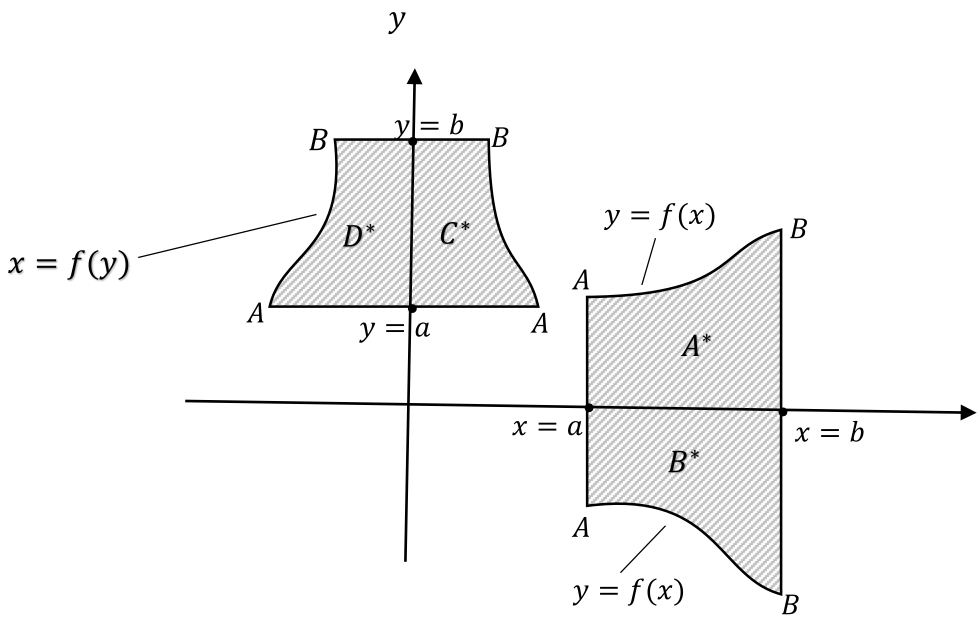
Application of integration

Area under a curve:

Consider the shaded areas A^* ,B^*, C^* and D^* below.



From the diagram above, we have the following important remarks;

1) The area under the curve $y = f(x)$, above the $x - axis$ between the ordinates $x = a$ and $x = b$ (i.e. A^*) is given by

$$\int_a^b y \cdot dx = \int_a^b f(x)dx$$

2) The area bounded by the curve $y = f(x),$ below the $x - axis$ between the ordinates $x = a$ and $x = b$(i.e. shaded area B^*) is given by

$$\int_a^b -y \cdot dx = -\int_a^b ydx = -\int_a^b f(x)dx$$

3) The area bounded by the curve $x = f(y)$, $y - axis$ between the abscissae $y = a$ and $y = b$ (Area C^*) is given by

$$\int_a^b xdy = \int_a^b f(y)dy$$

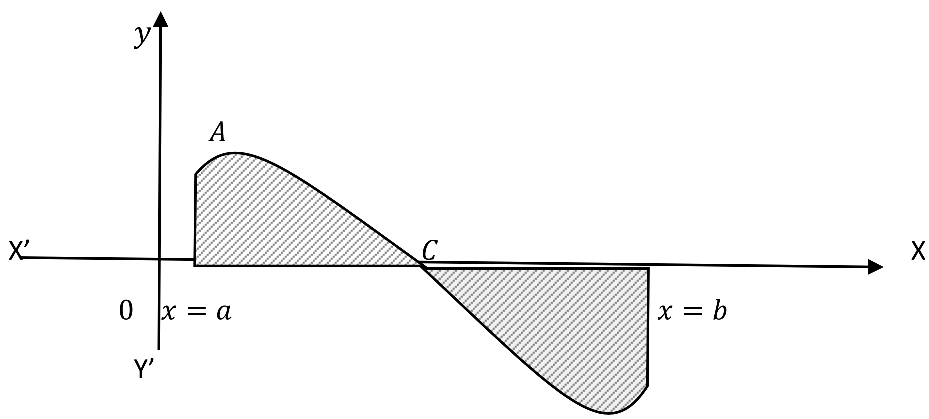
4) The area bounded by the curve $x = f(y)$, $y - axis$ between abscissae $y = b$ (Area D^*) is given by

$$\int_a^b -x\,dy = -\int_a^b f(y)dy$$

5) If $f(x) \geq 0$ for $a \leq x \leq c$ and $f(x) \leq 0$ for $c \leq x \leq b$,(figure below), then the area bounded by the curve $y = f(x)$, $x - axis$ and the ordinates $x = a$ and $x = b$ is given by

$$S^* = \int_a^c f(x)dx + \int_c^b -f(x)dx$$

$$= \int_a^c f(x)dx - \int_c^b f(x)dx$$



Note: it is necessary to draw a rough sketch in order to see whether the region is above the $x - axis$ or is below the $x - axis$ or is partly above and partly below the $x - axis$. If it is difficult to draw the sketch of a function or it is not specifically asked in the question, then we can avoid drawing the sketch and observe the sign of the function on the interval under consideration.

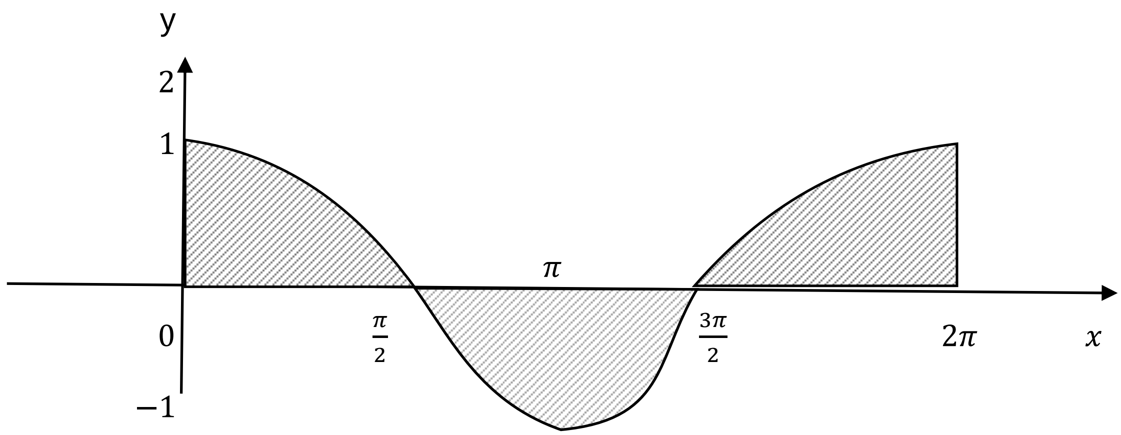
N.B:

Note that

$$\int_a^c f(x)dx - \int_c^b dx \neq \int_a^b f(x)dx$$

Example 1

Find the area bounded by the curve $y = \cos x$, $x - axis$ and the ordinates $x = 0$ and $x = 2\pi$



Now

$\cos x > 0$ When $x \in \left(0, \frac{\pi}{2}\right)$

$\cos x < 0$ when $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

$\cos x > 0$ when $x \in \left(\frac{3\pi}{2}, 2\pi\right)$

∴ the required Area= $Area$ of shaded ergion

$$= \int_0^{\frac{\pi}{2}} |y| dy = \int_0^{\frac{\pi}{2}} |y| dx + \int_{\frac{3\pi}{2}}^{\frac{\pi}{2}} |y| dx + \int_{2\pi}^{\frac{3\pi}{2}} |y| dx$$

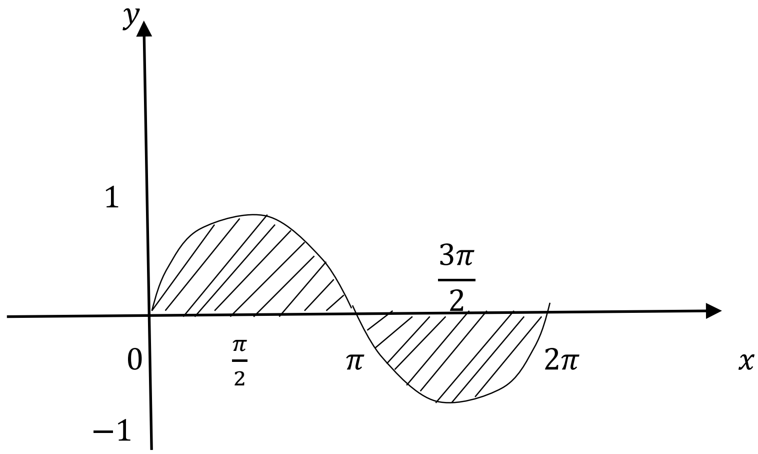
$$= \int_0^{\frac{\pi}{2}} \cos x dx + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -\cos x\, dx + \int_{\frac{3\pi}{2}}^{2\pi} \cos x\, dx$$

$$= \sin x|_0^{\frac{\pi}{2}} - \sin x|_{\frac{3\pi}{2}}^{\frac{\pi}{2}} + \sin x|_{\frac{3\pi}{2}}^{2\pi}$$

$$= 4\ sq \cdot units$$

Example

Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$



Required area = area of shaded region

$$= \int_0^{2\pi} |y| dx = \int_{\pi}^{2\pi} |y| dx$$

$$= \int_0^{\pi} y dx + \int_{\pi}^{2\pi} -y \, dx$$

$$= \int_0^{\pi} \sin x \, dx - \int_{\pi}^{2\pi} \sin x \, dx$$

$$= [-\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi}$$

$$= 4 \, sq. units$$

Exercise (first do the example behind this page)

3) Draw a rough sketch of $y = \sin 2x$ and determine the area(bounded) enclosed by the curve, $x - axis$ and the lines $x = \frac{\pi}{4}$ and $x = 3\pi/4$

4) Make a rough sketch of the function $y = \cos 3x$, $0 \leq x \leq \frac{\pi}{6}$ and determine the area enclosed between the curve and the coordinate axis.

5) find the area bounded by the curve $y = x$,axis and the ordinates $x = 1,\ x = 2$

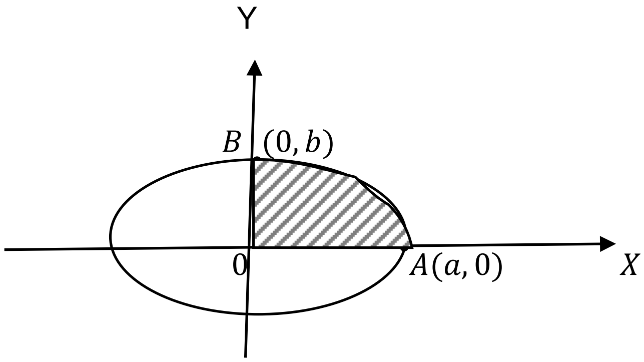
6) find the area in the first quadrant bounded by the parabola $y = 4x^2$

and line $x = 0, y = 1 , y = 4$

7)find the area bounded by the curve $y^2 = 4a^2(x - 3)$ and the lines $x = 3 , y = 4a$

$$Ans\left(\frac{16a}{3}\right)$$

8) find the area under the ellipse: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



Required area = $4\times$ area of ellipse in the first quadrant

$$= 4\times \text{ area } OAB$$

$$= 4\int_0^a y\cdot dx$$

Where y is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$y^2=\frac{b^2}{a^2}\left(a^2-x^2\right)$$

$$y=\frac{b}{a}\sqrt{a^2-x^2}$$

$\therefore$ required area $=\frac{4b}{a}\int_0^a\sqrt{a^2-x^2}\,dx$

Let $x=a\sin\theta$

$\Rightarrow dx = acos\theta d\theta$, $\theta=0|_{x=0}$ and $\theta=\frac{\pi}{2}\Big|_{x=a}$

$$=\frac{4b}{a}\int_0^{\frac{\pi}{2}}\sqrt{a^2-a^2\sin^2\theta}\cdot a\cos\theta\,d\theta$$

$$=4ab\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta$$

$=4ab\int_0^{\frac{\pi}{2}}\left(\frac{1+cos2\theta}{2}\right)d\theta$ ($\because 1+cos2\theta=2\cos^2\theta$, from $\cos(\theta+\theta)$)

$$=2ab\int_0^{\frac{\pi}{2}}(1+\cos 2\theta)\,d\theta$$

$=\pi ab$ sq. units.

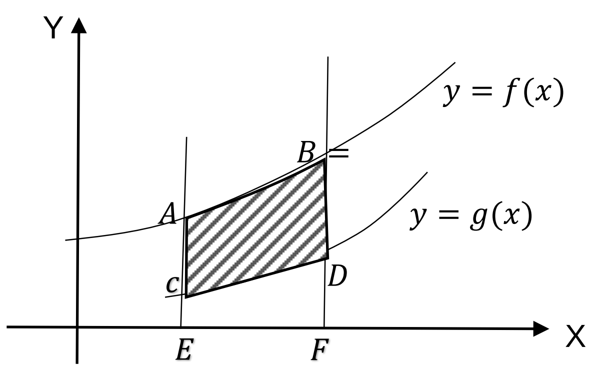
Area between two curves

Let $y = f(x)$ and $y = g(x)$ be two functions such that $0 \le g(x) \le f(x)$ for $a \le x \le b$

I.e. both the curves lie above the axis and the curve $y = f(x)$

Lies above the curve $y = g(x)$

The area between $y = f(x)$ and $y = g(x)$ for $a \le x \le b$ is shown in the figure:



$\therefore$ required area =Area of shaded region

=Area ABDC =Aea ABFE−Area CDFE

$$= \int_a^b f(x)dx - \int_a^b g(x)dx$$

$$= \int_a^b [f(x) - g(x)]dx$$

$\therefore$ Area between $y = f(x)$ and $g(x) = y, a \le x \le b$

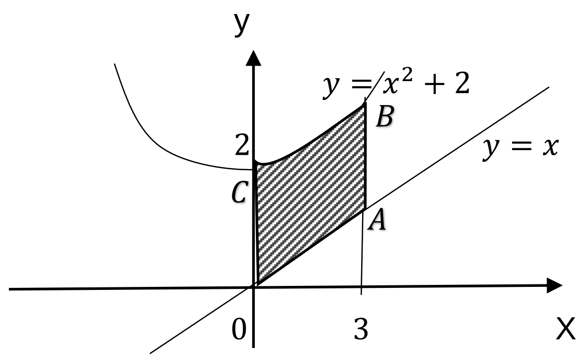
$$= \int_a^b \left(y_{upper\ curve} - y_{lower\ curve}\right)dx$$

Example 1

Using integration, find the area of the region bounded by the curves

$y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$

Solution



$y = x^2 + 2$ ……….. (1)

$x = y$ ………….. (2)

$x = 0$ ………….. (3)

$x = 3$ …………. (4)

From (1); we have

$x^2 = y - 2$

x	0	1	-1	2	-2
y	2	3	3	4	4

Required area = area of shaded region

= area of OABC

$$= \int_0^3 \left(y_{upper\ curve} - y_{lower\ curve}\right) dx$$

$$= \int_0^3 (x^2 + 2 - x) dx$$

$$= \left[\frac{x^3}{3} + 2x - \frac{x^2}{2}\right]_0^3$$

$$= \left(9 + 6 - \frac{9}{2}\right) - 0$$

$$= \frac{21}{2} \text{ or } 10\tfrac{1}{2} \text{ sq. units}$$

Example

Find the area of the region enclosed by the parabola

$y^2 = 4ax$ and the chord $y = mx$.

Solution

$y^2 = 4ax$ ………… (1): right handed parabola with vertex $(0,0)$ and symmetric to $x - axis$

$y = mx$ ………….. (2) straight line passing through the origin with slope m

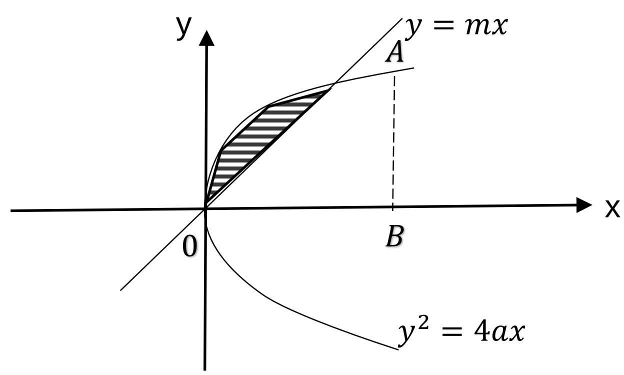
Solving (1) and (2) simultaneously, we have;

$$(mx)^2 = 4ax$$

$$m^2x^2 - 4ax = 0$$

$$x(m^2x - 4s) = 0$$

$$x = 0,\ x = \frac{4a}{m^2}$$



∴

$$y = m(0) = 0 \text{, when } x = 0$$

$$y = m\left(\frac{4a}{m^2}\right) = \frac{4a}{m} \text{, when } x = \frac{4a}{m^2}$$

Thus we have $(0,0)$ and $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$ as points of intersection

Shaded area

$$= \int_0^{\frac{4a}{m^2}} \left(Y_{upper\ curve} - Y_{lower\ curve}\right) dx$$

$$= \int_0^{\frac{4a}{m^2}} \left(\sqrt{4ax} \ \ - mx\right) dx$$

$$= \left[\sqrt{4a} \ \cdot \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{mx^2}{2}\right]_0^{\frac{4a}{m^2}}$$

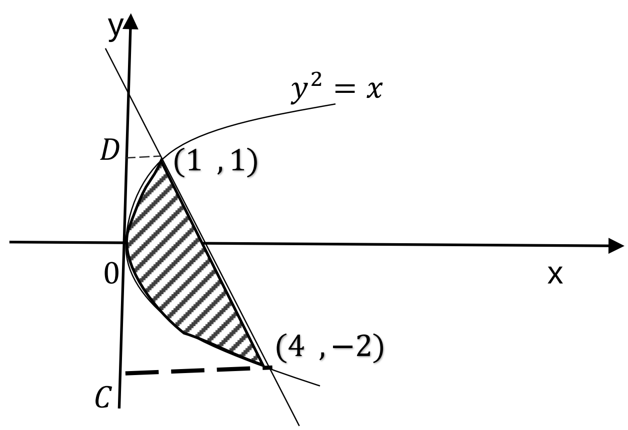
$$= \left[\frac{\sqrt{4a}}{3} \cdot x^{\frac{3}{2}} - \frac{mx^2}{2}\right]_0^{\frac{4a}{m^2}} = \frac{8}{3}\left(\frac{a^2}{m^3}\right) \text{ sq. units}$$

Example

Find the area of the region induced between the parabola

$y^2 = x$ and the line $x + y = 2$

Solution



$y^2 = x$

$\Rightarrow y^2 + y = 2$

$y^2 + y - 2 = 0$

$(y + 2)(y - 1) = 0$

$y = -2\ , y = 1$

$\therefore \quad x = (-2)^2 = 4\ , y = -2$

$x = (1)^2 = 1\ , y = 1$

$\Rightarrow$ points of intersection $(1,1)$ and $(4, -2)$

Shaded area = Area OABO

$$= \int_{-2}^{1} \left(Y_{upper\ curve} - Y_{lower\ curve}\right) dx$$

$$= \int_{-2}^{1} [(2 - y) - y^2] dx$$

$$= \int_{-2}^{1} (2 - y - y^2) dx$$

$$= \left[2y - \frac{y^2}{2} - \frac{y^3}{3}\right]_{-2}^{1}$$

$$= \frac{9}{2} \text{ sq. units.}$$

Exercise

1) Find the area of the region bounded by the parabola $y^2 = 16x$ and the line $x = 4$

2) $x^2 = 4ay$ and $y^2 = 4as$ (Ans: $\frac{16a^2}{3}$ sq. units)

3) $y = x^2$ and the lines $y = |x|$. (Ans:$\frac{1}{3}$ sq . units)

4) $y^2 = 2x + 1$ and the line $x - y - 1 = 0$ (Ans:$\frac{16}{3}$ sq.units)

5) Curves $y = x$ and $y = x^3$ (Ans:$\frac{1}{2}$ sq.units)

6) Find the area enclosed by the curve $y^2 = 4a^2(x - 1)$ and the line $y = 4a$

7) Enclosed by the parabola $x^2 = 6y$ and the circle $x^2 + y^2 = 16$

   (Ans:$\left(\frac{4}{\sqrt{3}} + \frac{16\pi}{3}\right)$ sq.units)

8) Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $y^2 = 4x$. (Ans: $\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$ sq.units)

9) The tangets at $x = 0$ and $x = 3$ on the curve $y = 2x - x^2 - 1$ meet at T

   a) Find the equations of those tangets and the coordinates of T.

   b) Calculate the area of the region bounded by the curve and the tangents.

   (Ans: $\frac{9}{4}$ sq.units).

   I.e. A= $\int_0^{\frac{3}{2}}[(2x - 1) - (2x - x^2 - 1)]dx$

   B= $\int_{\frac{3}{2}}^{3}[(-4x + 8) - (2x - x^2 - 1)]dx$

   $\Rightarrow$ Area $= A + B$