In this Chapter we treat the subspace identification of purely deterministic systems, with no measurement nor process noise (vk wk 0 in Figure 1.4).

1. Block Hankel matrices and state sequences

$$
U_{0|2i-1} \stackrel{\text{def}}{=} \begin{array}{c} i \\ \\ i \end{array}
\begin{array}{c} j \\
\left(\begin{array}{ccccc}
u_0 & u_1 & u_2 & \dots & u_{j-1} \\
u_1 & u_2 & u_3 & \dots & u_j \\
\dots & \dots & \dots & \dots & \dots \\
u_{i-1} & u_i & u_{i+1} & \dots & u_{i+j-2} \\
\hline
u_i & u_{i+1} & u_{i+2} & \dots & u_{i+j-1} \\
u_{i+1} & u_{i+2} & u_{i+3} & \dots & u_{i+j} \\
\dots & \dots & \dots & \dots & \dots \\
u_{2i-1} & u_{2i} & u_{2i+1} & \dots & u_{2i+j-2}
\end{array}\right)
\end{array}
\begin{array}{l} \text{"past"} \\ \\ \text{"future"} \end{array}
$$

$$
\stackrel{\text{def}}{=} \left(\frac{U_{0|i-1}}{U_{i|2i-1}}\right) \stackrel{\text{def}}{=} \left(\frac{U_p}{U_f}\right)
$$

‘i’ is user defined and should at least be larger than the maximum order

Following the notation of Willems [Wil 86], we define the block Hankel matrices consisting of inputs and outputs as W_0| i-1 :

$$
\begin{aligned}
W_{0|i-1} &\stackrel{\text{def}}{=} \left(\begin{array}{c} U_{0|i-1} \\ Y_{0|i-1} \end{array}\right) \\
&= \left(\begin{array}{c} U_p \\ Y_p \end{array}\right) \\
&= W_p \ .
\end{aligned}
$$

Deterministic state sequence :

$$
X_i^d \stackrel{\text{def}}{=} \left(\begin{array}{ccccc} x_i^d & x_{i+1}^d & \dots & x_{i+j-2}^d & x_{i+j-1}^d \end{array} \right) \quad \in \mathbb{R}^{n \times j} \ ,
$$

The extended (i > n) observability matrix ; (subscript i denotes the number of block rows)

$$\Gamma_i \stackrel{\text{def}}{=} \begin{pmatrix} C \\ CA \\ CA^2 \\ \dots \\ CA^{i-1} \end{pmatrix} \quad \in \mathbb{R}^{li \times n} \ .$$

controllability matrix :(where the subscript i denotes the number of block columns)

$$\Delta_i^d \stackrel{\text{def}}{=} \begin{pmatrix} A^{i-1}B & A^{i-2}B & \dots & AB & B \end{pmatrix} \quad \in \mathbb{R}^{n \times mi} \ .$$

We assume the pair {A,B } to be controllable. The controllable modes can be either stable or unstable. The lower block triangular Toeplitz matrix is defined as:

$$H_i^d \stackrel{\text{def}}{=} \begin{pmatrix} D & 0 & 0 & \dots & 0 \\ CB & D & 0 & \dots & 0 \\ CAB & CB & D & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ CA^{i-2}B & CA^{i-3}B & CA^{i-4}B & \dots & D \end{pmatrix} \quad \in \mathbb{R}^{li \times mi} \ .$$

2.2 GEOMETRIC PROPERTIES OF DETERMINISTIC SYSTEMS

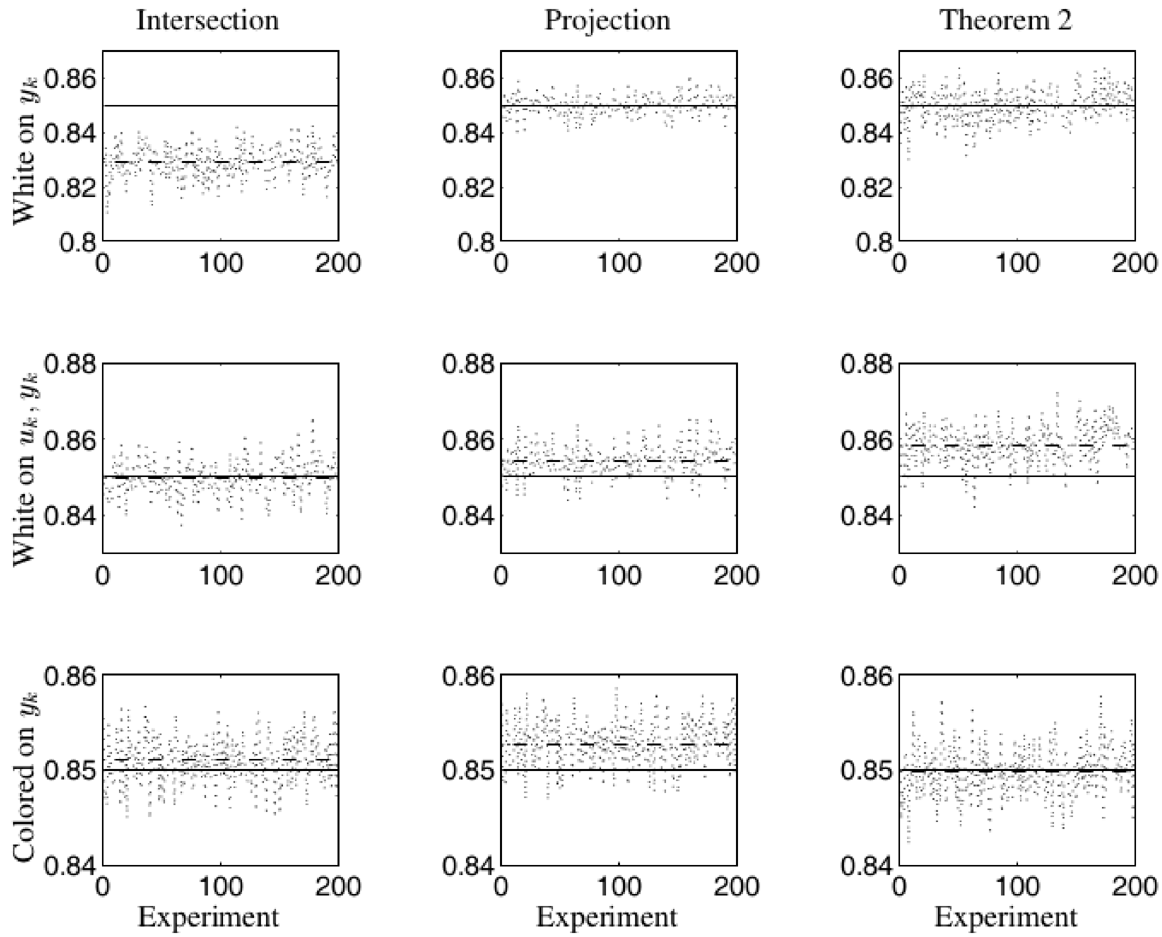
Deterministic state sequence :

$$X_f^d = \begin{pmatrix} B & AB & A^2B & \dots & A^{j-1}B \end{pmatrix}$$

Observability Matrix:

$$\mathcal{O}_i = \begin{pmatrix} C \\ CA \\ \dots \\ CA^{i-1} \end{pmatrix} \begin{pmatrix} B & AB & A^2B & \dots & A^{j-1}B \end{pmatrix}$$

Three algorithms :



200 Monte Carlo estimates of matrix A

Deterministic identification problem:

Given: s measurements of the input $u_k \in \mathbb{R}^m$ and the output $y_k \in \mathbb{R}^l$ generated by the unknown deterministic system of order n:

$$x^d_{k+1} = Ax^d_k + Bu_k \; , \quad (2.1)$$
$$y_k = Cx^d_k + Du_k \; . \quad (2.2)$$

Determine:

- The order n of the unknown system
- The system matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{l\times n}, D \in \mathbb{R}^{l\times m}$ (up to within a similarity transformation).

Theorem 1 Matrix input-output equations

$$Y_p = \Gamma_i X^d_p + H^d_i U_p \; , \quad (2.5)$$
$$Y_f = \Gamma_i X^d_f + H^d_i U_f \; , \quad (2.6)$$
$$X^d_f = A^i X^d_p + \Delta^d_i U_p \; . \quad (2.7)$$

main Theorem which
states how the state sequence and the extended observability matrix can be extracted from the given input-output data. After having treated the three Theorems for the three different cases (deterministic, stochastic and combined deterministic-stochastic identification), it will become clear that they are very similar.)

The state sequence Xfd can be determined directly from the given data u k and yk , without knowledge of the system matrices A B C D
The extended observability matrix can be determined directly from the given input-output data.

A B C D can
be extracted from these intermediate results Xfd and ;i .

In the main deterministic identification Theorem, we introduce two weighting matrices W1 and W2 .

specific choices of the matrices
lead to different identification algorithms
the choice of the
weights determines the state space basis in which the final model is obtained

Definition 5 Persistency of excitation

The input sequence $u_k \in \mathbb{R}^m$ is persistently exciting of order $2i$ if the input covariance matrix

$$R^{uu} \stackrel{def}{=} \Phi_{[U_{0|2i-1}, U_{0|2i-1}]}$$

has full rank, which is $2.m.i$.

Theorem 2 Deterministic identification
Under the assumptions that:

- The input uk is persistently exciting of order 2i (Definition 5).
- The intersection of the row space of Uf (the future inputs) and the row space of Xpd (the past states) is empty.
- The user-defined weighting matrices W1 and W2 are such that W1 is of full rank and W2 obeys: rank (Wp) = rank (Wp.W2), where Wp is the block Hankel matrix containing the past inputs and outputs.

And with $\mathcal{O}_i$ defined as the oblique projection:

$$\mathcal{O}_i \stackrel{def}{=} Y_f /_{U_f} \boldsymbol{W}_p \ , \qquad (2.8)$$

and the singular value decomposition:

$$\begin{aligned} W_1 \mathcal{O}_i W_2 &= \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1^T \\ V_2^T \end{pmatrix} & (2.9) \\ &= U_1 S_1 V_1^T \ , & (2.10) \end{aligned}$$

we have:

1. *The matrix $\mathcal{O}_i$ is equal to the product of the extended observability matrix and the states:*

$$\mathcal{O}_i = \Gamma_i . X_f^d \ . \tag{2.11}$$

2. *The order of the system (2.1)-(2.2) is equal to the number of singular values in equation (2.9) different from zero.*

3. *The extended observability matrix Γ_i is equal to*[5]*:*

$$\Gamma_i = W_1^{-1} U_1 S_1^{1/2} . T \ . \tag{2.12}$$

4. *The part of the state sequence X_f^d that lies in the column space of W_2 can be recovered from:*

$$X_f^d W_2 \quad = \quad T^{-1} . S_1^{1/2} V_1^T \ . \tag{2.13}$$

5. *The state sequence X_f^d is equal to:*

$$X_f^d = \Gamma_i^{\dagger} . \mathcal{O}_i \ . \tag{2.14}$$

Equation (2.10) can be split into two parts (where T is an arbitrary non-singular matrix representing a similarity transformation)

$$\begin{aligned} W_1 \Gamma_i &= U_1 S_1^{1/2} . T \ , \\ X_f^d W_2 &= T^{-1} . S_1^{1/2} V_1^T \end{aligned}$$

Summarized theorem 2

$$\boxed{\begin{aligned} \text{rank } (Y_f /_{U_f} \boldsymbol{W}_p) &= n \\ \text{row space } (Y_f /_{U_f} \boldsymbol{W}_p) &= \text{row space } (X_f^d) \\ \text{column space } (Y_f /_{U_f} \boldsymbol{W}_p) &= \text{column space } (\Gamma_i) \end{aligned}}$$

This summary is the essence of why these algorithms are called subspace algorithms: they retrieve system related matrices as subspaces of projected data matrices.

2.4 COMPUTING THE SYSTEM MATRICES

2.4.1 Algorithm 1 using the states

n this Section, we explain how the system matrices A B C and D can be computed from the results of Theorem 2 in two different ways.

Order n from theorem 2 svd(2.9) , observability matrix(2.12) The state sequence(2.14)

Through a similar reasoning and proof as in Theorem 2, it is easy to show that the following holds:

$$\begin{aligned} \mathcal{O}_{i-1} &\stackrel{\text{def}}{=} Y_f^- /_{U_f^-} \boldsymbol{W}_p^+ \\ &= \Gamma_{i-1}.X_{i+1}^d \ . \end{aligned}$$

It is also easy to check that if we strip the last l (number of outputs) rows of Γ_i (calculated from 2.12), we find Γ_{i-1}:

$$\Gamma_{i-1} = \underline{\Gamma_i} \ ,$$

where $\underline{\Gamma_i}$ denotes the matrix Γ_i without the last l rows. Now X_{i+1}^d can be calculated as:

$$X_{i+1}^d = \Gamma_{i-1}^\dagger \mathcal{O}_{i-1} \ .$$

At this moment, we have calculated X_i^d and X_{i+1}^d, using only input-output data. The matrices A, B, C, D can be solved from:

$$\underbrace{\begin{pmatrix} X_{i+1}^d \\ Y_{i|i} \end{pmatrix}}_{\text{known}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \underbrace{\begin{pmatrix} X_i^d \\ U_{i|i} \end{pmatrix}}_{\text{known}} , \tag{2.20}$$

Deterministic algorithm 1:

1. Calculate the oblique projections:

$$\mathcal{O}_i = Y_f /_{U_f} \mathbf{W}_p \,,$$
$$\mathcal{O}_{i-1} = Y_f^- /_{U_f^-} \mathbf{W}_p^+ \,.$$

2. Calculate the **SVD** of the weighted oblique projection:

$$W_1 \mathcal{O}_i W_2 = U S V^T \,.$$

3. Determine the order by inspecting the singular values in S and partition the **SVD** accordingly to obtain U_1 and S_1.

4. Determine Γ_i and Γ_{i-1} as:

$$\Gamma_i = W_1^{-1} U_1 S_1^{1/2} \quad , \quad \Gamma_{i-1} = \underline{\Gamma_i} \,.$$

5. Determine X_i^d and X_{i+1}^d as:

$$X_i^d = \Gamma_i^\dagger \mathcal{O}_i \quad , \quad X_{i+1}^d = \Gamma_{i-1}^\dagger \mathcal{O}_{i-1} \,.$$

6. Solve the set of linear equations for A, B, C and D:

$$\begin{pmatrix} X_{i+1}^d \\ Y_{i|i} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} X_i^d \\ U_{i|i} \end{pmatrix} \,.$$

Figure 2.7 A schematic overview of the first deterministic identification algorithm. See Section 6.1 for implementation issues. This algorithm has been implemented in the Matlab function det_stat.m.

2.4.2 Algorithm 2 using the extended observability matrix

The system matrices are determined in two separate steps: As a first step, A and C are determined from i ; In a second step B and D are computed.

Least square ;