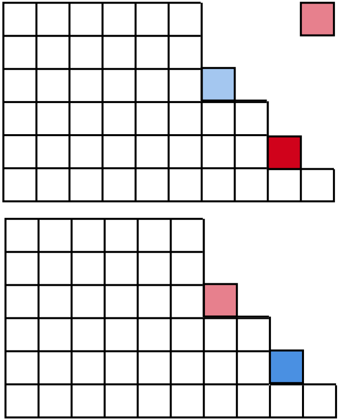


Chomp

Strategy stealing for Chomp.



Tic-Tac-Toe

Tic-Tac-Toe

If you don't know this game, then you've never lived

Tic-Tac-Toe

It's a draw, right ?

Tic-Tac-Toe

- 3×3 is indeed a draw (enumerate !)
- And it stays that way in $n \times n$ from 5 onward, due to *pairing strategies*:

$$\begin{bmatrix} 11 & 1 & 8 & 1 & 12 \\ 6 & 2 & 2 & 9 & 10 \\ 3 & 7 & * & 9 & 3 \\ 6 & 7 & 4 & 4 & 10 \\ 12 & 5 & 8 & 5 & 11 \end{bmatrix}$$

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- What about ... in 3D ?

Tic-Tac-Toe

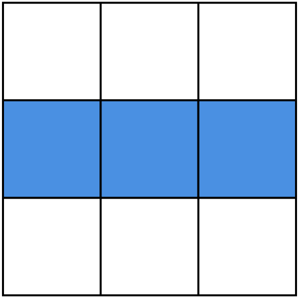
Combinatorial lines in dimension d on a cube of length n:

- image of a sequence of points
- coordinate sequences are of for $1, 2, \ldots, n$ or $n, n-1, \ldots, 1$ or $x, x, \ldots, x$ for some $1 \leqslant x \leqslant n$
- not all sequences are constant (else, it's a point)

Tic-Tac-Toe

Combinatorial lines in dimension d on a cube of length n:

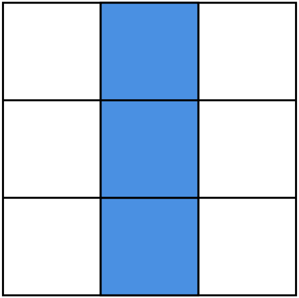
$$(1,2),(2,2),(3,2) \text{ or } (3,2),(2,2),(1,2)$$



Tic-Tac-Toe

Combinatorial lines in dimension d on a cube of length n:

$$(2,1),(2,2),(2,3) \text{ or } (2,3),(2,2),(2,1)$$



Tic-Tac-Toe

Combinatorial lines in dimension d on a cube of length n:

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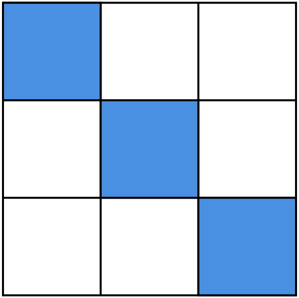

Tic-Tac-Toe

Combinatorial lines in dimension d on a cube of length n:

$$(1, 3, 2), (2, 2, 2), (3, 1, 2)$$

... left to the imagination

Tic-Tac-Toe

The theorem

If $n \geqslant 3^d - 1$*, then Tic-Tac-Toe admits pairing strategies, and is therefore a draw.*

Tic-Tac-Toe

The theorem

If $n \geqslant 3^d - 1$, then Tic-Tac-Toe admits pairing strategies, and is therefore a draw.

Sketch of the proof

- Double-counting : at most $\frac{3^d-1}{2}$ lines incident to a point
- Consider bipartite graph with one vertex per point, two per line, and an edge between them if the point is on the line.
- If $\frac{n}{3^d-1} \geqslant 1$, *Hall's theorem* applies and we get a matching