CS 3510 HW#1
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1.

a) Using the definition of Big-O, if $f(n) \in O(g(n))$, there exists a c, $n_0 > 0$ such that $f(n) \leq c g(n)$ for all $n \geq n_0$. Similarly, $g(n) \leq d\, h(n)$
Given that $f(n) \leq c g(n)$ and $g(n) \leq d\, h(n)$, it follows

$$f(n) \leq c d\, h(n)$$
$$f(n) \leq c' h(n)$$
$$f(n) \leq O(h(n))$$

b. $\Theta(n \log n^2)$ → [n → complexity of innermost loop being incremented by constant
log n → complexity of both outer loops being multiplied and divided by constant
$n \times \log n \times \log n$]

c. For $f(n) \geq \Omega(g(n))$, there exist c_0, n_0 such that $f(n) \geq c_0 g(n)$ for all $n > n_0$
For $f(n) \leq O(g(n))$, there exist c_1, n_1 such that $f(n) \leq c_1 g(n)$ for all $n \geq n_1$
Thus, the statement $f(n) \leq O(g(n))$, $f(n) \geq \Omega(g(n))$ holds if $f(n) = g(n)$

d. We know that
$$(\log n)^a <= n^b \quad \text{for any } a \text{ and } b > 0$$
And,
$$n^2 \leq O(n^2)$$
Combining the above with $(\log n)^{10} \leq O(n^{0.1})$, where $a = 10$, $b = 0.1$
we get $n^2 \log^{10} n \leq O(n^{2.1})$

e.
$$2^{2n} \leq O(2^n) \rightarrow 2^{2n} \leq c 2^n$$
$$\ln 2 \cdot 2n \leq \ln c + \ln 2n$$
$$2n \leq \ln c + n$$
$$n \leq \ln c$$
But we know that $n > \ln c$, so the statement is not true

2.

a) $T(n) = 3T(n/4) + O(n)$

$$\frac{a}{b^d} = \frac{3}{4} < 1 \rightarrow \text{case } 1 \rightarrow \Theta(n)$$

b) $T(n) = 8T(n/4) + O(n^{1.5})$

$$\frac{a}{b^d} = \frac{8}{4^{1.5}} = 1 \rightarrow \text{case } 2 \rightarrow \Theta(\log n)$$

c) From the given problem statement we get

$$T(n) = 3T(n/\cancel{4}) + O(n^0)$$

$$\frac{a}{b^d} = 3 > 1 \rightarrow n^{\log_4 3}$$

d) From the given problem statement we get

$$T(n) = 2^n T(n/2) + O(n^1)$$

Master theorem doesn't apply directly since a is not constant.

3.

a) Bozzsort → $T(n) = 3T(n/5) + \Theta(1)$

$\frac{a}{b^d} = \frac{3}{5^0} \cancel{=} > 1$

↳ Case 3 → $n^{\log_5 3}$

O(1) → Bubble sort for constant n

$3T(n/5)$ → 3 sub problems of size n/5

b) Example → 7, 8, 9, 1, 2

First sort → 7 8 9 1 2

2^{nd} → 7 8 1 2 9

last → 1 7 8 2 9

4.

a) Given that the coefficient for x^i for multiplying degree n polynomials p, q with coefficients $a_0 \dots a_n$, $b_0 \dots b_n$ is

$$\sum_{\max\{0, i-n\} \le j \le \min\{i, n\}} a_j \cdot b_{i-j}$$

For x^n

$$\rightarrow \sum_{0 \le j \le n} a_j \, b_{i-j}$$

Given that the range of coefficients is $[0, n]$, the range of coefficients in the product will be

$$[0, O(n^3)]$$

min $\rightarrow 0 \rightarrow$ when all a_s and b_s are $0 \rightarrow 0$

max $\rightarrow n^3 \rightarrow$ when all a_s and b_s are $n \rightarrow$ For $x^n \rightarrow n$ terms of n^2

n^3

c) We know that coefficients for product of polynomials are

$$\sum_{\max\{0, i-d\} \le j \le \min\{i, d\}} a_j \cdot b_{i-j}$$

For x^{n-s} $\rightarrow$

$$\sum_{\max\{0, i-n+s\} \le j \le \min\{i, n-s\}} a_j \cdot b_{i-j} \rightarrow \sum a_j \, b_{n-s-j}$$

there $a_0 = y_0 \quad a_i = y_i \quad \dots \quad a_n = y_n$

$b_0 = z_n \quad b_1 = z_{n-1} \quad \dots \quad b_n = z_0 \qquad b_{n-s} = z_s$

$$\rightarrow \sum_{0 \le j \le n-s} y_j \, z_{j+s}$$

which is equal to the s shifted dot product of $y_0 \dots y_n$, $z_0 \dots z_n$

$$\sum_{j=0}^{n-s} y_j \, z_{j+s}$$

b) Taking two polynomials p and q, we can divide each of them into two parts, $A + Bx$ and $C + Dx$ with A and C containing even power terms and B containing odd powers

Now for $(A + Bx)(C + Dx)$ we can just apply the original Karatsuba algorithm to get

$$T(n) = 3T(n/2) + O(n)$$

$$\text{with } T(n) = O(n^{\log_2 3}) = O(n^{1.585})$$

(case 3 of the master theorem)