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Semester -IV **ENGG.Mathematics-IV**

BAS-403 ENGG- Mathematics-IV

UNIT-1 : ONE SHOT

Partial Differential Equations

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- Topic Wise Entire Syllabus
- Long - Short Questions Covered
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Gateway Classes

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BAS-403 ENGG. MATHEMATICS-IV

Unit-1-ONE SHOT

Introduction to Partial Differential Equations

Syallbus

Origin of Partial Differential Equations, Linear and Non-Linear Partial Differential Equations of first order, Lagrange's Equations method to solve Linear Partial Differential Equations, Charpit's method to solve Non-Linear Partial Differential Equations, Solution of Linear Partial Differential Equation of Higher order with constant coefficients, Equations reducible to linear partial differential equations with constant coefficients.

MATHS-4

ONE SHOT

Unit-I

Partial Diff. Equations

Live 8 March 10 AM

Gulshan sir

DR. A.P.J. ABDUL KALAM TECHNICAL UNIVERSITY, UTTAR PRADESH • LUCKNOW •

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GW GATEWAY CLASSES

UNIT-1 Partial Differential Equations

GATEWAY CLASSES

Syllabus

Module-I: Partial Differential Equations

- Origin of Partial Differential Equations, Linear and Non-Linear Partial Differential Equations of first order
- Lagrange's Equations method to solve Linear Partial Differential Equations
- Charpit's method to solve Non-Linear Partial Differential Equations,
- Solution of Linear Partial Differential Equation of Higher-order with constant coefficients
- Equations reducible to linear partial differential equations with constant coefficients.

Engineering Mathematics-IV

Unit: Partial Differential Equations (PDE)

Today's Target

- Introduction to Partial Differential Equations
- Univ. Questions
- DDPs

By Gulshan sir

Differential Equation

An equation containing the dependent variable, independent variable and the derivative of the dependent variable with respect to the independent variable is known as a differential equation.

$y = f(x)$

$\frac{dy}{dx}$

$\frac{d^2y}{dx^2}$

$\vdots$

$\frac{d^ny}{dx^n}$

$z = f(x, y)$

$\frac{\partial z}{\partial x}$

$\frac{\partial z}{\partial y}$

$\frac{\partial^2 z}{\partial x^2}$

$\frac{\partial^2 z}{\partial y^2}$

$\frac{\partial^2 z}{\partial x \partial y}$

Differential Equation

Ordinary Differential Equation — Partial Differential Equation

Ordinary Differential Equation (ODE)

A differential equation containing ordinary derivatives :

(i) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

(ii) $\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = x^2$

Partial Differential Equation (PDE)

A differential equation containing partial derivatives :

(i) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x$

(ii) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = xy$

(iii) $\left(\frac{\partial z}{\partial y}\right)^2 = 2x\frac{\partial^3 z}{\partial x^3}$

(iv) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz$

Note :

$$\boxed{\begin{aligned} \frac{\partial z}{\partial x} &= p \\ \frac{\partial z}{\partial y} &= q \end{aligned}}$$

Partial Derivative of first order

$$\boxed{\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= r \\ \frac{\partial^2 z}{\partial x \partial y} &= s \\ \frac{\partial^2 z}{\partial y^2} &= t \end{aligned}}$$

Partial Derivative of second order

Order and Degree of a PDE

Order of PDE

Order of PDE is the order of highest order derivatives :

(i) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ $\quad O = 1$

(ii) $\frac{\partial^2 z}{\partial x^2} + x\frac{\partial z}{\partial y}$ $\quad O = 2$

(iii) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = xy$ $\quad O = 2$

(iv) $\left(\frac{\partial z}{\partial y}\right)^2 = 2x\frac{\partial^3 z}{\partial x^3}$ $\quad O = 3$

Degree of PDE

Degree of PDE is degree of highest order derivatives in equation, provided it is free from radical and fractional power.

(i) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = xy$ $\qquad O = 1 \qquad D = 1$

(ii) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^3 z}{\partial y^3} = 2x\,\frac{\partial z}{\partial x}$ $\qquad O = 3 \qquad D = 1$

(iii) $\frac{\partial^2 z}{\partial x^2} = \left(1 + \frac{\partial z}{\partial y}\right)^{1/3}$ $\qquad O = 2 \qquad D = 3$

(iv) $\frac{\partial^3 z}{\partial y^3} = \left(1 - \frac{\partial z}{\partial x}\right)^{1/2}$ $\qquad O = 3 \qquad D = 2$

$$\left(\frac{\partial^2 z}{\partial x^2}\right)^3 = \left(1 + \frac{\partial z}{\partial y}\right)^{\cancel{3}/\cancel{3}}$$

$$\left(\frac{\partial^2 z}{\partial x^2}\right)^3 = 1 + \frac{\partial z}{\partial y}$$

$$\left(\frac{\partial^3 z}{\partial y^3}\right)^2 = \left(1 - \frac{\partial z}{\partial x}\right)^{\cancel{2}/\cancel{2}}$$

Linear PDE

A PDE is said to be linear, if dependent variable and its partial derivatives present only in first degree and not multiplied together.

Example

(A) $\boxed{\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = xyz}$ Linear

(B) $\boxed{yp + xq = x}$ Linear

$p = \dfrac{\partial z}{\partial x}, \quad q = \dfrac{\partial z}{\partial y}$

(C) $\boxed{\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = z + xy}$ Linear

$$\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial y^2} + z\frac{\partial^2 u}{\partial z^2} = xyz$$

Non-Linear PDE

A PDE which is not linear is called non-linear PDE

Example

(1) $\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial x}\right)$ Non-Linear

(2) $z\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} = x$ Non-Linear

(3) $u\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x$ Non-Linear

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = xy + z^2$

Types of first order PDE

(i) Linear
(ii) Semi – Linear
(iii) Quasi – Linear

$p = \frac{\partial z}{\partial x}$

$q = \frac{\partial z}{\partial y}$ $\qquad z$

(1) Linear :

A first order PDE $f(x, y, z, p, q) = 0$ is known as linear if it is linear in p, q, and z and is of the form

$$\boxed{P(x,y)\,p + Q(x,y)\,q = R(x,y)\,z + s(x,y)}$$

Example

$$\boxed{xp + yq = z + xy}$$

$$\boxed{y^2x^2 p + x^3y^2 q = xyz + x^3 y^3}$$

(2) Semi – Linear:

A first order PDE $f(x, y, z, p, q) = 0$ is known as semi-linear if it is linear in p, q, and is of the form

$$\boxed{P(x,y)\,p + Q(x,y)\,q = R(x,y,z)}$$

Example

$$\boxed{x^2 y p + x^2 y^2 q = x^2 y^2 z^2}$$

(3) Quasi – Linear :

A first order PDE $f(x, y, z, p, q) = 0$ is known as Quasi-linear if it is linear in p and q and is of the form

$$\boxed{P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z)}$$

Example

$$\boxed{x^2 z^2 p + y^2 x q = x y z}$$

$$\boxed{(x^2 - yz)\,p + (y^2 - zx)\,q = z^2 - xy}$$

Engineering Mathematics

Unit: Partial Differential Equations (PDE)

Lec-2

Today's Target

- Formation of PDE [Part-1]
- Univ. Questions
- DDPs

By Gulshan sir

Note : $z = f(x, y)$

$$\boxed{\begin{aligned} \frac{\partial z}{\partial x} &= p \\ \frac{\partial z}{\partial y} &= q \end{aligned}}$$

Partial Derivative of first order

$$\boxed{\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= r \\ \frac{\partial^2 z}{\partial x \partial y} &= s \\ \frac{\partial^2 z}{\partial y^2} &= t \end{aligned}}$$

Partial Derivative of second order

Formation of PDE

By elimination of arbitrary constants

By elimination of arbitrary functions

By elimination of arbitrary constants

$$\boxed{\text{Let } F(x, y, z, a, b) = 0} \quad \text{—— ①}$$

be the given function, where a and b are arbitrary constants

Step 1 - *Find* $\boxed{\dfrac{\partial F}{\partial x} = 0}$ —— ②

Step 2 - *Find* $\boxed{\dfrac{\partial F}{\partial y} = 0}$ —— ③

Step 3 - *Eliminate a, b (arbitrary constants) from equations ①, ② and ③*

Now, we get required PDE

CASE - I

If No. of arbitrary constants < No. of independent variables

*Then we get **more than one** PDE of order one*

Q.1 Find PDE by eleminating a from $z = ax + y$.

Given

$z = ax + y$ — (1)

Partially Diff. wrt x and y

$\frac{\partial z}{\partial x} = a$ — (2)

$\frac{\partial z}{\partial y} = 1$ — (3)

From eqn (1) and (2)

$z = x\frac{\partial z}{\partial x} + y$

$$\left.\begin{array}{l}\boxed{z = px + y} \\ \boxed{q = 1}\end{array}\right\} \text{Required PDE}$$

CASE - II

If No. of arbitrary constants = No. of independent variables

Then we get only one PDE of order one

Q.2 Form partial differential equation by eliminating the arbitrary constants:

$$z = ax + a^2 + b^2 + by$$

Given

$$z = ax + a^2 + b^2 + by \quad \text{---} (1)$$

Partially diff. wrt x and y

$$\frac{\partial z}{\partial x} = a + 0 + 0 + 0$$

$$\frac{\partial z}{\partial x} = a \quad \text{---} (2)$$

$$\frac{\partial z}{\partial y} = 0 + 0 + 0 + b$$

$$\frac{\partial z}{\partial y} = b \quad \text{---} (3)$$

From eqn (1), (2) and (3)

$$z = x\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + y\frac{\partial z}{\partial y}$$

$$\boxed{z = px + p^2 + q^2 + qy}$$

Q.3 Form partial differential equation by eliminating the arbitrary constants:

$$z = (x+a)(y+b)$$

Given

$z = (x+a)(y+b)$ — (1)

Partially diff wrt x and y

$\frac{\partial z}{\partial x} = (y+b)(1+0)$

$\frac{\partial z}{\partial x} = (y+b)$ — (2)

$\frac{\partial z}{\partial y} = (x+a)(1+0)$

$\frac{\partial z}{\partial y} = (x+a)$ — (3)

From (1), (2) and (3)

$z = \frac{\partial z}{\partial y} \times \frac{\partial z}{\partial x}$

$$\boxed{z = pq}$$

Q.4 Form the partial differential equation by eliminating a and b from $z=(x^2+a)(y^2+b)$. [2012]

Given

$z=(x^2+a)(y^2+b)$ — (1)

Partially diff. wrt x and y

$\frac{\partial z}{\partial x}=(y^2+b)(2x+0)$

$\frac{\partial z}{\partial x}=2x(y^2+b)$

$(y^2+b)=\frac{1}{2x}\times\frac{\partial z}{\partial x}$ — (2)

$\frac{\partial z}{\partial y}=(x^2+a)(2y+0)$

$\frac{\partial z}{\partial y}=2y(x^2+a)$

$(x^2+a)=\frac{1}{2y}\times\frac{\partial z}{\partial y}$ — (3)

From (1), (2) and (3)

$z=\frac{1}{2y}\times\frac{\partial z}{\partial y}\times\frac{1}{2x}\times\frac{\partial z}{\partial x}$

$4xyz=\frac{\partial z}{\partial y}\times\frac{\partial z}{\partial x}$

$\boxed{4xy=pq}$

Q.5 Find the partial differential equation of all spheres whose centers lie on z-axis and given by equations $x^2 + y^2 + (z - a)^2 = b^2$; a and b being constants. [2017]

Given

$$x^2+y^2+(z-a)^2 = b^2 \quad \text{—} (1)$$

Partially diff. wrt x and y

$$2x + 0 + 2(z-a)\frac{\partial z}{\partial x} = 0$$

$$2x + 2(z-a)\frac{\partial z}{\partial x} = 0$$

$$\cancel{2}(z-a)\frac{\partial z}{\partial x} = -\cancel{2}x$$

$$(z-a)p = -x$$

$$(z-a) = -\frac{x}{p} \quad \text{—} (2)$$

$$0 + 2y + 2(z-a)\frac{\partial z}{\partial y} = 0$$

$$\cancel{2}(z-a)q = -\cancel{2}y$$

$$z-a = -\frac{y}{q} \quad \text{—} (3)$$

From equation (2) and (3)

$$\cancel{-}\frac{x}{p} = \cancel{-}\frac{y}{q}$$

$$\boxed{qx - py = 0}$$

CASE - III

If No. of arbitrary constants > No. of independent variables

more than one

Then we get ***higher order*** *PDE*

Q.6 Find PDE by eleminating a, b, c : $z = ax + by + cxy$

$z = ax + by + cxy$ — (1)

Partially diff. w.r.t x

$\frac{\partial z}{\partial x} = a + 0 + cy$

$\frac{\partial z}{\partial x} = a + cy$ — (2)

Again partially diff. w.r.t x

$\boxed{\frac{\partial^2 z}{\partial x^2} = 0}$ — (3)

partially diff. (1) w.r.t y

$\frac{\partial z}{\partial y} = 0 + b + cx$ — (4)

Again partially diff w.r.t y

$\boxed{\frac{\partial^2 z}{\partial y^2} = 0}$ — (5)

Multiply eqn (2) by x and (4) by y

$x\frac{\partial z}{\partial x} = ax + cxy$ — (6)

$y\frac{\partial z}{\partial y} = by + cxy$ — (7)

Adding (6) and (7)

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = ax + cxy + by + cxy$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + cxy \quad - (8)$$

From (2)

$$\frac{\partial z}{\partial x} = a + cy$$

Partially diff wrt y

$$\frac{\partial^2 z}{\partial x \partial y} = c$$

Put c in (8)

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + \frac{\partial^2 z}{\partial x \partial y} \times xy$$

$$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z + xy\frac{\partial^2 z}{\partial x \partial y}}$$

Engineering Mathematics

Unit: Partial Differential Equations (PDE)

Lec-3

Today's Target

- Formation of PDE [Part-2]
- Univ. Questions
- DDPs

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Formation of PDE

By elimination of arbitrary constants

By elimination of arbitrary functions

Note :

$$\frac{\partial z}{\partial x} = p \qquad \frac{\partial z}{\partial y} = q$$

$$\frac{\partial^2 z}{\partial x^2} = r \qquad \frac{\partial^2 z}{\partial x \partial y} = s \qquad \frac{\partial^2 z}{\partial y^2} = t$$

By elimination of arbitrary function

CASE - 1

When given function is of the form of

$$\boxed{z = f(x, y)}$$

When single function is given

NOTE : $\boxed{\text{Order of PDE} = 1}$

Q.1 Form the partial differential equation by eliminating the arbitrary function from
$Z = f(x^2 - y^2)$ (Sem. 2015)

$z = f(x^2-y^2)$ —— ①

Partially Diff. w.r.t x

$$\frac{\partial z}{\partial x} = f'(x^2-y^2) \times \frac{\partial}{\partial x}(x^2-y^2)$$

$$\frac{\partial z}{\partial x} = f'(x^2-y^2) \times (2x-0)$$

$$p = 2x\, f'(x^2-y^2) \quad \text{—— ②}$$

Partially Diff. w.r.t y

$$\frac{\partial z}{\partial y} = f'(x^2-y^2) \times \frac{\partial}{\partial y}(x^2-y^2)$$

$$\frac{\partial z}{\partial y} = f'(x^2-y^2) \times (0-2y)$$

$$q = -2y\, f'(x^2-y^2) \quad \text{—— ③}$$

Divide ② ÷ ③

$$\frac{p}{q} = \frac{\cancel{2}x\, \cancel{f'(x^2-y^2)}}{-\cancel{2}y\, \cancel{f'(x^2-y^2)}}$$

$$\frac{p}{q} = \frac{x}{-y}$$

$$qx = -py$$

$$\boxed{qx + py = 0}$$

CASE - 2

When given function is of the form of

$$\boxed{z = f_1(x, y) + f_2(x, y)}$$

Note: Order of PDE > 1

Q.2 ***Form the partial differential equation by eliminating the arbitrary function from the following : $z = f(x + it) + g(x - it)$.*** **(Sem. 2011)**

$$z = f(x+it) + g(x-it) \quad \text{— (1)}$$

Partially diff. wrt x

$$\frac{\partial z}{\partial x} = f'(x+it)\times 1 + g'(x-it)\times 1$$

Again partially diff. wrt x

$$\frac{\partial^2 z}{\partial x^2} = f''(x+it) + g''(x-it) \quad \text{— (2)}$$

Partially Diff. wrt t

$$\frac{\partial z}{\partial t} = f'(x+it)\times i + g'(x-it)\times(-i)$$

Again partially diff wrt t

$$\frac{\partial^2 z}{\partial t^2} = f''(x+it)\times i^2 + g''(x-it)\times i^2$$

$$\frac{\partial^2 z}{\partial t^2} = i^2\left[f''(x+it) + g''(x-it)\right]$$

$$\frac{\partial^2 z}{\partial t^2} = -\left[f''(x+it) + g''(x-it)\right] \quad \text{— (3)}$$

Adding equation 2 and 3

$$\boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0}$$

order $= 2$

Q.3 Form the partial differential equation by eliminating the arbitrary function from the following : $Z = x + y + f(xy)$.

$$z = x + y + f(xy) \quad \text{—— } (1)$$

Partially diff. w.r.t. x

$$\frac{\partial z}{\partial x} = 1 + 0 + f'(xy) \times y$$

$$p - 1 = f'(xy) \times y \quad \text{— } (2)$$

Partially diff. w.r.t. y

$$\frac{\partial z}{\partial y} = 0 + 1 + f'(xy) \times x$$

$$q - 1 = f'(xy) \times x \quad \text{—— } (3)$$

Divide eqn (2) by (3)

$$\frac{p-1}{q-1} = \frac{\cancel{f'(xy)} \times y}{\cancel{f'(xy)} \times x}$$

$$\frac{p-1}{q-1} = \frac{y}{x}$$

$$px - x = qy - y$$

$$\boxed{px - qy = x - y}$$

order $= 1$

Q.4 Form the partial differential equation by eliminating the arbitrary function from the following : $z = \varphi(x) . \Psi(y)$

$$z = \phi(x) \times \psi(y) \quad \text{— (1)}$$

Partially diff wrt x

$$\frac{\partial z}{\partial x} = \phi'(x) \times \psi(y) \quad \text{— (2)}$$

Partially diff wrt y

$$\frac{\partial z}{\partial y} = \phi(x) \times \psi'(y) \quad \text{— (3)}$$

Partially diff eqn (3) wrt x

$$\frac{\partial^2 z}{\partial x \partial y} = \phi'(x) \times \psi'(y) \quad \text{— (4)}$$

Multiply eqn (2) and (3)

$$\frac{\partial z}{\partial x} \times \frac{\partial z}{\partial y} = \phi'(x) \times \psi(y) \times \phi(x) \times \psi'(y)$$

$$p \times q = \phi'(x) \times \psi'(y) \times \phi(x) \times \psi(y)$$

$$p \times q = \frac{\partial^2 z}{\partial x \partial y} \times z$$

$$\boxed{p \times q = sz}$$

order $= 2$

CASE - 3

When given function is of the form of $\boxed{\varphi(u, v) = 0}$

$p = \frac{\partial z}{\partial x}$ $q = \frac{\partial z}{\partial y}$

Where u and v are the function of x, y, z

Steps - (1) Given $\varphi(u, v) = 0$ Where $u = u(x, y, z)$ $v = v(x, y, z)$

(2) Then PDE can be formed by $\boxed{Pp + Qq = R}$ —— (1)

$$P = \frac{\partial(u, v)}{\partial(y, z)} = \begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix}$$

$$Q = \frac{\partial(u, v)}{\partial(z, x)} = \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix}$$

$$R = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

Put P, Q, R in (1) to get required PDE

Q. 5 Formulate the PDE by eliminating the arbitrary function from $\boxed{\varphi(x^2+y^2,\ y^2+z^2)=0.}$

$$\phi(x^2+y^2,\ y^2+z^2)=0$$

Where $u=x^2+y^2$

$v=y^2+z^2$

Then required PDE is

$$\boxed{Pp+Qq=R} \quad \text{— (1)}$$

$$P=\frac{\partial(u,v)}{\partial(y,z)}=\begin{vmatrix}\frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\end{vmatrix}$$

$$P=\begin{vmatrix}2y & 0\\ 2y & 2z\end{vmatrix}=4yz$$

$$Q=\frac{\partial(u,v)}{\partial(z,x)}=\begin{vmatrix}\frac{\partial u}{\partial z} & \frac{\partial u}{\partial x}\\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x}\end{vmatrix}$$

$$Q=\begin{vmatrix}0 & 2x\\ 2z & 0\end{vmatrix}=-4xz$$

$$R=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}$$

$$R=\begin{vmatrix}2x & 2y\\ 0 & 2y\end{vmatrix}=4xy$$

Put P, Q and R in eqn (1)

$$4yzp-4xzq=4xy$$

$$\boxed{yzp-xzq=xy}$$

order $=1$

Engineering Mathematics

Unit: Partial Differential Equations (PDE)

Lec-4

Today's Target

- Lagrange's Method [Part-1]
- Univ. Questions
- Practice Questions

By Gulshan sir

Lagrange's Method

First order Quasi-Linear PDE in standard form is called Lagrange's Equation

$$\boxed{Pp + Qq = R}$$

Where P, Q, R are function of x, y, z and $\boxed{p = \frac{\partial z}{\partial x}\ ,\ q = \frac{\partial z}{\partial y}}$

Step to Solve questions

Step-1 Write down the PDE in standard form

Step-2 Write down the Auxiliary Equation

$$\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}} \quad \text{--- (1)}$$

(I) (II) (III)

Step-3 Solve above auxiliary Equation by

(i) Method of Grouping

(ii) Method of Multipliers

Now we get two independent solutions in the form of

$$u(x, y, z) = c_1 \quad \text{--- } (2)$$

$$v(x, y, z) = c_2 \quad \text{--- } (3)$$

Step-4 General Solution

$\boxed{\phi(u, v) = 0}$ OR $\boxed{u = \phi(v)}$ OR $\boxed{v = \phi(u)}$

Method of Grouping

Type-1

Q.1 Solve : $p + q = z$

$p + q = z$

Here $P = 1$

$Q = 1$

$R = z$

Aux. Eq^n

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\underset{\text{I}}{\frac{dx}{1}} = \underset{\text{II}}{\frac{dy}{1}} = \underset{\text{III}}{\frac{dz}{z}} \quad \text{--- (1)}$$

(I, II)

$$\frac{dx}{1} = \frac{dy}{1}$$

Integrate both side

$$\int dx = \int dy$$

$$x = y + c_1$$

$$x - y = c_1 \quad \text{--- (2)}$$

(I, III)

$$\frac{dx}{1} = \frac{dz}{z}$$

Integrate both side

$$\int dx = \int \frac{dz}{z}$$

$$x = \log z + c_2$$

$$x - \log z = c_2 \quad \text{--- (3)}$$

General solution

$$\boxed{\phi(x - y,\ x - \log z) = 0}$$

Q.2 Solve : $px + qy = z$

$$Px + qy = z$$

Here

$P = x, \quad Q = y, \quad R = z$

Aux. Eq^n

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

I $\qquad$ II $\qquad$ III

(I, II)

$$\frac{dx}{x} = \frac{dy}{y}$$

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log c_1$$

$$\log x = \log(y \times c_1)$$

$$x = y \times c_1$$

$$\frac{x}{y} = c_1 \quad \text{--- (1)}$$

(II, III)

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\log y = \log z + \log c_2$$

$$\log y = \log(z \times c_2)$$

$$y = z \times c_2$$

$$\frac{y}{z} = c_2 \quad \text{--- (2)}$$

General solution

$$\boxed{\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0}$$

Q.3 Solve the following partial differential equations : $yzp - xzq = xy$

$$yzp - xzq = xy$$

Here

$$P = yz, \quad Q = -xz, \quad R = xy$$

Aux. Eq^n

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$$

$$\mathrm{I} \qquad \mathrm{II} \qquad \mathrm{III}$$

$(\mathrm{I}, \mathrm{II})$

$$\frac{dx}{y\cancel{z}} = \frac{dy}{-x\cancel{z}}$$

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$\frac{y^2}{2} + \frac{x^2}{2} = C$$

$$x^2 + y^2 = 2C$$

$$x^2 + y^2 = c_1 \quad \text{--- (1)}$$

$(\mathrm{II}, \mathrm{III})$

$$\frac{dy}{-\cancel{x}z} = \frac{dz}{\cancel{x}y}$$

$$\frac{dy}{-z} = \frac{dz}{y}$$

$$y\,dy = -z\,dz$$

$$\int y\,dy = -\int z\,dz$$

$$\frac{y^2}{2} = -\frac{z^2}{2} + c$$

$$y^2 + z^2 = c_2 \quad \text{--- (2)}$$

General solution

$$\boxed{\phi(x^2+y^2,\ y^2+z^2) = 0}$$

$(\mathrm{I}, \mathrm{III})$

$$\frac{dx}{\cancel{y}z} = \frac{dz}{x\cancel{y}}$$

$$x\,dx = z\,dz$$

$$\frac{x^2}{2} = \frac{z^2}{2} + c_2$$

$$x^2 - z^2 = c_2$$

Q.4 Solve the following partial differential equations : $p\tan x + q\tan y = \tan z$

$p\tan x + q\tan y = \tan z$

Here

$P = \tan x,\ Q = \tan y,\ R = \tan z$

Aux. Eq^n

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z}$$
$$\text{I} \qquad \text{II} \qquad \text{III}$$

(I, II)

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

$\cot x\, dx = \cot y\, dy$

$\int \cot x\, dx = \int \cot y\, dy$

$\log \sin x = \log \sin y + \log c_1$

$\log \sin x = \log \sin y \times c_1$

$\sin x = \sin y \times c_1$

$$\frac{\sin x}{\sin y} = c_1 \quad \text{—} \ (1)$$

(II, III)

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

$\cot y\, dy = \cot z\, dz$

$\int \cot y\, dy = \int \cot z\, dz$

$\log \sin y = \log \sin z + \log c_2$

$\log \sin y = \log \sin z \times c_2$

$$\frac{\sin y}{\sin z} = c_2 \quad \text{—} \ (2)$$

$$\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$$

Q.5 Solve the following differential equations : $\frac{y^2 z}{x} p + xzq = y^2$

$$\frac{y^2 z}{x} p + xzq = y^2$$

$$\frac{y^2 z p + x^2 z q}{x} = y^2$$

$$y^2 z p + x^2 z q = x y^2$$

Here

$P = y^2 z \ , Q = x^2 z \ , R = x y^2$

Aux. Eq^n

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{x y^2}$$

(I, II)

$$\frac{dx}{y^2 \cancel{z}} = \frac{dy}{x^2 \cancel{z}}$$

$$x^2 dx = y^2 dy$$

$$\int x^2 dx = \int y^2 dy$$

$$\frac{x^3}{3} = \frac{y^3}{3} + c$$

$$\frac{x^3 - y^3}{3} = c$$

$$x^3 - y^3 = c_1 \quad \text{— (1)}$$

(I, III)

$$\frac{dx}{\cancel{y^2} z} = \frac{dz}{x \cancel{y^2}}$$

$$x dx = z dz$$

$$\int x dx = \int z dz$$

$$\frac{x^2}{2} = \frac{z^2}{2} + c$$

$$\frac{x^2 - z^2}{2} = c$$

$$x^2 - z^2 = c_1 \quad \text{— (2)}$$

General solution

$$\boxed{\phi(x^3 - y^3, x^2 - z^2) = 0}$$

Q.6 Solve the following partial differential equations : $y^2 p - xyq = x(z-2y)$ (Sem. 2014)

$$y^2 p - xyq = x(z-2y)$$

Here

$$P = y^2, \quad Q = -xy, \quad R = x(z-2y)$$

Aux. Eq^n

$$\underset{\text{I}}{\frac{dx}{y^2}} = \underset{\text{II}}{\frac{dy}{-xy}} = \underset{\text{III}}{\frac{dz}{x(z-2y)}}$$

(I, II)

$$\frac{dx}{y^{\cancel{2}}} = \frac{dy}{-x\cancel{y}}$$

$$y\,dy = -x\,dx$$

$$\int y\,dy = -\int x\,dx$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c$$

$$\frac{x^2+y^2}{2} = c$$

$$x^2+y^2 = c_1 \quad \text{—} (1)$$

II, III

$$\frac{dy}{-\cancel{x}y} = \frac{dz}{+\cancel{x}(z-2y)}$$

$$\frac{dy}{-y} = \frac{dz}{z-2y}$$

$$(z-2y)\,dy = -y\,dz$$

$$z\,dy - 2y\,dy = -y\,dz$$

$$z\,dy + y\,dz = 2y\,dy$$

$$d(yz) = 2y\,dy$$

$$\int d(yz) = \int 2y\,dy$$

$$yz = \cancel{2}\frac{y^2}{\cancel{2}} + c_2$$

$$yz = y^2 + c_2$$

$$\boxed{yz - y^2 = c_2}$$

$$\boxed{y^2 - yz = c_2}$$

General solution

$$\phi\left(x^2+y^2,\ y^2-yz\right) = 0$$

Type-2

Q.7 Solve : $p + 3q = 5z + \tan(y-3x)$ **(Sem. 2017)**

$P + 3q = 5z + \tan(y-3x)$

Here

$P = 1$

$Q = 3$

$R = 5z + \tan(y-3x)$

Aux. Eq^n

$$\underset{\mathrm{I}}{\frac{dx}{1}} = \underset{\mathrm{II}}{\frac{dy}{3}} = \underset{\mathrm{III}}{\frac{dz}{5z + \tan(y-3x)}}$$

$(\mathrm{I}, \mathrm{II})$

$$\frac{dx}{1} = \frac{dy}{3}$$

$$\int dx = \frac{1}{3}\int dy$$

$$x = \frac{y}{3} + c$$

$$3x = y + 3c$$

$$3x - y = 3c$$

$$-1(y-3x) = 3c$$

$$y - 3x = \frac{3c}{-1}$$

$$y - 3x = c_1 \quad -(1)$$

$(\mathrm{I}, \mathrm{III})$

$$\frac{dx}{1} = \frac{dz}{5z + \tan(y-3x)}$$

Put $y - 3x = c_1$

$$\frac{dx}{1} = \frac{dz}{5z + \tan c_1}$$

$$\int dx = \int \frac{dz}{5z + \tan c_1}$$

Put $5z + \tan c_1 = t$

$$5\,dz = dt$$

$$dz = \frac{dt}{5}$$

$$\int dx = \frac{1}{5}\int \frac{dt}{t}$$

$$x = \frac{1}{5}\log t + c$$

$$x = \frac{\log t + 5c}{5}$$

$$5x = \log(5z + \tan c_1) + 5c$$

$$5x - \log(5z + \tan(y-3x)) = c_2$$

$$5x - \log[5z + \tan(y-3x)) = c_2$$

General solution

$$\boxed{\phi\left[y - 3x,\ 5x - \log\{5z + \tan(y-3x)\}\right] = 0}$$

Q.8 Solve the following differential equations : $pz - qz = z^2 + (x+y)^2$

$pz - qz = z^2 + (x+y)^2$

Here

$P = z$, $Q = -z$

$R = z^2 + (x+y)^2$

Aux. Eq^n

$$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$$

I II III

(I, II)

$$\frac{dx}{\cancel{z}} = \frac{dy}{-\cancel{z}}$$

$dy = -dx$

$\int dy = -\int dx$

$y = -x + c_1$

$x + y = c_1$

(I, III)

$$\frac{dx}{z} = \frac{dz}{z^2+(x+y)^2}$$

Put $x + y = c_1$

$$\frac{dx}{z} = \frac{dz}{z^2+c_1^2}$$

$$\int dx = \int \frac{z\,dz}{z^2+c_1^2}$$

Put $z^2 + c_1^2 = t$

$2z\,dz = dt$

$z\,dz = \frac{dt}{2}$

$$\int dx = \frac{1}{2}\int \frac{dt}{t}$$

$$x = \frac{1}{2}\log t + c$$

$$2x = \log(z^2 + c_1^2) + 2c$$

$$2x - \log(z^2 + c_1^2) = c_2$$

$$2x - \log\left(z^2 + (x+y)^2\right) = c_2$$

$$2x - \log\left(z^2 + x^2 + y^2 + 2xy\right) = c_2$$

$$\log\left(z^2 + x^2 + y^2 + 2xy\right) - 2x = c_2$$

<u>General solution</u>

$$\phi\left[x+y,\ 2x - \log(z^2 + x^2 + y^2 + 2xy)\right]$$

OR

$$\phi\left[x+y,\ \log(z^2 + x^2 + y^2 + 2xy) - 2x\right] = 0$$

Q.9 Solve the following differential equations : $zxp + yzq = xy$

Engineering Mathematics

Unit: Partial Differential Equations (PDE)

Lec-5

Today's Target

- Lagrange's Method [Part-2]
- Univ. Questions
- Practice Questions

By Gulshan sir

Lagrange's Method

First order Quasi-Linear PDE in standard form is called Lagrange's Equation

$$\boxed{Pp + Qq = R}$$

Where P, Q, R are function of x, y, z and $\boxed{p = \frac{\partial z}{\partial x}, \ q = \frac{\partial z}{\partial y}}$

Step to Solve questions

Step-1 Write down the PDE in standard form

Step-2 Write down the Auxiliary Equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad \text{—— } (1)$$

(I) (II) (III)

Step-3 Solve above auxiliary by

(i) Method of Grouping (I, II) , (II, III) , (I, III)

$\boxed{\text{(ii) Method of Multipliers}}$

Let two independent solutions are

$$u(x, y, z) = c_1 \quad \text{—— } 2$$

$$v(x, y, z) = c_2 \quad \text{— } 3$$

Step-4 General Solution

$\phi(u, v) = 0$ OR $u = \phi(v)$ OR $v = \phi(u)$

Method of Multiplier

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Let Multipliers (P_1, Q_1, R_1)

Each fraction = $\dfrac{P_1 dx + Q_1 dy + R_1 dz}{PP_1 + QQ_1 + RR_1}$

Type - 1

Select Multipliers in such a way that

$$\boxed{PP_1 + QQ_1 + RR_1 = 0}$$

Hence each fraction = $\dfrac{P_1 dx + Q_1 dy + R_1 dz}{0}$

$$\boxed{P_1 dx + Q_1 dy + R_1 dz = 0}$$

Integrate above equation to get solution

Type - 2

Select Multiplier in such a way that

Each fraction = $\dfrac{P_1 dx + Q_1 dy + R_1 dz}{PP_1 + QQ_1 + RR_1}$

Numerator = Derivative of Denominator

Type - 1

Q.1 Solve the partial differential equation $(y^2+z^2)p - xyq = -zx$

$(y^2+z^2)p - xyq = -zx$

Here

$P = y^2+z^2$

$Q = -xy$

$R = -zx$

Aux. Eqn

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$\underset{\mathrm{I}}{\frac{dx}{y^2+z^2}} = \underset{\mathrm{II}}{\frac{dy}{-xy}} = \underset{\mathrm{III}}{\frac{dz}{-zx}}$ — (1)

(II, III)

$\frac{dy}{+\cancel{x}y} = \frac{dz}{+z\cancel{x}}$

$\frac{dy}{y} = \frac{dz}{z}$

$\int \frac{dy}{y} = \int \frac{dz}{z}$

$\log y = \log z + \log c_1$

$\log y - \log z = \log c_1$

$\log \frac{y}{z} = \log c_1$

$\boxed{\frac{y}{z} = c_1}$ — (2)

Taking x, y, z as Multiplier

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dy}{\cancel{xy^2} + \cancel{xz^2} - \cancel{xy^2} - \cancel{z^2x}}$$

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dy}{0}$$

$$x\,dx + y\,dy + z\,dz = 0$$

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c$$

$$x^2 + y^2 + z^2 = 2c$$

$$x^2 + y^2 + z^2 = c_2 \quad \text{—— } 3$$

General solution

$$\boxed{\phi\left(\frac{y}{z},\, x^2 + y^2 + z^2\right) = 0}$$

Q.2 Solve : $(y+zx)p-(x+yz)q=x^2-y^2$

(Sem. 2022)

$$(y+zx)p-(x+yz)q=x^2-y^2$$

Aux. Eq^n

$$\frac{dx}{y+zx}=\frac{dy}{-x-yz}=\frac{dz}{x^2-y^2} \quad \text{— (1)}$$

Taking $(x, y, -z)$ as a Multipliers

$$\text{each fraction}=\frac{x\,dx+y\,dy-z\,dz}{\cancel{xy}+\cancel{x^2z}-\cancel{xy}-\cancel{y^2z}-\cancel{zx^2}+\cancel{zy^2}}$$

$$\text{each fraction}=\frac{x\,dx+y\,dy-z\,dz}{0}$$

$$x\,dx+y\,dy-z\,dz$$

$$\int x\,dx+\int y\,dy-\int z\,dz=0$$

$$\frac{x^2}{2}+\frac{y^2}{2}-\frac{z^2}{2}=c$$

$$\boxed{x^2+y^2-z^2=c_1} \quad \text{— (2)}$$

Taking $(y, x, 1)$ as a Multiplier

$$\text{each fraction} = \frac{y\,dx + x\,dy + dz}{\cancel{y^2} + \cancel{xyz} - \cancel{x^2} - \cancel{xyz} + \cancel{x^2} - \cancel{y^2}}$$

$$\text{each fraction} = \frac{y\,dx + x\,dy + dz}{0}$$

$$y\,dx + x\,dy + dz = 0$$

$$d(xy) + dz = 0$$

$$\int d(xy) + \int dz = 0$$

$$xy + z = c_2$$

General solution

$$\boxed{\phi(x^2 + y^2 - z^2,\ xy + z) = 0}$$

Q.3 Solve the following differential equations $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$.

(Sem. 2013)

$$x^2(y-z)p + y^2(z-x)q = z^2(x-y)$$

Aux. Eq^n

$$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \quad \text{--- (1)}$$

Taking $\left(\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}\right)$ as a multiplier

$$\text{each fraction} = \frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{\cancel{y}-\cancel{z}+\cancel{z}-\cancel{x}+\cancel{x}-\cancel{y}}$$

$$\text{each fraction} = \frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{0}$$

$$\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

$$\int \frac{dx}{x^2} + \int \frac{dy}{y^2} + \int \frac{dz}{z^2} = 0$$

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = c$$

$$\boxed{\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = c_1} \quad \text{--- (2)}$$

Taking $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$ as a Multiplier

$$\text{each fraction} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{\cancel{xy}-\cancel{xz}+\cancel{yz}-\cancel{yx}+\cancel{zx}-\cancel{zy}}$$

$$\text{each fraction} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

$$\int\frac{dx}{x}+\int\frac{dy}{y}+\frac{dz}{z}$$

$$\log x+\log y+\log z=\log c_2$$

$$\log(xyz)=\log c_2$$

$$xyz=c_2 \quad \text{—} \ 2$$

General solution

$$\boxed{\phi\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z},\ xyz\right)=0}$$

Type - 2

Q.4 Solve the following partial differential equations : $x^2 p + y^2 q = (x+y)z$ **(Sem. 2015)**

$$x^2 p + y^2 q = (x+y)z$$

Aux. Eq^n

$$\underset{\mathrm{I}}{\frac{dx}{x^2}} = \underset{\mathrm{II}}{\frac{dy}{y^2}} = \underset{\mathrm{III}}{\frac{dz}{(x+y)z}} \quad \text{--- (1)}$$

(I, II)

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

$$-\frac{1}{x} = -\frac{1}{y} + c$$

$$\frac{1}{y} - \frac{1}{x} = c$$

$$\boxed{\frac{1}{x} - \frac{1}{y} = c_1} \quad \text{--- (2)}$$

Taking $(1, -1, 0)$

$$\text{each fraction} = \frac{dx - dy}{x^2 - y^2}$$

$$\text{each fraction} = \frac{dx - dy}{(x-y)(x+y)}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} = \frac{dx-dy}{(x-y)(x+y)}$$

I II III IV

(III, IV)

$$\frac{dz}{\cancel{(x+y)}z} = \frac{dx-dy}{(x-y)\cancel{(x+y)}}$$

$$\frac{dz}{z} = \frac{dx-dy}{x-y}$$

$$\int\frac{dz}{z} = \int\frac{dx-dy}{x-y}$$

$$\log z = \log(x-y) + \log c_2$$

$$\log z = \log(x-y) \times c_2$$

$$z = (x-y)\, c_2$$

$$\boxed{\frac{z}{x-y} = c_2} \quad \text{--- } (3)$$

General solution

$$\boxed{\phi\left(\frac{1}{x}-\frac{1}{y}, \frac{z}{x-y}\right) = 0}$$

Q.5 Solve the following differential equations : $(y^2 + z^2 - x^2)p - 2xyq + 2xz = 0$

$$(y^2+z^2-x^2)p - 2xyq + 2xz = 0$$

$$(y^2+z^2-x^2)p - 2xyq = -2xz$$

Aux. Eq^n

$$\underset{\mathrm{I}}{\frac{dx}{y^2+z^2-x^2}} = \underset{\mathrm{II}}{\frac{dy}{-2xy}} = \underset{\mathrm{III}}{\frac{dz}{-2xz}} \quad \text{— (1)}$$

(II, III)

$$\frac{dy}{\cancel{+2x}y} = \frac{dz}{\cancel{+2x}z}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\int\frac{dy}{y} = \int\frac{dz}{z}$$

$$\log y = \log z + \log c_1$$

$$\log y = \log z \times c_1$$

$$y = z \times c_1$$

$$\boxed{\frac{y}{z} = c_1} \quad \text{— (2)}$$

Taking (x, y, z) as a multiplier

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dz}{xy^2 + xz^2 - x^3 - 2xy^2 - 2xz^2}$$

$$\text{each fraction} = \frac{x\,dx + y\,dy + z\,dz}{-xy^2 - xz^2 - x^3}$$

$$\underset{\mathrm{I}}{\frac{dx}{y^2+z^2-x^2}} = \underset{\mathrm{II}}{\frac{dy}{-2xy}} = \underset{\mathrm{III}}{\frac{dz}{-2xz}} = \underset{\mathrm{IV}}{\frac{x\,dx + y\,dy + z\,dz}{-x^3 - xy^2 - xz^2}}$$

III, IV

$$\frac{dz}{\cancel{-2x}\,z} = \frac{x\,dx + y\,dy + z\,dz}{\cancel{-x}(x^2+y^2+z^2)}$$

$$\frac{dz}{z} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$$

$$\int \frac{dz}{z} = \int \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$$

$$\log z = \log(x^2+y^2+z^2) + \log c$$

$$\log z = \log(x^2+y^2+z^2) \times c$$

$$z = (x^2+y^2+z^2) \times c$$

$$c = \frac{z}{x^2+y^2+z^2}$$

$$\frac{1}{c} = \frac{x^2+y^2+z^2}{z}$$

$$c_2 = \frac{x^2+y^2+z^2}{z}$$

General solution

$$\boxed{\phi\left(\frac{y}{z}, \frac{x^2+y^2+z^2}{z}\right) = 0}$$

Q. 6 Solve : $(y+z)p+(z+x)q=x+y$

$$(y+z)p+(z+x)q=x+y$$

Aux. Eqⁿ

$$\frac{dx}{y+z}=\frac{dy}{z+x}=\frac{dz}{x+y} \quad \text{—} (1)$$

Taking $(1,-1,0)$ as a multiplier

$$\text{each fraction}=\frac{dx-dy}{y+\cancel{z}-\cancel{z}-x}$$

$$=\frac{dx-dy}{-(x-y)}$$

Taking $(0,1,-1)$

$$\text{each fraction}=\frac{dy-dz}{z+\cancel{x}-\cancel{x}-y}$$

$$=\frac{dy-dz}{-(y-z)}$$

Taking $(1,1,1)$ as a multiplier

$$\text{each fraction}=\frac{dx+dy+dz}{y+z+z+x+x+y}$$

$$=\frac{dx+dy+dz}{2(x+y+z)}$$

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} = \frac{dx-dy}{-(x-y)} = \frac{dy-dz}{-(y-z)} = \frac{dx+dx+dz}{2(x+y+z)}$$

I II III IV V VI

(IV, V)

$$\frac{dx-dy}{+(x-y)} = \frac{dy-dz}{+(y-z)}$$

$$\int \frac{dx-dy}{x-y} = \int \frac{dy-dz}{y-z}$$

$$\log(x-y) = \log(y-z) + \log c_1$$

$$\log(x-y) = \log(y-z) \times c_1$$

$$x-y = (y-z)\, c_1$$

$$\boxed{\frac{x-y}{y-z} = c_1} \quad —(2)$$

(IV, VI)

$$\frac{dx-dy}{-(x-y)} = \frac{dx+dy+dz}{2(x+y+z)}$$

$$\frac{dn-dy}{(n-y)} + \frac{dn+dy+dz}{2(n+y+z)} = 0$$

$$\int \frac{dn-dy}{n-y} + \frac{1}{2}\int \frac{dn+dy+dz}{n+y+z} = 0$$

$$\log(n-y) + \frac{1}{2}\log(n+y+z) = \log c$$

$$2\log(n-y) + \log(n+y+z) = 2\log c$$

$$\log(n-y)^2 + \log(n+y+z) = \log c_2$$

$$\log(n-y)^2 \times (n+y+z) = \log c_2$$

$$\boxed{(n-y)^2(n+y+z) = c_2} \quad - (3)$$

General solution

$$\boxed{\phi\left[\left(\frac{n-y}{y-z}, (n-y)^2(n+y+z)\right)\right] = 0}$$

Practice Q.1 Solve the following differential equation : $9mz - ny)p + (nx - lz)q = ly - mx$

Practice Q.2 Solve the following differential equations: $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

UNIT : Partial Differential Equation

Today's Target

- Non Linear Partial Differential Equation (Some Standard Forms)
- PYQs
- DPP

Non – Linear Partial Differential Equations

Some Standard Forms

Type – I

Equation containing p and q only

$$f(p,\ q) = 0$$

Steps to solve

Step-I $f(p,\ q) = 0$ —— 1

Step-II Replace p by a and q by b

$$f(a,\ b) = 0 \quad \text{—— } 2$$

Step-III Find b from (2) in terms of a

Step-IV Complete solution is given by

$$\boxed{z = ax + by + c} \quad \text{—— } 3$$

Step – V Put b in 3

Q.1 Solve : $p+q=1$

$p+q=1$ —— ①

Replace p by a and q by b

$a+b=1$ —— ②

$b=1-a$

complete solution is

$z=ax+by+c$ — ③

$$\boxed{z=ax+(1-a)y+c}$$

Q.2 Solve : $\sqrt{p}+\sqrt{q}=1$

$\sqrt{p}+\sqrt{q}=1$ —— ①

Replace p by a and q by b

$\sqrt{a}+\sqrt{b}=1$ —— ②

$\sqrt{b}=1-\sqrt{a}$

$b=(1-\sqrt{a})^2$

complete solution is

$z=ax+by+c$

$$\boxed{z=ax+(1-\sqrt{a})^2y+c}$$

Q.3 Solve the partial differential equation : $p^2 + p = q^2$

$$p^2 + p = q^2 \quad \text{——} \ (1)$$

Replace p by a and q by b

$$a^2 + a = b^2 \quad \text{——} \ (2)$$

$$b^2 = a^2 + a$$

$$b = \sqrt{a^2 + a}$$

Complete solution is

$$z = ax + by + c \quad \text{—} \ (3)$$

Put b in (3)

$$\boxed{z = ax + \left(\sqrt{a^2 + a}\right) y + c}$$

Type – II

Clairaut Form $\quad z = px + qy + f(p, q)$

Steps to solve

Step-I $\quad z = px + qy + f(p, q) \quad \text{——} \quad (1)$

Step-II Replace p by a and q by b in (1)

Step-III Complete solution is given by

$$\boxed{z = ax + by + f(a, b)}$$

Q.4 Solve : $z = px + qy + \sqrt{1 + p^2 + q^2}$.

$$z = px + qy + \sqrt{1+p^2+q^2} \quad \text{—— (1)}$$

Replace p by a and q by b

$$\boxed{z = ax + by + \sqrt{1+a^2+b^2}}$$

Q.5 Solve : $z = px + qy + \frac{p}{p+q}$.

$$z = px + qy + \frac{p}{p+q} \quad \text{—— (1)}$$

Replace p by a and q by b

$$\boxed{z = ax + by + \frac{a}{a+b}}$$

Type – III

Equations not containing x and y

$$f(z,\ p,\ q) = 0$$

Steps to solve

Step-I Let $u = x + ay$ so that $p = \frac{dz}{du}$ $q = a\frac{dz}{du}$

Step-II Put the values of p and q in the given equation

Step-III Solve the ordinary differential equation in z and u

Step-IV Replace u by $x + ay$ in the solution of ODE

$u = x + ay$

$\frac{\partial u}{\partial x} = 1$ $\frac{\partial u}{\partial y} = a$

We know that

$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \times \frac{\partial u}{\partial x} = \frac{\partial z}{\partial u} = \frac{dz}{du}$

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \times \frac{\partial u}{\partial y} = a\frac{\partial z}{\partial u} = \frac{a\,dz}{du}$

Q.6 Solve the partial differential equation : p + q = z

$$p+q=z \quad \text{—} \ (1)$$

Let $u = x + ay$ so that

$$p = \frac{dz}{du} \quad \text{and} \quad q = a\frac{dz}{du}$$

Put p and q in (1)

$$\frac{dz}{du} + a\frac{dz}{du} = z$$

$$(1+a)\frac{dz}{du} = z$$

$$(1+a)\frac{dz}{z} = du$$

Integrate both side

$$(1+a)\int\frac{dz}{z} = \int du$$

$$(1+a)\log z = u + b$$

Put $u = x + ay$

$$\boxed{(1+a)\log z = x + ay + b}$$

Q.7 Solve : $p(1+q^2) = q(z-a)$

$$p(1+q^2) = q(z-a) \quad \text{--- (1)}$$

Let $u = x + by$ so that

$$p = \frac{dz}{du} \quad \text{and} \quad q = b\frac{dz}{du}$$

Put p and q in (1)

$$\frac{\cancel{dz}}{du}\left[1+\left(b\frac{dz}{du}\right)^2\right] = b\frac{\cancel{dz}}{\cancel{du}}(z-a)$$

$$1+b^2\left(\frac{dz}{du}\right)^2 = bz-ab$$

$$b^2\left(\frac{dz}{du}\right)^2 = bz-ab-1$$

$$b\frac{dz}{du} = \sqrt{bz-ab-1}$$

$$\frac{dz}{du} = \frac{1}{b}\sqrt{bz-ab-1}$$

$$\frac{b\,dz}{\sqrt{bz-ab-1}} = du$$

Integrate both side

$$\int\frac{b\,dz}{\sqrt{bz-ab-1}} = \int du$$

$$b\int(bz-ab-1)^{-\frac{1}{2}} = u+c$$

$$\frac{\cancel{b}(bz-ab-1)^{\frac{1}{2}}}{\frac{1}{2}\times\cancel{b}} = u+c$$

$$2(bz-ab-1)^{\frac{1}{2}} = u+c$$

$$4(bz-ab-1) = (u+c)^2$$

$$\boxed{4(bz-ab-1) = (x+by+c)^2}$$

Type – IV

Equations of the form

$$f_1(x,\ p) = f_2(y,\ q)$$

In these equations, z is absent and the terms containing x and p can be written on one side and terms containing y and q can be written on other side.

Steps to solve

Step-I $f_1(x,\ p) = f_2(y,\ q) = a$ —— 1

Step-II $f_1(x,\ p) = a$ Solve it for p ~~Let $p = f_1(x)$~~

Step-III $f_2(y,\ q) = a$ Solve it for q ~~Let $q = f_2(y)$~~

Step-IV We know that $\boxed{dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy = p\, dx + q\, dy}$

Step-V Put p and q in (2) and integrate it $z = \int \underset{p}{\cancel{F_1(x)}}dx + \int \underset{q}{\cancel{F_2(y)}}dy + C$

Q.8 **Solve :** $p^2 - q^2 = x - y$

$$p^2 - q^2 = x - y$$

$$p^2 - x = q^2 - y$$

Let $p^2 - x = q^2 - y = a$ —— (1)

$p^2 - x = a$	$q^2 - y = a$
$p^2 = x + a$	$q^2 = y + a$
$p = \sqrt{x+a}$	$q = \sqrt{y+a}$

We know that

$$dz = p\,dx + q\,dy$$

Put p and q in above equation

$$dz = \sqrt{x+a}\,dx + \sqrt{y+a}\,dy$$

Integrate both side

$$z = \int (x+a)^{1/2} dx + \int (y+a)^{1/2} dy$$

$$\boxed{z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + c}$$

Engineering Mathematics by Gulshan Sir

PDE : DPP-6

Topic : Non Linear Partial Differential Equation(Some Standard Forms)

Q .1 *Solve :* $p + q = pq$

Q .2 *Solve the partial differential equation :* $p^2 + q^2 = 2$

Q .3 *Solve the partial differential equation* $z = px + qy + \sin(p + q)$

Q .4 *Solve :* $9(p^2z + q^2) = 4$

Q .5 *Solve :* $p - 3x^2 = q^2 - y$

UNIT : Partial Differential Equation

Today's Target

- Solution of $\boxed{\text{Non Linear Partial Differential Equation}}$ by **CHARPIT METHOD**
- PYQs
- DPP

CHARPIT METHOD

It is a general method to solve non-linear first order partial differential equation

STEPS TO SOLVE

Step-I Write the given PDE in the form given below

$$f(x, y, z, p, q) = 0 \quad \text{---------- (1)}$$

Step-II Partially differentiate equation (1) w.r.t. x, y, z, p and q to find,

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial p}, \frac{\partial f}{\partial q}$$

Step-III Write Charpit Auxiliary equation

$$\boxed{\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}}$$

Step-IV *Select two proper fraction so that resulting integral may be simplest relation involving at least one of p and q*

Step-V *With the help of relation in step 4 and equation (1) find p and q in terms of x, y, z*

Step-VI *Put the value of p and q in* $\boxed{dz = p\,dx + q\,dy}$

Step-VII *Integrate above equation to get required Answer.*

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

p q

Q.1 *By Charpit's method, find the complete solution of PDE :* $px + qy = pq$

$$px + qy = pq$$

$$px + qy - pq = 0 \quad \text{—— (1)}$$

Let $f(x, y, z, p, q) = px + qy - pq = 0$

$$\frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial y} = q, \quad \frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial p} = x - q, \quad \frac{\partial f}{\partial q} = y - p$$

Charpit Aux. Eq^n

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

$$\frac{dx}{-(x-q)} = \frac{dy}{-(y-p)} = \frac{dz}{-p(x-q)-q(y-p)} = \underbrace{\frac{dp}{p+0} = \frac{dq}{q+0}}$$ — (2)

$$\frac{dp}{p} = \frac{dq}{q}$$

Integrate both side

$$\int \frac{dp}{p} = \int \frac{dq}{q}$$

$$\log p = \log q + \log a$$

$$\log p = \log aq$$

$$p = aq \quad \text{— (3)}$$

Put p in (1)

$$px + qy - pq = 0$$

$$aqx + qy - aq^2 = 0$$

$$(ax+y)\cancel{q} = aq^{\cancel{2}}$$

$$aq = ax+y$$

$$\boxed{q = \frac{ax+y}{a}}$$

Put q in (3)

$$p = \cancel{a} \times \frac{(ax+y)}{\cancel{a}}$$

$$\boxed{p = ax+y}$$

We know that

$$dz = p\,dx + q\,dy$$

$$dz = (ax+y)\,dx + \left(\frac{ax+y}{a}\right)dy$$

$$\int dz = \int (ax+y)\,dx + \left(\frac{ax+y}{a}\right)dy$$

$$z = \int (ax+y)\left(dx + \frac{dy}{a}\right)$$

$$z = \int (ax+y)\frac{(a\,dx+dy)}{a}$$

$$z = \frac{1}{a}\int (ax+y)(a\,dx+dy)$$

Put $ax+y = t$

$a\,dx + dy = dt$

$$z = \frac{1}{a}\int t\,dt = \frac{1}{a}\frac{t^2}{2} + b$$

$$\boxed{az = \frac{(ax+y)^2}{2} + b}$$

Q.2 Solve the following equation by Charpit's method : $pxy + pq + qy = yz$

$pxy + pq + qy = yz$

$pxy + pq + qy - yz = 0$ —— (1)

Let $f(x, y, z, p, q) = pxy + pq + qy - yz = 0$

$\frac{\partial f}{\partial x} = py$, $\frac{\partial f}{\partial y} = px + q - z$, $\frac{\partial f}{\partial z} = -y$, $\frac{\partial f}{\partial p} = xy + q$, $\frac{\partial f}{\partial q} = p + y$

Charpit Aux. Eq^n

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

$$\frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)} = \frac{dz}{-p(xy+q)-q(p+y)} = \underbrace{\frac{dp}{\cancel{py}-\cancel{py}} = \frac{dq}{px+q-z-qy}}$$

$$\frac{dp}{0} = \frac{dq}{px+q-z-qy}$$

$$dp = 0$$

$$\int dp = 0$$

$$\boxed{p = a} \quad —(3)$$

Put p in (1)

$$pxy + pq + qy - yz = 0$$

$$axy + \underline{aq + qy} - yz = 0$$

$$(a+y)q = yz - axy$$

$$\boxed{q = \frac{y(z-ax)}{a+y}}$$

We know that

$$dz = p\,dx + q\,dy$$

$$dz = a\,dx + \frac{y(z-ax)}{a+y}dy$$

$$dz - a\,dx = \frac{y(z-ax)}{a+y}dy$$

$$\int \frac{dz - a\,dx}{z-ax} = \int \frac{y}{a+y}dy$$

$$\int \frac{dz - a\,dx}{z-ax} = \int \frac{a+y-a}{a+y}dy$$

$$\int \frac{dz - a\,dx}{z-ax} = \int \left(1 - \frac{a}{a+y}\right)dy$$

$$\boxed{\log(z-ax) = y - a\log(a+y) + b}$$

Q.3 *Find the complete integral of PDE :* $(p^2+q^2)y = qz$

$(p^2+q^2)y = qz$

$(p^2+q^2)y - qz = 0 \quad \text{—— } (1)$

Let $f(x,y,z,p,q) = (p^2+q^2)y - qz = 0$

$\frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = p^2+q^2, \quad \frac{\partial f}{\partial z} = -q, \quad \frac{\partial f}{\partial p} = 2py, \quad \frac{\partial f}{\partial q} = 2qy - z$

Charpit Aux. Eq^n

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

$$\frac{dx}{-2py} = \frac{dy}{-(2qy-z)} = \frac{dz}{-2p^2y-q(2qy-z)} = \underbrace{\frac{dp}{-pq} = \frac{dq}{p^2+\cancel{q^2}-\cancel{q^2}}}$$

$$\frac{dp}{-\cancel{p}q} = \frac{dq}{p^{\cancel{2}}}$$

$$\frac{dp}{-q} = \frac{dq}{p}$$

$$p\,dp = -q\,dq$$

$$p\,dp + q\,dq = 0$$

$$\int p\,dp + \int q\,dq = 0$$

$$\frac{p^2}{2} + \frac{q^2}{2} = \frac{a^2}{2}$$

$$p^2 + q^2 = a^2 \quad \text{--- (3)}$$

Put eqn (3) in (1)

$$(p^2+q^2)y - qz = 0$$

$$a^2y - qz = 0$$

$$a^2y - qz = 0$$

$$\boxed{q = \frac{a^2y}{z}}$$

Put q in (3)

$$p^2 + \left(\frac{a^2y}{z}\right)^2 = a^2$$

$$p^2 = a^2 - \frac{a^4y^2}{z^2}$$

$$p^2 = \frac{a^2z^2 - a^4y^2}{z^2}$$

$$p^2 = \frac{a^2}{z^2}(z^2 - a^2y^2)$$

$$\boxed{p = \frac{a}{z}\sqrt{z^2 - a^2y^2}}$$

We know that

$$dz = p\,dx + q\,dy$$

$$dz = \frac{a}{z}\sqrt{z^2 - a^2y^2}\,dx + \frac{a^2y}{z}\,dy$$

$$z\,dz = a\left(\sqrt{z^2 - a^2y^2}\right)dx + a^2y\,dy$$

$$z\,dz - a^2y\,dy = a\left(\sqrt{z^2 - a^2y^2}\right)dx$$

$$\int \frac{z\,dz - a^2y\,dy}{\sqrt{z^2 - a^2y^2}} = \int a\,dx$$

$$z^2 - a^2y^2 = t$$

$$2z\,dz - 2a^2y\,dy = dt$$

$$z\,dz - a^2y\,dy = \frac{dt}{2}$$

$$\frac{1}{2}\int \frac{dt}{\sqrt{t}} = ax$$

$$\frac{1}{2}\int t^{-1/2}\,dt = ax$$

$$\frac{t^{1/2}}{\cancel{2}\cdot\frac{1}{\cancel{2}}} = ax + b$$

$$t^{1/2} = ax + b$$

$$(z^2 - a^2y^2)^{1/2} = ax + b$$

$$\boxed{(z^2 - a^2y^2) = (ax + b)^2}$$

PDE : DPP-7

Topic : CHARPIT METHOD

Q .1 *Solve :* $2zx - px^2 - 2pxy + pq = 0$

Ans $z = ay + b(x^2 - a)$

Q .2 *Solve the following equation by charpit's method :* $z^2 = pqxy$

Ans $z = ax^b y^{1/b}$

Q .3 *Solve the following equation by charpit's method :* $z = p^2x + q^2x$

Ans $\sqrt{(1+a)z} = \sqrt{ax} + \sqrt{y} + b$

UNIT : Partial Differential Equation

Today's Target

- Homogeneous Linear Partial Differential Equation with constant coefficient (Complementary function-CF)

- PYQs
- DPP

Unit-1

L-1
1. PDE
2. Order and degree of PDE
3. Types of PDE L2
4. Formation of PDE L3

Solution of PDE

1. Lagrange Method L4, L5
2. Some standard form of non linear PDE
3. Charpit Method
4. Linear PDE with constant coefficient

(a) Homogeneous Linear PDE with constant coefficient

(b) Non-Homogeneous Linear PDE with constant coefficient

8. Equation reducible to Linear PDE with constant coefficient

Linear Partial Differential Equation with constant coefficient

Linear PDE with constant coefficient

Homogeneous Linear PDE with constant coefficient

(All partial derivatives are of same order)

Complete solution = CF + PI

Non-Homogeneous Linear PDE with constant coefficient

(All partial derivatives are not of same order)

Complete solution = CF + PI

Homogeneous Linear Partial Differential Equation with constant coefficient

<u>*General equation*</u>

$$a_0\frac{\partial^n z}{\partial x^n}+a_1\frac{\partial^n z}{\partial x^{n-1}\partial y}+\cdots\cdots\cdots\cdots+a_n\frac{\partial^n z}{\partial y^n}=F(x,y)$$

Put $\frac{\partial}{\partial x}=D$ *and* $\frac{\partial}{\partial y}=D'$

$$(a_0D^n+a_1D^{n-1}D'+\cdots\cdots\cdots\cdots a_nD'^n)z=F(x,y)$$

$$f(D,D')z=F(x,y)$$

CF | *PI*

Complementary Function

Number of arbitrary function = order of PDF

Particular integral

Does not contain any arbitrary function or constant

Steps to find CF

STEP-1: Convert the given PDE in the form

$$f(D,\ D')z = F(x,\ y) \quad \text{---} \quad 1$$

Using $\frac{\partial}{\partial x} = D,\quad \frac{\partial^2}{\partial x^2} = D^2,\quad \frac{\partial}{\partial y} = D',\quad \frac{\partial^2}{\partial x \partial y} = DD'\quad \frac{\partial^2}{\partial y^2} = D'^2$ etc.

STEP-2: Put $D = m$ and $D' = 1$ in 1

$$f(m,\ 1)z = F(x,\ y) \quad \text{---} \quad 2$$

STEP-3: Form Aux. Equation

$$f(m,\ 1) = 0$$

STEP-4: Solve to find roots of Aux. Equation

CASE-1: When roots are different

S. No.	Roots of A E	C. F.
1.	m_1, m_2	$f_1(y+m_1x)+f_2(y+m_2x)$
2.	m_1, m_2, m_3	$f_1(y+m_1x)+f_2(y+m_2x)+f_3(y+m_3x)$
3.	$m_1, m_2, m_3\ \ m_4$	$f_1(y+m_1x)+f_2(y+m_2x)+f_3(y+m_3x)+f_4\,(y+m_4x)$

Note: Where $f_1,\ f_2,\ f_3$ .... are arbitrary function

Q.1 Solve: $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$$

$$(D^2 - D'^2)z = 0$$

Put $D = m$, $D' = 1$

$$(m^2 - 1)z = 0$$

Aux. Eq^n

$$m^2 - 1 = 0$$

$$m = 1, -1$$

$$CF = f_1(y + m_1 x) + f_2(y + m_2 x)$$

$$CF = f_1(y + x) + f_2(y - x)$$

$$PI = 0$$

complete solution

$$z = CF + PI$$

$$\boxed{z = f_1(y + x) + f_2(y - x)}$$

Q.2 :- Solve the following partial differential equation : $\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial x \partial y} - 5\frac{\partial^2 z}{\partial y^2} = 0$

$$\frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial x \partial y} - 5\frac{\partial^2 z}{\partial y^2} = 0$$

$$(D^2 + 4DD' - 5D'^2)z = 0$$

Put $D = m,\ D' = 1$

$$(m^2 + 4m - 5)z = 0$$

Aux. Eq^n

$$m^2 + 4m - 5 = 0$$

$m = 1$

$m = -5$

$$CF = f_1(y + m_1 x) + f_2(y + m_2 x)$$

$$CF = f_1(y + x) + f_2(y - 5x)$$

$$PI = 0$$

complete solution

$$z = CF + PI$$

$$\boxed{z = f_1(y + x) + f_2(y - 5x)}$$

Q.3 :- Solve the following partial differential equation $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$

$(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$

Put $D = m$ and $D' = 1$

$(m^3 - 6m^2 + 11m - 6)z = 0$

Aux. Eq^n

$m^3 - 6m^2 + 11m - 6 = 0$

$m = 3, 1, 2$

$CF = f_1(y + m_1x) + f_2(y + m_2x) + f_3(y + m_3x)$

$CF = f_1(y + 3x) + f_2(y + x) + f_3(y + 2x)$

$PI = 0$

Complete solution

$z = CF + PI$

$$\boxed{z = f_1(y + 3x) + f_2(y + x) + f_3(y + 2x)}$$

Q.4 :- Solve : $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0.$

$$\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$$

$$(D^4 - D'^4)z = 0$$

Put $D = m,\ D' = 1$

$$(m^4 - 1)z = 0$$

Aux. Eq^n

$$m^4 - 1 = 0$$

$$(m^2-1)(m^2+1) = 0$$

$$m = \pm 1$$

$$m = \pm i$$

$$CF = f_1(y+m_1x) + f_2(y+m_2x) + f_3(y+m_3x) + f_4(y+m_4x)$$

$$CF = f_1(y+x) + f_2(y-x) + f_3(y+ix) + f_4(y-ix)$$

$$PI = 0$$

complete solution

$$\boxed{z = CF + PI}$$

Q.5 :- Solve the linear partial differential equation $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 0$

$$\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 0$$

$$(D^4 + D'^4)z = 0$$

Put $D = m$, $D' = 1$

$$(m^4+1)z = 0$$

Aux. Eq^n

$$m^4+1=0$$

$$m^4+1=0$$

$$m^4+2m^2+1 = 2m^2$$

$$(m^2)^2+2m^2+1 = 2m^2$$

$$(m^2+1)^2 = 2m^2$$

$$(m^2+1)^2 - 2m^2 = 0$$

$$(m^2+1)^2 - (\sqrt{2}\,m)^2 = 0$$

$$(m^2+\sqrt{2}m+1)(m^2-\sqrt{2}m+1) = 0$$

$$m^2+\sqrt{2}m+1=0$$

$$m = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2}$$

$$m = -\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}$$

$$m = -\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$$

$$m = z_1, \bar{z}_1$$

$$m^2-\sqrt{2}m+1=0$$

$$m = \frac{\sqrt{2} \pm \sqrt{2-4}}{2}$$

$$m = \frac{\sqrt{2}}{2} \pm \frac{i\sqrt{2}}{2}$$

$$m = \frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$$

$$m = z_2, \bar{z}_2$$

$$CF = f_1(y+m_1x) + f_2(y+m_2x) + f_3(y+m_3x) + f_4(y+m_4x)$$

$$CF = f_1(y+z_1x) + f_2(y+\bar{z}_1x) + f_3(y+z_2x) + f_4(y+\bar{z}_2x)$$

$$PI = 0$$

complete solution

$$\boxed{Z = CF + PI}$$

CASE-2: When roots are repeated

S. No.	Roots of A E	C. F.
1.	m, m	$f_1(y+mx) + xf_2(y+mx)$
2.	m, m, m	$f_1(y+mx) + xf_2(y+mx) + x^2 f_3(y+mx)$

Note: Where f_1, f_2, f_3 ……… are arbitrary function

Q.6 :- Solve $25r - 40s + 16t = 0$

$$25r - 40s + 16t = 0$$

$$25\frac{\partial^2 z}{\partial x^2} - 40\frac{\partial^2 z}{\partial x \partial y} + 16\frac{\partial^2 z}{\partial y^2} = 0$$

$$(25D^2 - 40DD' + 16D'^2)z = 0$$

Put $D = m$, $D' = 1$

$$(25m^2 - 40m + 16)z = 0$$

Aux Eq^n

$$25m^2 - 40m + 16 = 0$$

$$(5m - 4)^2 = 0$$

$$(5m-4)(5m-4) = 0$$

$$m = \frac{4}{5}, \frac{4}{5}$$

$$CF = f_1(y + mx) + x f_2(y + mx)$$

$$CF = f_1\left(y + \frac{4}{5}x\right) + x f_2\left(y + \frac{4}{5}x\right)$$

complete solution

$$\boxed{z = CF + PI}$$

Q.7 :- Solve: $(D^3 - 6D^2D' + 12DD'^2 - 8D'^3)z = 0$

$(D^3 - 6D^2D' + 12DD'^2 - 8D'^3)z = 0$

Put $D = m$, $D' = 1$

$(m^3 - 6m^2 + 12m - 8)z = 0$

Aux. Eq^n

$m^3 - 6m^2 + 12m - 8 = 0$

$m = 2, 2, 2$

$CF = f_1(y+mx) + x f_2(y+mx) + x^2 f_3(y+mx)$

$CF = f_1(y+2x) + x f_2(y+2x) + x^2 f_3(y+2x)$

$PI = 0$

complete solution

$$\boxed{z = CF + PI}$$

Q.8 :- Solve: $(D + 2D')(D - 3D')^2 z = 0$

$(D + 2D')(D - 3D')^2 z = 0$

Put $D = m,\ D' = 1$

$(m+2)(m-3)^2 z = 0$

Aux. Eq^n

$(m+2)(m-3)^2 = 0$

$m = -2$

$m = 3, 3$

$CF = f_1(y + m_1 x) + \underbrace{f_2(y + mx) + x f_3(y + mx)}$

$CF = f_1(y - 2x) + f_2(y + 3x) + x f_3(y + 3x)$

$PI = 0$

complete solution

$$\boxed{z = CF + PI}$$

CASE-3: When D , D' , both D and D' are common factor

(i) C F corresponding to $D' z = f_1(x)$

(ii) C F corresponding to $D'^2 z = f_1(x) + y f_2(x)$

(iii) C F corresponding to $D'^3 z = f_1(x) + y f_2(x) + y^2 f_3(x)$

(ii) C F corresponding to $Dz = f_1(y)$

(iii) C F corresponding to $D^2 z = f_1(y) + x f_2(y)$

(iv) C F corresponding to $D^3 z = f_1(y) + x f_2(y) + x^2 f_3(y)$

Q.9 :- Solve: $\dfrac{\partial^3 z}{\partial y^3} - \dfrac{\partial^3 z}{\partial x^2 \partial y} = 0$

$$\frac{\partial^3 z}{\partial y^3} - \frac{\partial^3 z}{\partial x^2 \partial y} = 0$$

$$(D'^3 - D^2 D')z = 0$$

$$D'(D'^2 - D^2)z = 0$$

Aux. Eq^n

$$D'(D'^2 - D^2) = 0$$

$$D' = 0$$

$$CF = f_1(x)$$

$$D'^2 - D^2 = 0$$

Put $D = m$

$$D' = 1$$

$$1 - m^2 = 0$$

$$m = \pm 1$$

$$CF = f_2(y + m_1 x) + f_3(y + m_2 x)$$

$$CF = f_2(y + x) + f_3(y - x)$$

Final

$$CF = f_1(x) + f_2(y + x) + f_3(y - x)$$

$$PI = 0$$

Complete solution

$$z = CF + PI$$

$$\boxed{z = f_1(x) + f_2(y + x) + f_3(y - x)}$$

Q.10 :- Solve: $(D^3D'^2 + D^2D'^3)z = 0$

$(D^3D'^2 + D^2D'^3)z = 0$

$D^2D'^2(D + D')z = 0$

Aux Eq^n

$D^2 = 0$

$CF = f_1(y) + x f_2(y)$

$D'^2 = 0$

$CF = f_3(x) + y f_4(x)$

$D + D' = 0$

Put $D = m$

$D' = 1$

$m + 1 = 0$

$m = -1$

$CF = f_5(y + mx)$

$CF = f_5(y - x)$

$CF = f_1(y) + x f_2(y) + f_3(x) + y f_4(x) + f_5(y - x)$

complete solution

$$\boxed{z = CF + PI}$$

PDE : DPP-8

Topic : Homogeneous Linear Partial Differential Equation with constant coefficient (Complementary function)

Q.1 Solve : $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$

Ans $z = f_1(y+3x) + f_2(y-2x)$

Q.2 Solve $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 2\frac{\partial^3 z}{\partial x \partial y^2} = 0$

Ans $z = f_1(y) + f_2(y+x) + f_3(y+2x)$

Q.3 Solve: $4r - 12s + 9t = 0$

Ans $z = f_1\left(y+\frac{3}{2}x\right) + x f_2\left(y+\frac{3}{2}x\right)$

Q.4 Solve $(D^3D' - 4D^2D'^2 + 4DD'^3)z = 0$

Ans

$z = f_1(y) + f_2(x) + f_3(y+2x) + x f_4(y+2x)$

Engineering Mathematics

UNIT : Partial Differential Equation

Today's Target

- $\boxed{\text{Homogeneous Linear Partial Differential Equation with constant coefficient}}$

  Particular Integral(PI) : Case-I (TYPE-1)

- PYQs
- DPP

CASE-1: When roots are different

S. No.	Roots of A E	C. F.
1.	m_1, m_2	$f_1(y+m_1x)+f_2(y+m_2x)$
2.	m_1, m_2, m_3	$\underbrace{f_1(y+m_1x)+f_2(y+m_2x)}+f_3(y+m_3x)$
3.	$m_1, m_2, m_3\ \ m_4$	$\underbrace{f_1(y+m_1x)+f_2(y+m_2x)+f_3(y+m_3x)}+f_4(y+m_4x)$

Note: Where $f_1,\ f_2,\ f_3$ ……… are arbitrary function

CASE-2: When roots are repeated

S. No.	Roots of A E	C. F.
1.	m, m	$f_1(y+mx) + xf_2(y+mx)$
2.	m, m, m	$f_1(y+mx) + xf_2(y+mx) + x^2 f_3(y+mx)$

Note: Where f_1, f_2, f_3 .... are arbitrary function

CASE-3: When D, D', both D and D' are common factor

(i) C F corresponding to $D' z = f_1(x)$

(ii) C F corresponding to $D'^2 z = f_1(x) + y f_2(x)$

(iii) C F corresponding to $D'^3 z = f_1(x) + y f_2(x) + y^2 f_3(x)$

(ii) C F corresponding to $Dz = f_1(y)$

(iii) C F corresponding to $D^2 z = f_1(y) + x f_2(y)$

(iv) C F corresponding to $D^3 z = f_1(y) + x f_2(y) + x^2 f_3(y)$

Particular Integral

Case – 1

$$F(D, D') = \boxed{\phi(ax + by)}$$

$$\text{PI} = \frac{1}{F(D, D')}\phi(ax + by)$$

Where $F(D, D')$ is a Homogeneous function of degree n

Replace

$$D = a\,, D' = b$$

TYPE-1 if $F(a, b) \neq 0$

put $ax + by = u$,

$$\boxed{\text{PI} = \frac{1}{F(a,b)}\iiint \phi(u)du\,du \;\ldots\ldots n \text{ times}}$$

TYPE-2

$If\ F(a,b) = 0$ (Next Lecture)

Type - 1	Type - 2
(i) When power of D $\geq$ power of D′	When power of D′ > power of D′
(ii) $\mathrm{PI} = x\left[\dfrac{1\times\phi(ax+by)}{\frac{\partial}{\partial D}[F(D,D')]}\right]$	$\mathrm{PI} = y\left[\dfrac{1\times\phi(ax+by)}{\frac{\partial}{\partial D'}[F(D,D')]}\right]$
Again Repeat if $\frac{\partial}{\partial D}[F(D,D')] = 0$	Again Repeat if $\frac{\partial}{\partial D'}[F(D,\ D')] = 0$

Important Formulae

1. $2\sin A\cos B = \sin(A+B) + \sin(A-B)$
2. $2\cos A\sin B = \sin(A+B) - \sin(A-B)$
3. $2\cos A\cos B = \cos(A+B) + \cos(A-B)$
4. $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

Q.1. Solve the linear partial differential equation $\frac{\partial^3 u}{\partial x^3} - 3\frac{\partial^3 u}{\partial x^2 \partial y} + 4\frac{\partial^3 u}{\partial y^3} = e^{x+2y}$.

$$\frac{\partial^3 u}{\partial x^3} - 3\frac{\partial^3 u}{\partial x^2 \partial y} + 4\frac{\partial^3 u}{\partial y^3} = e^{x+2y}$$

$$\underbrace{(D^3 - 3D^2D' + 4D'^3)}u = e^{x+2y}$$

Put $D = m,\ D' = 1$

$$(m^3 - 3m^2 + 4)u = e^{x+2y}$$

Aux. Eq^n

$$m^3 - 3m^2 + 4 = 0$$

$$m = -1, 2, 2$$

$$\boxed{CF = f_1(y-x) + f_2(y+2x) + x f_3(y+2x)}$$

$$PI = \frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$$

Put $D = 1,\ D' = 2,\ x + 2y = u$

$$PI = \frac{1}{1 - 3\times1\times2 + 4\times8} \iiint e^u \, du\, du\, du$$

$$PI = \frac{1}{27} e^{u}$$

$$\boxed{PI = \frac{1}{27} e^{x+2y}}$$

complete solution

$$u = CF + PI$$

$$\boxed{u = f_1(y-x) + f_2(y+2x) + x f_3(y+2x) + \frac{1}{27} e^{x+2y}}$$

Q.2 Solve the linear partial differential equation $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = sin\,(2x + 3y)$.

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin(2x+3y)$$

$$(D^2 - 2DD' + D'^2)z = \sin(2x+3y)$$

Put $D = m,\ D' = 1$

$$(m^2 - 2m + 1)z = \sin(2x+3y)$$

Aux. Eq^n

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0$$

$$m = 1, 1$$

$$\boxed{CF = f_1(y+x) + x f_2(y+x)}$$

$$PI = \frac{1}{(D^2 - 2DD' + D'^2)}\sin(2x+3y)$$

Put $D = 2,\ D' = 3,\ 2x + 3y = u$

$$PI = \frac{1}{4 - 12 + 9}\iint \sin u\,du\,du$$

$$PI = \iint \sin u \, du \, du$$

$$PI = -\int \cos u \, du$$

$$PI = -\sin u$$

$$PI = -\sin(2x+3y)$$

complete solution

$$z = CF + PI$$

$$\boxed{z = f_1(y+x) + x f_2(y+x) - \sin(2x+3y)}$$

Q.3 Solve: $r + s - 2t = \sqrt{2x + y}$. (UPTU-2015)

$$r + s - 2t = \sqrt{2x+y}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = (2x+y)^{1/2}$$

$$(D^2 + DD' - 2D'^2)z = (2x+y)^{1/2}$$

Put $D = m$, $D' = 1$

$$(m^2 + m - 2)z = (2x+y)^{1/2}$$

Aux. Eqn

$$m^2 + m - 2 = 0$$

$$m = 1, -2$$

$$\boxed{CF = f_1(y+x) + f_2(y-2x)}$$

$$PI = \frac{1}{(D^2 + DD' - 2D'^2)}(2x+y)^{1/2}$$

Put $D = 2$, $D' = 1$, $2x + y = u$

$$PI = \frac{1}{(4+2-2)}\iint u^{1/2}\, du\, du$$

$$PI = \frac{1}{4}\iint u^{1/2}\,du\,du$$

$$PI = \frac{1}{4}\int \frac{u^{3/2}}{\frac{3}{2}}\,du$$

$$PI = \frac{1}{4}\times\frac{2}{3}\times\frac{u^{5/2}}{\frac{5}{2}}$$

$$PI = \frac{1}{\cancel{4}}\times\frac{\cancel{2}}{3}\times\frac{\cancel{2}}{5}\times u^{5/2}$$

$$\boxed{PI = \frac{1}{15}u^{5/2}}$$

complete solution

$$z = CF + PI$$

$$\boxed{z = f_1(y+x) + f_2(y-2x) + \frac{1}{15}(2x+y)^{5/2}}$$

Q.4 *Solve the linear partial differential equation* $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$

(AKTU-2018)

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$$

$$(D^2 - 2DD')z = \sin x \cos 2y$$

$$D(D - 2D')z = \sin x \cos 2y$$

$$Dz = 0$$

$$CF = f_1(y)$$

$$(D - 2D')z = 0$$

Put $D = m$, $D' = 1$

$$(m-2)z = 0$$

$$m - 2 = 0$$

$$m = 2$$

$$CF = f_2(y+2x)$$

$$\boxed{CF = f_1(y) + f_2(y+2x)}$$

$$PI = \frac{1}{(D^2 - 2DD')} \sin x \cos 2y$$

$$PI = \frac{1}{2(D^2 - 2DD')} 2\sin x \cos 2y$$

$$PI = \frac{1}{2(D^2-2DD')}\left[\sin(x+2y) + \sin(x-2y)\right]$$

$$PI = \frac{1}{2(D^2-2DD')}\sin(x+2y) + \frac{1}{2(D^2-2DD')}\sin(x-2y)$$

$$PI = P_1 + P_2$$

$$P_1 = \frac{1}{2(D^2-2DD')}\sin(x+2y)$$

Put $D=1$, $D'=2$, $x+2y=u$

$$P_1 = \frac{1}{2(1-4)}\iint \sin u \, du \, du$$

$$P_1 = -\frac{1}{6}(-\sin u)$$

$$P_1 = \frac{1}{6}\sin(x+2y)$$

$$P_2 = \frac{1}{2(D^2-2DD')}\sin(x-2y)$$

Put $D=1$, $D'=-2$

$x-2y=u$

$$P_2 = \frac{1}{2(1+4)}\iint \sin u \, du \, du$$

$$P_2 = \frac{1}{10}(-\sin u)$$

$$P_2 = -\frac{1}{10}\sin(x-2y)$$

$$\boxed{PI = \frac{1}{6}\sin(x+2y) - \frac{1}{10}\sin(x-2y)}$$

Complete solution

$$z = CF + PI$$

$$\boxed{z = f_1(y) + f_2(y+2x) + \frac{1}{6}\sin(x+2y) - \frac{1}{10}\sin(x-2y)}$$

Q.5 *Solve the linear partial differential equation* $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx \cos ny + 30(2x + y)$.

(GBTU-2011, 2012)

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx \cos ny + 30(2x+y)$$

$$(D^2 + D'^2)z = \cos mx \cos ny + 30(2x+y)$$

Put $D = m$, $D' = 1$

$$(m^2+1)z = \cos mx \cos ny + 30(2x+y)$$

Aux. Eq^n

$$m^2 + 1 = 0$$

$$m = \pm i$$

$$\boxed{CF = f_1(y+ix) + f_2(y-ix)}$$

$$PI = \frac{1}{D^2+D'^2}\left[\cos mx \cos ny + 30(2x+y)\right]$$

$$PI = \frac{1}{(D^2+D'^2)}\left[\frac{2}{2}\cos mx \cos ny + 30(2x+y)\right]$$

$$PI = \frac{1}{(D^2+D'^2)}\left[\frac{1}{2}\{\cos(mx+ny) + \cos(mx-ny)\} + 30(2x+y)\right]$$

$$PI = \frac{1}{2(D^2+D'^2)}\cos(mx+ny) + \frac{1}{2(D^2+D'^2)}\cos(mx-ny) + \frac{1}{(D^2+D'^2)} \times 30(2x+y)$$

$$PI = P_1 + P_2 + P_3$$

$$P_1 = \frac{1}{2(D^2+D'^2)}\cos(mx+ny)$$

Put $D = m$, $D' = n$, $mx+ny = u$

$$P_1 = \frac{1}{2(m^2+n^2)}\iint \cos u \, du \, du$$

$$P_1 = \frac{-1}{2(m^2+n^2)}\cos u$$

$$\boxed{P_1 = \frac{-1}{2(m^2+n^2)}\cos(mx+ny)}$$

$$P_2 = \frac{1}{2(D^2+D'^2)}\cos(mx-ny)$$

Put $D = m,\ D' = n,\ mx-ny = u$

$$P_2 = \frac{1}{2(m^2+n^2)}\iint \cos u\, du\, du$$

$$P_2 = \frac{-1}{2(m^2+n^2)}\cos u$$

$$\boxed{P_2 = \frac{-1}{2(m^2+n^2)}\cos(mx-ny)}$$

$$P_3 = \frac{1}{D^2+D'^2}30(2x+y)$$

Put $D = 2,\ D' = 1,\ 2x+y = u$

$$P_3 = \frac{1}{4+1}\times 30\iint u\, du\, du$$

$$P_3 = \frac{1}{\cancel{5}}\times \overset{6}{\cancel{30}}\times \int \frac{u^2}{2}\, du$$

$$P_3 = \cancel{6}\times \frac{u^3}{\cancel{6}}$$

$$\boxed{P_3 = (2x+y)^3}$$

$$PI = P_1 + P_2 + P_3$$

$$PI = \frac{-1}{2(m^2+n^2)}\cos(mx+ny) - \frac{1}{2(m^2+n^2)}\cos(mx-ny) + (2x+y)^3$$

$$PI = \frac{-1}{2(m^2+n^2)}\left[\cos(mx+ny) + \cos(mx-ny)\right] + (2x+y)^3$$

$$PI = -\frac{1}{\cancel{2}(m^2+n^2)}\left[\cancel{2}\cos mx \cos ny\right] + (2x+y)^3$$

$$PI = \frac{-1}{m^2+n^2}(\cos mx \cos ny) + (2x+y)^3$$

complete solution

$$\boxed{Z = CF + PI}$$

Q.6 *Solve the linear partial differential equation* $(D^2 + 7DD' + 12D'^2)z = \sinh x$.

(AKTU-2017)

$(D^2 + 7DD' + 12D'^2)z = \sinh x$

Put $D = m,\ D' = 1$

$(m^2 + 7m + 12)z = \sinh(x + 0y)$

Aux. Eq^n

$m^2 + 7m + 12 = 0$

$m = -3, -4$

$(F = f_1(y - 3x) + f_2(y - 4x))$

$PI = \dfrac{1}{(D^2 + 7DD' + 12D'^2)} \sinh(x + 0y)$

Put $D = 1,\ D' = 0,\ x + 0y = u$

$PI = \dfrac{1}{(1 + 0 + 0)} \iint \sinh u \, du\, du$

$PI = \int \cosh u \, du$

$PI = \sinh u$

$PI = \sinh x$

complete solution

$Z = CF + PI$

$$\boxed{Z = f_1(y-3x) + f_2(y-4x) + \sinh x}$$

PDE : DPP-9

Topic : Homogeneous Linear Partial Differential Equation with constant coefficient

Particular Integral(PI) : Case-I

Q.1 *Solve the linear partial differential equation* $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x + y.$

$$z = f_1(y-x) + f_2(y-2x) + \frac{(x+y)^3}{36}$$

Q.2 *Solve :* $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin x.$ *(MTU-2013)*

$$z = f_1(y+x) + x f_2(y+x) - \sin x$$

Engineering Mathematics

UNIT : Partial Differential Equation

Today's Target

- Homogeneous Linear Partial Differential Equation with constant coefficient

  Particular Integral(PI) : Case-I(Type-2)
- PYQs
- DPP

Homogeneous Linear PDE with constant coefficient
(All partial Derivatives are of same order)

Particular Integral

Case – 1

$$F(D, D')z = \phi(ax + by)$$

$$\mathrm{PI} = \frac{1}{F(D, D')}\phi(ax + by)$$

Where $F(D, D')$ *is a Homogeneous function of degree* n

Replace

$$D = a\,, D' = b$$

TYPE-1

if $F(a, b) \neq 0$

put $ax + by = u,$

$$\boxed{\mathrm{PI} = \frac{1}{F(a,b)}\iiint \phi(u)du\,du\ \ldots\ldots n\ times}$$

Type - 2

$\boxed{If\ F(a,b)=0}$ case of failure

(i) When power of D ≥ power of D′	When power of D′ > power of D
(ii) $\mathrm{PI} = x\left[\dfrac{1\times\phi(ax+by)}{\frac{\partial}{\partial D}[F(D,D')]}\right]$	$\mathrm{PI} = y\left[\dfrac{1\times\phi(ax+by)}{\frac{\partial}{\partial D'}[F(D,D')]}\right]$
Again Repeat if $\frac{\partial}{\partial D}[F(D,D')]=0$	Again Repeat if $\frac{\partial}{\partial D'}[F(D,D')]=0$

Q.1 Solve the linear partial differential equation $(D - D')^2 z = \tan(y + x)$

$(D - D')^2 z = \tan(y+x)$

Put $D = m$, $D' = 1$

$(m-1)^2 z = \tan(y+x)$

Aux. Eqⁿ

$(m-1)^2 = 0$

$m = 1, 1$

$$\boxed{CF = f_1(y+x) + x f_2(y+x)}$$

$PI = \dfrac{1}{(D-D')^2}\tan(y+x)$ {case of failure}

$PI = x \times \dfrac{1}{2(D-D')}\tan(y+x)$ {case of failure}

$PI = x^2 \times \dfrac{1}{2\times 1}\tan(y+x) = \dfrac{x^2}{2}\tan(y+x)$

complete solution is

$$\boxed{z = CF + PI = f_1(y+x) + x f_2(y+x) + \frac{x^2}{2}\tan(y+x)}$$

Q.2 Solve $4r - 4s + t = 16\log(x+2y)$

$$4r - 4s + t = 16\log(x+2y)$$

$$4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y} = 16\log(x+2y)$$

$$(4D^2 - 4DD' + D'^2)z = 16\log(x+2y)$$

$$(2D - D')^2 z = 16\log(x+2y)$$

Put $D = m,\ D' = 1$

$$(2m-1)^2 z = 16\log(x+2y)$$

Aux. Eq^n

$$(2m-1)^2 = 0$$

$$m = \frac{1}{2}, \frac{1}{2}$$

$$\boxed{CF = f_1\left(y + \frac{x}{2}\right) + x f_2\left(y + \frac{x}{2}\right)}$$

$$PI = \frac{1}{(2D - D')^2} 16\log(x+2y)$$

{ case of failure }

$$PI = x \times \frac{1}{2(2D-D') \times 2} \times 16 \log(x+2y)$$

{Again case of failure}

$$PI = x^2 \times \frac{1}{\cancel{4 \times 2}} \times \cancel{16}^{\,2} \log(x+2y)$$

$$\boxed{PI = 2x^2 \log(x+2y)}$$

Complete solution is

$$z = CF + PI$$

$$\boxed{z = f_1\left(y+\frac{x}{2}\right) + x f_2\left(y+\frac{x}{2}\right) + 2x^2 \log(x+2y)}$$

Q.3 Solve the differential equation : $\dfrac{\partial^3 z}{\partial x^2 \partial y} - 2\dfrac{\partial^3 z}{\partial x \partial y^2} + \dfrac{\partial^3 z}{\partial y^3} = \dfrac{1}{x^2}$.

$$\frac{\partial^3 z}{\partial x^2 \partial y} - 2\frac{\partial^3 z}{\partial x \partial y^2} + \frac{\partial^3 z}{\partial y^3} = \frac{1}{x^2}$$

$$(D^2 D' - 2 D D'^2 + D'^3) z = \frac{1}{x^2}$$

$$D'(D^2 - 2DD' + D'^2) z = \frac{1}{x^2}$$

Aux. Eqn

$D' = 0$

$CF = f_1(x)$

$D^2 - 2DD' + D'^2 = 0$

$m^2 - 2m + 1 = 0$

$(m-1)^2 = 0$

$m = 1, 1$

$CF = f_2(y+x) + x f_3(y+x)$

$$\boxed{CF = f_1(x) + f_2(y+x) + x f_3(y+x)}$$

$$PI = \frac{1}{(D^2 D' - 2DD'^2 + D'^3)} \times \frac{1}{(x+0y)^2}$$

{case of failure}

$$PI = y \times \frac{1}{(D^2 - 4DD' + 3D'^2)} \times \frac{1}{(x+0y)^2}$$

Put $D = 1$, $D' = 0$, $(x + 0y) = u$

$$PI = y \times \frac{1}{1} \times \iint \frac{1}{u^2}\, du\, du$$

$$PI = y \times \int -\frac{1}{u}\, du$$

$$PI = -y \log u$$

$$\boxed{PI = -y \log x}$$

Complete solution is

$$z = CF + PI$$

$$\boxed{z = f_1(x) + f_2(y+x) + x f_3(y+x) - y \log x}$$

Q.4 Solve $\left(D^3 - 4D^2D' + 5DD'^2 - 2D'^3\right)z = e^{y+2x} + (y+x)^{1/2}$ **(AKTU-2017)**

$$(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{y+2x} + (y+x)^{1/2}$$

Put $D = m,\ D' = 1$

$$(m^3 - 4m^2 + 5m - 2)z = e^{y+2x} + (y+x)^{1/2}$$

Aux. Eq^n

$$m^3 - 4m^2 + 5m - 2 = 0$$

$$m = 1, 1, 2$$

$$\boxed{CF = f_1(y+x) + x f_2(y+x) + f_3(y+2x)}$$

$$PI = \frac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3}\left[e^{y+2x} + (y+x)^{1/2}\right]$$

$$PI = \underbrace{\frac{1}{(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)} e^{y+2x}}_{P_1} + \underbrace{\frac{1}{(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)} (y+x)^{1/2}}_{P_2}$$

$$PI = P_1 + P_2$$

$$P_1 = \frac{1}{(D^3-4D^2D'+5DD'^2-2D'^3)} e^{y+2x}$$

Put $D=2,\ D'=1$

$$P_1 = \frac{1}{(8-16+10-2)} e^{y+2x}$$

{case of failure}

$$P_1 = x \times \frac{1}{(3D^2-8DD'+5D'^2)} e^{y+2x}$$

Put $D=2,\ D'=1,\ y+2x=u$

$$P_1 = \frac{x}{12-16+5} \iint e^u \,du\,du$$

$$P_1 = x\, e^u$$

$$\boxed{P_1 = x\, e^{y+2x}}$$

$$P_2 = \frac{1}{D^3-4D^2D'+5DD'^2-2D'^3)} (y+x)^{1/2}$$

{case of failure}

$$P_2 = x \times \frac{1}{(3D^2-8DD'+5D'^2)} (y+x)^{1/2}$$

{Case of failure}

$$P_2 = x^2 \times \frac{1}{(6D - 8D')}(y+x)^{1/2}$$

Put $D = 1,\ D' = 1,\ (y+x) = u$

$$P_2 = x^2 \times \frac{1}{(6-8)} \int u^{1/2}\, du$$

$$P_2 = \frac{x^2}{-\cancel{2}} \times \frac{u^{3/2}}{\frac{3}{\cancel{2}}}$$

$$P_2 = -\frac{x^3}{3} u^{3/2}$$

$$\boxed{P_2 = -\frac{x^3}{3}(y+x)^{3/2}}$$

$$PI = x e^{y+2x} - \frac{x^3}{3}(y+x)^{3/2}$$

Complete solution is

$$z = CF + PI$$

$$\boxed{z = f_1(y+x) + x f_2(y+x) + f_3(y+2x) + x e^{y+2x} - \frac{x^3}{3}(y+x)^{3/2}}$$

Q.5 Solve the linear partial differential equation : $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos y$

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos y$$

$$(D^2 - DD')z = \sin x \cos y$$

$$D(D-D')z = \sin x \cos y$$

Aux. Eq^n

$D = 0$

$CF = f_1(y)$

$D - D' = 0$

$m - 1 = 0$

$m = 1$

$CF = f_2(y+x)$

$$PI = \frac{1}{(D^2 - DD')} \sin x \cos y$$

$$PI = \frac{1}{2(D^2 - DD')}(2 \sin x \cos y)$$

$$PI = \frac{1}{2(D^2 - DD')}\{\sin(x+y) + \sin(x-y)\}$$

$$PI = \frac{1}{2(D^2 - DD')}\sin(x+y) + \frac{1}{2(D^2 - DD')}\sin(x-y)$$

$$PI = P_1 + P_2$$

$$P_1 = \frac{1}{2(D^2 - DD')} \sin(x+y)$$

{case of failure}

$$P_1 = x \times \frac{1}{2(2D - D')} \sin(x+y)$$

Put $D = 1$, $D' = 1$, $x+y = u$

$$P_1 = x \times \frac{1}{2(2-1)} \int \sin u \, du$$

$$P_1 = \frac{x}{2}(-\cos u)$$

$$\boxed{P_1 = -\frac{x}{2}\cos(x+y)}$$

$$P_2 = \frac{1}{2(D^2 - DD')} \sin(x-y)$$

Put $D = 1$, $D' = -1$, $x-y = u$

$$P_2 = \frac{1}{2(1+1)} \iint \sin u \, du \, du$$

$$P_2 = \frac{1}{4}(-\sin u)$$

$P_2 = -\frac{1}{4}\sin(x-y)$

$PI = -\frac{x}{2}\cos(x+y) - \frac{1}{4}\sin(x-y)$

complete solution

$Z = CF + PI$

$$\boxed{Z = f_1(y) + f_2(y+x) - \frac{x}{2}\cos(x+y) - \frac{1}{4}\sin(x-y)}$$

PDE : DPP-10

Topic : Homogeneous Linear Partial Differential Equation with constant coefficient

Particular Integral(PI) : Case-I (Type-2)

Q .1 *Solve the linear partial differential equation* $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 4\sin(2x+y)$.

Ans $f_1(y) + f_2(y+2x) + x f_3(y+2x) - x^2\cos(2x+y)$

Q .2 *Solve :* $2r - s - 3t = 5\frac{e^x}{e^y}$

Ans $z = f_1(y-x) + f_2\left(y + \frac{3}{2}x\right) + xe^{x-y}$

Q .3 *Solve :* $\frac{\partial^3 z}{\partial x^3} - 7\frac{\partial^3 z}{\partial x \partial y^2} - 6\frac{\partial^3 z}{\partial y^3} = \sin(x+2y) + e^{3x+y}$. *(UPTU-2014)*

Ans $z = f_1(y-x) + f_2(y-2x) + f_3(y+3x) - \frac{1}{75}\cos(x+2y) + \frac{x}{20}e^{3x+y}$

Engineering Mathematics

UNIT : Partial Differential Equation

Today's Target

- Homogeneous Linear Partial Differential Equation with constant coefficient

  Particular Integral(PI) : Case-II and Case-III
- PYQs
- DPP

Homogeneous Linear PDE with constant coefficient

(All partial Derivatives are of same order)

Particular Integral

Case-II

$$F(x\,,\ y) = x^m y^n$$

$$PI = \frac{1}{f(D,\ D')}\, x^m y^n$$

$$\boxed{PI = [1 + \phi(D,\ D')]^{-1}\, x^m y^n}$$

Note :-

(i) If $n = m$ (Taking either D or D' as common factor)

(ii) If $n > m$ (Taking D' as a common factor)

(iii) If $m > n$ (Taking D as a common factor)

After taking common, Apply the formulae given below

(1) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \ldots..$

(2) $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots..$

(3) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

(4) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

Q.1 Solve : $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 12xy.$ **(UPTU-2015)**

$(D^2 + 3DD' + 2D'^2)z = 12xy$

Put $D = m$, $D' = 1$

$(m^2 + 3m + 2)z = 12xy$

Aux. Eq^n

$m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$m = -1, -2$

$$\boxed{CF = f_1(y-x) + f_2(y-2x)}$$

$$PI = \frac{1}{(D^2 + 3DD' + 2D'^2)} 12xy$$

$$PI = \frac{1}{D^2\left[1 + 3\frac{D'}{D} + 2\frac{D'^2}{D^2}\right]} 12xy$$

$$PI = \frac{1}{D^2}\left[1 + \left(3\frac{D'}{D} + 2\frac{D'}{D^2}\right)\right]^{-1} \times 12xy$$

$$PI = \frac{1}{D^2}\left[1 - \left(3\frac{D'}{D} + 2\frac{D'^2}{D^2}\right) + \left(3\frac{D'}{D} + 2\frac{D'^2}{D^2}\right)^2 - \cdots\right] 12xy$$

✓ ✓ ✗ ✗

$$PI = \frac{12}{D^2}\left[1 - \frac{3D'}{D}\right] xy$$

$$PI = \frac{12}{D^2}\left[xy - \frac{3D'(xy)}{D}\right]$$

$$PI = \frac{12}{D^2}\left[xy - 3 \times \frac{1}{D} x\right]$$

$$PI = \frac{12}{D^2}\left[xy - \frac{3x^2}{2}\right]$$

$$PI = \frac{12}{D}\left[\int\left(xy - \frac{3x^2}{2}\right)dx\right]$$

$$PI = \frac{12}{D}\left[\frac{x^2}{2}y - \frac{\cancel{3}x^3}{\cancel{3}\times 2}\right]$$

$$PI = 12\int\left(\frac{x^2}{2}y - \frac{x^3}{2}\right)dx$$

$$PI = 12\left(\frac{x^3}{6}y - \frac{x^4}{8}\right)$$

$$\boxed{PI = 2x^3y - \frac{3x^4}{2}}$$

complete solution

$$\boxed{z = CF + PI}$$

Q.2 Solve : $\frac{\partial^3 u}{\partial x^3} - \frac{\partial^3 u}{\partial y^3} = x^3y^3$.

$$\frac{\partial^3 u}{\partial x^3} - \frac{\partial^3 u}{\partial y^3} = x^3y^3$$

$$(D^3 - D'^3)u = x^3y^3$$

Put $D = m$, $D' = 1$

$$(m^3 - 1)u = x^3y^3$$

Aux. Equation

$$m^3 - 1 = 0$$

$$(m-1)(m^2+m+1) = 0$$

$$m = 1, \omega, \omega^2$$

$$\boxed{CF = f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x)}$$

$$PI = \frac{1}{(D^3 - D'^3)} x^3y^3$$

$$PI = \frac{1}{D^3\left[1 - \frac{D'^3}{D^3}\right]} x^3y^3$$

$$PI = \frac{1}{D^3}\left[1 - \frac{D'^3}{D^3}\right]^{-1} x^3y^3$$

$$PI = \frac{1}{D^3}\left[1 + \frac{D'^3}{D^3}\right] x^3y^3$$

$$PI = \frac{1}{D^3}\left[x^3y^3 + \frac{D'^3}{D^3} x^3y^3\right]$$

$$PI = \frac{1}{D^3}\left[x^3y^3 + 6 \times \frac{1}{D^3} x^3\right]$$

$$PI = \frac{1}{D^3}\left[x^3y^3 + \frac{\cancel{6} \times x^6}{4\times5\times\cancel{6}}\right]$$

$$PI = \frac{1}{D^3}\left[x^3y^3 + \frac{x^6}{20}\right]$$

$$PI = \frac{x^6y^3}{4\times5\times6} + \frac{x^9}{20\times7\times8\times9}$$

$$\boxed{PI = \frac{x^6y^3}{120} + \frac{x^9}{10080}}$$

Complete solution is

$$u = CF + PI$$

$$\boxed{u = f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x) + \frac{x^6y^3}{120} + \frac{x^9}{10080}}$$

CASE- III :- General case

Any Function of $(x,\ y)$

$$PI = \frac{1}{F(D, D')}\ \phi(x,\ y)$$

$$PI = \frac{1}{(D-m_1D')(D-m_2D')\ ....(D-m_nD)}\ \phi(x,\ y)$$

$$\boxed{\frac{1}{(D-m_1D')}\ \phi(x,\ y) = \int \phi(x,\ c-m_1x)\, dx}$$

Put $\boxed{y = c - m_1 x}$ $\quad y = (c + m_1 x)$

Q.3 Solve : $(D^2 - DD' - 2D'^2)z = (y-1)e^x.$ **(UPTU-2014)**

$(D^2 - DD' - 2D'^2)z = (y-1)e^x$

Put $D = m,\ D' = 1$

$(m^2 - m - 2)z = (y-1)e^x$

Aux. Eq^n

$m^2 - m - 2 = 0$

$(m+1)(m-2) = 0$

$m = -1, 2$

$$\boxed{CF = f_1(y-x) + f_2(y+2x)}$$

$$PI = \frac{1}{D^2 - DD' - 2D'^2}(y-1)e^x$$

$$PI = \frac{1}{(D+D')(D-2D')}(y-1)e^x$$

$$PI = \frac{1}{(D-2D')}\left[\frac{1}{D+D'}(y-1)e^x\right]$$

$$PI = \frac{1}{(D-2D')}\int (\underset{1}{c+x-1})\,\underset{2}{e^x}\,dx$$

$$PI = \frac{1}{(D-2D')}\left[(c+x-1)e^x - 1\times e^x\right]$$

$$PI = \frac{1}{D-2D'}\left[(c+x-1-1)e^x\right]$$

$$PI = \frac{1}{D-2D'}\left[(c+x-2)e^x\right]$$

$$PI = \frac{1}{D-2D'}(y-2)e^x$$

$$PI = \int (c-2x-2)e^x\,dx$$

$$PI = (c-2x-2)e^x - (-2)e^x$$

$$PI = (c-2x-\cancel{2}+\cancel{2})e^x$$

$$PI = (c-2x)e^x$$

$$PI = ye^x$$

complete solution

$$\boxed{z = CF + PI}$$

Q.4 Solve the linear partial deferential equation $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$.

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$$

$$(D^2 + DD' - 6D'^2)z = y\cos x$$

Put $D = m$, $D' = 1$

$$(m^2 + m - 6)z = y\cos x$$

Aux. Eqn

$$m^2 + m - 6 = 0$$

$$(m-2)(m+3) = 0$$

$$m = 2, -3$$

$$\boxed{CF = f_1(y+2x) + f_2(y-3x)}$$

$$PI = \frac{1}{(D^2 + DD' - 6D'^2)} y\cos x$$

$$PI = \frac{1}{(D-2D')(D+3D')} y\cos x$$

$$PI = \frac{1}{(D-2D')}\left[\frac{1}{D+3D'} y\cos x\right]$$

$$PI = \frac{1}{(D-2D')} \int (c+3x)\cos x \, dx$$

$$PI = \frac{1}{(D-2D')} \int (c x \cos x + 3x\cos x) \, dx$$

$$PI = \frac{1}{D-2D'} \left[c\int \cos x \, dx + 3\int \underset{1}{x} \underset{2}{\cos x} \, dx \right]$$

$$PI = \frac{1}{D-2D'} \left[c \sin x + 3\{ x(\sin x) - 1 \times (-\cos x) \} \right]$$

$$PI = \frac{1}{D-2D'} \left[c \sin x + 3x \sin x + 3\cos x \right]$$

$$PI = \frac{1}{D-2D'} \left[\underline{(c+3x)} \sin x + 3\cos x \right]$$

$$PI = \frac{1}{D-2D'} \left[y \sin x + 3\cos x \right]$$

$$PI = \int \left[(c-2x)\sin x + 3\cos x \right] dx$$

$$PI = \int (c \sin x - 2x \sin x + 3\cos x) \, dx$$

$$PI = c\int \sin x \, dx - 2\int x \sin x \, dx + 3\int \cos x \, dx$$

$$PI = -cx\cos x - 2\{x(-\cos x) - 1\times(-\sin x)] + 3\sin x$$

$$= -cx\cos x + 2x\cos x - 2\sin x + 3\sin x$$

$$= -\cos x(c - 2x) + \sin x$$

$$PI = -y\cos x + \sin x$$

<u>complete solution</u>

$$z = CF + PI$$

$$\boxed{z = f_1(y+2x) + f_2(y-3x) - y\cos x + \sin x}$$

PDE : DPP-11

Topic : Homogeneous Linear Partial Differential Equation with constant coefficient

Particular Integral(PI) : Case-II and Case-III

Q.1 Solve $\left[D^2 + (a+b)DD' + ab\,D'^2\right]z = xy$ (GBTU-2011 & 2013)

$$z = f_1(y-ax) + f_2(y-bx) + \frac{1}{6}x^3y - \frac{(a+b)}{24}x^4$$

Q.2 Solve : $\left(D^3 + 2D^2D' - DD'^2 - 2D'^3\right)z = (y+2)e^x.$

$$z = f_1(y+x) + f_2(y-x) + f_3(y-2x) + ye^x$$

Q. Solve : $\left(D^2 + DD' - 6\ D'^2\right)z = y\sin x.$ (MTU-2011 & 2012)

$$z = f_1(y+2x) + f_2(y-3x) - y\sin x - \cos x$$

UNIT : Partial Differential Equation

Today's Target

- $\boxed{\text{Non-Homogeneous Linear Partial Differential Equation with constant coefficient}}$

  Complementary Function (CF)

- PYQs
- DPP

Non - Homogeneous Linear Partial Differential Equation with constant coefficient
(All Partial Derivatives are not of same order)

Given PDE

$$\phi(D,\ D') = F(x,\ y) \text{ --------} (1)$$

Methods to find CF

CASE-1:- When $\phi(D, D')$ resolve in to linear factors

$$(D - m_1 D' - a_1)(D - m_2 D' - a_2)(D - m_3 D' - a_3)z = F(x, y)$$

$$\boxed{CF = e^{a_1 x} f_1(y + m_1 x) + e^{a_2 x} f_2(y + m_2 x) + e^{a_3 x} f_3(y + m_3 x)}$$

NOTE : (i) CF corresponding to D = $f_1(y)$ (ii)CF corresponding to D' = $f_1(x)$

Q.1 *Solve the linear partial differential equation* $(D + D' - 1)(D + 2D' - 2)z = 0.$

$$(D + D' - 1)(D + 2D' - 2)z = 0$$

$$CF = e^{x} f_1(y - x) + e^{2x} f_2(y - 2x)$$

$$PI = 0$$

complete solution

$$z = CF + PI$$

$$\boxed{z = e^{x} f_1(y - x) + f_2(y - 2x)}$$

Q.2 Solve: $DD'(D+2D'+1)z=0$.

$DD'(D+2D'+1)z=0$

CF corresponding to $D = f_1(y)$

CF " " $D' = f_2(x)$

CF " $(D+2D'+1) = e^{-x} f_3(y-2x)$

$CF = f_1(y) + f_2(x) + e^{-x} f_3(y-2x)$

$PI = 0$

Complete solution

$\boxed{z = CF + PI}$

Q.3 Solve : $r - t + p - q = 0.$ (UPTU-2013,2018)

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$$

$$(D^2 - D'^2 + D - D')z = 0$$

$$[(D - D')(D + D') + (D - D')]z = 0$$

$$(D - D')(D + D' + 1)z = 0$$

$$(D - D' + 0)(D + D' + 1)z = 0$$

$$CF = e^{0x} f_1(y+x) + e^{-x} f_2(y-x)$$

$$CF = f_1(y+x) + e^{-x} f_2(y-x)$$

$$PI = 0$$

complete solution

$$z = CF + PI$$

$$\boxed{z = f_1(y+x) + e^{-x} f_2(y-x)}$$

CASE-2:-

When $\phi(D,\ D')$ contains repeated factors

$$(D - m\,D' - a)^3\ z = F(x, y)$$

$$CF = e^{ax} f_1(y + mx) + xe^{ax} f_2(y + mx) + x^2 e^{ax} f_3(y + mx)$$

Q.4 Solve : $(D + 4D' + 5)^2 z = 0$. (AKTU-2018)

$$(D + 4D' + 5)^2 z = 0$$

$$CF = e^{-5x} f_1(y-4x) + x e^{-5x} f_2(y-4x)$$

$$PI = 0$$

Complete solution

$$z = CF + PI$$

$$\boxed{z = e^{-5x} f_1(y-4x) + x f_2(y-4x)}$$

Q.5 Solve: $r + 2s + t + 2p + 2q + z = 0$.

$$r + 2s + t + 2p + 2q + z = 0$$

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} + 2\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} + z = 0$$

$$\left[(D^2 + 2DD' + D'^2) + 2D + 2D' + 1\right]z = 0$$

$$\left[(D + D')^2 + 2(D + D') + 1\right]z = 0$$

$$(D + D' + 1)^2 z = 0$$

$$CF = e^{-x} f_1(y - x) + x e^{-x} f_2(y - x)$$

$$PI = 0$$

complete solution

$$z = CF + PI$$

$$\boxed{z = e^{-x} f_1(y - x) + x e^{-x} f_2(y - x)}$$

CASE-3:-

When Function $\phi(D,\ D')$ con not be factorized into linear factor

Take a trial solution

$$z = Ae^{hx+ky}$$

Where A, h, k *are constants*.

Q. 6 **Solve:** $(D^2 + D'^2 - p^2)z = 0.$

$(D^2 + D'^2 - p^2)z$ —— (1)

consider a trial solution

$z = Ae^{hx+ky}$

From eqn (1)

$(D^2 + D'^2 - p^2)Ae^{hx+ky} = 0$

$A[D^2 e^{hx+ky} + D'^2 e^{hx+ky} - p^2 e^{hx+ky}] = 0$

$A[h^2 e^{hx+ky} + k^2 e^{hx+ky} - p^2 e^{hx+ky}] = 0$

$Ae^{hx+ky}(h^2 + k^2 - p^2) = 0$

But $Ae^{hx+ky} \neq 0$

$\therefore h^2 + k^2 - p^2 = 0$

$\boxed{CF = \sum Ae^{hx+ky}}$

$\boxed{z = CF + PI}$

Where $h^2 + k^2 - p^2 = 0$

$PI = 0$

PDE : DPP-12

Topic : Non-Homogeneous Linear Partial Differential Equation with constant coefficient

Complementary Function (CF)

Q .1 *Solve* $2s + t - 3q = 0$

Q .2 Solve $DD'(2D + D' + 11)^2 z = 0$

UNIT : Partial Differential Equation

Today's Target

- Non-Homogeneous Linear Partial Differential Equation with constant coefficient

  Particular Integral (PI) : Case-1 and Case-2

- PYQs
- DPP

Course Details(Paid) : All Subjects

1	Recorded Video Lectures (100 % Syllabus Coverage)
2	Pdf Notes
3	Lecture wise DPP with Answers
4	DPP Solutions
5	Unit wise set of PYQs

Given PDE

$$\phi(D,\ D') = F(x,\ y)$$

$$\phi(D, D') \to CF \qquad F(x, y) \to PI$$

Methods to find PI

CASE-1:- When $F(x,\ y) = e^{ax+by}$

$$PI = \frac{1}{\phi(D, D')} e^{ax+by}$$

Put $D = a$

$D' = b$

$$\boxed{PI = \frac{1}{\phi(a, b)} e^{ax+by}}$$

When $\phi(a, b) \neq 0$

Q.1 Solve $(D^2 - 4DD' + 4D'^2 - D + 2D')z = e^{3x+4y}$

$$(D^2 - 4DD' + 4D'^2 - D + 2D')z = e^{3x+4y}$$

$$\left[(D-2D')^2 - (D-2D')\right]z = e^{3x+4y}$$

$$(D-2D')(D-2D'-1)z = e^{3x+4y}$$

$$(D-2D'-0)(D-2D'-1)z = e^{3x+4y}$$

$$CF = e^{0x} f_1(y+2x) + e^{-x} f_2(y+2x)$$

$$\boxed{CF = f_1(y+2x) + e^{+x} f_2(y+2x)}$$

$$PI = \frac{1}{(D^2-4DD'+4D'^2-D+2D')} e^{3x+4y}$$

Put $D = 3$ and $D' = 4$

$$PI = \frac{1}{9-48+64-3+8} e^{3x+4y}$$

$$PI = \frac{1}{30} e^{3x+4y}$$

Complete solution

$$z = CF + PI$$

$$\boxed{z = f_1(y+2x) + e^{x} f_2(y+2x) + \frac{1}{30} e^{3x+4y}}$$

Q.2 Solve: $D(D-2D'-3)z = e^{x+2y}$.

$$D(D-2D'-3)z = e^{x+2y}$$

CF corresponding to $D = f_1(y)$

CF " $(D-2D'-3) = e^{3x} f_2(y+2x)$

$$\boxed{CF = f_1(y) + e^{3x} f_2(y+2x)}$$

$$PI = \frac{1}{D(D-2D'-3)} e^{x+2y}$$

Put $D = 1$

$D' = 2$

$$PI = \frac{1}{1(1-2\times 2-3)} e^{x+2y}$$

$$PI = -\frac{1}{6} e^{x+2y}$$

complete solution

$$z = CF + PI$$

$$\boxed{z = f_1(y) + f_2(y+2x) - \frac{1}{6} e^{x+2y}}$$

Non - Homogeneous Linear Partial Differential Equation with constant coefficient
(All Partial Derivatives are not of same order)

Given PDE

$$\phi(D,\ D') = F(x,\ y)$$

Where

$$\phi(D, D') \neq 0$$

Methods to find PI

CASE-2:- When $F(x,\ y) = sin(ax+by)/Cos(ax+by)$

$$PI = \frac{1}{\phi(D, D')} \sin(ax+by) / \cos(ax+by)$$

Put $D^2 = -a^2$

$DD' = -ab$

$D'^2 = -b^2$

Q.3 Solve: $(D - D' - 1)(D - D' - 2)z = \sin(2x + 3y)$. (AKTU-2021-22)

$(D - D' - 1)(D - D' - 2)z = \sin(2x + 3y)$

$CF = e^{x} f_1(y + x) + e^{2x} f_2(y + x)$

$PI = \frac{1}{(D - D' - 1)(D - D' - 2)} \sin(2x + 3y)$

$PI = \frac{1}{D^2 - DD' - 2D - D'D + D'^2 + 2D' - D + D' + 2} \sin(2x + 3y)$

$PI = \frac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2} \sin(2x + 3y)$

Put $D^2 = -4$

$DD' = -6$

$D'^2 = -9$

$PI = \frac{1}{-4 + 12 - 9 - 3D + 3D' + 2} \sin(2x + 3y)$

$PI = \frac{1}{-3D + 3D' + 1} \sin(2x + 3y)$

$PI = \frac{1}{-[3D - 3D' - 1]} \sin(2x + 3y)$

$$PI = \frac{-1 \times [(3D-3D')+1]}{[(3D-3D')-1][(3D-3D')+1]} e^{2x+3y}$$

$$PI = \frac{-[(3D-3D')+1]}{(3D-3D')^2 - 1^2} e^{2x+3y}$$

$$PI = \frac{-[3D-3D'+1]}{9D^2+9D'^2-18DD'-1} e^{2x+3y}$$

Put $D^2 = -4$

$D'^2 = -9$

$DD' = -6$

$$PI = \frac{-(3D-3D'+1)}{-36-81+108-1} \sin(2x+3y)$$

$$PI \frac{+(3D-3D'+1)}{+10} \sin(2x+3y)$$

$$PI = \frac{1}{10}\Big[3D\sin(2x+3y) - 3D'\sin(2x+3y)$$

$$+ \sin(2x+3y)\Big]$$

$$PI = \frac{1}{10}\Big[3\times 2\cos(2x+3y) - 9\cos(2x+3y)$$

$$+\sin(2x+3y)\Big]$$

$$PI = \frac{1}{10}\left[\sin(2x+3y) - 3\cos(2x+3y)\right]$$

complete solution

$$\boxed{z = CF + PI}$$

Q.4 Solve the following partial differential equations:

$$(D^2 - DD' - 2D'^2 + 2D + 2D')z = \sin(2x + y)$$ **(AKTU-2016)**

$(D^2 - DD' - 2D'^2 + 2D + 2D')z = \sin(2x+y)$

$(D^2 + DD' - DD' - DD' - 2D'^2 + 2D + 2D')z = \sin(2x+y)$

$(D^2 + DD' - 2DD' - 2D'^2 + 2D + 2D')z = \sin(2x+y)$

$[D(D+D') - 2D'(D+D') + 2(D+D')]z = \sin(2x+y)$

$(D+D')(D-2D'+2)z = \sin(2x+y)$

$CF = e^{0x} f_1(y-x) + e^{-2x} f_2(y+2x)$

$CF = f_1(y-x) + f_2(y+2x)$

$$PI = \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'} \sin(2x+y)$$

Put $D^2 = -4$

$D'^2 = -1$

$DD' = -2$

$$PI = \frac{1}{-4 + \cancel{2} + \cancel{2} + 2D + 2D'} \sin(2x+y)$$

$$PI = \frac{1}{2(D+D')} \sin(2x+y)$$

$$PI = \frac{D-D'}{2(D+D')(D-D')} \sin(2x+y)$$

$$PI = \frac{D-D'}{2(D^2-D'^2)} \sin(2x+y)$$

Put

$D^2 = -4$

$D'^2 = -1$

$$PI = \frac{(D-D')}{2(-4+1)} \sin(2x+y)$$

$$PI = -\frac{1}{6}\left[D\sin(2x+y) - D'\sin(2x+y)\right]$$

$$PI = -\frac{1}{6}\left[2\cos(2x+y) - \cos(2x+y)\right]$$

$$PI = -\frac{1}{6}\cos(2x+y)$$

complete solution

$$\boxed{Z = CF + PI}$$

PDE : DPP-13

- **Topic :** Non-Homogeneous Linear Partial Differential Equation with constant coefficient

Particular Integral (PI) : Case-1 and Case-2

Q .1 *Solve:* $(D^3 - 3DD' + D' + 4)z = e^{2x+y}$.

Q .2 *Solve* $(D^2 - D'^2 - 3D + 3D')z = e^{x-2y}$

Q .3 *Find the particular integral of* $2s + t - 3q = 5\cos(3x - 2y)$.

Q .4 *Solve the following partial differential equations:* $(D^2 - DD' + D' - 1)z = \cos(x + 2y)$

UNIT : Partial Differential Equation

Today's Target

- Non-Homogeneous Linear Partial Differential Equation with constant coefficient

  Particular Integral (PI) : Case-3 and Case-4

- PYQs
- DPP

Course Details(Paid) : All Subjects

1	Recorded Video Lectures (100 % Syllabus Coverage)
2	Pdf Notes
3	Lecture wise DPP with Answers
4	DPP Solutions
5	Unit wise set of PYQs

Non - Homogeneous Linear Partial Differential Equation with constant coefficient
(All Partial Derivatives are not of same order)

Given PDE

$$\underset{\downarrow \atop CF}{\phi(D,\ D')} = \underset{PI}{F(x,\ y)}$$

Methods to find PI

CASE-3:- When $F(x,\ y) = x^m y^n$

$$PI = \frac{1}{\phi(D, D')} x^m y^n$$

$$\boxed{PI = [1 + \phi(D, D')]^{-1} \cdot x^m y^n}$$

Note :-

(i) If a separate constant term is present , take it common

(ii) If $n = m$ (Taking either D or D' as common factor)

(iii) If $n > m$ (Taking D' as a common factor)

(iv) If $m > n$ (Taking D as a common factor)

After taking common, Apply the formulae given below

(1) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \ldots\ldots$

(2) $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots\ldots$

(3) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots$

(4) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

Q.1 Solve : $s + p - q = z + xy$.

$$s + p - q = z + xy$$

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} - z = xy$$

$$(DD' + D - D' - 1)z = xy$$

$$(DD' - D' + D - 1)z = xy$$

$$[D'(D-1) + 1(D-1)]z = xy$$

$$\underline{(D-1)(D'+1)}z = xy$$

CF corresponding to $D-1 = e^{x} f_1(y)$

CF $\quad$ " $\quad$ " $\quad$ " $\underset{\sim\sim\sim}{(D'+1)} = e^{-y} f_2(x)$

$$\boxed{CF = e^{x} f_1(y) + e^{-y} f_2(x)}$$

$$PI = \frac{1}{(D-1)(D'+1)} xy$$

$$PI = \frac{-1}{(1-D)(1+D')} xy$$

$$PI = -1\left[(1-D)^{-1}(1+D')^{-1}\right] xy$$

$$PI = -\left[\left(1+D+D^2+\cdots\right)\left(1-D'+D'^2-\cdots\right)\right]xy$$

$$PI = -\left[1-D'+D-DD'+\cdots\right]xy$$

$$PI = -\left[xy - D'(xy) + D(xy) - DD'(xy)\right]$$

$$PI = -(xy - x + y - 1)$$

$$PI = -xy + x - y + 1$$

complete solution

$$Z = CF + PI$$

$$\boxed{Z = e^{x} f_1(y) + e^{-y} f_2(x) - xy + x - y + 1}$$

Q.2 Solve: $D(D + D' - 1)(D + 3D' - 2)z = x^2 - 4xy + 2y^2$.

$$D(D+D'-1)(D+3D'-2)z = x^2-4xy+2y^2$$

$$\boxed{CF = f_1(y) + e^{x} f_2(y-x) + e^{2x} f_3(y-3x)}$$

$$PI = \frac{1}{D(D+D'-1)(D+3D'-2)}(x^2-4xy+2y^2)$$

$$PI = \frac{1}{D(-1)\left[1-(D+D')\right](-2)\left[1-\left(\frac{D}{2}+\frac{3}{2}D'\right)\right]}(x^2-4xy+2y^2)$$

$$PI = \frac{1}{2D}\left[1-(D+D')\right]^{-1}\left[1-\left(\frac{D}{2}+\frac{3}{2}D'\right)\right]^{-1}(x^2-4xy+2y^2)$$

$$PI = \frac{1}{2D}\left[1 + (D+D') + (D+D')^2 + \dots\right]\left[1 + \left(\frac{D}{2} + \frac{3}{2}D'\right) + \left(\frac{D}{2} + \frac{3}{2}D'\right)^2 + \dots\right] x^2 - 4xy + 2y^2$$

$$PI = \frac{1}{2D}\left[1 + D + D' + D^2 + D'^2 + 2DD'\right]\left[1 + \frac{D}{2} + \frac{3}{2}D' + \frac{D^2}{4} + \frac{9}{4}D'^2 + \frac{3}{2}DD'\right](x^2 - 4xy + 2y^2)$$

$$PI = \frac{1}{2D}\left[1 + \frac{D}{2} + \frac{3}{2}D' + \frac{D^2}{4} + \frac{9}{4}D'^2 + \frac{3}{2}DD' + D + \frac{D^2}{2} + \frac{3}{2}DD' + D' + \frac{DD'}{2} + \frac{3}{2}D'^2 + D^2 + D'^2 + 2DD'\right](x^2 - 4xy + 2y^2)$$

$$PI = \frac{1}{2D}\left[1 + \frac{3}{2}D + \frac{5D'}{2} + \frac{7D^2}{4} + \frac{19D'^2}{4} + \frac{11}{2}DD'\right](x^2 - 4xy + 2y^2)$$

$$PI = \frac{1}{2D}\left[x^2 - 4xy + 2y^2 + \frac{3}{2}D(x^2 - 4xy + 2y^2) + \frac{5}{2}D'(x^2 - 4xy + 2y^2) + \frac{7}{4}D^2(x^2 - 4xy + 2y^2) + \frac{19}{4}D'^2(x^2 - 4xy + 2y^2) + \frac{11}{2}DD'(x^2 - 4xy + 2y^2)\right]$$

$$PI = \frac{1}{2D}\left[x^2 - 4xy + 2y^2 + \frac{3}{2}(2x-4y) + \frac{5}{2}(-4x+4y) + \frac{7}{4}(2) + \frac{19}{4}(4) + \frac{11}{2}(-4)\right]$$

$$PI = \frac{1}{2D}\left[x^2 - 4xy + 2y^2 + 3x - 6y - 10x + 10y + \frac{7}{2} + 19 - 22\right]$$

$$PI = \frac{1}{2D}\left[x^2 - 4xy + 2y^2 - 7x + 4y + \frac{1}{2}\right]$$

$$\boxed{PI = \frac{1}{2}\left[\frac{x^3}{3} - 2x^2y + 2xy^2 - \frac{7x^2}{2} + 4xy + \frac{1}{2}x\right]}$$

complete solution

$$\boxed{Z = CF + PI}$$

Non - Homogeneous Linear Partial Differential Equation with constant coefficient

(All Partial Derivatives are not of same order)

Given PDE

$$\phi(D,\ D') = F(x,\ y)$$

Methods to find PI

CASE-4:- When $F(x,\ y) = e^{ax+by}\,V$

$$PI = \frac{1}{\phi(D, D')} e^{(ax+by)} V(x,y)$$

$$\boxed{PI = e^{ax+by} \times \frac{1}{\phi(D+a,\ D'+b)} V(x,y)}$$

Q.3 *Solve* : $(D - 3D' - 2)^2 z = 2e^{2x} \tan(y + 3x)$.

$$(D-3D'-2)^2 z = 2e^{2x}\tan(y+3x)$$

$$\boxed{CF = e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x)}$$

$$PI = \frac{1}{(D-3D'-2)^2} 2e^{2x}\tan(y+3x)$$

$$PI = 2 \times \frac{1}{(D-3D'-2)^2} e^{2x+0y}\tan(y+3x)$$

$$PI = 2e^{2x+0y} \times \frac{1}{(D+\cancel{2}-3D'-\cancel{2})^2}\tan(y+3x)$$

$$PI = 2e^{2x} \times \frac{1}{(D-3D')^2}\tan(y+3x)$$

Put $D \to 3,\ D' \to 1$

case of failure

$$PI = 2e^{2x} \times x \times \frac{1}{2(D-3D')}\tan(y+3x)$$

Again case of failure

$$PI = \cancel{2} x^2 e^{2x} \times \frac{1}{\cancel{2}}\tan(y+3x)$$

$$PI = x^2 e^{2x} \tan(y+3x)$$

complete solution

$$\boxed{z = CF + PI}$$

Q.4 *Solve the linear partial differential equation :* $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^y$.

$$(D^2 - DD' + D' - 1)z = \cos(x+2y) + e^y$$

$$[(D^2-1) + D' - DD']z = \cos(x+2y) + e^y$$

$$[(D-1)(D+1) - D'(D-1)]z = \cos(x+2y) + e^y$$

$$(D-1)(D+1-D')z = \cos(x+2y) + e^y$$

$$\boxed{CF = e^x f_1(y) + e^{-x} f_2(y+x)}$$

$$PI = \frac{1}{(D^2 - DD' + D' - 1)}\{\cos(x+2y) + e^y\}$$

$$PI = \frac{1}{(D^2 - DD' + D' - 1)}\cos(x+2y) + \frac{1}{(D^2 - DD' + D' - 1)}e^y$$

$$PI = P_1 + P_2 \quad \text{——— (1)}$$

$$P_1 = \frac{1}{(D^2 - DD' + D' - 1)}\cos(x+2y)$$

Put $D^2 = -1$

$DD' = -2$

$$P_1 = \frac{1}{-\cancel{1}+\cancel{2}+D'-\cancel{1}} \cos(x+2y)$$

$$P_1 = \frac{1}{D'} \cos(x+2y)$$

$$\boxed{P_1 = -\frac{\sin(x+2y)}{2}}$$

$$P_2 = \frac{1}{(D-1)(D-D'+1)} e^{y}$$

$$P_2 = \frac{1}{(D-D'+1)} \left[\frac{1}{D-1} e^{0x+y}\right]$$

$$P_2 = \frac{1}{(D-D'+1)} \left(-e^{0x+y}\right)$$

$$P_2 = -e^{0x+y} \left[\frac{1}{D-(D'+1)+1}\right]$$

$$P_2 = -e^{y} \left(\frac{1}{D-D'-\cancel{1}+\cancel{1}}\right)$$

$$P_2 = -e^{y} \times \frac{1}{D-D'} e^{0x+0y}$$

$$P_2 = -e^y \times \frac{1}{0-0} e^{0x+0y}$$

case of failure

$$P_2 = -xe^y \times \frac{1}{1} e^{0x+0y}$$

$$P_2 = -xe^y$$

Put P_1 and P_2 in (1)

$$PI = -\frac{\sin(x+2y)}{2} - xe^y$$

complete solution

$$\boxed{z = CF + PI}$$

PDE : DPP-14

➢ Topic : Non-Homogeneous Linear Partial Differential Equation with constant coefficient

Particular Integral (PI) : Case-3 and Case-4

Q .1 *Solve the linear partial differential equation :* $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$. (GBTU-2011)

Q .2 *Solve* $(D + D' - 1)^2 z = xy$ (GBTU-2011)

Q .3 *Solve :* $(D - 3D' - 2)^3 z = 6e^{2x} \sin(3x + y)$.

Q .4 *Solve :* $r - 4s + 4t + p - 2q = e^{x+y}$.

UNIT : Partial Differential Equation

Today's Target

- Equation Reducible to Linear PDE with constant coefficient
- PYQs
- DPP

Course Details(Paid) : All Subjects

1	Recorded Video Lectures (100 % Syllabus Coverage)
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Equation Reducible to Linear PDE with constant coefficient

General Form

$$a_0 x^2 \frac{\partial^2 z}{\partial x^2} + a_1 xy \frac{\partial^2 z}{\partial x \partial y} + a_2 y^2 \frac{\partial^2 z}{\partial y^2} + b_1 \frac{x\partial z}{\partial x} + b_2 y \frac{\partial z}{\partial y} = F(x, y)$$

Standard substitution to convert above equation in Linear PDE with constant coefficient

(1) Put $x = e^u \quad \Rightarrow \quad u = \log x$

$y = e^v \quad \Rightarrow \quad v = \log y$

(2) Put $x \frac{\partial z}{\partial x} = DZ$,

$x^2 \frac{\partial^2 z}{\partial x^2} = D(D-1)\, Z,$ where, $D \equiv \frac{\partial}{\partial u}$

$x^3 \frac{\partial^2 z}{\partial x^3} = D(D-1)\,(D-2)Z$

(3) : $y\frac{\partial z}{\partial y} = D'Z$

$$y^2\frac{\partial^2 z}{\partial y^2} = D'(D'-1)\,Z,$$ where $D' \equiv \frac{\partial}{\partial v}$

$$y^3\frac{\partial^3 z}{\partial y^3} = D'(D'-1)\,(D'-2)Z$$

(4) : $xy\frac{\partial^2 z}{\partial x\,\partial y} = DD'Z$

(5): After above substitution given PDE is converted in to $\boxed{\text{Linear PDE with constant coefficient}}$

Linear Partial Differential Equation with constant coefficient

Linear PDE with constant coefficient

Homogeneous Linear PDE with constant coefficient

(All partial derivatives are of same order)

Complete solution = CF + PI

Non-Homogeneous Linear PDE with constant coefficient

(All partial derivatives are not of same order)

Complete solution = CF + PI

Q.1 *Solve the differential equation :* $x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = xy$. **AKTU-2022-23**

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = xy$$

Put $x = e^u \Rightarrow u = \log x$

$y = e^v \Rightarrow v = \log y$

$$x^2\frac{\partial^2 z}{\partial x^2} = D(D-1)z$$

$$y^2\frac{\partial^2 z}{\partial y^2} = D'(D'-1)z$$

$$D(D-1)z - D'(D'-1)z = e^u \times e^v$$

$$[D(D-1) - D'(D'-1)]z = e^{u+v}$$

$$(D^2 - D - D'^2 + D')z = e^{u+v}$$

$$(D^2 - D'^2 + D' - D)z = e^{u+v}$$

$$[(D-D')(D+D') - (D-D')]z = e^{u+v}$$

$$(D-D')(D+D'-1)z = e^{u+v}$$

$$(D-D')(D+D'-1)z = e^{u+v}$$

$$CF = e^{0u} f_1(v+u) + e^{u} f_2(v-u)$$

$$CF = f_1(v+u) + e^{u} f_2(v-u)$$

$$CF = f_1(\log y + \log x) + x f_2(\log y - \log x)$$

$$CF = f_1\{\log xy\} + x f_2(\log y/x)$$

$$\boxed{CF = g_1(xy) + x\, g_2(y/x)}$$

$$PI = \frac{1}{(D-D')\underbrace{(D+D'-1)\quad}} e^{u+v}$$

$$PI = \frac{1}{(D-D')}\left[\frac{1}{(D+D'-1)} e^{u+v}\right]$$

$$PI = \frac{1}{D-D'}\left[\frac{1}{1+\cancel{1}-\cancel{1}} e^{u+v}\right]$$

$$PI = \frac{1}{D-D'} e^{u+v}$$

case failure

$$PI = \frac{1}{D-D'} e^{u+v}$$

$$PI = u \times \frac{1}{1-0} e^{u+v}$$

$$PI = u e^{u+v}$$

$$PI = u \times e^{u} \times e^{v}$$

$$\boxed{PI = xy \log x}$$

complete solution

$$z = CF + PI$$

$$\boxed{z = g_1(xy) + x\, g_2(y/x) + xy \log x}$$

Q.2 *Solve the linear partial differential equation :* $x^2\frac{\partial^2 z}{\partial x^2} - 4xy\frac{\partial^2 z}{\partial x \partial y} + 4y^2\frac{\partial^2 z}{\partial y^2} + 6y\frac{\partial z}{\partial y} = x^3y^4$.

AKTU-2021-22

$$x^2\frac{\partial^2 z}{\partial x^2} - 4xy\frac{\partial^2 z}{\partial x \partial y} + 4y^2\frac{\partial^2 z}{\partial y^2} + 6y\frac{\partial z}{\partial y} = x^3y^4$$

Put $x = e^u \Rightarrow u = \log x$

$y = e^v \Rightarrow v = \log y$

$$x^2\frac{\partial^2 z}{\partial x^2} = D(D-1)z$$

$$xy\frac{\partial^2 z}{\partial x \partial y} = DD'z$$

$$y^2\frac{\partial^2 z}{\partial y^2} = D'(D'-1)z$$

$$y\frac{\partial z}{\partial y} = D'z$$

$$D(D-1)z - 4DD'z + 4D'(D'-1)z + 6D'z = (e^u)^3(e^v)^4$$

$$[D^2 - D - 4DD' + 4D'^2 - 4D' + 6D']z = e^{3u} \times e^{4v}$$

$$[\underline{D^2} - D - \underline{4DD'} + \underline{4D'^2} + 2D']z = e^{3u+4v}$$

$$[(D^2 - 4DD' + 4D'^2) - D + 2D']z = e^{3u+4v}$$

$$\left[(D-2D')^2-(D-2D')\right]z = e^{3u+4v}$$

$$(D-2D')(D-2D'-1)z = e^{3u+4v}$$

$$CF = e^{0u} f_1(v+2u) + e^{u} f_2(v+2u)$$

$$CF = f_1(\log y + 2\log x) + x f_2(\log y + 2\log x)$$

$$CF = f_1(\log y + \log x^2) + x f_2(\log y + \log x^2)$$

$$CF = f_1(\log yx^2) + x f_2(\log yx^2)$$

$$\boxed{CF = g_1(yx^2) + x g_2(yx^2)}$$

$$PI = \frac{1}{(D-2D')(D-2D'-1)} e^{3u+4v}$$

Put $D = 3$ and $D' = 4$

$$PI = \frac{1}{(3-8)(3-8-1)} e^{3u+4v}$$

$$PI = \frac{1}{30} e^{3u} \times e^{4v}$$

$$PI = \frac{1}{30}(e^{u})^{3} \times (e^{v})^{4}$$

$$\boxed{PI = \frac{1}{30}x^{3}y^{4}}$$

complete solution

$$Z = CF + PI$$

$$\boxed{Z = g_1(yx^2) + x\,g_2(yx^2) + \frac{1}{30}x^3y^4}$$

Q.3 Solve : $x^2r - y^2t + px - qy = \log x$ AKTU-2017-18

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \log x$$

Put $x = e^u \Rightarrow u = \log x$

$y = e^v \Rightarrow v = \log y$

$$x^2\frac{\partial^2 z}{\partial x^2} = D(D-1)z$$

$$y^2\frac{\partial^2 z}{\partial y^2} = D'(D'-1)z$$

$$x\frac{\partial z}{\partial x} = Dz$$

$$y\frac{\partial z}{\partial y} = D'z$$

$$D(D-1)z - D'(D'-1)z + Dz - D'z = u$$

$$[D^2 - \cancel{D} - D'^2 + \cancel{D'} + \cancel{D} - \cancel{D'}]z = u$$

$$(D^2 - D'^2)z = u$$

Aux. Eq^n

$$D^2 - D'^2 = 0$$

Put $D = m,\ D' = 1$

$m^2 - 1 = 0$

$m = \pm 1$

$CF = f_1(v+u) + f_2(v-u)$

$CF = f_1(\log y + \log x) + f_2(\log y - \log x)$

$CF = f_1(\log xy) + f_2(\log y/x)$

$$\boxed{CF = g_1(xy) + g_2(y/x)}$$

$PI = \frac{1}{(D^2 - D'^2)} u$

$PI = \frac{1}{(D^2 - D'^2)}(u + 0v)$

Put $D = 1$, $D' = 0$ and $u + 0v = t$

$PI = \frac{1}{1-0} \iint t\,dt\,dt$

$PI = \int \frac{t^2}{2} dt$

$$PI = \frac{t^3}{6} = \frac{u^3}{6}$$

$$PI = \frac{(\log x)^3}{6}$$

complete solution

$$z = CF + PI$$

$$\boxed{z = g_1(xy) + g_2(y/x) + \frac{1}{6}(\log x)^3}$$

PDE : DPP-15

- **Topic :** Non-Homogeneous Linear Partial Differential Equation with constant coefficient

Particular Integral (PI) : Case-3 and Case-4

Q .1 *Solve the linear partial differential equation :* $\left(x^2D^2 + 2xyDD' + y^2D'^2\right)z = x^m y^n$. AKTU-2020-21

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Thank You