Integration by parts

We know that if u and v are functions of x , then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\Rightarrow u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrating both side with respect to x, we obtain:

$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx} + c$

$\Rightarrow \int u\, dv = uv - \int v\, du + c$ $\cdots\cdots$ Integration by parts

Examples

a) $\int xe^x\, dx$ b) $\int \ln x\, dx$ c) $\int_0^{\frac{\pi}{2}} x^2 \sin 2x\, dx$

d) $\int e^{-2x}\cos 3x\ dx$ e) $\int_{-1}^{\frac{3}{2}} |xsin\pi x|\, dx$

Solution

a) $\int xe^x\, dx$

Let $u = x$ $du = 1dx$

$dv = e^x$, $v = e^x$

$$\Rightarrow \int xe^x dx = xe^x - \int e^x \cdot dx$$

$$= xe^x - e^x + c = e^{x(x-1)} + c$$

b) $\int \ln x \, dx$

Let $u = \ln x, \quad dv = 1$

$du = \frac{1}{x} \; , v = x$

$\Longrightarrow \ln \int \; x \, dx = x ln \, x - \int x \cdot \frac{1}{x} \, dx$

$= x \ln x - \int dx$

$= \text{x} \ln x - x + c$

$= x(\ln x - 1) + c$

c) $\int_0^{\frac{\pi}{2}} x^2 \sin 2x \, dx$

Let $u = x^2 \quad , \; dv = sin2x$

$\frac{du}{dx} = 2x \quad , \quad v = -\frac{1}{2}\cos 2x$

$\Longrightarrow \int_0^{\frac{\pi}{2}} x^2 \; \sin 2x \, dx = \left[x^2 \cdot (-\frac{1}{2}\cos 2x \,)\right]_0^{\frac{\pi}{2}} - \int -\frac{1}{2}\cos 2x \cdot 2x \, dx$

$= -\frac{1}{2}x^2 \cos 2x \,\Big|_0^{\frac{\pi}{2}} + \frac{1}{2} \cdot 2 \; \int_0^{\frac{\pi}{2}} x \cos 2x \, dx$

Now

$-\frac{1}{2}x^2 \cos 2x \,\Big|_0^{\frac{\pi}{2}} = -\frac{1}{2}\left(\frac{\pi}{2}\right)^2 \cos 2\left(\frac{\pi}{2}\right) - 0$

$= +\frac{\pi^2}{8}$ (as $\cos \pi = -1$)

And

$\int_0^{\frac{\pi}{2}} \cos 2x \, dx$

Let $u = x \qquad dv = \cos 2x$

$du = dx \; , \; v = \frac{1}{2} \cdot \sin 2x$

$\Longrightarrow \int_0^{\frac{\pi}{2}} x \cos 2x \, dx = \frac{x \cdot \sin 2x}{2}\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin 2x \, dx$

$= x \cdot \frac{\sin 2x}{2}\Big|_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2x \, dx$

$$\Longrightarrow \left[\left(\frac{\pi}{2}\right)\cdot\frac{1}{2}sin2\left(\frac{\pi}{2}\right)-0\right]-\frac{1}{2}\left[-\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}$$

$$=\frac{1}{4}[\cos\pi-\cos 0]=\frac{1}{4}(-1-1)=-\frac{1}{2}$$

$$\therefore \int_0^{\frac{\pi}{2}} x^2 sin2x\ dx = +\frac{\pi}{8}-\frac{1}{2}$$

d) $\int e^{-2x}\cos 3x\ dx$

Let us first evaluate the more general integral given by

$$I=\int e^{-2x}\cos 3x\ dx$$

Here $\quad u=e^{ax} \quad , \quad dv=\cos bx$

$$du=ae^{ax}dx\ ,\ v=\frac{1}{b}\sin bx$$

$$\therefore \quad I=\frac{1}{b}\ e^{ax}\sin bx-\frac{a}{b}\int e^{ax}\sin bx\ dx$$

$$(n=e^{ax}\ ,du=ae^{ax}\ ,dv=\sin bx\ ,v=-\frac{1}{b}\cos bx)$$

Evaluating the integral on the right side, we have;

$$I=\frac{1}{b}e^{ax}\sin bx-\frac{a}{b}\left[e^{ax}\left(-\frac{cosbx}{b}\right)-\int\ -\frac{cosbx}{b}\cdot ae^{a}dx\right]$$

$$I=\frac{1}{b}e^{ax}\sin bx-\frac{a}{b}(-\frac{1}{b}e^{ax}\cos bx+\frac{a}{b}\underbrace{\int e^{ax}cosbx\ dx)}_{I}$$

$$=\frac{1}{b}e^{ax}\sin bx+\frac{a}{b^2}e^{ax}\cos bx-\frac{a^2}{b^2}I+c$$

$$I+\frac{a^2}{b^2}I=\frac{1}{b}e^{ax}\sin bx+\frac{a}{b^2}e^{ax}\cos bx+c$$

$$I\left(1+\frac{a^2}{b^2}\right)=\frac{1}{b}e^{ax}\sin bx+\frac{a}{b^2}e^{ax}\cos bx+c$$

$$I\left(1+\frac{a^2}{b^2}\right)=\frac{e^{ax}}{b2}\ (b\sin bx+a\cos bx\ )+c$$

Divide $I\left(1+\frac{a^2}{b^2}\right)$ throughout;

$I = \frac{e^{ax}}{b^2} \cdot \frac{b^2}{a^2+b^2}\ (b \sin bx + a \cos bx) + c \cdot \frac{b^2}{a^2+b^2}$ $\Rightarrow$ this is still a constant , say k

$\therefore \quad I = \frac{e^{ax}}{a^2+b^2}\ (b \sin bx + a \cos bx) + k$

Thus;

$$\int e^{-2x} \cos 3x\ dx\ = \frac{e^{-2x}}{(-2)^2+(3)^2} (3 \sin 3x + (-2) \cos 3x) + k$$

$$= \frac{e^{-2x}}{13}\ (3 \sin 3x - 2 \cos 3x\ ) + k$$

e) $\int_{-1}^{\frac{3}{2}} |x \sin \pi x|\ dx$

We first split the range of integration into two parts;
and write the integral as;

$$I = \int_{-1}^{1} |xsin\pi x| dx + \int_{1}^{\frac{3}{2}} |x \sin \pi x\ |\ dx$$

(since the product $xsin\pi x$ changes sign at the point $x = 1.\ +v(-1 < x < 1)$
$-v(1 < x < \frac{3}{2}))$

$$I = \int_{-1}^{1} x \sin \pi x\ dx - \int_{1}^{\frac{3}{2}} x \sin \pi x\ dx$$

$$= \left[\frac{xcos\ \pi x}{-\pi} + \frac{sin\pi x}{\pi^2}\right]_{-1}^{1} - \left[\frac{xcos\pi x}{-\pi} + \frac{sin\pi x}{\pi^2}\right]_{1}^{\frac{3}{2}}$$

$$= \left(\frac{cos\pi}{-\pi} + \frac{sin\pi}{\pi^2}\right) - \left[\frac{(-1)\cos(-\pi)}{-\pi} + \frac{\sin(-\pi)}{\pi^2}\right]$$

$$- \left(\frac{\frac{3}{2}\cos\left(\frac{3\pi}{2}\right)}{-\pi} + \frac{\sin\left(\frac{3\pi}{2}\right)}{\pi^2}\right) + \left(\frac{cos\pi}{-\pi} + \frac{sin\pi}{\pi^2}\right)$$

$$= \left(\frac{1}{\pi} + 0\right) - \left(-\frac{1}{\pi} + 0\right) - \left(0 - \frac{1}{\pi^2}\right) + \left(\frac{1}{\pi} + 0\right)$$

$$= \frac{1}{\pi} + \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi}$$

$$= -\frac{2}{\pi} cos\pi - \frac{1}{\pi} \cos \pi + \frac{1}{\pi^2} = \frac{3}{\pi} + \frac{1}{\pi^2}$$

$(cos\pi = -1, \cos\left(\frac{3\pi}{2}\right) = 0\ , sin\pi = 0, \sin\left(\frac{3\pi}{2}\right) = -1\ , \cos(-\pi) = -1, \sin(-\pi) = 0)$

Reduction formulae

Suppose we wish to integrate

$$I = \int x^n e^{ax} dx$$

By parts. Taking $u = x^n$ and $dv = e^{ax}\, dx$, we obtain;

$$du = nx^{n-1}dx\ , v = \frac{e^{ax}}{a}$$

$$\Longrightarrow I_n = \frac{x^n e^{ax}}{a} - \frac{\int e^{ax}}{a} \cdot nx^{n-1}dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} \int x^{n-1} e^{ax} dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1} \qquad \cdots\cdots\cdots \qquad (1)$$

The above formula is called a reduction formula, as it can be used continuously to reduce the integral. E.g, when $n = 4$ and $a = 2$ we have the integral;

$$I_4 = \int x^4 e^{2x} dx$$

By repeated application of exp/eqn (1), we obtain;

$$I_4 = \frac{x^4 e^{2x}}{2} - \frac{4}{2}\ I_3 = \frac{x^4 e^{2x}}{2} - 2\ I_3$$

$$\frac{1}{2} x^4 e^{2x} - 2\left(\frac{x^3 e^{2x}}{2} - \frac{3}{2}\ I_2\right)$$

$$= \frac{1}{2}\ x^4 e^{2x} - x^3 e^{2x} + 3I_2$$

$$= \frac{1}{2} x^4 e^{2x} - x^3 e^{2x} + 3\left(\frac{x^2 e^{2x}}{2} - \frac{2}{2}\ I_1\right)$$

$$= \frac{1}{2} x^4 e^{2x} - x^3 e^{2x} + \frac{3}{2} x^2 e^{2x} - 3I_1$$

$$= \frac{1}{2} x^4 e^{2x} - x^3 e^{2x} + \frac{3}{2} x^2 e^{2x} - 3\left(\frac{x^2 e^{2x}}{2} - \frac{1}{2}\ I_0\right)$$

$$I_4 = \frac{1}{2} x^4 e^{2x} - x^3 e^{2x} + \frac{3}{2} x^2 e^{2x} - \frac{3}{2} x e^{2x} + \frac{3}{4} e^{2x} + c$$

A special case is obtained when $a = -1$

i.e. $I_n = \int x^n e^{-1}\, dx$, then the reduction formula is;

$$I_n = -x^n e^{-x} - \int \frac{e^{-x}}{-1}\, n x^{n-1} dx$$

$$= -x^n e^{-x} + n\int x^{n-1} e^{-x} dx$$

$$= -x^n e^{-x} + nI_{n-1} \qquad \cdots \quad \cdots \qquad (2)$$

Now let us consider the definite integral

$$I_n = \int_0^\infty x^n e^{-x}\, dx \qquad \cdots \qquad \cdots \qquad (3)$$

Formula gives;

$$I_n = [-x^n e^{-x}]_0^\infty + nI_{n-1} = nI_{n-1}$$

The function defined by the integral in (3) is known as the gamma function and is defined by $\Gamma(n+1)$, i.e.

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx$$

The reduction formula now gives;

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1)\cdots$$

$$= n(n-1)(n-2)\cdots\cdots 1 \cdot \Gamma(1)$$

$$= n! \int_0^\infty e^{-x}\, dx$$

$$= n! \left[\frac{e^x}{-1}\right]_0^\infty$$

$$= n!$$

Exercise

1) Find the reduction formula for $\int x^n e^x\, dx$ and use it to solve $\int x^4 e^x\, dx$

2) Establish the reduction formula for

   a) $\int (x+1)^n e^{2x} dx$ hence find $\int (1+x)^3 e^{2x}\, dx$

   b) $\int (1-x)^3 e^{4x}\, dx$ using reduction formula.

   c) $\int x^6 e^{-x} dx$

Recall

$$\int \cos^4 x\, dx$$

$$= \int (\cos^2 x)^2 \ dx = \int [\tfrac{1}{2}\cos 2x + 1]^2 \ \ dx$$

$$= \tfrac{1}{4}\int (\cos^2 2x + 2\cos 2x + 1)dx$$

$$= \tfrac{1}{4}\int \left[\tfrac{1}{2}(cos4x + 1) + 2cos2x + 1\right] dx$$

$$= \tfrac{1}{4}\int [\tfrac{1}{2}cos4x + \tfrac{1}{2} + 2\cos 2x + 1]dx$$

Complete the integration……..

But for $\int \cos^n x\, dx$ when n is large, we let

$$I_n = \int \cos^n x dx$$

$$I_n = \int \cos^{n-1} x \cdot cosx\, dx$$

Integrating the R.H.S by parts

Let $u = \cos^{n-1} x$, $dv = \cos x$, $du = -(n-1)\cos^{n-2} x \cdot \sin x\, dx$, $v = \sin x$

$$I_n = uv - \int vdu$$

$$\Longrightarrow \sin x \cdot \cos^{n-1} x + \int (n-1)\sin^2 xcos^{n-2}x\, dx$$

$$I_n = \sin x \cos^{n-1} x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\, dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int (\cos^{n-2} x - \cos^n x\ )\, dx$$

$$= \sin x \cos^{n-1} x + (n-1)\left[\int \cos^{n-2} x\, dx - \int \cos^n x\, dx\right]$$

$$= \sin xcos^{n-1}x + (n-1)[I_{n-2} - I_n]$$

$$\cancel{I}_n = \sin x \cos^{n-1} x + nI_{n-2} - I_{n-2} + \cancel{I}_n - nI_n$$

$$nI_n = \sin xcos^{n-1}x + (n-1)I_{n-2}$$

$$I_n = \frac{1}{n}\sin x \cos^{n-1} x + \frac{(n-1)}{n}I_{n-2}$$

or $$I_n = \frac{1}{n}[\sin x\, cos^{n-1} x + (n-1)I_{n-2}]$$

Example

$$\int \cos^5 x\, dx$$

$$I_5 = \tfrac{1}{5}\sin x \cos^4 x + \tfrac{4}{5}I_3$$

$I_3 = \frac{1}{3}\sin x \cos^2 x + \frac{2}{3} I_1$

$I_1 = \sin x$ [which is $\int \cos x dx = \sin x + c$]

Thus

$$I_5 = \frac{1}{5}\sin x \cos^4 x + \frac{4}{5}[\frac{1}{3}\sin x \cos^2 x + \frac{2}{3}(\sin x)] + c$$

$$I_5 = \frac{1}{5}\sin x \cos^4 x + \frac{4}{15}\sin x \cos^2 x + \frac{8}{15}\sin x + c$$

$$= \frac{1}{5}\sin x \ \left(\cos^4 x + \frac{4}{3}\cos^2 x + \frac{8}{3}\right) + c$$

Example

Establish the reduction formula for

$$\int \cos^n 2x \, dx$$

Now consider the integral

$$I_n = \int \sin^n x \, dx$$

$$I_n = \int \sin^{\mathrm{n}-1} x \sin x \, dx$$

Let $u = \sin^{n-1} x$, $dv = \sin x$

$du = (n-1)\sin^{n-2} x \cdot \cos x \, dx$, $v = -\cos x$

$$\Longrightarrow I_n = -\cos x \sin^{n-1} x + \cos \int x \cdot (n-1)\sin^{n-2} x \cdot \cos x \, dx$$

$$I_n = -\cos x sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x \, dx$$

$$I_n = -\cos x sin^{n-1} x + (n-1)\int (1-\sin^2 x)\sin^{n-2} dx$$

$$I_n = -\cos x sin^{n-1} x + (n-1)\left[\int \sin^{n-2} x \, dx - \int \sin^n x \, dx\right]$$

$$I_n = -\cos x sin^{n-1} x + (n-1)[I_{n-2} - I_n]$$

$$I_n = -\cos x sin^{n-1} x + (n-1)I_{n-2} - nI_n + I_n$$

$$I_n = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{(n-1)}{n} \ I_{n-2}$$

Find $\int \tan^n x \, dx$

$$I_n = \int \tan^n x dx$$

$$= \int \tan^{n-1} x tan \, x \, dx$$

$$= \int \tan^{n-2} x \cdot \tan^2 x \, dx$$

$$= \int \tan^{n-2} x(\sec^2 x - 1)dx$$

$$= \int (\sec^2 x \tan^{n-2} x - \tan^{n-2} x)dx$$

Let $u = \tan^{n-2} x$, $dv = \sec^2 x$ $du = (n-2)\tan^{n-3} x \cdot \sec^2 x dx$ $v = \tan x$

$$I_n = uv - \int v du$$

$$\Longrightarrow I_n = \tan x \cdot \tan^{n-2} x - (n-2)\tan \int x \cdot \tan^{n-3} x \cdot \sec^2 x\, dx - \int \tan^{n-2} x\, dx$$

$$\Longrightarrow I_n = \tan^{n-1} x - (n-2)\int \tan^{n-2} x(\tan^2 x + 1)dx - \int \tan^{n-2} x\, dx$$

$$I_n = \tan^{n-1} x - (n-2)\int \tan^{n} x + \tan^{n-2} x\, dx - \int \tan^{n-2} x dx$$

$$I_n = \tan^{n-1} x - (n-2)\,[I_n + I_{n-2}] - I_{n-2}$$

$$\Longrightarrow I_n = \tan^{n-1} x - (n-2)\, I_n - (n-2)I_{n-2} - I_{n-2}$$

$$I_n(1 + n - 2) = \tan^{n-1} x - I_{n-2}((n-2) + 1)$$

$$\mathrm{I_n}(n-1) = \tan^{n-1} x - (n-1)I_{n-2}$$

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

Example

Find $\int \tan^5 dx$

$$I_5 = \frac{\tan^4 x}{4} - I_3$$

$$I_3 = \frac{\tan^2 x}{2} - I_1$$

$$I_1 = \int \tan x dx = \ln|\sec x| + c$$

$$\therefore \quad I_3 = \frac{\tan^2 x}{2} - \ln|\sec x|$$

$$\Longrightarrow I_5 = \frac{\tan^4 x}{4} - \frac{\tan^2 x}{2} + \ln|\sec x| + c$$

1) Find using reduction formula

a) $\int x^5 sinx\, dx$

b) $\int_0^{\frac{\pi}{2}} \sin^n x dx$

c) $\int_0^{\frac{\pi}{2}} \sin^9 \theta d\theta$

d) $\int_0^{\pi} x \cos^5 x\, dx = \frac{3\pi^2}{16}$

e) $\int_0^{\frac{\pi}{2}} \sin^6 \theta\, d\theta = \frac{5\pi}{32}$

f) $\int_0^{\frac{\pi}{2}} xsin\, xcos^4 x\, dx = \frac{\pi}{5}$

g) $\int_0^{\infty} e^{-ax} \sin mx\, dx = \frac{m}{a^2+m^2}$

2) Obtain a reduction formula for $\int_0^{\frac{\pi}{2}} \sec^2 x dx$ and hence evaluate $\int_0^{\frac{\pi}{4}} \sec^5 x dx$

3) If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x\, dx$, prove that

$$I_n + n(n-1)I_{n-2} = \left(\frac{\pi}{2}\right)^n$$