Filtering in the Frequency Domain

- Fourier transform
- Filtering in the frequency domain
- Separable filters
- Filtering examples

1-D Discrete-Time Fourier Transform (DTFT)

- Given a 1-d sequence $s[k], k \in \{\ldots, -1, 0, 1, 2, 3, \ldots\}$
- Discrete-time Fourier transform

$$S(\omega) = \sum_{k=-\infty}^{\infty} s[k] e^{-j\omega k} \qquad \omega \in \mathcal{R}$$

- Discrete-time Fourier transform is periodic with 2π
- Inverse DTFT:

$$s[k] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) e^{j\omega k} d\omega$$

Sampling Interpretation of the 1-D DTFT

- How is the Fourier transform of a sequence s[k] related to the Fourier transform of the continuous signal?

$$\underbrace{s(t)}_{\text{continuous}} = \sum_{k=-\infty}^{\infty} \underbrace{s[k]}_{\text{discrete}} \underbrace{\delta(t-k)}_{\text{“continuous” delta function}}$$

- Continuous-time Fourier transform

$$S(\omega) = \int_{\mathcal{R}} \sum_{k=-\infty}^{\infty} s[k]\delta(t-k)e^{-j\omega t}dt = \sum_{k=-\infty}^{\infty} s[k]e^{-j\omega k}$$

2-D Discrete-Space Fourier Transform (DSFT)

- Given a 2-d field of image samples

$$s[m,n] \quad (m,n) \in \mathcal{Z}^2$$

- Discrete-space Fourier transform

$$S(\omega_x, \omega_y) = \sum_{(m,n)\in\mathcal{Z}^2} s[m,n] e^{-j\omega_x m - j\omega_y n} \quad (\omega_x, \omega_y) \in \mathcal{R}^2$$

- DSFT is 2π-periodic both in ω_x and ω_y
- Inverse DSFT:

$$s[m,n] = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} S(\omega_x, \omega_y) e^{j\omega_x m + j\omega_y n} d\omega_x d\omega_y$$

Sampling Interpretation of the 2-D DSFT

- How is the Fourier transform of the field s[m,n] related to the Fourier transform of the continuous signal?

$$s(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s[m,n]\delta(x-m, y-n)$$

continuous

discrete

"continuous"
2-d delta function

- Continuous-space 2-d Fourier transform

$$\begin{aligned} S(\omega_x, \omega_y) &= \int_{\mathcal{R}^2} \sum_{m,n} s[m,n]\delta(x-m, y-n) e^{-j\omega_x x - j\omega_y y} dx dy \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} s[m,n] e^{-j\omega_x m - j\omega_y n} \end{aligned}$$

Some Properties of the 2-D DSFT

- Translation

$$s[x - x_0, y - y_0] \; \overset{\mathcal{F}_*}{\circ\!\!-\!\!\!-\!\!\bullet} \; S(\omega_x, \omega_y) e^{-j(\omega_x x_0 + \omega_y y_0)}$$

$$s[x, y] e^{j(x\omega_{x,0} + y\omega_{y,0})} \; \overset{\mathcal{F}_*}{\circ\!\!-\!\!\!-\!\!\bullet} \; S(\omega_x - \omega_{x,0}, \omega_y - \omega_{y,0})$$

- Scaling

$$s[ax, by] \; \overset{\mathcal{F}_*}{\circ\!\!-\!\!\!-\!\!\bullet} \; \frac{1}{|ab|} S\left(\frac{\omega_x}{a}, \frac{\omega_y}{b}\right)$$

Filtering in the Frequency Domain

- Linear shift-invariant processing

$$g[m,n] = \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f[x,y] h_{siv}[m-x, n-y]$$

- Convolution theorem

$$g[x,y] = h[x,y] * f[x,y]$$

$$\circ\!\!-\!\!\bullet \; \mathcal{F}_*$$

$$G(\omega_x, \omega_y) = H(\omega_x, \omega_y) F(\omega_x, \omega_y)$$

Separable Filters

- For separable, shift-invariant, linear processing

$$h[x,y] = h_x[x]h_y[y]$$

- Separable filters

$$H(\omega_x,\omega_y) = H_x(\omega_x)H_y(\omega_y)$$

Filtering Examples - Revisited

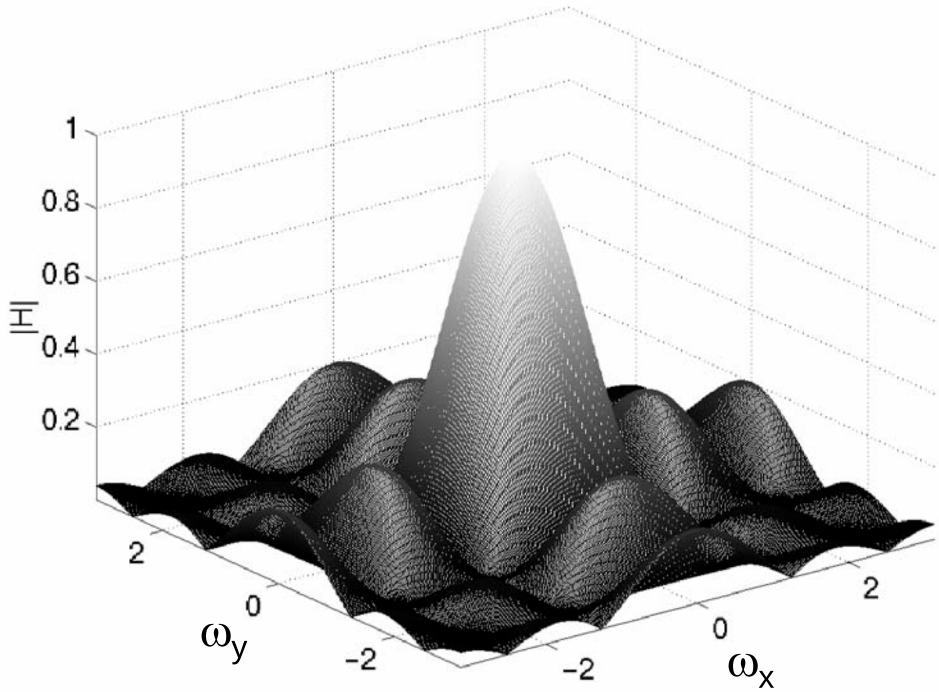
Cameraman
original

Cameraman
blurred by convolution
Filter impulse response:

$$\frac{1}{25}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & [1] & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Transfer Function 5x5 Low-Pass Filter

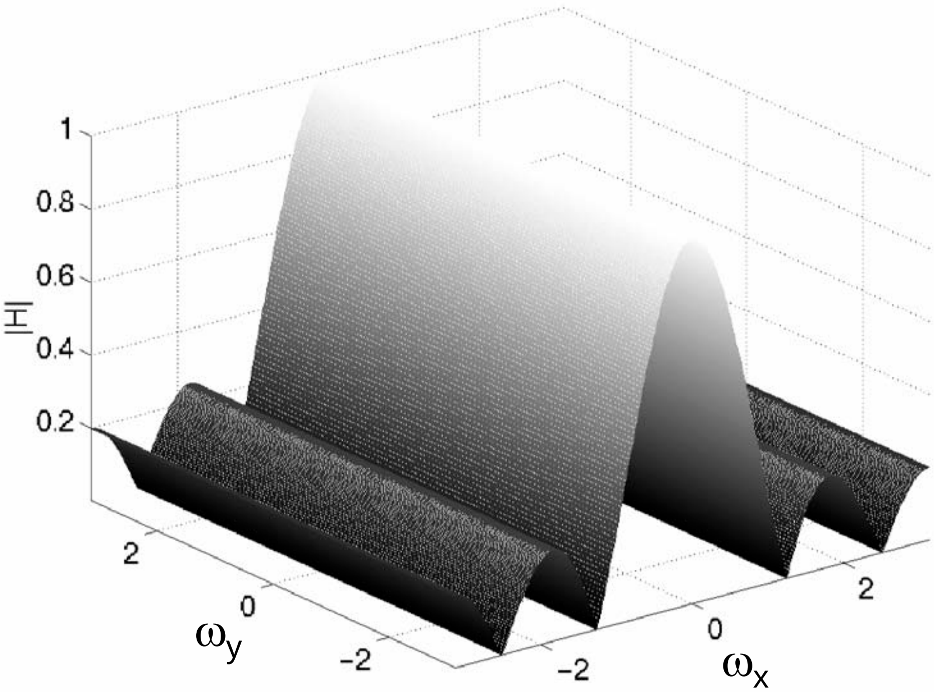
$$
\begin{aligned}
H(\omega_x, \omega_y) &= \frac{1}{25} \sum_{m=-2}^{2} \sum_{n=-2}^{2} e^{-j\omega_x m - j\omega_y n} = \frac{1}{25} \sum_{m=-2}^{2} e^{-j\omega_x m} \sum_{n=-2}^{2} e^{-j\omega_y n} \\
&= \frac{1}{25} [1 + 2\cos(\omega_x) + 2\cos(2\omega_x)][1 + 2\cos(\omega_y) + 2\cos(2\omega_y)]
\end{aligned}
$$



Filtering Examples

Cameraman
blurred horizontally
Filter impulse response:

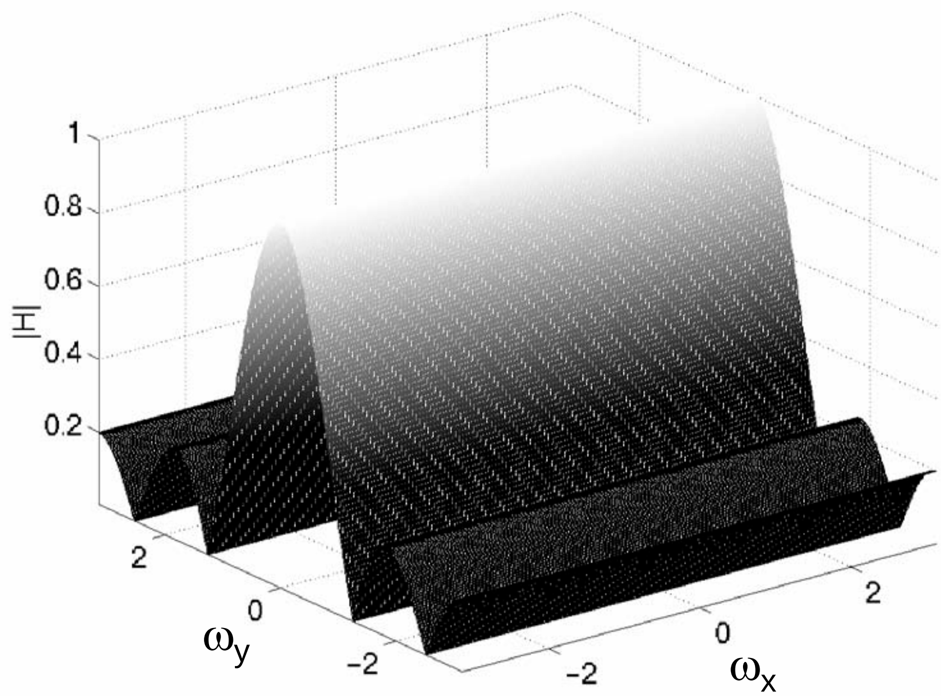
$$\frac{1}{5}\begin{bmatrix} 1 & 1 & [1] & 1 & 1 \end{bmatrix}$$



Filtering Examples

Cameraman
blurred vertically
Filter impulse response:

$$\frac{1}{5}\begin{bmatrix} 1 \\ 1 \\ [1] \\ 1 \\ 1 \end{bmatrix}$$



Filtering Examples

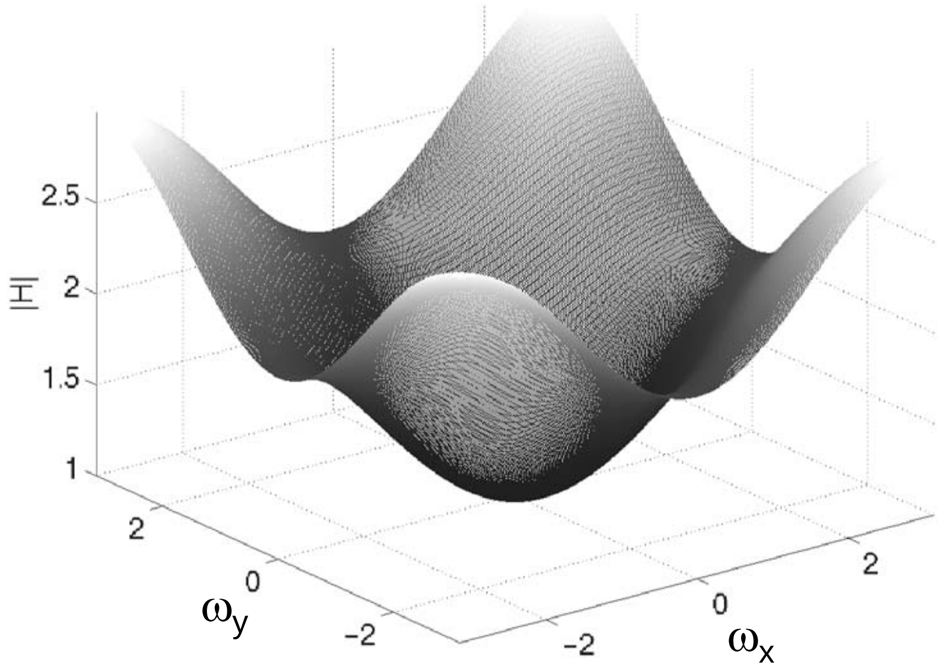
Cameraman
original

Cameraman
sharpened
Filter impulse response:

$$\frac{1}{4}\begin{bmatrix} 0 & -1 & 0 \\ -1 & [8] & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

Transfer Function High-Pass Filter

$$\begin{aligned} H(\omega_x, \omega_y) &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} h[m,n] e^{-j\omega_x m - j\omega_y n} = \frac{1}{4}\left[8 - e^{-j\omega_x} - e^{j\omega_x} - e^{-j\omega_y} - e^{j\omega_y}\right] \\ &= \frac{1}{4}\left[8 - 2\cos(\omega_x) - 2\cos(\omega_y)\right] \end{aligned}$$



Filtering Examples

Cameraman
sharpened
Filter impulse response:

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & [5] & -1 \\ 0 & -1 & 0 \end{bmatrix}$$