# 2. Synthesis of MIMO Control Systems

- For SISO systems, Bode and root locus methods are adequate for design of feedback control systems. The phase margin and sensitivity are well known concepts.
- However these are graph-based and difficult to be generalized to MIMO systems, which motivated state space approach to design of MIMO control systems.
- We begin with linear quadratic regulator (LQR) over the infinity time horizon aimed at minimizing the performance index

$$J = \int_0^\infty x(t)'Qx(t) + u(t)'Ru(t)\ dt, \tag{1}$$

where $R = R' > 0$ and $Q = Q' \geq 0$, and the state vector is governed by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \neq 0 \tag{2}$$

- For simplicity, the control law is restricted to the class of linear state feedback $u(t) = Fx(t)$ over all stabilizing state feedback gain $F$. It follows from (2) that

$$\dot{x}(t) = [A + BF]x(t), \quad x(0) = x_0 \neq 0.$$

## LQR Control

- The state trajectory $x(t)$ is given by $x(t) = e^{A_F t} x_0$. The following Lyapunov equation

$$A_F' X_F + X_F A_F + Q + F' R F = 0 \tag{3}$$

  has a unique solution $X_F \geq 0$, if $A_F := A + BF$ is a stability matrix, assuming $(A, B)$ is stabilizable.

- Substituting $u(t) = Fx(t)$ and $x(t) = e^{A_F t} x_0$ into performance index $J$ in (1) yields

$$\begin{aligned} J &= \int_0^\infty x_0' e^{A_F' t} (Q + FRF) e^{A_F t} x_0 \, dt \\ &= -\int_0^\infty x_0' e^{A_F' t} (A_F' X_F + X_F A_F) e^{A_F t} x_0 \, dt \\ &= -x_0' \left[ e^{A_F' t} X_F e^{A_F t} \right] x_0 \Big|_0^\infty = x_0' X_F x_0 \end{aligned}$$

- Since $x_0 \in \mathbb{R}^n$ can be arbitrary, minimizing $J$ is equivalent to minimizing "$X_F$" in some sense over the set of all stabilizing $F$, which is nonempty.

- Replacing $A_F$ by $(A + BF)$ in Lyapunov equation (3) gives

$$A' X_F + X_F A + X_F B F + F' B' X_F + Q + F' R F = 0$$

## Monotonicity of the ARE Solution

- A simple completion of square yields the following identity $\Longleftrightarrow$ (3):
$$A'X_F + X_F A - X_F BR^{-1}B'X_F + Q + (F + R^{-1}B'X_F)'R(F + R^{-1}B'X_F) = 0. \quad (4)$$
- It is now meaningful to consider the algebraic Riccati equation (ARE)
$$A'X + XA - XBR^{-1}B'X + Q = 0, \quad (5)$$
which is obtained from the first part of (4), and $X$ is independent of $F$.
- A solution $X$ to ARE (5) is called stabilizing solution, if $X \geq 0$ and $(A + BF)$ is a stability matrix where $F = -R^{-1}B'X$.
- **Lemma 1** Suppose that $X \geq 0$ is a stabilizing solution to ARE (5). Then $X$ is an increasing function of $Q \geq 0$.
- Proof: Let $X \geq 0$ be a stabilizing solution to ARE (5), and $X_1 \geq 0$ be a stabilizing solution to the ARE
$$A'X_1 + X_1 A - X_1 BR^{-1}B'X_1 + Q_1 = 0 \quad (6)$$
where $Q_1 \geq Q$, we need to show that $X_1 \geq X$.

## Proof of Lemma 1

- Subtracting ARE (5) from ARE (6) yields

$$A'\Delta_X + \Delta_X A - X_1 B R^{-1} B' X_1 + X B R^{-1} B' X + \Delta_Q = 0 \tag{7}$$

where $\Delta_X = X_1 - X$ and $\Delta_Q = Q_1 - Q \geq 0$ by $Q_1 \geq Q$.

- Adding and subtracting $XBR^{-1}B'X_1$ with rearrangement changes (7) into

$$\begin{aligned} 0 &= A'\Delta_X + \Delta_X A - \Delta_X B R^{-1} B' X_1 - X B R^{-1} B' \Delta_X + \Delta_Q \\ &= A'\Delta_X + \Delta_X A - \Delta_X B R^{-1} B' X_1 - X_1 B R^{-1} B' \Delta_X + \Delta_X B R^{-1} B' \Delta_X + \Delta_Q \end{aligned}$$

- Define $F_1 = -R^{-1}B'X_1$. The above equation can be in turn written as

$$(A + BF_1)'\Delta_X + \Delta_X (A + BF_1) + \Delta_X B R^{-1} B' \Delta_X + \Delta_Q = 0$$

- Since $X_1 \geq 0$ is a stabilizing solution, $(A + BF_1)$ is a stability matrix. It follows from the Lyapunov equation that $\Delta_X$ can be written as

$$\Delta_X = \int_0^\infty e^{(A+BF_1)'t} \left( \Delta_X B R^{-1} B' \Delta_X + \Delta_Q \right) e^{(A+BF_1)'t} \, dt \geq 0$$

by $\Delta_Q \geq 0$ and $\Delta_X B R^{-1} B' \Delta_X \geq 0$ that concludes $X_1 \geq X$.

# LQR Solution

- Coming back to the equation (4), we conclude that $X_F \geq X$ with $X \geq 0$ the stabilizing solution to ARE (5). Equality holds, if the stabilizing $F = -R^{-1}B'X$ is taken leading to $X = X_F$.
- In summary the LQR problem admits an optimal solution, if and only if the ARE (5) admits a stabilizing solution $X \geq 0$, and in this case $J$ is minimized by taking $F = -R^{-1}B'X$. The minimum index $J_{\min} = x_0'Xx_0$.
- We note that the ARE (5) can be written in form of Lyapunov equation:

$$(A + BF)'X + X(A + BF) + Q + F'RF$$

  that has the same form as (3). It follows that any stabilizing solution $X \geq 0$.
- **Lemma 2** The stabilizing solution to ARE (5) is unique, if it exists.
- Proof: The proof is similar to the proof of Lemma 1 by setting $Q_1 = Q$.
- Assume that two stabilizing solutions $X \geq 0$ and $X_1 \geq 0$ exist. Then (7) reduces to

$$A'\Delta_X + \Delta_X A - X_1BR^{-1}B'X_1 + XBR^{-1}B'X = 0$$

## Existence of the Stabilizing Solution

- By adding and subtracting $XBR^{-1}B'X_1$ again yields

$$\begin{aligned} 0 &= A'\Delta_X + \Delta_X A - \Delta_X BR^{-1}B'X_1 - XBR^{-1}B'\Delta_X \\ &= A'\Delta_X + \Delta_X A + \Delta_X BF_1 + F'B'\Delta_X = (A+BF)'\Delta_X + \Delta_X(A+BF_1) \end{aligned}$$

- Stability of $(A+BF)$ and $(A+BF_1)$ imply that $\Delta_X = 0$, i.e., $X = X_1$, concluding the proof for the uniqueness of the stabilizing solution.

- **Theorem 1** Let $Q = C_1'C_1$. There exists a stabilizing solution $X \geq 0$ to ARE (5), if and only if $(A, B)$ is stabilizable and

$$\operatorname{rank}\left\{\begin{bmatrix} A - j\omega I_n \\ C_1 \end{bmatrix}\right\} = n \ \ \forall\, \omega \in \mathbb{R}, \tag{8}$$

- Proof: Stabilizability of $(A, B)$ is clearly necessary for the existence of the stabilizing solution. We use a contradiction argument to show necessity of rank condition (8).

- If the stabilizing solution exists but the rank condition (8) fails, then there exists a nonzero vector $v \in \mathbb{R}^n$ such that $Av = j\omega_0 v$ and $C_1 v = 0$ for some $\omega_0 \in \mathbb{R}$.

## Proof of Theorem 1

- For convenience, we copy ARE (5) here: $A'X + XA - XBR^{-1}B'X + C_1'C_1 = 0$.
- Multiplying $v$ from right and $v^*$ from left to ARE (5) yields

$$(j\omega_0 - j\omega_0)v^*Xv - v^*XBR^{-1}B'Xv = 0 \quad \Longrightarrow \quad B'Xv = 0.$$

- It follows that $Fv = -R^{-1}B'Xv = 0$ and thus $(A+BF)v = Av = j\omega_0 v$, i.e., $j\omega_0$ remains to be an eigenvalue of $(A+BF)$, contradicting to that ARE (5) admits the stabilizing solution.
- For sufficiency, assume $(A, B)$ is stabilizable and the rank condition (8) is true. We will show that ARE (5) admits the stabilizing solution.
- Stabilizability of $(A, B)$ implies the existence of $F_1$ such that $(A + BF_1)$ is a stability matrix, and thus there exists $X_i \geq 0$ for $i = 1$ to the Lyapunov equation

$$(A + BF_i)'X_i + X_i(A + BF_i) + F_i'RF_i + Q = 0 \tag{9}$$

- We set $F_{i+1} = -R^{-1}B'X_i$ for $i = 1$, and then repeat the process of computing $X_i$ from (9) and setting $F_{i+1} = -R^{-1}B'X_i$ for $i = 2, 3, \cdots$.

## Proof of Theorem 1 (Continued)

- It is claimed that (i) $(A+BF_{i+1})$ is a stability matrix as long as $(A+BF_i)$ is a stability matrix, (ii) $X_i \geq X_{i+1}$ for each $i$, and (iii) $X_i \to X$ that is the stabilizing solution to ARE (5), if the rank condition (8) holds.

- Indeed $X_i \geq 0$ if $(A+BF_i)$ is a stability matrix. In addition the Lyapunov equation (9) has the same form as that in (3) as copied below:

$$A_F' X_F + X_F A_F + Q + F'RF = 0$$

except that $X_i$ and $F_i$ are interchanged with $X$, and $F$, respectively.

- An equivalent equation to (4) holds, which is the following (by noting $F_{i+1} = -R^{-1}B'X_i$):

$$A'X_i + X_iA - X_iBR^{-1}B'X_i + Q + (F_i - F_{i+1})'R(F_i - F_{i+1}) = 0. \qquad (10)$$

- Since $F_{i+1} = -R^{-1}B'X_i$, we have $-X_iBR^{-1}B'X_i = X_iBF_{i+1} = F_{i+1}'B'X_i$. The first part of (10), i.e., the ARE-similar part, can be rewritten, leading to

$$(A+BF_{i+1})'X_i + X_i(A+BF_{i+1}) + F_{i+1}'RF_{i+1} + Q + (F_i - F_{i+1})'R(F_i - F_{i+1}) = 0$$

## Proof of Theorem 1 (Continued)

- The fact that $(A + BF_{i+1}) + B(F_i - F_{i+1}) = (A + BF_i)$ is a stability matrix implies that the pair $\{(F_i - F_{i+1}), (A + BF_{i+1})\}$ is detectable.

- The fact that $X_i \geq 0$ implies that $(A + BF_{i+1})$ is a stability matrix, in light of the Lyapunov stability theory. Hence (i) is true.

- Denote $\Delta_i = X_i - X_{i+1}$. Subtracting (9) with $i$ replaced by $(i + 1)$ from the bottom equation in the previous page yields

$$(A + BF_{i+1})'\Delta_i + \Delta_i(A + BF_{i+1}) + (F_i - F_{i+1})'R(F_i - F_{i+1}) = 0$$

- Since (i) is true or $(A+BF_{i+1})$ is a stability matrix, there holds $\Delta_i \geq 0$, i.e., $X_i \geq X_{i+1}$ that proves (ii).

- We notice that the sequence $\{X_i\}$ is monotonically decreasing and bounded below by $X_i \geq 0 \ \forall \ i$, it admits a limit $X$ that clearly satisfies the ARE (5) that proves (iii).

- Finally we need to show stability of $(A + BF)$ where $F = -R^{-1}B'X$, which is not implied by stability of $(A + BF_i)$ for each $i$.

## Positive Real (Continued)

- It is possible that $(A+BF_i)$ approach the imaginary axis as $i \to \infty$. However the rank condition (8) prevents this from taking place, because any eigenvalue of $(A+BF)$ on the imaginary axis is observable.
- So if $(A+BF)$ has eigenvalue on the imaginary axis, the performance index $J$ will increase indefinitely, contradicting to the finite value of $J = x_0' X x_0$ in the limit.
- Next we will show an important robustness property for LQR control. Before doing this, we introduce a new notion, termed as *positive real* (PR).
- A square transfer matrix $T(s)$ is said to be PR, if $T(s)$ is stable and $T(s)+T(s)^* \geq 0$ for all $\mathrm{Re}[s] \geq 0$.
- **Claim:** Under LQR control, the transfer matrix $T_F(s) = -RF(sI - A - BF)^{-1}B$ is PR.
- To show this, rewrite the ARE (5) into the form of Lyapunov equation:

$$A_F' X + X A_F + F'RF + Q = 0, \quad A_F = A + BF$$

## Robustness of LQR Control

- For each $s \in \overline{\mathcal{H}}_+$, $s = \sigma + j\omega$ with $\sigma \geq 0$. The above Lyapunov equation is same as

$$(sI - A_F)^* X + X(sI - A_F) = F'RF + Q + 2\sigma X$$

- Multiplying the above equation by $B'(sI - A_F)^{*-1}$ from left, by $(sI - A_F)^{-1}B$ from right, and using the relation $B'X = -RF$ leads to

$$T_F(s)^* + T_F(s) = B'(sI - A_F)^{*-1}\left(F'RF + Q + 2\sigma X\right)(sI - A_F)^{-1}B \geq 0$$

for all $\sigma \geq 0$, thereby concluding the PR property.

- Define the return difference as (recall $1 + K(s)P(s)$ for SISO systems)

$$I - F(sI - A)^{-1}B$$

that measures the robustness of the feedback system. It is a fact that

$$[I - B'(-j\omega I - A')^{-1}F']R[I - F(j\omega I - A)^{-1}B] \geq R \tag{11}$$

for all $\omega \in \mathbb{R}$. In the case of single input ($m = 1$), the above is the same as

$$|1 - F(j\omega I - A)^{-1}B| \geq 1 \;\; \forall\, \omega \;\in\; \mathbb{R}.$$

## Proof of Inequality (11)

- The loop transfer function under state feedback is "$-F(j\omega I - A)^{-1}B$". So its Nyquist plot never intersects the unit disk centered at the critical point of $-1+j0$ in the complex plane, implying $60^o$ phase margin and infinite gain margin.
- To prove inequality (11), we note $XBR^{-1}B'X = F'RF$ and write the ARE as

$$X(sI - A) + (-sI - A')X + F'RF = C_1'C_1$$

- Multiplying $B'(-sI - A')^{-1}$ from left and $(sI - A)^{-1}B$ from right gives

$$\begin{aligned} \text{left} &= B'(-sI - A')^{-1}XB + B'X(sI - A)^{-1}B + B'(-sI - A')^{-1}F'RF(sI - A)^{-1}B \\ &= \text{right} = B'(-sI - A')^{-1}C_1'C_1(sI - A)^{-1}B \end{aligned}$$

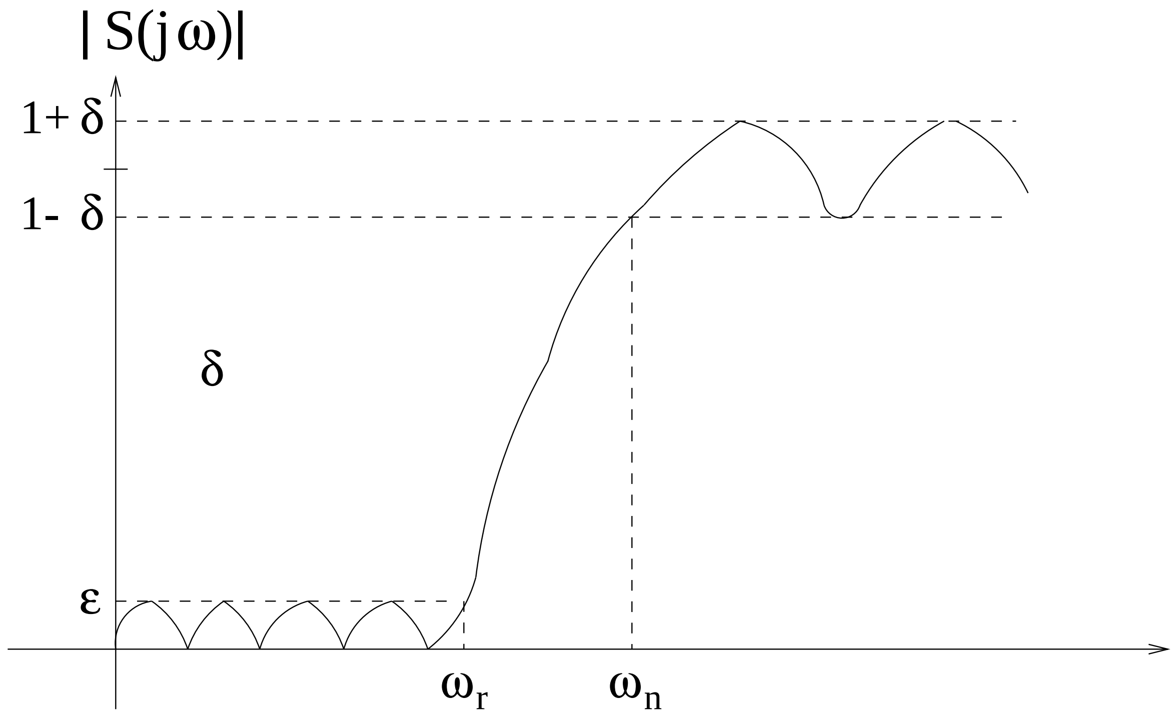
- For convenience denote $G_F(s) = F(sI - A)^{-1}B$. Then $B'X = -RF$ yields

$$\begin{aligned} R + \text{left} &= R - G_F(-s)R - RG_F(s) + G_F(-s)'RG_F(s) = [I - G_F(-s)]'R[I - G_F(s)] \\ &= R + \text{right} = R + B'(-sI - A')^{-1}C_1'C_1(sI - A)^{-1}B \geq R \;\forall\; \mathrm{Re}[s] = 0 \end{aligned}$$

  that is the same as inequality (11).

## Performance in Frequency Domain

- Let $S_F(s) = [I - F(sI - A)^{-1}B]^{-1} = I + F(sI - A - BF)^{-1}B$ be sensitivity under state feedback. We require the sensitivity to have the following magnitude response:



# Frequency Shape Specification

- In other words we require the "loop gain" $|F(j\omega I - A)^{-1}B|$ to be large at low frequency range, be small at high frequency range, and to admit good phase margin in the transition frequency band.

- A good measure for "loop gain" is the singular values of $F(j\omega I - A)^{-1}B$ at different frequencies. Given plant model $P(s) = C(sI - A)^{-1}B$ (assuming $D = 0$ for simplicity), we can relate the "loop gain" $F(j\omega I - A)^{-1}B$ to the loop gain of $P(j\omega)$.

- Consider $P_W(s) = P(s)W(s)$ with $W(s)$ an appropriate weighting function in frequency domain. Integrators, lead, and lag compensators can be used in selection of $W(s)$. We choose $W(s)$ so that $P_W(s)$ has desired frequency shape as required earlier.

- Let $P_W(s) = C_W(sI - A_W)^{-1}B_W$ with $(A_W, B_W)$ stabilizable and $(C_W, A_W)$ detectable. Then we can solve the stabilizing solution to the ARE

$$A'_W X_W + X_W A_W - X_W B_W B'_W X + C'_W C_W = 0 \tag{12}$$

The feedback gain $F_W = -B'_W X_W$ is stabilizing.

## All Pass Characterization

- It is claimed that the right coprime factors of $P_W(s) = N_W(s)M_W(s)^{-1}$ with realization

$$G_W(s) = \begin{bmatrix} M_W(s) \\ N_W(s) \end{bmatrix} = \left[\begin{array}{c|c} A_W + B_W F_W & B_W \\ \hline F_W & I \\ C_W & 0 \end{array}\right]$$

satisfy the normalized condition: $N_W(j\omega)^* N_W(j\omega) + M_W(j\omega)^* M_W(j\omega) = I$ for all $\omega \in \mathbb{R}$. In the SISO case, $|N_W(j\omega)|^2 + |M_W(j\omega)|^2 = 1$ for all $\omega$.

- **Lemma 3** Consider $G(s) = D_g + C_g(sI - A_g)^{-1}B_g$ with $A_g$ Hurwitz stable and $(A_g, B_g)$ controllable. Then $G(-s)'G(s) = I$ for all $s \in \overline{H}_+$, if and only if

$$\text{(i) } A_g'X + XA_g + C_g'C_g = 0, \quad \text{(ii) } D_g'C_g + B_g'X = 0$$

- Proof: We prove only the sufficiency. Rewrite the Lyapunov equation as

$$(-sI - A_g')X + X(sI - A_g) = C_g'C_g$$

- Multiplying $B_g'(-sI - A_g')^{-1}$ from left and $(sI - A_g)^{-1}B_g$ from right yields

$$B_g'X(sI - A_g)^{-1}B_g + B_g'(-sI - A_g')^{-1}XB_g = B_g'(-sI - A_g')^{-1}C_g'C_g(sI - A_g)^{-1}B_g$$

## Proof of Lemma 3

- Direct calculation and use (i) and (ii) show that

$$\begin{aligned} G(-s)'G(s) &= D_g'D_g + D_g'C_g(sI-A_g)^{-1}B_g + B_g'(-sI-A_g')^{-1}C_g'D_g \\ &\qquad + B_g'(-sI-A_g')^{-1}C_g'C_g(sI-A_g)^{-1}B_g \\ &= D_g'D_g + (D_g'C_g + B_g'X)(sI-A_g)^{-1}B_g \\ &\qquad + B_g'(-sI-A_g')^{-1}(D_g'C_g + B_g'X)' = I \;\forall\; s \end{aligned}$$

- For $G_W(s)$, we set $A_g = A_W + B_W F_W$, $B_g = B_W$, $C_g = \begin{bmatrix} F_W \\ C_W \end{bmatrix}$ and $D_g = \begin{bmatrix} I \\ 0 \end{bmatrix}$. Then ARE (12) can be put into the form of Lyapnov equation (recall (3) for ARE)

$$A_g'X_W + X_W A_g + C_g'C_g = 0, \quad C_g'C_g = F_W'F_W + C_W'C_W$$

- We also have $D_g'C_g + B_g'X_W = F_W + B_W X_W = 0$. Hence both (i) and (ii) in Lemma 3 are true, implying that $G_W(s)$ is all pass. Since $G_W(s)$ is also stable, $G_W(s)$ is called inner.

## Desired Frequency Shape

- Note that in the SISO case, we have that

$$|N_W(j\omega)| = \frac{|P_W(j\omega)|}{\sqrt{1+|P_W(j\omega)|^2}}, \quad |M_W(j\omega)| = \frac{1}{\sqrt{1+|P_W(j\omega)|^2}}.$$

- Hence if $|P_W(j\omega)|$ has good loop shape, then $F_W(sI - A_W)^{-1}B_W$ has a good frequency shape by $N_W(s) = C_W(sI - A_W - B_WF_W)^{-1}B_W$ and

$$M_W(s) = [I - F_W(sI - A_W)^{-1}B_W]^{-1} = I + F_W(sI - A_W - B_WF_W)^{-1}B_W$$

- So Bode design can be extended to MIMO control system design under state feedback.

- If the state feedback is not possible, then we estimate the state and use $u(t) = r(t) + F\hat{x}(t)$ with $r(t)$ the command input.

- If $P(s) = D + C(sI - A)^{-1}B$, then the estimator has the form

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - L[y(t) - \hat{y}(t)] = (A + LC)\hat{x}(t) + (B + LD)u(t) - Ly(t)$$

by $\hat{y}(t) = C\hat{x}(t) + Du(t)$. In practice $D = 0$.

## Observer-Based Controller

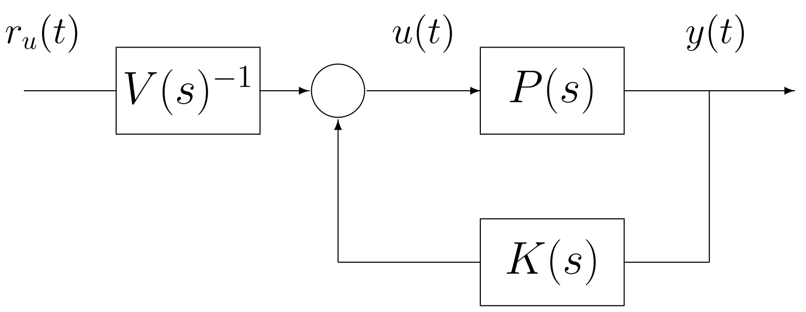
- Substituting $u(t) = r(t) + F\hat{x}(t)$ yields

$$\dot{\hat{x}}(t) = (A + BF + LC + LDF)\hat{x}(t) - Ly(t), \quad u(t) = r(t) + F\hat{x}(t)$$

- Hence the controller that is the transfer matrix from $y(t)$ to $u(t)$ is given by

$$\begin{aligned} K(s) = -V(s)^{-1}U(s) &= -F(sI - A - BF - LC - LDF)^{-1}L \\ \begin{bmatrix} V(s) & U(s) \end{bmatrix} &= \left[\begin{array}{c|cc} A + LC & -(B + LD) & L \\ \hline F & I & 0 \end{array}\right] \end{aligned}$$

- We use the following configured feedback system



## Observer-Based Controller (Continued)

- Implementation of the feed-forward and feedback controllers can be combined into the following single (two-degree-freedom) controller

$$K_a(s) = V(s)^{-1} \begin{bmatrix} I & -U(s) \end{bmatrix} = \left[ \begin{array}{c|cc} A + LC + BF + LDF & B + LD & -L \\ \hline F & I & 0 \end{array} \right] \tag{13}$$

with two-input $\{r_u(t), y(t)\}$ and one-output $u(t)$.

- The transfer matrix from $r_u(t)$ to $y(t)$ is identical to the case of state feedback:

$$T_{ry}(s) = D + (C + DF)(sI - A - BF)^{-1}B = C(sI - A - BF)^{-1}B \tag{14}$$

- We claim that the closed-loop system with transfer matrix $T_{ry}(s)$ is internally stable.

- Specifically, with $u(t) = F\hat{x}(t) + r_u(t)$ and $e_x(t) = x(t) - \hat{x}(t)$, we have state equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B[F\hat{x}(t) + r_u(t)] = (A + BF)x(t) - BFe_x(t) + Br_u(t), && (15) \\ \dot{\hat{x}}(t) &= (A + BF + LC + LDF)\hat{x}(t) + (B + LD)r_u(t) - Ly(t) \\ &= (A + BF)\hat{x}(t) - LCe_x(t) + Br_u(t). && (16) \end{aligned}$$

## Internal Stability

- Taking difference of (15) and (16) yields $\dot{e}_x(t) = (A + LC)e_x(t)$.
- Hence we obtain the state equation for the closed-loop system and output as

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{e}_x(t) \end{bmatrix} &= \begin{bmatrix} A + BF & -BF \\ 0 & A + LC \end{bmatrix} \begin{bmatrix} x(t) \\ e_x(t) \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} r_u(t), \\ y(t) &= \begin{bmatrix} C + DF & -DF \end{bmatrix} \begin{bmatrix} x(t) \\ e_x(t) \end{bmatrix} + Dr_u(t). \end{aligned}$$

- It follows that the closed-loop system is indeed internally stable, and the expression of transfer matrix $T_{ry}(s)$ in (14) holds.
- Unfortunately use of observer destroys the robustness. The 60 degree phase margin and infinity gain margin are not true anymore unless the estimation gain $L$ is designed in a smart way.
- We introduce loop transfer recovery technique: design of the state estimation gain is aimed at recover the loop transfer matrix in the sense that $P(s)K(s)$ is close to $-F(sI - A)^{-1}B$ where $K(s)$ is the observer-based controller in (13).

## Loop Transfer Recovery (LTR)

- We assume $D = 0$ and begin with Case (i): $P(s)$ is square and strictly minimum phase, satisfying $\det(CB) \neq 0$.

- In this case, $L = -Y_q C' J^{-1}$ with $Y_q \geq 0$ the stabilizing solution to ARE

$$AY_q + Y_q A' - Y_q C' J^{-1} C Y_q + q^2 BB' = 0, \tag{17}$$

where $J = J' > 0$. It is claimed that as $q \to \infty$, there holds

$$K(s)P(s) - \left[-F(sI - A)^{-1}B\right] \;\; \to \;\; 0.$$

- That is, the loop transfer matrix under state feedback is asymptotically recoverable.

- Notice that the ARE (17) can be written as

$$A\tilde{Y}_q + \tilde{Y}_q A' - \tilde{Y}_q C' \tilde{J}^{-1} C \tilde{Y}_q + BB' = 0$$

after multiplying with $q^{-2}$ where $\tilde{Y}_q = q^{-2} Y_q$ and $\tilde{J} = q^{-2} J$.

- Notice also that $L = \tilde{L} = -\tilde{Y}_q C' \tilde{J}^{-1}$. As $q \to \infty$, $\tilde{Y}_q \to 0$, and thus $\tilde{Y}_q C' \tilde{J}^{-1} C \tilde{Y}_q = \tilde{L} \tilde{J} \tilde{L} = q^{-2} L' J L = BB'$.

## **Loop Transfer Recovery (LTR)** (Continued)

- Let $V$ be square and orthogonal and thus $VV' = I$. Then as $q \to \infty$, there holds

$$BV = -q^{-1}Y_qC'J^{-1/2} = q^{-1}LJ^{1/2} \implies q^{-1}L \to BVJ^{-1/2} \tag{18}$$

- For Case (i), we claim that $(A + LC) = (A + qBVJ^{-1/2}C)$ is a stability matrix as $q \to \infty$. Indeed by $\det(CB) \neq 0$, there is no loss of generality to assume that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} I_m \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_s & 0 \end{bmatrix}$$

where $\det(CB) = \det(C_s) \neq 0$ and $A_{11}, C_s \in \mathbb{R}^{m\times m}$.

- The minimum phase assumption implies that $A_{22}$ is a stability matrix by

$$\text{rank}\left\{\begin{bmatrix} A - sI & B \\ C & 0 \end{bmatrix}\right\} = \text{rank}\left\{\begin{bmatrix} A_{11} - sI & A_{12} & I_m \\ A_{21} & A_{22} - sI_{n-m} & 0 \\ C_s & 0 & 0 \end{bmatrix}\right\} = n + m \ \ \forall\ \text{Re}[s] \geq 0$$

- There exists a square nonsingular matrix $\Omega$ such that $\hat{A}_{11} = (A_{11} - q\Omega C_s)$ is a stability matrix as $q \to \infty$, since $\det(C_s) \neq 0$. In fact we can make $\hat{A}_{11}$ negative definite.

## Loop Transfer Recovery (LTR) (Continued)

- The above implies that as $q \to \infty$, $A - qB\Omega C$ is Hurwitz stable. It also suggests that we choose $J$ such that $\Omega = -VJ^{-1/2}$ that is the same as $J = \Omega'\Omega$, implying that $A + qBVJ^{-1/2}C = A - qB\Omega C$ is a stability matrix as $q \to \infty$.

- Next we prove $K(s)P(s) \quad \to \quad -F(sI - A)^{-1}B$ as $q \to \infty$. Specifically we have

$$\begin{aligned}
-K(s) &= F(sI - A - BF - LC)^{-1}L = F(sI - A_F - LC)^{-1}L \\
&= F(sI - A_F)^{-1}\left[I + LC(sI - A_F)^{-1}\right]^{-1} L \\
&= F(sI - A_F)^{-1}Lq^{-1}\left[q^{-1}I + C(sI - A_F)^{-1}Lq^{-1}\right]^{-1}
\end{aligned}$$

- Taking limit $q \to \infty$, we have $q^{-1}L \to BVJ^{-1/2}$ by (18) and thus

$$\begin{aligned}
-K(s)P(s) &\to -F(sI - A_F)^{-1}BVJ^{-1/2}[C(sI - A_F)^{-1}BVJ^{-1/2}]^{-1}P(s) \\
&= -F(sI - A_F)^{-1}B[C(sI - A - BF)^{-1}B]^{-1}P(s) \\
&= -F(sI - A_F)^{-1}B\left[C(sI - A)^{-1}\left\{I - BF(sI - A)^{-1}\right\}^{-1} B\right]^{-1} P(s) \\
&= -F(sI - A_F)^{-1}B\left[C(sI - A)^{-1}B\left\{I - F(sI - A)^{-1}B\right\}^{-1}\right]^{-1} P(s)
\end{aligned}$$

## Loop Transfer Recovery (LTR) (Continued)

- Continue from the previous page, we have

$$
\begin{aligned}
-K(s)P(s) &= -F(sI - A_F)^{-1}B\left[P(s)\left\{I - F(sI - A)^{-1}B\right\}^{-1}\right]^{-1}P(s) \\
&= -F(sI - A_F)^{-1}B\left[I - F(sI - A)^{-1}B\right] \\
&= -F(sI - A - BF)^{-1}\left[I - BF(sI - A)^{-1}\right]B = -F(sI - A)^{-1}B
\end{aligned}
$$

- The above shows that the loop transfer matrix under state feedback can be fully recovered in Case (i). The infinity gain margin and sixty degree phase margin are preserved under output feedback.

## LTR for Case (ii)

- For Case (ii), $P(s)$ is assumed to be minimum phase but $\det(CB) = 0$. In this case $P(s) = H(s)^{-1}P_0(s)$ where $H(s)$ is a polynomial matrix and $P_0(s) = C_0(sI - A_0)^{-1}B_0$ is minimum phase and $\det(C_0B_0) \neq 0$.
- We now set $K(s) = H_w(s)^{-1}K_0(s)H(s)$ with $K_0(s)$ designed as the loop transfer recovery controller with the observer gain $L_0$ designed based on $P_0(s)$, and $H_w(s)^{-1}$ is chosen such that $K(s)$ is proper or strictly proper with the poles of $H_w(s)^{-1}$ far away from the $j\omega$-axis on LPH.
- The above choice of $K(s)$ leads to

$$K(s)P(s) \quad \rightarrow \quad -H_w(s)^{-1}F(sI - A)^{-1}B_0$$

- For the SISO case, a typical $H_w(s) = (\epsilon s + 1)$ with $\epsilon > 0$ sufficiently small. Thus the dynamics of $K(s)P(s)$ is dominated by $-F(sI - A)^{-1}B$ that is the desired one.
- We note that $B = B_0$ is true generally, and the state feedback is also design based on $P_0(s)$.

## LTR for Case (iii)

- For Case (iii), $P(s)$ does not satisfy $\det(CB) \neq 0$, nor being minimum phase.
- In this case we can factorize $P(s) = H(s)^{-1}P_0(s)B_a(s)$ where $H(s)$ is a polynomial matrix, $B_a(s)$ is a stable and all-pass transfer matrix, and $P_0(s) = C_0(sI - A_0)^{-1}B_0$ is minimum phase and $\det(C_0B_0) \neq 0$. By an abuse of notation we assume that $P(s) = P_0(s)B_a(s) = C(sI - A)^{-1}B = C(sI - A)^{-1}B_0B_a(s)$ and thus $\det(CB) \neq 0$.
- We design $K_0(s)$ as the loop transfer recovery controller with the observer gain $L_0$ designed based on $P_0(s)$. Note that $B_0B_a(s) \neq B$ for Case (iii).
- Setting $K(s) = H_w(s)^{-1}K_0(s)H(s)$ with similar $H_w(s)$ to Case (ii) leads to

$$K(s)P(s) \;\;\rightarrow\;\; -F(sI - A)^{-1}B_0B_a(s)$$

- It follows that the LTR cannot be achieved fully, and loop transfer function is dominated by $-F(sI - A)^{-1}B_0B_a(s)$.
- First it is noted that as $q \to \infty$, there holds $q^{-1}L \;\rightarrow\; B_0VJ^{-1/2}$ and $B_0 \neq B$ by (18), and the fact that design of $L$ is based on $P_0(s)$.

## LTR for Case (iii) (Continued)

- We can employ (**??**) with a simple modification when taking limit $q \to \infty$:

$$
\begin{aligned}
-K(s)P(s) &= F(sI - A_F)^{-1}q^{-1}L\left[q^{-1}I - C(sI - A_F)^{-1}q^{-1}L\right]P(s) \\
&\to -F(sI - A_F)^{-1}B_0 V J^{-1/2}[C(sI - A_F)^{-1}B_0 V J^{-1/2}]^{-1}P(s) \\
&= -F(sI - A - BF)^{-1}B_0[C(sI - A - BF)^{-1}B_0]^{-1}P(s)
\end{aligned}
$$

- Denote $G_{F0}(s) = -F(sI - A)^{-1}B_0$ and $G_F(s) = -F(sI - A)^{-1}B$. Then we have

$$
\begin{aligned}
-F(sI - A - BF)^{-1}B_0 &= -F\left[I - (sI - A)^{-1}BF\right]^{-1}(sI - A)^{-1}B_0 \\
&= -\left[I - F(sI - A)^{-1}B\right]^{-1}F(sI - A)^{-1}B_0 \\
&= [I + G_F(s)]^{-1}G_{F0}(s) \qquad (19)
\end{aligned}
$$

- Denote $T_{F0}(s) = C(sI - A - BF)^{-1}B_0$. The matrix inversion formula yields

$$
\begin{aligned}
T_{F0}(s) &= C\left[(sI - A)^{-1} + (sI - A)^{-1}B\{I - F(sI - A)^{-1}B\}^{-1}F(sI - A)^{-1}\right]B_0 \\
&= C(sI - A)^{-1}B_0\left[I + B_a(s)\{I - F(sI - A)^{-1}B\}^{-1}F(sI - A)^{-1}B_0\right] \\
&= C(sI - A)^{-1}B_0\left[I - B_a(s)\{I + G_F(s)\}^{-1}G_{F0}(s)\right]
\end{aligned}
$$

## LTR for Case (iii) (Continued)

- The above derivations imply

$$\begin{aligned} T_{\mathrm{tmp}}(s) &:= T_{F0}(s)^{-1}P(s) = [C(sI-A-BF)^{-1}B_0]^{-1}P(s) \\ &= [C(sI-A-BF)^{-1}B_0]^{-1}C(sI-A)^{-1}B \\ &= [C(sI-A-BF)^{-1}B_0]^{-1}C(sI-A)^{-1}B_0B_a(s) \\ &= \left[I - B_a(s)\{I+G_F(s)\}^{-1}G_{F0}(s)\right]^{-1}B_a(s) \end{aligned}$$

- Using the derivation in (19) and the above $T_{\mathrm{tmp}}(s)$ for $-K(s)P(s)$ leads to

$$\begin{aligned} -KP(s) &\rightarrow [I+G_F(s)]^{-1}G_{F0}(s)\left[I - B_a(s)\{I+G_F(s)\}^{-1}G_{F0}(s)\right]^{-1}B_a(s) \\ &= [I+G_F(s)]^{-1}\left[I - G_{F0}(s)B_a(s)\{I+G_F(s)\}^{-1}\right]^{-1}G_{F0}(s)B_a(s) \\ &= [I+G_F(s)-G_{F0}(s)B_a(s)]^{-1}G_{F0}(s)B_a(s) \\ &= [I+E(s)]^{-1}[G_F(s)-E(s)] \end{aligned}$$

by taking $E(s) = G_F(s) - G_{F0}(s)B_a(s) = F(sI-A)^{-1}[B_0B_a(s)-B] \neq 0$ as the LTR error and thus $G_{F0}(s)B_a(s) = G_F(s) - E(s)$. Note $C(sI-A)^{-1}[B_0B_a(s)-B] = 0$ by second bullet of page 25.