Definite integrals

If $f(x)$ is a continuous function defined on a closed interval$[a, b]$, and if $F(x)$ is the ant-derivative of $f(x)$ i.e. $\frac{d}{dx}\,[F(x)] = f(x)$, then the definite integral of $f(x)$ over$[a, b]$ is denoted by

$$\int_a^b f(x)dx$$

and is equal to

$$[\mathcal{F}(b) - \mathcal{F}(a)]$$

i.e.

$$\int_a^b f(x)dx = [\mathcal{F}(x)]_a^b = [(\mathcal{F}b) - \mathcal{F}(a)]$$

where "a" is the lower limit and "b" is the upper limit.

Note:

Taking $\mathcal{F}(x) + C$ instead of $\mathcal{F}(x)$ as the anti-derivative of $f(x)$, we have;

$$\int_a^b f(x)\,dx = [\mathcal{F}(x) + c]_a^b$$

$$= [\mathcal{F}(b) + c] - [\mathcal{F}(a) + c]$$

$$= \mathcal{F}(b) - \mathcal{F}(a)$$

As the constant of integration disappears, so

$$\int_a^b f(x)dx$$

has a definite value, which is:

$$[\mathcal{F}(b) - \mathcal{F}(a)].$$

Examples

Evaluate the following integrals:

a) $\int_{-4}^{-1} \frac{1}{x}\,dx$ b) $\int_2^5 (x^3 + x)dx$ c) $\int_0^1 \frac{1}{2x-3}\,dx$

Solution

a) $\int_{-4}^{-1} \frac{1}{x}\, dx = -\int_{-4}^{-1} x^{-1}\, dx$

$= \ln x|_{-1}^{-4}$

$= (\ln|-1| - \ln|(-4)|)$

$= 0 - \ln|(-4)|$

$= -\ln 4$

b) $\int_{2}^{5} (x^3 + x) dx$

$= \int_{2}^{5} x^3 dx + \int_{2}^{5} x\, dx$

$= \left.\frac{x^4}{4}\right|_2^5 + \left.\frac{x^2}{2}\right|_2^5$

$= \left(\frac{5^4}{4} - \frac{2^4}{4}\right) + \left(\frac{5^2}{4} - \frac{2^2}{2}\right)$

$= \frac{651}{4}$

c) $\int_{0}^{1} \frac{1}{2x-3}\, dx$

$= \int_{0}^{1} (2x-3)^{-1}\, dx$
$= \frac{1}{2} \cdot \ln(2x-3)|_0^1$
$= \frac{1}{2}[\ln|-1| - \ln|-3|]$
$= \frac{1}{2}[\ln 1 - \ln 3]$
$= -\frac{1}{2}\ln 3$

d) $\int_{0}^{4} \left(x + x^{\frac{3}{2}}\right)\, dx$

e) $\int_{1}^{2} \frac{x^2-3x+2}{x^4}\, dx$

f) $\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}\ dx$

g) $\int_0^{\frac{\pi}{2}} \cos 2x\ dx$

h) $\int_0^{\frac{\pi}{2}} \sin^2 x\ dx$

i) $\int_0^{\frac{\pi}{2}} \frac{sinx}{\sqrt{1+\cos x}}\ dx$ (hint: use half angle identities)

j) $\int_0^{\frac{\pi}{2}} \sin^2 \frac{x}{(1+cosx)^2}\ dx$ hint : $\Rightarrow \int_0^{\frac{\pi}{2}} \left(\frac{sinx}{1+cosx}\right)^2\ dx$

k) $\int_0^{\frac{\pi}{2}} \sin^4 x\ dx$

Integration by substitution

Indefinite integrals of the form $\int f(g(x) \cdot g'(x)dx$ can sometimes be evaluated by making the $u-$ substitution;

$$u = g(x)\ \ du = g'(x)dx \rightarrow \text{ Substitution (1),}$$

Which converts the integral to the form

$$\int f(u).$$

To apply this method for the effect that the substitution has on the x −limits of integration. We can do this in two ways:

Method 1

First evaluate the indefinite integral

$$\int f\big(g(x)\big) \cdot g'(x)dx$$

By substitution, and then use the relationship

$$\int_a^b f\big(g(x)\big)g'(x)dx = \left[\int f(g(x) \cdot g'(x)dx\right]_a^b$$

To evaluate the definite integral. This procedure does not require any modification of the $x-$ limits of integration.

Method 2

Make substitution (1) directly in the definite integral, and then use the relationship $u = g(x)$ to replace $x -$ limits, $x = a$ and $x = b$ by corresponding u –limits, $u = g(a)$ and $u = g(b)$. This produces a new definite integral

$$\int_{g(a)}^{g(b)} f(u)du$$

That is expressed entirely in terms of$u$. This method is sometimes refeered to as the ***change of limits*** method.

Examples

Use the two methods above to evaluate

$$\int_0^2 x(x^2+1)^3 \; dx$$

Method 1:

Let

$u = x^2 + 1$ so that

$du = 2xdx$

Then replacing we have;

$$\int x(x^2+1)^3 dx = \int x \cdot u^3 \cdot \frac{du}{2x}$$

$$= \int \frac{1}{2} \cdot u^3 \, du$$

$$= \frac{1}{2} \int u^3 \cdot du$$

$$= \frac{1}{2} \cdot \frac{u^4}{4} + c$$

$$= \frac{u^4}{8} + c = \frac{(x^2+1)^4}{8} + c$$

Thus;

$$\int_0^2 x(x^2+1)^3 dx = \left[\int x(x^2+1)^3 dx\right]_0^2 = \left.\frac{X^2+1}{8}\right|_0^2$$

$$= \frac{625}{8} - \frac{1}{8} = 78$$

Method 2

If we make the substitution

$$u = x^2 + 1$$

In (1), then

$$u = 1 \text{ if } x = 0$$

$$u = 5 \text{ if } x = 2$$

Thus;

$$\int_0^2 x(x^2+1)^3\,dx = \frac{1}{2}\int_1^5 u^3\,du = \left.\frac{u^4}{8}\right|_{u=1}^{5}$$

$$= \frac{625}{8} - \frac{1}{8}$$

$= 78$ (same as the result for method 1)

Example

Evaluate:

a) $\int_0^{\frac{3}{4}} \frac{dx}{1-x}$ b) $\int_0^{\frac{\pi}{8}} \sin^5 2x\, cos2x\, dx$

Solutions

a) Let $u = 1 - x$ so that $du = -dx$

$\Longrightarrow u = 1$ if $x = 0$

$u = 1/4$ if $x = \frac{3}{4}$

Thus:

$$\int_0^{\frac{3}{4}} \frac{dx}{1-x} = \int_1^{\frac{1}{4}} -\frac{du}{u} = -\int_1^{\frac{1}{4}} \frac{1}{u}\,du$$

$$= -\ln|u| \;\Big|_1^{\frac{1}{4}}$$

$$= -\left[\ln\left(\frac{1}{4}\right) - \ln(1)\right]$$

$$= -\ln\left(\frac{1}{4}\right)$$

$$= \ln 4$$

b) Let $u = \sin 2x$

So that $du = 2\cos 2x\, dx$

$\Longrightarrow \frac{1}{2} du = \cos 2x dx$

$\Longrightarrow u = \sin(0) = 0$ if $x = 0$

$u = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ if $x = \frac{\pi}{8}$

Therefore;

$$\int_0^{\frac{\pi}{8}} \sin^5 2x\, cos2x\, dx = \int_0^{\frac{1}{\sqrt{2}}} u^5 \cdot \cos 2x \cdot \frac{du}{2\cos 2x}$$

$$= \frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}} u^5 du = \frac{1}{2}\cdot\frac{u^6}{6}\Bigg|_0^{\frac{1}{\sqrt{2}}} = \frac{1}{96}$$

Determine:

a) $\int 6x(5x^2+1)^5 dx$

Let $u = 5x^2 + 1$

So that $du = 10xdx$

Thus;

$$\int 6x\,(5x^2+1)^5\, dx = \int 6\cancel{x} \cdot u^5 \cdot \frac{du}{10\cancel{x}}$$

$$= \frac{3}{5}\int u^5 du$$

$$= \frac{3}{5}\cdot\frac{u^6}{6} + c$$

$$= \frac{1}{10}(5x^2+1)^6 + c$$

b) Evaluate $\int_0^{\frac{\pi}{\sqrt{6}}} 24\sin^5\theta\cos\theta d\theta$

Let $u = \sin\theta$ then

$du = cos\theta d\theta$

Thus;

$$\int_0^{\frac{\pi}{6}} 24\sin^5\theta cos\theta d\theta = \int_0^{\frac{\pi}{6}} 24\cdot u^5 \cancel{cos\theta}\cdot\frac{du}{\cancel{cos\theta}}$$
$$= 24\int_0^{\frac{\pi}{6}} u^5 du$$
$$= 24\cdot\left[\frac{u^6}{6}\right]_0^{\frac{\pi}{6}} \qquad \left(\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}\right)$$
$$= \frac{1}{16}$$

c) $\frac{\int cos\sqrt{x}}{\sqrt{x}}\,dx$

d) $\int\frac{cosx}{\sin^4 x}\,dx$

e) $x^2\cot\int\ x^3\,dx$

f) $\int_{\pi^2}^{4\pi^2}\frac{1}{\sqrt{x}}\cdot sin\sqrt{x}\ \ dx$

g) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} sin\theta\sqrt{1-\cos^2\theta}\ \ d\theta \qquad (u = 2cos\theta)$

Integration by trigonometric substitution

Example

Integrate $\int\sqrt{1-x^2}\ \ dx$

Solution

Let $x = sin\theta$, $dx = cos\theta d\theta$

$$\Longrightarrow x^2 = \sin^2\theta$$

Therefore,

$$1-x^2 = 1-\sin^2\theta = \cos^2\theta$$

Thus;

$$\int\sqrt{1-x^2}\,dx = \int\sqrt{\cos^2\theta}\ d\theta\cdot cos\theta\ d\theta$$
$$= \int\ \cos^2\theta\ d\theta$$

$$= \int \frac{1}{2} (cos2\theta + 1)d\theta$$
$$= \frac{1}{4} sin2\theta + \frac{1}{2}\theta + c \qquad \cdots \quad (1)$$

But $x = sin\theta$

$\Longrightarrow \theta = \sin^{-1}(x)$ and $sin2\theta = 2cos\theta sin\theta = 2\sqrt{1-\sin^2\theta} \cdot sin\theta$

Replacing in (1) we have

$$\frac{1}{4} \cdot 2\sqrt{1-\sin^2\theta} \cdot sin\theta + \frac{1}{2}\sin^{-1}x + c$$

$$= \frac{1}{2}\sqrt{1-x^2} \cdot x + \frac{1}{2}\sin^{-1}x + c$$

$$= \frac{x}{2}\sqrt{1-x^2} + \frac{\sin^{-1}x}{2} + c$$

Summary table of integrals that require use of trigonometric substitution

	$f(x)$	$\int f(x)dx$	Method
1.	$\cos^2 x$	$\frac{1}{2}\left(x + \frac{1}{2}sin2x\right) + c$	use $\cos 2x = 2\cos^2 x - 1$
2.	$\sin^2 x$	$\frac{1}{2}\left(x - \frac{1}{2}sin2x\right) + c$	use $\cos 2x = 1 - 2\sin^2 x$
3.	$\tan^2 x$	$\tan x - x + c$	use $1 + \tan^2 x = \sec^2 x$
4.	$\cot^2 x$	$-\cot x - x + c$	use $\cot^2 x + 1 = cosec^2 x$
5.	$\cos^m x sin^n x$	when m or n is odd but not both	use $\cos^2 x + \sin^2 x = 1$
		When m and n are both even	use either $cos2x = 2\cos^2 x - 1$ or $\cos 2x = 1 - \sin^2 x$
6.	$\sin AcosB$		use $\frac{1}{2}[\sin(A+B) + \sin(A-B)]$

7. $cosAsinB$ use $\frac{1}{2}[\sin(A+B)-\sin(A-B)]$

8. $cosAcosB$ use $\frac{1}{2}[\cos(A+B)+\cos(A-B)]$

9. $sinAsinB$ use $-\frac{1}{2}[\cos(A+B)-\cos(A-B)]$

10. $\frac{1}{\sqrt{a^2-x^2}}$ $\sin^{-1}\left(\frac{x}{a}\right)+c$ use $x=a\,sin\theta$ substitution

11. $\sqrt{a^2-x^2}$ $\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$ use $x=a\,sin\theta$ substitution

12. $\frac{1}{a^2+x^2}$ $\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$ use $x=\tan\theta$ substitution