Expected IL for Uniswap v3 positions

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Contents

1 Introduction

Impermament loss (IL) or divergence-loss, is a phenomena experienced by LP providers on almost all types of AMM platforms, especially when providing liquidity for non-mean reverting assets. The effects of IL have been studied extensively because of their negative impact on LP providers' returns [Loe+21]. When modelling an asset's price with a geometric brownian motion (GBM), IL can be estimated numerically by using iterative techniques such as the Monte Carlo method. The result of this estimation is referred to as "expected impermanent loss" (EIL). A fellow scholar wrongly concluded that the computation of EIL for Uniswap v3 positions is analytically intractable, due to the complexity of the math expressions involving the GBM [Dan22]. However, a recent paper provided a framework for the analytical computation of EIL for Uniswap v3 positions [HP22]. Previous work [DZ22] introduced the "unit impermanent loss per liquidity" (UIL) and static replication formulas. We implement these ideas and present the results obtained with Monte Carlo simulations.

1.1 Simulation parameters

- Number of simulations: 1000
- Initial price: 1000 (Units of numéraire asset, e.g. DAI or USDC)
- Drift (μ): 0.000075
- Standard deviation (σ): 0.001

2 Technical details

2.1 Expected IL

The integrals in the analytical computation of EIL are calculated using the "*integrate*" R function, which is based on QUADPACK routines dqags and dqagi by R. Piessens and E. deDoncker–Kapenga [PU83].

```
function(f, lb, ub, P, Pa, Pb, mu, sigma, T) {

    integral <- integrate(
        f,
        # Lower integration bound
        lb,
        # Upper integration bound
        ub,
        subdivisions = 5000,
        abs.tol = 1e-5,
        rel.tol = 1e-7,
        stop.on.error = F,
        T,
        P,
        Pa,
        Pb,
        mu,
        sigma
    )$value;

    coefficient <- (1 / (sigma * sqrt(2 * pi * T)))
    return(coefficient * integral)
}
```

2.2 Time in-the-money

The two-dimensional integrals for the computation of "*time in-the-money*" ($\mathscr{T}$) are calculated using the dblquad function from the pracma library.

```
function(P, Pa, Pb, mu, sigma, T) {

    lower_bound <- log(Pa / P)
    upper_bound <- log(Pb / P)

    f <- function(x, y, mu, sigma) {
        t <- y
        first_term <- -(x - (mu - (sigma^2 / 2)) * t)^2
        second_term <- (2 * t * (sigma^2))
        exp(first_term / second_term) / sqrt(t)
    }

    integral <- dblquad(
        f,
        # Lower integration bound
        log(Pa / P),
        # Upper integration bound
        log(Pb / P),
        0,
        T,
        tol=1e-5,
        subdivs=5000,
        mu=mu,
        sigma=sigma
    )

    result <- (1 / (sigma * (sqrt(2 * pi)))) * integral[1]

    return(result)
}
```

3 Monte Carlo simulation results

3.1 EIL for 168 hours (7 days)

Table 1: Results for setting μ = 7.5e-5, 1.5e-4 and σ = 0.001 and σ = 0.002 for 7 days (T = 168 hours) of providing liquidity with initial price $1000.

			$\sigma = 0.001$			$\sigma = 0.002$		
			Formula		Simulation	Formula		Simulation
μ	r	CE	$\mathscr{T}$	EIL	IL	$\mathscr{T}$	EIL	IL
7.50e-05	1.001	2000	16.92	-0.0072	-7.16e-03	9.72	-0.0112	-1.12e-02
	1.010	200	110.61	-0.0052	-5.23e-03	77.77	-0.0092	-9.21e-03
	1.100	21	168.00	-8.71e-04	-8.71e-04	167.99	-0.0022	-2.21e-03
	1.200	11	168.00	-4.66e-04	-4.66e-04	168.00	-0.0012	-1.18e-03
	2.000	3	168.00	-1.38e-04	-1.38e-04	168.00	-3.51e-04	-3.51e-04
	5.000	2	168.00	-7.20e-05	-7.20e-05	168.00	-1.85e-04	-1.85e-04
1.50e-04	1.001	2000	11.76	-0.0124	-1.24e-02	8.71	-0.0145	-1.45e-02
	1.010	200	78.75	-0.0103	-1.03e-02	69.16	-0.0124	-1.24e-02
	1.100	21	168.00	-0.0021	-2.14e-03	167.94	-0.0035	-3.46e-03
	1.200	11	168.00	-0.0011	-1.15e-03	168.00	-0.0019	-1.85e-03
	2.000	3	168.00	-3.41e-04	-3.41e-04	168.00	-5.50e-04	-5.50e-04
	5.000	2	168.00	-1.31e-04	-1.31e-04	168.00	-2.91e-04	-2.91e-04

3.2 EIL for 720 hours (30 days)

Table 2: Results for setting μ = 7.5e-5, 1.5e-4 and σ = 0.001 and σ = 0.002 for 30 days (T = 720 hours) of providing liquidity with initial price $1000.

			$\sigma = 0.001$			$\sigma = 0.002$		
			$\sigma = 0.001$			$\sigma = 0.002$		
μ	r	CE	$\mathscr{T}$	EIL	IL	$\mathscr{T}$	EIL	IL
7.50e-05	1.001	2000	21.27	-0.0130	-1.30e-02	13.36	-0.0175	-1.75e-02
	1.010	200	154.71	-0.0109	-1.09e-02	113.66	-0.0154	-1.54e-02
	1.100	21	336.00	-0.0026	-2.58e-03	334.31	-0.0052	-5.19e-03
	1.200	11	336.00	-0.0014	-1.38e-03	336.00	-0.0028	-2.79e-03
	2.000	3	336.00	-4.11e-04	-4.11e-04	336.00	-8.30e-04	-8.30e-04
	5.000	2	336.00	-2.18e-04	-2.18e-04	336.00	-4.40e-04	-4.40e-04
1.50e-04	1.001	2000	12.39	-0.0249	-2.49e-02	10.91	-0.0261	-2.61e-02
	1.010	200	86.72	-0.0226	-2.26e-02	91.18	-0.0239	-2.39e-02
	1.100	21	335.83	-0.0077	-7.67e-03	329.03	-0.0101	-1.01e-02
	1.200	11	336.00	-0.0041	-4.10e-03	336.00	-0.0055	-5.47e-03
	2.000	3	336.00	-0.0012	-1.22e-03	336.00	-0.0016	-1.63e-03
	5.000	2	336.00	-6.46e-04	-6.46e-04	336.00	-8.62e-04	-8.62e-04

3.3 Expected prices

Table 3: Expected price of the asset, for 7 ($\mathscr{T}$ = 168 hours) and 30 days ($\mathscr{T}$ = 720 hours).

μ	$\mathscr{T} = 168$	$\mathscr{T} = 720$
0.000075	1012.68	1055.485
0.000150	1025.52	1114.048

3.4 Unit impermanent loss per liquidity (UIL)

Table 4 - Unit impermanent loss per liquidity (UIL).

	Formula		Static replication	
sigma	UIL^R	UIL^L	UIL^R	UIL^L
0.001	-0.0733081429997792	-0.674402053309331	-0.0733081429997886	-0.674402053309333
0.005	-0.10354377675798	-0.674610528493785	-0.10354377675799	-0.67461052849379

4 Conclusion

The method introduced by [HP22] provides a reliable way to determine the expected value of impermanent loss under a geometric brownian motion assumption. The model is sensitive to parameters such as drift and volatility as seen in Table 1 and 2. The expected value of impermanent loss computed analytically converges with the result of the Monte Carlo simulation with low error rate. [DZ22] characterizes analytically the option-like payoff structure of impermanent loss for concentrated liquidity positions and provides formulas for static replication of the impermanent loss with a combination of European options. Table 4 reports the result of the computation obtained with the analytical approach and highlights the high accuracy of the replication formulas.

References

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