Deriving the General Relativistic Schwarzschild Metric

by
Mohit S. Kasal
Dept. of Physics, Fergusson College(Autonomous), Pune

A report submitted in the fulfillment of project work

Naxxatra Sciences and Collaborative Research,
Bengaluru

January 2023

ABSTRACT : We will briefly discuss some underlying principles of general relativity with the focus on a more geometric interpretation. We outline the Einstein Equations which describe the geometry of space-time, and from there derive the Schwarzschild metric, to describe the gravitational field outside a static black hole with no electric charge.

Contents

1 Introduction to relativity

Before the year 1905, it was assumed that universe was unbounded, infinite 3-dimensional space modeled by Euclidean geometry, which describes flat space. Thus, any event in the universe could be described by three spatial coordinates and time, generally written as (x, y, z) with the implied concept of an absolute time t.

In 1905, Albert Einstein introduced the Special Theory of Relativity in his paper 'On the Electrodynamics of Moving Bodies.' Special realtivity, as it is usually called, postulated two things. First, any physical law which is valid in one frame of reference is also valid for any other frame of reference moving uniformly relative to the first. Such a frame of reference is called Inertial reference frame. Second, the speed of light in vacuum is the same in all intertial reference frames, regardless of how the light source may be moving.

The first postulate implies there is no preferred set space and time coordinates. For instance, suppose you are sitting at rest in a car moving at constant speed. While looking straight out a side window, everything appears to be moving so quickly! Trees, buildings, and even people are flashing by faster than you can focus on them. However, an observer outside of your vehicle would say that you are the one who appears to be moving. In this case, how should we define the coordinates of you in your car and the observer outside of your car? We could say that the outside observer was simply mistaken, and that you were definitely not moving. Thus, his spatial coordinates were changing while you remained stationary. However, the observer could adamantly argue that you definitely were moving, and so it is your spatial coordinates that are changing. Hence, there is no absolute coordinate system that could describe every event in the universe for which all observers would agree and we see that each observer has their own way to measure distances relative to the frame of reference they are in. your car and the observer outside of your car? We could say that the outside observer was simply mistaken, and that you were definitely not moving. Thus, his spatial coordinates were changing while you remained stationary. However, the observer could adamantly argue that you definitely were moving, and so it is your spatial coordinates that are changing. Hence, there is no absolute coordinate system that could describe every event in the universe for which all observers would agree and we see that each observer has their own way to measure distances relative to the frame of reference they are in.

It is important to note that special relativity only holds for frames of reference moving uniformly relative to the other, that is, constant velocities and no acceleration. We can illustrate this with a simple example. Imagine a glass of water sitting on a table. According to special relativity, there is no difference in that glass sitting on a table in your kitchen and any other frame with uniform velocity, such as a car traveling at constant speed. The glass of water in the car, assuming a smooth, straight ride with no shaking, turning or bumps, will follow the same laws of physics as it does in your kitchen. In this case, the water in each glass is undisturbed within the glass as time goes on. However, if either reference frame underwent an acceleration, special relativity would no longer hold. For instance, if in your car, you were to suddenly stop, then the water in your glass would likely spill out and you would be forced forward against your seat belt.

1.1 Minkowski Space

Einstein's intuition motivated his formulation of special relativity, but his generalization to general relativity would not have occured without the mathematical formulation given by Hermann Minkowski. In 1907, Minkowski realized the physical notions of Einstein's special relativity could be expressed in terms of events occuring in a universe describe with a non-Euclidean geometry. Minkowski took the three dimensional space with an absolute time and transformed it into a four dimensional manifold that represented space-time. A manifold is a topological space that is described locally by Euclidean geometry; that is, around every point there is a neighborhood of surrounding points which is approximately flat. Thus, one can think of a manifold as a surface with many flat spacetimes covering it where all of the overlaps are smooth and continuous. A simple example of this is the Earth. Even though the world is known to be spherical, on small scales, such as those we see everyday, it appears to be flat.

We now formalize special relativity as follows, points in this space-time ('events') have coordinates x^μ $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$. The invariant distance, or 'seperation' between two events (in Cartesian coordinates), $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$, is written in the form

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$$

where the summation convention has been used: repeated indices are summed over the values $0, 1, 2, 3$. Thus above equation is short-hand for

$$ds^2 = \eta_{00}(dx^0)^2 + \eta_{01}dx^0dx^1 + \eta_{02}dx^0dx^2 + ...(16\ terms),$$

and $\eta\mu\nu$ has the following values, in Cartesian coordinates:

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

hence

$$\eta_{00} = 1, \quad \eta_{11} = \eta_{22} = \eta_{33} = -1, \quad \eta_{\mu\nu} = 0, \quad \mu \neq \nu;$$

or in matrix form

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The matrix $\eta_{\mu\nu}$ is referred to as the metric tensor for Minkowski space.

Special relativity was not extended to include acceleration until Einstein published 'The Foundation of the General Theory of Relativity' in 1916. Einstein desired to formulate gravity so that observers in any frame would agree on the definition, regardless of how they were moving in relation to each other. Einstein accomplished this by defining gravity as a curvature of spacetime rather than a force. We can then think of falling objects, planets in orbits, and rays of light as objects following the path in a curved spacetime know as geodesics, which are fully described in Section 1.3. As we will see, this then implies that gravity and acceleration are essentially equivalent.

Imagine a two dimensional flat world represented by a rubber sheet stretched out to infinity and imagine that a one dimensional line object of finite length inhabits this world. In this imaginary setting, there is no gravity, no notion of up or down, and there is no height dimension. The line object living in this world is flat, regardless of where or when it is. Now let some three dimensional being hit this flat world with a hammer, producing ripples in the rubber plane. As the ripples propagate through the region where the line object resides, they produce a geometric force pushing the line object with the curvature of the waves. That is, the line object would feel a force, and from our vantage point outside of the rubber sheet, we would see the line object bend and stretch. Similarly, the curvature in a four dimensional universe acts as a force that pushes three dimensional objects.

General relativity is often summarized with a quote by physicist John Wheeler:

"Spacetime tells matter how to move,
and matter tells spacetime how to curve."

The curvature of spacetime defines a gravitational field and that field acts on nearby matter, causing it to move. However, the matter, specifically it's mass, determines the geometric properties of spacetime, and thus it's curvature. So in general relativity, an object's position in spacetime is unaffected by the object's mass (assuming that it is not large enough to significantly alter the curvature of spacetime) and relies only on the geometry of the spacetime.

1.2 What is a black hole?

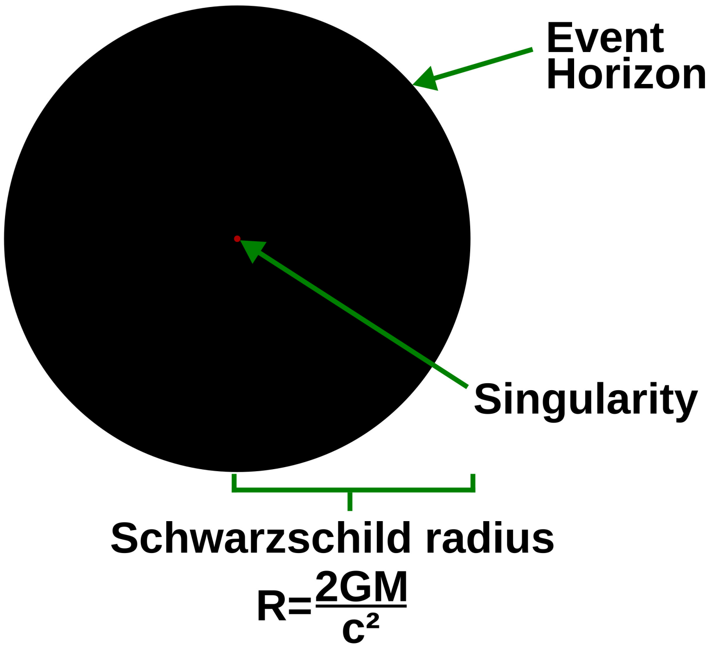

Figure 1: A simple illustration of a non-spinning black hole

Figure 1[1] is an illustration of a Black hole. A black hole is a region in space where the force of gravity is so strong, not even light, the fastest known entity in our universe, can escape. Idea of a body so massive that even light could not escape was proposed way back in the 18th century. Few months after Einstein developed his theory of general relativity, Karl Schwarzschild found a solution to the Einstein field equations that describe the gravitational field of a point mass and a spherical mass. The boundary of a black hole is called the event horizon, a point of no return, beyond which we trully cannot see. When something crosses the event horizon it collapses into the black hole's singularity, an infinitely small, infinitely dense point of spacetime where laws of physics don't apply.

A spacetime without mass can be interpreted as a large flat frictionless rubber sheet, similar to the surface of a trampoline. When a massive object is placed on the surface of rubber sheet, it flexes around the region of massive object. Now, if we slide a small spherical object towards the massive object, it either moves over the curved surface around the massive object and go out towards the flat surface or it moves on the curved surface, goes out and comes back towards the massive object. Which motion will occur depends on the escape velocity of the smaller object. Escape velocity is the velocity required by an object to leave the gravitational lock of a massive object. We know that a massive object with sufficiently high density exists so that no object can escape it's gravitational pull since nothing can travel faster than the speed of light. So, if an object is so dense that its escape velocity is greater than the speed of light, then nothing can escape the gravity of that object and it is called a black hole.

We can now take that geometrical interpretation and make a few predictions using classical Newtonian physics. The kinetic energy of an object of mass m with a velocity v is

[1] $https://upload.wikimedia.org/wikipedia/commons/8/82/Black\_hole\_details.svg$

given by

$$\frac{1}{2}mv^2$$

The gravitational potential energy for an object of mass M at a radius r is given by

$$-\frac{GMm}{r}$$

Then, for an object of mass m to escape from a mass M, it's kinetic energy must be greater than the magnitude of the gravitational potential energy. So if we had an object with a maximum velocity, that is $v = c$ and then set the kinetic energy for that equal to the magnitude of the gravitational potential energy, we would have a potential from which nothing could escape.

$$\frac{1}{2}mc^2 = \frac{GMm}{r}$$

We could then solve to find a radius r in terms of mass M for which nothing could escape.

$$r = \frac{2GM}{c^2}$$

We can set $c = G = 1$ known as geometrized units. We get

$$r = 2M$$

Thus, if a massive static object M is condensed into a spherical region with a radius r, as measured in mass, less than 2M, then that object is a black hole. As we will see later, this value of r, which we derived using classical arguments, happens to be the actual value for the Schwarzschild radius, which coincides with the event horizon of a static spherically symmetric black hole.

1.3 Geodesics and Christoffel Symbols

In general relativity, gravity is formulated as a geometric interpretation, and as such, we must discard the classical Newtonian view of gravity. Instead, we can think of an object in a gravitational field as travelling along a geodesic in the semi-Riemannian manifold that represents a 4-dimensional spacetime. A geodesic is commonly defined as the shortest distance between two points. We know that geodesic in a flat Euclidean space is a straight line between two points. However, in a 4-dimensional spacetime, it is not always that simple. In calculating the geodesic on a curved manifold, the curvature must be taken into account. For instance, let us return to our example of the bowling ball on the frictionless rub- ber sheet. Suppose we roll a marble towards the flexed region. Assume that the marble starts on a flat part of the surface, and that it does not run into the bowling ball. Then the marble would initially roll straight toward the flexed region, but upon entering the curvature, it would appear to bend with the surface and exit the region heading straight out in a different direction. This is analogous to the deflection of a comet's trajectory by the gravitational influence of the Sun. In this instance, the marble and the comet are both following "straight" paths on the curved surface which are both geodesics.

One way to describe geodesics is by a concept called parallel transport. In this description, a path is considered a geodesic if it parallel transports its own tangent vectors at all points on the path. The act of parallel transporting a tangent vector relies specifically on how the curvature changes from point to point. As such, there is a natural way in which we can define a geodesic based on the intrinsic properties of that curvature. Mathematically, we denote this as :

$$\frac{d^2x^\lambda}{d\rho^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\rho}\frac{dx^\nu}{d\rho} = 0 \tag{1}$$

which is called the geodesic equation. In the above equation, $\Gamma^\lambda_{\mu\nu}$ is a Christoffel symbol and is defined by :

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\beta}(\frac{\partial g_{\alpha\mu}}{\partial x^\nu} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\alpha}) \tag{2}$$

where $g_{\mu\nu}$ is a component of the metric tensor previously alluded to. In 4-dimensional spacetime, λ, μ and ν can be 0, 1, 2, or 3 and so, the above equation represents 64 values. However, in the specific case we will look at, most of those values will be zero. We'll return to these equations frequently in the next few sections, but let us consider this parallel transport idea without equations first. Let us begin with a geodesic in a flat geometry, which is a straight line. Then a tangent vector at any point on the line is identically parallel to the line, and thus trivially parallel to every other tangent vector. For curved geometry, let us make a simple analogy to riding in a car. In this case, suppose you are riding in the car and the direction of your gaze is a tangent vector. Now consider looking straight ahead as the car travels in a straight path. Your gaze does not change and is trivially parallel throughout. Now consider looking straight ahead while the car travels on a road with twists and turns. In this case, your gaze changes directions, but it is due to the change in direction of the car rather than the movement of your body. We could then find a relation between the change in your gaze and a change the car made. For example, if the car was initially going North and then turned East, we could simply "transport" your original gaze Eastward and it would be parallel to your gaze while the car headed East. Then by definition, we could consider your gaze to be the tangent vectors of a path that is a geodesic in a sense. You did not make any additional movements; your gaze was only altered due to the change in the car's direction. However, if we consider the case that the car travels in a straight path, but your gaze varies from left to right or any other way, then no matter how straight of a path the car took, your gaze would no longer represent the tangent vectors of a geodesic, because your gaze deviated without respect to the car's changing direction. Thus, a geodesic in a curved space is simply a path for which the tangent vectors only change due to the changing geometry of that space.

2 Einstein's Field Equations

In the introduction, we stated that Einstein formulated gravity as a geometry of spacetime (as seen in the Figure 2[2]). We now know that spacetime tells matter how to move, and matter tells spacetime how to curve. We even alluded to the metric tensor $g_{\mu\nu}$ and its role in characterizing the geometry of curved spacetime. However, we have not yet described in any detail in what way matter, and specifically mass, influences the curvature of spacetime. This relation will be described by Einstein's Field Equations.

[2]By Mysid - Own work. Self-made in Blender & $amp; Inkscape.$, CC $BY - SA$ 3.0, $https : //commons.wikimedia.org/w/index.php?curid =$ 45121761

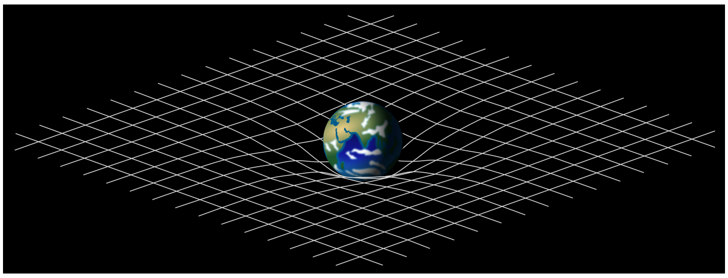

Figure 2: Spacetime curvature schematic

Einstein's Field Equations are given as follows :

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{3}$$

Here, $R_{\mu\nu}$ is defined as the Ricci Tensor and is given as :

$$R_{\mu\nu} = \frac{\partial \Gamma^{\lambda}_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \Gamma^{\beta}_{\mu\lambda}\Gamma^{\lambda}_{\nu\beta} - \Gamma^{\beta}_{\mu\nu}\Gamma^{\lambda}_{\beta\lambda} \tag{4}$$

Also, $T_{\mu\nu}$ is called the Energy-Momentum Tensor and Λ is called the cosmological constant. But for now we don't need to worry about the energy-momentum tensor and cosmological constant as we are considering a charge-free and local region of spacetime.

3 Derivation of the Schwarzschild Metric

The Schwarzschild metric is a spacetime metric for a mass that's spherically symmetric, non-rotating mass that has no electric charge. It's useful for describing the curvature of spacetime near non-rotating Black Holes. It predicts the existence of black holes with event horizon of size r_s, which is the Schwarzschild radius.

Now we are going to find the Schwarzschild solution to the Einstein Field Equation. So to solve the Einstein Field Equations we put in a Energy-Momentum Tensor $T_{\mu\nu}$ and solve for the metric $g_{\mu\nu}$ in equation (3). Hence, using the description of energy and momentum we can get a description of geometry of spacetime. Using the metric we can then solve the geodesic equation which will give us the paths of masses and light beams through curved spacetime. So we will know how gravity effects mass and light.

So the question is what type of Energy-Momentum Tensor will we use to get our spacetime solution for a spherically symmetric mass. Since we are trying to find solutions for a spherically symmetric, uncharged and non-rotating black hole, we can choose $T_{\mu\nu} = 0$. We are going to set the cosmological constant $\Lambda = 0$ since it's negligible unless we are working at a cosmological scale. Therefore, we get :

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 0 \tag{5}$$

Using inverse metric to raise an index :

$$R_{\mu\nu}g^{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}g^{\mu\nu} = 0 \tag{6}$$

$$R^{\mu}_{\mu} - \frac{1}{2}R\delta^{\mu}_{\mu} = 0 \tag{7}$$

Since the trace of the 4×4 Identity matrix is 4 and $R^{\mu}_{\mu} = R$ is the Ricci Scalar. We get :

$$R - 4 \times \frac{1}{2}R = 0 \tag{8}$$

$$R - 2R = 0 \tag{9}$$

$$R = 0 \tag{10}$$

So for vacuum region Einstein's Field Equation simplify to :

$$R_{\mu\nu} = 0 \tag{11}$$

This implies that Ricci Tensor is zero. This we call "Ricci flat" space-time. Now we will take the above equation and solve for components of 4×4 space time metric :

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ g_{xt} & g_{xx} & g_{xy} & g_{xz} \\ g_{yt} & g_{yx} & g_{yy} & g_{yz} \\ g_{zt} & g_{zx} & g_{zy} & g_{zz} \end{pmatrix}$$

$$g_{\mu\nu} = g_{\nu\mu}$$

This describes the curved space time near a body of mass M. We also assume that as we move far away from the mass the effect of gravity becomes negligible and space time becomes flat :

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

And in spherical coordinates the flat Minkowski metric looks as follows :

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2(sin\theta)^2 \end{pmatrix}$$

Close to the mass, metric components of the curved space time are unknown (In spherical coordinates) :

$$\begin{pmatrix} g_{tt} & g_{tr} & g_{t\theta} & g_{t\phi} \\ g_{rt} & g_{rr} & g_{r\theta} & g_{r}\phi \\ g_{\theta t} & g_{\theta r} & g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi t} & g_{\phi r} & g_{\phi\theta} & g_{\phi\phi} \end{pmatrix}$$

Now we can make use of some assumptions to narrow down the exact form the metric components should take. First let's assume that the space time is static meaning that the metric doesn't depend on time and metric is symmetric when we reverse the coordinate :

- $\partial_t g_{\mu\nu} = 0$
- $t \longrightarrow -t$ *doesn't change* $g_{\mu\nu}$

This is like mirror symmetry across the time axis. This guarantees that the black hole isn't rotating, because if we reverse time for a rotating blackhole the gravitational effects due to rotation would also reverse direction. Since basis vectors are just $\vec{e_t} = \frac{\partial}{\partial ct}$, then reversing the time coordinate also reverses the basis, i.e. $\vec{e_t} = \frac{\partial}{\partial ct} = \frac{\partial}{\partial c(-t)} = -\frac{\partial}{\partial ct} = -\vec{e_t}$. This does not change the g_{tt} components since $g_{tt} = \vec{e_t} \cdot \vec{e_t} \longrightarrow (-\vec{e_t}) \cdot (-\vec{e_t}) = +g_{tt}$. But it will change the g_{ti} components, i.e. $g_{ti} = \vec{e_t} \cdot \vec{e_i} \longrightarrow (-\vec{e_t}) \cdot \vec{e_i} = -g_{ti}$. Since, $g_{ti} = -g_{ti}$ these metric components go to zero, i.e. $g_{ti} = 0$. If we want spherical symmetry in space the θ and ϕ components should resemble the matrix of sphere of radius r. However, we also allow multiplication by a radial function $C(r)$ since this doesn't violate the sherical symmetry. We are also going to make these terms negative since spacelike vectors have negative metric components in the mostly minus metric convention. Hence, we have a metric that looks as follows :

$$\begin{pmatrix} g_{tt} & 0 & 0 & 0 \\ 0 & g_{rr} & g_{r\theta} & g_{r}\phi \\ 0 & g_{\theta r} & -C(r)r^2 & 0 \\ 0 & g_{\phi r} & 0 & -C(r)r^2(sin\theta)^2 \end{pmatrix}$$

Ifwe want the radial basis vector $\vec{e_r}$ to stick out normal to the sphere in the radial direction it must be perpendicular to $\vec{e_\theta}$ and $\vec{e_\phi}$, i.e. $\vec{e_\theta} \cdot \vec{e_r} = 0 = g_{\theta r}$ and $\vec{e_\phi} \cdot \vec{e_r} = 0 = g_{\phi r}$. So these dot products and metric compnents go to zero. So under our assumptions so far the metric is diagonal and the remaining g_{tt} and g_{rr} components should only depend on the radial coordinate r if we want to maintain spherical symmetry. So we will call them $A(r)$ and $-B(r)$ (negative sign since this metric component corresponds to a spacelike direction). Hence, the metric will be given as :

$$\begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -C(r)r^2 & 0 \\ 0 & 0 & 0 & -C(r)r^2(sin\theta)^2 \end{pmatrix}$$

To simplify we can define $\tilde{r} = \sqrt{C(r)}r$, therefore we get

$$\begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -\tilde{r}^2 & 0 \\ 0 & 0 & 0 & -\tilde{r}^2(sin\theta)^2 \end{pmatrix} \tag{12}$$

3.1 Evaluation of Christoffel Symbols

So now we have simplified the metric as much as possible. in order to solve for A(r) and B(r), we have to calculate the connection coefficients, calculate the Ricci tensor components and then force the metric to give us the results of Newtonian gravity in the limit of low velocity and weak gravity and this will give us the Schwarzschild metric. So let's start by calculating the connection coefficients. There are 13 non-zero connection

coefficients in the Schwarzschild solution and only nine of them are independent.
So here is the standard formula for connection coefficients :

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\alpha} \left(\frac{\partial g_{\alpha\mu}}{\partial x^{\nu}} + \frac{\partial g_{\alpha\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\alpha}} \right)$$

Since the metric is diagonal (from equation (12)) :

$$g_{\mu\nu} = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2(sin\theta)^2 \end{pmatrix}$$

we can easily get the inverse metric just by taking the reciprocal of all of the diagonal elements denoted by :

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{A(r)} & 0 & 0 & 0 \\ 0 & -\frac{1}{B(r)} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2(sin\theta)^2} \end{pmatrix}$$

So Now we know that our metric is diagonal it's two indices always needs to match if the components are to be non-zero, i.e. $g_{\mu\nu} = 0$ for $\mu \neq \nu$. In the standard form of connection coefficient we substitute $\alpha = \sigma$, such that we get :

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\sigma} \left(\frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} + \frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right) \tag{13}$$

Using $g_{\mu\nu}$ and $g^{\mu\nu}$ we can calculate the values of non-zero connection coefficients. They will be as follows :

- $\Gamma^0_{01} = \Gamma^0_{10} = \frac{1}{2}\frac{1}{A}(\partial_r A)$
- $\Gamma^1_{00} = \frac{1}{2}\frac{1}{B}(\partial_r A)$
- $\Gamma^1_{11} = \frac{1}{2}\frac{1}{B}(\partial_r B)$
- $\Gamma^1_{22} = -\frac{r}{B}$
- $\Gamma^1_{33} = -\frac{r(sin\theta)^2}{B}$
- $\Gamma^2_{33} = -sin\theta cos\theta$
- $\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$
- $\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$
- $\Gamma^3_{23} = \Gamma^3_{32} = cot\theta$

3.2 Ricci Tensor Components

The Ricci tensor represents how a volume in a curved space differs from a volume in Euclidean space. In particular, the Ricci tensor measures how a volume between geodesics changes due to curvature. In general relativity, the Ricci tensor represents volume changes due to gravitational tides. So now with the new found values of connection coefficients we are going to find the value of Ricci Tensor $R_{\mu\nu}$. Now we need to calculate the values of the following equations :

- $R_{00} = 0$
- $R_{11} = 0$
- $R_{22} = 0$

Then we will see that the above three equations are enough to solve for our $A(r)$ and $B(r)$ functions. We know that Ricci tensor is just the Riemann Tensor with it's upper and lower indices summed together. And Riemann Tensor is given as follows :

$$R^{\rho}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}_{\nu\sigma} - \partial_{nu}\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\alpha}_{\nu\sigma}\Gamma^{\rho}_{\mu\alpha} - \Gamma^{\beta}_{\mu\sigma}\Gamma^{\rho}_{\nu\beta}$$

So for R_{00} we get :

$$R_{00} = R^{\mu}_{0\mu0} = \partial_{\mu}\Gamma^{\mu}_{00} - \partial_{0}\Gamma^{\mu}_{\mu0} + \Gamma^{\alpha}_{00}\Gamma^{\mu}_{\mu\alpha} - \Gamma^{\beta}_{\mu0}\Gamma^{\mu}_{0\beta}$$

Using the connection coefficient values we get[3] :

$$R_{00} = 2rABA'' - rAA'B' + 4ABA' - rB(A')^2 = 0$$

Here, $\partial_r A = A'$ and $\partial_r B = B'$. Similarly for R_{11} we get :

$$R_{11} = R^{\mu}_{1\mu1} = \partial_{\mu}\Gamma^{\mu}_{11} - \partial_{1}\Gamma^{\mu}_{\mu1} + \Gamma^{\alpha}_{11}\Gamma^{\mu}_{\mu\alpha} - \Gamma^{\beta}_{\mu1}\Gamma^{\mu}_{1\beta}$$

$$R_{11} = -2rABA'' + rB(A')^2 + rAA'B' + 4A^2B' = 0$$

And for R_{22} we get :

$$R_{22} = R^{\mu}_{2\mu2} = \partial_{\mu}\Gamma^{\mu}_{22} - \partial_{2}\Gamma^{\mu}_{\mu2} + \Gamma^{\alpha}_{22}\Gamma^{\mu}_{\mu\alpha} - \Gamma^{\beta}_{\mu2}\Gamma^{\mu}_{2\beta}$$

$$R_{22} = -2AB + 2AB^2 - rA'B + rAB' = 0$$

3.3 Solving for $A(r)$ and $B(r)$

From the values of Ricci Tensors calculated in the above section we are going to deduce the values of $A(r)$ and $B(r)$. We know that the values of R_{00}, R_{11} and R_{22} are zero. Hence we can say that

$$R_{00} + R_{11} = 0$$

This implies that,

$$4ABA' + 4A^2B' = 0$$

[3] Calculations of each Ricci Tensor component has been skipped as it is basic calculus and algebra which one can check for themselves.

On further solving we get :

$$BA' + AB' = 0$$

which is the product rule :

$$\partial_r(AB) = 0$$
$$\Rightarrow AB = K$$

Here, K is some constant. Now we know that as the Schwarzschild metric goes out to $r \to \infty$ it will approach the flat Minkowski metric, where $A(r) = 1$ and $B(r) = 1$. So in that limit $A(r) \to 1$ and $B(r) \to 1$

$$\Rightarrow K = 1$$

and since $K = 1$ for all r values, this means

$$B(r) = \frac{1}{A(r)}; \quad \forall r$$

So knowing the R_{22} formula and the above result and that $B' = \partial_r(A^-1) = -\frac{A'}{A^2}$ we can substitute in for B and B', we will get :

$$R_{22} = -2AB + 2AB^2 - rA'B + rAB'$$

$$0 = -2A\frac{1}{A} + 2A(\frac{1}{A})^2 - rA'\frac{1}{A} + rA(-\frac{A'}{A^2})$$

$$\Rightarrow rA' = 1 - A$$

Solving the differential equation we get :

$$A(r) = 1 - \frac{k}{r}$$

Now we have solved for $A(r)$ and $B(r)$, so we have the Schwarzschild metric as :

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{k}{r} & 0 & 0 & 0 \\ 0 & -(1-\frac{k}{r})^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2(sin\theta)^2 \end{pmatrix}$$

Now we need to solve for the constant k which turns out to be

$$k = \frac{2GM}{c^2}$$

. This is called the Schwarzschild Radius a.k.a. the Event Horizon of the black hole. Hence we can write the above metric as follows :[4]

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{2M}{r} & 0 & 0 & 0 \\ 0 & -(1-\frac{2M}{r})^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2(sin\theta)^2 \end{pmatrix}$$

[4]We have considered G=c=1

or

$$ds^2 = (1 - \frac{2M}{r})dt^2 - \frac{dr^2}{(1 - \frac{2M}{r})} - r^2 d\phi^2 - r^2 sin^2\phi d\theta^2 \tag{14}$$

Here, is the Schwarzschild Metric which defines the way we measure invariant intervals around a mass M in a static spherically symmetric spacetime.

4 Conclusion

In conclusion, we briefly discussed about the underlying principles of general relativity and the equations governing the gravitational field, i.e. Einstein's Field Equations, in a local atlas. And using all the information we derived the General Relativistic Schwarzschild metric, which governs the geometry of space-time around a spherically symmetric, uncharged and non-rotating mass. One drawback of Schwarschild metric is that it doesn't hold when $r \leq 2M$. Therefore we have to go to some other coordinates which explains the geometry of space-time on and inside the horizon too. Such coordinate transformations can be easily done and one of the most famous coordinates which explain the geometry of space-time on the inside as well as outside of a mass (Black hole) are known as Eddington-Finkelstein coordinates.
Similarly one can go about constructing different metrics for spherically symmetric, rotating, charged masses.

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