Continuation of integration by substitution

Examples:

a) $\int \tan x \ dx$

$= \int \frac{sinx}{cosx} \ dx$

Let $u = \cos x$ so that $du = -\sin x \, dx$
$\Longrightarrow \sin x \, dx = -du$

$\therefore \tan \int \ x \, dx = \int \frac{sinx}{consx} \ dx = \int \frac{sinx}{u} \cdot -\frac{du}{sinx}$
$= -\int \frac{1}{u} \ du$
$= -\ln|u| + c = -\ln|\cos x| + c$
$= \ -\ln|secx| + c$

b) Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} \cot^2 2\theta \ d\theta$

since $\cot^2 2\theta = cosec^2\theta - 1$

$\Longrightarrow \cot^2 2\theta = cosec^2 2\theta - 1$

$\therefore \ \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} \cot^2 2\theta \ d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{2} (\operatorname{cosec}^2 2\theta - 1) \ d\theta$

$= \frac{1}{2} \left[-\frac{cot2\theta}{2} - \theta \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$
$= 0.0269$

Remarks:

When a power of a cosine is multiplied by a sine of power 1, or vise versa, the following holds;

1) $\int \cos^n \theta \sin \theta \ d\theta = -\cos^{n+1} \frac{\theta}{n+1} + c$

2) $\int \ \sin^n \theta \cos \theta \ d\theta = \sin^{n+1} \frac{\theta}{n+1} + c$

Example

Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x \, dx$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \cdot \cos^2 \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x(1 - \sin^2 x) \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} (\sin^2 x - \sin^4 x) \cos x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \cos x \, dx - \int_0^{\frac{\pi}{2}} \sin^4 x \cos x \, dx$$

$$= \left[\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5}\right]_0^{\frac{\pi}{2}}$$

$= \frac{2}{15}$ or 0.1333

Find the integral of the following

a) $\int \frac{1}{\sqrt{a^2 - x^2}} \, dx$

Let $x = a \sin\theta$, so that $dx = \mathrm{a}\cos\theta \, d\theta$

$$\therefore \quad \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \int \frac{1}{\sqrt{a^2 - (a sin\theta)^2}} \cdot a\cos\theta \, d\theta$$

$$= \int \frac{1}{\sqrt{a^2(1 - \sin^2\theta)}} \cdot a\cos\theta \, d\theta$$

$$= \int \frac{1}{a cos\theta} \cdot a\, cos\theta \;\; d\,\theta$$

$$= \int 1\theta \, d\theta = \theta$$

But $x = a \sin\theta$

$$\Longrightarrow \sin\theta = \frac{x}{a} \qquad \theta = \sin^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \quad \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

b) Find $\int \frac{1}{x^2\sqrt{x^2+4}} \, dx$

Let $x = 2\tan\theta$, so that $dx = 2\sec^2\theta \, d\theta$ and

$$\sqrt{x^2 + 4} = \sqrt{4(\tan^2\theta + 1)} = \sqrt{4\sec^2\theta} = 2|\sec\theta| = 2\sec\theta$$

Thus we have:

$$\int \frac{1}{x^2\sqrt{x^2+1}} \quad dx = \frac{\int 2\sec^2\theta\, d\theta}{4\tan^2\theta \cdot 2sec\theta} = \frac{1}{4}\int \frac{\sec^2\theta}{\tan^2\theta} \quad d\theta$$

$$\Longrightarrow \frac{1}{4}\int \frac{\sec^2\theta}{\tan^2\theta}\, d\theta = \frac{1}{4}\int \frac{1}{cos\theta} \cdot d\theta$$

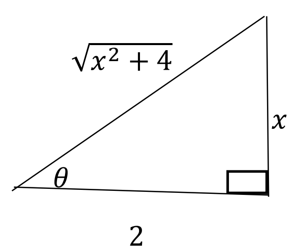
$= \frac{1}{4} \int \frac{cos\theta}{\sin^2\theta}\, d\,\theta$, let $u = sin\theta$, so that $du = \cos\theta\, d\theta$

Then $\frac{1}{4} \int \frac{cos\theta}{\sin^2\theta}\, d\theta = \frac{1}{4}\frac{du}{u^2}$

$$= \frac{1}{4}\left(-\frac{1}{u}\right) + c = \frac{1}{4}\left(-\frac{1}{sin\theta}\right) + c$$

But $x = 2\, tan\theta$

$$\Longrightarrow \tan\theta = \frac{x}{2}$$



$$sin\theta = \frac{x}{\sqrt{x^2+4}} \quad \Longrightarrow \frac{1}{sin\theta} = cosec\ \theta = \frac{\sqrt{x^2+4}}{x}$$

Therefore:

$$\int \frac{dx}{x^2\sqrt{x^2+4}} = -\frac{\sqrt{x^2+4}}{4x} + c$$

c) $\int \frac{x}{\sqrt{x^2+4}}\, dx$ (can use $x = 2tan\theta$)

But direct substitution $u = x^2 + 4$ is simpler, because $du = 2xdx$

and $\int \frac{x}{\sqrt{x^2+4}}\, dx = \int \frac{x}{\sqrt{u}} \cdot \frac{du}{2x}$

$$= \frac{1}{2} \int \frac{1}{\sqrt{u}}\, du = \sqrt{u} + c$$

$$= \sqrt{x^2+4} + c$$

Note: even when trig substitutions are possible, they may not give the easiest solution. We should look for a simpler method first.

Exercise

Evaluate

a) $\int \frac{dx}{\sqrt{x^2-a^2}}$, where $a > 0$ (let $x = a\, sec\theta$) Ans: $\ln\left|x + \sqrt{x^2 - a^2}\right| + c$

$(or\ x = a\cosh t)$ or $\cosh^{-1}\left(\frac{x}{a}\right) + c$

b) $\int_0^{\frac{13\sqrt{3}}{2}} \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}\, dx$ let $u = 2x$ and $x = \frac{3}{2}tan\theta$ Ans: $\frac{3}{32}$

c) $\int \frac{x}{\sqrt{3-2x-x^2}}\, dx$ (compare the square under the root sign to have $4 - (x+1)^2$)
Then let $u = x + 1$

d) $\int \frac{x^2}{(x^2+a^2)^{\frac{3}{2}}}\, dx$ Ans: $-\sqrt{3 - 2x - x^2} - \sin^{-1}\left(\frac{x+1}{2}\right) + c$

a) By trig substitution
b) By hyperbolic substitution $x = \operatorname{asinh} t$

Integration by partial fractions

To illustrate, we have;

$$\frac{2}{x-1} - \frac{1}{x+2} = 2(x+2) - \frac{1(x-1)}{(x-1)\cdot(X+2)} = \frac{x+5}{x^2+x-2}$$

Reversing the procedure, we have;

$$\int \frac{x+5}{x^2+x-2}\, dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx$$

$$= 2\ln|x-1| - \ln|x+2| + c$$

Recall the following forms of partial fractions below

Form of rational fraction	form of the partial fraction
1. $\frac{px+q}{(x-a)(x-b)}$, $a \neq b$	$\frac{A}{x-a} + \frac{B}{x-b}$
2. $\frac{px+q}{(x-b)^2}$	$\frac{A}{x-b} + \frac{B}{(x-b)^2}$
3. $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4. $\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
5. $\frac{px+q}{ax^2+bx+c}$	$\frac{Ax+B}{ax^2+bx+c}$
6. $\frac{px^2+qx+r}{(ax^2+bx+c)^2}$	$\frac{Ax+B}{(ax^2+bx+c)} + \frac{Cx+D}{(ax^2+bx+c)^2}$

Example

Evaluate the following

a) $\int \frac{x^3+x}{x-1} dx$

Since the degree of the numerator is greater than that of the denominator, we first perform long division.

$$
\begin{array}{r|l}
 & x^2 + x + 2 \\
\hline
x - 1 & x^3 + x \\
 & \underline{-x^3 - x^2} \\
 & x^2 + x \\
 & \underline{-x^2 - x} \\
 & 2x \\
 & \underline{-2x - 2} \\
 & 2
\end{array}
$$

$$\therefore \quad \int \frac{x^3+x}{x-1}\; dx = \int \left(x^2 + x + 2 + \frac{2}{x-1}\right)\; dx$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + c$$

b) $\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\; dx$ (no need to divide)

We factor the denominator as;

$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) = x(2x-1)(x+2)$$

Thus the partial fraction decomposition of the integral has the form;

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

$$\Longrightarrow x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

Solving we have;

$A = \frac{1}{2}$, $B = \frac{1}{5}$, $C = -\frac{1}{10}$ and so

$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\; dx = \int \left(\frac{1}{2}\cdot\frac{1}{x} + \frac{1}{5}\cdot\frac{1}{2x-1} - \frac{1}{10}\cdot\frac{1}{x+2}\right) dx$$

$\frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + c$

c) $\int \frac{dx}{x^2-a^2}\ dx$, when $a \neq 0$.

$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$ and therefore

$$A(x+a) + B(x-a) = 1 \Longrightarrow A = \frac{1}{2a}, B = -\frac{1}{2a}$$

Thus;

$$\int \frac{dx}{x^2-a^2}\ dx = \frac{1}{2a}\int\left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx$$

$$= \frac{1}{2a}[\ln|x-a| - \ln|x+a|] + c = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$$

d) $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\ dx$

The first step is to divide: the result of long division is:

$$\frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = x + 1 + \frac{4x}{x^3-x^2-x+1}$$

Then factoring the denominator we have:

$$x^3 - x^2 - x + 1 = (x-1)(x^2-1) = (x-1)(x-1)(x+1)$$

$$= (x-1)^2(x+1)$$

Thus the partial fraction decomposition is:

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$$

This gives

$A = 1\ \ , B = 2\ ,\ C = -1$, so

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\ dx = \int\left[x + 1 + \frac{1}{(x-1)} + \frac{2}{(x-1)^2} - \frac{1}{(x+1)}\right]dx$$

$$= \frac{x^2}{2} + x + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + c$$

$$= \frac{x^2}{2} + x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + c$$

e) $\int \frac{2x^2-x+4}{x^3+4x}\, dx$

Since $x^3 + 4x = x(x^2+4)$ cant be factored further,
We write

$$\frac{2x^2-x+4}{x^3+4x} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$\Longrightarrow 2x^2 - x + 4 = A(x^2+4) + (Bx+C)$$

Which gives
$A = 1$, $B = 1$ and $c = -1$ and so

$$\int \frac{2x^2-x+4}{x^3+4x}\, dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right) dx$$

$$\int \frac{1}{x} dx + \int \frac{x}{x^2+4}\, dx - \int \frac{1}{x^2+4}\, dx$$

$$= \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

f) $\int \frac{4x^2-3x+2}{4x^2-4x+3}\, dx$

We first divide to get:

$$\frac{4x^2-3x+2}{4x^2-4x+3} = 1 + \frac{x-1}{4x^2-4x+3}$$

Notice that $4x^2 - 4x + 3$ is irreducible because $b^2 - 4ac = -32 < 0$

Thus it cant be factored. But completing the square we have:

$$4x^2 - 4x + 3 = (2x-1)^2 + 2$$

Now let $u = 2x - 1$, so that $du = 2dx$ and $x = \frac{1}{2}(u+1)$,

$$\Longrightarrow \int \frac{4x^2-3x+2}{4x^2-4x+3}\ dx = \int \left(1+\frac{x-1}{4x^2-4x+3}\right)\ dx$$

$$= x + \int \frac{\frac{1}{2}(u+1)-1}{u^2+2} \cdot \frac{du}{2}$$

$$= x + \frac{1}{4}\int \frac{u-1}{u^2+2}\ du$$

$$= x + \frac{1}{4}\int \frac{u}{u^2+2}\ du - \frac{1}{4}\int \frac{1}{u^2+2}\ du$$

$$= x + \frac{1}{8}\ln(u^2+2) - \frac{1}{4}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + c$$

$$= x + \frac{1}{8}\ln(4x^2-4x+3 - \frac{1}{4\sqrt{2}}\tan^{-1}\left(\frac{2x-1}{\sqrt{2}}\right) + c$$

g) $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\ dx$

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$\Longrightarrow A = 1\ , B = -1\ , C = -1\ , D = 1$ and $E = 0$

Thus

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\ dx\ = \int (\frac{1}{x} - \frac{x+1}{x^2+1} + \frac{x}{(x^2+1)^2} dx$$

$$= \int \frac{1}{x}\ dx - \int \frac{x}{x^2+1}\ dx + \int \frac{1}{x^2+1}\ dx + \int \frac{x}{(x^2+1)^2}\ dx$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1} x - \frac{1}{2(x^2+1)^2} + c$$