Expected IL for Uniswap v3 positions

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1 Introduction

Impermament loss (IL) or divergence-loss, is a phenomena experienced by LP providers on almost all types of AMM platforms, especially when providing liquidity for non-mean reverting assets. This phenomena has been studied extensively because of its negative impact on LP providers' returns [Loe+21] [DZ22]. When modelling an asset's price with a geometric brownian motion (GBM), IL can be estimated numerically by using iterative techniques such as the Monte Carlo method. The result of this estimation is referred to as "expected impermanent loss" (EIL). Previous work wrongly concluded that the computation of EIL for Uniswap v3 positions is analytically intractable, due to the complexity of the math expressions involving the GBM [Dan22]. However, a recent paper provided a framework for the analytical computation of EIL for Uniswap v3 positions [HP22]. We implement the ideas presented in the paper and compare the results with the results obtained from the Monte Carlo simulation.

1.1 Simulation starting parameters

- Number of simulations: 100000
- Total time: 168 (Hours)
- Initial price: 1000 (Units of numéraire asset)
- Drift: 0.000075
- Standard deviation: 0.001

2 Technical details

2.1 Expected IL

The integrals in the analytical computation of EIL are calculated using the "*integrate*" R function, which is based on QUADPACK routines dqags and dqagi by R. Piessens and E. deDoncker–Kapenga [PU83].

```r
function(f, lb, ub, P, Pa, Pb, mu, sigma, T) {

    integral <- integrate(
        f,
        # Lower integration bound
        lb,
        # Upper integration bound
        ub,
        subdivisions = 5000,
        abs.tol = 1e-5,
        rel.tol = 1e-7,
        stop.on.error = F,
        T,
        P,
        Pa,
        Pb,
        mu,
        sigma
    )$value;

    coefficient <- (1 / (sigma * sqrt(2 * pi * T)))
    return(coefficient * integral)
}
```

2.2 Time in-the-money

The two-dimensional integrals for the computation of "*time in-the-money*" ($\mathscr{T}$) are calculated using the dblquad function from the pracma library.

```r
function(P, Pa, Pb, mu, sigma, T) {

    lower_bound <- log(Pa / P)
    upper_bound <- log(Pb / P)

    f <- function(x, y, mu, sigma) {
        t <- y
        first_term <- -(x - (mu - (sigma^2 / 2)) * t)^2
        second_term <- (2 * t * (sigma^2))
        exp(first_term / second_term) / sqrt(t)
    }

    integral <- dblquad(
        f,
        # Lower integration bound
        log(Pa / P),
        # Upper integration bound
        log(Pb / P),
        0,
        T,
        tol=1e-5,
        subdivs=5000,
        mu=mu,
        sigma=sigma
    )

    result <- (1 / (sigma * (sqrt(2 * pi)))) * integral[1]

    return(result)
}
```

3 Simulation results

3.1 Results for 7 days

Table 1: Results for setting μ = 7.5e-5, 1.5e-4 and σ = 0.001 and σ = 0.002 for 7 days (T = 168 hours) of providing liquidity with initial price \$1000.

		$\sigma = 0.001$				$\sigma = 0.002$		
			Formula		Simulation	Formula		Simulation
μ	r	CE	$\mathscr{T}$	EIL	IL	$\mathscr{T}$	EIL	IL
7.50e-05	1.001	2000	16.92	-0.0072	-0.0070	9.72	-0.0112	-0.0111
	1.010	200	110.61	-0.0052	-0.0048	77.77	-0.0092	-0.0089
	1.100	21	168.00	-8.71e-04	-8.71e-04	167.99	-0.0022	-0.0014
	1.200	11	168.00	-4.66e-04	-4.66e-04	168.00	-0.0012	-1.18e-03
	2.000	3	168.00	-1.38e-04	-1.38e-04	168.00	-3.51e-04	-3.51e-04
	5.000	2	168.00	-7.20e-05	-7.20e-05	168.00	-1.85e-04	-1.85e-04
1.50e-04	1.001	2000	11.76	-0.0124	-0.0127	8.71	-0.0145	-0.0140
	1.010	200	78.75	-0.0103	-0.0105	69.16	-0.0124	-0.0118
	1.100	21	168.00	-0.0021	-0.0018	167.94	-0.0035	-0.0022
	1.200	11	168.00	-0.0011	-1.15e-03	168.00	-0.0019	-0.0012
	2.000	3	168.00	-3.41e-04	-3.41e-04	168.00	-5.50e-04	-5.50e-04
	5.000	2	168.00	-1.31e-04	-1.31e-04	168.00	-2.91e-04	-2.91e-04

3.2 Results for 30 days

Table 2: Results for setting μ = 7.5e-5, 1.5e-4 and σ = 0.001 and σ = 0.002 for 30 days (T = 720 hours) of providing liquidity with initial price \$1000.

		$\sigma = 0.001$				$\sigma = 0.002$		
			Formula		Simulation	Formula		Simulation
μ	r	CE	$\mathscr{T}$	EIL	IL	$\mathscr{T}$	EIL	IL
7.50e-05	1.001	2000	21.27	-0.0130	-0.0131	13.36	-0.0175	-0.0154
	1.010	200	154.71	-0.0109	-0.0109	113.66	-0.0154	-0.0132
	1.100	21	336.00	-0.0026	-0.0019	334.31	-0.0052	-0.0026
	1.200	11	336.00	-0.0014	-0.0010	336.00	-0.0028	-0.0014
	2.000	3	336.00	-4.11e-04	-4.11e-04	336.00	-8.30e-04	-8.30e-04
	5.000	2	336.00	-2.18e-04	-2.18e-04	336.00	-4.40e-04	-4.40e-04
1.50e-04	1.001	2000	12.39	-0.0249	-0.0257	10.91	-0.0261	-0.0276
	1.010	200	86.72	-0.0226	-0.0235	91.18	-0.0239	-0.0254
	1.100	21	335.83	-0.0077	-0.0072	329.03	-0.0101	-0.0083
	1.200	11	336.00	-0.0041	-0.0039	336.00	-0.0055	-0.0044
	2.000	3	336.00	-0.0012	-0.0011	336.00	-0.0016	-0.0013
	5.000	2	336.00	-6.46e-04	-6.46e-04	336.00	-8.62e-04	-8.62e-04

Table 3: Expected price of the asset, for 7 ($\mathscr{T}$ = 168 hours) and 30 days ($\mathscr{T}$ = 720 hours).

μ	$\mathscr{T} = 168$	$\mathscr{T} = 720$
0.000075	1012.68	1055.485
0.000150	1025.52	1114.048

References

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