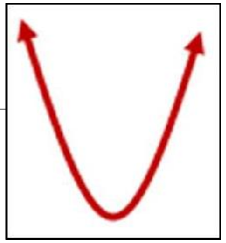
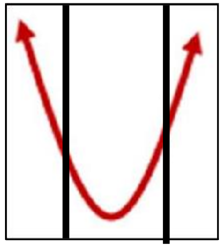
Week 5

THEORY

Constraints Optimization

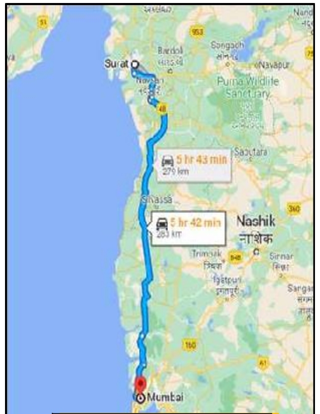


➢Un<u>constraint optimization</u>: In the last week, the functions we examined were unconstrained, meaning they either had ***no boundaries, or the boundaries were soft.***

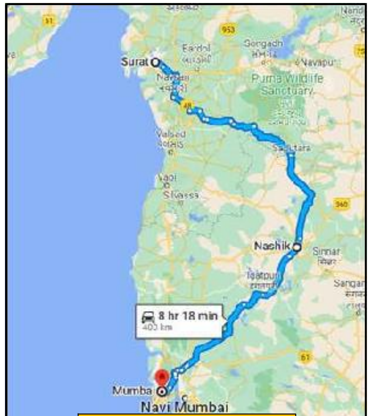


➢<u>Constraint optimization</u>: In this week, we will be examining the functions with constraint. A constraint is a ***hard limit placed on the value of a variable***, which ***prevents us from going forever in certain directions.***

➢ <u>Example</u>: Travel car from Surat to Mumbai Constraint → must visit the Nashik city

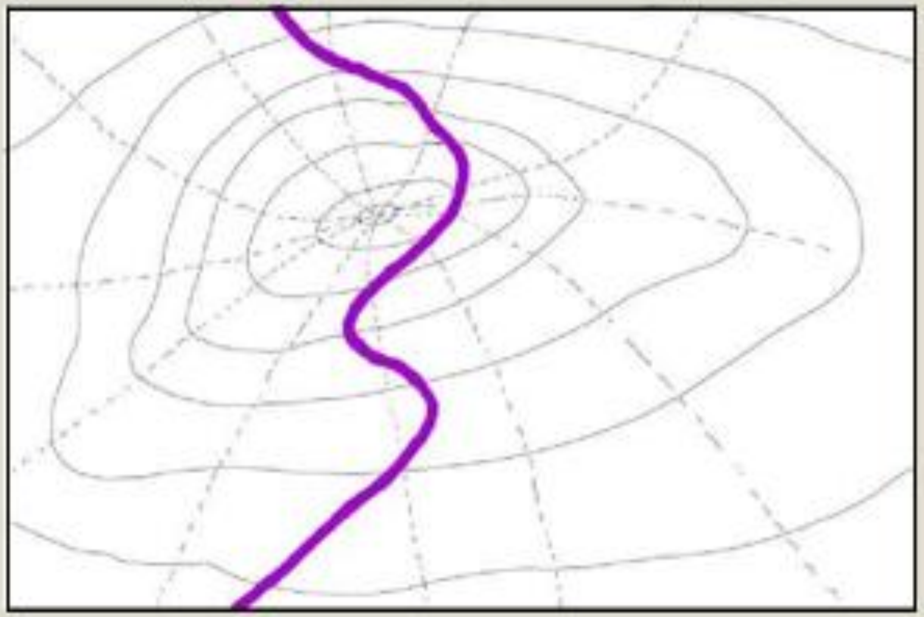


Unconstraint



Constraint

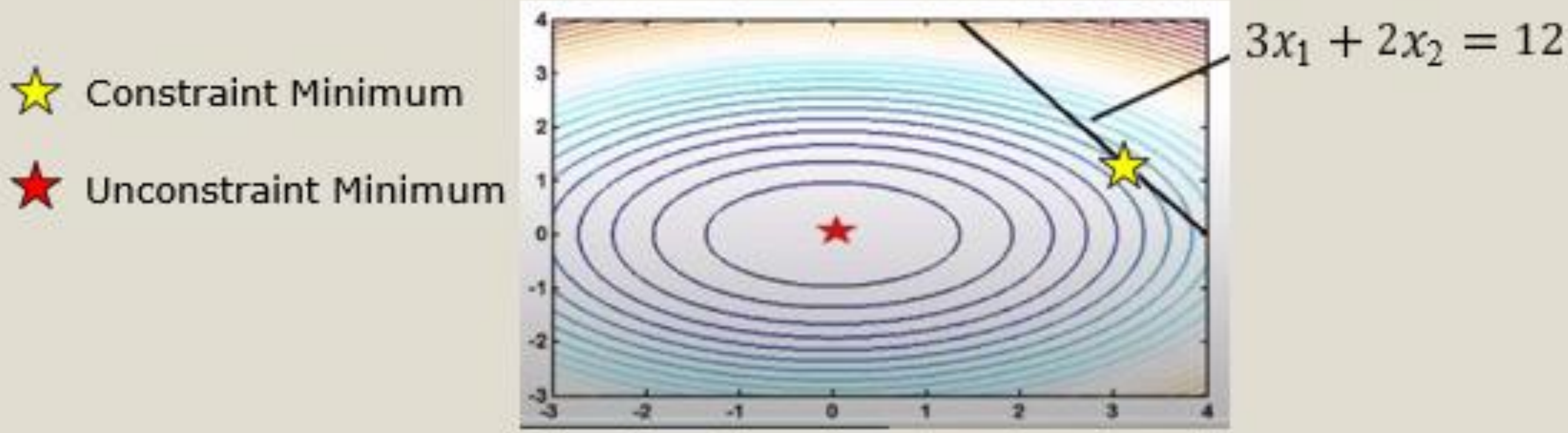
- ***Task/Problem:*** Climb as high as possible on the mountain to have a better view of the moon.
- ***Constraint***: The car must be on the road



Equality constraint optimization

$$\min_{x1,x2} \; 2x_1^2 + 4x_2^2$$

$$st \quad 3x_1 + 2x_2 = 12$$



Q: For a rectangle whose perimeter is 20m, find the dimensions that will maximize the area?

- Let consider breadth = b, length = l , Area = A and Perimeter = P

$$\text{Maximize } f(x) = A = l \times b$$

$$st \;\; g(x) = 2l + 2b = 20$$

$$-\nabla f = \lambda \nabla g$$

- Applying 1st order condition

$$\underline{\nabla f(l,b) = -\lambda \nabla g(l,b)}$$

$$\nabla f = \begin{bmatrix} b \\ l \end{bmatrix} \qquad \nabla g = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

- $b = -2\lambda$ (1)
- $l = -2\lambda$ (2)
- $2l + 2b = 20$ (3)

$b = -2\lambda$

$l = -2\lambda$

- Solving above equation (1) and (2) **gives** $\underline{l = b}$. Putting these value in equation (3) gives

$$l = 5m \text{ and } b = 5m$$

- Calculating the dimensions, the Area of the rectangle is $A = 5*5 = 25$

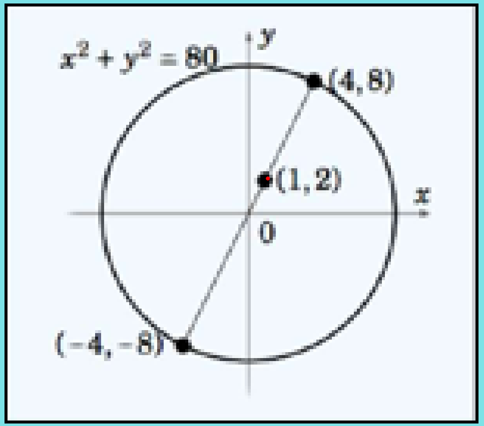
Q: Find the points on the circle $x^2 + y^2$ -80 which are closest to & farthest from the point (1,2)?

- Distance d from any point (x,y) to the point (1,2) is $d = \sqrt{(x-1)^2+(y-2)^2}$

$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

Maximize $f(x,y) = (x-1)^2+(y-2)^2$

$g(x) = x^2 + y^2 = 80$



- Applying 1st order condition

$\underline{\nabla f(x,y) = -\lambda \nabla g(x,y)}$

$\frac{\partial f}{\partial x} = 2(x-1)$ $\frac{\partial g}{\partial x} = 2x$

$\frac{\partial f}{\partial y} = 2(y-2)$ $\frac{\partial g}{\partial y} = 2y$

$x - 1 = -\lambda x$

$y - 2 = -\lambda y$

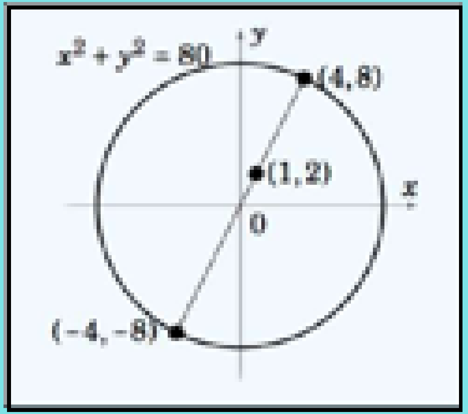
- Putting all derivatives into the main 1st order equation gives

$$\frac{x-1}{x} = -\lambda \quad (1) \qquad \frac{y-2}{y} = -\lambda \quad (2) \qquad x^2 + y^2 = 80 \quad (3)$$

- Solving equation (1) and (2) gives **y = 2x.** This value substituted in equation (3)

$$x^2 + y^2 = 80 \rightarrow x^2 + (2x)^2 = 80 \rightarrow x^2 + 4x^2 = 80 \rightarrow \boldsymbol{x = \pm 4}$$

- Putting x values in equation (3) gives $y = \pm 8$
- So we obtained the points where (4,8) is the nearest and (-4,-8) is the farthest point from (1,2)



Q: $f(x,y,z) = x + z$
$g(x,y,z) = x^2 + y^2 + z^2 - 1$

$\nabla f = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\nabla g = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$

$\nabla f = \lambda \nabla g$

$\nabla f = -\lambda \nabla g$

- Applying 1st order condition

$$\underline{\nabla f(x,y,z) = -\lambda\, \nabla g(x,y,z)}$$

$$1 = -2\lambda x \quad (1) \qquad 0 = -2\lambda y \quad (2) \qquad 1 = -2\lambda z \quad (3)$$

- Solving equation (1) and (3) we get **x = z**. And with equ. (2) we get **y = 0**
- Substituting above variables in the constraint equation

$$x^2 + y^2 + z^2 = 1 \rightarrow \qquad x^2 + x^2 = 1 \rightarrow \qquad 2x^2 = 1 \rightarrow \qquad x = \pm\sqrt{\frac{1}{2}}$$

- Calculating other point by putting x in the above equations gives

Pt.1$=\left(\sqrt{\frac{1}{2}}, 0, \sqrt{\frac{1}{2}}\right)$ **pt.2** $=\left(-\sqrt{\frac{1}{2}}, 0, -\sqrt{\frac{1}{2}}\right)$

Inequality constraint optimization

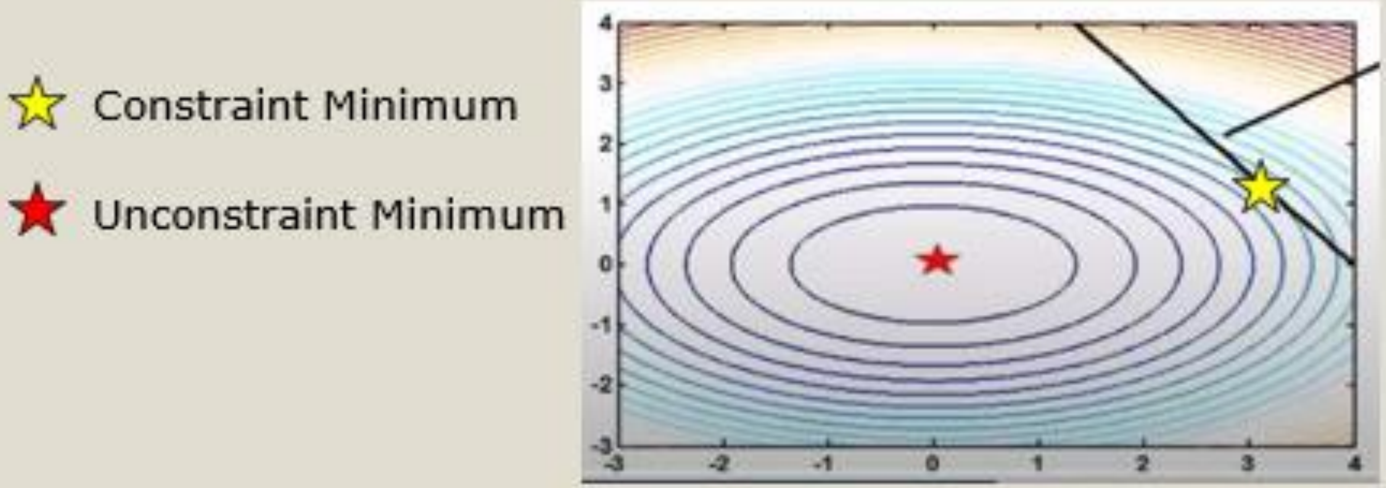
equality $\nabla f = -\lambda \nabla g$

$$\min_{x1,x2} \; 2x_1^2 + 4x_2^2$$

$$st \quad 3x_1 + 2x_2 \geq 12$$

Constraint Minimum

Unconstraint Minimum



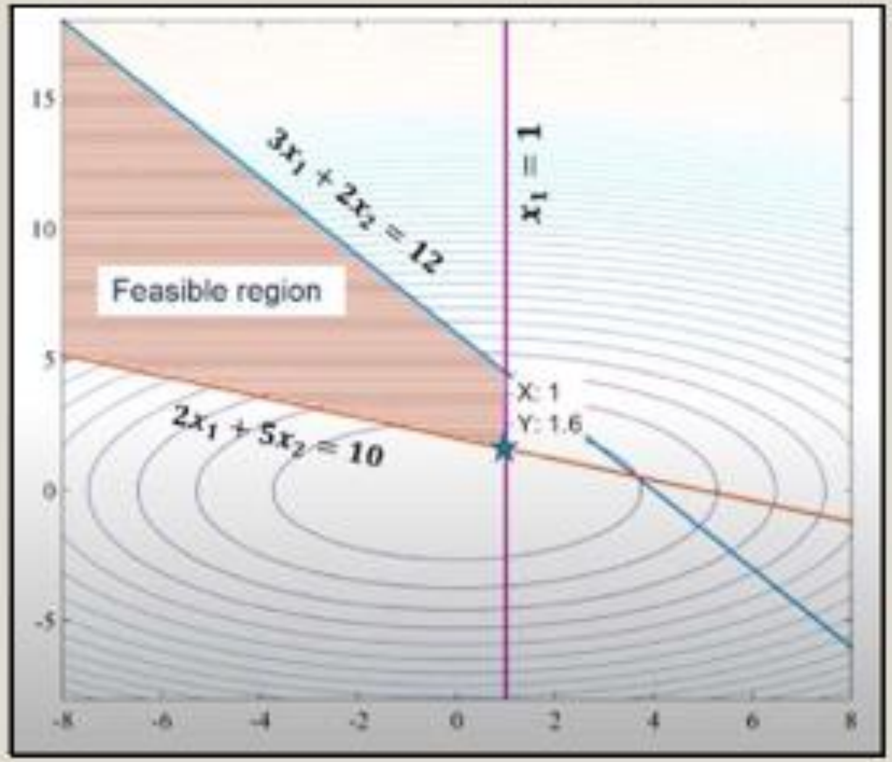
Multivariate inequality constraint

$$\min_{x1,x2} \quad {}_1^2 + 4x_2^2$$

$$st \;\; 3x_1 + 2x_2 \le 12$$

$$2x_1 + 5x_2 \le 10$$

$$x_1 \le 1$$



$$\min_{x_1, x_2} \quad 2x_1^2 + 4x_2^2$$

$$st$$

$$3x_1 + 2x_2 \le 12 \Rightarrow (a)$$

$$2x_1 + 5x_2 \ge 10 \Rightarrow (b)$$

$$x_1 \le 1 \Rightarrow (c) \qquad x_1 - 1 \le 0$$

$$(b) \rightarrow \quad 10 - 2x_1 - 5x_2 \le 0$$

$$(a) \rightarrow \quad 3x_1 + 2x_2 - 12 \le 0$$

- Lagrangian

$$L(x_1, x_2, \mu_1, \mu_2, \mu_3) = 2x_1^2 + 4x_2^2 + \mu_1(3x_1 + 2x_2 - 12)$$
$$+\mu_2(10 - 2x_1 - 5x_2) + \mu_3(x_1 - 1)$$

- First order KKT conditions

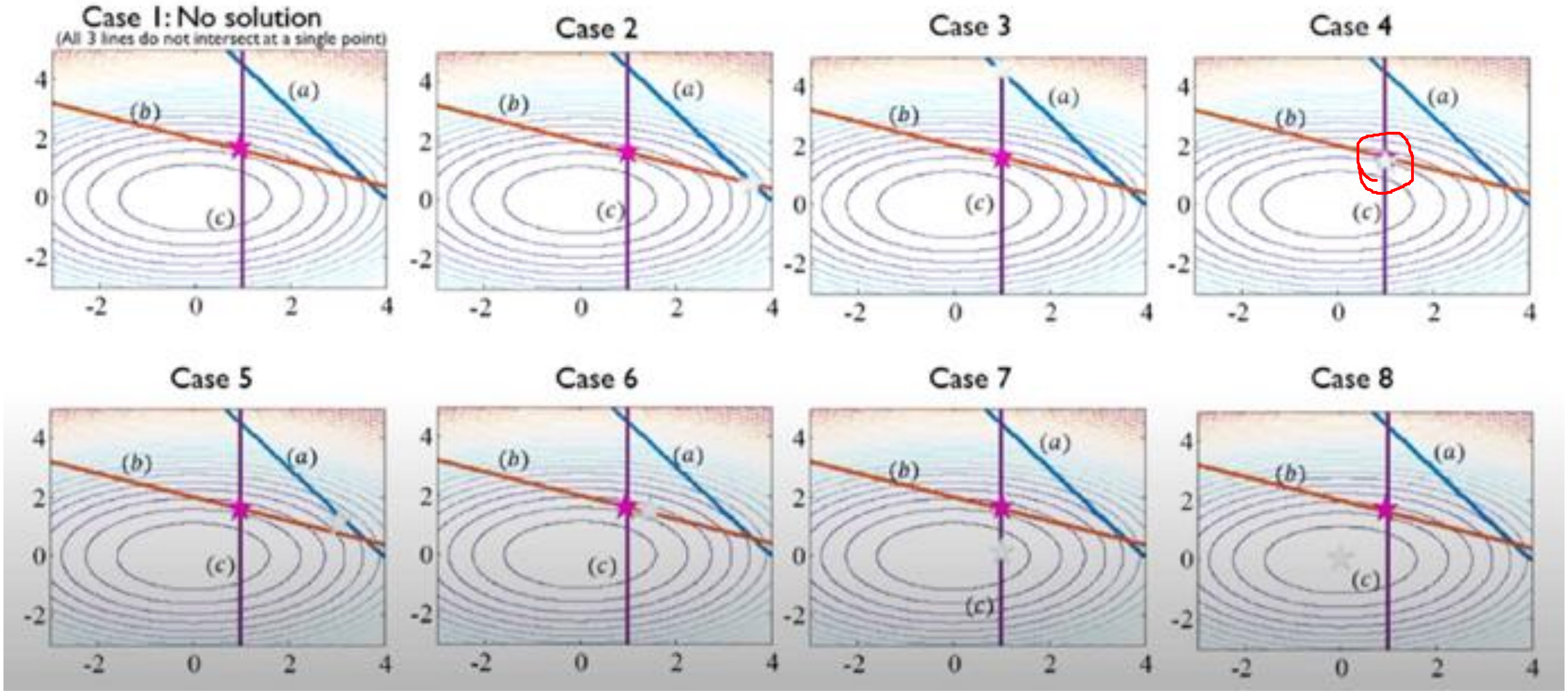
$$\frac{\partial L}{\partial x_1} = 4x_1 + 3\mu_1 - 2\mu_2 + \mu_3$$

$$4x_1 + 3\mu_1 - 2\mu_2 + \mu_3 = 0 \quad — ①$$

$$\frac{\partial L}{\partial x_2} \qquad 8x_2 + 2\mu_1 - 5\mu_2 = 0 \quad — ②$$

$$\mu_1(3x_1 + 2x_2 - 12) = 0$$
$$\mu_2(10 - 2x_1 - 5x_2) = 0$$
$$\mu_3(x_3 - 1) = 0$$
$$\mu_i \ge 0$$

Sl.no	Active (A) /Inactive (I) constraints (a)	(b)	(c)	Solution (x, μ)	Possible optima (Y/N)	Remark
1	A	A	A	Infeasible	N	Equations do not have a valid solution.
2	A	A	I	$x = [3.6364 \quad 0.5455]$ $\mu = [-5.2 \quad -1.45 \quad 0]$	N	$x_1 \leq 1$ is not satisfied, $\mu_1 < 0$, $\mu_2 < 0$
3	A	I	A	$x = [1 \quad 4.5]$ $\mu = [-18 \quad 0 \quad 50]$	N	$\mu_1 < 0$
4	I $\mu_1 = 0$	A	A	$x = [1 \quad 1.6]$ $\mu = [0 \quad 2.56 \quad 1.12]$	Y	All constraints and KKT conditions satisfied
5	A	I $\mu_2 = 0$	$\mu_3 = 0$ I	$x = [3.27 \quad 1.09]$ $\mu = [-4.36 \quad 0 \quad 0]$	N	$x_1 \leq 1$ is not satisfied
6	I	A	I	$x = [1.21 \quad 1.51]$ $\mu = [0 \quad 2.45 \quad 0]$	N	$x_1 \leq 1$ is not satisfied
7	I	I	A	$x = [1 \quad 0]$ $\mu = [0 \quad 0 \quad -4]$	N	$2x_1 + 5x_2 \geq 10$ is not satisfied
8	I	I	I	$x = [0 \quad 0]$ $\mu = [0 \quad 0 \quad 0]$	N	$2x_1 + 5x_2 \geq 10$ is not satisfied


Case 1: No solution
(All 3 lines do not intersect at a single point)
Case 2
Case 3
Case 4
Case 5
Case 6
Case 7
Case 8
(a)
(b)
(c)

Week 5

ASSIGNMENT QUESTIONS

1) Which of the following statements is/are not TRUE with respect to the multi variate optimization?

I - The gradient of a function at a point is parallel to the contours
II - Gradient points in the direction of greatest increase of the function
III - Negative gradient points in the direction of the greatest decrease of the function
IV - Hessian is a non-symmetric matrix

- ○ I
- ○ II and III
- ○ I and IV
- ○ III and IV

Accepted Answers:
I and IV

2) The solution to an unconstrained optimization problem is always the same as the solution to the constrained one.

- True
- False

Accepted Answers:
False

3) Gradient based algorithm methods compute

- only step length at each iteration
- both direction and step length at each iteration
- only direction at each iteration
- none of the above

Accepted Answers:
both direction and step length at each iteration

4) For an unconstrained multivariate optimization given $f(\bar{x})$, the necessary second order condition for $\bar{x}^*$ to be the minimizer of $f(x)$ is

a ○ $\nabla^2 f(\bar{x}^*)$ must be negative definite.

b ○ $\nabla^2 f(\bar{x}^*)$ must be positive definite.

c ○ $\nabla f(\bar{x}^*) = 0$

d ○ $f''(\bar{x}^*) > 0$

Accepted Answers:
$\nabla^2 f(\bar{x}^*)$ *must be positive definite.*

Use the following information to answer Q5, 6, 7 and 8

$$\min_{x_1, x_2 \in \mathbb{R}} f(x_1, x_2) = x_1^2 + 4x_2^2 - 2x_1 + 8x_2.$$

5) Which among the following is the stationary point for $f(x_1, x_2)$?

- $(0, 0)$
- ✓ $(1, -1)$
- $(-1, -1)$
- $(-1, 1)$

Accepted Answers:
$(1, -1)$

$f = x_1^2 + 4x_2^2 - 2x_1 + 8x_2$

$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 2x_1 - 2 \\ 8x_2 + 8 \end{bmatrix}$

$\nabla f = 0$

$2x_1 - 2 = 0 \quad x_1 = 1$

$8x_2 + 8 = 0 \quad x_2 = -1$

$\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$

$|A - \lambda I| = 0$

$\begin{vmatrix} 2-\lambda & 0 \\ 0 & 8-\lambda \end{vmatrix} = 0$

$(2-\lambda)(8-\lambda) = 0$

$\lambda = 2, 8$

6) Find the eigen values corresponding to Hessian matrix of f.

- $1, -1$
- $1, 1$
- $2, 8$
- $0, 2$

Accepted Answers:
$2, 8$

7) Find the minimum value of f.

- 0
- -5
- -1
- 1

Accepted Answers:
-5

$x_1, x_2 = 1, -1$

$x_1^2 + 4x_2^2 - 2x_1 + 8x_2$

$1 + 4 - 2 - 8 = -5$

$f(x_1, x_2)\Big|_{x_1 = 1,\ x_2 = -1}$

8) What is the minimum value of $f(x_1, x_2)$ subject to the constraint $x_1 + 2x_2 = 7$?

- -5
- -1
- 27
- 0

$f(x_1,x_2) = x_1^2 + 4x_2^2 - 2x_1 + 8x_2$

$g(x_1,x_2) = x_1 + 2x_2 = 7$ $\quad x_1 + 2x_2 - 7 = 0$ — 2

$\nabla f = \begin{bmatrix} 2x_1 - 2 \\ 8x_2 + 8 \end{bmatrix}$

$\nabla g = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\nabla f = -\lambda \nabla g$

$2x_1 - 2 = -\lambda(1)$

$2x_1 = 2 - \lambda$

$x_1 = \frac{2-\lambda}{2}$

$8x_2 + 8 = -\lambda(2)$

$8x_2 = -8 - 2\lambda$

$x_2 = \frac{-8-2\lambda}{8}$

Accepted Answers:
27

9) Find the maximum value of $f(x, y) = 49 - x^2 - y^2$ subject to the constraint $x + 3y = 10$.

- 49
- 46
- 59
- 39 ✓

$f(x,y) = 49 - x^2 - y^2$

$\nabla f = \begin{bmatrix} -2x \\ -2y \end{bmatrix}$

$x + 3y - 10 = 0$

$\nabla g = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Accepted Answers:
39

$\nabla f = -\lambda \nabla g$

$-2x = -\lambda \qquad -2y = -3\lambda$

$x = \frac{\lambda}{2} \qquad y = \frac{3\lambda}{2}$

$\frac{\lambda}{2} + 3\left(\frac{3\lambda}{2}\right) - 10 = 0$

$\frac{10\lambda}{2} - 10 = 0$

$\lambda = 2$

$x = 1$

$y = 3$

$f(x,y)\Big|_{(1,3)} = 49 - 1 - 9$

$= \underline{\underline{39}}$

10) Consider an optimization problem $\min\limits_{x_1, x_2} x^2 - xy + y^2$ subject to the constraints

$$2x + y \leq 1$$
$$x + 2y \geq 2$$
$$x \geq -1$$

$L(x, y, \mu_1, \mu_2, \mu_3) = x^2 - xy + y^2 + \mu_1(2x + y - 1)$
$+ \mu_2(2 - x - 2y)$
$+ \mu_3(-x - 1)$

Find the lagrangian function for the above optimization problem.

- ✓ $L(x, y, \mu_1, \mu_2, \mu_3) = x^2 - xy + y^2 + \mu_1(2x + y - 1) + \mu_2(2 - x - 2y) + \mu_3(-x - 1)$
- $L(x, y, \mu_1, \mu_2, \mu_3) = x^2 - xy + y^2 + \mu_1(2x + y - 1) + \mu_2(x + 2y - 2) + \mu_3(-x - 1)$
- $L(x, y, \mu_1, \mu_2, \mu_3) = x^2 - xy + y^2 + \mu_1(2x + y - 1) + \mu_2(x + 2y - 2) + \mu_3(x + 1)$
- $L(x, y, \mu_1, \mu_2, \mu_3) = x^2 - xy + y^2 + \mu_1(1 - 2x - y) + \mu_2(2 - x - 2y) + \mu_3(-x - 1)$

Accepted Answers:
$L(x, y, \mu_1, \mu_2, \mu_3) = x^2 - xy + y^2 + \mu_1(2x + y - 1) + \mu_2(2 - x - 2y) + \mu_3(-x - 1)$

Practice Questions

Consider the objective function $max\, f(x) = xy$ subject to $x + y^2 \leq 2$ and $x, y \geq 0$. The Lagrangian function is given by $L(x, y, \mu_1, \mu_2, \mu_3) = xy - \mu_1(-x - y^2 + 2) - \mu_2 x - \mu_3 y$

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1) Which of the following is not an apt representation of the constraint?

- $x + y^2 \leq 2$
- $-x \geq 0$
- $-x \leq 0$

1) Solution: b)

Feedback: The given constraint has not been mentioned as a part of the problem.

Consider the objective function $max\, f(x) = xy$ subject to $x + y^2 \leq 2$ and $x, y \geq 0$. The Lagrangian function is given by $L(x, y, \mu_1, \mu_2, \mu_3) = xy - \mu_1(-x - y^2 + 2) - \mu_2 x - \mu_3 y$

2) The values of μ_1, μ_2 and μ_3 while evaluating the Karush-Kuhn-Tucker (KKT) condition with all the constraints being inactive are

- $\mu_1 = \mu_2 = \mu_3 = 1$
- $\mu_1 = \mu_2 = \mu_3 = 0$
- $\mu_1 = \mu_3 = 0, \mu_2 = 1$
- $\mu_1 = \mu_2 = 0, \mu_3 = 1$

2) Solution: b)

Feedback: When all the constraints are inactive, $\mu_1 = \mu_2 = \mu_3 = 0$.

Consider the function $f(x, y) = 5x^2 + 3y^2 + 8xy + 12x + 6y$ as the function to be optimized and answer question 3.

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3) The saddle point of the function $f(x, y)$ exists in which of the following coordinates (x, y).

- ✓ $(6, -9)$
- $(5, 3)$
- $(2, -3)$
- There is no saddle point

$\frac{\partial f}{\partial x} = 10x + 8y + 12$ $\quad 10x + 8y = -12$

$\frac{\partial f}{\partial y}$ $\quad 6y + 8x + 6$ $\quad 6y + 8x = -6$

$y = -9$

$x = 6$

3) Solution: a)

Feedback: To find the saddle point, we have to solve the equations $\frac{\delta f(x,y)}{\delta x} = 0$ and $\frac{\delta f(x,y)}{\delta y} = 0$. This gives us the equations $10x + 8y = -12$ and $8x + 6y = -6$. Solving the equations, we get $x = 6$ and $y = -9$. Hence, the saddle point is $(6, -9)$.

4) A circle with center at origin is defined as: $x^2 + y^2 = 250$. Find the points on the circle at the minimum distance from the point (10, 1).

$\min f(x,y) = (x-10)^2 + (y-1)^2$ — ①

$\sqrt{(x-10)^2 + (y-1)^2}$

- [x] $x = 15.733, y = 1.573$
- [ ] $x = 10.911, y = 11.443$
- [ ] $x = 1.573, y = 15.733$
- [ ] $x = 4.564, y = 15.138$

$g(x,y) = x^2 + y^2 - 250 = 0$

$$\frac{\partial f(x,y)}{\partial x} * \frac{\partial g(x,y)}{\partial y} - \frac{\partial f(x,y)}{\partial y} \times \frac{\partial g(x,y)}{\partial x} = 0$$

$$2(x-10) \times 2y - 2(y-1) \times 2x = 0$$

$$4xy - 40y - 4xy + 4x = 0$$

$x = 10y$

$100y^2 + y^2 = 250$

$y = \pm 1.573$

$x = \pm 15.733$

✓ $(15.733, 1.573)$

$(-15.733, -1.573)$

4) Solution: a)

Feedback: The optimization problem to find out a given point $(x,\ y)$ can be formulated as follows

$$\min f(x, y) = (x - 10)^2 + (y - 1)^2,$$
$$s.t.\ g(x, y) = x^2 + y^2 - 250$$

The condition for finding the stationary point(s) in case of a multivariate optimization problem is

$$\frac{\delta f(x,y)}{\delta x} * \frac{\delta g(x,y)}{\delta y} - \frac{\delta f(x,y)}{\delta y} * \frac{\delta g(x,y)}{\delta x} = 0$$

$$\Rightarrow 2(x - 10) * 2y - 2(y - 1) * 2x = 0$$

$$\Rightarrow 4xy - 40y - 4xy + 4x = 0$$

Solving this equation, we get $x = 10y$. Substituting for x in the constraint, we get $y = \pm 1.573$, and hence $x = \pm 15.73$. Hence the optimum values are (15.73, 1.573) and (-15.73, -1.573)

Substituting the coordinates of the points, we can observe that the point (15.73, 1.573) is closer to the point (10, 1). Hence, $x = 15.73,\ y = 1.573$

5) A function is defined as, $f(x, y) = 10x + 5y + 29$. Find the maximum such that the constraint $5x + 5y^2 = 44$ is satisfied using a Lagrange multiplier.

- [x] $x = 8.738, y = 0.25$
- [ ] $x = 7.8, y = 1$
- [ ] $x = 8.55, y = 0.5$
- [ ] $x = 4.8, y = 2$

$$L(x,y,\lambda) = 10x + 5y + 29 - \lambda(5x + 5y^2 - 44)$$

$$L'(x,y,\lambda) = \begin{bmatrix} \frac{\partial L(x,y,\lambda)}{\partial x} \\ \frac{\partial L}{\partial y} \\ \frac{\partial L}{\partial \lambda} \end{bmatrix} = \begin{bmatrix} 10 - 5\lambda \\ 5 - 10\lambda y \\ -5x - 5y^2 + 44 \end{bmatrix} = 0$$

$\lambda = 2$

$5 - 20y = 0$

$y = 1/4$

$-5x = -44 + 5y^2$

$x = \frac{44 - 5y^2}{5}$

$=$

Solution: a)

Feedback: The Lagrangian function for the given problem can be defined as

$$L(x, y, \lambda) = 10x + 5y + 29 - \lambda(5x + 5y^2 - 44)$$

Differentiating the function w.r.t the decision variables, we get

$$L'(x, y, \lambda) = \begin{bmatrix} \frac{\partial L(x,y,\lambda)}{\partial x} \\ \frac{\partial L(x,y,\lambda)}{\partial y} \\ \frac{\partial L(x,y,\lambda)}{\partial \lambda} \end{bmatrix}$$

Equating $L'(x, y, \lambda)$ to 0, we get $\lambda = 2,\ y = 0.25,\ x = 8.7375$

6) A manufacturer incurs a monthly fixed cost of \$7350 and a variable cost, $C(m) = 0.001m^3 - 2m^2 + 324m$ dollars. The revenue generated by selling these units is, $R(m) = -6m^2 + 1065m$. How many units produced every month (m) will generate maximum profit?

- m = 46
- m = 90 ✓
- m = 231
- m = 125

$R(m) = -6m^2 + 1065m$

Total cost $TC(m) = 7350 + 0.001m^3 - 2m^2 + 324m$

max $P(m) = R(m) - TC(m)$

$= -0.001m^3 - 4m^2 + 741m - 7350$

$P(m) = -0.001m^3 - 4m^2 + 741m - 7350$

$\frac{\partial P}{\partial m} = -0.003m^2 - 8m + 741 = 0$

$m = -2756$

$m \approx 90$

89.61

6) Solution: b)

Feedback: The total cost (per month) incurred by the company for manufacturing **m** units is $TC(m) = 0.001m^3 - 2m^2 + 324m + 7350$, and the revenue is $R(m) = -6m^2 + 1065m$. Hence, the profit (objective function) upon manufacturing **m** units is given by

$$P(m) = -0.001m^3 - 4m^2 + 741m - 7350$$

Based on the first derivative necessary condition for the function $P(m)$, we get $m = -\underline{2756.28}, \underline{89.61\ (\sim 90)}$

7) An optimization problem, solved for N variables, with one equality constraint will have

- N equations in N variables
- N + 1 equations in N + 1 variables
- N equations in N + 1 variables
- None of these

Solution: b)

Feedback: This problem contains N + 1 variables: N decision variables and 1 multiplier variable λ corresponding to the constraint, which can be found using N + 1 equations: N equations by solving $\nabla f(x_i) = \lambda \nabla h(x_i)$ and 1 equation by solving the constraint $h(x_i) = 0$ $(where\ i = 1, 2, ..., n)$

Extra Questions

1) For a function $f(x, y) = 2x^2 - xy + y^2 - 3x - y$, the stationary point (x, y) is
(Hint: Stationary point is a solution to the first order necessary conditions for maxima or minima of $f(x, y)$)

- (0,1)
- (-1,0)
- (1,0)
- ✓ (1,1)

$$\frac{\partial f}{\partial x} = 4x - y - 3$$

$$\frac{\partial f}{\partial y} = -x + 2y - 1$$

2) The Hessian matrix of $f(x, y) = 2x^2 - xy + y^2 - 3x - y$ is

- $\begin{bmatrix} -4 & 1 \\ 1 & -2 \end{bmatrix}$
- $\begin{bmatrix} 1 & -4 \\ -2 & 1 \end{bmatrix}$
- $\begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$

$$\begin{vmatrix} 4-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(2-\lambda) - 1 = 0$$

4.414
1.585

3) The Eigenvalues of Hessian matrix of $f(x, y) = 2x^2 - xy + y^2 - 3x - y$ is

- ○ -1.585786, -4.414214
- ○ 3.828427, -1.828427
- ✓ 4.414214, 1.585786
- ○ -3.828427, 1.828427

4) The Hessian matrix of $f(x, y) = 2x^2 - xy + y^2 - 3x - y$ is

- ○ positive definite
- ○ positive semidefinite
- ○ negative definite
- ○ negative semidefinite

5) The function $f(x, y) = 2x^2 - 2y^2$

- ○ has no stationary point
- ○ has a stationary point at (1,1)
- ○ has a stationary point at (1,-1)
- ○ has a stationary point at (0,0)

∇f $4x$ $x=0$

$-4y$ $y=0$