COMPLEX NUMBERS

INTRODUCTION

1) <u>IMAGINARY NUMBERS</u>

Let us start our study with a question.

Q. Solve the equation:

$$x^2 + 1 = 0$$

Solution:

Given $x^2 + 1 = 0$

<=> $x^2 = -1$

<=> $\sqrt{x^2} = \pm\sqrt{-1}$

∴ $x = \pm\sqrt{-1}$

The imaginary number $\sqrt{-1}$ is denoted by the letter i

Note that:

$i = \sqrt{-1}$ ....(1)

=> $i^2 = \left(\sqrt{-1}\right)^2 = -1$ ....(2)

i.e. $i^2 = -1$

<u>POWERS OF i</u>

1. $i = \sqrt{-1}$
2. $i^2 = -1$
3. $i^3 = i^2 \times i = -1 \times i = -i$
4. $i^4 = i^2 \times i^2 = -1 \times (-1) = 1$
5. $i^5 = i^4 \times i = 1 \times i = i$
6. $i^6 = i^4 \times i^2 = 1 \times (-1) = -1$
7. $i^7 = i^4 \times i^2 \times i = 1 \times (-1) \times i = -i$
8. $i^8 = (i^4)^2 = 1^2 = 1$

EXAMPLE ONE

Simplify a) i^{29} b) i^{31} c) i^{-3}

Solution

a) $i^{29} = i^{28} \times i$

$= (i^4)^7 \times i$

$= 1^7 \times i$

$= 1 \times i = i$

b) $i^{31} = i^{28} \times i^2 \times i$

$= 1 \times (-1) \times i = -i$

c) $i^{-3} = \frac{1}{i^3} = \frac{1 \times i}{i^3 \times i} = \frac{i}{i^4} = \frac{i}{1} = i$

<u>Powers of i rule</u>

If n is a natural number that has a remainder of r when divided by 4, then

$$i^n = i^r$$

N.B: If n is divisible by 4, the remainder is 0 and $i^n = i^0 = 1$

EXAMPLE TWO

Simplify i^{107}

Solution:

$\frac{107}{4} = 26 \; and \; remainder \; of \; 3.$

Hence, $i^{107} = i^3 = i^2 \times i = (-1) \times i = -i$

IMAGINARY NUMBERS

A number of the form b**i, where** $\boldsymbol{i} = \sqrt{-\mathbf{1}}$ is known as an imaginary number.

PROPERTY OF RADICALS

If at least one of a and b is nonnegative, then

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

N.B This rule falls apart if both numbers a and b are negative. That is if both a and b are negative, then $\sqrt{ab} \neq \sqrt{a}\sqrt{b}$

EXAMPLE THREE

Simplify a) $\sqrt{-4}$ b) $\sqrt{-9}$ c) $\sqrt{-7}$ d) $\sqrt{-4}\sqrt{-9}$

Solution:

a) $\sqrt{-4} = \sqrt{4 \times (-1)}$

$= \sqrt{4}\sqrt{-1}$

$= 2i$

b) $\sqrt{-9} = \sqrt{9 \times (-1)}$

$= \sqrt{9}\sqrt{-1}$

$= 3i$

c) $\sqrt{-7} = \sqrt{7 \times (-1)}$

$= \sqrt{7}\sqrt{-1}$

$= \sqrt{7}i$

d) Note that in this case $\sqrt{-4}\sqrt{-9} \neq \sqrt{-4 \times (-9)}$ since both $-4 \; and - 9 \; are \; negative.$

Hence, we proceed as follows:

$\sqrt{-4}\sqrt{-9} = 2i \times 3i = 6i^2 = -6.$

2) THE CONCEPT OF A COMPLEX NUMBER

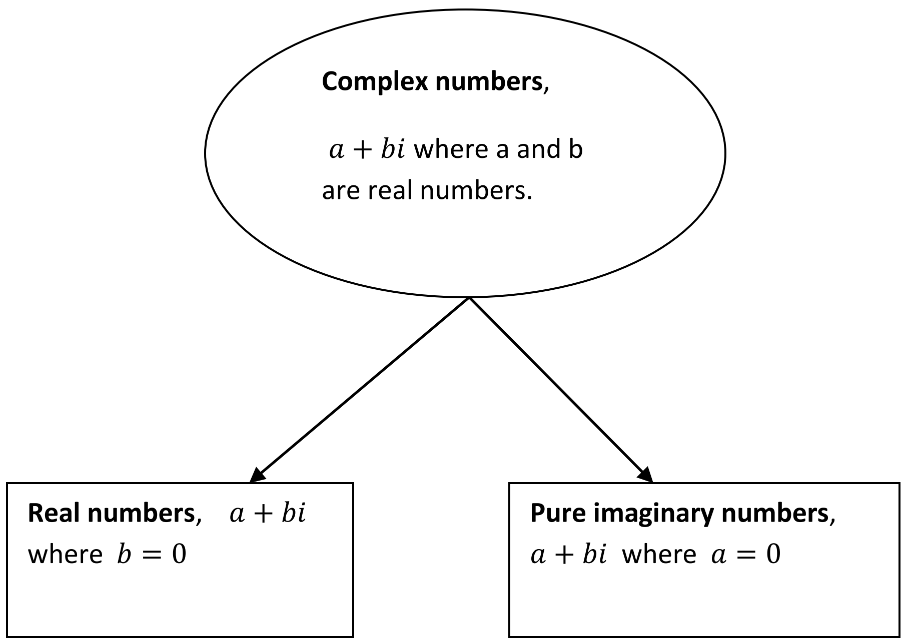
Definition:

A complex number is any number of the form: $z = a + bi,$ where a and b are real numbers and $i = \sqrt{-1}$.

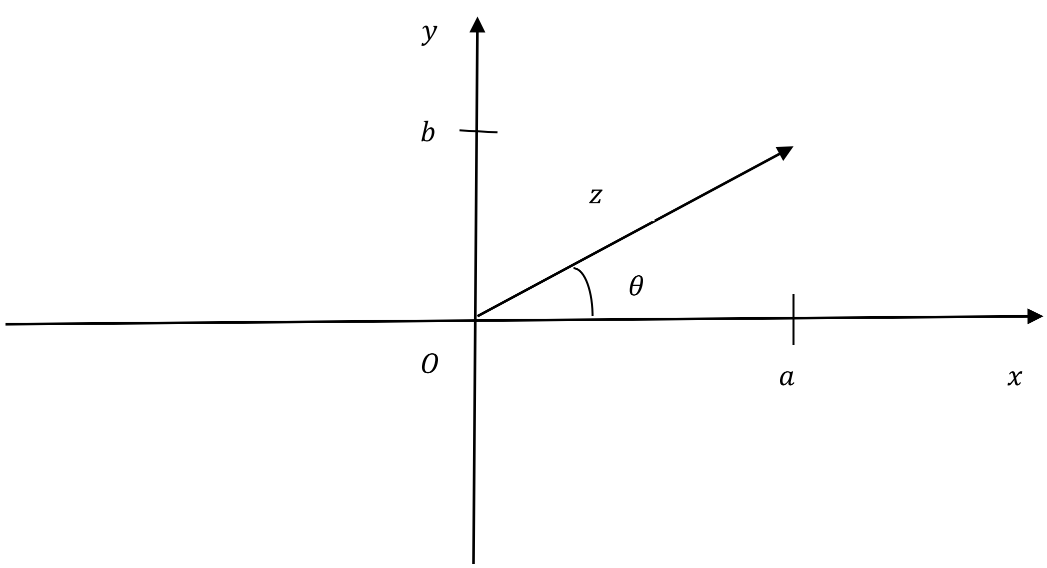
Take note of the following about complex numbers:

i. The complex number is denoted by letter z.
ii. In the complex numbe**r:** $\boldsymbol{z = a + bi}$, **a** is known as the **real part** of the complex number $\boldsymbol{z}$ and **b** is known as the **imaginary part** of the complex number $\boldsymbol{z}$**.**

iii. If $a = 0\ in\ z = a + bi$, we get a number of the form $\boldsymbol{bi}$ which is called an imaginary number. But, if instead $b = 0\ in\ z = a + bi$, we get a number of form $\boldsymbol{a}$ which is a real number. Hence, both the **set of real numbers** and the **set of imaginary numbers** are subsets of the **set of complex numbers**,



iv. A complex number can be represented on the **Argand diagram**, as below:



Here vertical axis is the **imaginary axis** ($y - axis$) and horizontal axis ($-axis$) is the **real axis**.

v. The **size** or the **magnitude** or the **absolute** value of the complex number, denoted as $|z|\ or\ sometimes\ as\ r$ is given by:

$$r = |z| = \sqrt{a^2 + b^2}$$

E.g. The absolute value or magnitude of the complex number $z = -3 + 2i$ is

$|z| = \sqrt{(-3)^2 + 2^2} = 5\ units.$

vi. The angle between a complex number z and the real axis $(x - axis)$ is called **argument.** On the **Argand diagram** above it is denoted **by θ. Its value is :**

$$\theta = tan^{-1}\left(\frac{b}{a}\right)$$

ARITHMETIC OPERATIONS ON COMPLEX NUMBERS

1) ADDITION AND SUBTRACTION OF TWO COMPLEX NUMBERS

Let $z_1 = a + bi\ \ and\ \ z_2 = c + di$ be two complex numbers, then:

i. addition two complex is defined as:

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)i$$

ii. subtraction two complex is defined as:

$$z_1 - z_2 = (a + bi) - (c + di) = (a - c) + (b - d)i$$

EXAMPLE 4

Given the complex numbers: $z_1 = 8 + 2i, z_2 = 12 + 5i, z_3 = 7 - 4i\ \ and\ z_4 = -4 + 3i$, find:

i. $z_1 + z_2$

ii. $z_1 - z_2$

iii. $z_3 + z_4$

iv. $z_4 - z_3$

Solution:

i. $z_1 + z_2 = (8 + 2i) + (12 + 5i)$

⇨ $= (8 + 12) + (2 + 5)i$ (Group like terms at this stage)

⇨ $= 20 + 7i$

$\therefore\ \boldsymbol{z_1 + z_2 = 20 + 7i}$

ii. $z_1 - z_2 = (8 + 2i) - (12 + 5i) = 8 + 2i - 12 - 5i$

⇨ $= (8 - 12) + (2 - 5)i$ (Group like terms at this stage)

⇨ $= -12 - 3i$

$\therefore\ \boldsymbol{z_1 - z_2 = -12 - 3i}$

iii. $z_3 + z_4 = (7 - 4i) + (-4 + 3i) = 7 - 4i - 4 + 3i$

⇨ $= (7 - 4) + (-4 + 3)i$ (Group like terms at this stage)

⇨ $= 3 - i$

$\therefore\ \boldsymbol{z_3 + z_4 = 3 - i}$

iv. $z_4 - z_3 = (-4+3i) - (7-4i) = -4 + 3i - 7 + 4i$

⇨ $= (-4-7) + (3+4)i$ (Group like terms at this stage)

⇨ $= -11 + 7i$

$\therefore\ \boldsymbol{z_1 - z_2 = -11 + 7i}$

2) MULTIPLICATION OF COMPLEX NUMBERS

Complex numbers are multiplied as if they were binomials, with $\boldsymbol{i^2 = -1}$.

Let $z_1 = a + bi\ \ and\ \ z_2 = c + di$ be two complex numbers, then:

$$z_1z_2 = (a+bi)(c+di) = a(c+di) + bi(c+di)$$

⇨ $= ac + adi + bci + bdi^2 = ac + adi + bci - bd$ (Since $i^2 = -1$)

⇨ $= ac - bd + adi + bci = (ac - bd) + (ad + bc)i$ (Group like terms at this stage)

$\therefore\ z_1z_2 = (ac - bd) + (ad + bc)i$

EXAMPLE 5

Given the complex numbers: $z_1 = 8 + 2i, z_2 = 12 + 5i, z_3 = 7 - 4i\ \ and\ z_4 = -4 + 3i$, find:

i. z_1z_2
ii. z_3z_4
iii. $(z_3)^2$

Solutions:

i. $z_1z_2 = (8+2i)(12+5i) = 8(12+5i) + 2i(12+5i)$

⇨ $= 96 + 40i + 24i - 10$

⇨ $= 86 + 64i$

$\therefore\ \boldsymbol{z_1z_2 = 86 + 64i}$

ii. $z_3z_4 = (7-4i)(-4+3i) = 7(-4+3i) - 4i(-4+3i)$

⇨ $= -28 + 21i + 16i + 12$

⇨ $= -16 + 37i$

$\therefore\ \boldsymbol{z_3z_4 = 86 + 37i}$

iii. $(z_3)^2 = (7-4i)^2 = (7-4i)(7-4i)$

⇨ $= 7(7-4i) - 4i(7-4i)$

⇨ $= 49 - 28i - 28i + 16i^2$

⇨ $= 49 - 28i - 28i - 16 = 33 - 56i$

$\therefore\ \boldsymbol{(z_3)^2 = 33 - 56i}$

EXAMPLE 6

Given the complex numbers: $z_1 = 8 + 2i, z_2 = 8 - 2i, z_3 = -4 - 3i \ \ and \ z_4 = -4 + 3i$, find:

i. $z_1 z_2$
ii. $z_3 z_4$

Solutions:

i. $z_1 z_2 = (8 + 2i)(8 - 2i) = 8(8 - 2i) + 2i(8 - 2i)$
⇨ $= 64 - 16i + 16i - 4i^2 = 64 + 4 = 68.$

$\therefore \ \boldsymbol{z_1 z_2 = 68}$

ii. $z_1 z_2 = (-4 - 3i)(-4 + 3i) = -4(-4 + 3i) - 3i(-4 + 3i)$
⇨ $= 16 - 12i + 12i - 9i^2 = 16 + 9 = 25.$

$\therefore \ \boldsymbol{z_1 z_2 = 25}$

Definition:

The complex number $\bar{z} = a - bi$ is a conjugate of the complex number $z = a + bi$.

Note the following:

- **The product of the complex number $z = a + bi$ and its conjugate $\bar{z} = a - bi$ is the real number $a^2 + b^2$, as the following work shows:**
  $z\bar{z} = (a + bi)(a - bi) = a(a - bi) + bi(a - bi)$
  ⇨ $= a^2 - abi + abi - b^2 i^2$
  ⇨ $= a^2 + b^2$

  $\therefore \quad z\bar{z} = a^2 + b^2$

- Remember that the absolute or magnitude of a complex number $z = a + bi$ is given by $|z| = \sqrt{a^2 + b^2}$, hence the absolute of the complex number can also be written in terms of the product between a complex number and its conjugate as:
  $|z| = \sqrt{z\bar{z}}$...........(1)
  Squaring both sides of (1):
  $|z|^2 = z\bar{z} = a^2 + b^2$ ....(2)

EXAMPLE 7

Fill in the missing information in the table below:

Complex number (z)	Its conjugate
$3-7i$	$3+7i$
$-3+5i$	$-3-5i$
$6+i$	$6-i$
i	$-i$
4	4

3) DIVISION OF TWO COMPLEX NUMBERS

To divide complex numbers, we often realize the denominator as follows:

Let $z_1 = a + bi \ and \ z_2 = c + di$ be two complex numbers, then:

$$\frac{z_2}{z_1} = \frac{c+di}{a+bi} = \frac{c+di}{a+bi}\left(\frac{a-bi}{a-bi}\right)$$

⇨ $= \frac{ac-bci+adi-bdi^2}{a^2+b^2}$

⇨ $= \frac{(ac+bd)+(ad-bc)i}{a^2+b^2}$

⇨ $= \frac{(ac+bd)}{a^2+b^2} + \frac{(ad-bc)}{a^2+b^2}i$

Let $p = \frac{(ac+bd)}{a^2+b^2} \ and \ q = \frac{(ad-bc)}{a^2+b^2}$, then we get:

$= p + qi$

Hence, to divide two complex use the conjugate of the denominate to realize the denominator.

EXAMPLE 8

Express each of the following in the form $p + qi, where\ p\ and\ q\ are\ rational\ numbers$:

i. $\frac{1}{3+i}$

ii. $\frac{5+3i}{6-2i}$

iii. $\frac{3+i}{i}$

Solutions:

i. $\frac{1}{3+i} = \frac{1}{3+i}\left(\frac{3-i}{3-i}\right) = \frac{3-i}{3^2+1^2}$

⇨ $= \frac{3-i}{10} = \frac{3}{10} - \frac{1}{10}i$

$\therefore \frac{1}{3+i} = \frac{3}{10} - \frac{1}{10}i$

ii. $\frac{5+3i}{6-2i} = \frac{5+3i}{6-2i}\left(\frac{6+2i}{6+2i}\right)$

$= \frac{5(6+2i)+3i(6+2i)}{6^2+(-2)^2} = \frac{30+10i+18i-6}{40}$

$= \frac{24+28i}{40} = \frac{24}{40} + \frac{28}{40}i$

$\therefore\ \frac{5+3i}{6-2i} = \frac{3}{5} + \frac{7}{10}i$

iii. $\frac{3+i}{i} = \frac{3+i}{i} \times \frac{i}{i}$

$= \frac{-1+3i}{-1} = 1 - 3i$

EXAMPLE 9

Find the real part and imaginary part of the complex number:

$$z = \frac{2+9i}{5-2i}$$

Solutions:

$\frac{2+9i}{5-2i} = \frac{(2+9i)(5+2i)}{(5-2i)(5+2i)}$

⇨ $= \frac{2(5+2i)+9i(5+2i)}{5^2+(-2)^2} = \frac{10+4i+45i-18}{29}$

⇨ $= \frac{-8+49i}{29} = -\frac{8}{29} + \frac{49}{29}i$

⇨ $z = -\frac{8}{29} + \frac{49}{29}i$

$\therefore\ Re(z) = -\frac{8}{29}$ and $Im(z) = \frac{49}{29}$.

THE ZERO COMPLEX NUMBER

A complex number is zero if and only if the real term and the imaginary term are each zero,

i.e. $x + yi = 0 \iff x = 0\ and\ y = 0.$

EQUAL COMPLEX NUMBERS

Now, consider the case $a + bi = c + di$…………(1)

Or $(a+bi) - (c+di) = 0$

⇨ $(a-c) + (b-d)i = 0$

$\iff a - c = 0\ and\ b - d = 0$

$\iff a = c\ and\ b = d.$

Thus two complex numbers are equal if and only if their real terms and their imaginary terms are separately equal.

EXAMPLE 10

Solve the following equations for x and y:

$$x + yi = \frac{3 - 2i}{5 + i}$$

Solution:

$$x + yi = \frac{3-2i}{5+i}\left(\frac{5-i}{5-i}\right)$$

⇨ $x + yi = \frac{3(5-i)-2i(5-i)}{5^2+1^2}$

⇨ $x + yi = \frac{15-3i-10i-2}{26}$

⇨ $x + yi = \frac{13-13i}{26}$

⇨ $x + yi = \frac{13}{26} - \frac{13}{26}i = \frac{1}{2} - \frac{1}{2}i$

Hence, $x = \frac{1}{2}$ and $y = -\frac{1}{2}$

EXAMPLE 11

Find the square roots of the complex number $15 + 8i$.

Solution:

Let $\sqrt{15 + 8i} = a + bi$, where a and b are real numbers……….(1)

Squaring both sides of (1), we get:

$$15 + 8i = (a + bi)^2$$

$$15 + 8i = a^2 + 2abi - b^2$$

⇨ $15 + 8i = (a^2 - b^2)$+2abi……………….(2)

Equating the real and imaginary parts gives:

$a^2 - b^2 = 15$ …………………(3)

$2ab = 8 \;\; or \; b = \frac{4}{a}$.............(4)

Substituting (4) in (3) gives:

$$a^2 - \frac{16}{a^2} = 15$$

⇨ $a^4 - 15a^2 - 16 = 0$

⇨ $(a^2 - 16)(a^2 + 1) = 0$

But a is a real number so $a^2 + 1$ gives no suitable values.

But, $a^2 - 16 = 0$,

So $a = \pm 4$

Now, when $\boldsymbol{a = -4}$:

Then $b = \frac{4}{a} = \frac{4}{-4} = -1$