Computational Physics Project 21

Orbits in a Solar Wind

In this project you will study the effect of the force due to the solar wind on a satellite orbiting the Earth. The results are quite surprising and somewhat counter intuitive. Your task is to write a computer program that demonstrates graphically the effects of this force on planetary motion. Investigate the effects of drag on the orbit of objects.

Ref.

A. Luehrmann “Orbits in the Solar Wind – a Mini Research Problem””

Orbits in the Solar Wind—a Mini-Research Problem

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(Received 22 May 1973; revised 8 August 1973)

A remarkable motion occurs when a classical inverse-square-law orbit is perturbed by a weak uniform force field in the orbital plane. The bizarre nature of the motion, together with the fact that most physicists and physics teachers have never seen or been aware of it, make it an ideal topic for independent research at the undergraduate level. Several modes of attack on the problem are suggested. They proceed from experimentation on a computer simulation, through analytic approximations, to closed-form solutions. A quantum mechanical treatment is also suggested.

I. INTRODUCTION

The education of the majority of physicists can be characterized as having a sharp discontinuity. During the first phase physics is presented for the most part as a sort of geometry—a logically deductive system. The relevant general laws of physics are presented in chapter one of the textbook, and the next twenty chapters extract inferences that can be compared with real-world observations. In phase two, which for many students starts when they begin to search for a thesis problem, the rules of the game are changed. The goal is to confront an untidy, seemingly disorganized physical reality and to make the inductive leap to a new general principle that will serve to bring order out of the apparent confusion. All students sense the change of educational goals; but it can be particularly painful to those who decide upon a career in physics because they like its orderly mathematical structure and because they excel at the phase-one task of inference and derivation. It is not surprising that they feel misled when phase two arrives and they discover that induction is the principal useful activity of practicing physicists.

What is lacking in the undergraduate curriculum is what might be called "mini-research" problems that give beginning students an intellectual experience similar in quality to that of the researcher. Ideally a mini-research problem begins with a confusing or surprising observation that fairly cries out for "explanation." The explanation (or explanations) should not be readily findable by literary research in textbooks. In fact, most practicing physicists should be hard pressed themselves to find a satisfactory explanation. Furthermore, the problem should be rich enough to support several different types of attack and alternative treatments. Finally, of course, all of this must be accomplished without excessive demands on the student's mathematical skills or physical sophistication.

What follows is an example of a mini-research problem that can be, and indeed has been, presented successfully to freshmen, upper-level physics majors, and beginning graduate students. I believe that it satisfies the above conditions remarkably well.

II. THE "SOLAR WIND" PROBLEM

The most important motion problem analyzed in elementary mechanics courses is that due to the attractive inverse-square central force law of gravitation or electrostatics. Textbooks for nonscience as well as those for science and engineering students deal at length with this problem since it plays a pivotal role in the conceptual history of the mechanical view of the universe. Students emerge with varying levels of awareness that such a force gives rise to elliptical orbits, that motion is faster near the focus containing the force center and slower at the opposite end of the ellipse, and the like. If the course is a modern one in which students learn Newton's second law as an algo-

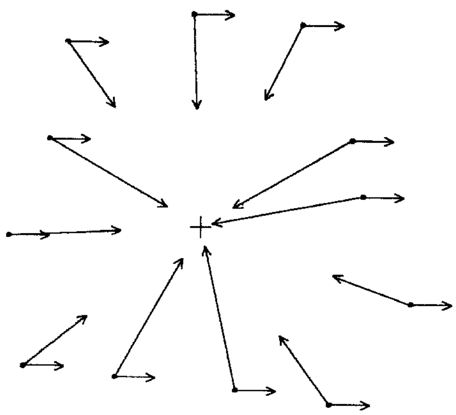

FIG. 1. Superposition of a uniform force field acting to the right and an inverse-square force field acting toward the cross in the center.

rithmic procedure for determining the motion of a particle, then they will have actually computed gravitational orbits for a large variety of initial conditions and they will have verified the Keplerian results directly.

In most courses the gravitational problem is dropped here, to be taken up in somewhat more mathematical detail in the intermediate mechanics course. Yet even there the problem is the pure, unperturbed inverse-square problem. What would happen, a student surely asks, if the force were just a bit different from the pure inverse-square law? One line of inquiry involves varying the exponent in the power law and studying the effect on the motion. That would form a suitable mini-research problem but for the fact that there is too much literature on the subject. The faculty and all the graduate assistants already know the answer to that problem, and the student is bound to be made to feel that he has been handed just another textbook problem.

There is, however, an equally simple variation on the pure gravitational problem which does not suffer from that defect. In fact, based on an informal poll of friends, my observation is that practically no working physicists or physics teachers have ever thought about the following problem. Nor, when it is posed to them, have they succeeded at all well in arriving at the correct behavior of the orbit. If you have not already read ahead or looked at the figures, you can see whether my observation is sound by closing this journal immediately after finishing this section and formulating your own answer to the problem.

Here then, is the problem. The *standard* gravitational problem is to find the motion of a particle for which Newton's second law is

$$m\mathbf{a} = -GMm\mathbf{r}/r^3. \tag{1}$$

More specifically, let us write Eq. (1) as a pair of differential equations for the Cartesian coordinates r_1 and r_2 in the plane of the orbit:

$$\ddot{r}_1 = -r_1/r^3 \tag{2a}$$

$$\ddot{r}_2 = -r_2/r^3 \tag{2b}$$

where $r = (r_1^2 + r_2^2)^{1/2}$ and where we have elected to rescale length and time so that $GM \equiv 1$, though this step is by no means necessary. (The reason for labeling the coordinates r_1, r_2, r_3 rather than the usual x, y, z will become clear in later sections when these differential equations are translated into algorithms written in computer languages.)

Now, what sort of motion would occur if in *addition* to the gravitational force there were also present a *uniform force, constant in magnitude and direction*, acting on the particle? For example, one might treat the pressure of the solar particle wind as a uniform force acting on the moon during its orbital motion about the earth. What is the effect? What would happen if the solar wind were stronger? Or one might think of a classical hydrogen atom between the plates of a capacitor. What happens to the electron orbit when the voltage is turned up?

To narrow the problem somewhat, consider the restricted case in which the direction of the uniform force lies in the plane of the unperturbed orbit. (This approximates the solar wind example.) If the r_1 axis is taken to lie in the direction of the uniform force, whose magnitude is k, then Eqs. (2) become modified as follows:

$$\ddot{r}_1 = -r_1/r^3 + k \tag{3a}$$

$$\ddot{r}_2 = -r_2/r^3. \tag{3b}$$

Figure 1 shows the corresponding vector force field as a pair of components, one central-inverse-square, and the other uniform and to the right.

In mathematical terms then, the problem is this: What are the solutions to Eqs. (3) and how do they differ from the solutions to Eqs. (2)? Since the change introduced is about the most trivial generalization imaginable for the inverse-square problem, it would seem reasonable that one ought to be able to reflect for a time and draw conclusions that are qualitatively correct at the very least—particularly if k is small relative to the inverse-square terms at all points in the orbit. I invite you to close the Journal now for a few minutes and make a conjecture before going on to Sec. III.

III. THE METHOD OF SOLUTION

To ask beginners for closed-form solutions to Eqs. (3) is obviously unreasonable. After all, except for the *shape* of the orbits, solutions to Eqs. (2) in terms of elementary functions are not known, a fact which textbooks usually gloss over. For this reason a conventional course in elementary, intermediate or even graduate mechanics never asks students to find the *motion* caused by a gravitational force—i.e., the position as a function of *time*. And this may be the reason why most readers of this journal will be at a loss in dealing with the solar-wind problem.

During the past five years or so, however, many physics teachers have taken a new approach to classical mechanics. Far from presenting $F=ma$ as an algebraic equality and then reinforcing its algebraicity by assigning homework problems that merely exercise the skill of transforming symbolic equations, the new approach has emphasized the dynamic nature of Newton's second law. This approach makes it perfectly clear that a knowledge of the force function plus the current state variables (position and velocity) uniquely determines the new state variables an instant later. The earliest significant expression of this approach in a textbook can be found in the Feynman lectures.[1] The treatment there was hampered by the author's assumption that students would carry out the finite-difference algorithm by means of hand calculations.

The advent and increasing availability of the computer changed that assumption. In a slim but very influential volume, Bork[2] showed how one might easily introduce freshmen to the minimum amount of computer programming necessary for them to "teach the computer"

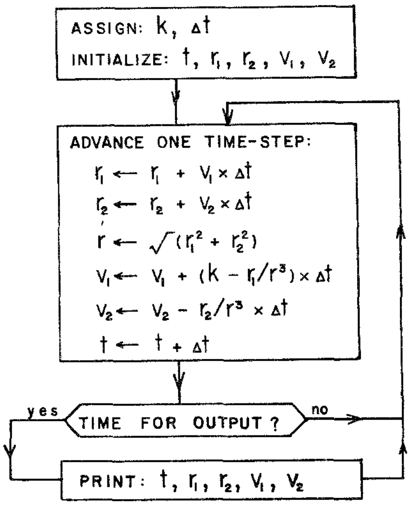

FIG. 2. Block diagram showing the steps of the Newtonian algorithm for determining the motion defined by Eqs. (3) for the solar-wind problem.

the steps for carrying out the Newtonian algorithm. With the calculational effort now minimized, it was possible for students to examine motions due to whatever forces seemed interesting, not just the three or four textbook examples.

In the intervening years several textbooks and monographs[3–5] incorporating this approach have appeared and several others[6–9] are pending publication. They should be examined for a detailed look at what has come to be called the "computer-based" treatment of classical mechanics and other areas of physics. For our purposes here, it must suffice to state that the pedagogical thrust of computer-based mechanics is to motivate a simple Euler method for integrating the equations of motion.

Now let us return to the solution of Eqs. (3). From the computer-based viewpoint, Eqs. (3) may be represented equivalently by the algorithm, or procedure, shown in the block diagram in Fig. 2. The top block establishes k, the strength of the uniform force, initializes the state variables r_1, r_2 and v_1, v_2 and sets a value for the time step Δt. The central block uses the previous velocity components to compute the new position components. Then it computes the force at that position and from that the new velocity. Finally the time is advanced by one time step. The third block prints or plots state variables and then returns to the top of the second block.

The logical equivalence of this representation

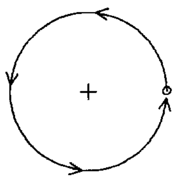

FIG. 3. Initial conditions. These would give rise to the circular orbit shown if the solar-wind force were absent.

and Eqs. (3) occurs in the limit as $\Delta t \to 0$. While for "small" Δt the algorithm yields only approximate results, it has the advantage of being a completely general method for any force law. Setting $k=0$ reduces to the pure inverse-square problem of Eqs. (2), while a non-zero k is equivalent to Eqs. (3). The method of solution is the same in both cases.

IV. SOLAR-WIND ORBITS

So much for the method of solution. Now let us see what motion results. One must, of course, make some assumption regarding initial conditions. For the present we initialize position and velocity such that, if k were equal to zero, circular motion would occur. In fact, as shown in Fig. 3, r_1 is set equal to one, as is v_2; while r_2 and v_1 are zero. In our scaled units this means that the inverse-square force has unit magnitude along the circle.

Now let us set k to be much less than one, keeping in mind that it always acts to the right. Figures 4–6 show the evolving motion with $k=.03$ as computed by the algorithm of Fig. 2.

I believe that for the vast majority of you these

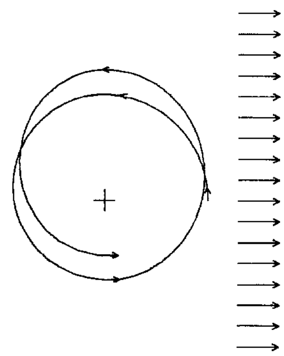

FIG. 5. Two laps of the solar-wind orbit. Solar wind force as in Fig. 4.

results will come as a complete surprise, as they did for me when I first produced them. In fact I would be especially pleased to hear from anyone who conjectured approximately this result without having computed the detailed motion.

While the result is surprising and indeed bizarre, there is considerable order here and it cries out for some sort of *explanation.* The bare computed result is no more satisfying to a physicist than the bare numbers that come out of the run of an experiment. One wants insight. Why doesn't the elongation occur in the same direction as the perturbing force? Why is it exactly ninety degrees away from that direction? Is the motion bounded? Why is the effect so big when the cause is so small? What happened at the conclusion of Fig. 6? It is these questions and many others that make this an interesting mini-research problem that can be attacked on many levels of sophistication. In the next four sections four distinct modes of analysis are sketched. For the most part the

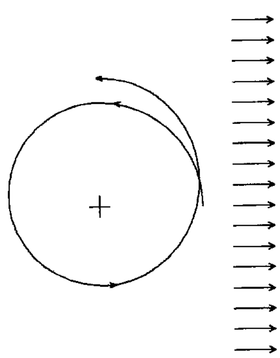

FIG. 4. First lap of the motion computed by the algorithm in Fig. 2. The solar-wind force is only 3% as strong as the inverse-square force.

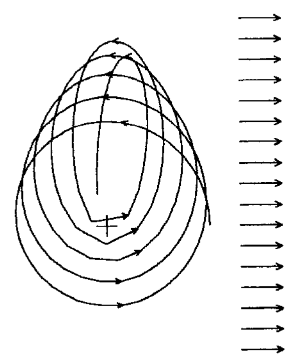

FIG. 6. Several laps of the solar-wind orbit. Computational errors appear to dominate the final stages. Solor wind force as in Fig. 4.

sketches are incomplete, questioning, open-ended—so as not to give it all away in these pages and cancel its usefulness.

V. AN OBSERVATIONAL ATTACK

The observational approach is accessible to students at all levels of sophistication and might be the first step toward a deeper analysis. In effect, the computer-based model becomes the object of a set of experiments and the student-researcher adopts the attitude of the experimental physicist. He can vary the parameters of the model and observe the effects. He can observe the "raw" state variables or else he can "process" them somewhat and observe for example, the angular momentum as a function of time. Below are a few questions that are relatively easy to ask of the model, most of them unanswered here.

(a) What does increasing or decreasing k do to the motion?

(b) Can you find a functional dependence on k among the observables?

(c) Does the effect depend on the initial angular position of the particle in the unperturbed circular orbit?

(d) Suppose k were turned on slowly, from zero to its full value in the space of several orbits. Would that change the effect very much?

(e) Is it true that the major axis of the quasi-ellipses stays about the same length throughout?

(f) Can you define a "period" well enough to ask whether the period of the "quasi-ellipses" is constant or changes?

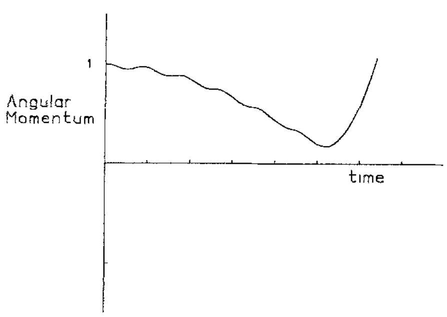

FIG. 7. Angular momentum vs time for the solar-wind motion. Each tick interval on the horizontal axis is equal to one period of the unperturbed circular inverse-square orbit. Computational error appears to dominate toward the end.

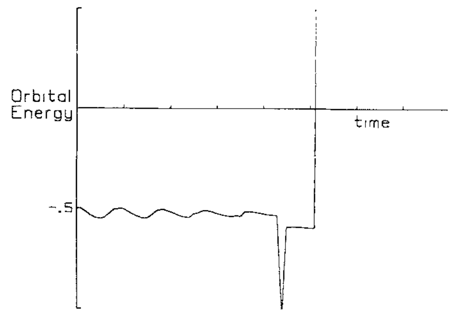

FIG. 8. Orbital energy (K.E. plus gravitational P.E.) vs time for the solar-wind motion. Ticks on the time axis have the same meaning as in Fig. 7. Again, computational error seems to dominate at the end.

(g) What does a graph of angular momentum vs time look like? (See Fig. 7.)

(h) What does a graph of "orbital energy" (K.E. plus gravitational P.E.) vs time look like? (See Fig. 8.)

(i) Can you improve the computational algorithm and examine the situation when the particle comes near the force center?

(j) Is the phenomenon characteristic of all central forces, or is it unique to the inverse-square force?

(k) What if the initial conditions were such that the unperturbed orbit was a fairly long ellipse lying along the horizontal axis? What motion would occur?

(l) Instead of plotting r_1 vs r_2, plot v_1 vs v_2. Bearing in mind that force changes the "velocity state," *not* the "position state," explain why the velocity-space picture is more intuitively clear.

VI. A QUALITATIVE ANALYSIS

The following approach is one that might be pursued by a student in an intermediate or graduate mechanics course, since it assumes familiarity with the concept of an "effective radial potential energy." Since this particular analysis of the problem is not likely to be found by very many students, it is perhaps justifiable to expose it somewhat more openly in these pages.

Figure 9 shows a graph of the effective potential as a function of r for the unperturbed gravitational problem and for the circular initial conditions shown in Fig. 3. For these conditions the angular momentum J equals 1 and the orbital energy

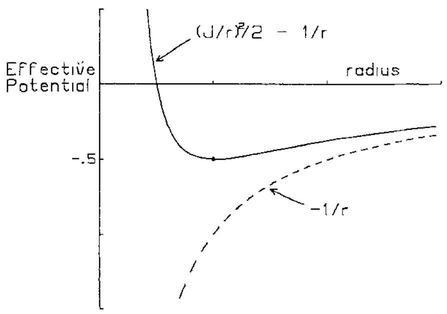

FIG. 9. Graph of the effective radial potential energy as a function of radius, for the inverse-square force with angular momentum $J=1$. The unperturbed circular orbit is represented by the minimum of the well at $r=1$. The energy there is $-\frac{1}{2}$.

equals $-\frac{1}{2}$. That is, the circular orbit corresponds to the minimum in the effective potential well.

Now, everyone knows that if a particle in a well is displaced from equilibrium it will oscillate with some frequency determined by the shape of the well. Set that fact aside for the moment and now examine (Fig. 10) the *radial component* of the solar-wind force on the particle during one circular orbit. Note that the radial component is large and positive at first. Ninety degrees later it is zero. Another ninety degrees has it large and negative. Another brings it to zero once more. And finally it returns to its original large positive value. Conclusion: The *radial component* of this external force is oscillating in time. With what frequency? Well, evidently with the *orbital* frequency appropriate to a gravitational orbit.

Now by a remarkable coincidence, for inverse-square forces and *only* those, the orbital frequency is exactly the same as the frequency of those radial oscillations in the effective well of Fig. 9. (This is what leads to closed gravitational orbits, of course.) Therefore, we see that from the viewpoint of the one-dimensional radial equation, what we have is a classic case of an oscillator driven in

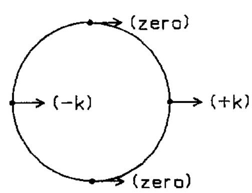

FIG. 10. Note the variation in the radial component of the solar-wind force during one circular orbit.

resonance by an external oscillating force. That is why a 3% cause can lead to such a colossal effect.

Further confirmation of the theory of resonance can be found. Again it is well known that at exact resonance the maximum *response* of the system comes 90 degrees out of phase and later than the maximum in the driving force. And that is exactly what was seen in the orbits of Figs. 4–6. The maximum external radial force was along the horizontal axis, but the maximum radial amplitude occurred ninety degrees later.

The resonance theory needs a great deal more testing before one can accept it. For example, it implies that the resonant effect should disappear if the driving frequency differs from the natural radial frequency. Since this is the case for all

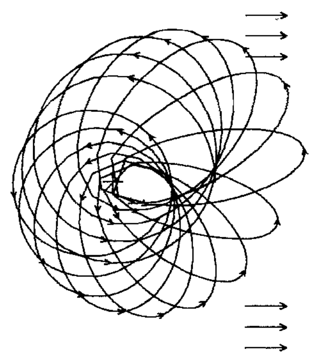

FIG. 11. Drastic change in the solar-wind effect when the exponent in the central-force law is changed from -2 to -1.9. Solar wind strength is 3% of "gravity."

forces *except* the inverse square, one is encouraged to vary the exponent slightly and see what happens. Figure 11 shows the effect of changing the power law from -2 to -1.9, and indeed the effect is strongly altered.

Nevertheless, the resonance theory is not quite right. In the standard version of resonance the external force pumps energy into the oscillator and in our problem would sooner or later drive the particle into an unbound orbit. But the fact is (see Fig. 8) that the orbital energy of the particle varies hardly at all during the many orbital laps. Perhaps it is best to leave this as a mystery and put it as the first question for deeper investigation:

(a) If the energy is essentially constant, how do the radial oscillations occur? Should not the particle merely remain at the bottom of the well?

(b) Relate the constancy of the energy to the behavior of the period of each lap and to the behavior of the major axis of the quasi-ellipses.

(c) Hooke's law, $\mathbf{F}=-\kappa\mathbf{r}$, also leads to closed orbits. Does a similar phenomenon occur for it? Explain your observations by means of the resonance theory.

(d) Compute and plot the radius as a function of time. On the same graph compute and plot the radial component of the solar-wind force. Compare phases.

VII. A QUANTITATIVE ANALYSIS

So far we have not seen how to say anything about the *rate* at which the orbit evolves from a circle to a long ellipse. For this we need a quantitative theory. The only dynamic variable we have observed that undergoes a systematic large-scale change is the angular momentum (Fig. 7). At the beginning it has the circular orbit value of one, in our scaled units. Subsequently it tends toward zero and then diverges sharply, presumably because of errors in our simple computational algorithm. The inverse-square force yields enormous accelerations near the center and, for a fixed time step, must finally contribute to large errors in the velocity calculation (see Fig. 2).

If one wants to have a closer look at the angular momentum variation, then a better algorithm is needed. We have two alternatives. One is to go to the numerical-methods books and search for a well advertised method, such as a fourth-order Runge–Kutta calculation. It turns out that this alternative leads to more complicated programming, yet never really eliminates the problem, which is inherent in the inverse-square law.

The second alternative is to stick with a simple Euler method, but to recast the differential equations (3) so that they contain a *scaled* time τ. The relationship between t and τ is determined by the requirement that equal steps in τ shall cause steps in t to become small as the radius becomes small. For example, we might define τ as follows:

$$dt=r^2d\tau. \tag{4}$$

The exponent of r might as easily have been one. In fact the ideal exponent is $\frac{3}{2}$ (not a surprising result in view of Kepler's third law), but that leads to computational inefficiencies.

Taking Eq. (4) as a definition, we rewrite Eqs. (3) first as a set of first-order equations,

$$dr_1/dt=v_1,$$
$$dr_2/dt=v_2,$$
$$dv_1/dt=-(r_1/r^3)+k,$$
$$dv_2/dt=-r_2/r^3,$$

and then we rewrite these in terms of τ:

$$dr_1/d\tau=v_1r^2, \tag{5a}$$
$$dr_2/d\tau=v_2r^2, \tag{5b}$$
$$dv_1/d\tau=-(r_1/r)+kr^2, \tag{5c}$$
$$dv_2/d\tau=-r_2/r, \tag{5d}$$
$$dt/d\tau=r^2. \tag{5e}$$

Note that Eqs. (5c, d) do not blow up as the radius becomes small. Furthermore, since the orbits we have observed remain well bounded, there is no new large-radius problem with Eqs. (5a, b).

Conversion of Eqs. (5) to a computer algorithm is quite straightforward and only slightly different from the algorithm in Fig. 2. Nevertheless, this new version is so useful in gravitational problems and so little known that a BASIC computer program incorporating it is included here as Fig. 12.

Figure 13 shows the result of running the program of Fig. 12, but with a graphical output. Now one can see that the large-scale motion does not stop at the long thin ellipse, but instead it reverses itself and unwinds into a clockwise circular orbit and goes on to a downward long ellipse and so on cyclically.

Perhaps as interesting is the computed angular momentum J during the cycle, which is shown as Fig. 14. Here again is an observation that cries out for explanation. Apart from fine structure, the shape of the curve is clearly sinusoidal. Should not it be possible to derive a harmonic oscillator equation for J?

$$\ddot{J}=-\Omega^2J. \tag{6}$$

If so, what is Ω and how does it depend on the strength of the solar wind?

In fact, approximations can indeed be made that lead to Eq. (6), but this is certainly best left some-

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100'  ALGORITHM DERIVED FROM EQS. (5)
110'
120'      K = SOLAR-WIND STRENGTH
130'  R1,R2 = POSITION VECTOR
140'  V1,V2 = VELOCITY VECTOR
150'      T = TIME
160'      D = DELTA-TAU
170'     D1 = DELTA-T
180'     T4 = PRINTOUT TIME STEP
190'     T5 - FOR PRINTOUT TEST
200'
210'  INITIALIZATION
220'
230  LET R1=V2=1
240  LET R2=V1=T=T5=0
250  LET T4=1/8
260  LET D=1/32
270'
280'  INITIAL VELOCITY HALF-STEP
290'
300  LET R8=R1*R1+R2*R2
310  LET R=SQR(R8)
320  LET R9=R*R8
330  LET D1=R8*D
340  LET V1=V1+(K-R1/R9)*D1/2
350  LET V2=V2-R2/R9*D1/2
360'
370'  START ITERATION WITH A PRINTOUT
380'
390  GOTO 920
400'
500'    ITERATIVE LOOP STARTS BELOW.
510'    R5,R6 = HALF-STEP ESTIMATE OF R1,R2
520'       R8 = ESTIMATE OF R-SQUARED
530'
540       LET R5=R1+V1*D1/2
550       LET R6=R2+V2*D1/2
560       LET R8=R5*R5+R6*R6
570'
580'    FULL-STEP CALCULATION OF R1,R2
590'
600       LET D1=R8*D
610       LET R1=R1+V1*D1
620       LET R2=R2+V2*D1
630'
640'    FULL-STEP CALCULATION OF V1,V2
650'
660       LET R8=R1*R1+R2*R2
670       LET R=SQR(R8)
680       LET V1=V1+(K*R8-R1/R)*D
690       LET V2=V2-R2/R*D
700'
710'    STEP TIME; TEST FOR PRINTOUT
720'
730       LET T=T+D1
740       IF T<T5 THEN 540
750'
900'  TIME FOR PRINTOUT
910'
920   LET T5=T5+T4
930   PRINT T,R1,R2,V1,V2
940   GOTO 540
950'
999  END
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FIG. 12. BASIC computer program for solar wind orbit, incorporating the improved algorithm based on Eqs. (5).

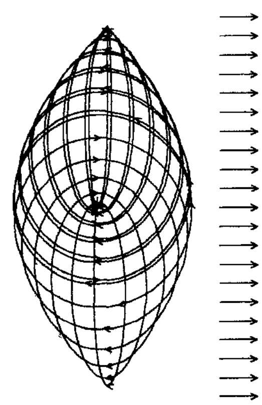

FIG. 13. Application of the program of Fig. 12 to plot many laps of the orbit. Note that the entire motion is cyclic, beginning with a counter-clockwise circle, moving to a thin ellipse at the top, then back to a clockwise circle, and on to a thin ellipse at the bottom. Three-fourths of the entire cycle is shown.

what vague and unresolved in these pages. The point to emphasize here is that this line of attack on the problem can yield quantitative results, which can be compared with runs of the computer model. It provides a different sort of "explanation" of the solar-wind phenomenon.

VIII. A QUANTUM MECHANICAL APPROACH

In a graduate mechanics course it may be appropriate to ask students to explore the quantum mechanical implications of these classical observations of the solar-wind phenomenon. What is the analog of the change in orbit from circular to elongated elliptical? Why in quantum mechanics should the phenomenon occur only for the inverse-square forces? Can one derive the Ω of Eq. (6) by quantum mechanics?

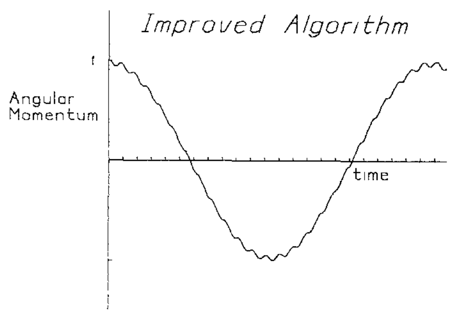

FIG. 14. Application of a modification of the program of Fig. 12 to plot the angular momentum vs time during a full cycle.

Such questions do indeed have simple answers. The quantum mechanical treatment of the central-force problem begins by separating the Hamiltonian into an angular part and a radial part. The radial wave functions depend on the form of the central potential. The angular wave functions, which are independent of the form of the potential, are usually labeled according to their symmetry and total angular momentum, $|s\rangle$, $|p\rangle$, $|d\rangle$, $|f\rangle$, etc. In general the energy eigenvalues are *different* for different angular momentum states. It is a unique feature of the inverse-square force, however, that within a given energy level all angular momentum states are degenerate.

Since the ground state has angular momentum equal to zero this degeneracy does not arise, and so one ought not expect to see the solar-wind (or Stark) effect there. But consider the first excited state, which consists of one $|s\rangle$ state and three $|p\rangle$ states:

$$\begin{aligned}
|s\rangle &= (4\pi)^{-1/2}(2-r)\exp(-r/2)\\
|p_1\rangle &= (4\pi)^{-1/2}r\sin\theta\cos\phi\exp(-r/2)\\
&= (4\pi)^{-1/2}r_1\exp(-r/2),\\
|p_2\rangle &= (4\pi)^{-1/2}r\sin\theta\sin\phi\exp(-r/2)\\
&= (4\pi)^{-1/2}r_2\exp(-r/2),\\
|p_3\rangle &= (4\pi)^{-1/2}r\cos\theta\exp(-r/2)\\
&= (4\pi)^{-1/2}r_3\exp(-r/2),
\end{aligned}$$

where r_1, r_2, and r_3 stand for x, y, and z as before and where the units are such that the Bohr radius has unit length. Note that $|s\rangle$ is spherically symmetric, while $|p_1\rangle$, $|p_2\rangle$, and $|p_3\rangle$ have exactly the same symmetries as the coordinates r_1, r_2, and r_3 themselves.

What combination of these states most closely represents the classical circular orbit, which was the starting point for observing the solar wind effect? One might try the combination with maximum angular momentum along the r_3-axis, which is

$$|p_+\rangle = -(|p_1\rangle + i|p_2\rangle)/\sqrt{2}.$$

Figure 15 shows a computer-generated graph of the probability density associated with $|p_+\rangle$, where the "probability cloud" is viewed from above, looking down the r_3-axis toward the r_1r_2-plane. Indeed, the probability density is circularly symmetric and has a ring-like structure about the origin. (A view along the r_1-axis or the r_2-axis would show that the probability cloud is rather flat with small vertical extent in the r_3 direction.) So $|p_+\rangle$ appears to be a good analog to the undisturbed classical circular orbit with maximum angular momentum.

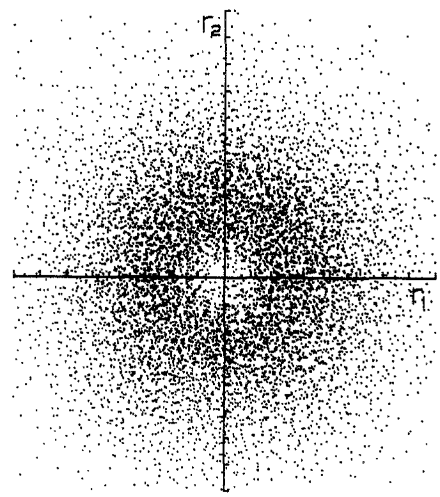

FIG. 15. Computer-generated plot of the probability density associated with $|p_+\rangle$, the first excited state of hydrogen with orbital angular momentum equal to 1 and r_3-component equal to $+1$. This state corresponds to the classical circular orbit.

Now, what happens in the quantum-mechanical situation when the solar-wind force is turned on? The answer is remarkably simple and beautiful. The perturbed Hamiltonian contains a single additional term, $-kr_1$, whose negative gradient is the solar-wind force parallel to r_1. The question in quantum mechanics then amounts to asking whether the perturbation $-kr_1$ causes transitions between $|p_+\rangle$ and any other states degenerate with it.

Recalling the symmetry of $|p_1\rangle$, $|p_2\rangle$, and $|p_3\rangle$, it should be evident that $\langle p_i|-kr_1|p_j\rangle \equiv 0$ for any $i, j = 1, 2$ or 3. Thus there can be no transitions among any of the $|p\rangle$ states. For most central-force cases that would be the end of the discussion. But for the inverse-square force there is also the degenerate $|s\rangle$ state to be examined. And, indeed

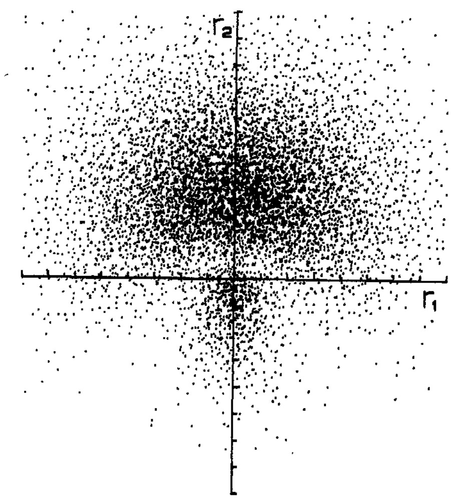

FIG. 16. Computer-generated plot of the probability density associated with $-|s\rangle+|p_2\rangle$, the state into which $|p_+\rangle$ evolves after one quarter of the solar wind cycle. This state corresponds to the classical elongated ellipse.

$\langle s|-kr_1|p_j\rangle\neq 0$ only if $j=1$. The perturbation will, therefore, cause the state $|p_1\rangle$ to transform, in a certain amount of time, to the state $|s\rangle$ (multiplied by some phase factor). Later the $|s\rangle$ state will transform to $|p_1\rangle$ again but with a minus sign—then back to $|s\rangle$ with a minus sign—finally back to $|p_1\rangle$.

Since $|p_+\rangle$ is a combination of $|p_1\rangle$ and $|p_2\rangle$, it is of interest to ask what Fig. 15 evolves into at the time when $|p_1\rangle$ has turned into $|s\rangle$. The result is shown in Fig. 16, and it has some features of the long elliptical orbit (see Fig. 6) that occur at the end of the first quarter cycle of the classical solar-wind motion: (a) the circular symmetry has vanished, (b) the probability cloud is most dense at a point some distance along the positive r_2-axis, and (c) the angular momentum expectation value is zero.

A quarter cycle later the probability cloud will again have the same appearance as Fig. 15. But now the angular momentum is directed in the *negative* r_3 direction. Such a state would correspond rather closely with the *clockwise* classical circular orbit that occurs at the midpoint of the solar-wind cycle.

After another quarter cycle the probability cloud looks like the reflection of Fig. 16 across the r_1 axis. Now the densest part of the cloud is displaced in the *negative* r_2 direction, corresponding to the elongated classical ellipse at the bottom of Fig. 13.

Another quarter cycle completes the entire cycle and returns the system to the original $|p_+\rangle$ state of Fig. 15.

While this is a very pretty example of the Correspondence principle, one can actually go further and compute the frequency Ω of the solar-wind cycle using time-dependent perturbation methods. With the assumption that only states in the first excited level need be considered, one may write

$$\psi(x,t)=a_0(t)\,|s\rangle+a_1(t)\,|p_1\rangle+a_2(t)\,|p_2\rangle+a_3(t)\,|p_3\rangle.$$

The initial $|p_+\rangle$ state corresponds to initial values $a_0(0)=0$, $a_1(0)=-1/\sqrt{2}$, $a_2(0)=-i/\sqrt{2}$, $a_3(0)=0$. By means of the time-dependent Schrödinger equation one can solve for the four coefficients as functions of time. The result is that, apart from the usual eigenstate oscillation, $\exp(iEt/\hbar)$, the only time dependent coefficients are a_0 and a_1. They oscillate with a frequency

$$\Omega=|\langle s|-kr_1|p_1\rangle|/\hbar,$$

the desired quantity. This can be computed and compared with the classical solar-wind frequency defined in Sec. VII.

The point to emphasize here is that this line of attack on the problem can yield both qualitative and quantitative results which can be compared with runs of the computer model of the classical problem. It provides still another "explanation" for the solar-wind phenomenon.

IX. FURTHER QUESTIONS

By no means have we exhausted the possible lines of inquiry into this curious problem. Here are several additional questions and suggestions for exploration:

(a) If the real solar wind produced the above effect the result would be catastrophic for earthlings. Why has it not happened?

(b) To a good approximation the magnitude of the sun's gravitational force on the moon is constant over all parts of the moon's orbit around the earth and is vastly stronger than the solar wind. Why does it not cause the result seen here?

(c) In nature inverse-square forces are as common as weeds. Why is the solar-wind effect not a common observation?

(d) Can one perform an experimental observation of the solar-wind effect? Consider a gas of hydrogen atoms between capacitor plates. Some of the atoms must be in excited states for the effect to occur. The initial state of the gas must be such that a significant population is in the $| p_+ \rangle$ state. The capacitor field has to be strong enough to produce the effect in a time short compared with collision times. Can all of this be done? If so, what would one observe?

(e) What about a planet in a tight orbit around one of a pair of binary stars? Will the gravitational pull of the other star cause the planet to undergo the solar-wind effect?

(f) For the advanced student: Is a closed-form analysis of Eqs. (3) possible? What about non-standard coordinate systems? Classical perturbation theory? Compare the insights gained from closed-form analysis with the ones gained from the methods suggested here.

X. A FINAL PLEA

As stated at the outset, the entire purpose of this article is to offer a useful example of a mini-research problem suitable for independent undergraduate exploration. It is for this reason that the presentation here is deliberately tentative, incomplete, and, hopefully, provocative and enticing. It would be a great misfortune if the chief result of this publication were to elicit a series of subsequent contributions to this Journal which proceeded to reveal the answers to all of the questions asked here and more, thereby converting a physics research problem into a library research problem and subverting its pedagogical utility.

I should like to conclude with a plea, therefore, that we teachers of physics restrain ourselves from that special pleasure of telling our students in class and each other in print all we know about this curious problem. Let us be content to smile knowingly, communicate privately, and, best of all, search out and publicize new and equally rich mini-research problems.

ACKNOWLEDGMENTS

The final form of this paper is a consequence of many fruitful communications and discussions with members of what came to be called the Solar Breeze Teasers Club: Richard Ballard, Alfred Bork, Tim Kelley, Herbert Peckham, Noah Sherman and Seymour Papert. Papert brought to my attention the idea of a velocity-space analysis of the problem and the reason why the results ought to be intuitively clearer in velocity space. Sherman called my attention to the existence of closed-form solutions to the problem, using parabolic coordinates and Hamilton–Jacobi theory, which was pointed out to him by David Judd. I am especially indebted to Tim Kelley for persuading me that the solar-wind problem was too useful as a "mini-research" problem (Kelley's phrase) to allow the whole truth to be told about it in print. I hope he will not think that I have already revealed too much.

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