Lista II - Física Matemática

Questão 3)

- Espaço L_p
  - → Imagem da Transf. Fourier
  - → Espaço de Sequências

$$\|x\| = \left(|x_1|^p + \cdots + |x_n|^p\right)^{\frac{1}{p}}$$

- $\mathbb{R}^n$
  - → Norma max

$$\|x\| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}$$

Questão 1)

$$T^0_2(V) \left\{ \begin{array}{l} T : V \otimes V \longrightarrow \mathbb{R} \\ \quad (v_1, v_2) \longmapsto T(v_1, v_2) \\ v_1, v_2 \in V \end{array} \right.$$

$\dim V = n$

A dimensão do tensor $T^0_2(V)$ é dada como $n \times n$ já que

$$\dim(V \otimes V) = \dim V \cdot \dim V$$

$$= n \cdot n = n^2 \rightarrow \text{dimensão total}$$

$$\overset{n^2}{\boxed{\dim T}} = \dim S + \dim A$$

$\dim S \Rightarrow$

$$n \left\{ \underbrace{\begin{bmatrix} 1 & 9 & 8 & 7 \\ & 2 & 10 & 6 \\ & & 3 & 5 \\ & & & 4 \end{bmatrix}}_{n} \right. \rightarrow \boxed{\frac{n(n+1)}{2}} \rightarrow \text{número de elementos}$$

Então a $\dim A = n^2 - \dfrac{n(n+1)}{2}$

$$= n^2 - \frac{n^2+n}{2} = \frac{n^2-n}{2} = \frac{n(n-1)}{2}$$

Então $$\boxed{\begin{aligned} \dim S &= n(n+1)/2 \\ \dim A &= n(n-1)/2 \end{aligned}}$$

Questão 2)

Métrica Real: $x = (x^1, \cdots, x^n) \in \mathbb{R}^n$
$y = (y^1, \cdots, y^n) \in \mathbb{R}^n$

$$d(x,y) = |x-y| = \left[(x^1-y^1)^2 + \cdots + (x^n-y^n)^2\right]^{1/2}$$

Métrica Complexa: $z = a+ib$ | $a,b,c,d \in \mathbb{R}$
$w = c+id$ | $z,w \in \mathbb{C}$

$$|z|^2 = a^2+b^2 \; ; \; |w|^2 = c^2+d^2$$

$$d(z,w) = |z-w| = \left[(a-c)^2 + (b-d)^2\right]^{1/2}$$

$\mathbb{C}$: span$\{(1,0), (0,i)\}$

$$z+w = Re(z) + R(w) + i\left[Im(z) + Im(w)\right]$$

Métrica no espaço Real $g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Métrica no espaço Complexo $g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$x, y \in \mathbb{R}^2$

$$\langle x, y\rangle = \underset{1\times 2}{(x^1 \; x^2)} \underset{2\times 2}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = x^T g \, y$$

1×2

$$= (x^1 \; x^2) \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = x^1 y^1 + x^2 y^2$$

Produto Escalar entre x e y

$$d(x,y) = \left[\langle x-y, x-y\rangle\right]^{1/2} \quad x, y \in \mathbb{R}^2$$

$$= \left[(x-y)^T g \, (x-y)\right]^{1/2}$$

$$= \left[(x^1-y^1)(x^1-y^1) + (x^2-y^2)(x^2-y^2)\right]^{1/2}$$

$$= \left[(x^1-y^1)^2 + (x^2-y^2)^2\right]^{1/2}$$

$$x = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} ; \quad y = \begin{bmatrix} y^1 \\ y^2 \end{bmatrix}$$

* Cálculo $x, y \in \mathbb{R}^2$
* Métrica euclidiana

$z, w \in \mathbb{C}$, $z = z^1 + i z^2 = \begin{bmatrix} z^1 \\ z^2 \end{bmatrix}$, $w = w^1 + i w^2 = \begin{bmatrix} w^1 \\ w^2 \end{bmatrix}$

$$\langle z, w \rangle = z^T g\, w$$

$$= \begin{pmatrix} z^1 & z^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

$$= \begin{pmatrix} z^1 & z^2 \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = z^1 w^1 + z^2 w^2$$

$$d(z, w) = \left[\langle z - w, z - w \rangle\right]^{1/2} \qquad z - w = \begin{pmatrix} z^1 - w^1 \\ z^2 - w^2 \end{pmatrix}$$

$$= \left[(z - w)^T g\, (z - w)\right]^{1/2}$$

$$= \left[\begin{pmatrix} z^1 - w^1 & z^2 - w^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z^1 - w^1 \\ z^2 - w^2 \end{pmatrix}\right]^{1/2}$$

$$= \left[\begin{pmatrix} z^1 - w^1 & z^2 - w^2 \end{pmatrix} \begin{pmatrix} z^1 - w^1 \\ z^2 - w^2 \end{pmatrix}\right]^{1/2}$$

$$= \left[(z^1 - w^1)^2 + (z^2 - w^2)^2\right]^{1/2}$$

* $\mathbb{C}$ é isomórfico a $\mathbb{R}^2$

$$\dim \mathbb{C} = \dim \mathbb{R}^2 = 2$$

A métrica euclidiana no plano real também pode ser utilizada como métrica para o plano imaginário.

Questão 1)

$$\mathbb{V} = (V, +, \mathbb{R})$$

$$\langle , \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{R} \qquad \dim \mathbb{V} = n$$
$$(v_1, v_2) \mapsto \langle v_1, v_2 \rangle$$

$$\text{dist}(x,y) = \langle x-y, x-y \rangle^{\frac{1}{2}}$$

Desigualdade de Cauchy:

$$|x-y|^2 \leq |x|^2 |y|^2$$

Positivo normado:

$$|x-y|^2 \geq 0 \iff (x-y)^T (x-y) \geq 0$$

$$(x^T - y^T)(x-y) \geq 0 \iff x^T x - 2x^T y + y^T y \geq 0$$

$$x^T x - 2x^T y + y^T y \geq 0$$

$$\Updownarrow$$

$$-(2x^T y)^2 - 4x^T x\, y^T y \leq 0$$

$$-(x^T y)^2 - |x|^2 |y|^2 \leq 0$$

$$\langle x, y \rangle^2 - |x|^2 |y|^2 \leq 0$$

$$\boxed{\langle x, y \rangle^2 \leq |x|^2 |y|^2}$$

Questão 5)

$$V = (V, +, \mathbb{R})$$

$$\dim V = n$$

$$g: V \otimes V \longrightarrow \mathbb{R}$$

$$(v_1, v_2) \longmapsto g(v_1, v_2)$$

$$g(v_1, v_2) = g_{ij}\, \phi^i \otimes \phi^j (v_1, v_2)$$

$$= g_{ij}\, v_1^k v_2^l\, \phi^i \otimes \phi^j (\phi_k, \phi_l)$$

$$= g_{ij}\, v_1^i v_2^j$$

$$\boxed{\phi_j = g_{ij}\, \phi^i}$$

Partindo da definição de Tensores,

$$g : V \otimes V \longrightarrow \mathbb{R}$$
$$(v_1, v_2) \longrightarrow g(v_1, v_2)$$

Fazendo para a base ϕ_i

$$g(\phi_i, \;) = g_{ij}\, \phi^i \otimes \phi^j (\phi_l, \;)$$
$$= g_{ij}\, \phi^i(\phi_l)\;\; \phi^j(\;)$$
$$g(\phi_i, \;) = \underbrace{g_{ij}\, \delta^i_l \qquad \phi^j(\;)}$$
$$\boxed{\phi_i = g_{ij}\, \phi^j(\;)}$$

Com isso obtemos a base no espaço V a partir da métrica.

$$\phi_i = g_{ij}\, \phi^j$$

De forma análoga também é possível obter,

$$\phi^j = g_{ij}^{-1}\, \phi_i = g^{ij}\, \phi_i$$

$\hookrightarrow g_{ij}^{-1} = g^{ij}$

Essa operação também com "subir e descer índices".

Questão 6)

$$\begin{cases} A \times B = \varepsilon_{ijk} A^i B^j \phi_k \\ A . B = \delta^i_j A_i B^j = A_i B^i \end{cases}$$

ε_{ijk} → Levi-Civita

δ^i_j → Delta de Kronecker

1º) Propriedade (Produto Misto)

$$A = A^i \phi_i \;;\; B = B^i \phi_i \;;\; C = C^i \phi_i$$

$$A \cdot (B \times C) = A . (B^i C^j \varepsilon_{ijk} \phi_k)$$

$$= A \circ (B^i C^j \varepsilon_{ijk} \phi_k)$$

$$= (A \circ \phi_k) B^i C^j \varepsilon_{ijk}$$

$$= A^l \phi_l \circ \phi_k \, B^i C^j \varepsilon_{ijk}$$

$$= \delta^l_k A^l B^i C^j \varepsilon_{ijk} = B^i C^j A^l \varepsilon_{ijl}$$

$$A \times (B \times C) = A \times (B^i C^j \varepsilon_{ijk})$$

$$= A^l \phi_l \times \varepsilon_{ijk} B^i C^j$$

$$= A^l B^i C^j \varepsilon_{lmk} \varepsilon_{ijk}$$

$$= A^l B^i C^j \varepsilon_{lmk} \varepsilon_{ijk}$$

$$= A^l B^i C^j (-\delta_{li} \delta_{mj} + \delta_{lj} \delta_{mi})$$

$$= -A^l B^i C^j \delta_{li} \delta_{mj} + A^l B^i C^j \delta_{lj} \delta_{mi}$$

$$= -(A \cdot B) C + (A \cdot C) B$$

$$= (A \cdot C) B - (A \cdot B) C$$

$$A \times (B \times C) = -(B \times C) \times A$$

2ª Propriedade (Identidade de Jacobi)

$$A \times (B \times C) + C \times (A \times B) + B \times (C \times A) = 0$$

$$\text{1} \qquad\qquad \text{2} \qquad\qquad \text{3}$$

Para 1:

$$A \times (B \times C) = (A \cdot C)\,B - (A \cdot B)\,C$$

Para 2:

$$\overset{A}{C} \times (\overset{B}{A} \times \overset{C}{B}) = (C \cdot B)\,A - (C \cdot A)\,B$$

Para 3:

$$\overset{A}{B} \times (\overset{B}{C} \times \overset{C}{A}) = (B \cdot A)\,C - (B \cdot C)\,A$$

Somando as contribuições

$$\cancel{(A \cdot C)}B - \cancel{(A \cdot B)}C + \cancel{(C \cdot B)}A - \cancel{(C \cdot A)}B + \cancel{(B \cdot A)}C - \cancel{(B \cdot C)}A$$

$$\therefore \; = 0$$

$\blacksquare$

3ª Propriedade (Binet - Cauchy Identity)

$$(A \times B) \cdot (C \times D)$$

$$(A^i B^j \varepsilon_{ijk}) \cdot (C^l D^m \varepsilon_{lmz})$$

$$A^i B^j C^l D^m \varepsilon_{ijk} \varepsilon_{lmz} \delta_{kz}$$

$$A^i B^j C^l D^m \varepsilon_{ijk} \varepsilon_{lmk}$$

$$A^i B^j C^l D^m (-\delta_{im} \delta_{jl} + \delta_{il} \delta_{jm})$$

$$-(A \cdot D)(B \cdot C) + (A \cdot C)(B \cdot D)$$

Questão 7)

Calculando $\Lambda^r(\mathbb{R}^4)$ para diferentes valores de r,

$r = 1$; $\Lambda^1(\mathbb{R}^4) = \mathbb{R}^4$

$$\text{Base} = \{\phi_1, \phi_2, \phi_3, \phi_4\}$$

$r = 2$; $i, j = \{1, 2, 3, 4\}$

$$\Lambda^2(\mathbb{R}^4) \Rightarrow \phi_i \wedge \phi_j = \quad \boxed{\beta_{ij} = -\beta_{ji}}$$

$$\beta = \beta^{ij}\, \phi_i \otimes \phi_j$$

Antissimetrizando temos,

impondo que $\beta^{ij} = -\beta^{ji}$

$$\beta = \frac{1}{2}\left[\beta^{ij}\, \phi_i \otimes \phi_j + \beta^{ji}\, \phi_j \otimes \phi_i\right]$$

$$= \frac{1}{2}\left[\beta^{ij}\, \phi_i \otimes \phi_j - \beta^{ij}\, \phi_j \otimes \phi_i\right]$$

$$= \frac{1}{2}\beta^{ij}\left[\phi_i \otimes \phi_j - \phi_j \otimes \phi_i\right]$$

$$= \beta^{ij}\, \phi_i \wedge \phi_j$$

$$\beta = \beta^{11}\, \cancel{\phi_1 \wedge \phi_1}^{\,0} + \beta^{12}\, \cancel{\phi_1 \wedge \phi_2} + \beta^{13}\, \cancel{\phi_1 \wedge \phi_3}$$
$$+ \beta^{14}\, \cancel{\phi_1 \wedge \phi_4} + \beta^{21}\, \cancel{\phi_2 \wedge \phi_1} + \beta^{22}\, \cancel{\phi_2 \wedge \phi_2}^{\,0}$$

$$+\beta^{23}\cancel{\phi_2\wedge\phi_3} + \beta^{24}\cancel{\phi_2\wedge\phi_4} + \beta^{31}\cancel{\phi_3\wedge\phi_1}$$

$$+\beta^{32}\cancel{\phi_3\wedge\phi_2} + \beta^{33}\cancel{\phi_3\wedge\phi_3}\xrightarrow{} 0 + \beta^{34}\cancel{\phi_3\wedge\phi_4}$$

$$+\beta^{41}\cancel{\phi_4\wedge\phi_1} + \beta^{42}\cancel{\phi_4\wedge\phi_2} + \beta^{43}\cancel{\phi_4\wedge\phi_3}$$

$$+\beta^{44}\cancel{\phi_4\wedge\phi_4}\xrightarrow{} 0$$

Assim,

$$\beta = 2\beta^{11}\cancel{\phi_1\wedge\phi_1}\xrightarrow{} 0 + 2\beta^{12}\phi_1\wedge\phi_2 + 2\beta^{13}\phi_1\wedge\phi_3$$

$$+2\beta^{14}\phi_1\wedge\phi_4 + 2\beta^{22}\cancel{\phi_2\wedge\phi_2}\xrightarrow{} 0 + 2\beta^{23}\phi_2\wedge\phi_3$$

$$+2\beta^{24}\phi_2\wedge\phi_4 + 2\beta^{33}\phi_3\wedge\phi_3 + 2\beta^{43}\phi_4\wedge\phi_3$$

$$+2\beta^{44}\cancel{\phi_4\wedge\phi_4}\xrightarrow{} 0$$

Ou seja, $\Lambda^2(\mathbb{R}^4)$ é gerado pela base,

$$\Lambda^2(\mathbb{R}^4) = \text{span}\{\phi_3\wedge\phi_4, \phi_1\wedge\phi_2, \phi_1\wedge\phi_3, \phi_1\wedge\phi_4, \phi_2\wedge\phi_4, \phi_2\wedge\phi_3\}$$

$$\dim \Lambda^2(\mathbb{R}^4) = \binom{4}{2} = \frac{4!}{2!(4-2)!} = 6$$

$r = 3$

$\begin{cases} 1,2,3 \\ 2,1,3 \\ 6 \end{cases}$

Para construir $\Lambda^3(\mathbb{R}^4)$ escrevemos um vetor no espaço e exigimos a anti-simetrização, chegando a expressão

$$\phi_i \wedge \phi_j \wedge \phi_k = \frac{1}{3!} \varepsilon_{ijk}\, \phi_i \otimes \phi_j \otimes \phi_k$$

Ou seja, temos a base $(i,j < 1,2,3,4)$

$$\phi_1 \wedge \phi_2 \wedge \phi_3 = \frac{1}{3!}\Big[\phi_1 \otimes \phi_2 \otimes \phi_3 + \phi_3 \otimes \phi_1 \otimes \phi_2 + \phi_2 \otimes \phi_3 \otimes \phi_1 - \phi_2 \otimes \phi_1 \otimes \phi_3 - \phi_3 \otimes \phi_2 \otimes \phi_1 - \phi_1 \otimes \phi_3 \otimes \phi_2 \Big]$$

1,2,3
2,1,3

$$\phi_1 \wedge \phi_2 \wedge \phi_4 = \frac{1}{3!}\Big[\phi_1 \otimes \phi_2 \otimes \phi_4 + \phi_4 \otimes \phi_1 \otimes \phi_2 + \phi_2 \otimes \phi_4 \otimes \phi_1 - \phi_2 \otimes \phi_1 \otimes \phi_4 - \phi_4 \otimes \phi_2 \otimes \phi_1 - \phi_1 \otimes \phi_4 \otimes \phi_2 \Big]$$

$\begin{cases} 1,2,4 \\ 2,1,4 \end{cases}$

$$\phi_3 \wedge \phi_4 \wedge \phi_1 = \frac{1}{3!}\Big[\phi_3 \otimes \phi_4 \otimes \phi_1 + \phi_1 \otimes \phi_3 \otimes \phi_4 + \phi_4 \otimes \phi_1 \otimes \phi_3 - \phi_4 \otimes \phi_3 \otimes \phi_1 - \phi_1 \otimes \phi_4 \otimes \phi_3 - \phi_3 \otimes \phi_1 \otimes \phi_4 \Big]$$

$$\begin{cases} 3 \; 4 \; 1 \\ 4 \; 3 \; 1 \end{cases}$$

$$\phi_3 \wedge \phi_4 \wedge \phi_2 = \frac{1}{3!}\Big[\phi_3 \otimes \phi_4 \otimes \phi_2 + \phi_2 \otimes \phi_3 \otimes \phi_4 + \phi_4 \otimes \phi_2 \otimes \phi_3 - \phi_4 \otimes \phi_3 \otimes \phi_2 - \phi_2 \otimes \phi_4 \otimes \phi_3 - \phi_3 \otimes \phi_2 \otimes \phi_4 \Big]$$

$$\begin{cases} 3 \; 4 \; 2 \\ 4 \; 3 \; 2 \end{cases}$$

Fazendo com que,

$$\Lambda^3(\mathbb{R}^4) = \text{span}\{ \phi_1 \wedge \phi_2 \phi_3, \; \phi_1 \wedge \phi_2 \wedge \phi_4, \; \phi_3 \wedge \phi_4 \wedge \phi_1, \; \phi_3 \wedge \phi_4 \wedge \phi_2 \}$$

O que também é confirmado por

$$\dim \Lambda^3(\mathbb{R}^4) = \binom{4}{3} = \frac{4!}{3!\,1!} = 4$$

$$\dim \Lambda^4(\mathbb{R}^4) = \binom{4}{4} = \frac{4!}{4!\,0!} = 1$$

$r = 4$

Partindo da definição,

$$\phi_i \wedge \phi_j \wedge \phi_k \wedge \phi_l = \frac{1}{4!} \varepsilon_{ijkl} \phi_i \otimes \phi_j \otimes \phi_k \otimes \phi_l$$

$$\Lambda^4(\mathbb{R}^4) = \text{span}\{\phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4\}$$

Ou seja, $\dim \Lambda^4(\mathbb{R}^4) = \binom{4}{4} = \frac{4!}{4!\,0!} = 1$

→ Impossibilidade de se definir um produto vetorial em $\mathbb{R}^4$

*Propriedades

- $v \times w$ deve ser ortogonal a v e w simultaneamente

- $(v \times w) \cdot v = 0$ e $(v \times w) \cdot w = 0$

$$\Lambda^r(\mathbb{R}^4) \longrightarrow \Lambda^{r+1}(\mathbb{R}^4)$$

$$\alpha_r : \Lambda^r(\mathbb{R}^4) \times \Lambda^1(\mathbb{R}^4) \longrightarrow \Lambda^{r+1}(\mathbb{R}^4)$$

$$\alpha_1 : \Lambda^1(\mathbb{R}^4) \times \Lambda^1(\mathbb{R}^4) \longrightarrow \Lambda^{1+1}(\mathbb{R}^4)$$

$$(v, w) \longmapsto v \wedge w$$

isso daqui
vira uma
espeço perpendi
cular

$$\alpha_r(v, w) = v \wedge w$$

$$= \left(\frac{1}{r!} v_{i_1 \cdots i_r} \phi^{i_1} \wedge \cdots \wedge \phi^{i_r}\right) \wedge w_j \phi^j$$

$$= \frac{1}{r!} w_j v_{i_1 \cdots i_r} \phi^{i_1} \wedge \cdots \wedge \phi^{i_r} \wedge \phi^j$$

Fazendo a igualdade

$$\Lambda^{r+1}(\mathbb{R}^4) = \frac{1}{(r+1)!} \phi^{k_1} \wedge \cdots \phi^{k_{r+1}}$$

$v_{i_1 \cdots i_{r+1}}$

$$\phi^j = \phi^{i_{r+1}}$$

$$v_{i_1 \cdots i_r} w_j = \frac{v_{i_1 \cdots i_{r+1}}}{(r+1)}$$

$$v_{i_1 \cdots i_r} w_j (r+1) = \underline{v_{i_1 \cdots i_{r+1}}}$$