INTEGRATION

Integration is the reverse process of differentiation. In differentiation, we are given a function and we are required to find its derivative or differential coefficient. In integration the derivative of some function is given and we are required to find the function. Integration is sometimes referred to as anti-differentiation or ant-derivative.

An elongated S, denoted as $\int$, is used to replace the words 'the integral of'. ($\int$ Represents the sum).

Generally, a function F(x) is said to be the anti-derivative of the function $f(x)$ on the internal [a, b] if

$$\frac{d}{dx}[F(x)] = f(x)\,,\ \forall\ x \in [a,b].$$

Thus if $\quad \frac{d}{dx}[F(x)] = f(x)\,,\ then$

$$\Rightarrow\ \int f(x)\,dx = F(x)$$

Read as integral of f(x) $\ w.r.t\ x$.

Example 1

$$\frac{d}{dx}\,(8x^3) = 24x^2$$

Hence $\quad \int 24x^2 = 8x^3.$

Similarly,

$$\frac{d}{dx}\,(sin\,x) = cos\,x$$

Hence $\quad \int cos\,x\ \ dx = sin\,x$

However, notice that the differential coefficient of $sin\,x + 9$ is also $cos\,x$, i.e.

$$\frac{d}{dx}\,(sin\,x + 9) = cosx\ .$$

Therefore, $\int\ cos\,x\,dx$ should also equal $sin\,x + 9$. To allow for the possible presence of a constant, whenever the process of integration is performed, a constant "C" is added to the result. Therefore in general;

$$\int cos\,x\,dx = sin\,x + C$$

This implies that different values of C will give different integrals and hence a given function may have an indefinite number of integrals.

Remarks:

The presence of an indefinite constant C justifies the name ***indefinite integral.***

Thus we conclude that

$$\frac{d}{dx}[F(x)] = f(x)$$

$$\Rightarrow \int f(x)dx = F(x) + C$$

where C is the arbitrary constant of integration.

Integrals of the form ax^n:

In general,

$$\int ax^n dx = \frac{ax^{n+1}}{(n+1)} + C$$

(This is sometimes referred to as the power formula).

Note: the power formula is true when n is fractional, zero, or a $\pm$ integer, with the exception of $n = -1$.

Addition and subtraction of integrals:

Let $f(x)$ and $g(x)$ be two functions of x, and let a and b be constants, then

$$\int [af(x) \pm bg(x)]dx = \int af(x)dx \pm \int bg(x)dx$$

$$= a\int f(x)dx \pm b\int g(x)dx$$

This is referred to as integrating term by term.

Examples:

i) $\int 4x^4\, dx = \frac{4x^{4+1}}{4+1} = \frac{4}{5}x^5 + c$

ii) $\int x^{\frac{3}{2}} dx = \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + c = \frac{\frac{2}{5}x^5}{2} + c$

iii) $\int \frac{1}{x^{\frac{2}{3}}}\, dx = \int x^{-\frac{2}{3}}\, dx = \frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + c = \frac{x^{\frac{1}{3}}}{\frac{1}{3}} + c = 3x^{\frac{1}{3}} + c$

iv) $\int \frac{2x^3-3x}{4x}\, dx$

Solutions:

$$= \int \left(\frac{2x^3}{4x} - \frac{3x}{4x}\right) dx$$

$$= \int \left(\frac{x^2}{2} - \frac{3}{4}\right) dx = \int \frac{x^2}{2}\, dx - \int \frac{3}{4} dx$$

$$= \frac{1}{2} \int x^2\, dx - \frac{3}{4} \int dx$$

$$= \frac{1}{2} \cdot \frac{x^{2+1}}{2+1} - \frac{3}{4} \cdot \frac{x^{0+1}}{0+1} + C$$

$$= \frac{1}{6}\, x^3 - \frac{3}{4} x + C$$

v) $\int (1-t)^2\, dt$

vi) $\int (x-3)2 \cdot \sqrt{x}\ \ dx$

vii) $\int \frac{x^2-4}{x+1}\, dx$

viii) $\int \sqrt{x}\ \ dx$

ix) $\int \frac{x^2}{1+x}\, dx$

x) $\int \frac{x^6+1}{x^2+1}\, dx$

Recall:

$$\frac{d}{dx}\,(\ln x) = \frac{1}{x}$$

$$\Rightarrow \int \frac{1}{x} dx = \int x^{-1} dx$$

$$= \ln x + c$$

Standard integrals

1) $\int 1\, dx = x + c$

2) $\int k dx = kx + c$

3) $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1} \cdot (1+n)x^n = x^n$

$$\Rightarrow \int x^n\, dx = \frac{x^{n+1}}{n+1} + c\,, n \neq -1$$

4) $\int \frac{1}{x}\, dx = \ln x + C$

5) $\int e^x\, dx = e^x + C$

6) $\int a^x\, dx = \frac{a^x}{\ln a} + C\ \ ; a > 0\ \ , a \neq 1$

7) $\int \cos ax\, dx = \frac{1}{a} \sin ax + C$

8) $\int \sin ax\, dx = \ -\frac{1}{a} \cos ax + C$

9) $\int \sec^2 ax = \frac{1}{a} \tan ax + C$

10) $\int cosec^2\, ax\, dx = \ -\frac{1}{a} \cot ax + C$

11) $\int cosec\, ax \cot ax\, dx = \ -\frac{1}{a}\ cosec\, ax + C$

12) $\int sec\, ax\, tan\, ax\, dx = \frac{1}{a} sec\, ax + C$

13) $\int \frac{1}{\sqrt{1-x^2}}\, dx = sin^{-1} x + C$

14) $\int \frac{1}{1+x^2}\, dx = tan^{-1} x + C$

15) $\int \frac{1}{x\sqrt{x^2-1}}\, dx = sec^{-1} x + C$

16) $\int -\frac{1}{\sqrt{1-x^2}}\, dx = cos^{-1} x + C$

17) $\int -\frac{1}{1+x^2}\, dx = cot^{-1} + C$

18) $\int -\frac{1}{x\sqrt{x^2-1}}\, dx = cosec^{-1} + C$

Examples

Evaluate the following integrals:

a) $\int 8\, cos\, 3x\, dx$

$$= 8 \int cos3x\, dx = 8 \cdot \frac{1}{3} sin\, 3x + C$$
$$= \frac{8}{3} sin\, 3x + C$$

b) $\int \frac{1+cosx}{sin^2 x}\, dx$

$$= \int \frac{1}{sin^2 x}\, dx + \int \frac{cosx}{sin^2 x}\, dx$$

$$= \int cosec^2 x\, dx + \int cot\, x\, cosec\, x\, dx$$

$$= -cot\, x - cosec\, x + C$$

c) $\int \frac{2}{3e^{4t}}\, dt$

$$= \frac{2}{3} \int \frac{1}{e^{4t}}\, dt = \frac{2}{3} \int e^{-4t} dt$$

$$= \frac{2}{3} \cdot \left(-\frac{1}{4}\right) \cdot e^{-4t} + C$$

$$= -\frac{1}{6}\, e^{-4t} + C$$

$$= -\frac{1}{6e^{4t}} + C$$

d) $\int \frac{1}{\sqrt[n]{x}}\, dx$

e) $\int e^{3lnx}\, dx$ *note ;* $e^{lnf(x)} = f(x)$ and $m\, ln\, n = ln\, n^m$

f) $\int \sqrt[3]{p^2}\; dp$

g) $\int \frac{(x+1)^2}{x\sqrt{x}}\, dx$

Examples

a) $\int tan^2 x\, dx$

$$= \int (sec^2 x - 1)\, dx$$

$$\int sec^2 x\, dx - \int 1 \cdot dx = tan\, x - x + C$$

b) $\int \sqrt{1 - sin2x}\;\; dx = \int [(cos^2 x + sin^2 x) - 2\, sin\, x\, cos\, x\,]^{\frac{1}{2}}\, dx$

$$= \int [\,(cos\, x - sin\, x)^2]^{\frac{1}{2}}\, dx$$

$$= \int (cos\, x - sin\, x\,)\, dx$$

$$= \int cosx\, dx - \int sin\, x\, dx$$

$$= sin\, x - (-\, cos\, x) + C$$

$$= sin\, x + cos\, x + C$$

c) $\int \frac{1 - sin\, x}{cos^2 x}\;\, dx$

d) $\int cos^4\ 2x\, dx$

Recall; $\cos 2x\ = \frac{1}{2}\ (cos2x + 1)$

$$\Longrightarrow \cos^2 2x = \frac{1}{2}\ (\cos 4x + 1)$$

Proof: $[\cos 4x = \cos(2x + 2x) = \cos^2 x - \sin^2 x]$

Solution

$$= \int (\cos^2 2x)^2\ dx$$

$$= \int (\frac{1}{2}(\cos 4x + 1)^2\ dx$$

$$= \frac{1}{4}\ \int (\cos^2 4x + 2\cos 4x + 1\ )dx$$

Integrating term by term we have;

$$\int\ \cos^{4x} dx$$

$$\cos^2 4x = \frac{1}{2}\ (\cos 8x + 1)$$

$$\Longrightarrow \frac{1}{4}\ \int (\frac{1}{2}(\cos 8x + 1) + 2\cos 4x + 1)\ dx$$

$$\frac{1}{4}\ \left[\frac{1}{16}\sin 8x + \frac{1}{2}\sin 4x + \frac{1}{2}x + x\ \right] = \frac{1}{64}\sin 8x + \frac{1}{8}\sin 4x + \frac{3}{8}x + C$$