Notes of LA

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Chapter 1

Determinants

1.1 Determinant of Matrix. laa

2 by 2, $\mathbf{det}(A) = a_{11}a_{22} - a_{12}a_{21}$
3 by 3, $\mathbf{det}(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}$

Definition 1.1.1 (minor and cofactor)**.** *Let $A = (a_{ij})$ be an $n \times n$ matrix, and let M_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting the row and column containing a_{ij}. The determinant of Mij is called the **minor** of a_{ij}. We define the **cofactor** A_{ij} of a_{ij} by $A_{ij} = (-1)^{i+j}\mathbf{det}(M_{ij})$*

Thus, Mij is a matrix, Aij is a scale.

Definition 1.1.2 (determinant)**.** *The **determinant** of an $n \times n$ matrix A, denoted $\mathbf{det}(A)$, is a scalar associated with the matrix A that is defined inductively as:*

$$\mathbf{det}(A) = \begin{cases} a_{11}, & n = 1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}, & n > 1 \end{cases}$$

where

$$A_{1j} = (-1)^{1+j}\mathbf{det}(M_{1j})$$

are the cofactors associated with the entries in the first row of A.

Chapter 2

Eigenvalues and Eigenvectors

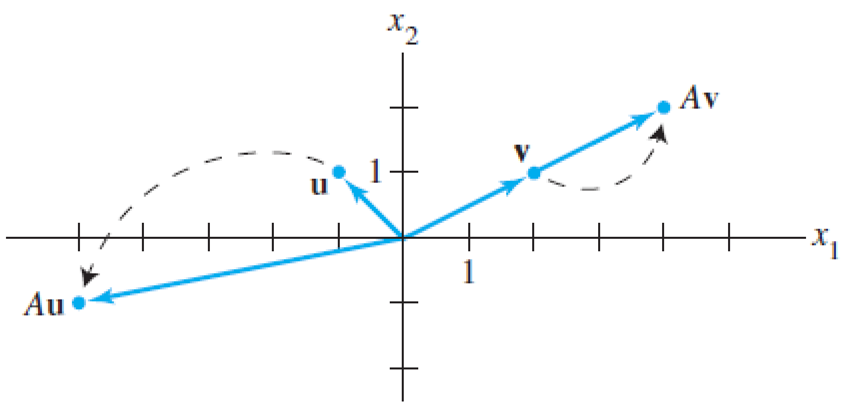
2.1 Eigenvectors and Eigenvalues. laia

Although a transformation $x| - > Ax$ may move vectors in a variety of directions, it often happens that there are special vectors on which the action of A is quite simple.

Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. T

cation by A are shown in Fig. 1. In fact, $A\mathbf{v}$ is ju

lates, $\mathbf{v}$.



Definition 2.1.1 (Eigenvector and Eigenvalue). *if* $A\mathrm{x} = \lambda \mathrm{x}$*,* λ *is called an eigenvalue of* A*,* x *is called an eigenvector corresponding to* λ*. (*A *is* $n \times n$*).*

Definition 2.1.2 (Eigensapce). *The subspace of* R^n *and is called the eigenspace of* A *corresponding to* λ*.*

Example 1. Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, Prove: $\lambda_1 = -4$ and $\lambda_2 = 7$ are eigenvalues, and show the eigenspace.

Proof. The scalar -4 is an eigenvalue of A if and only if the equation

$$(A + 4I)\mathrm{x} = 0 \tag{2.1}$$

To solve this homogeneous equation, from the matrix

$$(A + 4I)\mathrm{x} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}$$

It is obviously linearly dependent. so (1.1) has nontrivial solutions. Thus -4 is an eigenvalue of A. To find the corresponding eigenvectors, use row operations:

$$\begin{bmatrix} 5 & 6 & 0 \\ 5 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 5 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, the general solution has the form $x_1\mathbf{u}$. Each vector of this form with $x_1 \neq 0$ is an eigenvector corresponding to $\lambda_1 = -4$.
The scalar 7 is an eigenvalue of A if and only if the equation

$$(A - 7I)\mathrm{x} = 0 \tag{2.2}$$

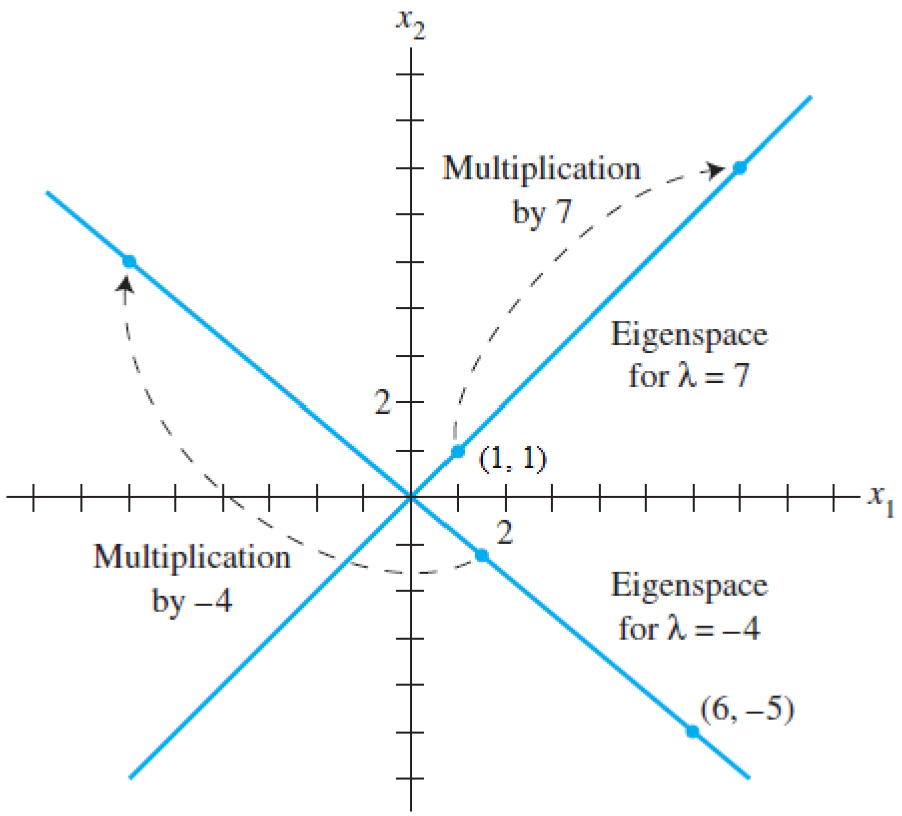
To solve this homogeneous equation, from the matrix

$$(A - 7I)\mathrm{x} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

It is obviously linearly dependent. so (1.2) has nontrivial solutions. Thus 7 is an eigenvalue of A. To find the corresponding eigenvectors, use row operations:

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the general solution has the form $x_2\mathbf{v}$. Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to $\lambda_2 = 7$.
The eigenspace showed in the figure:



□

Example 2

Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the coresponding eigenspace.
Solution. Form

$$A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

and row reduce the augmented matrix for

$$(A - 2I)\mathrm{x} = 0 \tag{2.3}$$

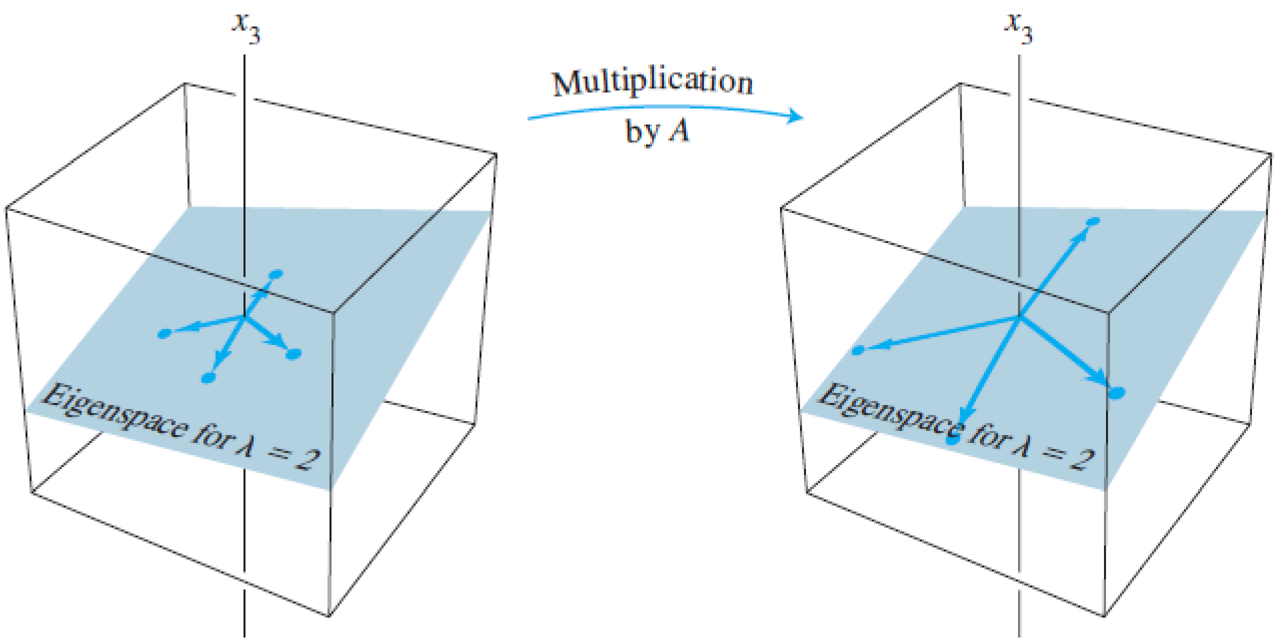
$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

At this point, it is clear that 2 is indeed an eigenvalue of A because the equation (1.2) has free variables. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

x_2 and x_3 free. The eigenspace, shown in figure, is a tow-dimensinal subspace of R^3. A basis is

$$\left\{ \begin{bmatrix} 1/2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$$



Theorem 2.1.1. *The eigenvalues of a triangular matrix are the entries of entries on its main diagonal.*

Proof.

$$A - \lambda I = \begin{bmatrix} a_{1,1} - \lambda & a & a \\ 0 & a_{2,2} - \lambda & a \\ 0 & 0 & a_{3,3} - \lambda \end{bmatrix}$$

when $\lambda = a_{11} | a_{22} | a_{33}$, has non-trivial solution. □

Theorem 2.1.2. *if $\mathbf{v_1}, ..., \mathbf{v_r}$ are eigenvectors coresppoding to distinct eigenvalues $\lambda_1, ..., \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v_1}, ..., \mathbf{v_r}\}$ are linear independent.*

Proof. Suppose $\{\mathbf{v_1}, ..., \mathbf{v_r}\}$ are linear dependent,

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + ... + c_{p-1}\mathbf{v_{p-1}} = \mathbf{v_p} \qquad (p \leq r) \tag{2.4}$$

$$c_1A\mathbf{v_1} + c_2A\mathbf{v_2} + ... + c_{p-1}A\mathbf{v_{p-1}} = A\mathbf{v_p}$$

$$c_1\lambda_1\mathbf{v_1} + c_2\lambda_2\mathbf{v_2} + ... + c_{p-1}\lambda_{p-1}\mathbf{v_{p-1}} = \lambda_p\mathbf{v_p} \tag{2.5}$$

(1.4)$\times\lambda_p$ - (1.5) :

$$c_1(\lambda_1 - \lambda_p)\mathbf{v_1} + c_2(\lambda_2 - \lambda_p)\mathbf{v_2} + ... + c_{p-1}(\lambda_{p-1} - \lambda_p)\mathbf{v_{p-1}} = \mathbf{0}$$

So, $c_i = 0$, then, $\mathbf{v_p} = \mathbf{0}$, but impossible. □

2.2 Equivalent Conditions. laa

Theorem 2.2.1. *Let A be an $n \times n$ matrix and λ be a scalar. The following statements are equivalent:*

1. *λ is an eigenvalue of A.*
2. *$(A - \lambda I) = \mathbf{0}$ has a nontrivial solution.*
3. *$N(A - \lambda I) \neq \{\mathbf{0}\}$*
4. *$(A - \lambda I)$ is singular.*
5. *$det(A - \lambda I) = 0$*

using 5 to find eigenvalue: write the determinants and solve the equation about λ.

2.3 Eigenvectors and Difference Equations. laia

$$\mathbf{x_{k+1}} = A\mathbf{x_k} \qquad (k = 0, 1, 2...) \tag{2.6}$$

An eigenvector $\mathbf{v_0}$ correspoding eigenvalue λ of A

$$\mathbf{x_{k+1}} = \lambda^k\mathbf{x_0} \tag{2.7}$$

are solution of (1.6).

2.4 Complex Eigenvalues. laa

Definition 2.4.1 (real matrix)**.** *: $A = \bar{A}$*

1. $\overline{A} = \overline{a_{ij}}$
2. If $\lambda = a + bi(b \neq 0)$ is eigenvalue of A, then $\overline{\lambda} = a - bi$ is also eigenvalue of A.
3. $\overline{AB} = \overline{AB}$
4. if $\mathbf{z}$ is eigenvector belonging to λ, $\overline{\mathbf{z}}$ is eigenvector belonging to $\overline{\lambda}$. $A\mathbf{z} = \overline{A}\overline{\mathbf{z}} = \overline{A\mathbf{z}} = \overline{\lambda\mathbf{z}} = \overline{\lambda}\overline{\mathbf{z}}$

2.4.1 The Product and Sum of the Eigenvalues. laa

1. $\lambda_1 \times \lambda_2 \times \cdots \times \lambda_n = p(0) = \mathbf{det}(A)$
2. $\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = \mathbf{tr}(A)$

2.4.2 Similar Matrices. laa

Definition 2.4.2 (Similar Matrix)**.** *matrix B is said to be similar to a matrix A if there exists a non-singular matrix S such that $B = S^{-1}AS$*

Theorem 2.4.1. *Let A and B be $n \times n$ matrices. If B is similar to A, then the two matrices have the same characteristic polynomial and, consequently, the same eigenvalues.*

Proof. Let pA(x) and pB(x) denote the characteristic polynomials of A and B.

$$\begin{aligned} p_B(\lambda) &= \mathbf{det}(B - \lambda I) \\ &= \mathbf{det}(S^{-1}AS - \lambda I) \\ &= \mathbf{det}(S^{-1}(A - \lambda I)S) \\ &= \mathbf{det}(S^{-1})\mathbf{det}(A - \lambda I)\mathbf{det}(S) \\ &= p_A(\lambda) \end{aligned}$$

The eigenvalues of a matrix are the roots of the characteristic polynomial. Since the two matrices have the same characteristic polynomial, they must have the same eigen- values. □

2.5 Matlab eig Syntax

For further references see Eigenvalues and Eigenvectors.

```
e = eig(A)
[V,D] = eig(A)
[V,D,W] = eig(A)
e = eig(A,B)
[V,D] = eig(A,B)
[V,D,W] = eig(A,B)
[___] = eig(A,balanceOption)
[___] = eig(A,B,algorithm)
[___] = eig(___,eigvalOption)
```

Listing 1: Matlab. eig syntax

Example

```
A = [1 7 3; 2 9 12; 5 22 7];
% Calculate the right eigenvectors, V, the eigenvalues, D,
% and the left eigenvectors, W.
[V,D,W] = eig(A)
V =

   -0.2610   -0.9734    0.1891
   -0.5870    0.2281   -0.5816
   -0.7663   -0.0198    0.7912
D =

   25.5548         0         0
         0   -0.5789         0
         0         0   -7.9759
W =

   -0.1791   -0.9587   -0.1881
   -0.8127    0.0649   -0.7477
   -0.5545    0.2768    0.6368
```

Listing 2: Matlab. eig example

2.6 Solutions of Exercises 5.1 of LAIA

1. yes. 2. yes. 3. yes, -2. 4. no.

2.7 Solutions of Exercises 6.1 of LAA

1. Find the eigenvalues and the corresponding eigenspaces for each of the following matrices

(a) $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$, $\mathbf{det}(A - \lambda I) = (3-\lambda)(1-\lambda) - 8 = \lambda^2 - 4\lambda - 5 = 0$

$\implies \lambda_1 = 5, \lambda_2 = -1.$

for $\lambda = 5$, $\begin{bmatrix} 3-5 & 2 & 0 \\ 4 & 1-5 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 0 \\ 4 & -4 & 0 \end{bmatrix}$, so eigenspace is $\{(1,1)^{\mathbf{T}}\}$

for $\lambda = -1$, $\begin{bmatrix} 3+1 & 2 & 0 \\ 4 & 1+1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ 4 & 2 & 0 \end{bmatrix}$, so eigenspace is $\{(1,-2)^{\mathbf{T}}\}$

(b) $A = \begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$, $\mathbf{det}(A - \lambda I) = (6-\lambda)(-1-\lambda) + 12 = \lambda^2 - 5\lambda + 6 = 0$

$\implies \lambda_1 = 2, \lambda_2 = 3.$

for $\lambda = 2$, $\begin{bmatrix} 6-2 & -4 & 0 \\ 3 & -1-2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -4 & 0 \\ 3 & -3 & 0 \end{bmatrix}$, so eigenspace is $\{(1,1)^{\mathbf{T}}\}$

for $\lambda = 3$, $\begin{bmatrix} 6-3 & -4 & 0 \\ 3 & -1-3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 0 \\ 4 & -4 & 0 \end{bmatrix}$, so eigenspace is $\{(4,3)^{\mathbf{T}}\}$

List of source codes

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