How To Prove It
Chapter 04 - Summary & Exercises

Math 2155

3.1

3.2

3.3

3.4

3.5 Existence and Uniqueness Proofs

1. **Theorem.** $\forall x \exists y\,(x^2 y = x - y)$

   Proof. Let x be an arbitrary real number, and suppose $y = \frac{x}{x^2+1}$
   $(Existence)\, Substitute in the new value of y into the first equation and verify.$
   $(Uniqueness)\, Suppose z satisfied the constraints. For this, z must equal y. Hence done.$
   □

2. **Theorem.** Let $a, b \in \mathbb{R}$. If $a < b < 0$, then $a^2 > b^2$.

   Proof. Suppose $a < b < 0$. Then $|a| > |b|$. Multiplying the inequality by $|a|$ gives $a^2 > ab$. Multiplying the inequality by $|b|$ gives $ab > b^2$. Therefore, $a^2 > ab > b^2$, so $a^2 > b^2$, as required. Thus, if $a < b < 0$ then $a^2 > b^2$. □

3. **Theorem.** Let $a, b \in \mathbb{R}$. If $0 < a < b$, then $\frac{1}{b} < \frac{1}{a}$.

   Proof. Suppose $0 < a < b$. Multiplying the inequality by $\frac{1}{ab}$ gives $\frac{1}{b} < \frac{1}{a}$, as required. □

4. **Theorem.** Let $a \in \mathbb{R}$. If $a^3 > a$ then $a^5 > a$.

   Proof. Suppose $a^3 > a$. Then $a^3 - a > 0$. Multiplying the inequality by $a^2 + 1$ gives

   $$\begin{aligned} &(a^3 - a)(a^2 + 1) > 0 \\ &\implies a^5 - a^3 + a^3 - a > 0 \\ &\implies a^5 - a > 0 \end{aligned}$$

   Thus we have $a^5 > a$, as required. □

5. **Theorem.** Let $A \setminus B \subseteq B \cap D$ and $x \in A$. If $x \notin D$ then $x \in B$.

Proof. From $A \setminus B \subseteq B \cap D$ we have $\forall y(y \in A \land y \notin B \to y \in C \land y \in D)$. Suppose $y = x$ and $x \notin D$. Then, $x \in C \land x \notin D$ is false which implies $x \in A \land x \notin B$ is false. Since $x \in A$, $x \in A$ is true and so $x \in B$ must be false, as required. □

6. **Theorem.** Let $a, b \in \mathbb{R}$. If $a < b$ then $\frac{a+b}{2} < b$.

   Proof. Suppose $a < b$. Adding b to the inequality gives $a + b < 2b$. Dividing the inequality by 2 gives $\frac{a+b}{2} < b$, as required. □

7. **Theorem.** Let $x \in \mathbb{R}$ and $x \neq 0$. If $\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x}$ then $x \neq 8$.

   Proof. We prove the contrapositive. Suppose $x = 8$. Then, $\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{7}{70} = \frac{1}{10} \neq \frac{1}{x} = \frac{1}{8}$. Therefore, if $\frac{\sqrt[3]{x}+5}{x^2+6} = \frac{1}{x}$ then $x \neq 8$. □

8. **Theorem.** Let $a, b, c, d \in \mathbb{R}$, $0 < a < b$, and $d > 0$. If $ac \geq bd$ then $c > d$.

   Proof. We prove the contrapositive. Suppose $c \leq d$. Then, multiplying this inequality by a gives $ac \leq ad$. Also, multiplying the inequality by b gives $bc \leq bd$. Since $a < b$, $ac < bc \leq bd$ and $ac < bd$. Therefore, if $ac \geq bd$ then $c > d$. □

9. **Theorem.** Let $a, b, c, d \in \mathbb{R}$, $0 < a < b$, and $d > 0$. If $ac \geq bd$ then $c > d$.

   Proof. We prove the contrapositive. Suppose $c \leq d$. Then, multiplying this inequality by a gives $ac \leq ad$. Also, multiplying the inequality by b gives $bc \leq bd$. Since $a < b$, $ac < bc \leq bd$ and $ac < bd$. Therefore, if $ac \geq bd$ then $c > d$. □

3.6 Proofs involving Negations and Conditionals

1. (a) *Proof.* Suppose P. Then, since $P \to Q$ it follows that Q. And, since $Q \to R$, it follows that R. Thus, $P \to R$. □

   (b) *Proof.* Suppose P and Q. From the contrapositive of $\neg R \to (P \to \neg Q)$, we have $\neg(P \to \neg Q) \to R$. Since P and Q, it follows that because $\neg(P \to \neg Q)$, we have R. Thus, $P \to (Q \to R)$. □

2. (a) *Proof.* Suppose P. Then, from $P \to Q$ we have Q and from the contrapositive $Q \to \neg R$ we have $\neg R$. Thus, $P \to \neg R$. □

   (b) *Proof.* Suppose Q. Then, since P, it follows that $\neg(Q \to \neg P)$. Thus, $Q \to \neg(Q \to \neg P)$. □

3. *Proof.* Suppose $x \in A$. Since $A \subseteq C$, we have that $x \in C$. Also, since $B \cap C = \emptyset$, $x \notin B$. Thus, $x \in A \to x \notin B$. □

6. *Proof.* Suppose $a \notin C$. Since $a \in A$ and $A \subseteq B$, it follows that $a \in B$. Then, it follows that $a \in B \setminus C$. However, this contradicts the given $a \notin B \setminus C$. Therefore, $a \in C$. □

3.7 Proofs Involving Quantifiers

1. *Proof.* Suppose $\exists x(P(x) \to Q(x))$. Then, we can choose x_0 such that $P(x_0) \to Q(x_0)$. Suppose also that $\forall x P(x) \to \exists x Q(x)$. In particular, we have $P(x_0) \to Q(x_0)$. Since x_0 is a value for x for which $Q(x_0)$ holds, $\exists x Q(x)$, as required. □

3. *Proof.* Suppose $x \in A$ and $A \subseteq B \setminus C$. Then, $x \in B$ and $x \notin C$. But, since x is arbitrary, $\forall x(x \in A \to x \notin C)$, or $A \cap C = \emptyset$, as required. □

7. *Proof.* Suppose $x > 2$. Let $y = \frac{x+\sqrt{x^2-4}}{2}$ which is defined since $x > 2$. Then,
$$\begin{aligned} y + \frac{1}{y} &= \frac{x+\sqrt{x^2-4}}{2} + \frac{2}{x+\sqrt{x^2-4}} \\ &= \frac{(x+\sqrt{x^2-4})^2+4}{2(x+\sqrt{x^2-4})} \\ &= x \end{aligned}$$
□

9. *Proof.* Suppose $x \in \cap\mathcal{F}$ and $A \in \mathcal{F}$. Since $x \in \cap\mathcal{F}$, x belongs to all the sets in $\mathcal{F}$, including A. It follows that $x \in A$. Thus, $x \in \cap\mathcal{F} \to x \in A$. □

12. *Proof.* Suppose $\mathcal{F} \subseteq \mathcal{G}$. Let $x \in \cup\mathcal{F}$ and $A \in \mathcal{G}$. Since $x \in \cup\mathcal{F}$, there exists a set $B \in \mathcal{F}$ such that $x \in B$. Also, since $\mathcal{F} \subseteq \mathcal{G}$, $B \in \mathcal{G}$. It follows that $x \in \cup\mathcal{G}$. Since x is arbitrary, $\cup\mathcal{F} \subseteq \mathcal{G}$, as required. □

14. *Proof.* Suppose $X \in \cup_{i\in I}\mathscr{P}(A_i)$. Suppose $X \in \mathscr{P}(A_j)$, where $j \in I$. Since $X \in \mathscr{P}(A_j)$, $X \subseteq A_j$. It follows that $X \subseteq \cup_{i\in I}A_i$. Thus, $X \in \mathscr{P}(\cup_{i\in I}A_i)$. Since X is arbitrary, $\cup_{i\in I}\mathscr{P}(A_i) \subseteq \mathscr{P}(\cup_{i\in I}A_i)$, as required. □

17. *Proof.* Suppose $x \in \cup\mathcal{F}$. Then, there exists $A \in \mathcal{F}$ where $x \in A$. Suppose $B \in \mathcal{G}$. Then, $A \subseteq B$ as given. It follows that $x \in B$. Since B is arbitrary, $x \in \cap\mathcal{G}$. Since x is arbitrary, $\cup\mathcal{F} \subseteq \cap\mathcal{G}$. □

20. The original goal of the proof is to prove $\forall x \in \mathbb{R}(x^2 \geq 0)$. The proof is by contradiction. However, the goal is incorrectly negated as $\forall x \in \mathbb{R}(x^2 < 0)$, when it should be $\exists x \in \mathbb{R}(x^2 < 0)$ (note the change in quantifier).

22. A correct proof must be valid for arbitrary values of y from a given value of x. However, the given proof defines x in terms of y, meaning that the choice of y is no longer arbitrary once the value of x is assigned.

25. *Proof.* Suppose $x \in \mathbb{R}$. Let $y = 2x$ and $z \in \mathbb{R}$. Then,
$$\begin{aligned} (x+z)^2 - (x^2+z^2) &= x^2 + 2xz + z^2 - x^2 - z^2 \\ &= 2xz \\ &= yz \end{aligned}$$
□