Feedback Control System Design

- We consider control system design for the feedback system in Fig. 1. For simplicity, we start with SISO systems.

- The performance on sensitivity and tracking requires that

$$|S(j\omega)| = \left|\frac{1}{1 + K(j\omega)P(j\omega)}\right| \leq |W_{\mathrm{p}}(j\omega)| \quad \forall\ |\omega| \leq \omega_{\mathrm{L}}. \tag{1}$$

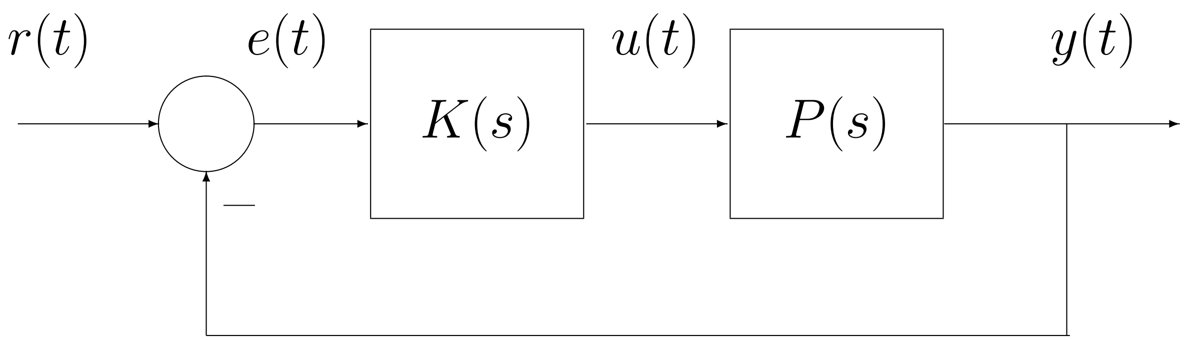

Fig. 1 Feedback control system

Conversion to Loop Gain at Low Frequencies

- The magnitude response of $W_{\mathrm{p}}(s)$ represents the performance in low frequency range. One may consider $W_{\mathrm{p}}(s) = R(s)^{-1}$ for the tracking performance.
- However, this may limit the performance weighting function $W_{\mathrm{p}}(s)$. For instance, unit step would give $W_{\mathrm{p}}(s) = s$ that increases as frequency increases, leading to the difficulty to impose $|W_{\mathrm{p}}(j\omega)| = 0.1$ at $\omega = 1$ and $|W_{\mathrm{p}}(j\omega)| = 0.2$ at $\omega = 3$.
- We assume that $W_{\mathrm{p}}(s)$ is a lowpass filter that has very small values at low frequency range, and greater or equal to one at high frequency range, and finitely valued.
- Denote $L(s) = P(s)K(s)$ that is the loop transfer function. Then constraint (1) is ensured by $|1 + L(j\omega)| \geq |L(j\omega)| - 1 \geq |W_{\mathrm{p}}(j\omega)|^{-1}$, bringing us the first condition:

$$|L(j\omega)| \geq 1 + |W_{\mathrm{p}}(j\omega)|^{-1} \quad \forall\ |\omega| \leq \omega_{\mathrm{L}}. \tag{2}$$

- Because model errors are mostly in high frequency range, and output measurement noises are also concentrated in high frequencies, these impose the constraint

$$|T(j\omega)| = \left| \frac{L(j\omega)}{1 + L(j\omega)} \right| \leq |W_{\mathrm{s}}(j\omega)| \quad \forall\ |\omega| \geq \omega_{\mathrm{H}}. \tag{3}$$

Conversion to Loop Gain at High Frequencies

- The frequency shape of $|W_{\rm s}(j\omega)|$ is of lowpass type. Inequality (3) is equivalent to

$$\frac{1}{|1+L(j\omega)^{-1}|} \leq |W_{\rm s}(j\omega)| \quad \Longleftrightarrow \quad \left|1+L(j\omega)^{-1}\right| \geq |W_{\rm s}(j\omega)|^{-1} \;\; \forall\ |\omega| \geq \omega_{_{\rm H}}.$$

- Assuming $|W_{\rm s}(j\omega)| << 1$ for $\omega \geq \omega_{_{\rm H}}$, the above inequality is ensured, if for all $\omega \geq \omega_{_{\rm H}}$, there holds $|L(j\omega)|^{-1} - 1 \geq |W_{\rm s}(j\omega)|^{-1}$ that is in turn equivalent to

$$|L(j\omega)| \leq \frac{|W_{\rm s}(j\omega)|}{1+|W_{\rm s}(j\omega)|} \;\; \forall\ |\omega| \geq \omega_{_{\rm H}}. \tag{4}$$

- Hence we obtain the desired loop gain schematically illustrated in Fig. 2 next page.
- The lower-left shaded area represents the performance requirement on sensitivity, and tracking.
- The upper-right shaded area represents the performance requirement on noise rejection and the closed-loop stability condition.
- The two shaded areas cannot overlap in order for control system design to be feasible.

Loop Gain Frequency Shape

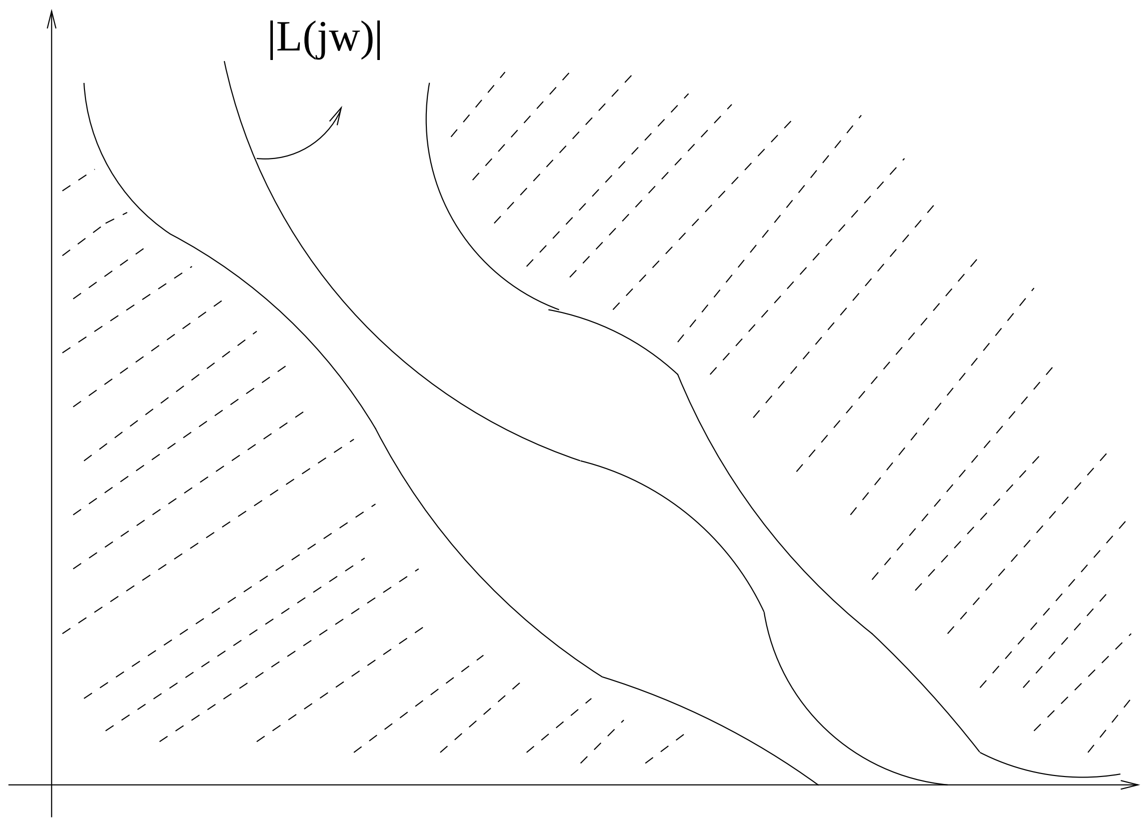

Fig. 2 Desired loop gain in dB

Shape of the Plant Model

- Because it is difficult to design the feedback controller $K(s)$ to ensure both feedback stability and ideal loop gain shape, we choose to shape the plant model directly, removing the constraint of the feedback stability.
- Specifically, we would like to have $P_{\mathrm{W}}(s) = P(s)W(s)$ the same desired frequency shape as the ideal $L(s)$ by choosing some stable or semi-stable $W(s)$.
- We can then set $K(s) = W(s)K_W(s)$ and thus $P(s)K(s) = P_W(s)K_W(s)$, thereby converting design of $K(s)$ into that of $K_W(s)$.
- We assume that we find some $W(s)$ such that

$$\begin{aligned} \frac{1}{\sqrt{1+|P_W(j\omega)|^2}} &\le |W_{\mathrm{p}}(j\omega)| \quad \forall\, |\omega| \le \omega_{\mathrm{L}}, \\ \frac{|P_W(j\omega)|}{\sqrt{1+|P_W(j\omega)|^2}} &\le |W_{\mathrm{s}}(j\omega)| \quad \forall\, |\omega| \ge \omega_{\mathrm{H}}. \end{aligned} \tag{5}$$

- The above inequalities are due to the gain frequency shape for $P_W(s)$.

State-Space Formulation

- Then we find a state-space realization for

$$P_W(s) = D_W + C_W(sI - A_W)^{-1}B_W = \left[\begin{array}{c|c} A_W & B_W \\ \hline C_W & D_W \end{array}\right].$$

- If we have state measurement for $P_W(s)$, then we can use state feedback control $u_w(t) = F_W x_w(t)$ that gives the loop transfer function $L_W(s) = F_W(sI - A_W)^{-1}B_W$.

- The sensitivity and the closed-loop transfer functions are given respectively by

$$S_W(s) = \left[\begin{array}{c|c} A_W + B_W F_W & B_W \\ \hline F_W & I \end{array}\right], \qquad T_W(s) = \left[\begin{array}{c|c} A_W + B_W F_W & B_W \\ \hline C_W + D_W F_W & D_W \end{array}\right].$$

- Indeed, under state feedback control with $u_w(t) = r(t) + F_w x_w(t)$, we have

$$\dot{x}_w(t) = (A_W + B_W F_W)x_w(t) + B_W r(t), \quad y(t) = (C_W + D_W F_W)x_w(t) + D_W r(t).$$

- The sensitivity $S_W(s) = [I + L_W(s)]^{-1}$ with $L_W(s) = -F_W(sI - A_W)^{-1}B_W$, yielding the expression of $S_W(s)$. We also note that $u_w(t)$ represents the tracking error.

Achieving Desired Frequency Shape Under State Feedback

- More importantly, $P_W(s) = T_W S_W(s)^{-1}$, i.e., $\{S_W(s), T_W(s)\}$ is RCF of $P_W(s)$.
- Suppose that $X_W \geq 0$ is the stabilizing solution to ARE

$$A_W' X_W + X_W A_W - X_W B_W B_W' X_W + C_W' C_W = 0. \tag{6}$$

- Then with $F_W = -B_W' X_W$, there holds (Problem 3 of Homework 5):

$$|S_W(j\omega)|^2 + |T_W(j\omega)|^2 = 1.$$

- We thus have both inequalities in (5) satisfied in light of

$$\begin{aligned} \frac{1}{\sqrt{1 + |P_W(j\omega)|^2}} &= |S_W(j\omega)| \leq |W_{\rm p}(j\omega)| \quad \forall\ |\omega| \leq \omega_{\rm L}, \\ \frac{|P_W(j\omega)|}{\sqrt{1 + |P_W(j\omega)|^2}} &= |T_W(j\omega)| \leq |W_{\rm s}(j\omega)| \quad \forall\ |\omega| \geq \omega_{\rm H}. \end{aligned}$$

- If the state is not measurable, then we need to estimate the state, and employ the LTR technique to recover the frequency shape of the loop gain.

Design Example 1

- Consider the plant model given by

$$P(s) = \frac{1 - \delta s}{s(s-1)}, \quad \delta > 0,\ \delta \approx 0.$$

Find $W(s)$ such that $P_W(s) = P(s)W(s)$ satisfies $|P_W(5j)| \approx 0\text{dB}$, and

$$|P_W(j\omega)| \geq 20\text{dB} \ \ \forall\ |\omega| \leq 1, \qquad |P_W(j\omega)| \leq -35\text{dB} \ \ \forall\ |\omega| \geq 100. \tag{7}$$

In addition, design the output feedback controller using the LQR/LTR method.

- We begin with $\delta = 0$, in which case $W = 35$, a constant satisfying the requirement in (7), and $|P_W(5j)| \approx 0\text{dB}$. See the Bode plot in the next page.

- While we can proceed to design state feedback and estimate gains using LQR/LTR, we notice that $P_W(s)$ does not satisfy $C_W B_W \neq 0$, due to the degree difference equal to 2 in the case of $\epsilon = 0$. In addition, the slop of -40dB per decade at crossover frequency near 5 may cause problem. These two issues may cause problem in the LTR procedure.

Bode Plot with Constan Weighting

The following Matlab plot shows the Bode plot of $P_W(s)$ with $W = 35$.

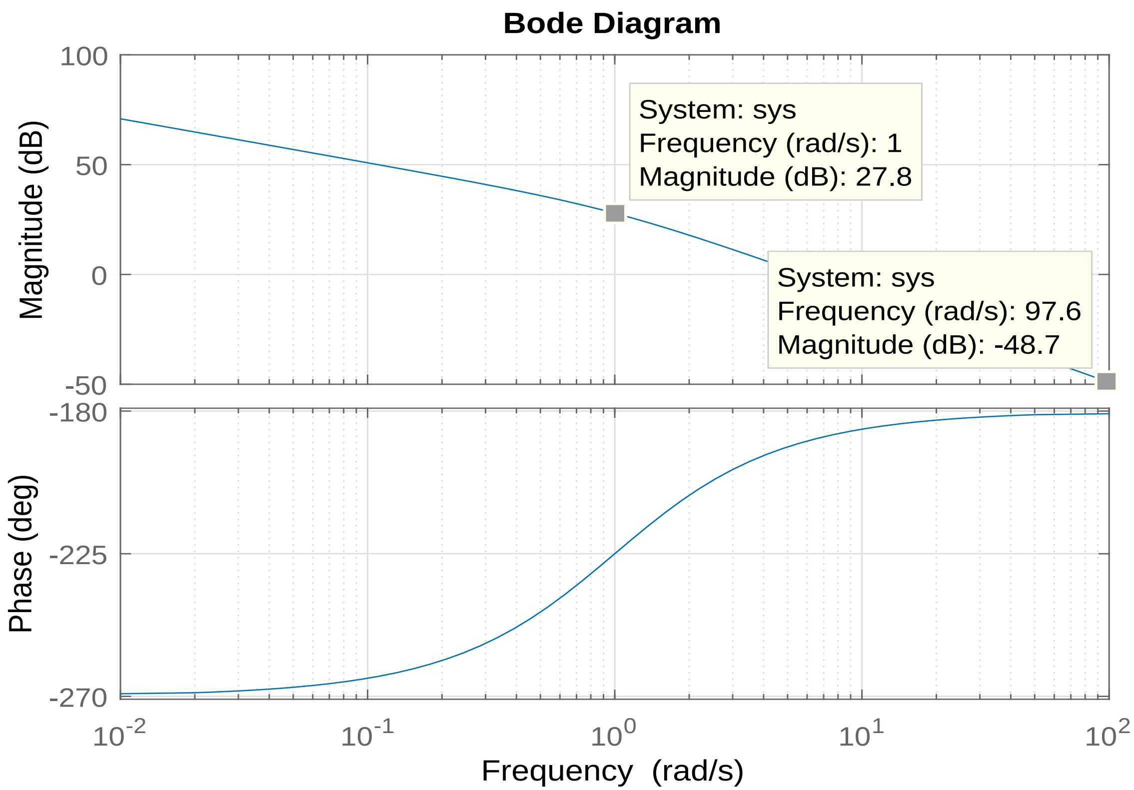

Fig. 3 Frequency response of $P_W(s)$ with $W = 35$

LQR Design

- We thus consider transfer function weighting with $W(s)$ chosen such that the slope at the crossover frequency is about -20dB/dec.
- After a few trials, we decide to take $W(s) = \dfrac{160(s+3)}{s+30}$ with Bode plot in next page.
- The crossover frequency is about 5.77 with slope -20dB/dec, implying that the pole at -30 has small impact to the LTR in the low frequency range.
- Computing the stabilizing solution to ARE (6) yields

$$X_W = \begin{bmatrix} 8.4261 & 279.86 & 480.00 \\ 279.86 & 974.12 & 17965 \\ 480.00 & 1.7965 & 43132 \end{bmatrix}, \quad F_W = -\begin{bmatrix} 8.4261 & 279.86 & 480.00 \end{bmatrix},$$

  that will give desired frequency shape to $S_W(s)$ and $T_W(s)$.
- The step response under state feedback gives $t_r = 0.361$, $t_s = 0.568$, and PO $= 1\%$ for $T_W(s) = C_W(sI - A_W - B_W F_W)^{-1} B_W$, which are very decent transient response, as expected by the frequency shape of $P_W(s)$. See Fig. 5.

Bode Plot with Dynamic Weighting

The following Matlab plot shows the Bode plot of the new $P_W(s)$.

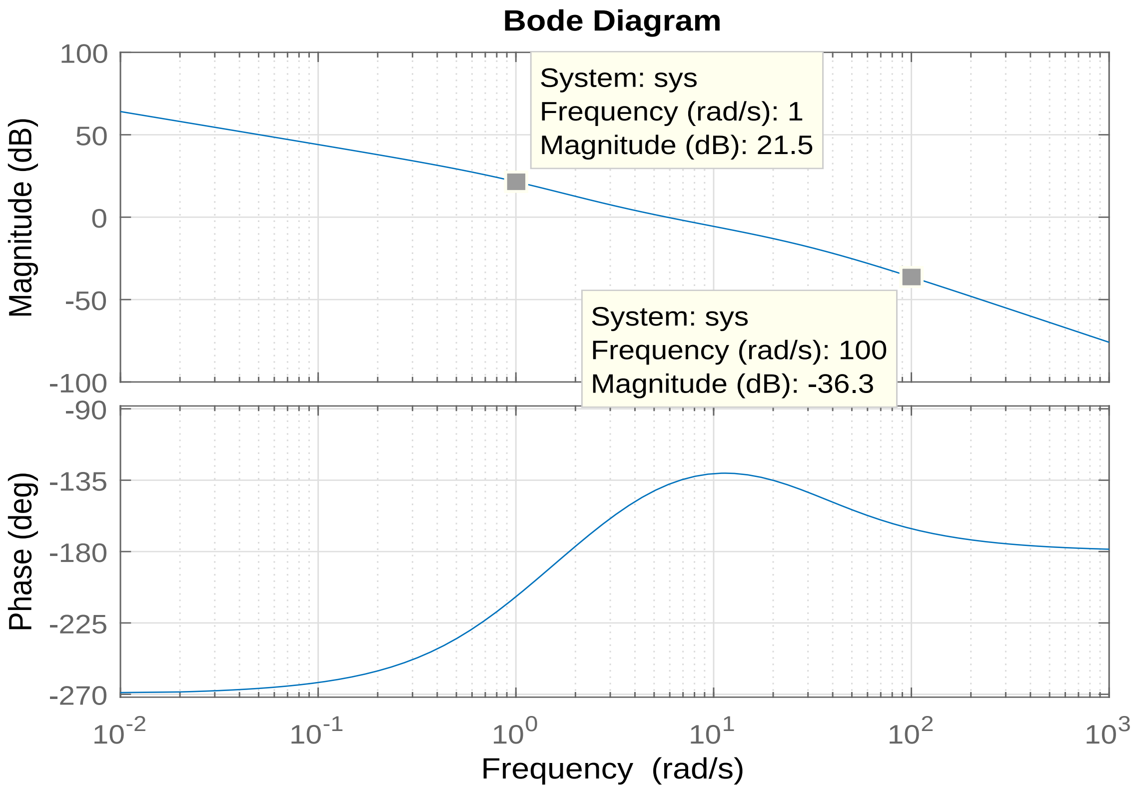

Fig. 4 Frequency response of $P_W(s)$ with dynamic weighting

Step Response Under State Feedback

The following Matlab plot shows the expected step response for $T_W(s)$.

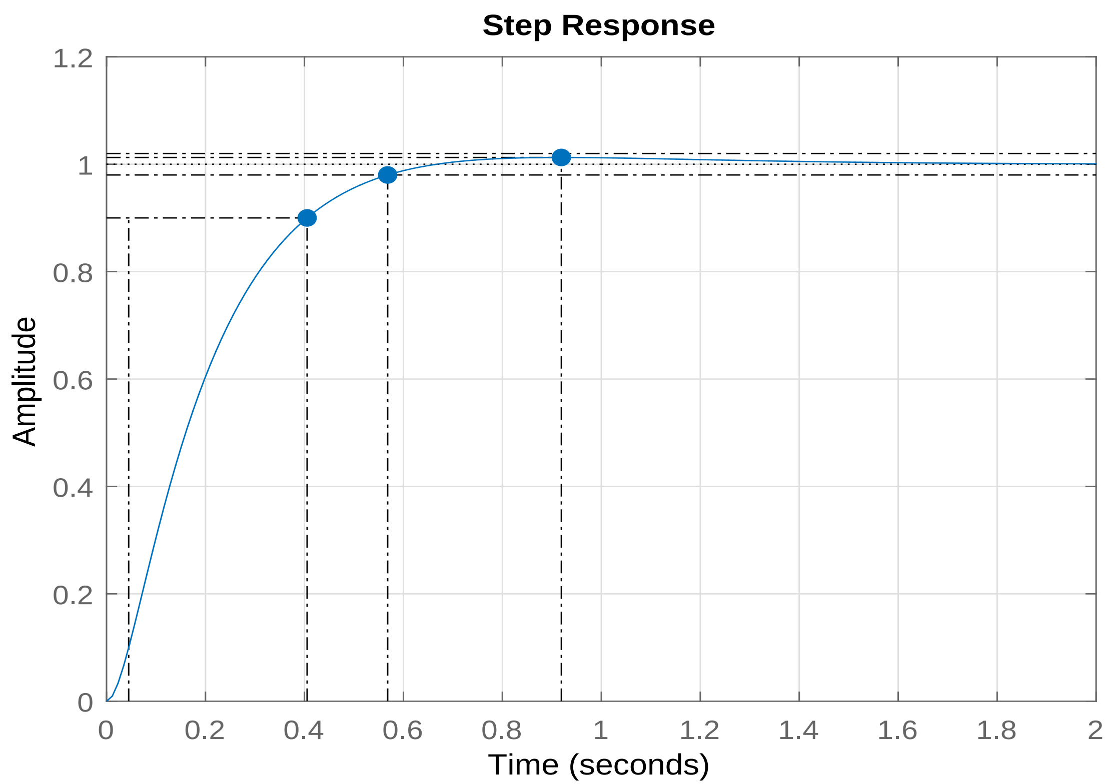

Fig. 5 Step response of $T_W(s)$

Loop Transfer Recovery

- For the LTR part, we simply compute the stabilizing solution to

$$A_W Y_q + Y_q A_W' - Y_q C_W' C_W Y_q + q^2 B_W B_W' = 0.$$

- By taking $q^2 = 1000$, we have

$$Y_q = \begin{bmatrix} 11.7445 & 0.11265 & 3.969 \times 10^{-5} \\ 0.11265 & 0.00291 & 1.929 \times 10^{-5} \\ 3.969 \times 10^{-5} & 1.929 \times 10^{-5} & 6.510 \times 10^{-6} \end{bmatrix}, \quad L_q = - \begin{bmatrix} 18.044 \\ 0.4747 \\ 0.0062 \end{bmatrix}.$$

- The magnitude plot of $|K_W(j\omega)P_W(j\omega)|$ is shown in Fig. 6 in solid line, together with that of $|P_W(j\omega)|$ (dashed line) and with

$$|L_W(j\omega)| = |F_W(j\omega I - A_W)^{-1} B_W|$$

with dot-dashed line.

- Three magnitude responses are very close to each until after $\omega = 10$.

Loop Magnitude Responses

The following Matlab plot shows the three loop gain curves.

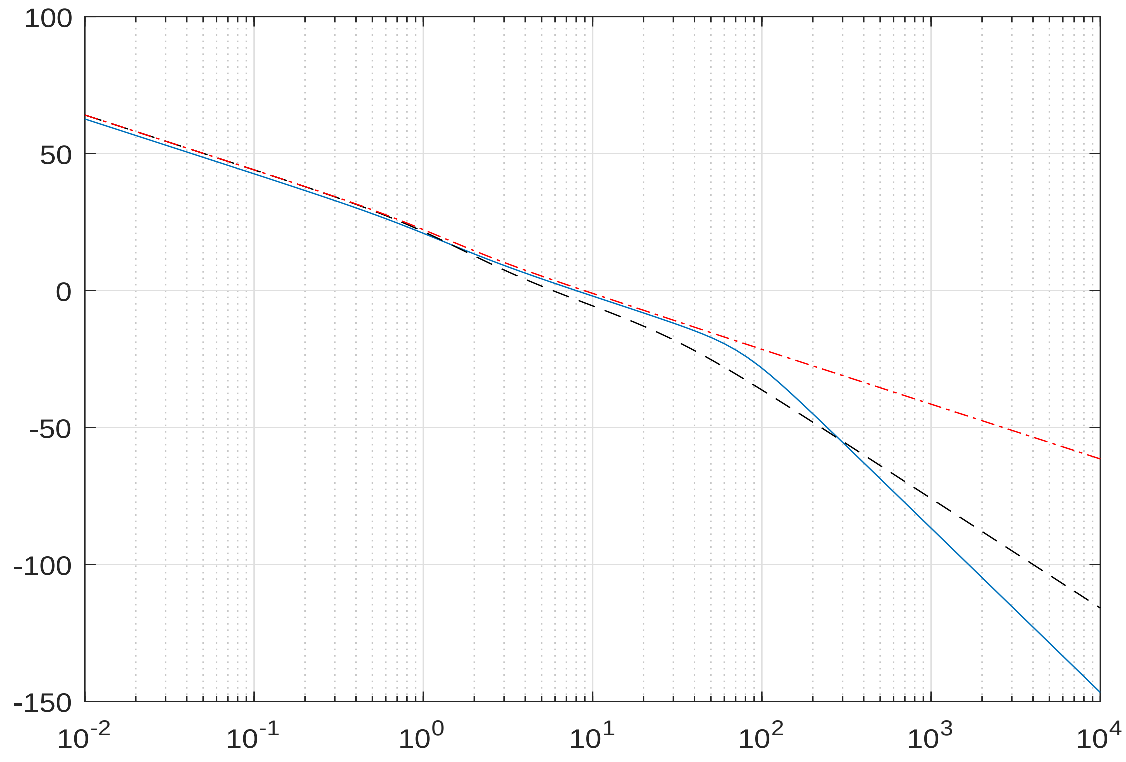

Fig. 6 Loop gain plots for $K_W P_W$ (solid), P_W (dashed), and L_W (dot-dashed)

Step Responses Under State and Output feedback

Because the closed-loop system has the same transfer function as $T_W(s)$, the two step responses are identical as shown next:

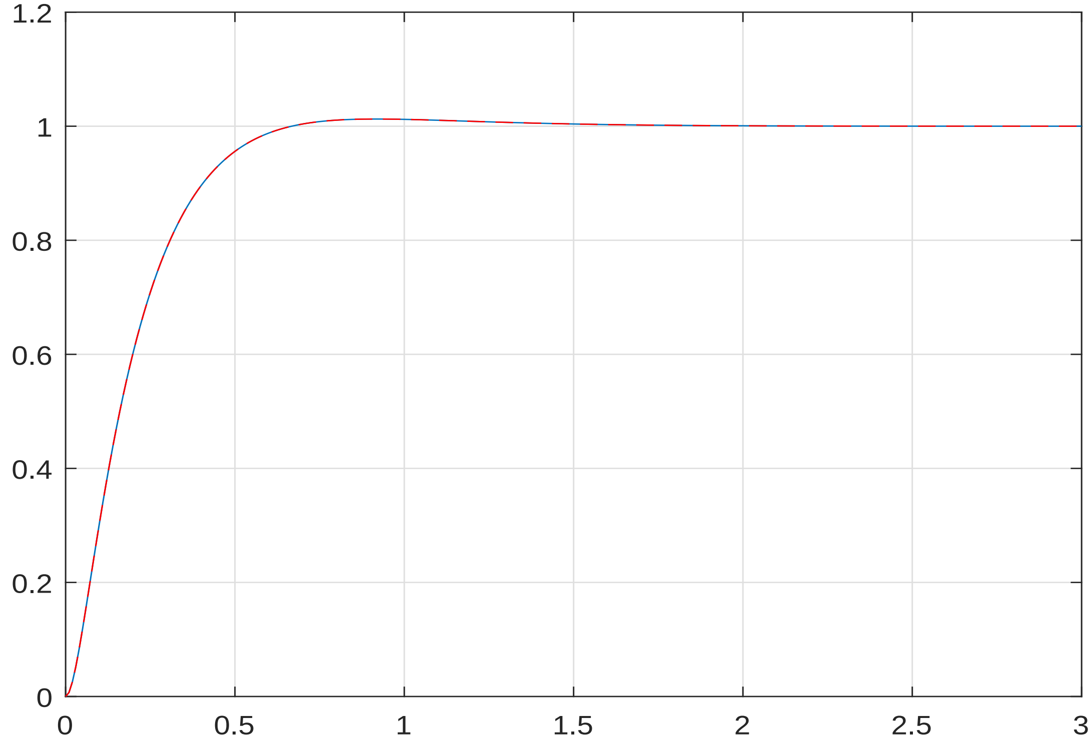

Fig. 7 Step responses under state feedback (solid) and under output feedback (dashed)

Output Controller

- The output feedback controller has the form of

$$K_W(s) = -F_W(sI - A_{K_W})^{-1}L_q, \quad A_{K_W} = A_W + B_W F_W + L_q C_W + L_q D_W F_W.$$

- However, we do not implement the output feedback controller under the unity feedback system. Instead, we implement the following two-degree-freedom controller:

$$K_{a_W}(s) = \left[\begin{array}{c|cc} A_{K_W} & B_W + L_q D_W & -L_q \\ \hline F_W & I & 0 \end{array}\right]$$

that has two-input $\{r(t), y(t)\}$ and one-output $u(t)$.

- It can be verified that the nominal feedback system has the transfer matrix identical to that under state feedback:

$$\begin{aligned} T_{ry}(s) &= D_W + (C_W + D_W F_W)(sI - A_W - B_W F_W)^{-1} B_W \qquad (8) \\ &= C_W(sI - A_W - B_W F_W)^{-1} B_W, \quad \text{if } D_W = 0. \end{aligned}$$

But in validating the design, we need to take the model uncertainties into account.

Perturbation Analysis

- If $\delta \neq 0$, then the system performance can differ greatly. We assume that the actual value of $\delta > 0$ is unknown but small.

- Suppose that $\delta = 0.1$. Then we have a different $P_{\delta_W}(s)$ with a different realization:

$$P_{\delta_W}(s) = \frac{160(s+3)(1-0.1s)}{s(s-1)(s+30)} = \left[\begin{array}{c|c} A_{\delta_W} & B_{\delta_W} \\ \hline C_{\delta_W} & D_{\delta_W} \end{array}\right].$$

- We study how the closed-loop system behaves under the same feedback controller.

- By Page 18-19 of Note2_XNJD2014.pdf, we have $u_w(t) = F_W \hat{x}_W(t) + r(t)$ and thus

$$\begin{aligned} \dot{x}_{\delta_W}(t) &= A_{\delta_W} x_{\delta_W}(t) + B_{\delta_W} F_W \hat{x}_W(t) + B_{\delta_W} r(t), \\ \dot{\hat{x}}_W(t) &= A_{K_W} \hat{x}_W(t) + (B_W + L_q D_W) r(t) - L_q y(t), \\ y(t) &= C_{\delta_W} x_{\delta_W} + D_{\delta_W} F_W \hat{x}_W(t) + D_{\delta_W} r(t), \end{aligned}$$

  where $A_{K_W} = A_W + B_W F_W + L_q C_W + L_q D_W F_W$.

- Note the differences of $(A_{\delta_W}, B_{\delta_W}, C_{\delta_W}, D_{\delta_W})$ from (A_W, B_W, C_W, D_W).

Realization of Closed-Loop System

- Putting together, we obtain the state-space description for the closed-loop system:

$$\begin{bmatrix} \dot{x}_{\delta_W}(t) \\ \dot{\hat{x}}_W(t) \end{bmatrix} = \begin{bmatrix} A_{\delta_W} & B_{\delta_W}F_W \\ -L_q C_{\delta_W} & A_K - L_q D_{\delta_W} F_W \end{bmatrix} \begin{bmatrix} x_{\delta_W}(t) \\ \hat{x}_W(t) \end{bmatrix} + \begin{bmatrix} B_{\delta_W} \\ B_W \end{bmatrix} r(t),$$

$$y(t) = \begin{bmatrix} C_{\delta_W} & D_{\delta_W}F_W \end{bmatrix} \begin{bmatrix} x_{\delta_W}(t) \\ \hat{x}_W(t) \end{bmatrix} + D_{\delta_W} r(t).$$

- The above gives realization for closed-loop system (it differs from (8) on Page 16):

$$T_{ry}(s) = \left[\begin{array}{c|c} A_T & B_T \\ \hline C_T & D_T \end{array}\right] := \left[\begin{array}{cc|c} A_{\delta_W} & B_{\delta_W}F_W & B_{\delta_W} \\ -L_q C_{\delta_W} & A_K - L_q D_{\delta_W} F_W & B_W \\ \hline C_{\delta_W} & D_{\delta_W}F_W & D_{\delta_W} \end{array}\right].$$

- In the Matlab environment, we can use the following commands:

  Try=ss(A_T,B_T,C_T,D_T); % Set up closed-loop state-space system

  y=lsim(Try,ones(1,N),t,zeros(2*n,1)); % Compute step response

  where t is a column vector consisting of time samples over $[0,\ t_{\max}]$, and N=length(t).

Step Response Under $\delta = 0.1$

The closed-loop step response is plotted next with solid line for $\delta = 0.1$ and dashed line for $\delta = 0$. We see that the presence of $\delta = 0.1$ worsens greatly the step response. Why?

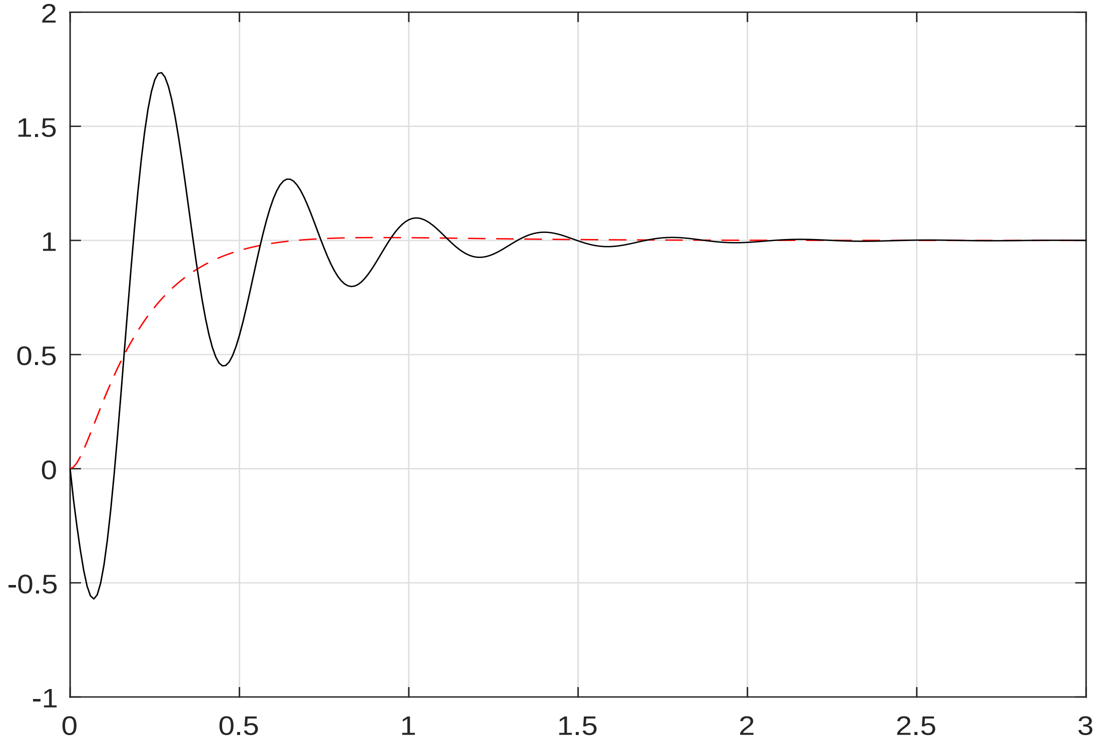

Fig. 8 Step response under perturbation

Two Different Ways to Achieving Better Overshot

- In the case when the overshot specification cannot be compromised, then we can have two different ways to deal with the design.
- The first method chooses to sacrifice the settling time and rise time, i.e., sacrifice the bandwidth by taking smaller value of ω_c.
- This method can violate the design specifications on the open-loop gain at $\omega = 1$.
- Approximately, we have $\omega_{\rm B}\tau_c \approx 1$ with $\omega_{\rm B}$ the bandwidth of the closed-loop system, which is close to crossover frequency ω_c of the loop transfer function, and τ_c the time constant of the closed-loop system, which is close to rise time.
- Since $0 < \delta \le 0.1$ can cause phase lag at $\omega = 5$, and at $\omega = 1$ as large as 26.57^o and 11.31^o, respectively, we make a trial by taking $\omega_c \approx 1$, which violates the gain requirement at $\omega = 1$. We may have to seek other ways to reduce both t_r and t_s.
- The second method chooses to design the controller for $P(s) = \dfrac{1-0.1s}{s(s-1)}$ that is the worst-case plant model, in hope that the performance does not degrade for $0 < \delta < 0.1$.

Controller Implementation

- The feedback controller is implemented by taking (see Page 16)

$$W(s)K_{a_W}(s) = W(s)V(s)^{-1}\begin{bmatrix} I & -U(s) \end{bmatrix} = W(s)\left[\begin{array}{c|cc} A_{K_W} & B_W + L_q D_W & -L_q \\ \hline F_W & I & 0 \end{array}\right]$$

  with input $r(t)$ and $y(t)$, where $A_{K_W} = A_W + B_W F_W + L_q C_W + L_q D_W F_W$.

- Recall that the plant model is $P_{\text{true}}(s)$, and hence the weighting $W(s)$ should be really implemented by the controller.

- This is the two-degree-freedom controller as we talked about in other lecture notes.

- The step response performance of the feedback system under this two-degree-freedom controller is identical to that under the state feedback, if the plant model does not have uncertainties, i.e., if $P_{\text{true}}(s) = P(s)$. Recall the expression of $T_{ry}(s)$ in (8).

- However, if $P_{\text{true}}(s) \neq P(s)$, then $T_{ry}(s)$ differs from that in (8). See Page 17-18 for realization of $T_{ry}(s)$ and simulation in Matlab codes.