Week-8
Mathematics for Data Science - 2
Rank of a matrix and Linear Transformation
Graded Assignment Solution

1. A function $T : V \to W$ between two vector spaces V and W is said to be a linear transformation if the following conditions hold:
   - **Condition 1:** $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in V$.
   - **Condition 2:** $T(cv) = cT(v)$ for all $v \in V$ and $c \in \mathbb{R}$.

   Consider the following function:
   $$T : \mathbb{R}^2 \to \mathbb{R}$$
   $$T(x, y) = \begin{cases} 3x & \text{if } y = 0 \\ 4y & \text{if } y \neq 0 \end{cases}$$
   Which of the following statements is true?
   - ○ Option 1: Condition 1 holds.
   - ○ **Option 2:** Condition 1 does not hold.
   - ○ **Option 3:** Condition 2 holds.
   - ○ Option 4: Condition 2 does not hold.

Solution: Given map is
$$T(x,y) = \begin{cases} 3x & \text{if } y=0 \\ 4y & \text{if } y\neq 0. \end{cases}$$

★ Condition 1: $T(v_1+v_2) = T(v_1)+T(v_2)$ for all $v_1, v_2 \in V$

Here $V = \mathbb{R}^2$ and $W = \mathbb{R}$.

Take $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2) \in \mathbb{R}^2$.

We want to know whether
$$T(v_1+v_2) = T((x_1,y_1)+(x_2,y_2)) = T(x_1+x_2, y_1+y_2) \overset{?}{=} T(v_1)+T(v_2)$$

$= T(x_1, y_1) + T(x_2, y_2)$, For all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. — (1)

If the above the above equality is true For all $v_1, v_2 \in \mathbb{R}^2$ then it should satisfy for

$$V_1 = (1,0), \quad V_2 = (0,1),$$

that is $T((1,0) + (0,1)) \stackrel{?}{=} T(1,0) + T(0,1)$

$$\Rightarrow \quad T(1,1) \stackrel{?}{=} T(1,0) + T(0,1). \quad \text{— (2)}$$

L.H.S of eq (2) is $T(1,1) = 4$ $\left[\begin{array}{l}\text{by definition} \\ T(x,y) = 4y \ \text{ if } y \neq 0\end{array}\right]$

R.H.S of eq (2) is $T(1,0) + T(0,1) = 3 + 4 = 7$.

$\left\{\begin{array}{l}\text{by definition} \\ T(x,y) = 3x \\ \text{if } y = 0\end{array}\right\}$ $\left\{\begin{array}{l}\text{by definition} \\ T(x,y) = 4y \\ \text{if } y \neq 0\end{array}\right\}$

L.H.S of eq (2) $\neq$ R.H.S of eq (2)

$\Rightarrow$ equality in eq (1) does not hold.

Hence the condition (1) is not satisfied by T.

★ Condition 2: $T(cv) = c\,T(v)$ for all $v \in V$ and $c \in \mathbb{R}$.

$\cdot$ Take $v = (x, y) \in \mathbb{R}^2$

Want to check:

$$T(cv) = T(cx, cy) \overset{?}{=} c\,T(x, y).$$

Subcase-1. $y = 0$.

$$T(cv) = T(cx, 0) = 3cx = c\,3x = c\,T(x, 0) = c\,T(v)$$

[If $y = 0$
$T(x, y) = 3x$.]

Subcase-2. $y \neq 0$

$$T(cv) = T(cx, cy) = 4cy = c\,4y = c\,T(x, y).$$
$$= c\,T(v)$$

[If $y \neq 0$
$T(x, y) = 4x$].

So condition-2 holds.

2. Suppose the matrix representation of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to ordered bases $\beta = \{(1,0,1),(0,1,0),(0,0,1)\}$ for the domain and $\gamma = \{(1,0,0),(0,1,0),(1,0,1)\}$ for the range, is $I_{3\times 3}$, i.e., the identity matrix of order 3. Let A denote the matrix representation of the linear transformation T with respect to the standard ordered basis of $\mathbb{R}^3$ for both domain and range. Which of the following are true?

- ○ Option 1: $A = I_{3\times 3}$ i.e., identity matrix of order 3.
- ○ Option 2: A is a singular matrix.
- ○ Option 3: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$
- ○ **Option 4:** $det(A) = 1$.
- ○ Option 5: $det(A) = -1$.
- ○ **Option 6:** $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

Solution: Given information in the question is:

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the matrix representation of T wrt the ordered bases $\beta = \{(1,0,1),(0,1,0),(0,0,1)\}$ for the domain and $\gamma = \{(1,0,0),(0,1,0),(1,0,1)\}$ for the range.

Using the above information we can write the explicit algebraic expression for T

$$ (1) - \begin{cases} T(1,0,1) = 1(1,0,0) + 0(0,1,0) + 0(1,0,1) = (1,0,0) \\ T(0,1,0) = 0(1,0,0) + 1(0,1,0) + 0(1,0,1) = (0,1,0) \\ T(0,0,1) = 0(1,0,0) + 0(0,1,0) + 1(1,0,1) = (1,0,1) \end{cases} $$

From (1):

$$\left.\begin{aligned} T(1,0,0) &= T((1,0,1)-(0,0,1)) = T(1,0,1) - T(0,0,1) \\ &= (1,0,0) - (1,0,1) = (0,0,-1) \end{aligned}\right\} - (2)$$

From (1) & (2):

$$T(x,y,z) = T(x(1,0,0) + y(0,1,0) + z(0,0,1))$$

$$= T(x(1,0,0)) + T(y(0,1,0)) + T(z(0,0,1))$$

$$= xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$$

$$= x(0,0,-1) + y(0,1,0) + z(1,0,1)$$

$$= (z, y, z-x).$$

$$T(1,0,0) = (0,0,-1) = 0(1,0,0) + 0(0,1,0) + -1(0,0,-1)$$

$$T(0,1,0) = (0,1,0) = 0(1,0,0) + 1(0,1,0) + 0(0,0,1)$$

$$T(0,0,1) = (1,0,1) = 1(1,0,0) + 0(0,0,1) + 1(0,0,1)$$

The matrix representation of T wrt the standard bases for both domain and co-domain is

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and } \det(A) = 1.$$

3. Match the linear transformations and sets of vectors in column A with the images of those sets under the linear transformation in column B and the geometric representations of both sets in column C.

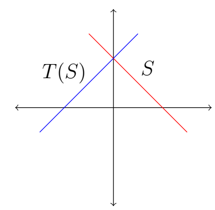
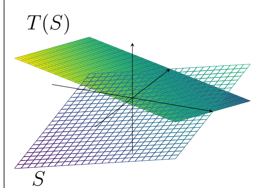
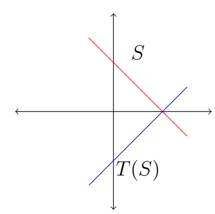
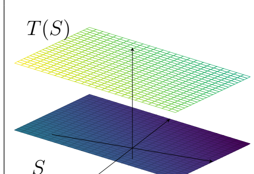
	Matrix form of linear transformation (Column A)		Image of the given set (Column B)		Geometric representations (Column C)
i)	$T : \mathbb{R}^3 \to \mathbb{R}^3$ $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}$ Set: $S = \{(x, y, z) \mid x + y + z = 1\}$	a)	$T(S) = \{(x, y) \mid x - y = 1\}$	1)	
ii)	$T : \mathbb{R}^3 \to \mathbb{R}^3$ $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ Set: $S = \{(x, y, z) \mid x + y + z = 1\}$	b)	$T(S) = \{(x, y, z) \mid x + y + z = 3\}$	2)	
iii)	$T : \mathbb{R}^2 \to \mathbb{R}^2$ $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Set: $S = \{(x, y) \mid x + y = 1\}$	c)	$T(S) = \{(x, y) \mid x - y = -1\}$	3)	
iv)	$T : \mathbb{R}^2 \to \mathbb{R}^2$ $T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ Set: $S = \{(x, y) \mid x + y = 1\}$	d)	$T(S) = \{(x, y, z) \mid x + y - z = 1\}$	4)	

Table: M2W8G1

Choose the correct options from the following.

- ◯ Option 1: i → d → 2, ii → b → 4.
- ◯ **Option 2:** i → b → 4, ii → d → 2.
- ◯ Option 3: iii → a → 1, iv → c → 3.
- ◯ **Option 4:** iii → a → 3, iv → c → 1.

<u>Solution:</u> i) Matrix form of the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}$$

Therefore, $T(x,y,z) = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right]^T = \begin{pmatrix} x \\ y \\ 2x+2y+3z \end{pmatrix}^T = (x, y, 2x+2y+3z)$

Given Set is $S = \{(x,y,z) \mid x+y+z=1\}$

$$x+y+z=1 \text{ iff } z = 1-x-y$$

$$\Rightarrow S = \{(x,y,1-x-y) \mid x,y \in \mathbb{R}\}$$

Let $v = (x,y,1-x-y) \in S$

$$T(v) = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1-x-y \end{pmatrix}\right]^T = \begin{pmatrix} x \\ y \\ 2x+2y+3-3x-3y \end{pmatrix}^T = (x, y, 3-x-y) \in T(S).$$

Let $a = x,\ b = y\ \ c = 3 - x - y.$

$$\Rightarrow a + b + c = 3$$

$$T(S) = \{(x, y, 3 - y - x) \mid x, y \in \mathbb{R}\} = \{(a, b, c) \mid a + b + c = 3\ \&\ a, b, c \in \mathbb{R}\}$$

Therefore $T(S)$ and S represent two distinct parallel planes.

$$\boxed{\text{i)} \Rightarrow \text{b)} \Rightarrow 4).}$$

ii) $T(x, y, z) = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right]^T$

Here $S = \{(x, y, z) \mid x + y + z = 1\} = \{(x, y, 1 - x - y) \mid x, y \in \mathbb{R}\}$

Let $v = (x, y, 1 - x - y) \in S$

Then $T(v) = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1-x-y \end{pmatrix}\right]^T = (x, y, x + y - 1).$

$$T(S) = \{(x, y, x + y - 1) \mid x, y \in \mathbb{R}\} = \{(a, b, c) \mid a + b - c = 1\}.$$

Both S and $T(S)$ represent two distinct planes and they are not parallel. Therefore they will inerset each other in a line.

$$\boxed{\text{ii)} \Rightarrow d \Rightarrow 2}$$

III) $T(x,y)=\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\right]^T$

$S=\{(x,y)\mid x+y=1\}=\{(x,1-x)\mid x\in\mathbb{R}\}$

Let $v=(x,1-x)\in S$

$T(v)=(x,x-1)\in T(S)$

$T(S)=\{(x,x-1)\mid x\in\mathbb{R}\}=\{(a,b)\mid a-b=1\}$

S and T(S) represent two distinct lines and they intersect at the point $(1,0)$.

$$\text{III)}\Rightarrow a)\Rightarrow 3)$$

IV) $T(x,y)=\left[\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\right]^T$.

$S=\{(x,y)\mid x+y=1\}=\{(x,1-x)\mid x\in\mathbb{R}\}$

Take $v=(x,1-x)\in S$

$T(v)=(x,1+x)\in T(S)$

$T(S)=\{(x,1+x)\mid x\in\mathbb{R}\}=\{(a,b)\mid a-b=-1\}$

S and T(S) two distinct lines. They interset at $(0,1)$.

$$\boxed{\text{IV)}\Rightarrow c\Rightarrow 1}$$

4. Consider two linear transformations T and S from $\mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x, y) = (2x + y, x + y)$ and $S(x, y) = (x + cy, x + 2y)$. Let A and B be matrix representations of linear transformations T and S with respect to the standard bases of $\mathbb{R}^2$ respectively. Consider the following statements:

- **P:** If $c = 1$, then A and B are similar matrices.
- **Q:** If $c = 2$, then A and B are similar matrices.
- **R:** If $c = 1$ and $P^{-1}AP = B$, then P can be the matrix $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$.
- **S:** If $c = 1$ and $P^{-1}AP = B$, then P can be the matrix $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$.
- **T:** If $c = 1$, then there are infinitely many P satisfying the equation $P^{-1}AP = B$.

Which of the following options are true?

○ **Option 1:** P is true but Q is false.

○ Option 2: Both P and Q are true.

○ Option 3: Both R and S are true.

○ **Option 4:** R is false but S is true.

○ **Option 5:** T is true.

Solution: $T(x,y) = (2x+y, x+y)$

$T(1,0) = (2,1) = 2(1,0) + 1(0,1)$

$T(0,1) = (1,1) = 1(1,0) + 1(0,1)$

The matrix representation of T wrt the standard basis is

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

- $S(x,y) = (x+cy, x+2y)$

$S(1,0) = (1,1) = 1(1,0) + 1(0,1)$

$S(0,1) = (c,2) = c(1,0) + 2(0,1)$

The matrix representation of S w.r.t to the standard basis is

$$B=\begin{bmatrix}1 & c\\ 1 & 2\end{bmatrix}$$

Def: Two matrices A and B are similar if $\exists\, P$ such that $P^{-1}AP=B \Rightarrow AP=BP$.

If $c=1$ then $A=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$

Let $P=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ such that $AP=PB$

$$\Leftrightarrow \begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix}2a+c & 2b+d\\ a+c & b+d\end{bmatrix}=\begin{bmatrix}a+b & a+2b\\ c+d & c+2d\end{bmatrix}$$

$$\Rightarrow \begin{aligned}2a+c&=a+b\\ 2b+d&=a+2b\\ a+c&=c+d\\ b+d&=c+2d\end{aligned} \quad\Rightarrow\quad \left.\begin{aligned}a-b+c&=0\\ a+d&=0\\ b-c-d&=0\end{aligned}\right\} \text{——} (1)$$

System 1 has infinitely many solutions. In particular if $a=1, b=2, c=1, d=1$ then $P=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}$ satisfy the relation $P^{-1}AP=B$.

The System (1) has infinitely many Solutions therefore we have infinitely many possible choices for "P".

$\boxed{\text{Statements } P, S \,\&\, T \text{ are Correct.}}$

* If $C=1$ & $P=\begin{bmatrix}1 & 0\\ 1 & 2\end{bmatrix}$ then $AP \neq PB \Rightarrow P^{-1}AP \neq B$

Hence statement "R" is not true.

* If $C=2$ then $A=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix}$

If A and B are Similar matrix then $\det(A)=\det(B)$.

Here $\det(A)=1$ and $\det(B)=0$

$\Rightarrow$ A and B are not similar.

Statement Q is not true.

5. Consider a linear transformation $S : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ such that $S(A) = A^T$. Let B be the matrix representation of S with respect to the ordered bases:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

of $M_2(\mathbb{R})$. Choose the set of correct options:

- ○ Option 1: The order of the matrix B is 2×2.
- ○ **Option 2:** The order of the matrix B is 4×4.
- ○ **Option 3:** The dimension of the row space of the matrix B is 4.
- ○ Option 4: The dimension of the column space of the matrix B is 3.
- ○ Option 5: The nullity of the matrix B is 1.
- ○ **Option 6:** The rank of the matrix B is 4.
- ○ **Option 7:** S is surjective.

Solution. $M_{2\times2}(\mathbb{R}) :=$ The set of all 2×2 matrices.

$$:= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{R} \right\}$$

Any arbitary element $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2}(\mathbb{R})$ can be written

as $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ Every element of $M_{2\times2}(\mathbb{R})$ can be written as a linear combination of the elements of the

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$\Rightarrow$ $Span(\mathcal{B}) = M_{2\times 2}(\mathbb{R})$.

Claim :- $\mathcal{B}$ is a basis of $M_{2\times2}(\mathbb{R})$, That is

1) $Span(\mathcal{B}) = M_{2\times2}(\mathbb{R})$ (We have proved this).

2) $\mathcal{B}$ - is Linearly independent.

Suppose

$$\alpha\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \beta\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + \gamma\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} + w\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\alpha & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & \beta\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ \gamma & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & w\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\alpha & \beta\\ \gamma & w\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \Rightarrow \alpha = \beta = \gamma = w = 0$$

Linear combination of the elements of $\mathcal{B}$ is 0 (Here $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ is the Zero element of the Vector Space $M_{2\times2}(\mathbb{R})$)

Iff all the scalars in the Linear Combination are Zero.

Hence $\mathcal{B}$ - is a basis for $M_{2\times2}(\mathbb{R})$.

and $\dim(M_{2\times2}(\mathbb{R})) = 4$.

Now:

$$T\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&0\end{bmatrix}^T=\begin{bmatrix}1&0\\0&0\end{bmatrix}=1\begin{bmatrix}1&0\\0&0\end{bmatrix}+0\begin{bmatrix}0&1\\0&0\end{bmatrix}+0\begin{bmatrix}0&1\\0&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)=\begin{bmatrix}0&1\\0&0\end{bmatrix}^T=\begin{bmatrix}0&0\\1&0\end{bmatrix}=0\begin{bmatrix}1&0\\0&0\end{bmatrix}+0\begin{bmatrix}0&1\\0&0\end{bmatrix}+1\begin{bmatrix}0&0\\1&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&1\end{bmatrix}.$$

$$T\left(\begin{bmatrix}0&0\\1&0\end{bmatrix}\right)=\begin{bmatrix}0&0\\1&0\end{bmatrix}^T=\begin{bmatrix}0&1\\0&0\end{bmatrix}=0\begin{bmatrix}1&0\\0&0\end{bmatrix}+1\begin{bmatrix}0&1\\0&0\end{bmatrix}+0\begin{bmatrix}0&0\\1&0\end{bmatrix}+0\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0&0\\0&1\end{bmatrix}\right)=\begin{bmatrix}0&0\\0&1\end{bmatrix}=0\begin{bmatrix}1&0\\0&0\end{bmatrix}+0\begin{bmatrix}0&1\\0&0\end{bmatrix}+0\begin{bmatrix}0&0\\1&0\end{bmatrix}+1\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

So the matrix representation of T wrt the ordered basis $\mathcal{B}$ is

$$B=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$$

Row reduced echelon form of B is the Identity matrix of order 4.

$\Rightarrow$ option 2, option 3 and option 6 are correct.

✪ <u>Claim</u>: $S: M_2(\mathbb{R}) \longrightarrow M_2(\mathbb{R})$

$$A \longmapsto A^T$$

is both one-one and on-to.

Suppose $S(A) = O$ (Zero matrix).

$\Rightarrow A^T = O$

$\Rightarrow A = O$ (i.e if transpose of a matrix is a zero matrix then the original matrix is also a zero matrix).

$S(A) = O$ Iff $A = O \Rightarrow S$ is one-one.

☆ Let $B \in \text{Codomain}(S) = M_2(\mathbb{R})$.

∴ Take $B^T \in \text{domain}(S) = M_2(\mathbb{R})$.

$S(B^T) = (B^T)^T = B \Rightarrow S$ is surjective.

Hence S is an isomorphism.

option-7 is correct.

6. Let L and L' be affine subspaces of $\mathbb{R}^3$, where $L = (0,1,1)+U$ and $L' = (0,1,0)+U'$, for some vector subspaces U and U' of $\mathbb{R}^3$. Let a basis for U be given by $\{(1,1,0),(1,0,1)\}$ and a basis for U' be given by $\{(1,0,0)\}$. Suppose there is a linear transformation $T : U \to U'$ such that $(1,0,1) \in \ker(T)$ and $T(1,1,0) = (1,0,0)$. An affine mapping $f : L \to L'$ is obtained by defining $f((0,1,1)+u) = (0,1,0)+T(u)$, for all $u \in U$. Which of the following options are true?

- ◯ **Option 1:** $L = \{(x, y+1, x-y+1) \mid x, y \in \mathbb{R}\}$.
- ◯ Option 2: $L' = \{(x, y+1, 0) \mid x, y \in \mathbb{R}\}$.
- ◯ Option 3: $L = \{(x-y, y+1, x-y+1) \mid x, y \in \mathbb{R}\}$.
- ◯ Option 4: $L = \{(x, x+1, y+1) \mid x, y \in \mathbb{R}\}$.
- ◯ **Option 5:** $f(x, y+1, x-y+1) = (y, 1, 0)$
- ◯ Option 6: $f(x-y, y+1, x-y+1) = (x, y+1, 0)$
- ◯ Option 7: $f(x, x+1, y+1) = (y, 1, 0)$
- ◯ Option 8: $f(x, y+1, x-y+1) = (0, 1, y)$

Solution:

★ $L = (0,1,1) + U$, where basis of U is $\{(1,1,0),(1,0,1)\}$.

$$U = \text{span}\{(1,1,0),(1,0,1)\}.$$
$$= \{\alpha(1,1,0) + \beta(1,0,1) \mid \alpha, \beta \in \mathbb{R}\}.$$
$$= \{(\alpha+\beta, \alpha, \beta) \mid \alpha, \beta \in \mathbb{R}\}$$

Take $x = \alpha+\beta$, $y = \alpha$, $z = \beta \Rightarrow z = x - y$

So we can write $U = \{(x, y, x-y) \mid x, y \in \mathbb{R}\}$

$$L = (0,1,1) + U = \{(x, y, x-y) \mid x, y \in \mathbb{R}\}$$

$$= \{(x, 1+y, 1+x-y) \mid x, y \in \mathbb{R}\}.$$

☆ $L' = (0,1,0) + U'$ where basis of U' is $\{(1,0,0)\}$

$$U = \text{span}\{(1,0,0)\} = \{x(1,0,0) \mid x \in \mathbb{R}\}$$
$$= \{(x,0,0) \mid x \in \mathbb{R}\}.$$

$$L' = (0,1,0) + U' = \{(x,0,0) \mid x \in \mathbb{R}\}.$$
$$= \{(x,1,0) \mid x \in \mathbb{R}\}.$$

☆ We have a linear transformation $T: U \rightarrow U'$ such that $(1,0,1) \in \ker(T)$ and $T(1,1,0) = (1,0,0)$.

basis of U is $\{(1,1,0), (1,0,1)\}$

$\Rightarrow$ every element $(x,y,z) \in U$ can be writtens as a linear combination of $(1,1,0)$ & $(1,0,1)$.

Suppose $(x,y,z) = \alpha(1,1,0) + \beta(1,0,1)$

$\Rightarrow$ $(x,y,z) = (\alpha+\beta, \alpha, \beta)$.

$\Rightarrow$ $\alpha = y$, $\beta = z$

Hence, $(x,y,z) = y(1,1,0) + z(1,0,1)$

Apply T on both the sides

$$T(x,y,z) = T(y(1,1,0) + z(1,0,1))$$

$$= yT(1,1,0) + zT(1,0,1).$$
$$= y(1,0,0) + z(0,0,0) = (y,0,0).$$

* The affine mapping $f: L \to L'$ is defined as

$$f((0,1,1)+u) = (0,1,0) + T(u) \quad \forall\, u \in U$$

$U = \{(x, y, x-y) \mid x, y \in \mathbb{R}\}$ & take $u = (x, y, x-y) \in U$.

$$f((0,1,1)+(x,y,x-y)) = (0,1,0) + T(x,y,x-y).$$

$$\Rightarrow f(0, 1+y, 1+x-y) = (y, 1, 0).$$

7. Consider the following statements:

- **Statement 1:** Consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ such that T is not injective. Then $rank(T) < 3$.
- **Statement 2:** If $T : V \to W$ is a linear transformation, whose matrix representation with respect to some ordered bases is given by the matrix $\begin{bmatrix} 0 & \alpha & \gamma \\ 1 & 0 & \gamma \\ 0 & \beta & \frac{\gamma\beta}{\alpha} \end{bmatrix}$, where $\alpha, \beta, \gamma \in \mathbb{R} \setminus \{0\}$, then the rank of the linear transformation T is 3.
- **Statement 3:** If $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation such that $T(x, y, z) = (2x - z, 3y - 2z, z, 0)$, then $\{$(-3, 1, 1, 0), (1, -5, 1, 0), (3, 5, -1, 0)$\}$ is a basis of the image space.
- **Statement 4:** If $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ is a linear transformation such that $T(A) = PA$, where $A \in M_2(\mathbb{R})$ and $P = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$, then $\left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$ is a basis of the kernel.

Write down the statement numbers corresponding to the correct statements in increasing order.

[Note: Suppose Statement 1, Statement 2, and Statement 4 are correct then your answer should be 124. Similarly, if Statement 2 and Statement 3 are correct then your answer should be 23. In this list one or more than one statement can be correct. Do not add any space between the digits.] [Ans : 13]

Solution: Rank-Nullity theorem:

If $T: V \to W$ is a linear transformation then

$$\underset{\dim(\mathrm{Im}(T))}{\underset{\parallel}{\mathrm{Rank}(T)}} + \underset{\dim(\mathrm{Ker}(T))}{\underset{\parallel}{\mathrm{Nullity}(T)}} = \dim(V).$$

Statement 1: $T: \mathbb{R}^3 \to \mathbb{R}^4$ and T is not injective

$$\mathrm{Ker}(T) \neq 0 \Rightarrow \dim(\mathrm{Ker}(T)) = \mathrm{Nullity}(T) \geq 1$$

by rank-nullity, $\mathrm{Rank}(T) + \mathrm{Nullity}(T) = \dim(\mathbb{R}^3) = 3.$

$\Rightarrow \text{Rank}(T) = 3 - \text{Nullity}(T) < 3 \; (\because \text{Nullity}(T) \geq 1).$

- Statement-1 is correct.

☆ $T: V \to W$ is a linear transformation & $\dim(V) = n$, $\dim(W) = m$. Let B be the matrix representation of T with respect to some ordered basis V and ordered basis W. Then the order of the matrix B is $m \times n$. —— ☆

Statement 2: $T: V \to W$ and the matrix representation of T is

$$\begin{bmatrix} 0 & \alpha & \gamma \\ 1 & 0 & \delta \\ 0 & \beta & \gamma\beta/\alpha \end{bmatrix} = B.$$

order of B is 3×3.

From ☆, $\dim V = \dim W = 3$.

$$\boxed{\text{Now } \det(B) = 0 \Rightarrow T\text{-is not invertible.}}$$

By rank-nullity, $\text{Rank}(T) + \text{Nullity}(T) = 3 = \dim(V)$.

- If $\text{Rank}(T) = 3 \Rightarrow \text{Nullity}(T) = 3 - 3 = 0$.

$\text{Rank}(T) = 3$ & $\dim(W) = 3 \Rightarrow T$-is surjective

$\text{Nullity}(T) = 0 \Rightarrow T$-is one-one

$\Rightarrow T$ is an isomorphism.

which is not true.

$\Rightarrow$ Rank(T) can not be 3.

Statement: 2 is not correct.

<u>Statement 3</u>: $T: \mathbb{R}^3 \to \mathbb{R}^4$ $\quad T(-2,1,1) = (-3,1,1,0)$.

$$T(x,y,z) = (2x-z,\ 3y-2z,\ z,\ 0)$$

$$Im(T) = \{(2x-z,\ 3y-2z,\ z,0) \mid x,y,z \in \mathbb{R}\}$$

$$= \{(2x,0,0,0) + (0,3y,0,0) + (-z,-2z,z,0) \mid x,y,z \in \mathbb{R}\}$$

$$= \{x(2,0,0,0) + y(0,3,0,0) + z(-1,-2,1,0) \mid x,y,z \in \mathbb{R}\}.$$

$$= \text{Span}\{(2,0,0,0),(0,3,0,0),(-1,-2,1,0)\}.$$

One can check that the set $S = \{(2,0,0,0),(0,3,0,0),(-1,-2,1,0)\}$ is linearly independent.

$\Rightarrow$ S is a basis for $Im(T)$. $\Rightarrow \boxed{\dim(Im(T)) = 3.}$

<u>Want to know</u>.

Whether $B = \{(-3,1,1,0),\ (1,-5,1,0),\ (3,5,-1,0)\}$ is a basis for $Im(T)$ or not?

Construct a matrix using the elements of B

$$\begin{bmatrix} -3 & 1 & 3 \\ 1 & -5 & 5 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -5 & 5 \\ -3 & 1 & 3 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3+3R_1]{R_3-R_1} \begin{bmatrix} 1 & -5 & 5 \\ 0 & -14 & 18 \\ 0 & 6 & -6 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{1}{6}R_3]{-\frac{1}{14}R_2} \begin{bmatrix} 1 & -5 & 5 \\ 0 & 1 & \frac{-18}{14} \\ 0 & 0 & 4/14 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3-R_2} \begin{bmatrix} 1 & -5 & 5 \\ 0 & 1 & -18/14 \\ 0 & 0 & 4/14 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{14}{4}R_3} \begin{bmatrix} 1 & -5 & 5 \\ 0 & 1 & -18/14 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$ ← all columns are non-zero

$\Rightarrow$ B is a linearly independent set

$\Rightarrow \dim(\text{Span}(B)) = 3.$

__next__: $B \subset \text{Im}(T)$.

$T(-2,1,1,0) = (-3,1,1,0) \in \text{Im}(T)$.

$T(1,-1,1,0) = (1,-5,1,0) \in \text{Im}(T)$.

$T(1,1,-1,0) = (3,5,-1,0) \in \text{Im}(T)$

$\Rightarrow B \subset \text{Im}(T)$

$\Rightarrow$ Span(B) $\subset$ Im(T), but both have same dimension

$\Rightarrow$ Im(T) = Span(B).

Since B is linearly independent $\Rightarrow$ B is a basis for Im(T).

<u>Statement 4:</u>

$$T(A) = PA \text{ where } P = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \notin \text{Ker}(T).$$

$\Rightarrow$ So $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right\}$ can not be a basis of Ker(T).

8. Let T be a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^7$. Suppose a basis for the null space of T has 2 vectors. How many linearly independent vectors are needed to form a basis for the range of T? [Answer : 2]

Solution. $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^7$

by rank-nullity : $\text{Rank}(T) + \text{Nullity}(T) = \dim(\mathbb{R}^4) = 4$

A basis for the null space of T has 2 vectors

$$\Rightarrow \text{Nullity}(T) = 2$$

$$\Rightarrow \text{Rank}(T) = 4 - \text{Nullity}(T) = 2$$

Therefore two linearly independent vectors are needed to form a basis for range of T.

9. Let $\mathcal{M}$ be the set of all skew-symmetric matrices of order 3. Then $\mathcal{M}$ forms a vector space under matrix addition and scalar multiplication. Let T be a linear transformation from $\mathcal{M}$ to $\mathbb{R}$ defined by $T(A) = c\, tr(A)$, where $tr(A)$ =trace of A. What is the nullity of T? [Answer : 3]

Solution: A is Skew-Symmetric Iff $A^T = -A$.

General form of 3x3 Skew-Symmetric matrices is $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.

So $M = \left\{ \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \;\middle|\; a, b, c \in \mathbb{R} \right\}$

So the basis of M would consists of $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$. Therefore $\dim(M) = 3$.

$$A \in M \Rightarrow Tr(A) = 0.$$

So the map $T: M \to \mathbb{R}$ is the zero map $\Rightarrow Im(T) = 0$

by rank-nullity: $Rank(T) + Nullity(T) = \dim(M) = 3$.

$$\Rightarrow 0 + Nullity(T) = 3.$$

$$\Rightarrow Nullity(T) = 3.$$

10. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined by $T(x, y, z) = (2x + y, z)$. Choose the correct options about T.

- ○ Option 1: The matrix of T is $\begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- ○ **Option 2:** The matrix of T is $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
- ○ **Option 3:** A basis for the null space of T is $\{(1, -2, 0)\}$.
- ○ Option 4: A basis for the null space of T is $\{(1, -2, 0), (0, 0, 1)\}$.
- ○ **Option 5:** A basis for the range of T is $\{(2, 1), (0, 1)\}$.
- ○ Option 6: A basis for the range of T is $\{(2, 2)\}$.

Solution: • Standard ordered basis for $\mathbb{R}^3 = \{(1,0,0),(0,1,0),(0,0,1)\}$

" " " " $\mathbb{R}^2 = \{(1,0),(0,1)\}$

$$T(x,y,z) = (2x+y, z).$$

$$T(1,0,0) = (2,0) = 2(1,0) + 0(0,1)$$

$$T(0,1,0) = (1,0) = 1(1,0) + 0(0,1)$$

$$T(0,0,1) = (0,1) = 0(1,0) + 1(0,1)$$

The matrix representation is $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

• $T(x,y,z) = (2x+y, z)$

$$Im(T) = \{(2x+y, z) \mid x,y,z \in \mathbb{R}\}$$

$$= \{(2x,0) + (y,0) + (0,z) \mid x,y,z \in \mathbb{R}\}$$

$$= \{x(2,0) + y(1,0) + z(0,1) \mid x,y,z \in \mathbb{R}\}$$

$\text{Im}(T) = \text{Span}\{(2,0), (1,0), (0,1)\} = \text{Span}\{(1,0), (0,1)\} = \mathbb{R}^2$

$\therefore \{(2,1), (0,1)\}$ is also a basis of $\text{Im}(T) = \mathbb{R}^2$.

Since $\text{Im}(T) = \mathbb{R}^2 \Rightarrow \text{Rank}(T) = 2$

- by rank-nullity: $\text{Rank}(T) + \text{Nullity}(T) = \dim(\mathbb{R}^3) = 3$.

$\Rightarrow \text{Nullity}(T) = 3 - 2 = 1$

$(-1, 2, 0) \in \ker(T)$ & $\dim(\ker(T)) = 1$

Therefore $\{(-1,2,0)\}$ is a basis for nullspace of T.

1 Comprehension Type Question:

Suppose a bread-making machine B makes 6 breads from 2 eggs, 3 (in hundreds) grams of wheat, and 1 (in hundred) grams of sugar. B also makes 8 breads from 3 eggs, 4 (in hundreds) grams of wheat, and 2 (in hundreds) grams of sugar, and 10 breads from 5 eggs, 5 (in hundreds) grams of wheat, and 3 (in hundreds) grams of sugar. Suppose the production of breads is a linear function of the amount of eggs, wheat (in hundreds), and sugar (in hundreds) used as raw ingredients. Based on the above data answer the following questions. Suppose x eggs, y (in hundreds) grams of wheat, and z (in hundreds) grams of sugar are used as the raw materials to produce $ax + by + cz$ number of breads. We can express this as follows:

$$T : \mathbb{R}^3 \to \mathbb{R}$$

$$T(x, y, z) = ax + by + cz$$

where the co-ordinates in $\mathbb{R}^3$ denote the number of eggs, amount (in grams) of wheat (in hundreds), and amount (in grams) of sugar (in hundreds). Observe that T is a linear transformation.

11. Choose the correct set of options from the the following. (MSQ)

- ◯ Option 1: $Nullity(T) = 1$
- ◯ **Option 2:** $Rank(T) = 1$
- ◯ **Option 3:** $Nullity(T) = 2$
- ◯ Option 4: $Rank(T) = 2$
- ◯ Option 5: $Nullity(T) = 3$
- ◯ Option 6: $Rank(T) = 3$
- ◯ Option 7: T is neither one to one nor onto.
- ◯ Option 8: T is one to one but not onto.
- ◯ **Option 9:** T is onto but not one to one.
- ◯ Option 10: T is an isomorphism.

Solution: expression for $T: \mathbb{R}^3 \to \mathbb{R}$ is

$T(x, y, z) = ax + by + cz$ which represents breads produced by the machine.

Given that the machine makes 6 breads from 2 eggs, 3 (in hundreds) grams of wheat, & 1 (in hundreds) gram of sugar, that is,

$$T(2,3,1) = 6 \Rightarrow 2a + 3b + c = 6 \quad \text{—} (1)$$

Similarly using the other information we get

$$3a + 4b + 2c = 8 \quad \text{—} (2)$$

$$5a+5b+3c=10 \quad (3)$$

From (1), (2), (3) we get $a=0,\ b=2,\ c=0$

$$\Rightarrow T(x,y,z)=2y$$

T is a linear map from $\mathbb{R}^3$ to $\mathbb{R}$.

$$\mathrm{Im}(T)=\mathbb{R}. \Rightarrow \mathrm{Rank}(T)=1=\dim(\mathbb{R}). \Rightarrow T\text{-is on-to}$$

by rank-nullity: $\quad \mathrm{nullity}(T)=2. \Rightarrow T$-is not one-one.

12. Choose the set of correct statements. (MSQ)

- ◯ Option 1: If 4 eggs and 2 (in hundreds) grams of sugar is used, and no wheat is used, then 9 breads are produced.
- ◯ **Option 2:** If 4 eggs and 2 (in hundreds) grams of sugar is used, and no wheat is used, then no bread is produced.
- ◯ **Option 3:** If only 3 (in hundreds) grams of wheat is used, then 6 breads are produced.
- ◯ Option 4: If 3 (in hundreds) grams of wheat and 1 (in hundred) grams of sugar is used, and no egg is used, then no bread is produced.
- ◯ **Option 5:** If 3 (in hundreds) grams of wheat and 1 (in hundred) grams of sugar is used, and no egg is used, then 6 breads are produced.

Solution. $T(x,y,z) = 2y$.

Option-1&2 $T(4,0,2) = 0 \Rightarrow$ no bread is produced
option 1 - is not correct & option-2 is correct.

Option-3. $T(0,3,0) = 6 \Rightarrow$ 3 gram (in hundreds) is used to produce 6 breads.

Option 4&5 $T(0,3,1) = 6 \Rightarrow$ Option-4 is not correct.
& option-5 is correct.

13. How many breads are produced by the machine from 6 eggs, 10 (in hundreds) grams of wheat, and 5 (in hundreds) grams of sugar?
(NAT) [Answer: 20]

Solution :- $T(x, y, z) = 2y \Rightarrow T(6, 10, 5) = 20$

Therefore 20 breads can be produced using the materials given in the question.