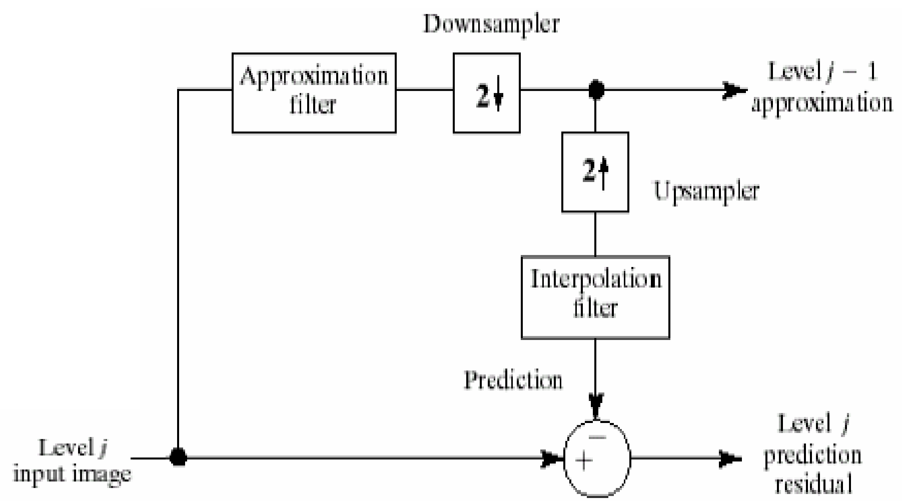
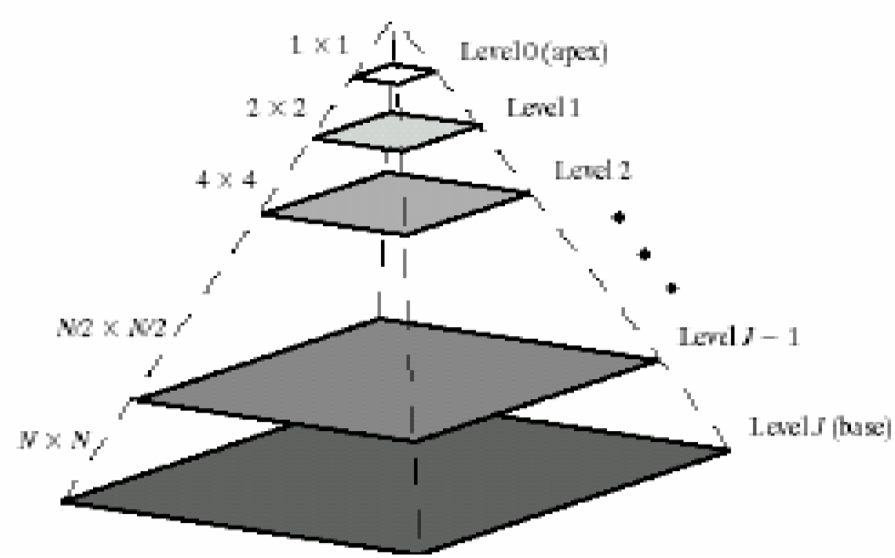
Multiresolution Processing

- Gaussian and Laplacian pyramid
- Discrete Wavelet Transform (DWT)
  - Two-channel filterbank with perfect reconstruction
  - Lifting implementation
  - Conjugate quadrature filters
- Wavelet theory
  - Wavelet basis
  - Scaling function and wavelet function

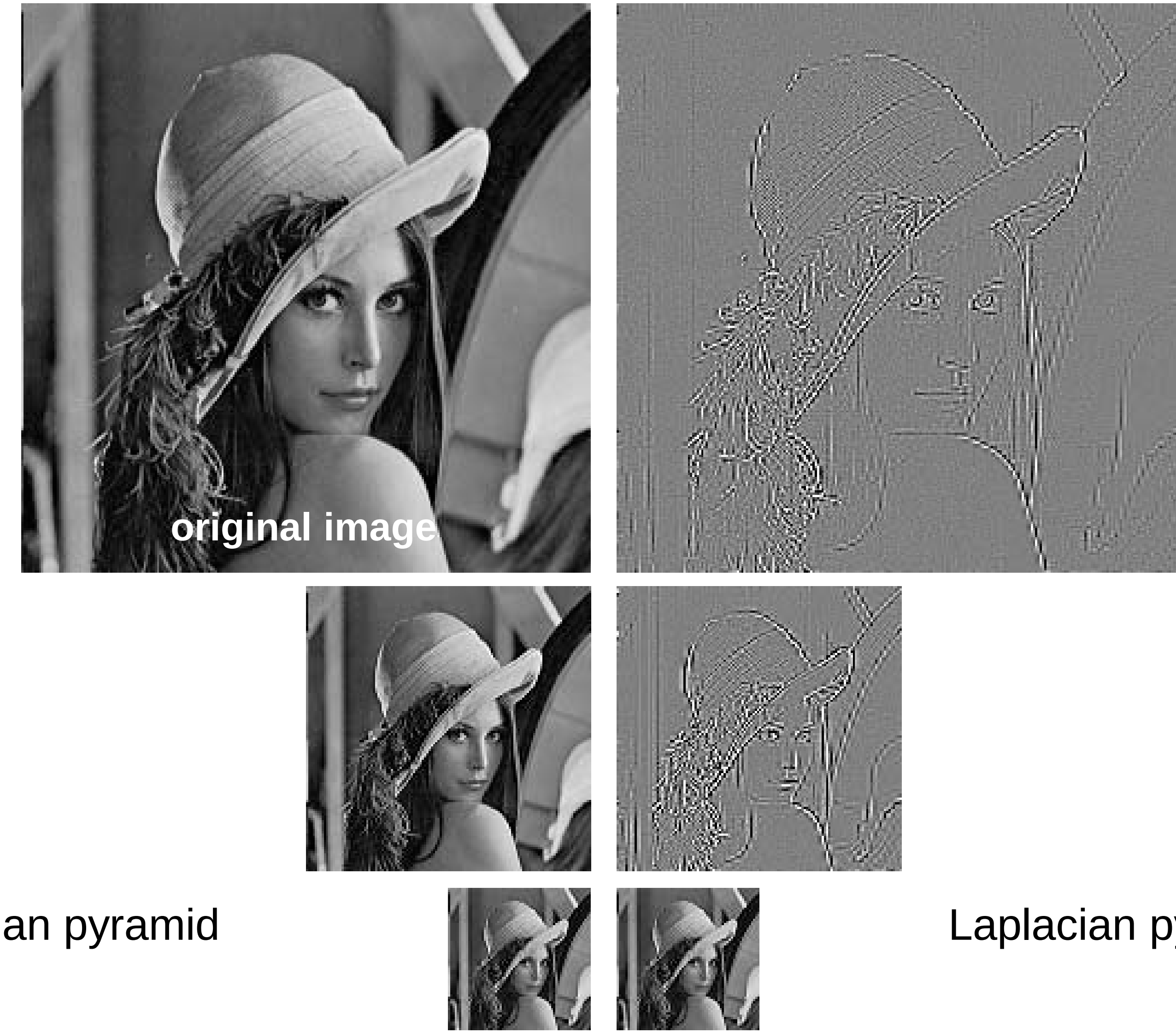
Image Pyramids





[Burt, Adelson, 1983]

Image Pyramid Example



Gaussian pyramid

Laplacian pyramid

Overcomplete Representation

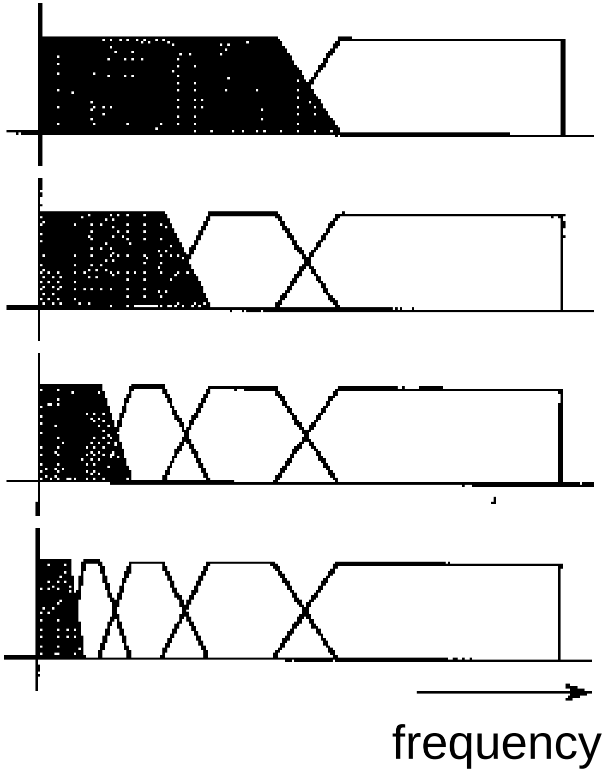
Number of samples to be encoded =

$$\left(1+\frac{1}{N}+\frac{1}{N^2}+\ldots\right)=\frac{N}{N-1}$$ x number of original image samples

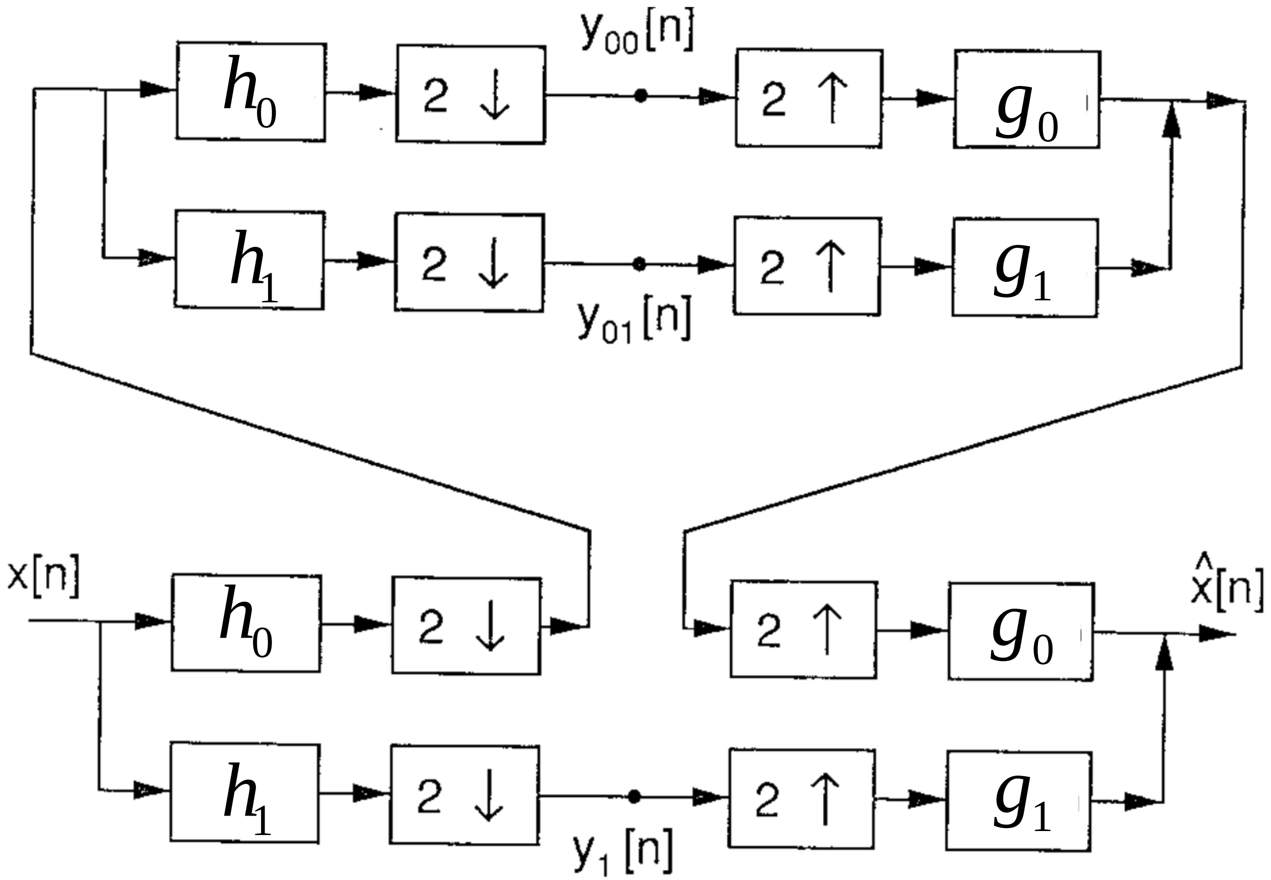
subsampling factor

1-D Discrete Wavelet Transform (DWT)

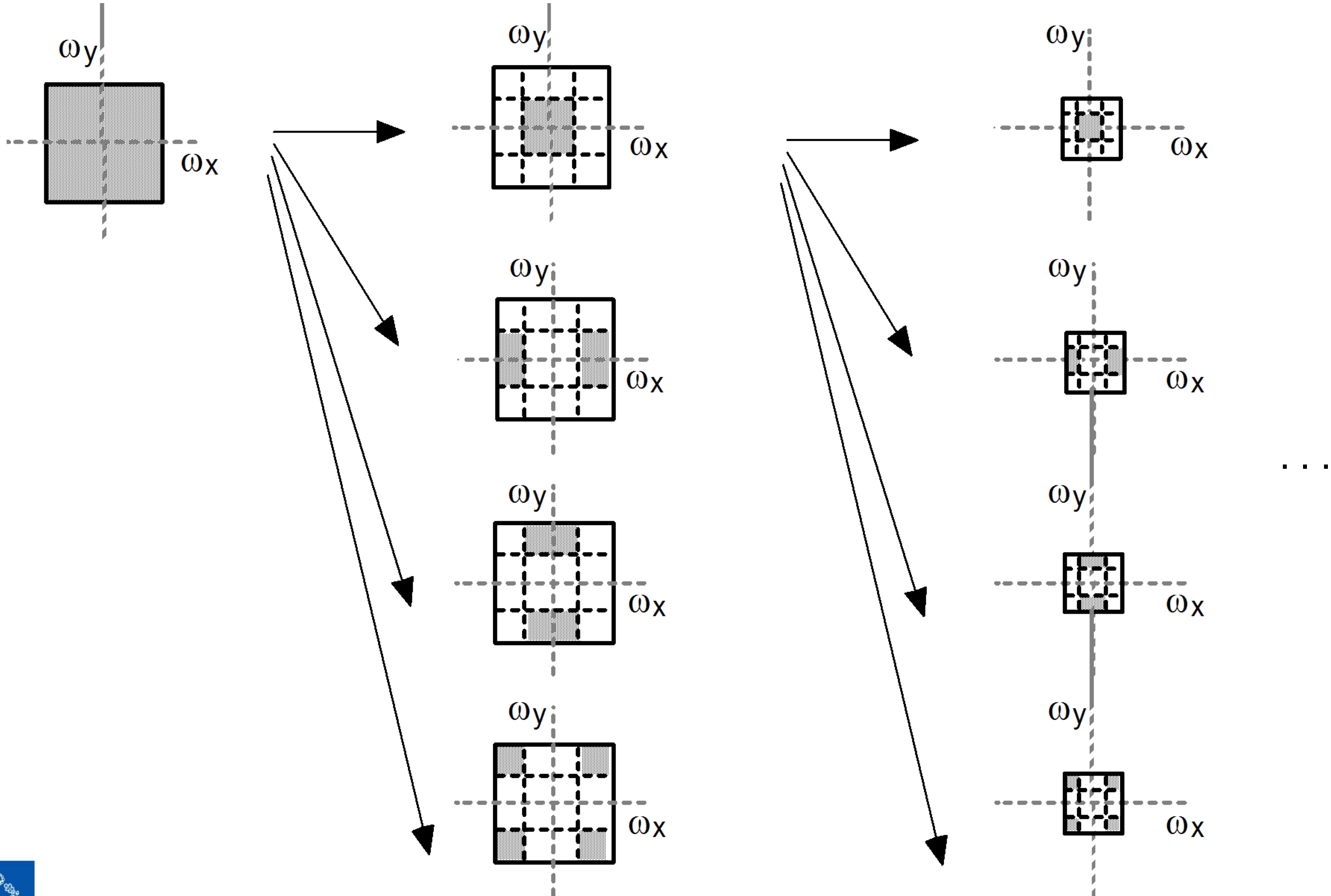
- Recursive application of a two-band filter bank to the lowpass band of the previous stage yields octave band splitting:



Cascaded Analysis / Synthesis Filterbanks



2-D Discrete Wavelet Transform



2-D Discrete Wavelet Transform Example

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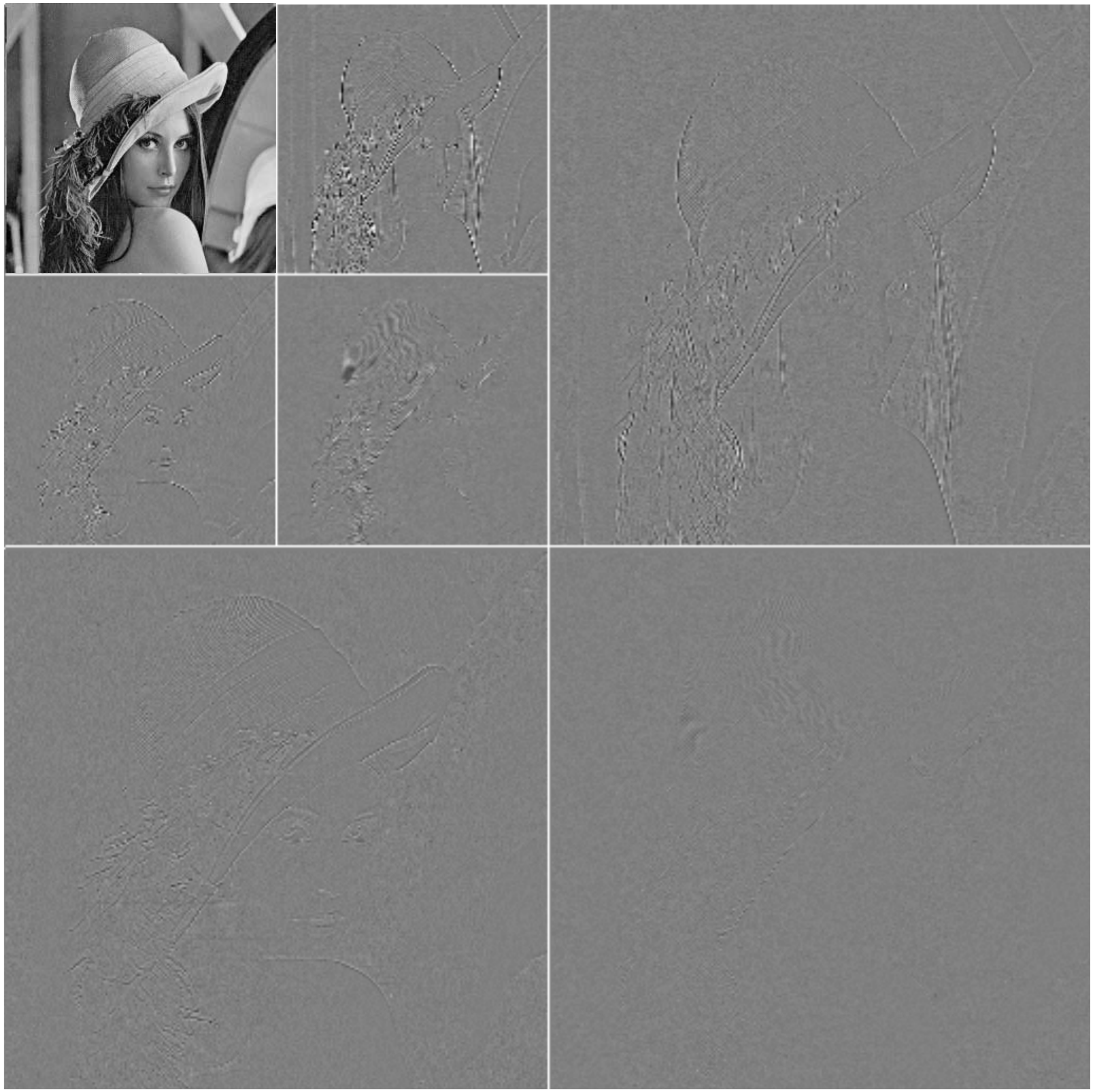
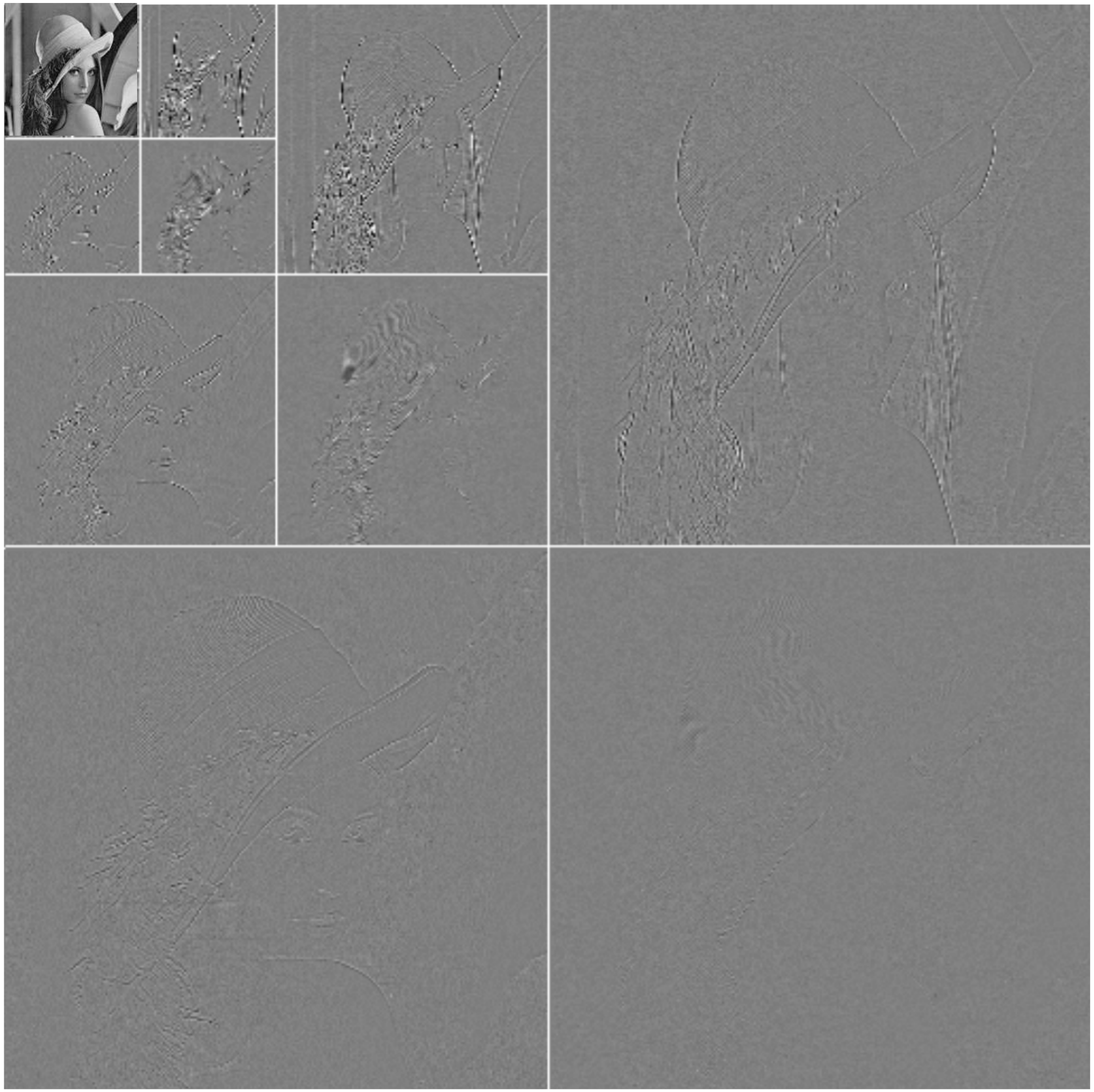
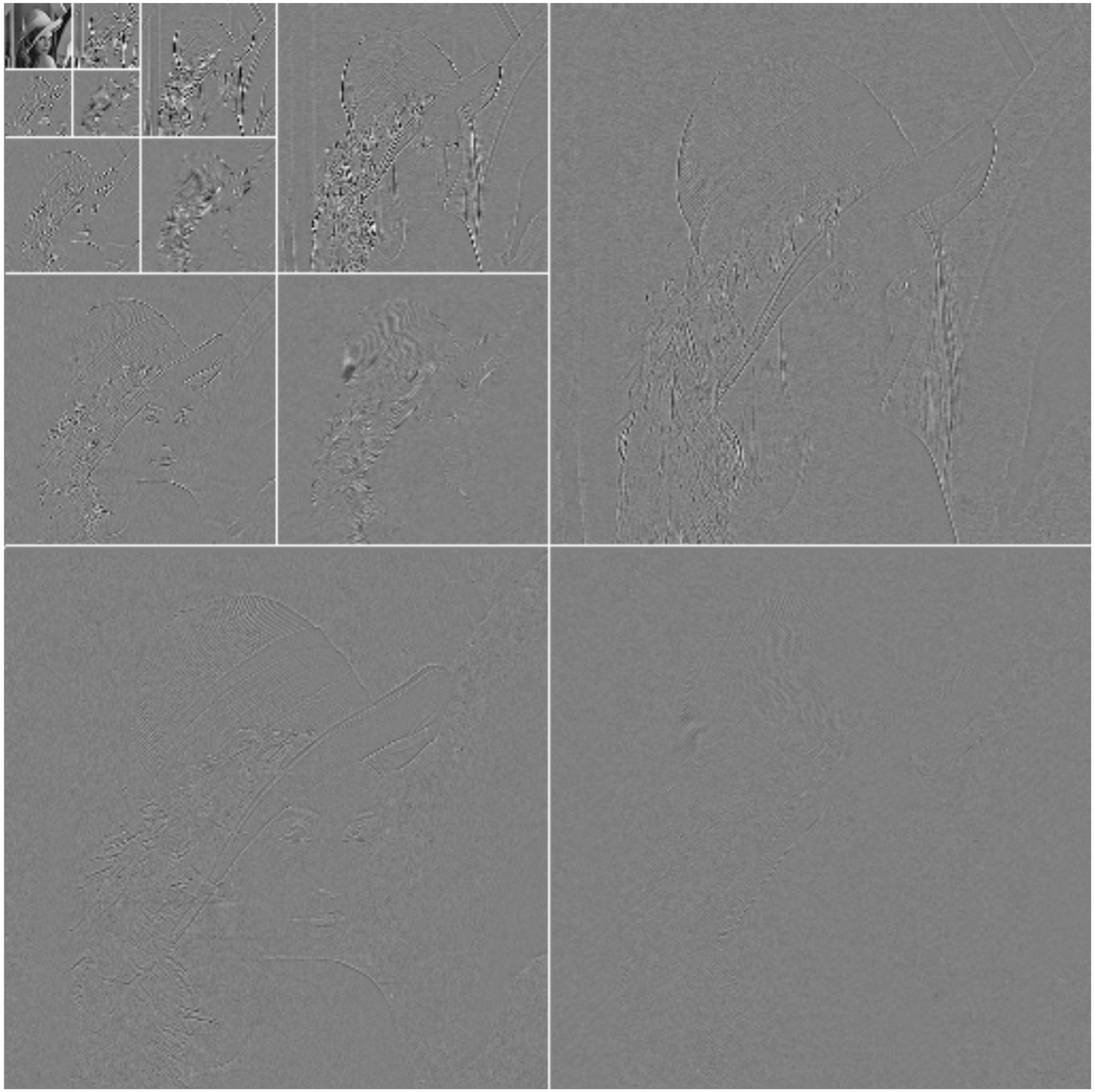


2-D Discrete Wavelet Transform Example



2-D Discrete Wavelet Transform Example

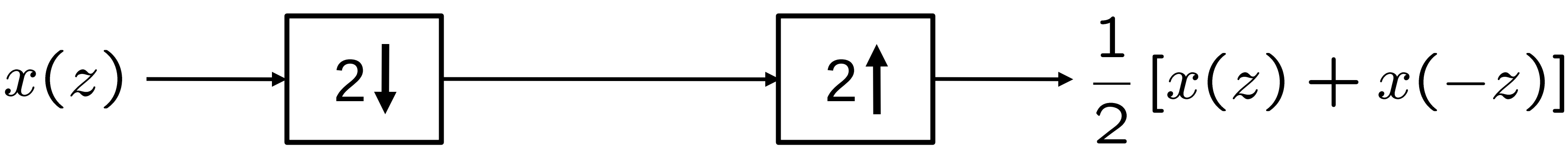


Review: Z-Transform and Subsampling

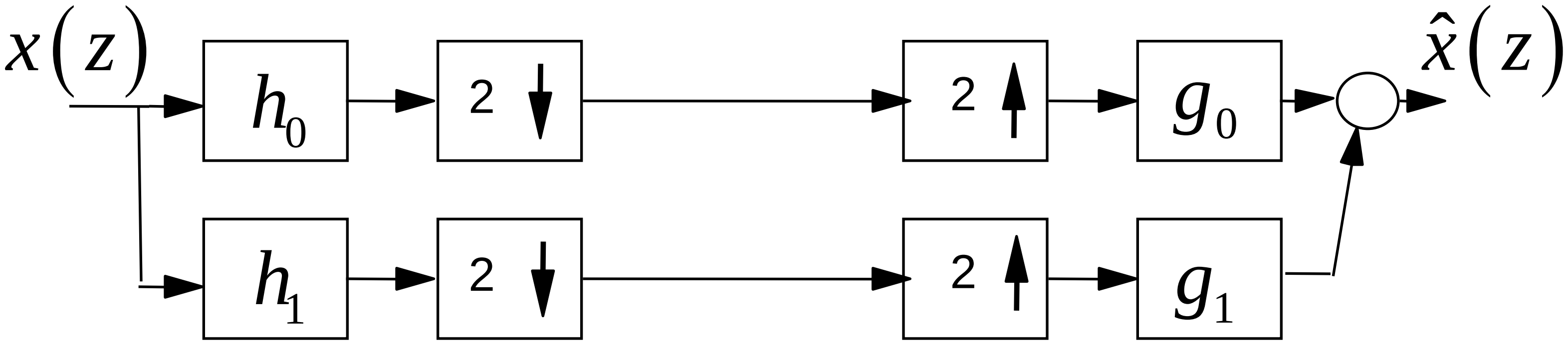
- Generalization of the discrete-time Fourier transform

$$x(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad z \in \mathcal{C} \qquad r^- < |z| < r^+$$

- Fourier transform on unit circle: Substitute $z=e^{j\omega}$
- Downsampling and upsampling by factor 2



Two-Channel Filterbank



$$\hat{x}(z) = \frac{1}{2}\left[h_0(z)g_0(z) + h_1(z)g_1(z)\right]x(z)$$

$$+\frac{1}{2}\left[h_0(-z)g_0(z) + h_1(-z)g_1(z)\right]x(-z)$$

Aliasing

- Aliasing cancellation if :

$$g_0(z) = h_1(-z)$$
$$-g_1(z) = h_0(-z)$$

Example: 2-Channel FB with Perfect Reconstruction

- Impulse responses, analysis filters:

  Lowpass highpass

$$\left(\frac{-1}{4},\frac{1}{2},\frac{3}{2},\frac{1}{2},\frac{-1}{4}\right) \qquad \left(\frac{-1}{4},\frac{1}{2},\frac{-1}{4}\right)$$

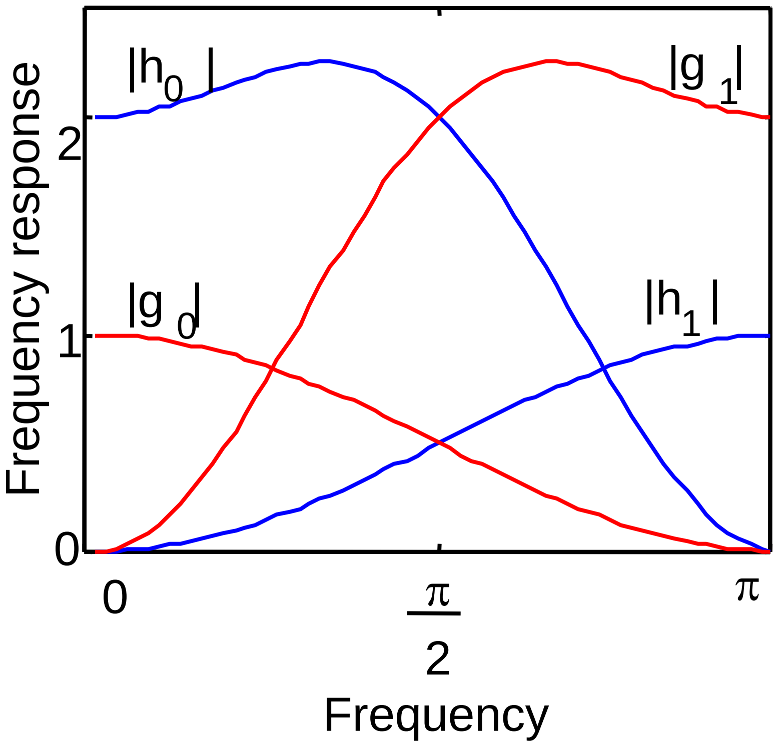
- Impulse responses, synthesis filters

  Lowpass highpass

$$\left(\frac{1}{4},\frac{1}{2},\frac{1}{4}\right) \qquad \left(\frac{1}{4},\frac{1}{2},\frac{-3}{2},\frac{1}{2},\frac{1}{4}\right)$$

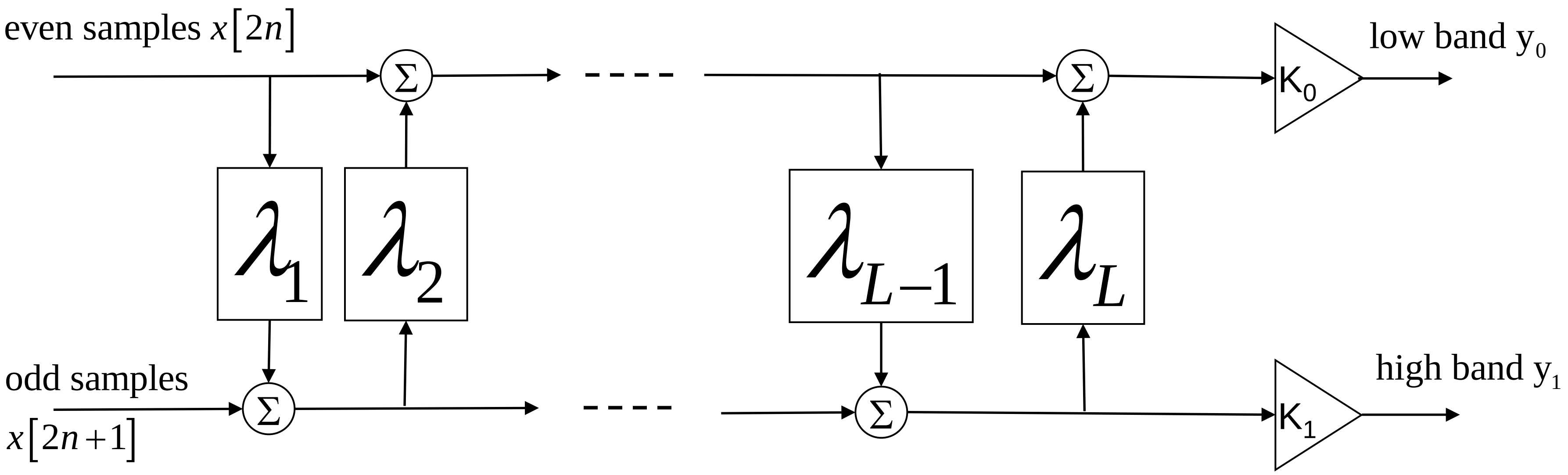
“Biorthogonal 5/3 filters”
“LeGall filters”

- Mandatory in JPEG2000
- Frequency responses:



Lifting

- Analysis filters

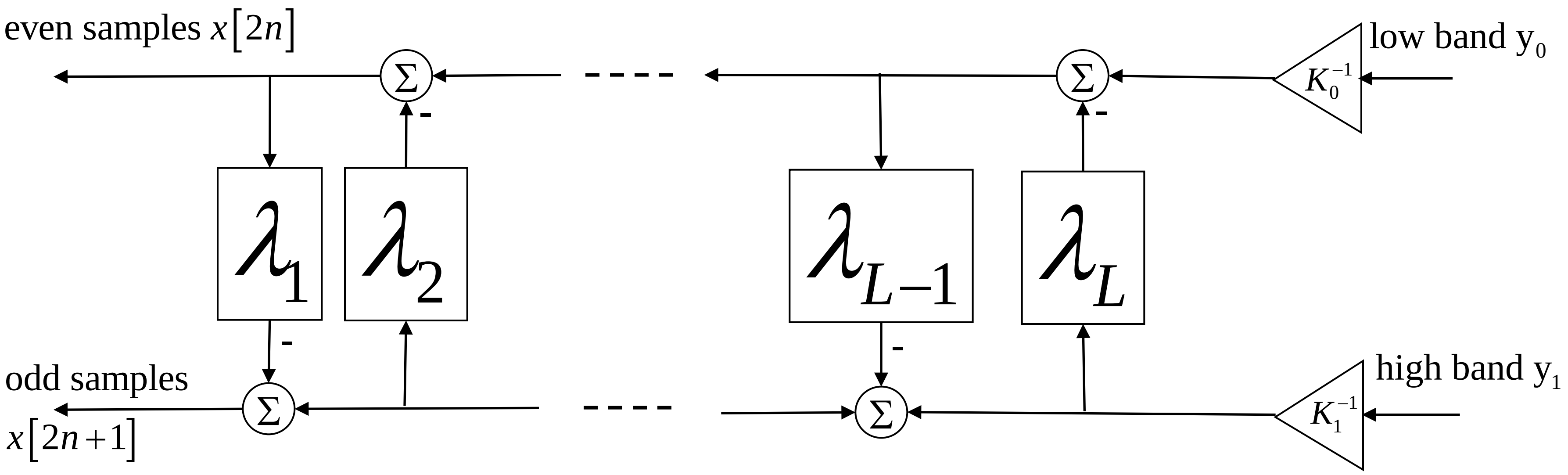


[Sweldens, 1996]

- L "lifting steps"
- First step can be interpreted as prediction of odd samples from the even samples

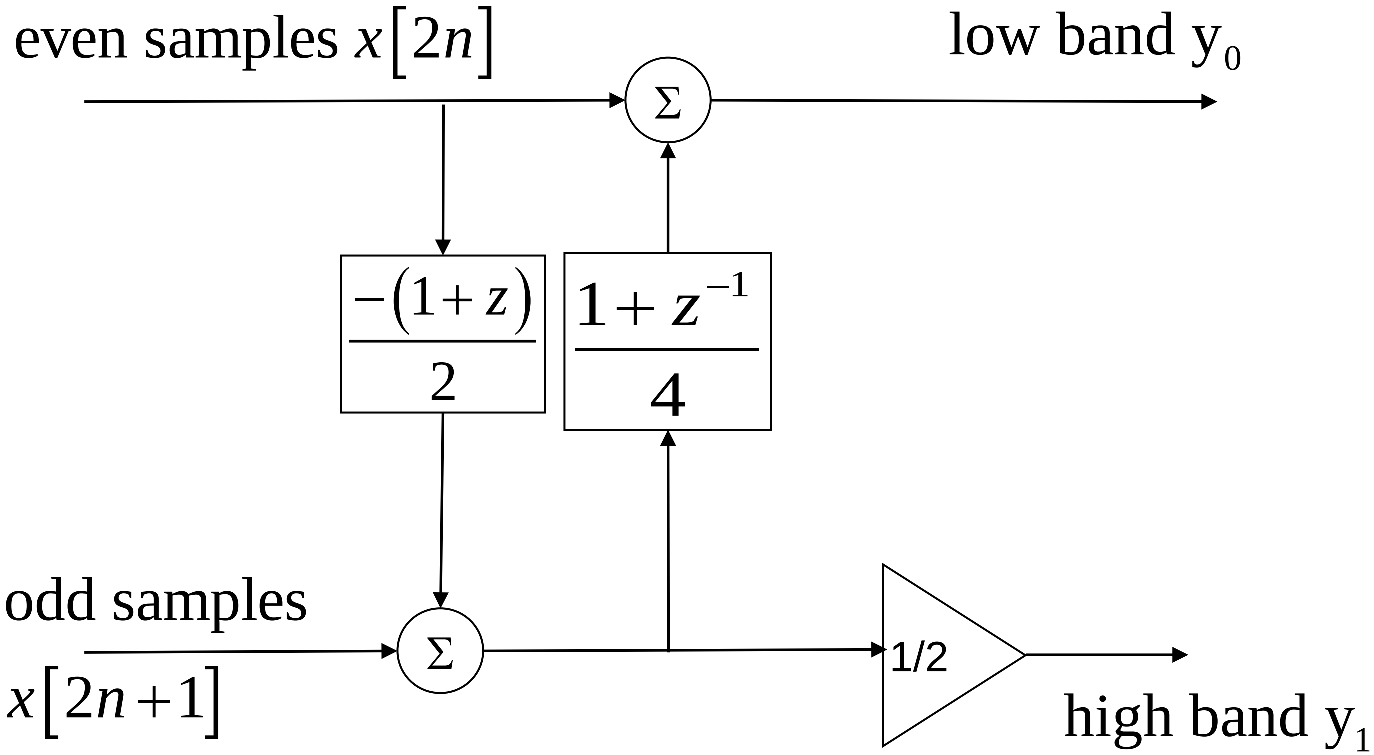
Lifting

- Synthesis filters



- Perfect reconstruction (biorthogonality) is directly build into lifting structure
- Powerful for both implementation and filter/wavelet design

Example: Lifting Implementation of 5/3 Filters



Verify by considering response to unit impulse in even and odd input channel.

Conjugate Quadrature Filters

- Achieve aliasing cancelation by

Prototype filter

$$
\boxed{\begin{aligned} h_0(z) &= g_0(z^{-1}) \equiv f(z) \\ h_1(z) &= g_1(z^{-1}) = zf(-z^{-1}) \end{aligned}}
$$

[Smith, Barnwell, 1986]

- Impulse responses

$$
\boxed{\begin{aligned} h_0[k] &= g_0[-k] = f[k] \\ h_1[k] &= g_1[-k] = (-1)^{k+1} f[-(k+1)] \end{aligned}}
$$

- With perfect reconstruction: Orthonormal subband transform!
- Perfect reconstruction: Find power complementary prototype filter

$$
|F(\omega)|^2 + |F(\omega \pm \pi)|^2 = 2
$$

Wavelet Bases

- Consider Hilbert space $\mathcal{L}^2(\mathcal{R})$ of finite-energy functions **x** = x(t)
- Wavelet basis for $\mathcal{L}^2(\mathcal{R})$: Family of linearly independent functions

"mother wavelet"

$$\psi_n^m(t) = \sqrt{2^{-m}}\psi(2^{-m}t - n)$$

that span $\mathcal{L}^2(\mathcal{R})$. Hence, any signal $\mathbf{x} \in \mathcal{L}^2(\mathcal{R})$ can be written as

$$\mathbf{x} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y^{(m)}[n]\psi_n^{(m)}$$

Multiresolution Analysis

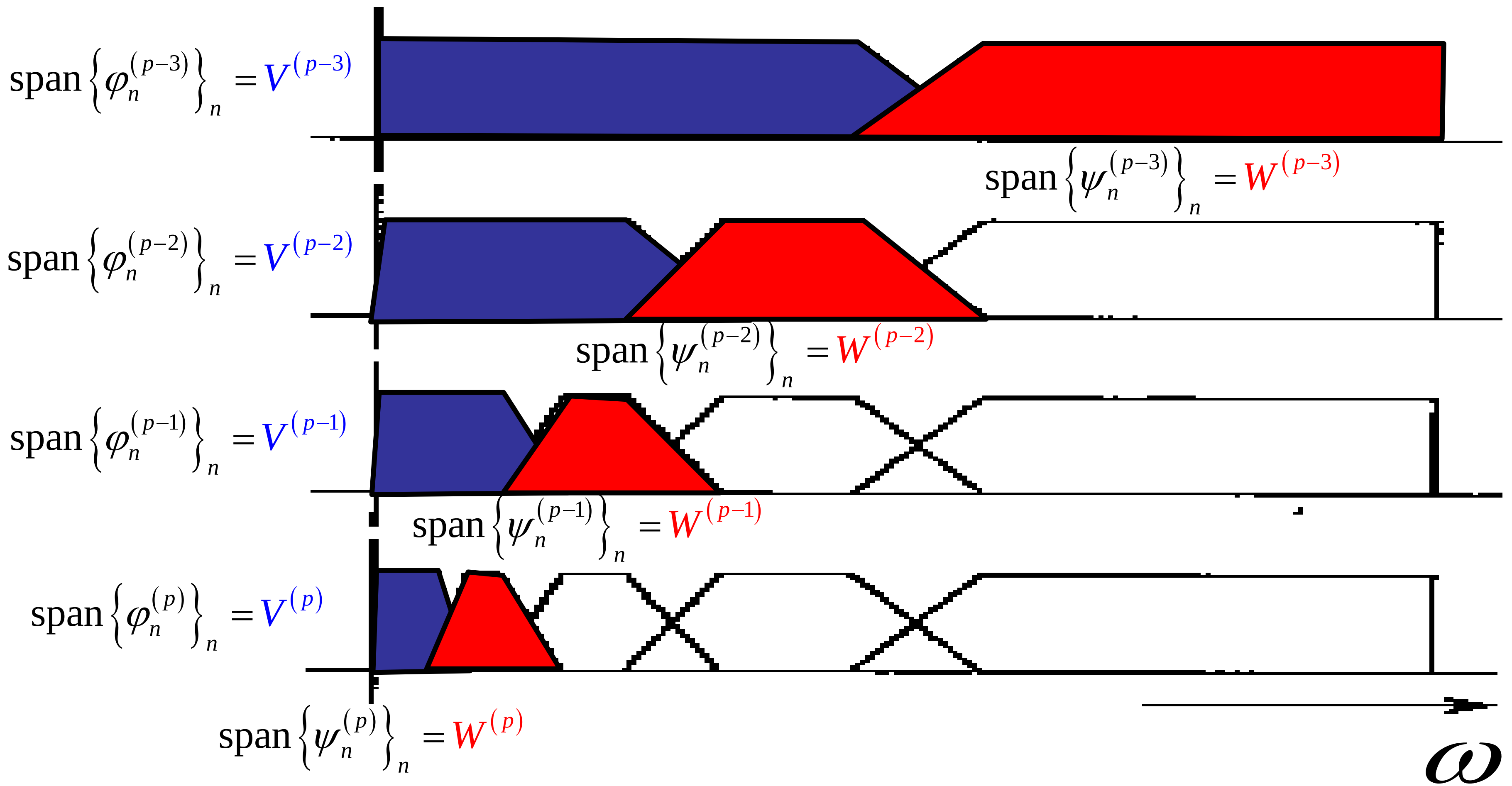
- Nested subspaces

$$\cdots \subset V^{(2)} \subset V^{(1)} \subset V^{(0)} \subset V^{(-1)} \subset V^{(-2)} \subset \cdots \subset \mathcal{L}^2(\mathcal{R})$$

Upward completeness	$\bigcup_{m\in\mathcal{Z}} V^{(m)} = \mathcal{L}^2(\mathcal{R})$
Downward completeness	$\bigcap_{m\in\mathcal{Z}} V^{(m)} = \{0\}$
Self-similarity	$x(t) \in V^{(0)}$ iff $x(2^{-m}t) \in V^{(m)}$
Translation invariance	$x(t) \in V^{(0)}$ iff $x(t-n) \in V^{(0)} \quad \forall n \in \mathcal{Z}$

- There exists a scaling function $\varphi(t)$ with integer translates $\varphi_n(t) = \varphi(t-n)$ such that $\{\varphi_n\}_{n\in\mathcal{Z}}$ forms an orthonormal basis for $V^{(0)}$

Multiresolution Fourier Analysis



Relation to Subband Filters

Since $V^{(0)} \subset V^{(-1)}$, recursive definition of scaling function

$$\varphi(t) = \underbrace{\sum_{n=-\infty}^{\infty} g_0[n]\varphi_n^{(-1)}(t)}_{\substack{\text{linear combination} \\ \text{of wavelets in } V^{(-1)}}} = \sqrt{2}\sum_{n=-\infty}^{\infty} g_0[n]\varphi_n(2t-n)$$

Orthonormality

$$\delta[\mathrm{n}] = \left\langle \varphi_0^{(0)}, \varphi_n^{(0)} \right\rangle$$

$$= 2\int_{-\infty}^{\infty} \left(\sum_i g_0[i]\varphi_n(2t-i) \sum_j g_0[j]\varphi_n(2(t-n)-j) \right) dt$$

$$= \sum_{i,j} g_0[i] g_0[j-2n] \left\langle \varphi_i^{(-1)}, \varphi_j^{(-1)} \right\rangle = \underbrace{\sum_i g_0[i] g_0[i-2n]}_{\substack{\text{unit norm and orthogonal} \\ \text{to its 2-translates: corresponds} \\ \text{to synthesis lowpass filter of} \\ \text{orthonormal subband transform}}}$$

Wavelets from Scaling Functions

$W^{(p)}$ is orthonormal complement of $V^{(p)}$ in $V^{(p-1)}$

$$W^{(p)} \perp V^{(p)} \quad \text{and} \quad W^{(p)} \cup V^{(p)} = V^{(p-1)}$$

Orthonormal wavelet basis $\left\{\psi_n^{(0)}\right\}$ for $W^{(0)} \subset V^{(-1)}$

$$\psi(t) = \underbrace{\sum_{n=-\infty}^{\infty} g_1[n]\varphi_n^{(-1)}(t)}_{\substack{\text{linear combination}\\ \text{of wavelets in } V^{(-1)}}} = \sqrt{2}\sum_{n=-\infty}^{\infty} g_1[n]\varphi_n(2t-n)$$

Using conjugate quadrature high-pass synthesis filter

$$g_1[n] = (-1)^{n+1} g_0\left[-(n-1)\right]$$

The mutually orthonormal functions, $\left\{\psi_n^{(0)}\right\}_{n\in\mathbf{Z}}$ and $\left\{\varphi_n^{(0)}\right\}_{n\in\mathbf{Z}}$ together span $V^{(-1)}$.

Easy to extend to dilated versions of $\psi(t)$ to construct orthonormal wavelet basis

$$\left\{\psi_n^{(m)}\right\}_{n,m\in\mathbf{Z}} \quad \text{for } \mathcal{L}^2(\mathcal{R})$$

Calculating Wavelet Coefficients for a Continuous Signal

- Signal synthesis by discrete filter bank

Suppose continuous signal $x^{(0)}(t) = \sum_{n \in \mathbb{Z}} y_0^{(0)}[n]\, \varphi(t-n) = \sum_{n \in \mathbb{Z}} y_0^{(0)}[n]\, \varphi_n^{(0)} \in V^{(0)}$

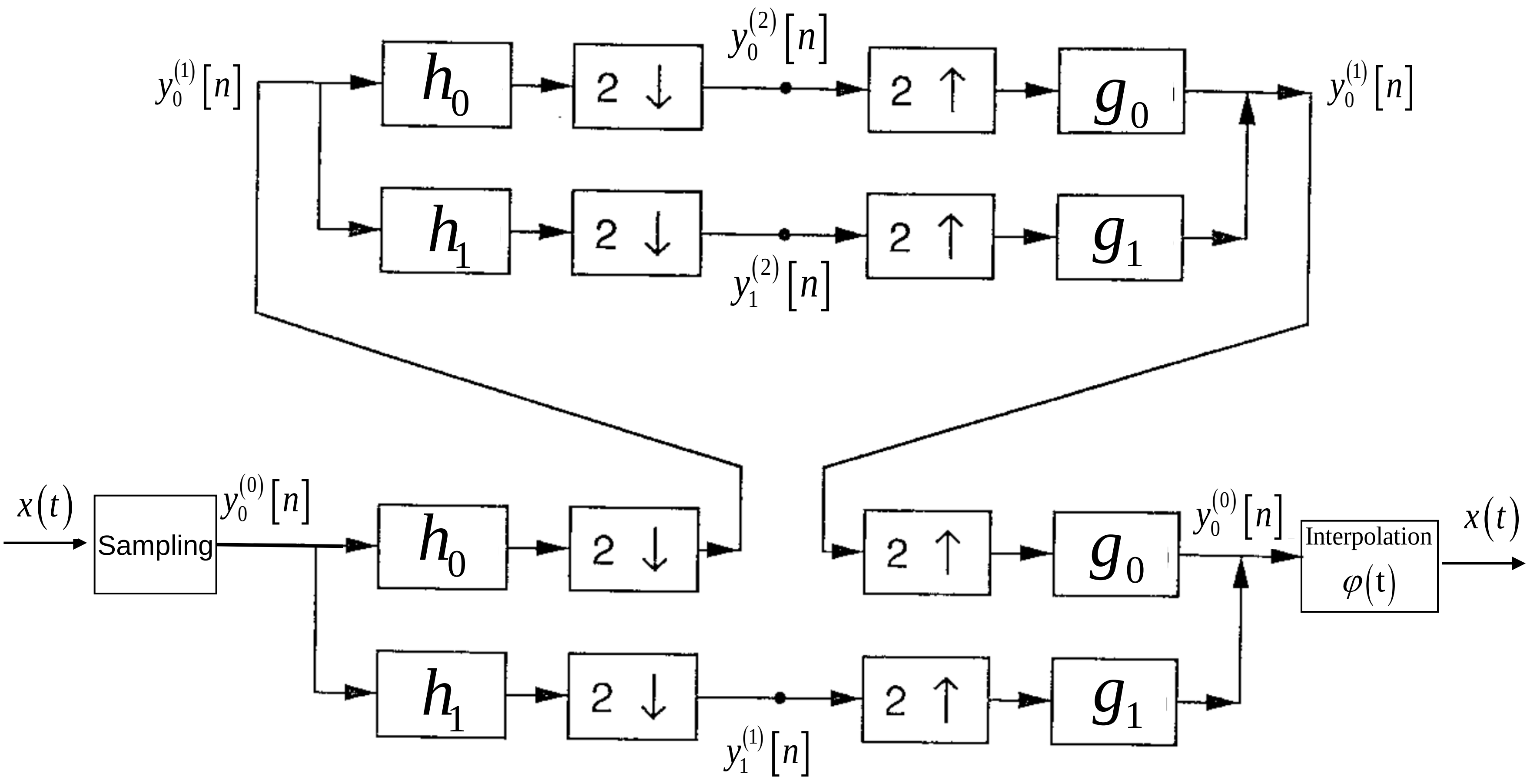
Write as superposition of $x^{(1)}(t) \in V^{(1)}$ and $w^{(1)}(t) \in W^{(1)}$

$$x^{(0)}(t) = \underbrace{\sum_{i \in \mathbb{Z}} y_0^{(1)}[i]\, \varphi_n^{(1)}}_{x^{(1)}(t) \in V^{(1)}} + \underbrace{\sum_{j \in \mathbb{Z}} y_1^{(1)}[j]\, \psi_n^{(1)}}_{w^{(1)}(t) \in W^{(1)}}$$

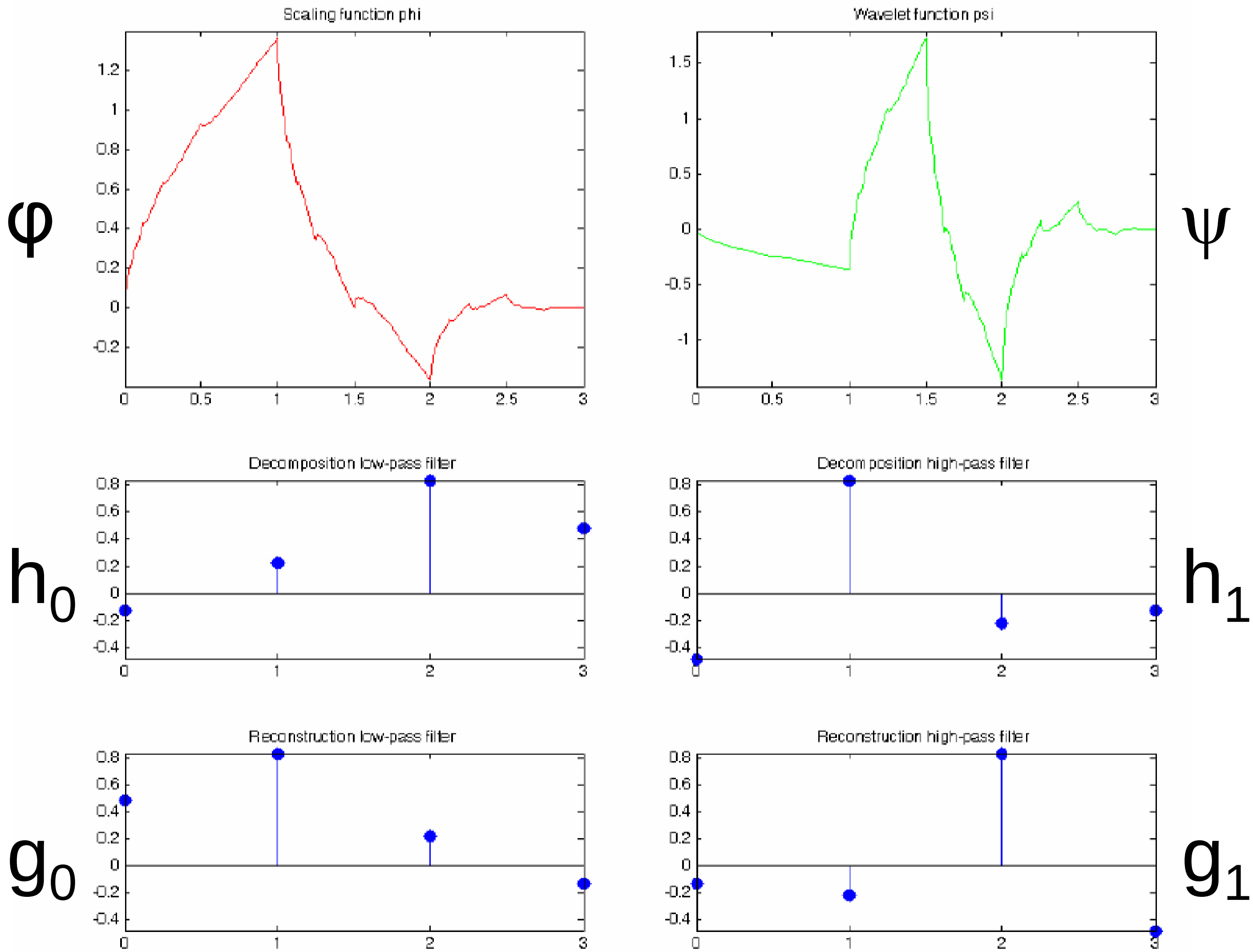
$$= \sum_{n \in \mathbb{Z}} \varphi_n^{(0)} \underbrace{\left(\sum_{i \in \mathbb{Z}} y_0^{(1)}[n]\, g_0[n-2i] + \sum_{j \in \mathbb{Z}} y_1^{(1)}[j]\, g_1[n-2i] \right)}_{y_0^{(0)}[n]}$$

- Signal analysis by analysis filters $h_0[k]$, $h_1[k]$
- Discrete wavelet transform

1-D Discrete Wavelet Transform



Example: Daubechies Wavelet, Order 2



Example: Daubechies Wavelet, Order 9