DATA SCIENCE FOR ENGINEERS

WEEK 2

Highlights

Topic	Details
Matrix	Introduction of Matrix/ Independent and Dependant Variables
Solving Linear Equations	Solving system of linear equations
Geometric view of Linear Algebra	Vector/Norm/Orthogonal Vector/Basis vector, Half space
Eigen Values and Eigen vectors	Eigen values, vectors, symmetric matrix, null space

Matrix

- Organisation of data into rows and columns
- Rows: Samples; Columns: Variables or Features
- Example: Biomedical dataset of 1000 patients

Sl No	Weight	Height	Age
1	55	140	20
.	.	.	.
.	.	.	.
.	.	.	.
1000	75	165	60

Independent and Dependant Variable

- Independent Variable: The variables that store unique information
- Dependant Variable: The variable formed by combination of already existing Variables BMI
- Example: BMI in the dataset is a dependant variable

$$BMI = \frac{W}{H^2}$$

Sl No	Weight	Height	Age	BMI
1	55	140	20	28.06
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
1000	75	165	60	27.55

Rank and Nullity

- Rank: Number of independent variables or samples
- Nullity: Number of linear relationship

Nullity + **Rank** = **Total number of Variables**

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

Number of Variables = 3
Rank of A = 2
Nullity of A = 1

Note: Col_2 = 2 * Col_1

3×3

$R_3 \to R_3 - R_2$

2

2×3

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 0 \\ 3 & 6 & 0 \end{bmatrix} = 2\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 0 \\ 3 & 3 & 0 \end{bmatrix}$$

Rank = 2

$n = 3$
$r = 2$ 1

1. Find the rank of the matrix $A = \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 6 \\ 4 & 5 & 9 \end{matrix}$

Rank 3

- Rank of matrix

  - 0
  - 1
  - 2
  - 3

$|A| = / \neq 0$

Linear dependent.

$|A| \neq 0 \quad = 1[27-30] - 2[18-24] + 3[10-12]$

$= -3 + 12 - 6 = 3$

2. Let M be a 2 × 2 matrix, with trace equal to 4 and determinant equal to 3. Find the eigen values of M

- −1, −3
- −1, 3
- 1, −3
- 1, 3

λ_1, λ_2

Trace $= 4$

~~det~~ $|A| = 3$

$\lambda_1 + \lambda_2 = 4$

$\lambda_1 \cdot \lambda_2 = |A|$

$\lambda_2 = 4 - \lambda_1$

$\lambda_1(4 - \lambda_1) = 3$

$4\lambda_1 - \lambda_1^2 = 3$

$\lambda_1^2 - 4\lambda_1 + 3 = 0$

$(\lambda_1 - 3)(\lambda_1 - 1) = 0$

$\lambda_1 = 3, 1$

$\lambda_1 = 3$

$\lambda_2 = 4 - \lambda_1$

$= 1$

$4\lambda_1 - \lambda_1^2 = 3$

$4\lambda_1 - \lambda_1^2 - 3 = 0$

$\lambda_1^2 - 4\lambda_1 + 3 = 0$

$\lambda_1 = 1$

$\lambda_2 = 4 - \lambda_1$

$= 4 - 1$

$= 3$

$Ax = \lambda x$ $\quad n \times n \quad x$

$A_{2\times 2}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad \lambda$

3. Which among the following can be the eigen vector for the matrix $A = \begin{matrix} 1 & 2 \\ 4 & 3 \end{matrix}$

$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $\quad |A - \lambda I| = 0$ $\quad \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$

- [x] $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ✓
- [ ] $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✗
- [ ] $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ✗
- [x] $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ✓

$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$ $\quad (1-\lambda)(3-\lambda) - 8 = 0$ $\quad 3 - 3\lambda - \lambda + \lambda^2 - 8 = 0$

$\lambda^2 - 4\lambda - 5 = 0$

$(\lambda - 5)(\lambda + 1) = 0$

$\lambda = 5, -1$

$\lambda = 5$

$Ax = \lambda x$

$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_1 + 2x_2 = 5x_1 \Rightarrow x_2 = 2x_1$

$2x_2 = 4x_1 \quad x_2 = 2x_1$

$4x_1 + 3x_2 = 5x_2$

$4x_1 = 2x_2$

$2x_1 = x_2$

$x_1 = 1$

$x_2 = 2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\lambda = -1$

$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_1 + 2x_2 = -x_1$

$4x_1 + 3x_2 = -x_2$

$2x_1 + 2x_2 = 0$

$x_1 = -x_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad x_2 = 1 \quad x_1 = -1$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$x_2 = 1$

$x_1 = -1$ $\quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

4. Let $A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$. Suppose the eigen values corresponding to AA^T are a, b and c, then find the value of $ab + bc + ac$ $= 0$

- 9
- 0
- 81
- 18

$A = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}_{3\times 1}$

$A^T = \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}_{1\times 3}$

$AA^T = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} \Rightarrow$ dependent

$|AA^T| = 0$

Rank of $AA^T = 1$

$-2 \times \begin{bmatrix} 1 & -2 & -2 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{bmatrix}$

Rank $= 1$

Nullity $= 2$ $\rightarrow$ 2 eigen values are 0

$\begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = 9$

$a = ?$ 9

$b = 0$

$c = 0$

5. Which among the following is true for an n×n matrix A

$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$

$A \Rightarrow \lambda$ $\qquad \lambda = \frac{1}{\lambda}$

- If λ is an eighen value of A, then $1/\lambda$ is an eighen value of A^{-1}.
- $\det(A^{-1}) = (\det(A))^{-1}$
- Let B be an $n \times n$ matrix. Then $\det(AB) = \det(A)\det(B)$
- All of the above

$|A^{-1}| =$

6. Check if the following set of vectors are linearly independent.

$$v_1 = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, v_3 = \begin{bmatrix} 3 \\ -3 \\ 8 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ -1 & 1 & -3 \\ 3 & -2 & 8 \end{bmatrix}$$

A

$|A| = 0$

$2(8-6) - 1(-8+9) + 3(2-3)$

$= +4 - 1 - 3 = 0$

$|A| = 0 \rightarrow$ linearly dependent

7. Which of the following set of vectors are orthogonal?

A

$$\begin{bmatrix} 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2 + 0 - 2 = 0$$

- [x] $\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

- [ ] $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$
- [ ] $\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 2 \end{bmatrix}$
- [x] $\begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix}$

$$\begin{bmatrix} 5 & -2 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \\ 6 \end{bmatrix} = 0$$

8. Suppose the eigen values of matrix A are −3,−4,−4,−3, then the determinant of the matrix $(A^{-1})^T$ is

$A \quad |A| = \lambda \quad A^{-1} = \frac{1}{\lambda}$

- 0.06
- -14
- 0
- ✓ 0.0069

Pdt of eigen values $= |A|$

$|A^T| = \lambda$

$|(A^{-1})^T|$

$|A| = \lambda$
$= -3 \times -4 \times -4 \times -3$
$= 144$

$|(A^{-1})| = \frac{1}{144}$

$|(A^{-1})^T| = \frac{1}{144}$

Extra Questions

Q1. Trace of a 3×3 matrix is 9. A has an eigenvalue 9 of multiplicity 2. Find the determinant of A

$\lambda_1, \lambda_2 = 9, 9$

- Trace of a matrix = Sum of its eigenvalues
- Given eigen values: 9,9
- Let third eigen value be λ_3
- Trace = $\lambda_1 + \lambda_2 + \lambda_3 = 9$
- $\lambda_3 = -9$
- Determinant of A = $\lambda_1 * \lambda_2 * \lambda_3 = -729$

Q2. Which of the following vector sets is/are orthogonal

a) $V1 = [5 \ \ -2 \ \ 3]^T, V2 = [-2 \ \ 4 \ \ 6]^T$
b) $V1 = [10 \ \ -2]^T, V2 = [14 \ \ 5]^T$
c) $V1 = [1 \ \ -2 \ \ 4]^T, V2 = [2 \ \ 5 \ \ 2]^T$
d) $V1 = [1 \ \ -2 \ \ 4]^T, V2 = [-1 \ \ 4]^T$

- Find the dot product of V1 and V2
- The dot product equal to 0 implies orthogonal vectors
- Dot product = $V1^T V2$

$V_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$ $V_2 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$

$V_1^T \cdot V_2$

$= \begin{bmatrix} 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ \cdot \end{bmatrix}$

$= -1 - 8 + 0 = -9$

Q3. If $A = \begin{bmatrix} 7 & 4 \\ 6 & 5 \end{bmatrix}$, then the value of $|A^5 - A^4|$ is

$$A^5 - A^4 = \begin{bmatrix} 96631 & 64420 \\ 96630 & 64421 \end{bmatrix} - \begin{bmatrix} 8785 & 5856 \\ 8784 & 5857 \end{bmatrix} = \begin{bmatrix} 87846 & 58564 \\ 87846 & 58564 \end{bmatrix}$$

$A^5 - A^4$ is singular

$|A^5 - A^4|$ would be zero

Q4. The inverse of the matrix $M = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 3 & 9 & 8 \end{bmatrix}$ is

- $1/15 \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 5 & 10 & 8 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \\ 5 & 10 & 8 \end{bmatrix}$
- $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 3 & 9 & 8 \end{bmatrix}$
- Inverse does not exist

$|A| = 0$

- Inverse does not exist since columns are ______________

Q5. Which of the following is true about orthonormal vectors?

- [ ] All orthonormal vectors are not orthogonal
- [ ] All orthonormal vectors are orthogonal
- [ ] Two vectors are orthonormal to each other if their dot product is 0
- [ ] Orthonormal vectors are orthogonal vectors with unit magnitude

- A set of vectors are said to be orthonormal, if each vector is ______ with ____________________

Q6. Which of the following sets of column vectors form a basis for R^4?

$$\begin{bmatrix} 13 & 10 & 13 & 5 \\ 12 & 10 & 16 & 3 \\ 13 & 10 & 10 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 15 & 11 & 2 & 1 \\ 6 & 1 & 10 & 7 \\ 11 & 2 & 14 & 11 \\ 14 & 12 & 14 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 14 & 2 & 6 \\ 6 & 84 & 12 & 36 \\ 11 & 154 & 22 & 66 \\ 4 & 56 & 8 & 24 \end{bmatrix}$$

None of the options

- Option a)
- Option c)
- Option b)
- Ans.

$$\begin{bmatrix} 13 & 10 & 13 & 5 \\ 12 & 10 & 16 & 3 \\ 13 & 10 & 10 & 0 \end{bmatrix} \quad 3 \times 4$$

$$\begin{bmatrix} 1 & 14 & 2 & 6 \\ 6 & 84 & 12 & 36 \\ 11 & 154 & 22 & 66 \\ 4 & 56 & 8 & 24 \end{bmatrix}$$

$C2 = 14 \times C_1$

$C3 = 2 \times C_1$

$C4 = 6 \times C_1$

$$\begin{bmatrix} 15 & 11 & 2 & 1 \\ 6 & 1 & 10 & 7 \\ 11 & 2 & 14 & 11 \\ 14 & 12 & 14 & 2 \end{bmatrix}$$

7) Let $\lambda_1 = 1$ and $\lambda_2 = 2$ be the eigenvalues, and $v_1 = [2 \; -2]^T$ and $v_2 = [3 \;\; 1]^T$ be the eigenvectors of a real matrix A. Let B be a projection matrix given as $B = [v1 \;\; v2]$. Compute $B^{-1}AB$.

- $\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$
- $\begin{bmatrix} 7 & 0 \\ 0 & 14 \end{bmatrix}$
- ✓ $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$
- $\begin{bmatrix} -2 & 0 \\ 0 & -4 \end{bmatrix}$

$B = [v_1 \; v_2]$

$\underbrace{A = PDP^{-1}}$

$P = \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, and B = P

$= (P^{-1}P)D(PP^{-1}) = $ D (Since, $P^{-1}P = PP^{-1} = I_{2\,x\,2}$)

$B^{-1}AB = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$B = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$

$B = \begin{bmatrix} 2 & 3 \\ -2 & 1 \end{bmatrix} = P$

$B = P$

$D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$A = PDP^{-1}$

$P^{-1}A = \underbrace{P^{-1}P}_{I} D P^{-1}$

$P^{-1}AP = I \times D \underbrace{P^{-1}P}_{I}$

$P^{-1}AB = D$

$P = B$

$D = B^{-1}AB$

$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Q8. How many solutions does the system of linear equations $15x + 20y = 25, 21x + 28y = 35$ have?

$$\begin{bmatrix} 15 & 20 & 25 \\ 21 & 28 & 35 \end{bmatrix}$$

$$\begin{matrix} 3 & 4 & 5 \\ 3 & 4 & 5 \end{matrix}$$

- Based on the given set of equations, it can be observed that the coefficients are linearly dependent. Hence, the given system of equations can have infinite number of solutions

Q9. Given the matrix $\begin{bmatrix} -31 & 12 \\ 12 & 14 \end{bmatrix}$, the eigenvectors are

Ch eqn : $|A \quad -\lambda I| = 0;\ |\lambda^2 + 17\lambda - 578| = 0$

$\lambda_1 = 17; \lambda_2 = -34$

Assume v_i to initially be a vector $[x\ y]^T$

Substituting the eigenvalues in the equation $(A \ - \ \lambda_i I)v_i \ = \ 0$ and solving for x and y, the eigenvector is obtained

The eigenvector for eigenvalue -34 is $[-\ 4\ 1]^T$, and the eigenvector for eigenvalue 17 is $[1\ 4]^T$

Q10. Consider the matrix A. The columns of matrix A form a basis for the null space of another square matrix B. Find the rank of the matrix B.

$$A = \begin{bmatrix} -9 & 3 & 8 \\ 3 & 8 & 7 \\ 8 & -5 & 6 \\ 6 & 5 & 8 \\ -3 & -5 & -9 \end{bmatrix}$$

- The rank of the given matrix is 3. Since the columns of matrix A form the basis for the null space of B, the nullity of B is 3. Also, the nullspace of B (y) should solve the equation $By = 0$. Since the basis of nullspace of B is of size 5 x 3 and B is a square matrix, B should be of size 5 x 5.
- According to rank-nullity theorem,
- Number of columns = Rank + Nullity
- Hence, Rank = 5 - 3 = 2