How to Prove It
Chapter 3 Summary & Exercises

Math 2155

Summary

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3.1 Proofs

1. (a) Done in OneNote here.

3.2 Proofs with Negations and Conditionals

1. (a) Given & Goal Table:

Given	Goal
$P \to Q$	R
$Q \to R$	
P	

Suppose P. $P \to Q$, Q. And since $Q \to R$, R.
$P \to R$.

(b) Given & Goal Table:

Given	Goal
$\neg R \to (P \to \neg Q)$	$\cancel{R}$
P	Q
Q	
$\neg R$	

Suppose $\neg R$ and P, then $\neg Q$. Contradicts Q.
It follows that $P \to (Q \to R)$.

2. (a) Given & Goal Table:

Given	Goal
$P \to Q$	$\neg R$
$R \to \neg Q$	
P	
$Q \to \neg R$	

Suppose P. $P \to Q$, Q. Taking the contrapositive, $Q \to \neg R$, $\neg R$.
$\therefore P \to \neg R$.

(b) Using logical equivalences, $\neg(Q \rightarrow \neg P)$ is the same as $\neg(\neg Q \vee \neg P)$ and of course $Q \wedge P$. Since P, the whole thing simplifies to $Q \rightarrow Q$.

3. Suppose $x \in A$, Since $A \subseteq C$, $x \in C$. Of course, $B \cap C = \emptyset$, and naturally $x \notin B$.

4. Suppose $x \in C$, Since $A \setminus B \cap C = \emptyset$, $x \notin A \setminus B$. Logical equivalent is, $x \notin A \vee x \in B$. But, $x \in A$, hence $x \notin B$

5. Suppose $a \in A \setminus B$. Logical equivalent is, $a \in A \wedge a \notin B$. But, $a \in A \cap C \subseteq B$. Contradicts $a \notin B$. Hence $a \notin A \setminus B$.

7. Suppose $y = 0$. Then $x = -x$. This is impossible as $x \neq 0$. $\therefore y \neq 0$.

8. Immediately derive that $a{<}b \wedge b/a{<}1$. Multiplying by a, we have $b{>}a \iff a{<}0$. Now, from $a{<}1/a, a^2{<}1$ Which is only possible if the absolute value of $a > 1$. Hence $a < -1$.

9. Suppose $x = 0$, contradiction. hence $x \neq 0$.

10. Simple algebra and some factoring gives $y(x^2 + 2 - 2) = 3x^2$. It is then clear that $y = 3$.

11. (a) The negation is applied incorrectly. $\neg(x \neq 3 \wedge x \neq 8) = (x = 3 \vee x = 8.)$.
    (b) $x = 3 \wedge y = 7$

12. (a) $x \in C \wedge x \notin B$ is possible.
    (b) let $A = \{1\}, B = \{2\}, C = \{1, 2\}, x = 1$.

3.3 Proofs involving Quantifiers

1. $[\exists x(P(x) \to Q(x))] \to [\forall x P(x) \to \exists x Q(x)]$

   Suppose $\exists x(P(x) \to Q(x))$. Then we can choose some x_0 such that $P(x_0) \to Q(x_0)$. Now suppose that $\forall x\, P(x)$. Then in particular, $P(x_0)$, and since $P(x_0) \to Q(x_0)$, it follows that $Q(x_0)$. Since we have found a particular value of x for which $Q(x)$ holds, we can conclude that $\exists x\, Q(x)$. Thus $\forall x\, P(x) \to \exists x\, Q(x)$.

2. Proof by contradiction.
   Suppose $x \in A \cap B$, and $x \notin C$. Logically, $x \in B \setminus C$. But since $x \in A \wedge x \in B \setminus C$, $x \in A \cap B \setminus C = \emptyset$. $\therefore x \notin C$ is false. Hence if $x \in A \cap B$, then $x \in C$, or in other words $A \cap B \subseteq C$.

3. Trivial proof by contradiction.
   Suppose $x \in A \wedge x \in C$. Then $A \subseteq B \setminus C$ cannot be true. $\therefore x \in A \to x \neq C$.

4. Suppose $x \in P(A)$. Then $x \subseteq A \wedge x \subseteq P(A)$.
   Hence, $x \in P(P(A)) \wedge P(A) \subseteq P(P(A))$. Not tricky but clearly powersets have more to them than has been revealed haha.

5. (a) $\emptyset$

   (b) Nope, the null set is the only one.

6. (a) Solve to $y = \frac{2x+1}{x-1}$. Sub y into $\frac{y+1}{y-2}$. Reduces to x. $\therefore \exists y(\frac{y+1}{y-2}) = x$

   (b) Proof by contradiction. Suppose $x = 1$, then clearly $1 = -2$. Absurd.

7. Identify quadratic and solve for $y = \frac{x+\sqrt{x^2-4}}{2}$.
   Note that the discriminant is positive due to $x{>}2$. Plug back in, reduce to x.

9. $\forall x \in \cap F(A \in F \to x \in A)$. As x is arbitrary, $\cap F \subseteq A$.

12. Suppose $A_0 \in F \wedge x \in A_0$. Since $F \subseteq G$, $A_0 \in G$. It's now clear that $x \in \cup G$. As x is arbitrary, $\cup F \subseteq \cup G$.

14. Suppose $x \in \cup_{i \in I} P(A_i) \wedge x \in P(A_t) \wedge t \in I$. Then $x \subseteq A_t$ and naturally, $x \in P(\cup_{i \in I} A_i)$. x is arbitrary, so conclude $\cup_{i \in I} P(A_i) \subseteq P(\cup_{i \in I} A_i 0)$.

15. Yeah couldn't do this one. Went and read a nice article tho:
    Terence Tao 1984.

16. Suppose $x \in \cup F$, then $x \in B$. x is arbitrary, and thus $\cup F \subseteq B$.

25. $\forall x \exists y (\forall z (y * z = (x+z)^2 - (x^2 + z^2)))$
    Doing some expanding, arrive at $y = 2x$. Where all vars were arbitrary real numbers.

26. (a) Goals with $\forall x P(x)$ and givens with $\exists x P(x)$ require us to introduce a new variable to move forward with the proof.
    Introduce new variable.

    (b) Goals with $\exists x P(x)$ and givens with $\forall x P(x)$ require us to find an x for which P(x) is true.
    Find specific value.

    (c) Proof by contradiction requires us to assume $\neg\forall x P(x)$ when proving $\forall x P(x)$. Logical equivalent is $\exists x \neg P(x)$ Vice versa as well.

3.4 Proofs involving Conjunctions and Biconditionals

1. Trivial.

3. Suppose $x \in C \setminus B$. Equivalently, $x \in C \land x \notin B$. It follows that, $x \notin A$. Then $x \in C \land x \notin B \to x \in C \land x \notin A$. x is arbitrary, hence conclusion reached.

6. $x \in A \land \neg(x \in (B \cap C))$
   $= x \in A \land \neg(x \in B \land x \in C)$
   $= x \in A \land (x \notin B \lor x \notin C)$
   $= (x \in A \land x \notin B) \lor (x \in A \land x \notin C)$
   $= x \in (A \setminus B) \cup (A \setminus C))$

7. ($\rightarrow$) Given $x \subseteq (A \cap B)$, suppose $y \in x$, then $y \in (A \cap B)$.

   Logically equivalent is $y \in A \land y \in B$. Of course, y is arbitrary, and hence $x \subseteq A \land x \subseteq B$.

   Conclusion of $x \in P(A) \land x \in P(B)$ reached.

   ($\leftarrow$) Follow above steps in reverse.

9. Suppose *int* $x, y, z = 2i+1, 2j+1, 2ij+i+j$.
   Then $xy = 4ij + 2i + 2j + 1 = 2(2ij + i + j) + 1 = 2z + 1$. Hence, two odd numbers multiplied together will produce an odd number.

10. Suppose *int* $n, m = 2i, i^3$. Then $n^3 = 8i^3 = 2(4m)$, even.

12. Arrive at $\frac{1}{y} = \frac{x}{x-1}$. Solve for $x + y = x + \frac{x}{x-1} = xy$. Naturally, we also conclude that $x \neq 1$.

22. (a) 1. $P \leftrightarrow Q$
       2. $P \to Q$
       3. $P \land Q$
       4. $\forall x P(x)$

    (b) Suppose $x \in B \setminus (\cup_{i \in I} A_i)$

       $= x \in B \land \neg\exists i \in I(x \in A_i)$
       $= x \in B \land \forall i \in I(x \notin A_i)$
       $= \forall i \in I(x \in B \land x \notin A_i)$
       $= x \in \cap_{i \in I}(B \setminus A_i)$

    (c) Suppose $x \in B \setminus (\cap_{i \in I} A_i)$

       $= x \in B \land \neg\forall i \in I(x \in A_i)$
       $= x \in B \land \exists i \in I(x \notin A_i)$

$= \exists i \in I(x \in B \wedge x \notin A_i)$
$= \exists i \in I(x \in B \setminus A_i)$
$= x \in \cup_{i \in I}(B \setminus A_i)$
As always, note the brilliant and beautiful parallel. Math is fucking awesome.

26. (a) $(\rightarrow)$ Clearly, $n = 15 * k$ and naturally, $n = 3 * 5k = 5 * 3k$, thus, $3|n \wedge 5|n$.
$(\leftarrow)$ 5 and 3 are both prime, given $5|k \wedge 3|l$, we derive $5p = k, 3q = l$. Naturally, $n = 3k = 5l = 15p = 15q$, thus, $p = q \wedge 15|n$.

    (b) Counter example: $n = 150$

3.5 Proofs involving Disjunctions

1. Suppose $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$. Then $A \cup B \in \mathcal{P}(A \cup B)$. Two cases arise,

    1. Suppose $A \cup B \in \mathcal{P}(A)$. Then $A \cup B \subseteq A$, it follows that $B \subseteq A$.
    2. Similar logic leads to $B \subseteq A$.

    Thus, either $A \subseteq B \vee B \subseteq A$.

12. (a)

    (b)

    (c) I don't understand the proof.

18. Considering this the end of this document. Learned a lot about LaTeX, Overleaf, and proofs. 3.6 3.7 will be part of Chapter 4, done in VSCode instead. Summaries can be found in OneNote (link at start).

    Cheers.