Problem: 2

→ Consider a moon lander with state $[h, v, m]^T$ to have following dynamics.

$$\dot{h}(t) = v(t)$$
$$\dot{v}(t) = -g + a(t)/m(t)$$
$$\dot{m}(t) = -K a(t).$$

here h is the altitude, v is the velocity, and m is mass of the moon lander $a(t) \in [0,1]$ is thrust, and K is a constant fuel burning rate. Let the initial state be $[h_0, v_0, m_0]^T$ and target $h(t^*) = 0$ $v(t^*) = 0$

→ Some aditional constraints that

$h(t) \geq 0$

$m(t) \geq 0.$

The goal here is to minimize the fuel consumption which mean we need to maximizing the mass

$$\max m(T)$$

where T is the first time $h(T) = v(T) = 0$

we can see $a(t) = -\dfrac{\dot{m}(t)}{K}$, so if we minimize total applied thrust before landing is as equal as the maximizing the mass of moon lander which give as the minimal fuel consumption, so,

$$\min_{a(t)} \int_0^T a(t) \cdot dt = \frac{m_0 - m(T)}{K}$$

In, terms of general notation the state vectors

$$f = \begin{bmatrix} v \\ -g + g/m \\ -Ka \end{bmatrix} \qquad l = a$$

Hence the Hamiltonian

$$H = -L + \lambda^T f \Rightarrow -a + \lambda_1 v + \lambda_2 \left(-g + \frac{a}{m}\right) + \lambda_3 (-Ka)$$

$$a^* = \arg\max_{a \in [0,1]} H$$

$$a^* = \arg\max_{a \in [0,1]} \left(-1, \frac{\lambda_2}{m} - \lambda_3 K\right) a + \lambda_1 v - \lambda_2 g$$

Thus the

$$a(t) = \begin{cases} 0 & b \leq 0 \\ 1 & b > 0 \end{cases}$$

where $b = -1 + \dfrac{\lambda_2}{m} - \lambda_3 K$

the guess policy $a(t) = \begin{cases} 0 & \text{for } t \in [0, t^*) \\ 1 & \text{for } t \in [t^*, T] \end{cases}$

In order to prove the guess of optimal policy we need to show that the b is either monotonically increasing or descreasing

so $\dot{b} = \dfrac{\dot{\lambda}_2}{m} - \dfrac{\lambda_2 \dot{m}}{m^2} - \dot{\lambda}_3 K$

and we know acc^n to adjoint equation

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial h} = 0 \qquad \dot{\lambda}_2 = -\frac{\partial H}{\partial v} = -\lambda_1$$

$$\dot{\lambda}_3 = -\frac{\partial H}{\partial m} = \frac{\lambda_2 a}{m^2}$$

putting all the λ' value in $\dot{b}$

$$\dot{b} = \frac{-\lambda_1}{m} - \frac{\lambda_2}{m^2}(-Kq) - \left(\frac{\lambda_2 q}{m^2}\right)K$$

$$\boxed{\dot{b} = \frac{-\lambda_1}{m}}$$

(A)

=) Now as we have switching condition when to engine (thrust start or when to stop acc^n to our guess, so here first I defined the dynamics when the engin is off no thrust, $t \in (0, t^*)$

thus, the dynamics become

$$\dot{h}(t) = v(t)$$
$$\dot{v}(t) = -g$$
$$\dot{m}(t) = 0$$

here $\dot{m}(t) = 0 \Rightarrow \int_0^{t^*} \dot{m}(t)\,dt = \int_0^{t^*} 0\,dt$

$$m = 0 + c_1$$
$$m = c_1$$

and we know at starting $m = m_0$

and since there is no thrust untill t^*

so at t^* $m = m_0$, so $c_1 = m_0$

$$\boxed{m = m_0}$$

$$\dot{v} = -g$$

$\int \dot{v} = \int -g$

$v = -gt + C_2$

we know at time $t=0$ $v = v_0$

$\boxed{C_2 = v_0}$ $\boxed{v = -gt + v_0}$

$\dot{h} = v$

$\int \dot{h} = \int v = \int -gt + v_0$

$h = -g\frac{t^2}{2} + v_0 t + C_3$

at $t=0$ $h = h_0$ $h_0 = C_3$

$\therefore h = -\frac{1}{2}gt^2 + tv_0 + h_0$

So,

$$\begin{cases} h_{free}(t) = -\frac{1}{2}gt^2 + tv_0 + h_0 \\ v_{free}(t) = -gt + v_0 \\ m_{free}(t) = m_0 \end{cases}$$

now we are considering the another condition when $a=1$, mean engine is on

$\therefore$ the following dynamics at $t \in [t^*, T]$

$$\begin{cases} \dot{h}(t) = v(t) \\ \dot{v}(t) = -g + 1/m(t) \\ \dot{m}(t) = -K \end{cases}$$

$\int \dot{m}(t) = \int -K$

$m = -Kt + C_1$

(3)

we know at $t = t^*$, $m = m_0$

$$m_0 = -kt^* + c_1$$

$$c_1 = m_0 + kt^*$$

$$m = m_0 + k(-t + t^*)$$

$$\boxed{m = m_0 + k(t^* - t)}$$

$$\int \dot{v} = \int -g + \frac{1}{m}$$

$$\int \dot{v} = \int -g + \frac{1}{m_0 + k(t^* - t)}$$

$$v = -gt + \frac{\log(m_0 + k(t^* - t))}{k} + c_1$$

we know at $t = T$, $v(T) = 0$

$$0 = -g(T) + \frac{\log(m_0 + k(t^* - T))}{k} + c_1$$

$$c_1 = gT - \frac{\log(m_0 + k(t^* - T))}{k}$$

so $$v = -gt + \frac{\log(m_0 + k(t^* - t))}{k} + g(T) - \frac{\log(m_0 + k(t^* - T))}{k}$$

$$v = g(T - t) + \frac{1}{k}\log\left[\frac{m_0 + k(t^* - T)}{m_0 + k(t^* - t)}\right]$$

$$\dot{h} = v$$

$$\int \dot{h} = \int v$$

$$h = \int g(T - t) + \frac{1}{k}\log\left[\frac{m_0 + k(t^* - T)}{m_0 + k(t^* - t)}\right]$$

$$g\left(\tau t - \frac{t^2}{2}\right) + \frac{1}{K} B$$

→ For B part

$$B = \int \log\left[\frac{(m_0 + K(t^* - \tau))}{m_0 + K(t^* - t)}\right]$$

$$= \int \log(m_0 + K(t^* - \tau)) - \log(m_0 + K(t^* - t))$$

$$= \log(m_0 + K(t^* - \tau))\, t - \left(-t - \frac{\log(K(t^* - t) + m_0)\, y_1 K(t^* - t) + m_0}{K}\right)$$

$\to C_1$

→ Due to limited space in my note book calculate it on side and of ton calculation

$$h = g\left(\tau t - \frac{t^2}{2}\right) + \left(-\frac{\tau - t}{K} - \frac{1}{K^2}\log\left(\frac{A}{e}\right)(m_0 - K(t - t^*)) - \frac{1}{2}\right)\tau^2$$

where $A = m_0 + K(t^* - \tau)$
$e = m_0 + K(t^* - t)$

So
$$\begin{cases} h_p = g\left(\tau t - \frac{t^2}{2}\right) - \frac{\tau - t}{K} - \frac{1}{K^2}\log\left(\frac{A}{e}\right)(m_0 + K(t^* - t)) - \frac{1}{2} g \tau^2 \\ v_p = g(\tau - t) + \frac{1}{K}\log\left[\frac{m_0 + K(t^* - \tau)}{m_0 + K(t^* - t)}\right] \\ m_p = m_0 + K(t^* - t) \end{cases}$$

now let put $t = t^*$ in equation when $d \cdot l$ an $a = 0$

$$h_{free}(t^*) = -\frac{1}{2} g t^{*2} + t^* v_0 + h_0$$

$$v_{free}(t^*) = -g t^* + v_0$$

$$m_{free}(t^*) = m_0$$

when $u = 1$

$$h_p(t^*) = -g(t^* - t^2) + \frac{t^* - T}{K} - \frac{m_0}{K^2}\log\left(\frac{m_0 + K(t^* - t)}{m_0}\right)$$

$$v_p(t^*) = g(T - t^*) + \frac{1}{K}\log\left[\frac{m_0 + K(t^* - T)}{m_0}\right]$$

$$m_p(t^*) = m_0$$

By using the above equations equatating the $v_{free}(t^*) = v_p(t^*)$ and $h_{free}(t^*) = h_p(t^*)$ we can determine the t^* and T value

→ now we can vizulize the system as

→ now we can return back to the b^0 equation

$$b^0 = -\frac{\lambda_1}{m}$$

→ if we consider the λ is a -ve so the b is monatonically increasing function. because b^0 is become +ve.

→ The reason λ is not choose +ve because we can not use thrust before the switching point if this is a case so the velocity at the end can not be zero

$$\text{so } u(t) = \begin{cases} 1 & \text{if } b > 0 \text{ on } (t^*, T) \\ 0 & \text{if } b < 0 \text{ on } [0, t^*) \end{cases}$$

→ earlier guess of $\alpha(t)$ does indeed the pontry again, maximum principle.

→ also the optimal control just change once from a to 1, and the $b<0$ on $[0, t^*)$ $b>0$ $(t^*, T]$.