TUTORIAL - 3
COMPLEX NO.S

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Question 1

$\sin^{-1}(e^{i\theta})$

$\because$ $e^{i\theta}$ can also be written as $\cos\theta + i\sin\theta$ in polar form :

$\therefore \sin^{-1}(\cos\theta + i\sin\theta)$

Let's asume $\sin^{-1}(\cos\theta + i\sin\theta)$ to be another complex number : $\alpha + i\beta$

$\therefore \sin^{-1}(\cos\theta + i\sin\theta) = \alpha + i\beta$

$(\cos\theta + i\sin\theta) = \sin(\alpha + i\beta)$.

$\cos\theta + i\sin\theta = \sin\alpha\cos i\beta + \sin i\beta\cos\alpha$

$\because \cos i\beta = \cosh\beta$ & $\sin i\beta = i\sinh\beta$

$\therefore \cos\theta + i\sin\theta = \sin\alpha\cosh\beta + i\sinh\beta\cos\alpha$

By Splitting real and imaginary parts :

$\cos\theta = \sin\alpha\cosh\beta$ & $\sin\theta = \sinh\beta\cos\alpha$.

Squaring and adding bothe the equations :

$\sin^2\theta + \cos^2\theta = \sin^2\alpha\cosh^2\beta + \sinh^2\beta\cos^2\alpha$.

$\because$ we know that $\sin^2\theta + \cos^2\theta = 1$.

and $\cosh^2\theta - \sinh^2\theta = 1$,

$1 = \sin^2\alpha(1 + \sinh^2\beta) + \sinh^2\beta\cos^2\alpha$.

$= \sin^2\alpha + \sin^2\alpha\sinh^2\beta + \sinh^2\beta\cos^2\alpha$

$= \sinh^2\beta(\sin^2\alpha + \cos^2\alpha) + \sin^2\alpha$

$1 = \sinh^2\beta(1) + \sin^2\alpha$.

$\therefore \sinh^2\beta = \cos^2\alpha$

$\therefore \sinh\beta = \cos\alpha$ —— (3)

~~cos α~~ Multiplying $\cos\alpha$ on both sides:

$\cos\alpha \sinh\beta = \cos^2\alpha$.

$\because \cos\alpha \sinh\beta = \sin\theta$

$\therefore \sin\theta = \cos^2\alpha$.

$$\boxed{\cos^{-1}\sqrt{\sin\theta} = \alpha.}$$

Putting value of α in (3).

$\sinh\beta = \cos\left(\cos^{-1}\left(\sqrt{\sin\theta}\right)\right)$

$\sinh\beta = \sqrt{\sin\theta}$

$$\boxed{\beta = \sinh^{-1}\sqrt{\sin\theta}.}$$

Ans.) The real value of $\sin^{-1}(e^{i\theta})$ is $\cos^{-1}\sqrt{\sin\theta}$ and imaginary value of $\sin^{-1}(e^{i\theta})$ is $\sinh^{-1}\sqrt{\sin\theta}$.

Question 2

Equation : $x^4 - 3x^3 + 8x^2 - 7x + 5 = 0$.

$\therefore$ one root is $(1+2i)$ anothe root must be $(1-2i)$

Because quadratic equation's one root is irrational, it's conjugete is its anothe root.

$\therefore \quad x^2 - (a+b)x + ab = 0$.

$\hookrightarrow$ ~~eq~~ $x^2 - [(1+2i)+(1-2i)]x + (1+2i)(1-2i)$.

$\Rightarrow x^2 - 2x + 5$

$\therefore$

$$\begin{array}{r|l} & x^2 - x + 1 \\ \hline x^2 - 2x + 5 & x^4 - 3x^3 + 8x^2 - 7x + 5 \\ & -\cancel{x^4} + 2x^3 \mp 5x^2 \\ \hline & -\cancel{x^3} + 3x^2 - 7x \\ & -\cancel{+x^3} \mp 2x^2 \mp 5x \\ \hline & \cancel{x^2} - 2x + 5 \\ & \cancel{x^2} - 2x + 5 \\ \hline & 0 \end{array}$$

$\therefore (x^2 - 2x + 5)(x^2 - x + 1) = 0$.

$\therefore$ Roots of equation $x^2 - x + 1 = 0$ are $\dfrac{1 \pm \sqrt{1-4}}{2} = \dfrac{1}{2} \pm \dfrac{i\sqrt{3}}{2}$.

Ans.) Therefore $x^4 - 3x^2 + 8x^2 - 7x + 5 = 0$ has roots as $1+2i$, $1-2i$, $\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}$

and $\dfrac{1}{2} - \dfrac{i\sqrt{3}}{2}$.

Question 3

~~To prove:~~

a) $x + \frac{1}{x} = 2\cos\theta$ b) $y + \frac{1}{y} = 2\cos\phi$

c) $z + \frac{1}{z} = 2\cos\varphi$

(i) $x + \frac{1}{x} = 2\cos\theta$.

$\frac{x^2+1}{x} = 2\cos\theta$.

$x^2 - 2x\cos\theta + 1 = 0$. $\therefore \boxed{x = \cos\theta + i\sin\theta}$

similarly for (ii) and (iii)

$\boxed{y = \cos\phi + i\sin\phi}$ and $\boxed{z = \cos\varphi + i\sin\varphi}$

$\therefore xyz = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)(\cos\varphi + i\sin\varphi)$ — (i)

$= \cos(\theta + \phi + \varphi) + i\sin(\theta + \phi + \varphi)$

Using De-moivres law.

$\frac{1}{xyz}$ = inverse of (xyz) expression.

$\therefore \frac{1}{xyz} = \left[\cos(\theta + \phi + \varphi) + i\sin(\theta + \phi + \varphi)\right]^{-1}$

Using De-moivre's law:

$\cos-(\theta + \phi + \varphi) + i\sin-(\theta + \phi + \varphi)$

$\because \cos(-x) = \cos x$ & $\sin(-x) = -\sin x$.

$\therefore \frac{1}{xyz} = \cos(\theta + \phi + \varphi) - i\sin(\theta + \phi + \varphi)$ — (ii)

PAGE NO.:

DATE: / /

Adding (i) and (ii) :

$$xyz + \frac{1}{xyz} = \cos(\theta+\phi+\psi) + \cancel{i\sin(\theta+\phi+\psi)} + \cos(\theta+\phi+\psi) \cancel{- i\sin(\theta+\phi+\psi)}$$

$$= 2\cos(\theta+\phi+\psi)$$ Hence proved.

$$\therefore \boxed{xyz + \frac{1}{xyz} = 2\cos(\theta+\phi+\psi)}$$

(ii) $x = \cos\theta + i\sin\theta$

$y = \cos\phi + i\sin\phi$

$z = \cos\psi + i\sin\psi$

$$\sqrt[m]{x} = \cos\left(\frac{\theta}{m}\right) + i\sin\left(\frac{\theta}{m}\right)$$

$$\sqrt[n]{y} = \cos\left(\frac{\phi}{n}\right) + i\sin\left(\frac{\phi}{n}\right)$$

$$\cancel{\sqrt[m]{z} = \cos\frac{\psi}{m}}$$

$$\therefore \frac{\sqrt[m]{x}}{\sqrt[n]{y}} = \frac{\cos(\theta/m) + i\sin(\theta/m)}{\cos(\theta/n) + i\sin(\theta/n)}$$

$$= \cos\left(\frac{\theta}{m} - \frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{m} - \frac{\theta}{n}\right)$$

By De-moivre's theorem.

$$\frac{\sqrt[n]{y}}{\sqrt[m]{x}} = \left[\cos\left(\frac{\theta}{m} - \frac{\theta}{n}\right) + i\sin\left(\frac{\theta}{m} - \frac{\theta}{n}\right)\right]^{-1}$$

$$= \cos\left(\frac{\theta}{m} - \frac{\theta}{n}\right) - i\sin\left(\frac{\theta}{m} - \frac{\theta}{n}\right)$$

By adding both equations :

$$\frac{\sqrt[m]{y}}{\sqrt[n]{x}} + \frac{\sqrt[n]{x}}{\sqrt[n]{y}} = \cos\left(\frac{\theta}{m} - \frac{\theta}{n}\right) + \cancel{i\sin\left(\frac{\theta}{m} - \frac{\theta}{n}\right)} + \cos\left(\frac{\theta}{m} - \frac{\theta}{n}\right) - \cancel{i\sin\left(\frac{\theta}{m} - \frac{\theta}{n}\right)}$$

$$= \boxed{2\cos\left(\frac{\theta}{m} - \frac{\theta}{n}\right)}$$

Hence proved