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6. Find all the values of $(1+i)^{2/3}$ and find the continued product of these values. (M.U. 1992)

[Ans. : $2^{1/3}\left(\cos\dfrac{8\pi k+\pi}{6}+i\sin\dfrac{8\pi k+\pi}{6}\right)$, $k=0, 1, 2$. Product $=2i$.]

7. Find all the values of $(1+i)^{1/3}$.

[Ans. : $2^{1/6}\left(\cos\dfrac{r\pi}{12}+i\sin\dfrac{r\pi}{12}\right)$ where $r=1, 9, 17$.]

8. Find the four fourth roots of unity. [Ans. : $\pm 1, \pm i$]

9. Solve the equations

(i) $x^9+8x^6+x^3+8=0$

(ii) $x^5+1=0$ (M.U. 2005)

(iii) $x^4-x^3+x^2-x+1=0$ (M.U. 1988, 2003)

(iv) $(x+1)^8+x^8=0$ (M.U. 1990, 97)

(v) $x^3=i(x-1)^3$ (M.U. 2008)

(vi) $x^6-x^5+x^4-x^3+x^2-x+1=0$ (M.U. 2012)

[Ans. : (i) $\cos(2k+1)\dfrac{\pi}{6}+i\sin(2k+1)\dfrac{\pi}{6}$, $k=0, 1, 2, 3, 4, 5$ and

$$2^{1/3}\left[\cos(2k+1)\frac{\pi}{3}+i\sin(2k+1)\frac{\pi}{3}\right],\quad k=0, 1, 2.$$

(ii) $\cos(2k+1)\dfrac{\pi}{5}+i\sin(2k+1)\dfrac{\pi}{5}$, $k=0, 1, 2, 3, 4$.

(iii) Multiply by $x+1$ $\therefore x^5+1=0$.

Answer as above, *i.e.*, (ii).

(iv) $$x=\frac{1}{[\cos(2k+1)(\pi/8)+i\sin(2k+1)(\pi/8)-1]}$$

where $k=0, 1, 2, 3, 4, 5, 6, 7$.

(v) $$x=\frac{\cos(4k+1)\pi/6+i\sin(4k+1)\pi/6}{\cos(4k+1)\pi/6+i\sin(4k+1)\pi/6-1}.$$

(vi) Multiply by $x+1$; $x^7+1=0$

$$\cos(2k+1)\frac{\pi}{7}+i\sin(2k+1)\frac{\pi}{7};\quad k=0, 1, 2, 3, 4, 5, 6.\ ]$$

10. Show that the continued product of all the values of

(a) $i^{2/3}$ is -1, (b) $(-i)^{2/3}$ is -1

11. If one root of $x^4-6x^3+15x^2-18x+10=0$ is $1+i$, find all other roots.

[Ans. : $1-i$, $2\pm i$]

12. If one root of $x^4-6x^3+18x^2-24x+16=0$ is $1+i$, find all other roots.

[Ans. : $1-i$, $2(1\pm i)$]

13. Solve the equations

(i) $x^{12} - 1 = 0$ **(M.U. 2003)**

(ii) $x^7 + x^4 + x^3 + 1 = 0$ **(M.U. 1988, 95, 2002)**

(iii) $x^7 - x^4 + x^3 - 1 = 0$

(iv) $x^7 - x^4 - x^3 + 1 = 0$

(v) $x^7 + x^4 + i x^3 + i = 0$ **(M.U. 1999)**

(vi) $x^9 - x^5 + x^4 - 1 = 0$ **(M.U. 2000, 01)**

(vii) $x^7 + 64x^4 + x^3 + 64 = 0$ **(M.U. 1985, 95)**

[**Ans. :** (i) $\pm 1, \pm i, \pm\left(\cos\frac{\pi}{6} \pm i\sin\frac{\pi}{6}\right), \pm\left(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}\right)$

(ii) $\pm\left(\frac{1}{\sqrt{2}} \pm i \cdot \frac{1}{\sqrt{2}}\right), \frac{1}{2} \pm i \cdot \frac{\sqrt{3}}{2}, -1$

(iii) $1, -\frac{1}{2}(1 \pm i\sqrt{3}), \pm\left(\frac{1}{\sqrt{2}} \pm i \cdot \frac{1}{\sqrt{2}}\right)$

(iv) $\pm 1, \pm i, -\frac{1}{2}(1 \pm i\sqrt{3})$

(v) $\pm\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right), \pm\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right), \frac{1}{2} \pm i\frac{\sqrt{3}}{2}, -1$

(vi) $\cos(2k+1)\frac{\pi}{5} + i\sin(2k+1)\frac{\pi}{5}; \quad k = 0, 1, 2, 3, 4$

$\cos\frac{2k\pi}{4} + i\sin\frac{2k\pi}{4}, \quad nk = 0, 1, 2, 3.$

(vii) $-4, \; 2(1 \pm i\sqrt{3}), \pm\left(\frac{1}{\sqrt{2}}\right)(1 \pm i)$.]

14. Find all the value of $\left(\frac{2+3i}{1+i}\right)^{1/4}$. **(M.U. 1985)**

[**Ans. :** $(13)^{1/4}\left[\cos\left(\frac{2\pi k + \theta}{4}\right) + i\sin\left(\frac{2\pi k + \theta}{4}\right)\right]$

where, $\theta = \tan^{-1}(1/5), \quad k = 0, 1, 2, 3.$]

15. If $\alpha, \alpha^2, \alpha^3, \ldots, \alpha^6$ are the roots of $x^7 - 1 = 0$, find them and prove that

$(1-\alpha)(1-\alpha^2) \ldots (1-\alpha^6) = 7.$ **(M.U. 2010)**

16. Use De Moivre's Theorem to solve the equation

(i) $x^4 - x^2 + 1 = 0$ **(M.U. 1996)**

(ii) $x^4 + x^2 + 1 = 0$

[**Ans. :** (i) $\cos(2k+1)\frac{\pi}{6} + i\sin(2k+1)\frac{\pi}{6}, \quad k = 0, 1, 2, 3, 4, 5.$

(ii) $x = \cos\frac{2k\pi}{6} + i\sin\frac{2k\pi}{6}, \quad k = 0, 1, 2, 3, 4, 5.$]

17. Solve $x^{14} + 127x^7 - 128 = 0$. **(M.U. 1998)**

[**Ans. :** $x = 2\left[\cos(2k+1)\dfrac{\pi}{7} + i\sin(2k+1)\dfrac{\pi}{7}\right]$, $x = 0, 1, 2, 3, 4, 5, 6$

and $x = \cos\dfrac{2k\pi}{7} + i\sin\dfrac{2k\pi}{7}$, $k = 0, 1, 2, 3, 4, 5, 6.$]

18. Show that the roots of $(x-1)^5 = 32(x+1)^5$ are given by

$$x = \left(-3 + 4i\sin\frac{2n\pi}{5}\right) \Big/ \left(5 - 4\cos\frac{2n\pi}{5}\right).$$

19. Find all the roots of $x^{12} - 1 = 0$ and identify the roots which are also the roots of $x^4 - x^2 + 1 = 0$.

[**Ans. :** $x = \cos(2m+1)\dfrac{\pi}{6} + i\sin(2m+1)\dfrac{\pi}{6}$,

$m = 0, 2, 3, 5$ give common roots.]

20. If $(1+x)^6 + x^6 = 0$ show that $x = -\dfrac{1}{2} - i\cot\dfrac{\theta}{2}$ where $\theta = (2k+1)\dfrac{\pi}{6}$, $k = 0, 1, 2, 3, 4, 5$. **(M.U. 2003)**

21. If $(x+1)^6 = x^6$, show that $x = -\dfrac{1}{2} - i\cot\dfrac{\theta}{2}$, where $\theta = \dfrac{2k\pi}{6}$, $k = 0, 1, 2, 3, 4, 5$.

22. Show that the continued product of all the values of

(i) $(1+i)^{1/8}$ is $-(1+i)$ (ii) $(1-i)^{1/8}$ is $-(1-i)$

(iii) $(1+i)^{1/5}$ is $1+i$ (iv) $(1-i)^{1/5}$ is $1-i$.

23. Find the cube roots of unity. Prove further that if α, β are complex roots then $\alpha^{3n} + \beta^{3n} = 2$ where n is any integer.

24. Find the cube roots of $1 - \cos\theta - i\sin\theta$.

[**Ans. :** $\left(2\sin\dfrac{\theta}{2}\right)^{1/3}\left[\cos\left(\dfrac{(2n-1)\pi - \theta}{6}\right) + i\sin\left(\dfrac{(2n-1)\pi - \theta}{6}\right)\right]$

25. Find all the roots of the equation $z^n = (z+1)^n$ and show that the real part of all the roots is $-1/2$.

6. Use of Exponential Form of a Complex Number

We shall now solve some problems based on Exponential form of a Complex Number. We know that

$z = x + iy$ **(Cartesian Form)**
$z = r(\cos\theta + i\sin\theta)$ **(Polar Form)**
$z = re^{i\theta}$ **(Exponential Form)**

Note ....

Note the following representations :

(i) $1 = \cos 2n\pi + i\sin 2n\pi = e^{i2n\pi}$ (ii) $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

(iii) $\sqrt{i} = \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]^{1/2} = \left[e^{i\pi/2}\right]^{1/2} = e^{i\pi/4}$

Class (a) : 3 Marks

Example 1 (a) : Prove that i^i is real and find the value of $\sin \log_e i^i$.

Sol. : We have $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

$\therefore \quad i^i = (e^{i\pi/2})^i = e^{i^2\pi/2} = e^{-\pi/2}$

$\therefore \quad i^i = e^{-\pi/2}$

This shows that i^i is real.

Now, $\sin\log i^i = \sin\log_e(e^{-\pi/2}) = \sin\left(-\frac{\pi}{2}\right) = -1$

$\therefore\ 1 + \sin\log i^i = 0.$

Similarly, we get $\sin\log i^{-i} = 1.$ **(M.U. 2003)**

Cor. : $(i^i)^n = (e^{-\pi/2})^n = e^{-n\pi/2}$

e.g., $i^{3i} = e^{-3\pi/2};\ i^{-5i} = e^{5\pi/2}.$

Example 2 (a) : Separate into real and imaginary parts $(\sqrt{i})^{\sqrt{i}}$. **(M.U. 2004)**

Sol. : We shall use exponential form for base $\sqrt{i}$ and standard form for exponent $\sqrt{i}$.

We have $\sqrt{i} = i^{1/2} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2}$

$$= \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}$$

Also $\sqrt{i} = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{1/2} = (e^{i\pi/2})^{1/2} = e^{i\pi/4}$

$$\therefore \quad (\sqrt{i})^{\sqrt{i}} = \left\{e^{i\pi/4}\right\}^{\left(\frac{1}{\sqrt{2}} + i\cdot\frac{1}{\sqrt{2}}\right)} = e^{i\pi/4\sqrt{2} - \pi/4\sqrt{2}}$$

$$= e^{-\pi/4\sqrt{2}} \cdot e^{i\pi/4\sqrt{2}} = e^{-\pi/4\sqrt{2}}\left(\cos\frac{\pi}{4\sqrt{2}} + i\sin\frac{\pi}{4\sqrt{2}}\right)$$

$\therefore$ Real part $= e^{-\pi/4\sqrt{2}} \cos\left(\dfrac{\pi}{4\sqrt{2}}\right)$

Imaginary part $= e^{-\pi/4\sqrt{2}} \sin\left(\dfrac{\pi}{4\sqrt{2}}\right)$

(For another method see Ex. 3 on page 4-12.)

Example 3 (a) : Separate into real and imaginary parts

$$z^z \text{ where } z = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Sol. : We have, $z = \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2} = \cos 60^\circ + i \sin 60^\circ$

$$= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = e^{i\pi/3}$$

$$\therefore \quad z^z = (e^{i\pi/3})^{(1/2)+i(\sqrt{3}/2)} = e^{(i\pi/6)-(\pi/2\sqrt{3})}$$

$$= e^{-(\pi/2\sqrt{3})+i(\pi/6)} = e^{-(\pi/2\sqrt{3})} \cdot e^{i\pi/6}$$

$$= e^{-\pi/2\sqrt{3}}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = e^{-\pi/2\sqrt{3}}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right).$$

$\therefore$ Real part $= e^{-\pi/2\sqrt{3}} \cdot \dfrac{\sqrt{3}}{2}$; Imaginary part $= e^{-\pi/2\sqrt{3}} \cdot \dfrac{1}{2}$.

Example 4 (a) : If $i^{i^{i \ldots\text{ad. inf.}}} = A + iB$, prove that

$$A^2 + B^2 = e^{i\pi B} \text{ and } \tan\left(\frac{\pi}{2}A\right) = B. \qquad \textbf{(M.U. 1999, 2002)}$$

Sol. : We have by data $i^{A+iB} = A + iB$

$$\therefore \quad \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{A+iB} = A + iB$$

$$\therefore \quad e^{(i\pi/2)(A+iB)} = A + iB \quad \therefore \quad e^{-\pi(B/2)} \cdot e^{i\pi A/2} = A + iB$$

$$\therefore \quad e^{-\pi B/2}\left[\cos\left(\frac{\pi}{2}A\right) + i\sin\left(\frac{\pi}{2}A\right)\right] = A + iB.$$

$$\therefore \quad e^{-\pi B/2} \cdot \cos\left(\frac{\pi}{2}A\right) = A \text{ and } e^{-\pi B/2} \cdot \sin\left(\frac{\pi}{2}A\right) = B$$

$$\therefore \quad A^2 + B^2 = e^{-\pi B} \text{ and } \tan\left(\frac{\pi}{2}A\right) = \frac{B}{A}.$$

Example 5 (a) : Prove that

$$\sqrt{1 + \operatorname{cosec}(\theta/2)} = (1 - e^{i\theta})^{-1/2} + (1 - e^{-i\theta})^{-1/2}. \qquad \textbf{(M.U. 2004, 05, 06)}$$

Sol. : We have to show that

$$\sqrt{\left(1+\operatorname{cosec}\frac{\theta}{2}\right)}=\frac{1}{\sqrt{(1-e^{i\theta})}}+\frac{1}{\sqrt{(1-e^{-i\theta})}}$$

Squaring both sides, we get

$$1+\operatorname{cosec}\frac{\theta}{2}=\frac{1}{1-e^{i\theta}}+\frac{1}{1-e^{-i\theta}}+\frac{2}{\sqrt{(1-e^{i\theta})(1-e^{-i\theta})}}$$

We shall prove this result.

$$\text{Now,}\quad \text{r.h.s.}=\frac{1-e^{-i\theta}+1-e^{i\theta}}{1-e^{-i\theta}-e^{i\theta}+1}+\frac{2}{\sqrt{1-e^{-i\theta}-e^{i\theta}+1}}$$

$$=1+\frac{2}{\sqrt{2-(e^{i\theta}+e^{-i\theta})}}$$

$$=1+\frac{2}{\sqrt{2-2\cos\theta}}\qquad \text{[See (6), page 1-11]}$$

$$=1+\frac{2}{\sqrt{2(1-\cos\theta)}}+\frac{2}{\sqrt{4\sin^2(\theta/2)}}$$

$$=1+\frac{2}{2\sin(\theta/2)}=1+\operatorname{cosec}\frac{\theta}{2}=\text{l.h.s.}$$

Example 6 (a) : If $i^z=z$ where $z=x+iy$, prove that $|i^z|^2=e^{-(4n+1)\pi y}$ where $n=0,1,2,\ldots$ **(M.U. 1989, 9**

Sol. :
$$i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=\cos\left(2n\pi+\frac{\pi}{2}\right)+i\sin\left(2n\pi+\frac{\pi}{2}\right)$$

$$=e^{i(2n\pi+(\pi/2))},\qquad n=0,1,2,\ldots$$

$$\therefore\quad i^z=e^{i(2n\pi+(\pi/2))\cdot z}=e^{i(2n\pi+(\pi/2))(x+iy)}$$

$$=e^{i(2n\pi+(\pi/2))\cdot x}\cdot e^{-(2n\pi+(\pi/2))y}$$

$$\therefore\quad |i^z|=e^{-(4n+1)\pi y/2}\qquad \therefore\quad |i^z|^2=e^{-(4n+1)\pi y}.$$

EXERCISE - V

Class (a) : 3 Marks

1. Prove that i^{-i} is real and hence show that $\sin\log(i^{-i})=1$. **(M.U. 200**

2. Separate into real and imaginary parts $(\sqrt{-i})^{\sqrt{-i}}$.

$$\left[\text{Ans. : } e^{-\pi/4\sqrt{2}}\left(\cos\frac{\pi}{4\sqrt{2}}-i\sin\frac{\pi}{4\sqrt{2}}\right.\right.$$

3. Prove that

(i) $\sqrt{1-\operatorname{cosec}(\theta/2)} = (1-e^{i\theta})^{-1/2} - (1-e^{-i\theta})^{-1/2}$

(ii) $\sqrt{1+\sec(\theta/2)} = (1+e^{i\theta})^{-1/2} + (1+e^{-i\theta})^{-1/2}$

(iii) $\sqrt{1-\sec(\theta/2)} = (1+e^{i\theta})^{-1/2} - (1+e^{-i\theta})^{-1/2}$

EXERCISE - VI

Short Answer Questions : Class (a) : 3 Marks

1. Find the value of i^i. [Ans. : $e^{-\pi/2}$]
2. Find the roots of $x^4 = 1$. [Ans. : $\pm 1, \pm i$]
3. If $x = \cos\theta + i\sin\theta$, find the value of $x^n - \dfrac{1}{x^n}$. [Ans. : $2i\sin n\theta$]
4. If $x = \cos\theta + i\sin\theta$, then find the value of $x^6 + \dfrac{1}{x^6}$. [Ans. : $2\cos 6\theta$]
5. If $x = e^{i\theta}$, $y = e^{-i\theta}$, then find the value of $x^n - y^n$. [Ans. : $2i\sin n\theta$]
6. If $x = \cos\theta + i\sin\theta$, $y = \cos\theta - i\sin\theta$, then find the value of $x^n + y^n$. [Ans. : $2\cos n\theta$]
7. Find the modulus and amplitude of $1 + i$. [Ans. : $\sqrt{2}, \pi/4$]
8. Find the modulus of $\tan\alpha + i$. [Ans. : $\sec\alpha$]
9. Find the modulus and amplitude of $\sqrt{3} + i$. [Ans. : $2, \pi/6$]
10. Find $\left[\sin\dfrac{\pi}{3} + i\cos\dfrac{\pi}{3}\right]^2$. [Ans. : $\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$]
11. Find $\left(\sqrt{i}\right)^i$. [Ans. : $e^{-\pi/4}$]
12. Find the real part of $e^{2+i\pi}$. [Ans. : $-e^2$]
13. Find the real part of $\sqrt{i}$. [Ans. : $1/\sqrt{2}$]
14. Find the value of $\log(i^i)$. [Ans. : $-\dfrac{\pi}{2}$]
15. Find the value of $i^{60} + i^{62}$. [Ans. : 0]

Summary

1. **De Moivre's Theorem**

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta$$

$$\frac{1}{(\cos\theta + i\sin\theta)^n} = \cos n\theta - i\sin n\theta$$

$$(\cos\theta - i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-n}$$

(2) $$(\cos\theta + i\sin\theta)^n = {}^nC_0\cos^n\theta + {}^nC_1\cos^{n-1}\theta\,(i\sin\theta) + {}^nC_2\cos^{n-2}\theta\,(i\sin\theta)^2 + \ldots\ldots$$

$$= [\,{}^nC_0\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + \ldots\ldots\,] + i[\,{}^nC_1\cos^{n-1}\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots\,]$$

(3) $$(\cos\theta + i\sin\theta)^{1/n} = \cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)$$

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