

Notes of LA

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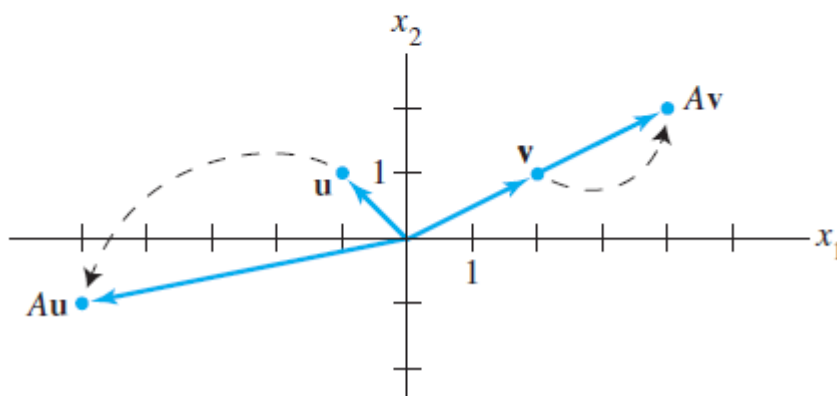
Chapter 1

Eigenvalues and Eigenvectors

1.1 Eigenvectors and Eigenvalues. laia

Although a transformation $x \mapsto Ax$ may move vectors in a variety of directions, it often happens that there are special vectors on which the action of A is quite simple.

Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Transformations by A are shown in Fig. 1. In fact, $A\mathbf{v}$ is just a scalar multiple of \mathbf{v} .



Definition 1.1.1 (Eigenvector and Eigenvalue). If $A\mathbf{x} = \lambda\mathbf{x}$, λ is called an eigenvalue of A , \mathbf{x} is called an eigenvector corresponding to λ . (A is $n \times n$).

Definition 1.1.2 (Eigenspace). The subspace of \mathbb{R}^n consisting of all eigenvectors corresponding to λ is called the eigenspace of A corresponding to λ .

Example 1. Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, Prove: $\lambda_1 = -4$ and $\lambda_2 = 7$ are eigenvalues, and show the eigenspace.

Proof. The scalar -4 is an eigenvalue of A if and only if the equation

$$(A + 4I)\mathbf{x} = \mathbf{0} \quad (1.1)$$

To solve this homogeneous equation, from the matrix

$$(A + 4I)\mathbf{x} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix} \mathbf{x}$$

It is obviously linearly dependent. so (1.1) has nontrivial solutions. Thus -4 is an eigenvalue of A. To find the corresponding eigenvectors, use row operations:

$$\begin{bmatrix} 5 & 6 & 0 \\ 5 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, the general solution has the form $x_1 \mathbf{u}$. Each vector of this form with $x_1 \neq 0$ is an eigenvector corresponding to $\lambda_1 = -4$.

The scalar 7 is an eigenvalue of A if and only if the equation

$$(A - 7I)\mathbf{x} = 0 \quad (1.2)$$

To solve this homogeneous equation, from the matrix

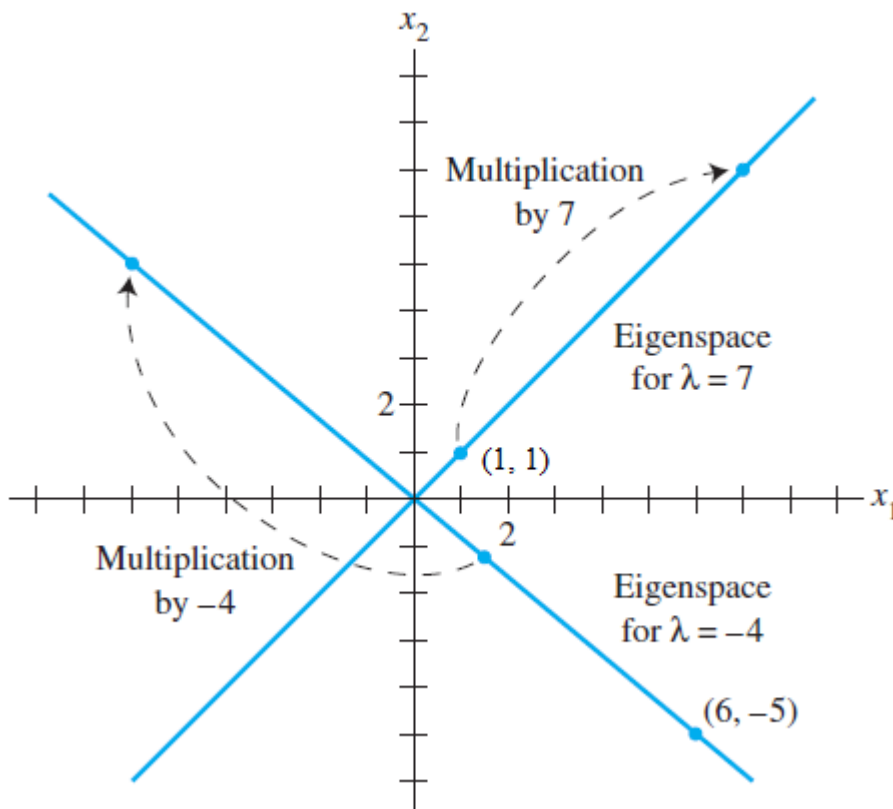
$$(A - 7I)\mathbf{x} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

It is obviously linearly dependent. so (1.2) has nontrivial solutions. Thus 7 is an eigenvalue of A. To find the corresponding eigenvectors, use row operations:

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the general solution has the form $x_2 \mathbf{v}$. Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to $\lambda_2 = 7$.

The eigenspace showed in the figure:



□

Example 2

Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

Solution. Form

$$A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

and row reduce the augmented matrix for

$$(A - 2I)x = 0 \quad (1.3)$$

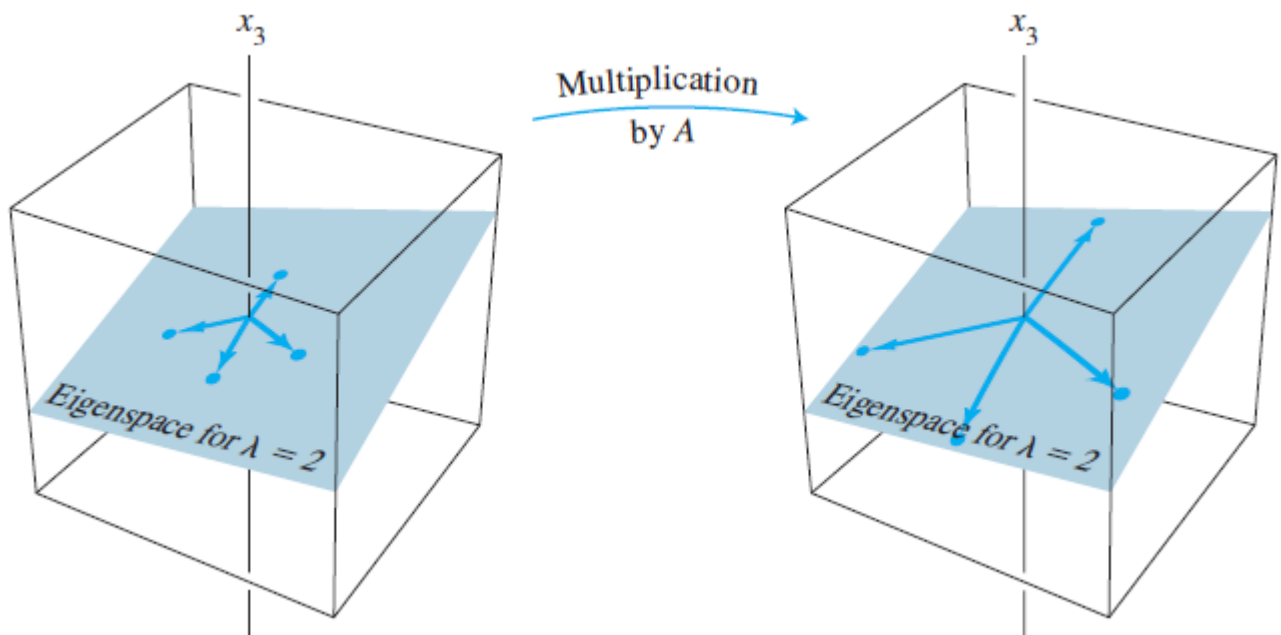
$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

At this point, it is clear that 2 is indeed an eigenvalue of A because the equation (1.2) has free variables. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

x_2 and x_3 free. The eigenspace, shown in figure, is a two-dimensional subspace of \mathbb{R}^3 . A basis is

$$\left\{ \begin{bmatrix} 1/2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$



Theorem 1.1.1. The eigenvalues of a triangular matrix are the entries of entries on its main diagonal.

Proof.

$$A - \lambda I = \begin{bmatrix} a_{1,1} - \lambda & a & a \\ 0 & a_{2,2} - \lambda & a \\ 0 & 0 & a_{3,3} - \lambda \end{bmatrix}$$

when $\lambda = a_{11} | a_{22} | a_{33}$, has non-trivial solution. □

Theorem 1.1.2. if $\mathbf{v}_1, \dots, \mathbf{v}_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A , then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ are linear independent.

Proof. Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ are linear dependent,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_{p-1}\mathbf{v}_{p-1} = \mathbf{v}_p \quad (p \leq r) \quad (1.4)$$

$$c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \dots + c_{p-1}A\mathbf{v}_{p-1} = A\mathbf{v}_p$$

$$c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \dots + c_{p-1}\lambda_{p-1}\mathbf{v}_{p-1} = \lambda_p\mathbf{v}_p \quad (1.5)$$

(1.4) $\times \lambda_p$ - (1.5) :

$$c_1(\lambda_1 - \lambda_p)\mathbf{v}_1 + c_2(\lambda_2 - \lambda_p)\mathbf{v}_2 + \dots + c_{p-1}(\lambda_{p-1} - \lambda_p)\mathbf{v}_{p-1} = \mathbf{0}$$

So, $c_i = 0$, then, $\mathbf{v}_p = \mathbf{0}$, but impossible. □

1.2 Equivalent Conditions. 1aa

Theorem 1.2.1. Let A be an $n \times n$ matrix and λ be a scalar. The following statements are equivalent:

1. λ is an eigenvalue of A .
2. $(A - \lambda I) = \mathbf{0}$ has a nontrivial solution.
3. $N(A - \lambda I) \neq \{\mathbf{0}\}$
4. $(A - \lambda I)$ is singular.
5. $\det(A - \lambda I) = 0$

using 5 to find eigenvalue: write the determinants and solve the equation about λ .

1.3 Eigenvectors and Difference Equations. 1aia

$$\mathbf{x}_{k+1} = A\mathbf{x}_k \quad (k = 0, 1, 2, \dots) \quad (1.6)$$

An eigenvector \mathbf{v}_0 corresponding eigenvalue λ of A

$$\mathbf{x}_{k+1} = \lambda^k \mathbf{x}_0 \quad (1.7)$$

are solution of (1.6).

1.4 Complex Eigenvalues. 1aa

Definition 1.4.1 (real matrix). : $A = \bar{A}$

1. $\bar{\bar{A}} = A$
2. If $\lambda = a + bi$ ($b \neq 0$) is eigenvalue of A , then $\bar{\lambda} = a - bi$ is also eigenvalue of A .
3. $\overline{AB} = \bar{A}\bar{B}$
4. if \mathbf{z} is eigenvector belonging to λ , $\bar{\mathbf{z}}$ is eigenvector belonging to $\bar{\lambda}$. $A\mathbf{z} = \bar{A}\bar{\mathbf{z}} = \overline{A\mathbf{z}} = \overline{\lambda\mathbf{z}} = \bar{\lambda}\bar{\mathbf{z}}$

1.4.1 The Product and Sum of the Eigenvalues. laa

1. $\lambda_1 \times \lambda_2 \times \cdots \times \lambda_n = p(0) = \mathbf{det}(A)$
2. $\sum_{i=1}^n \lambda_i = \sum_{i=1}^n a_{ii} = \mathbf{tr}(A)$

1.4.2 Similar Matrices. laa

Definition 1.4.2 (Similar Matrix). *matrix B is said to be similar to a matrix A if there exists a non-singular matrix S such that $B = S^{-1}AS$*

Theorem 1.4.1. *Let A and B be $n \times n$ matrices. If B is similar to A , then the two matrices have the same characteristic polynomial and, consequently, the same eigenvalues.*

Proof. Let $p_A(x)$ and $p_B(x)$ denote the characteristic polynomials of A and B .

$$\begin{aligned} p_B(\lambda) &= \mathbf{det}(B - \lambda I) \\ &= \mathbf{det}(S^{-1}AS - \lambda I) \\ &= \mathbf{det}(S^{-1}(A - \lambda I)S) \\ &= \mathbf{det}(S^{-1})\mathbf{det}(A - \lambda I)\mathbf{det}(S) \\ &= p_A(\lambda) \end{aligned}$$

The eigenvalues of a matrix are the roots of the characteristic polynomial. Since the two matrices have the same characteristic polynomial, they must have the same eigenvalues. \square

1.5 Matlab eig Syntax

For further references see [Eigenvalues and Eigenvectors](#).

```
e = eig(A)
[V,D] = eig(A)
[V,D,W] = eig(A)
e = eig(A,B)
[V,D] = eig(A,B)
[V,D,W] = eig(A,B)
[___] = eig(A,balanceOption)
[___] = eig(A,B,algorithm)
[___] = eig(___,eigvalOption)
```

Listing 1: Matlab. eig syntax

Example

```
A = [1 7 3; 2 9 12; 5 22 7];
% Calculate the right eigenvectors, V, the eigenvalues, D,
% and the left eigenvectors, W.
[V,D,W] = eig(A)
V =

    -0.2610    -0.9734     0.1891
    -0.5870     0.2281    -0.5816
    -0.7663    -0.0198     0.7912
D =

    25.5548         0         0
         0    -0.5789         0
         0         0    -7.9759
W =

    -0.1791    -0.9587    -0.1881
    -0.8127     0.0649    -0.7477
    -0.5545     0.2768     0.6368
```

Listing 2: Matlab. eig example

1.6 Solutions of Exercises 5.1 of LAIA

1. yes. 2. yes. 3. yes, -2. 4. no.

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