Direct and Inverse Spectral Problems for 2-dimensional Hamiltonian Systems

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These slides are available from my website
http://asc.tuwien.ac.at/index.php?id=woracek

Not all what is written on these slides is *strictly* correct.

We will occasionally neglect some
technical difficulties and/or exeptional cases.

Each such instance will be clearly marked.

We consider 2×2-Hamiltonian systems without potential:

$$y'(t) = zJH(t)y(t), \quad t \in (s_-, s_+).$$

Here the *Hamiltonian* H shall be subject to

- $H(t) : (s_-, s_+) \to \mathbb{R}^{2\times 2}$,
- $H(t) \geq 0$, $t \in (s_-, s_+)$,
- $H \in L^1_{\text{loc}}(s_-, s_+)$,
- H does not vanish identically on any set of positive measure,
- $z \in \mathbb{C}$ a parameter,
- $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

This equation is the eigenvalue equation of a differential operator.

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- Existence Theorems: Given some spectral data, does there exist a Hamiltonian H which leads to this data.

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Direct Problems: Given a Hamiltonian H, find information about spectral data of selfadjoint realizations.

Inverse Problems:

- Existence Theorems: Given some spectral data, does there exist a Hamiltonian H which leads to this data.
- Uniqueness Theorems: Which spectral data obtained from some Hamiltonian determine this Hamiltonian uniquely.

Outline

The Operator Model

Let H be a Hamiltonian on (s_-, s_+), let $(a, b) \subseteq (s_-, s_+)$ and $\phi \in \mathbb{R}$. Then (a, b) is *H-indivisible of type ϕ*, if

$$H(t) = h(t) \cdot \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \big(\cos\phi, \sin\phi \big), \quad t \in (a, b),$$

with some scalar function $h \in L^1_{\mathrm{loc}}(a, b)$.

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Definition (The model space $L^2(H)$)

The *model space* $L^2(H)$ is the space of all $f : (s_-, s_+) \to \mathbb{C}^2$ with

- $\|f\|_H^2 := \displaystyle\int_{s_-}^{s_+} f(t)^* H(t) f(t)\, dt < \infty.$
- If $(a,b) \subseteq (s_-, s_+)$ is indivisible of type ϕ, then

$$\big(\cos\phi, \sin\phi\big) f(t) = \text{ constant on } (a,b).$$

The Operator Model

In the definition of $L^2(H)$, we tacitly understand that two functions f, g with

$$H(t)f(t) = H(t)g(t), \quad t \in (s_-, s_+) \text{ a.e.},$$

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If endowed with the scalar product

$$(f, g)_H = \int_{s_-}^{s_+} g(t)^* H(t) f(t)\, dt, \quad f, g \in L^2(H),$$

the space $L^2(H)$ becomes a Hilbert space.

The Operator Model

Here we suppress some technical terms.

Definition (The maximal operator $T_{\max}(H)$)

The (graph of the) *maximal operator* $T_{\max}(H)$ is

$$T_{\max}(H) = \Big\{(f;g) \in L^2(H) \times L^2(H) : \\ f \text{ is locally absolutely continuous and } f' = JHg\Big\}$$

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The operator $T_{\max}(H)$ is closed.

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The operator $T_{\min}(H)$ is closed and symmetric. It is either selfadjoint, or completely nonselfadjoint.

Limit Circle vs. Limit Point Case

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- H is in *limit circle case at* s_-, if $(x_0 \in (s_-, s_+))$

$$\int_{s_-}^{x_0} \operatorname{tr} H(t)\, dt < \infty \qquad \Big(\Leftrightarrow H \in L^1_{\mathrm{loc}}([s_-, s_+)) \Big).$$

- H is in *limit point case at* s_-, if $(x_0 \in (s_-, s_+))$

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- H is in *limit point case at* s_-, if $(x_0 \in (s_-, s_+))$

$$\int_{s_-}^{x_0} \operatorname{tr} H(t)\, dt = \infty.$$

Similar: *limit circle case at* s_+ and *limit point case at* s_+.

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- Case I, lc $\leftrightarrow$ lc: $(2, 2)$.
- Case II, lc $\leftrightarrow$ lp or lp $\leftrightarrow$ lc: $(1, 1)$.
- Case III, lp $\leftrightarrow$ lp: $(0, 0)$.

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- Case III, lp $\leftrightarrow$ lp: $(0, 0)$.

In Case III, $T_{\min}(H) = T_{\max}(H)$. Hence, $T_{\min}(H)$ is selfadjoint and is the only selfadjoint realization.

In the Cases I and II, there are many different selfadjoint realizations.

Boundary Values

If s_- is in limit circle case, each $f = (f_1, f_2)^T \in \operatorname{dom} T_{\max}(H)$ has a continuous extension to s_-. Similar for s_+.

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- Assume Case lc ↔ lp. Then (for example)

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- Assume Case lc ↔ lc. Then (for example) for each $\tau \in \mathbb{R} \cup \{\infty\}$

$$A_{D,\tau} = \big\{(f;g) \in T_{\max}(H) : \; f_1(s_-) = 0, \tau f_1(s_+) + f_2(s_+) = 0\big\}$$

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Reparameterizations

Two Hamiltonians H_1 on H_2 defined on (s_-^1, s_+^1) and (s_-^2, s_+^2), respectively, are *reparameterizations of each other*, if there exists

$$\phi : (s_-^2, s_+^2) \to (s_-^1, s_+^1)$$

such that

- ϕ is bijective and monotonically increasing,
- ϕ and ϕ^{-1} are both absolutely continuous,
- $H_2(t) = H_1(\phi(t)) \cdot \phi'(t)$ for $t \in (s_-^2, s_+^2)$.

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If H_1 and H_2 are reparameterizations of each other, their operator models are unitarily equivalent.

Examples: 1. The Schrödinger Equation

Consider the equation $(0 < T < \infty)$

$$-y''(t) + V(t)y(t) = zy(t), \quad t \in [0, T],$$

where the *potential* $V(t)$ belongs to $L^1([0, T])$.

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Let y_1 and y_2 be the solutions of $-y''(t) + V(t)y(t) = 0$ with

$$y_1(0) = 0, y_1'(0) = 1, \quad y_2(0) = 1, y_2'(0) = 0,$$

and define

$$H(t) := \begin{pmatrix} y_1(t)^2 & y_1(t)y_2(t) \\ y_1(t)y_2(t) & y_2(t)^2 \end{pmatrix}, \quad t \in [0, T].$$

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Then H is a Hamiltonian which is lc $\leftrightarrow$ lc.

Examples: 1. The Schrödinger Equation

A function $y(t)$ solves the Schrödinger equation with potential $V(t)$ and parameter z, if and only if the function

$$u(t) = \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \cdot \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}$$

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The operator models of the Schrödinger equation and of the Hamiltonian system are unitarily equivalent.

Examples: 2. The String Equation

Here we ignore some technicalities.

Let $L > 0$, and μ be a positive Borel measure on $\mathbb{R}$ with $\operatorname{supp}\mu \subseteq [0, L]$ and $\mu(\{L\}) = 0$. Consider the integral equation boundary value problem with complex parameter z:

$$y'(t) + \int_{[0,t]} zy(u)d\mu(u) = 0, \quad y'(0-) = 0.$$

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Set $m(t) := \mu((-\infty, t))$, and

$$\hat{m}(x) = \begin{cases} \inf \{t \geq 0 : \; x \leq m(t)\}, & x \in [0, m(L)] \\ L, & x > m(L) \end{cases}$$

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Define

$$H(x) := \begin{pmatrix} \hat{m}(x)^2 & \hat{m}(x) \\ \hat{m}(x) & 1 \end{pmatrix}, \quad x \in (0, \infty).$$

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The operator models of the string equation and of the Hamiltonian system with this Hamiltonian are unitarily equivalent.

Examples: 3. The Hamburger Moment Problem

Let $(s_n)_{n\geq 0}$ be a sequence of real numbers. Is this sequence the sequence of power moments of some positive Borel measure on the real line ? That is, does there exist a positive Borel measure μ on $\mathbb{R}$ with

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The answer is *yes*, if and only if

$$\det\left[(s_{i+j})_{i,j=0}^{N}\right] \geq 0, \quad N \geq 0.$$

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- there exist infinitely many such measures.

If $D_N = 0$ for some $N \geq 0$, then the solution is unique and is a discrete measure with finitely many pointmasses.

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Set

$$E_N = \det\left[(s_{i+j+1})_{i,j=0}^N\right], \quad C_N = \det\left[(s_{i+j-1})_{i,j=0}^N\right]\ (s_{-1} = 0),$$

$$l_0 = 1, \quad l_N = (E_N^2 + C_N^2)\big(D_{N-1}D_N\big)^{-1},\ N \geq 1,$$

$$t_0 = 0, \quad t_N = \sum_{n=0}^{N-1} l_n,\ n \geq 1, \qquad T = \lim_{N\to\infty} t_n,$$

$$\theta_0 = \frac{\pi}{2}, \quad \theta_N = \begin{cases} \arctan\left(-\frac{E_N}{C_N}\right), & C_N \neq 0 \\ \frac{\pi}{2} & , \quad C_N = 0 \end{cases}$$

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$$H(t) = \begin{pmatrix} \cos\theta_N \\ \sin\theta_N \end{pmatrix} \left(\cos\theta_N, \sin\theta_N \right), \; t \in [t_N, t_{N-1}), \quad N \geq 0.$$

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Then H is a Hamiltonian which is

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Then H is a Hamiltonian which is

- lc ↔ lp, if the solution is unique,
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The set of solutions of the Hamburger moment problem coincides with the set of all *spectral measures* of the Hamiltonian H.

Case lc $\leftrightarrow$ lc. Spectral Measures

Denote by $W(t,z) = (w_{ij}(t,z))_{i,j=1}^{2}$ the solution of

$$\frac{d}{dt}W(t,z)J = zW(t,z)H(t), \quad t \in [s_-, s_+], \qquad W(s_-, z) = I.$$

Case lc ↔ lc. Spectral Measures

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Then, for each $\tau \in \mathbb{R} \cup \{\infty\}$, the function

$$q_{H,\tau}(z) = \frac{w_{11}(s_+,z)\tau + w_{12}(s_+,z)}{w_{21}(s_+,z)\tau + w_{22}(s_+,z)}$$

belongs to the *Nevanlinna class*, that is,

- q_H is analytic in $\mathbb{C} \setminus \mathbb{R}$ and $q_H(\overline{z}) = \overline{q_H(z)}$,
- $\operatorname{Im} q_H(z) \geq 0$ for $\operatorname{Im} z > 0$.

Case lc ↔ lc. Spectral Measures

We can represent $q_{H,\tau}$ as (*Herglotz integral representation*)

$$q_{H,\tau}(z) = a_{H,\tau} + b_{H,\tau} z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu_{H,\tau}(t), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

with

- $a_{H,\tau} \in \mathbb{R}$, $b_{H,\tau} \geq 0$,
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The measure $\mu_{H,\tau}$ can be computed from $q_{H,\tau}$ by means of the Stieltjes Inversion Formula, and the constant $b_{H,\tau}$ from the behaviour of $q_{H,\tau}$ towards $i\infty$.

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Each measure $\mu_{H,\tau}$ obtained in this way is called a *spectral measure of H*.

Case lc $\leftrightarrow$ lc. Fourier Transforms

Here we ignore the exceptional case $b_{H,\tau} > 0$ and technicalities.

Theorem (Integral transforms, lc $\leftrightarrow$ lp)

Let $\tau \in \mathbb{R} \cup \{\infty\}$. A unitary map $U_\tau : L^2(H) \to L^2(\mu_{H,\tau})$ is defined by

$$(U_\tau f)(x) = \int_{s_-}^{s_+} \big(w_{21}(t,x), w_{22}(t,x)\big) H(t) f(t) dt.$$

Its inverse $U_\tau^{-1} : L^2(\mu_{H,\tau}) \to L^2(H)$ is given as

$$(U_\tau^{-1} F)(t) = \int_{-\infty}^{\infty} \begin{pmatrix} w_{21}(t,x) \\ w_{22}(t,x) \end{pmatrix} F(x) d\mu_{H,\tau}(x).$$

Case lc ↔ lc. Fourier Transforms

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Theorem (Unitary equivalence)

Let $\tau \in \mathbb{R} \cup \{\infty\}$. The selfadjoint realization

$$A_{D,\tau} = \left\{(f;g) \in T_{\max}(H) : \; f_1(s_-) = 0, \tau f_1(s_+) + f_2(s_+) = 0\right\}$$

is unitarily equivalent to the the multiplication operator M_x in $L^2(\mu_{H,\tau})$ via U_τ, that is,

$$U_\tau \circ A_{D,\tau} = M_x \circ U_\tau.$$

Case lc ↔ lc. A Direct Theorem

Theorem (Direct Spectral Theorem)

- *Each selfadjoint realization has compact resolvents.*
- *The spectrum of each selfadjoint realization defined by separated boundary conditions is simple.*
- *Let* (λ_n^+) *and* (λ_n^-) *denote the sequences of positive and negative, respectively, eigenvalues of a selfadjoint realization arranged according to increasing modulus. Then*

$$\lim \frac{n}{\lambda_n^+} = \lim \frac{n}{\lambda_n^-} = \frac{1}{\pi} \int_{s_-}^{s_+} \sqrt{\det H(t)}\,dt.$$

Case lc ↔ lc. A Uniqueness Theorem

A Hamiltonian H is not uniquely determined by the spectrum of one of its selfadjoint realizations. It may happen that H_1 and H_2 are different (not reparameterizations of each other), and still

$$\sigma(A^1_{D,0}) = \sigma(A^2_{D,0}).$$

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Theorem (Inverse Theorem / Uniqueness)

Assume that two Hamiltonians H_1 and H_2 satisfy

$$\sigma(A^1_{D,0}) = \sigma(A^2_{D,0}) \quad \textit{and} \quad \sigma(A^1_{D,\infty}) = \sigma(A^2_{D,\infty}),$$

then H_1 and H_2 are equal (up to a reparameterization).

Case lc ↔ lc. Existence Theorems

Theorem (Characterization of spectra)

Let (λ_n) be a sequence of pairwise different real numbers. Then there exists a Hamiltonian H in lc ↔ lc *with $\{\lambda_n\} = \sigma(A_{D,0})$, if and only if all λ_n are nonzero, and*

- *the limits $\lim \dfrac{n}{\lambda_n^+}$ and $\lim \dfrac{n}{\lambda_n^-}$ exist in $[0, \infty)$ and are equal, where (λ_n^+) and (λ_n^-) denote the sequences of positive and negative elements of (λ_n),*
- $\displaystyle\lim_{R\to\infty} \sum_{|\lambda_n|\le R} \frac{1}{\lambda_n}$ *exists in $\mathbb{R}$,*
- $\displaystyle\sum_n \frac{1}{|\lambda_n|^2 |A'(\lambda_n)|} < \infty$ *where $A(z) = \displaystyle\lim_{R\to\infty} \prod_{|\lambda_n|\le R} \left(1 - \frac{z}{\lambda_n}\right)$.*

Case lc ↔ lc. Existence Theorems

Theorem (Characterization of pairs of spectra)

Let (λ_n) and (μ_n) be two sequences of pairwise different real numbers. Then there exists a Hamiltonian H in lc ↔ lc *with $\{\lambda_n\} = \sigma(A_{D,0})$ and $\{\mu_n\} = \sigma(A_{D,\infty})$, if and only if all λ_n are nonzero, the point zero is among the μ_n's, and*

- *the sequences (λ_n) and (μ_n) interlace,*
- $\lim \frac{n}{\lambda_n^+} = \lim \frac{n}{\lambda_n^-} \in [0, \infty)$ *and* $\lim_{R\to\infty} \sum_{|\lambda_n|\le R} \frac{1}{\lambda_n}$ *exists in* $\mathbb{R}$,
- $\sum_n \frac{1}{|\lambda_n|^2 |A'(\lambda_n)B(\lambda_n)|} < \infty$ *where*

$$A(z) = \lim_{R\to\infty} \prod_{|\lambda_n|\le R} \left(1 - \frac{z}{\lambda_n}\right), \; B(z) = z \lim_{R\to\infty} \prod_{0<|\mu_n|\le R} \left(1 - \frac{z}{\mu_n}\right).$$

Case lc $\leftrightarrow$ lp. The Weyl Construction

The cases lc $\leftrightarrow$ lp and lp $\leftrightarrow$ lc are fully analogous. We confine attention to lc $\leftrightarrow$ lp.

Case lc $\leftrightarrow$ lp. The Weyl Construction

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Theorem (Weyl coefficient. Existence)

Denote by $W(t,z) = (w_{ij}(t,z))_{i,j=1}^2$ *the solution of*

$$\frac{d}{dt}W(t,z)J = zW(t,z)H(t), \quad t \in [s_-, s_+), \qquad W(s_-,z) = I.$$

Then, for each $\tau \in \mathbb{R} \cup \{\infty\}$ *the limit*

$$q_H(z) = \lim_{t \nearrow s_+} \frac{w_{11}(t,z)\tau + w_{12}(t,z)}{w_{21}(t,z)\tau + w_{22}(t,z)}$$

exists locally uniformly on $\mathbb{C} \setminus \mathbb{R}$. *It does not depend on* τ.

Case lc ↔ lp. The Weyl Construction

Theorem (Weyl coefficient. Properties)

The function q_H belongs to the Nevanlinna class, that is,

- q_H *is analytic in* $\mathbb{C} \setminus \mathbb{R}$ *and* $q_H(\overline{z}) = \overline{q_H(z)}$,
- $\operatorname{Im} q_H(z) \geq 0$ *for* $\operatorname{Im} z > 0$.

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- $\operatorname{Im} q_H(z) \geq 0$ *for* $\operatorname{Im} z > 0$.

We can therefore represent q_H as

$$q_H(z) = a_H + b_H z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu_H(t), \quad z \in \mathbb{C} \setminus \mathbb{R},$$

with

- $a_H \in \mathbb{R}$, $b_H \geq 0$,
- μ_H positive Borel measure with $\int_{\mathbb{R}} \frac{d\mu_H(t)}{1+t^2} < \infty$.

Case lc ↔ lp. The Weyl Construction

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- μ_H positive Borel measure with $\int_{\mathbb{R}} \frac{d\mu_H(t)}{1+t^2} < \infty$.

The measure μ_H obtained in this way is called the *spectral measure of H*.

Case lc $\leftrightarrow$ lp. A Direct Theorem

Here we ignore the exceptional case $b_H > 0$ and technicalities.

Theorem (Direct Spectral Theorem)

A unitary map $U : L^2(H) \to L^2(\mu_H)$ is defined by

$$(Uf)(x) = \int_{s_-}^{s_+} \big(w_{21}(t,x), w_{22}(t,x)\big) H(t) f(t) dt.$$

It intertwines A_D and the multiplication operator M_x in $L^2(\mu_H)$:

$$U \circ A_D = M_x \circ U.$$

Its inverse $U^{-1} : L^2(\mu_H) \to L^2(H)$ is given as

$$(U^{-1}F)(t) = \int_{-\infty}^{\infty} \begin{pmatrix} w_{21}(t,x) \\ w_{22}(t,x) \end{pmatrix} F(x) d\mu_H(x).$$

Case lc $\leftrightarrow$ lp. The Inverse Theorem

The following Existence and Uniqueness Theorem is *the* major result in the spectral theory of Hamiltonian systems.

Case lc $\leftrightarrow$ lp. The Inverse Theorem

The following Existence and Uniqueness Theorem is *the* major result in the spectral theory of Hamiltonian systems.

Theorem (Inverse Spectral Theorem)

Let a function q in the Nevanlinna class be given. Equivalently, let $a \in \mathbb{R}$, $b \geq 0$, and a positive Borel measure μ with $\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2}$ be given.

Then there exists a (up to reparameterization) unique Hamiltonian in lc $\leftrightarrow$ lp *whose Weyl coefficient equals q.*

Case lc ↔ lp. A Local Uniqueness Theorem

Theorem (Local uniqueness)

Let H_1 and H_2 be Hamiltonians defined on (s_-^1, s_+^1) and (s_-^2, s_+^2), respectively. For $a > 0$ set

$$s_a^j = \sup\left\{t \in [s_-^j, s_+^j) : \int_{s_-^j}^t \sqrt{\det H_j(x)}\, dx < a\right\}, \quad j = 1, 2\,.$$

Then the following are equivalent.

- $H_1 \big|_{(s_-^1, s_a^1)}$ *and* $H_2 \big|_{(s_-^2, s_a^2)}$ *are reparameterizations of each other.*
- $q_{H_1}(z) - q_{H_2}(z) = O\big((\operatorname{Im} z)^3 e^{-2a \operatorname{Im} z}\big), \quad z \hat{\to} i\infty.$

Case lc ↔ lp. Semibounded Spectrum

Theorem (Consequence of semibounded spectrum)

Let H be given with $\inf \operatorname{supp} \mu_H > -\infty$. Then there exist unique $L \in (0, \infty]$ and $\nu : [0, L) \to [0, +\infty)$, such that

- *ν is nondecreasing, right-continuous, and normalized by $\nu(0) \in [0, \pi)$ and $\nu(t) - \nu(t-) < \pi$,*
- *H is (a reparameterization of)*

$$H_\nu(x) = \begin{cases} \begin{pmatrix} [\cot \nu(t)]^2 & -\cot \nu(t) \\ -\cot \nu(t) & 1 \end{pmatrix} & \text{if } \nu(t) \notin \pi\mathbb{Z} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \text{if } \nu(t) \in \pi\mathbb{Z} \end{cases}$$

Case lc $\leftrightarrow$ lp. Semibounded Spectrum

The question converse to the above theorem is: Given ν with the properties stated in the theorem. Is the spectral measure of H_ν semibounded from below ?

Case lc $\leftrightarrow$ lp. Semibounded Spectrum

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The answer is *unknown*.

Case lc $\leftrightarrow$ lp. Semibounded Spectrum

Theorem (The case of finite negativity)

The following are equivalent:

- H *is (up to a reparameterization) equal to* H_ν *with* ν *being (in addition) bounded.*
- $(-\infty, 0) \cap \operatorname{supp} \mu_H$ *is a finite set.*

Case lc $\leftrightarrow$ lp. Discrete Spectrum

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What is easy to see is that for each discrete subset M of $\mathbb{R}$, there exist (infinitely many) Hamiltonians H which are lc ↔ lp and such that

$$\sigma(A_D) = M.$$

Case lc ↔ lp. Discrete Spectrum

Write $H(t) = (h_{ij}(t))_{i,j=1}^2$, and set $B(t) = \int_{s_-}^t h_{12}(x)dx$.

For $r > 0$ set

$$M_1(r) = \Big\{\lambda \in \mathbb{R} \setminus \{0\} : \limsup_{t \nearrow s_+} \Big(\int_{s_-}^{t} h_{22}(x) e^{-2\lambda B(x)} dx \int_{t}^{s_+} h_{11}(x) e^{2\lambda B(x)} dx \Big) \leq \frac{r}{\lambda^2} \Big\},$$

$$M_2(r) = \Big\{\lambda \in \mathbb{R} \setminus \{0\} : \limsup_{t \nearrow s_+} \Big(\int_{s_-}^{t} h_{11}(x) e^{2\lambda B(x)} dx \int_{t}^{s_+} h_{22}(x) e^{-2\lambda B(x)} dx \Big) \leq \frac{r}{\lambda^2} \Big\}.$$

Case lc ↔ lp. Discrete Spectrum

In the literature this theorem is only stated. We have not seen a proof.

Theorem (Discreteness of spectrum)

If μ_H is discrete, then

$$\mathbb{R} \setminus \{0\} \subseteq M_1(1) \cup M_2(1).$$

If there exist sequences $\lambda_i \to +\infty$ and $\mu_i \to -\infty$ with

$$\{\lambda_i : i \in \mathbb{N}\} \cup \{\mu_i : i \in \mathbb{N}\} \subseteq \bigcup_{r<\frac{1}{4}} \big(M_1(r) \cup M_2(r)\big),$$

then μ_H is discrete.

Case lc ↔ lp. Discrete Spectrum

Theorem (The diagonal case)

Assume that H is diagonal, that is,

$$H(x) = \begin{pmatrix} h_1(x) & 0 \\ 0 & h_2(x) \end{pmatrix}, \quad x \in (s_-, s_+) \text{ a.e.}$$

Then μ_H is discrete, if and only if either

$$\int_{s_-}^{s_+} h_1(x)\,dx < \infty \text{ and } \lim_{x \nearrow s_+} \Big(\int_x^{s_+} h_1(t)\,dt \cdot \int_{s_-}^{x} h_2(t)\,dt \Big) = 0,$$

or

$$\int_{s_-}^{s_+} h_2(x)\,dx < \infty \text{ and } \lim_{x \nearrow s_+} \Big(\int_x^{s_+} h_2(t)\,dt \cdot \int_{s_-}^{x} h_1(t)\,dt \Big) = 0.$$

Case lc $\leftrightarrow$ lp. Hilbert-Schmidt Property

Contrasting discreteness of the spectrum, the property that the spectrum is discrete with square summable eigenvalues can be characterized in general.

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Theorem (Characterization of Hilbert-Schmidt class)

The resolvents of selfadjoint realizations belong to the Hilbert-Schmidt class, if and only if there exists an angle $\phi \in [0, \pi)$ *such that (here* $\xi_\alpha = \binom{\cos\alpha}{\sin\alpha}$*)*

- $\displaystyle\int_{s_-}^{s_+} \xi_\phi^T H(t) \xi_\phi \, dt < \infty,$
- $\displaystyle\int_{s_-}^{s_+} \left(\int_{s_-}^{t} \xi_{\phi+\frac{\pi}{2}}^T H(u) \xi_{\phi+\frac{\pi}{2}} \, du \right) \xi_\phi^T H(t) \xi_\phi \, dt < \infty.$

Case lp $\leftrightarrow$ lp. Vector-Valued L^2-Spaces

Let $\Omega = (\Omega_{ij})_{i,j=1}^n$ be a positive $n \times n$-matrix valued Borel measure on $\mathbb{R}$, that is, a map of Borel sets to positive semidefinite $n \times n$-matrices which is σ-additive and satisfies $\Omega(\emptyset) = 0$.

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Set $\rho(\Delta) = \operatorname{tr}\Omega(\Delta)$, then ρ is a finite positive Borel measure on $\mathbb{R}$ and each entry Ω_{ij} is absolutely continuous w.r.t. ρ. The *symmetric derivative* of Ω w.r.t. ρ is

$$\frac{d\Omega}{d\rho}(x) = \lim_{\varepsilon \downarrow 0} \frac{\Omega\big((x-\varepsilon, x+\varepsilon)\big)}{\rho\big((x-\varepsilon, x+\varepsilon)\big)}, \quad x \in \mathbb{R} \ \rho\text{-a.e.}$$

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$L^2(\Omega)$ is the space of all $f : \mathbb{R} \to \mathbb{C}^n$ which are ρ-measurable and $\big(f(x), \frac{d\Omega}{d\rho}(x) f(x)\big)_{\mathbb{C}^n} \in L^1(\rho)$. It is endowed with

$$(f,g)_{L^2(\Omega)} = \int_{\mathbb{R}} \big(f(x), \frac{d\Omega}{d\rho}(x) g(x)\big)_{\mathbb{C}^n} \, d\rho(x).$$

Case lp $\leftrightarrow$ lp. Matrix Weyl Function

Choose a point $s_0 \in (s_-, s_+)$. Then

- $H_+ = H|_{(s_0,s_+)}$ is a Hamiltonian lc $\leftrightarrow$ lp.
- $H_- = H|_{(s_-,s_0)}$ is a Hamiltonian in lp $\leftrightarrow$ lc,

Case lp ↔ lp. Matrix Weyl Function

Choose a point $s_0 \in (s_-, s_+)$. Then

- $H_+ = H|_{(s_0, s_+)}$ is a Hamiltonian lc ↔ lp.
- $H_- = H|_{(s_-, s_0)}$ is a Hamiltonian in lp ↔ lc,

The 2×2-matrix valued function

$$Q_H(z) = \frac{1}{q_{H_+}(z) + q_{H_-}(z)} \begin{pmatrix} q_{H_+}(z) q_{H_-}(z) & -q_{H_+}(z) \\ -q_{H_+}(z) & -1 \end{pmatrix}$$

belongs to the 2×2-*Nevanlinna class*, that is,

- Q_H is analytic in $\mathbb{C} \setminus \mathbb{R}$ and $Q_H(\overline{z}) = Q_H(z)^*$,
- $\operatorname{Im} q_H(z) = \frac{1}{2i}(Q_H(z) - Q_H(z)^*)$ is positive semidefinite for each z with $\operatorname{Im} z > 0$.

Case lp ↔ lp. Matrix Weyl Function

We can represent Q_H as $(z \in \mathbb{C} \setminus \mathbb{R})$

$$Q_H(z) = a_H + b_H z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \cdot (1+t^2) d\Omega_H(t),$$

where

- $a_H, b_H \in \mathbb{C}^{2\times 2}$, with $a_H = a_H^*$ and b_H positive semidefinite,
- Ω_H positive 2×2-matrix valued Borel measure.

Case lp ↔ lp. The Titchmarsh-Kodaira formula

Here we ignore the exceptional case $b_H \neq 0$.

Theorem (Unitary equivalence)

The selfadjoint operator $T_{min}(H)$ is unitarily equivalent to the multiplication operator in the space $L^2(\Omega_H)$. That is, there exists a unitary operator $U : L^2(H) \to L^2(\Omega_H)$ which intertwines $T_{\min}(H)$ and M_x:

$$U \circ T_{\min}(H) = M_x \circ U.$$

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The action of U can again be described as an integral transform.

Corollary

The spectral multiplicity of $T_{\min}(H)$ cannot exceed 2.

Case lp $\leftrightarrow$ lp. Local Spectral Multiplicity

Let E be the projection valued spectral measure of $T_{\min}(H)$, and let σ_1, σ_2 be scalar positive Borel measures with $\sigma_2 \ll \sigma_1 \sim E$ such that $T_{\min}(H)$ is unitarily equivalent to the multiplication operator in $L^2(\sigma_1) \oplus L^2(\sigma_2)$.

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Set (*layers of spectrum*)

$$Y_l = \Big\{x \in \mathbb{R} : \frac{d\sigma_l}{d\sigma_1}(x) \in (0, \infty]\Big\}, \quad l = 1, 2,$$

and (*spectral multiplicity function*)

$$N_H(x) := \#\big\{l \in \{1, 2\} : x \in Y_l\big\}, \quad x \in \mathbb{R},\ \sigma_1\text{-a.e.}$$

Case lp $\leftrightarrow$ lp. Local Spectral Multiplicity

Theorem

We have $N_H(x) = \operatorname{rank} \dfrac{d\Omega_H}{d\rho}(x)$ *for* $x \in \mathbb{R}$, ρ*-a.e. (notice* $\rho \sim E$*).*

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Denote by λ the Lebesgue measure, and set $\mu := \mu_{H_+} + \mu_{H_-}$.

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Denote by λ the Lebesgue measure, and set $\mu := \mu_{H_+} + \mu_{H_-}$.

Decompose

$$E = E_{ac} + E_s \quad \text{with} \quad E_{ac} \ll \lambda,\ E_s \perp \lambda,$$

and further

$$E_s = E_{s,ac} + E_{s,s} \quad \text{with} \quad E_{s,ac} \ll \mu,\ E_{s,s} \perp \mu.$$

Case lp ↔ lp. Local Spectral Multiplicity

Theorem

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and further

$$E_s = E_{s,ac} + E_{s,s} \quad \text{with} \quad E_{s,ac} \ll \mu,\ E_{s,s} \perp \mu.$$

Moreover, decompose

$$\mu = \mu_{ac} + \mu_s \quad \text{with} \quad \mu \ll \lambda,\ \mu \perp \lambda.$$

Case lp $\leftrightarrow$ lp. Local Spectral Multiplicity

Set ($x \in \mathbb{R}$, μ-a.e.)

$$r(x) = \begin{cases} 2\,, & \frac{d\mu_{H_+}}{d\mu}(x), \frac{d\mu_{H_-}}{d\mu}(x) \in (0,\infty] \\ 1\,, & \text{exactly one of } \frac{d\mu_{H_+}}{d\mu}(x), \frac{d\mu_{H_-}}{d\mu}(x) \text{ is nonzero} \\ 0\,, & \frac{d\mu_{H_+}}{d\mu}(x), \frac{d\mu_{H_-}}{d\mu}(x) = 0 \end{cases}$$

Case lp ↔ lp. Local Spectral Multiplicity

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Theorem (Computation of the multiplicity function)

We have

- $E_{ac} \sim \mu_{ac}$ *and* $N_H(x) = r(x)$, E_{ac}*-a.e.*
- $E_{s,ac} \sim \mathbb{1}_{\{r(x)=2\}} d\mu_s$ *and* $N_H(x) = 1$, $E_{s,ac}$*-a.e.*
- $N_H(x) = 1$, $E_{s,s}$*-a.e.*

Case lp $\leftrightarrow$ lp. Simple Spectrum

Corollary

The singular spectrum of $T_{\min}(H)$ is always simple.

Case lp ↔ lp. Simple Spectrum

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The singular spectrum of $T_{\min}(H)$ is always simple.

Theorem (Characterization of simplicity)

The operator $T_{\min}(H)$ has simple spectrum if and only if the set

$$\left\{x \in \mathbb{R} : \lim_{\epsilon \downarrow 0} \operatorname{Im} q_{H_+}(x + i\epsilon) \text{ exists in } (0, \infty)\right\} \\ \cap \left\{x \in \mathbb{R} : \lim_{\epsilon \downarrow 0} \operatorname{Im} q_{H_-}(-x + i\epsilon) \text{ exists in } (0, \infty)\right\}$$

has Lebesgue measure zero.

Case lp ↔ lp. Simple Spectrum

An explicit sufficient condition for simplicity is:

Theorem

Assume that H_+ has the Hilbert-Schmidt property, i.e., that there exists $\phi \in [0, \pi)$ with

- $\displaystyle\int_{s_0}^{s_+} \xi_\phi^T H(t) \xi_\phi \, dt < \infty$,
- $\displaystyle\int_{s_0}^{s_+} \Big(\int_{s_0}^{t} \xi_{\phi+\frac{\pi}{2}}^T H(u) \xi_{\phi+\frac{\pi}{2}} \, du \Big) \, \xi_\phi^T H(t) \xi_\phi \, dt < \infty.$

Then the spectrum of $T_{\min}(H)$ is simple.