Data Science for Engineers

WEEK 4 ASSIGNMENT

1) Let $f(x) = x^3 + 6x^2 - 3x - 5$. Select the correct options from the following:

- [ ] $-2 + \sqrt{5}$ will give the maximum for $f(x)$.
- [x] $-2 + \sqrt{5}$ will give the minimum for $f(x)$.
- [x] The stationary points for $f(x)$ are $-2 + \sqrt{5}$ and $-2 - \sqrt{5}$.
- [ ] The stationary points for $f(x)$ are −4 and 0.

$f'(x) = 3x^2 + 12x - 3$

$f'(x) = 0$

$x^2 + 4x - 1 = 0$

$x = -2 \pm \sqrt{5}$

$x_1 = -2 + \sqrt{5}$

$x_2 = -2 - \sqrt{5}$

$f''(x) = 6x + 12$

$f''(x)\big|_{x_1} = 6(-2+\sqrt{5}) + 12 = 6\sqrt{5} \rightarrow \text{min}$

$f''(x)\big|_{x_2} = -6\sqrt{5} \rightarrow \text{max}$

$-2 + \sqrt{5}$ will give the minimum for $f(x)$.
The stationary points for $f(x)$ are $-2 + \sqrt{5}$ and $-2 - \sqrt{5}$.

Use the following information to answer Q2 and Q3.

Consider the following optimization problem:

$$\max_{x \in \mathbb{R}} f(x)$$

, where

$$f(x) = x^4 + 7x^3 + 5x^2 - 17x + 3$$

Let x^* be the maximizer of $f(x)$.

$f'(x) = 4x^3 + 21x^2 + 10x - 17$

$f'(x) = 0$

$x = -1.432, -4.48, 0.662$

2) What is the second order sufficient condition for x^* to be the maximizer of the function $f(x)$?

- $4x^3 + 21x^2 + 10x - 17 = 0$
- $12x^2 + 42x + 10 = 0$
- $12x^2 + 42x + 10 > 0$
- ✓ $12x^2 + 42x + 10 < 0$

$f''(x) = 12x^2 + 42x + 10$

Accepted Answers:
$12x^2 + 42x + 10 < 0$

$f''(0.662) = 12 \times (0.662)^2 + 42 \times 0.662 + 10 = 43$

$f''(-1.432) = -74.59 \quad < 0$

$f''(-4.48) = -419 \quad < 0$

3) Find the value of x^*.

- -4.48
- 0.66
- ✓ -1.43
- 4.45

Accepted Answers:
-1.43

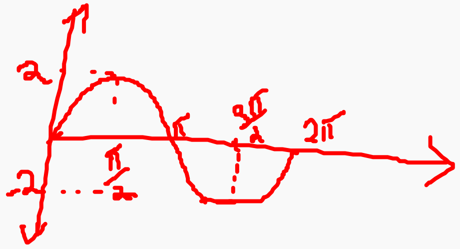
$-1.43\cancel{7}, -4.48$

$f(-1.43) = -1.43$ max

$f(-4.48) = -47.07$

4) Let $f(x) = 2\sin x, 0 \leq x \leq 2\pi$. Select the correct options from the following:

- [ ] $\frac{\pi}{2}$ is the global maximum of $f(x)$.
- [ ] π is the global minimum of $f(x)$.
- [ ] $\frac{3\pi}{2}$ is the global maximum of $f(x)$.
- [ ] $\frac{3\pi}{2}$ is the global minimum of $f(x)$.



Accepted Answers:

$\frac{\pi}{2}$ is the global maximum of $f(x)$.

$\frac{3\pi}{2}$ is the global minimum of $f(x)$.

Use the following information to answer Q5, Q6, Q7 and Q8.

Let $f(x) = 2x_1^2 + 3x_1x_2 + 3x_2^2 + x_1 + 3x_2$.

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 4x_1 + 3x_2 + 1 \\ 3x_1 + 6x_2 + 3 \end{bmatrix}$$

5) Find the gradient for $f(x)$.

- $\nabla f = \begin{bmatrix} 4x_1 + 3x_2 + 1 \\ 3x_1 + 6x_2 + 3 \end{bmatrix}$
- $\nabla f = \begin{bmatrix} 3x_1 + 6x_2 + 3 \\ 4x_1 + 3x_2 + 1 \end{bmatrix}$
- $\nabla f = \begin{bmatrix} 4x_1 + 3x_2 \\ 3x_1 + 6x_2 \end{bmatrix}$
- $\nabla f = \begin{bmatrix} 4x_2 + 3x_1 + 1 \\ 3x_2 + 6x_1 + 3 \end{bmatrix}$

Accepted Answers:

$\nabla f = \begin{bmatrix} 4x_1 + 3x_2 + 1 \\ 3x_1 + 6x_2 + 3 \end{bmatrix}$

Use the following information to answer Q5, Q6, Q7 and Q8.

Let $f(x) = 2x_1^2 + 3x_1x_2 + 3x_2^2 + x_1 + 3x_2$.

$$\nabla f = \begin{matrix} 4x_1 + 3x_2 + 1 \\ 3x_1 + 6x_2 + 3 \end{matrix} = 0$$

6) Find the stationary point for $f(x)$.

- 0.6, 0.4
- −0.6, −0.4
- ✓ 0.2, −0.6
- 0.2, 0.6

$$\frac{\partial f}{\partial x_1} = 4x_1 + 3x_2 + 1 = 0$$

$$\frac{\partial f}{\partial x_2} = 3x_1 + 6x_2 + 3 = 0$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.2 \\ -0.6 \end{bmatrix}$$

Accepted Answers:
0.2, −0.6

$$\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \frac{\partial^2 f}{\partial x_1 x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 3 & 6 \end{bmatrix}$$

Use the following information to answer Q5, Q6, Q7 and Q8.

Let $f(x) = 2x_1^2 + 3x_1x_2 + 3x_2^2 + x_1 + 3x_2$.

7) Find the Hessian matrix for $f(x)$.

- [ ] $\nabla^2 f = \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}$
- [ ] $\nabla^2 f = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$
- [x] $\nabla^2 f = \begin{bmatrix} 4 & 3 \\ 3 & 6 \end{bmatrix}$
- [ ] $\nabla^2 f = \begin{bmatrix} 6 & 3 \\ 3 & 4 \end{bmatrix}$

Accepted Answers:
$\nabla^2 f = \begin{bmatrix} 4 & 3 \\ 3 & 6 \end{bmatrix}$

$|A - \lambda I|$ $\begin{vmatrix} 4-\lambda & 3 \\ 3 & 6-\lambda \end{vmatrix}$ $(4-\lambda)(6-\lambda) - 9 = 0$

$\cancel{\lambda =}$

$\lambda^2 - 10\lambda + 15 = 0$

λ_1 8.16

λ_2 1.84

Use the following information to answer Q5, Q6, Q7 and Q8.

Let $f(x) = 2x_1^2 + 3x_1x_2 + 3x_2^2 + x_1 + 3x_2$.

8) The stationary point obtained in Q6 is a

- maxima
- minima
- saddle point

Accepted Answers:
minima

9) Let $f(x_1, x_2) = 4x_1^2 - 4x_1x_2 + 2x_2^2$. Select the correct options from the following:

- [ ] (2, 4) is a stationary point of $f(x)$.
- [x] (0, 0) is a stationary point of $f(x)$.
- [x] The Hessian matrix $\nabla^2 f$ is positive definite.
- [ ] The Hessian matrix $\nabla^2 f$ is not positive definite.

$$\nabla f = \begin{matrix} 8x_1 - 4x_2 \\ -4x_1 + 4x_2 \end{matrix} \qquad (x_1, x_2) = 0, 0$$

$$\nabla^2 f = \begin{bmatrix} 8 & -4 \\ -4 & 4 \end{bmatrix} \qquad \begin{vmatrix} 8-\lambda & -4 \\ -4 & 4-\lambda \end{vmatrix}$$

$$(8-\lambda)(4-\lambda) - 16 = 0$$

$$\begin{matrix} 10.47 \\ 1.52 \end{matrix} > 0$$

Accepted Answers:
(0, 0) is a stationary point of $f(x)$.
The Hessian matrix $\nabla^2 f$ is positive definite.

10) In optimization problem, the function that we want to optimize is called

- Decision function
- Constraints function
- Optimal function
- Objective function

Accepted Answers:
Objective function

11) The optimization problem $\min_{x} f(x)$ can also be written as $\max_{x} f(x)$.

- True
- False ✓

Accepted Answers:
False

12) In the gradient descent algorithm, the step size should always be same for each iteration.

- ○ True
- ○ False

Accepted Answers:
False

Extra Questions

Answer questions 1 to 4 using the given objective function $f(x) = 3x^4 - 2x^3 - 3x^2 + 6$

1) The first order necessary condition for either maxima or minima of f(x) is

- $6x^3 - 3x^2 - 6x = 0$
- $12x^3 - 6x^2 - 6x = 0$
- $12x^3 - 9x^2 - 6x = 0$
- None of these

Answer questions 1 to 4 using the given objective function $f(x) = 3x^4 - 2x^3 - 3x^2 + 6$

1) **Solution: b)**

Feedback: The first order necessary condition for finding the stationary point(s) of a given function $f(x)$ is

$$f'(x) = \frac{\delta f(x)}{\delta x} = 0$$

Differentiating the function w.r.t x and equating to 0 gives the equation

$$12x^3 - 6x^2 - 6x = 0$$

Answer questions 1 to 4 using the given objective function $f(x) = 3x^4 - 2x^3 - 3x^2 + 6$

2) Which of the following point(s) is/are stationary point(s) of $f(x)$?

- [x] $\frac{-1}{2}$
- [x] 0
- [x] 1
- [ ] $\frac{1}{2}$

$12x^3 - 6x^2 - 6x = 0$

$x = 0, 1, -\frac{1}{2}$

Answer questions 1 to 4 using the given objective function $f(x) = 3x^4 - 2x^3 - 3x^2 + 6$

2) Solution: a) b) c)

Feedback: Solving the equation $12x^3 - 6x^2 - 6x = 0$, one of the roots of x is 0.

This means that, $x(12x^2 - 6x - 6) = 0$. Solving for the equation $12x^2 - 6x - 6 = 0$ gives you the other 2 roots.

Solving the above equation, we get the roots $x = 1, \frac{-1}{2}$

Hence, the stationary points are $x = 0, 1, \frac{-1}{2}$

Answer questions 1 to 4 using the given objective function $f(x) = 3x^4 - 2x^3 - 3x^2 + 6$

3) The stationary point(s) which maximize(s) the value of $f(x)$ is

- $\frac{-1}{2}$
- 0
- 1
- $\frac{1}{2}$

$f''(x) = \frac{d}{dx}(12x^3 - 6x^2 - 6x)$

$= 36x^2 - 12x - 6$

$f''(0) = -6 \longrightarrow \max$

$f''(1) = 18$

$f''(-\frac{1}{2}) = 36 \times \frac{1}{4} - 12 \times -\frac{1}{2} - 6$

$= 9 + 6 - 6 = 9$

Answer questions 1 to 4 using the given objective function $f(x) = 3x^4 - 2x^3 - 3x^2 + 6$

3) Solution: b)

Feedback: The condition for a stationary point to be a local maximum is

$$f''(x) = \frac{\delta^2 f(x)}{\delta x^2} < 0$$

Differentiating the function twice w.r.t x gives us $f''(x) = 36x^2 - 12x - 6$

We can observe that by substituting the values of stationary points, we can see that the points $x = 0$ to satisfy the second-order sufficient condition for local maximum

Answer questions 1 to 4 using the given objective function $f(x) = 3x^4 - 2x^3 - 3x^2 + 6$

4) The stationary point(s) which minimize(s) the value of $f(x)$ is

- [x] $\frac{-1}{2}$
- [ ] 0
- [x] 1
- [ ] $\frac{1}{2}$

4) Solution: a) c)

Feedback: The condition for a stationary point to be a local minimum is

$$f''(x) = \frac{\delta^2 f(x)}{\delta x^2} > 0$$

Differentiating the function twice w.r.t x gives us $f''(x) = 36x^2 - 12x - 6$

We can observe that by substituting the values of stationary points, we can see that the points $x = 1$ and $x = \frac{-1}{2}$ to satisfy the second-order sufficient condition for local minimum

5) Find the minima of the function $f(x) = (x-5)^2 - 5$ using gradient search method starting from $x = -6$, and learning rate $\alpha = 0.5$ and choose the correct statement from the options given below

- [x] Minimum is $f(x) = -5$
- [ ] Minimum is $f(x) = -16$
- [ ] $x = -11$ yields the minimum of the function
- [x] $x = 5$ yields the minimum of the function

$f(x)\big|_5 = -5$

$f(x) = (x-5)^2 - 5$

$= x^2 - 10x + 25 - 5 = x^2 - 10x + 20$

$f'(x_k) = 2x - 10$

$x_{k+1} = x_k - \alpha f'(x_k)$

$x_0 = -6$

$\alpha = 0.5$

$x_1 = -6 - (0.5)(2 \times -6 - 10)$

$= -6 - 0.5(-22)$

$= -6 + 11 = 5$

$x_2 = x_1 - \alpha f'(x_1) = 5 - \frac{1}{2}(0)$

$= 5$

5) Find the minima of the function $f(x) = (x-5)^2 - 5$ using gradient search method starting from $x = -6$, and learning rate $\alpha = 0.5$ and choose the correct statement from the options given below

5) Solution: a) d)

Feedback: The formula for finding the next search point, given the initial search point and the learning rate are

$$x_{k+1} = x_k - \alpha f'(x_k)$$

$f'(x_k) = 2(x - 5)$. Substituting $x_0 = -6,\ \alpha = 0.5,$

$$x_1 = x_0 - \alpha \nabla f'(x_0) = -6 - 0.5 * 2 * (-6 - 5) = 5$$

$$x_2 = x_1 - \alpha \nabla f'(x_1) = 5 - 0.5 * 2 * (5 - 5) = 5$$

Since, all iterations give the same value for x, the minimum value for x is 5.

Substituting $x_{min} = 5,\ f(x_{min}) = (5 - 5)^2 - 5 = -5$

6) The gross domestic product (GDP) of a country in billion dollars following a crisis (at $t = 0$) is given by: $G(t) = -0.196t^3 + 3.244t^2 + 9.179$ for $0 \le t \le 28$. When is the GDP highest in the given time period?

- t = 4.903
- t = 11.034 ✓
- t = 3.269
- t = 8.295

$G'(t) = (-0.196 \times 3)t^2 + (3.244 \times 2)t = -0.588t^2 + 6.488t$

$t = 0$

$t = \frac{6.488}{-0.588} = 11.03$

$G''(t) = (-0.588 \times 2)t + 6.488$

$G''(0) = 6.488$

$G''(11.03) = -6.48 < 0 \rightarrow \text{max}$

6) The gross domestic product (GDP) of a country in billion dollars following a crisis (at $t = 0$) is given by: $G(t) = -0.196t^3 + 3.244t^2 + 9.179$ for $0 \leq t \leq 28$. When is the GDP highest in the given time period?

6) **Solution: b)**

Feedback: Differentiating the function $G(t)$ and equating it to zero gives the stationary point(s)

Solving $G'(t) = 0$ gives $t = 11.034$. Hence, the optimum value achieved by the function is at $t = 11.034$

7) A function is defined as: $7x^2 + 70x + 12$. Find the value of x at its stationary point.

- $x = 10$
- $x = 0.071$
- $x = -350$
- $x = -5$

$f'(x) = 14x + 70$

7) A function is defined as: $7x^2 + 70x + 12$. Find the value of x at its stationary point.

7) **Solution: d)**

Feedback: Differentiating the function and equating it to zero gives the stationary point(s). From the given equation, it can be verified that $x = -5$ is a stationary point of the function

Practice Questions

1) Class of optimization problems **WITH NO** constraints are known as

- constrained optimization problems
- unconstrained optimization problems
- linear constrained optimization problems
- none of the above

2) The optimum for a function $f(x)$ at x^*, exists if:

- If the first derivative at x^* is zero
- If the first derivative at x^* is positive
- If the first derivative at x^* is negative
- None of the above

3) When the feasibility regions defined by equality constraints and inequality constraints are compared,

- ○ The regions defined by both are exactly the same
- ○ The region defined by the inequality constraint is greater
- ○ The region defined by the equality constraint is greater
- ○ none of the above

4) If $f(x) = 12x^4 - 2x^3 + 9x^2 + 5$, then the first order necessary condition for either maxima or minima of $f(x)$ is

- $24x^2 + 4x - 6 = 0$
- $48x^3 - 6x^2 + 18x = 0$
- $36x^3 - 2x^2 - 6x = 0$
- $48x^2 - 4x - 6 = 0$

5) The restrictions on the possible values of the solution to the optimization problem are called:

- ○ objective functions
- ○ cost functions
- ○ equality/inequality constraints
- ○ none

Example using R

Q: A company wants to maximize the profit for two products A and B which are sold at $ 25 and $ 20 respectively. There are 1800 resource units available every day and product A requires 20 units while B requires 12 units. Both of these products require a production time of 4 minutes and total available working hours are 8 in a day. What should be the production quantity for each of the products to maximize profits?

Ans:

A B

Resource 20 12 8hrs

4 4

This is a maximization problem

The objective function is to be defined:

$$\max(Sales) = \max(25y_1 + 20y_2)$$

- y_1 is the units of Product A produced
- y_2 is the units of Product B produced
- y_1 and y_2 are called the decision variables
- 25 and 20 are the selling price of the products

$20y_1 + 12y_2 \le 1800$

$20y_1 + 12y_2 \leq 1800$ (***Resource Constraint***)

$4y_1 + 4y_2 \leq 8 * 60$ (***Time constraint***)