DATA SCIENCE FOR ENGINEERS

Week 2

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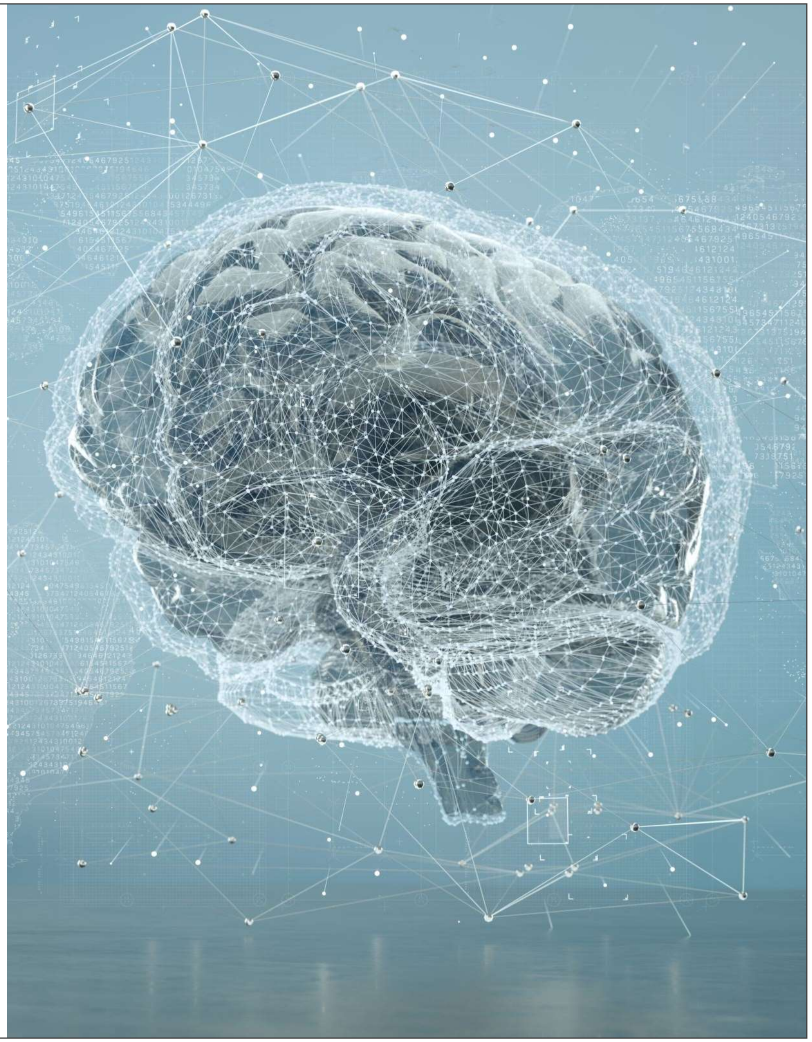


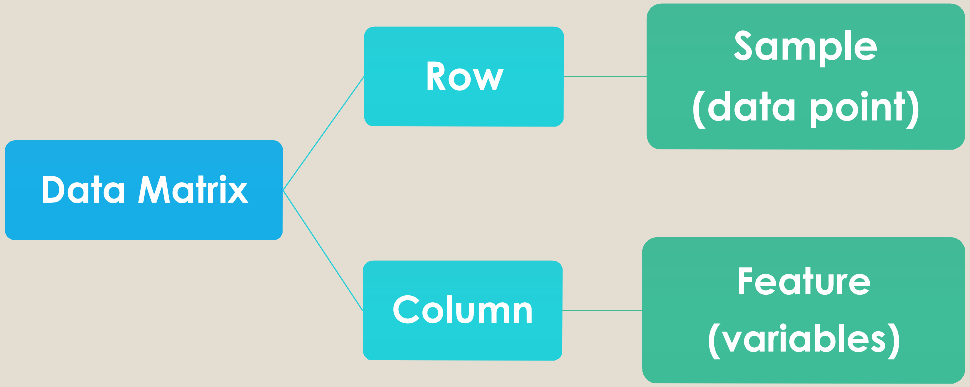
Highlights of Week 2

Topic	Details
Matrix	Introduction of Matrix/ Independent and dependent variables/ rank of matrix/ nullity of matrix
Solving Linear Equations	Solving system of linear equations for different conditions (no of variables and equations) / Pseudo inverse
Geometric View of Linear Algebra	Vector/ Norm/ orthogonal vector/ Basis vector/ Subspace/ Projection/ Hyperplane/ Eigenvalue/ Eigenvector

Matrix

- Matrix is a form of organizing data into rows and columns
- Generally we use rows for samples and columns for variables
- Example: Physical dataset of 1000 persons

$$
\begin{array}{cc}
 & \begin{array}{ccc} \text{Weight} & \text{Height} & \text{Age} \end{array} \\
\begin{array}{c} 1 \\ \vdots \\ 1000 \end{array} & \begin{bmatrix} 50 & 145 & 29 \\ \vdots & \vdots & \vdots \\ 65 & 160 & 25 \end{bmatrix}
\end{array}
$$



Independent and Dependent Variable

- Independent variable: The variables which stores the unique information
- Dependent Variable: The variable formed by combination of already existing variables
- Example: Body Mass Index (BMI) in the dataset is dependent variable

$$\begin{array}{c c} & \begin{array}{cccc} \text{Weight} & \text{Height} & \text{Age} & \text{BMI} \end{array} \\ \begin{array}{c} 1 \\ \vdots \\ 1000 \end{array} & \begin{bmatrix} 50 & 145 & 29 & 23.73 \\ \vdots & \vdots & \vdots & \vdots \\ 65 & 160 & 25 & 25.39 \end{bmatrix} \end{array}$$

$$BMI = \frac{Weight}{(Height)^2}$$

Rank & Nullity of matrix

How to find independent variables if there are many more samples than variables?

Rank	Number of independent variables or samples
Nullity	Number of Linear relationship

Nullity + Rank = Total number of Variables

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

Number of Variables = 3
Rank of A = 2
Nullity of A = 1

Note: Col_2 = 2 * Col_1

Determinant of matrix

- It is a **scalar value** that provides important information about the **properties of the matrix** (such as: inverse of matrix, linear independence, Eigenvectors etc.

- Determinant of matrix A is calculated as follows

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\det(A) = a.(e.i - h.f) - b(d.i - g.f) + c(d.h - g.e)$$

Step to calculate the reduce row echelon form for system of equations 3x+2y-1 = 1, 1x+2y+2z = 0, 2x+y-3z = -1?

$$\begin{pmatrix} 3 & 2 & -1 & 1 \\ 1 & 2 & 2 & 0 \\ 2 & 1 & -3 & -1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & -2 & 0 \\ 3 & 2 & -1 & 1 \\ 2 & 1 & -3 & -1 \end{pmatrix} \xrightarrow{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & -4 & -4 & 1 \\ 2 & 1 & -3 & -1 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 - 2R_1} \begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & -4 & -4 & 1 \\ 0 & -3 & -5 & -1 \end{pmatrix} \xrightarrow{R_2 = \frac{R_2}{-4}} \begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & 1 & -\frac{1}{4} \\ 0 & -3 & -5 & -1 \end{pmatrix}$$

$$\xrightarrow{R_3 = \frac{R_3}{-3}} \begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & 1 & -\frac{1}{4} \\ 0 & 1 & \frac{5}{3} & \frac{1}{3} \end{pmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & 1 & -\frac{1}{4} \\ 0 & 0 & \frac{2}{3} & \frac{7}{12} \end{pmatrix}$$

$$\xrightarrow{R_3 = \frac{3}{2} \cdot R_3} \begin{pmatrix} 1 & 2 & -2 & 0 \\ 0 & 1 & 1 & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{7}{8} \end{pmatrix}$$

Row echelon form

Questions

Q1)Calculate the determinant of matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 0 & 2 \\ 4 & 5 & 7 \end{bmatrix}$$

A) 12

B) 6

C) -20

D) 0

Explanation: calculating determinant of matrix

$$\det(A) = a(e.i - h.f) - b(d.i - g.f) + c(d.h - g.e)$$

$$\det(A) = 2(0 * 7 - 5 * 2) - 1(-1 * 7 - 4 * 2) + 3(-1 * 5 - 4 * 0)$$

$$\det(A) = 2(0 - 10) - 1(-7 - 8) + 3(-5 - 0)$$

$$\det(A) = -20 + 15 - 15$$

$$\det(A) = -20$$

Questions

Q1)What is the rank of a matrix?

A) The number of rows in the matrix

B) The number of columns in the matrix

C) The maximum number of linearly independent rows (or columns) in the matrix

D) The product of the number of rows and columns in the matrix

Q)If a matrix has full rank, what can we say about its determinant?

A) The determinant is always 0

B) The determinant is always 1

C) The determinant is always positive

D) The determinant is non-zero

Explanation: For a square matrix A of size (n*n) if the determinant is zero (det(mat)=0) then matrix rank is less than n

Questions

Q)For a square matrix A of size n × n, if the rank of A is less than n, what can we say about A?

A) A is a zero matrix

B) A is a diagonal matrix

C) A is a singular matrix

D) A is a symmetric matrix

Explanation: Singular matrix represents the matrix with determinant zero. Hence as per matrix properties answer is C.

Q) If a matrix A is of size m × n, where m < n, what can we say about its rank?

A) The rank is always m

B) The rank is always n

C) The rank is at most m

D) The rank is at most n

Question

Q)Find rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

a) 1

b) 2

c) 3

d) 0

Explanation: performing row operation

Step1) $R_2 \leftarrow \left(\frac{R_2}{4} - R_1\right); R_3 \leftarrow \left(\frac{R_3}{7} - R_1\right)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3/_4 & -3/_2 \\ 0 & -6/_7 & -12/_7 \end{bmatrix}$$

Step2) $R_2 \leftarrow (-4 * R_2); R_3 \leftarrow (-7 * R_3)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 6 & 12 \end{bmatrix}$$

Step3) $R_3 \leftarrow \left(\frac{R_3}{2} - R_2\right)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

Maximum number of linearly indepedent rows are 1 hence **rank(A)=2**

Question

Q)Find rank of matrix $A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 3 & 6 & 9 \end{bmatrix}$

a) 1

b) 2

c) 3

d) 0

Explanation: performing row operation

Step1) $R_2 \leftarrow (\frac{R_2}{2} - R_1)$

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \\ 3 & 6 & 9 \end{bmatrix}$$

Step2) $R_3 \leftarrow (\frac{R_3}{3} - R_1)$

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Maximum number of linearly indepedent rows are 1 hence **rank(A)=1**

Question

Q) Are these vectors independent or not? Calculate the rank and nullity of the set of vectors.

Vector A: (1, 2, 3, 4, 5)

Vector B: (-2, 0, 1, 3, -4)

Vector C: (3, 6, 9, 12, 15)

A) They are dependent; Rank = 3, Nullity = 0

B) They are dependent; Rank = 2, Nullity = 1

C) They are independent; Rank = 3, Nullity = 0

D) They are independent; Rank = 2, Nullity = 1

<u>**Explanation**</u>: performing row operation

$$P = \begin{bmatrix} 1, & 2, & 3, & 4, & 5 \\ -2, & 0, & 1, & 3, & -4 \\ 3, & 6, & 9, & 12\,, & 15 \end{bmatrix}$$

Step1) $R_3 \leftarrow (\frac{R_3}{3} - R_1)$

$$\begin{bmatrix} 1, & 2, & 3, & 4, & 5 \\ -2, & 0, & 1, & 3, & -4 \\ 0, & 0, & 0, & 0\,, & 0 \end{bmatrix}$$

Maximum number of linearly indepedent rows are 2 hence **rank(A)=2 and nullity=1**

Solving Linear Equation

- We consider the set of two equations that is written in matrix form

$$1x_1 + 3x_2 = 7 \qquad (1)$$

$$6x_1 + 2x_2 = 10 \qquad (2)$$

$$\begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

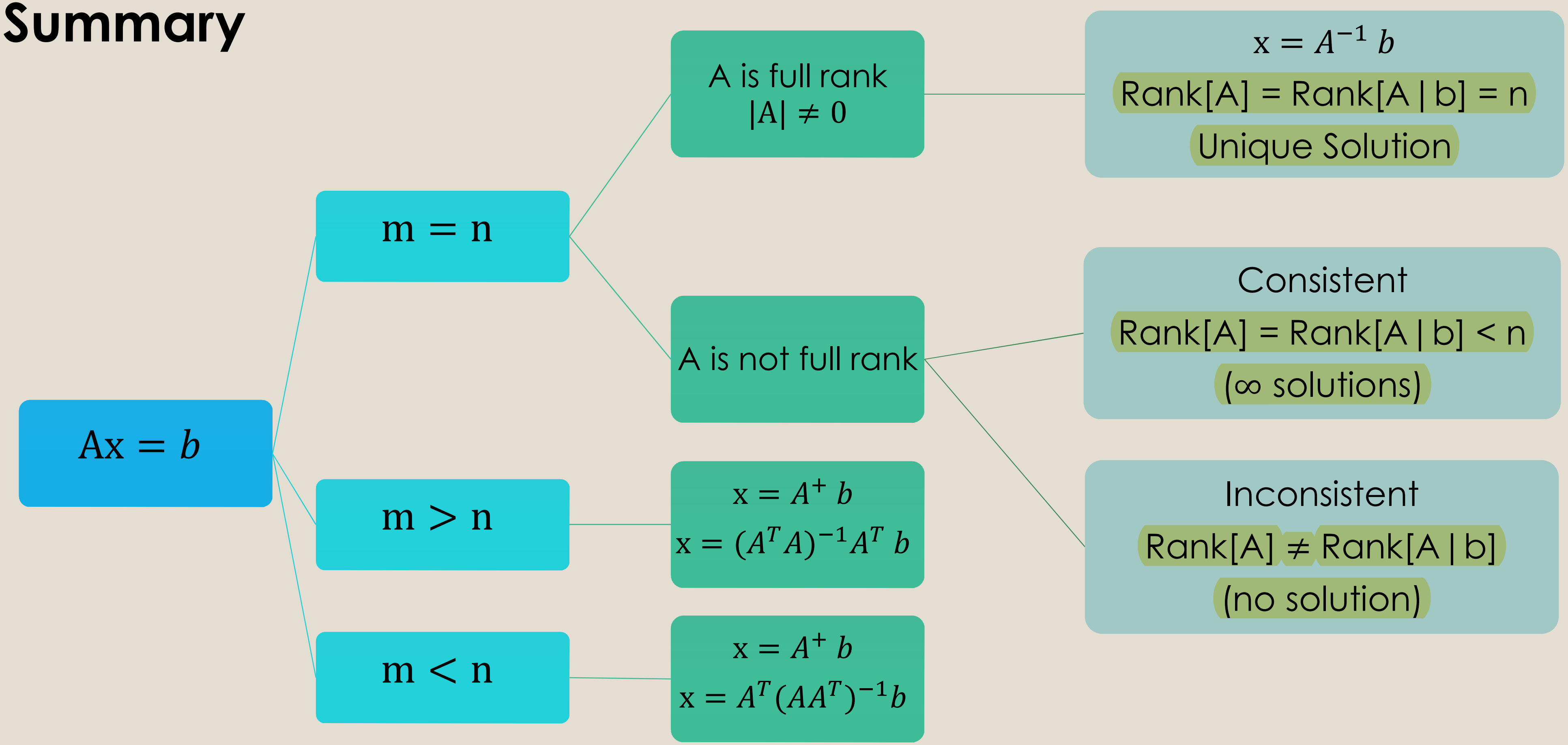
$$\underset{A}{\downarrow} \quad \underset{x}{\downarrow} \quad = \quad \underset{b}{\downarrow}$$

- The general form to represent the matrix form with **m rows** and **n columns**

$$A_{m\times n}\; x_{n\times 1} = b_{m\times 1}$$

- m = n → Easy to solve
- m > n → Usually no solution
- m < n → Multiple solution

Summary



Note: A^{+} is a pseudo matrix

Question

Q)Consider the following system of linear equations:

3x + 2y = 10

6x + 4y = 20

Which of the following statements is true about this system of equations?

A) The system has no solution.

B) The system has a unique solution.

C) The system has an infinite solutions.

Explanation: Convert equation into matrix form Ax=b

$$\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$$

Step1) Check m=n; If equal then calculate rank (A|b) and rank(A)

$$(\mathrm{A}|\mathrm{b}) = \begin{bmatrix} 3 & 2 & 10 \\ 6 & 4 & 20 \end{bmatrix}$$

Step2) $R_2 \leftarrow \left(\frac{R_2}{2} - R_1\right)$

$$\begin{bmatrix} 3 & 2 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, **rank(A|b)=1, rank(A)=1 and variables=2**

As **rank(A|b) = rank(A) ≠ n,** then it represents a system with **infinite solution**

Question

Q)Consider the following system of linear equations:

3x + 2y = 10

6x + 4y = 16

Which of the following statements is true about this system of equations?

A) The system has no solution.

B) The system has a unique solution.

C) The system has an infinite solutions.

Explanation: Convert equation into matrix form Ax=b

$$\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$$

Step1) Check m=n; If equal then calculate rank (A|b) and rank(A)

$$(\mathrm{A}|\mathrm{b}) = \begin{bmatrix} 3 & 2 & 10 \\ 6 & 4 & 16 \end{bmatrix}$$

Step2) $R_2 \leftarrow \left(\frac{R_2}{2} - R_1\right)$

$$\begin{bmatrix} 3 & 2 & 10 \\ 0 & 0 & -2 \end{bmatrix}$$

Here, **rank(A|b)=2, rank(A)=1 and variables=2**

As **rank(A|b) ≠ rank(A)**, then it represents a system with **no solution**

Question

Q)Consider the following system of linear equations:

3x + 2y = 10

6x + 2y = 16

Which of the following statements is true about this system of equations?

A) The system has no solution.

B) The system has a unique solution.

C) The system has an infinite solutions.

Explanation: Convert equation into matrix form Ax=b

$$\begin{bmatrix} 3 & 2 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$$

Step1) Check m=n; If equal then calculate rank (A|b) and rank(A)

$$(\mathrm{A}|\mathrm{b}) = \begin{bmatrix} 3 & 2 & 10 \\ 6 & 2 & 16 \end{bmatrix}$$

Step2) $R_2 \leftarrow \left(\frac{R_2}{2} - R_1\right)$

$$\begin{bmatrix} 3 & 2 & 10 \\ 0 & -1 & -2 \end{bmatrix}$$

Here, **rank(A|b)=2, rank(A)=2 and variables=2**

As **rank(A|b)** = **rank(A) = n,** then it represents a system with **unique solution**

Question

Q)Consider the system of equations:

2x + 3y = 8

4x + 6y = k

Find k for which equations have infinite number of solution?

A) k=12

B) k=4

C) k=16

D) K=8

<u>Explanation</u>: Convert equation into matrix form Ax=b

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \end{bmatrix}$$

Step1) Check m=n; If equal then calculate rank (A|b) and rank(A)

$$(\mathrm{A}|\mathrm{b}) = \begin{bmatrix} 2 & 3 & 8 \\ 4 & 6 & 16 \end{bmatrix}$$

Step2) $R_2 \leftarrow \left(\frac{R_2}{2} - R_1\right)$

$$\begin{bmatrix} 2 & 3 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

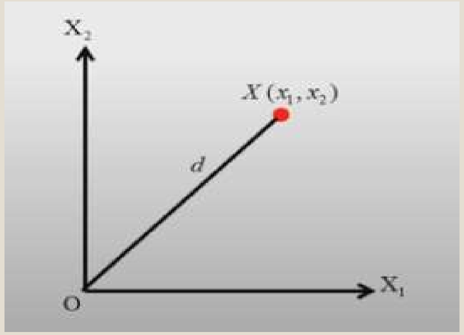
Here, **rank(A|b)=1, rank(A)=1 and variables=2**

As **rank(A|b)** = **rank(A) < n,** then it represents a system with **infinite solution**

Vector

- The vector X can be interpret as point in the 2D

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



- **Unit Vector:** The vector whose magnitude is unity

$$\hat{a} = \frac{A}{|A|}$$

- **Orthogonal Vectors:** The vectors whose dot product is zero

$$A^T B = 0$$

Que: Check whether $X1 = [1 \ -2 \ 0]^T$ and $X2 = [6 \ 3 \ 4]^T$ are perpendicular ?

A) Yes
B) No
C) Need more information

Explanation: Find the dot product of X1 and X2

$$X1^T X2 = [1 \ -2 \ 0] \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix} = 6 - 6 + 0 = 0$$

- The dot product equal to zero implies the vectors are orthogonal

Basis vectors

- Set of vectors that are independent and span the space
- Example: [1 0] and [0 1]

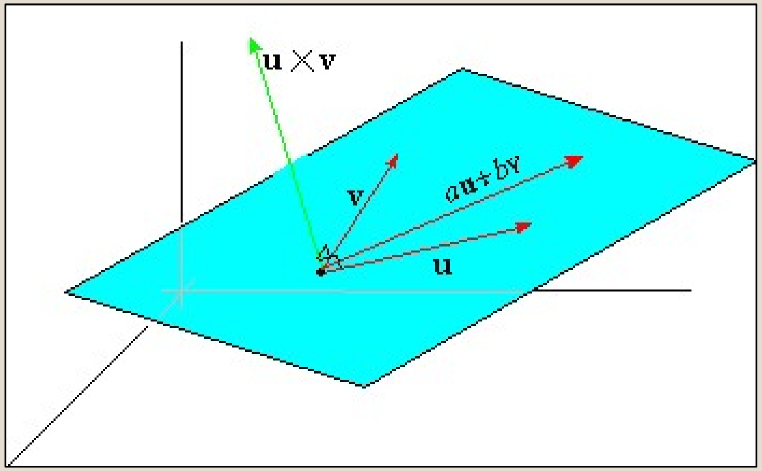
$$v_1 = \begin{bmatrix}1\\0\end{bmatrix} \quad v_2 = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\begin{bmatrix}2\\1\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\end{bmatrix} = 2v_1 + 1v_2$$

$$\begin{bmatrix}4\\4\end{bmatrix} = 4\begin{bmatrix}1\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\end{bmatrix} = 4v_1 + 4v_2$$

$$\begin{bmatrix}1\\3\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix} = 1v_1 + 3v_2$$

Basis vector u and v span the space



Question

Q) Consider two vectors in 3D space:

Vector A: (2, 1, 3)

Vector B: (-1, 2, 1)

Which of the following statements is true about the span of vectors A and B?

A) The span of A and B forms a line in 3D space.

B) The span of A and B forms a plane in 3D space.

C) The span of A and B forms a 3D space (the entire 3D space).

D) The span of A and B forms a point in 3D space.

Explanation: First find the rank of matrix [A B]

The rank([A B]) is 2, hence vectors limited to 2D space which is plane

Hence, it span **plane in 3D space**

Question

Q) Consider a 2D vector space V spanned by two basis vectors:

Vector A: (2, 1)

Vector B: (-1, 3)

Which of the following statements about the basis vectors is true?

A) The vectors A and B are linearly dependent.

B) The vectors A and B are linearly independent.

C) The vectors A and B are orthogonal to each other.

D) The vectors A and B form a basis for a 3D vector space.

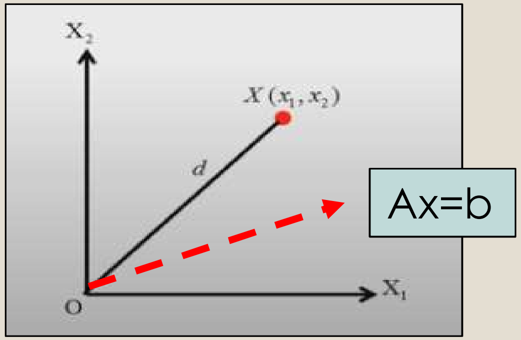
Explanation: First find the rank of matrix [A B]

The rank([A B]) is 2, hence vectors are linearly independent and span 2D

Eigen Vector and Eigen Value

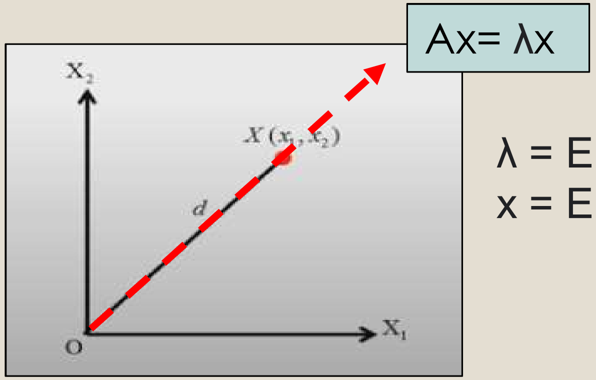
- Previously we see Ax=b, where A matrix transform vector x into vector b

$x \longrightarrow \boxed{A} \longrightarrow Ax = b$



- Eigen vector are the vectors whose direction not change after matrix transformation

$x \longrightarrow \boxed{A} \longrightarrow Ax = \lambda x$



λ = Eigen value
x = Eigen vector

Eigen Vector and Eigen Value

➢ Eigen values (λ) can be calculated as follows

$$A\,x = \lambda\,x$$

$$A\,x - \lambda \mathrm{I}\,x = 0$$

$$|\,A\; - \lambda I\,|\,x = 0$$

The eigen values (λ) can be obtained by solving below equation

$$|\,A\; - \lambda I| = 0$$

Finding the Eigen value and eigen vector of matrix $\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$?

- First we solve the equation $|A - \lambda I| = 0$ to find eigen values

$$|A - \lambda I| = \left| \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} -\lambda & 0 \\ -2 & -3-\lambda \end{bmatrix} \right| = \lambda^2 + 3\lambda + 2 = 0$$

$$\boldsymbol{\lambda_1 = -1} \qquad \boldsymbol{\lambda_2 = -2}$$

$[A - \lambda I]\, x = 0$ Putting $\lambda = -1$

$$[A - \lambda I]\, x = \left| \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right| x = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} x$$

$$\mathrm{x}_1 + \mathrm{x}_2 = 0 \;, \qquad -2\mathrm{x}_1 - 2\mathrm{x}_2 = 0$$

$$\mathrm{x}_1 = -\mathrm{x}_2$$

$$\mathrm{v}_1 = [\mathrm{x}_2 \; -\mathrm{x}_2] \rightarrow \mathbf{k}\,[\mathbf{1} \; -\mathbf{1}]$$

Question

Q)What are the eigenvalues of matrix A?

A) 1, 2

B) 4, 4

C) 2, 6

D) 4, 6

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$$

Explanation: we solve the equation $|A - \lambda I| = 0$

$$|A - \lambda I| = \left|\begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{bmatrix}\right| = (3-\lambda)(5-\lambda) - (-1) = 0$$

$$= 15 - 3\lambda - 5\lambda + \lambda^2 + 1 = \lambda^2 - 8\lambda + 16 = 0$$

$$= (\lambda - 4)^2 = 0$$

$$\lambda = 4,,4$$

Question

Q)What are the eigenvectors of matrix A?

A) [-1 1]

B) [1 1]

C) [1 0]

D) [0 1]

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}$$

Explanation: we solve the equation $|A - \lambda I| = 0$ and got λ =4, 4

$[A - \lambda I]\, x = 0$ Putting $\lambda = 4$

$$[A - \lambda I]\, x = \left| \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right| x = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} x = 0$$

$-x_1 - x_2 = 0$, $x_1 + x_2 = 0$

$x_1 = -x_2$

$v_1 = [-x_2\ x_2] \rightarrow x_2$ **[-1 1]**

Thank you!