Valuation of binaries

Notation and Definitions

Parameter	Description
S	Spot price of the underlying asset at time τ
K	Strike price of the option
r	Risk-free interest rate
σ	Volatility of the underlying asset
τ	Time to expiry
T	Maturity of the option
Φ	Cumulative distribution function of the standard normal distribution
C	Value of a call option
P	Value of a put option
V_{AC}	Value of an asset-or-nothing call option
V_{AP}	Value of an asset-or-nothing put option
V_{CC}	Value of a cash-or-nothing call option
V_{CP}	Value of a cash-or-nothing put option

Black-Scholes formula

The value of a call option for a non-dividend-paying underlying stock in terms of the Black–Scholes parameters is:

$$C = S\Phi(d_1) - e^{-r\tau} K\Phi(d_2)$$

$$d_2 = -\frac{\ln\left(\frac{K}{S}\right) - \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$$

$$d_1 = d_2 + \sigma\sqrt{\tau}$$

The formula can be interpreted as breaking down a call option into the difference between two distinct binary options: an asset-or-nothing call option and a cash-or-nothing call option. Essentially, it involves going long on an asset-or-nothing call option while simultaneously going short on a cash-or-nothing call option. A standard call option allows the holder to trade cash for an asset upon its expiration, whereas an asset-or-nothing call option grants the holder the asset without any cash exchanged, and a cash-or-nothing call option provides the holder with cash without transferring any asset. The Black-Scholes formula consists of the difference beteween the value of the two binary options. The payoff of the call option at maturity T is given by:

$$C_{(T)} = max\{0,\, S_{(T)} - K\} = \begin{cases} S_{(T)} - K, & S_{(T)} > K, \\ 0, & \text{otherwise.} \end{cases}$$

The first component is the payment of the exercise price contingent to the option ending in the money. The payoff of this component is:

$$C^1_{(T)} = \begin{cases} -K, & S_{(T)} > K, \\ 0, & \text{otherwise.} \end{cases}$$

The second component is the receipt of the asset, again contingent to the option ending in the money. The payoff of this component is:

$$C^2_{(T)} = \begin{cases} S_{(T)}, & S_{(T)} > K, \\ 0, & \text{otherwise.} \end{cases}$$

Asset-or-nothing Call

With this type of option, the payout is one unit of the underlying asset if the spot price S at maturity T ($S_{(T)}$) is above the strike price K or zero otherwise. The payoff of this option contingent to the receipt of the asset equals the expected value, computed using risk-adjusted probabilities and discounted ad the risk-less rate. The expected future value of this option is the product of the partial expectation of the asset price at exercise and the probability of the exercise event.

$$V_{AC_{(T)}} = \begin{cases} S_{(T)}, & S_{(T)} > K, \\ 0, & \text{otherwise.} \end{cases}$$

$$V_{AC} = \mathrm{e}^{-r\tau} \underbrace{\mathbb{E}[S_{(T)} \mathbb{1}_{[S_{(T)}>K]}]}_{\text{Partial expectation}} \underbrace{\mathbb{P}\{S_{(T)} > K\}}_{\text{Probability of exercise}}$$

$$\mathbb{E}[S_{(T)} \mathbb{1}_{[S_{(T)}>K]}] \mathbb{P}\{S_{(T)} > K\} = S\mathrm{e}^{r\tau}\Phi(d_1)$$

$$V_{AC} = \mathrm{e}^{-r\tau} S\mathrm{e}^{r\tau}\Phi(d_1) = S\mathrm{e}^{-r\tau}\Phi(d_1) \tag{1}$$

$\Phi(d_1)$ is a measure by which the discounted expected value of contingent receipt of the asset exceeds the present value of the asset.

Asset-or-nothing Put

With this type of option, the payout is one unit of the underlying asset if the spot price S at maturity is below the strike price K or zero otherwise.

$$V_{AP_{(T)}} = \begin{cases} S_{(T)}, & \text{if } S_{(T)} < K, \\ 0, & \text{otherwise.} \end{cases}$$

Using put-call parity, we can derive the value of an asset-or-nothing binary put option (V_{AP}) by substituting (1) in the formula for V_{AP}:

$$\begin{aligned} V_{AP} &= S\mathrm{e}^{-r\tau} - V_{AC} \\ &= S\mathrm{e}^{-r\tau} - S\mathrm{e}^{-r\tau}\Phi(d_1) \\ &= S\mathrm{e}^{-r\tau}(1-\Phi(d_1)) \\ &= S\mathrm{e}^{-r\tau}\Phi(-d_1) \end{aligned} \tag{2}$$

Cash-or-nothing Call

For cash-or-nothing call, the payout is one unit of risk-less asset if the spot price S at maturity T is above the strike price K or zero otherwise. Since this type of option does not involve receiving an asset, the payoff does not depend on the expectation of the asset price at exercise but rather on the probability of the exercise event.

$$V_{CC_{(T)}} = \begin{cases} 1, & \text{if } S_{(T)} > K, \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{aligned} V_{CC} &= \mathrm{e}^{-r\tau}\mathbb{E}[v_{CC}(T, S_{(T)})] \\ &= \mathrm{e}^{-r\tau}\mathbb{E}[\mathbb{1}_{[S_{(T)}>K]}] \\ &= \mathrm{e}^{-r\tau}\underbrace{\mathbb{P}[S_{(T)} > K]}_{\text{Tail probability}} \end{aligned}$$

$$\mathbb{P}[S_{(T)} > K] = \Phi(d_2)$$

$$V_{CC} = \mathrm{e}^{-r\tau}\Phi(d_2) \tag{3}$$

The value of a this option equals the risk-free compounding factor multiplied by $\Phi(d_2)$, the risk-adjusted probability that the option will be exercised.

Cash-or-nothing Put

With this type of option, the payout is one unit of risk-less asset if the spot price S at maturity T is below the strike price K or zero otherwise.

$$V_{CP_{(T)}} = \begin{cases} 1, & \text{if } S_{(T)} < K, \\ 0, & \text{otherwise.} \end{cases}$$

Using put-call parity, we can derive the value of an cash-or-nothing binary put option (V_{CP}) by substituing the value of V_{CC} in the formula for V_{CP}:

$$\begin{aligned} V_{CP} &= \mathrm{e}^{-r\tau} - V_{CC} \\ &= \mathrm{e}^{-r\tau} - \mathrm{e}^{-r\tau}\Phi(d_2) \\ &= \mathrm{e}^{-r\tau}(1 - \Phi(d_2)) \\ &= \mathrm{e}^{-r\tau}\Phi(-d_2) \end{aligned} \tag{4}$$