DATA SCIENCE FOR ENGINEERS

Week 3

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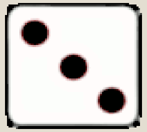
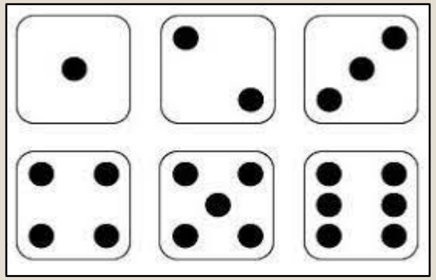
Random Phenomenon

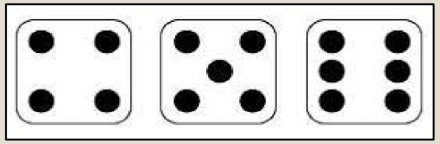
➢ The phenomenon/experiment whose outcomes are not predictable with certainty are called **random phenomenon**.



➢**Sample space:** The set of all possible outcome of an experiment.
Ex → S={1,2,3,4,5,6}



➢**Event:** Any subset of sample space
Ex → outcome is greater than 3. E={4,5,6}



Probability

- Probability is a measure that assign real value to every outcome of a random phenomenon
- The probability is ratio of number of ways an event can happen to the number of ways sample space can happen

$$\boxed{P(E) = \frac{n(E)}{n(S)}}$$

- Axioms of probability

- $\mathbf{0 \leq P(E) \leq 1}$ (Probability is non-negative and less than one)
- $\boldsymbol{P(S) = 1}$ (Probability of entire sample space in1)
- $\boldsymbol{P(A \cup B) = P(A) + P(B)}$ (For two mutually exclusive events)

Q) What is the probability of rolling an even number on a fair six-sided die?

A) 1/6

B) 1/3

C) 1/2

D) 2/3

Explanation:

Even numbers $(E) = \{2,4,6\}$; $n(E) = 3$
Sample space$(S) = \{1,2,3,4,5,6\}$; $n(S) = 6$

Therefore Prob of even no is $= \frac{n(E)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

Q) If two fair coins are flipped, what is the probability of getting exactly one head?

A) 1/4

B) 1/2

C) 3/4

D) 1/3

Explanation:
$\text{Getting exactly on head } (E) = \{HT, TH\}$;
$n(E) = 2$

Sample space$(S) = \{HH, HT, YH, TT\}$;
$n(S) = 4$

Therefore$\mathrm{P(E)} = \frac{n(E)}{n(S)} = \frac{2}{4} = \frac{1}{2}$

Q) A deck of playing cards contains 52 cards. What is the probability of drawing a red card (heart or diamond)?

a) 1/2

b) 39/52

c) 1/4

d) 1/3

Explanation:

Total no of red cards in deck: $n(R) = 13D + 13H = 26$

Total cards in deck: $n(S) = 52$

Therefore Prob of even no is $= \frac{n(R)}{n(S)} = \frac{26}{52} = \frac{1}{2}$

Q) A bag contains 5 red balls and 3 green balls. What is the probability of drawing a red ball and then a green ball (without replacement)?

a) 5/24

b) 15/56

c) 5/14

d) 5/16

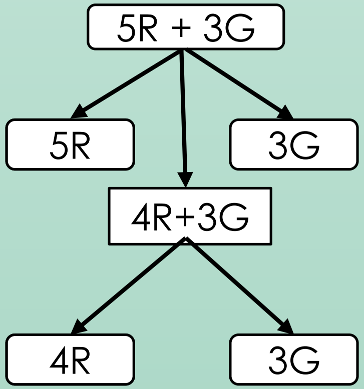
Explanation:

Prob of drawing red ball is:

$P(R) = \frac{5}{8}$

Prob of drawing green ball is:

$P(R) = \frac{3}{7}$



Combine prob$=P(R) * P(R) = \frac{5}{8} * \frac{3}{7} = \frac{15}{56}$

Q) C if out of all possible jumbles of the 'BIRD', a random word is picked, what is the probability, that this will start with a 'B'.

A)1/3

B)1/4

C)3/4

D)2/3

BIRD

↓

BIDR
BRID
BRDI
BDRI
BDIR
.
.

Explanation

- $n(S) = all\ possible\ jumbles\ of\ BIRD = 4! = 4 \times 3 \times 2 \times 1$
- $n(E) = jumbles\ starting\ with\ 'B' = 3! = 3 \times 2 \times 1$

$$\boxed{P(E) = \frac{n(E)}{n(S)} = \frac{3!}{4!} = \frac{1}{4} = \mathbf{0.25}}$$

Events

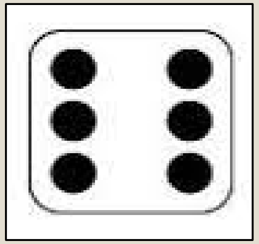
➢ **Mutually Exclusive Event:** The occurrence of one event implies that other event does not occur

Ex→ Coin toss gives either Head or Tail not both

➢ $\mathrm{P}(\mathrm{A} \cap B) = \mathbf{0}$

➢ $\mathrm{P}(\mathrm{A} \cup B) = P(A) + P(B) + P(A \cap B) = \boldsymbol{P(A) + P(B)}$

➢ **Independent Event:** two events are independent if occurrence of one has no influence on other

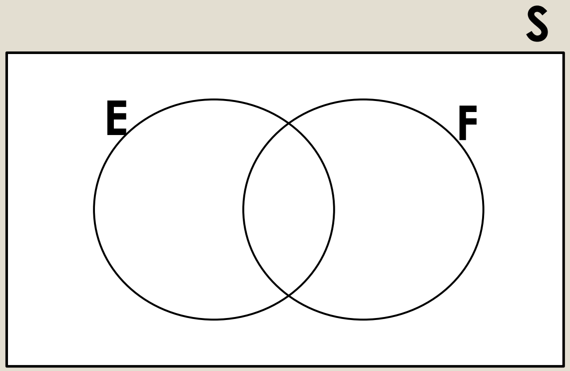


Ex→ Landing on heads after tossing a coin AND rolling a 5 on a single 6-sided die

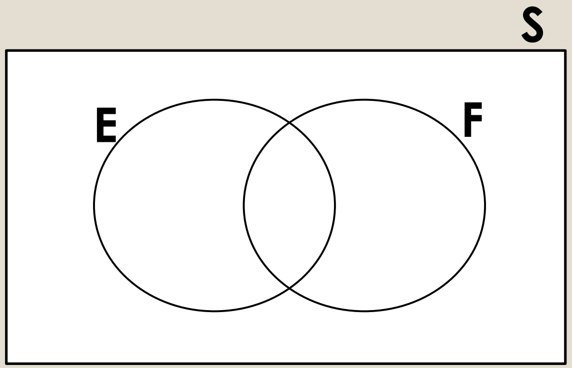
➢ $\mathrm{P}(\mathrm{A} \cap B) = \boldsymbol{P(A).P(B)}$

➢ $\mathrm{P}(\mathrm{A} \cup B) = P(A) + P(B) + P(A \cap B) = \boldsymbol{P(A) + P(B) + P(A).P(B)}$

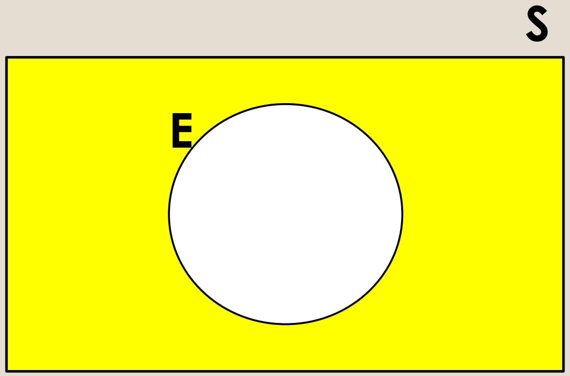
Venn Diagram



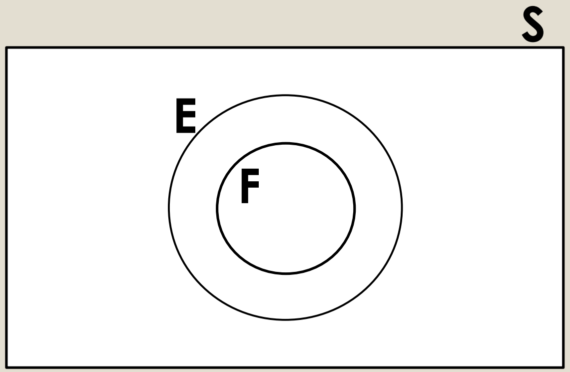
Colour region:$E \cup F$



Colour region:$E \cap F$



Colour region: E^c



$E \subseteq F$

Q) If two events A and B are mutually exclusive, what can be said about their intersection?

A) The intersection of A and B is the empty set.

B) The intersection of A and B contains only one element.

C) The intersection of A and B is equal to A.

D) The intersection of A and B is equal to B.

Explanation:

Mutually exclusive events have no common outcomes (coin toss and dice rolling), so the intersection of two mutually exclusive events is the empty set (∅)

Q) If events A and B are independent, which of the following is true about their joint probability?

A) P(A and B) = P(A) × P(B)

B) P(A and B) = P(A) + P(B)

C) P(A and B) = P(A) - P(B)

D) P(A and B) = P(A) / P(B)

Explanation:

For independent events A and B, the joint probability of both events occurring is given by the product of their individual probabilities: P(A and B) = P(A) × P(B).

Q) Are mutually exclusive events always dependent?

a) Yes

b) No

Explanation:

Mutually exclusive events cannot happen together. If one event occurs, it eliminates the possibility of the other event occurring.

Q) If events A and B are independent, and $P(A) = 0.4$ and $P(B) = 0.6$, what is $P(A \cup B)$?

a) 0.2

b) 0.6

c) 0.8

d) 0.98

Explanation:

For independent events, the probability of either event happening is given by

$P(A \cup B) = P(A) + P(B) - P(A) * P(B)$

$P(A \cup B) = 0.4 + 0.6 - (0.4 * 0.6) = 0.8$.

Conditional Probability

➢If two event A & B are not independent, then information available about the outcome of event A can influence the predictability of event B

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

➢ Example → Two fair coins are toss

Event A: First toss is Head = {HT,HH}

Event B: Two successive head = {HH}

P(A) = n(A)/n(S) = 2/4 = 0.5

P(B) = n(B)/n(S) = 1/4 = 0.25

If the event A is given then probability of event B is:

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{0.25}{0.5} = \mathbf{0.5}$$

Q) If events A and B are independent, which of the following is true about their conditional probabilities?

A) P(A|B) = P(A)

B) P(A|B) = P(B)

C) P(A|B) = P(A) + P(B)

D) P(A|B) = P(A) × P(B)

Explanation:

For independent events A and B, the occurrence of event B does not affect the probability of event A.

With formula: $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A).\ P(B)}{P(B)} = P(A)$

Q) In a group of people, 40% like ice cream, 30% like chocolate, and 20% like both ice cream and chocolate. What is the probability that a randomly selected person likes ice cream given that they like chocolate?

A) 1/2

B) 2/3

C) 4/5

D) 1/3

Explanation:

P(I) = 0.4

P(C) = 0.3

P(C ∩ I) = 0.2

Randomly selected person likes ice cream given that they like chocolate

$$P(I|C) = \frac{P(C \cap I)}{P(C)} = \frac{0.2}{0.3} = \frac{2}{3}$$

Random Variable

- A random variable (RV) is a map from sample space to a real line such that there is a unique real number corresponding to every outcome of sample space.
- Ex → coin toss sample space [H T] map to [0 1]

- **Discrete random variable:**

A variable that take one value from a discrete set of values

Ex → dice rolling [1 2 3 4 5 6]

- **Continuous random variable:**

The variable that can take continuous range of values

Ex → Temperature of the week [28.3, 21.0, 25.9, 26.1, 32.6, 30.0, 29.8]

Q) Which of the following is an example of a discrete random variable?

a) Height of individuals in a population.

b) Temperature in degrees Celsius.

c) Number of heads in three coin tosses.

d) Time taken to complete a marathon.

Explanation:

A discrete random variable takes on distinct, separate values with gaps in between, such as the number of heads in a coin toss, which can only be 0, 1, 2, or 3.

Q) The probability distribution of a discrete random variable must satisfy which of the following?

a) The probabilities must be negative.

b) The sum of the probabilities must be exactly 1.

c) The probabilities must be greater than 1.

d) The probabilities must be integers.

Explanation:

The probabilities assigned to each possible value of a discrete random variable must add up to 1, representing the entire probability space.

This is one of the axiom in probability

Probability mass/density function

➢ Probability mass/density function help to assign the probability to every outcome of a sample space

Probability Mass Function (PMF)

- PMF use for discrete random variable
- $\boldsymbol{P(x) = P[X = x]}$
- Ex → In coin toss

$P[X = 0] = 0.5, P[X = 1] = 0.5$

Probability Density Function (PDF)

- PDF use for continuous random variable
- $\boldsymbol{P(a < x < b) = \int_a^b F(x)\, dx}$
- Ex → height between than 20 and 40 C

$p(20 < T < 40) = \int_{20}^{40} F(x)\, dx$

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$$p(20 < T < 40) = \int_{20}^{40} F(x)\, dx$$

Que: The box contain 20 defective items and 80 non-defective items. If two items are selected at random without replacement , what will be the probability that both items are defective ?

A) 26/495

B) 36/495

C) 23/495

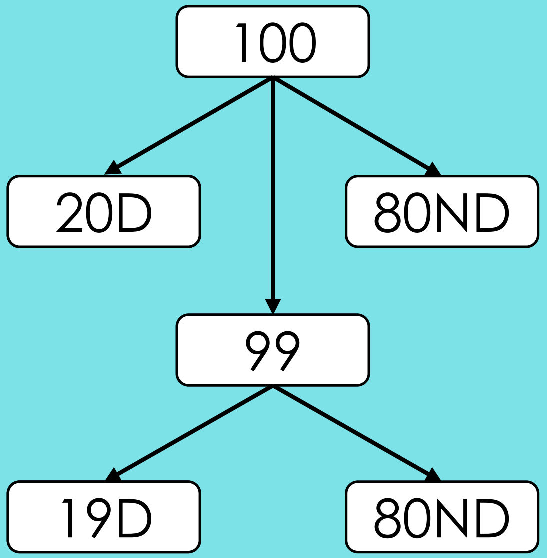
D) 19/495

Explanation

- P(both items are defective)= ?
- Probability of selecting first defective item:

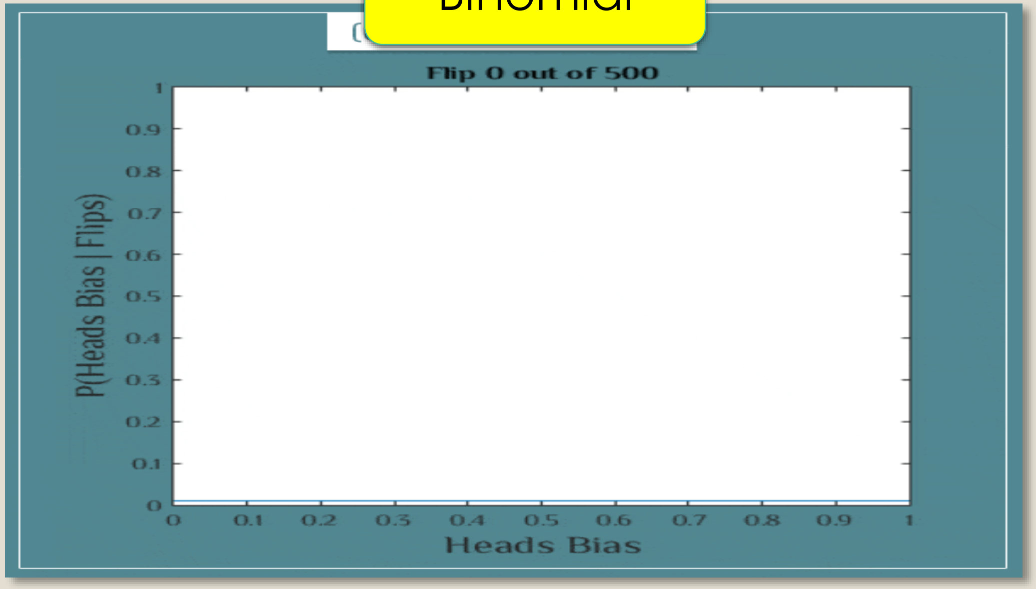
$$P(1D) = \frac{20}{100}$$

- Now we left with total 99 items with 19 defective
- Therefore, probability of selecting second defective item: $P(2D) = \frac{19}{99}$
- Combine prob= $P(1D) * P(2D) = \frac{20}{100} * \frac{19}{99} = \frac{19}{495}$

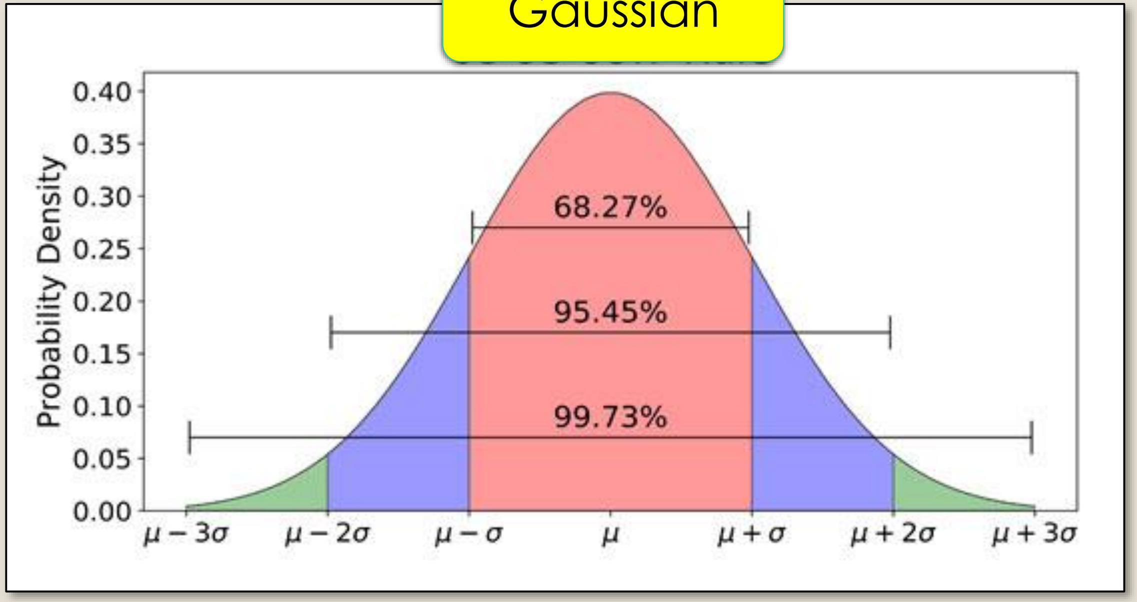


Distributions

Binomial



Gaussian



$$P(x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{(n-x)!\,x!} p^x q^{n-x}$$

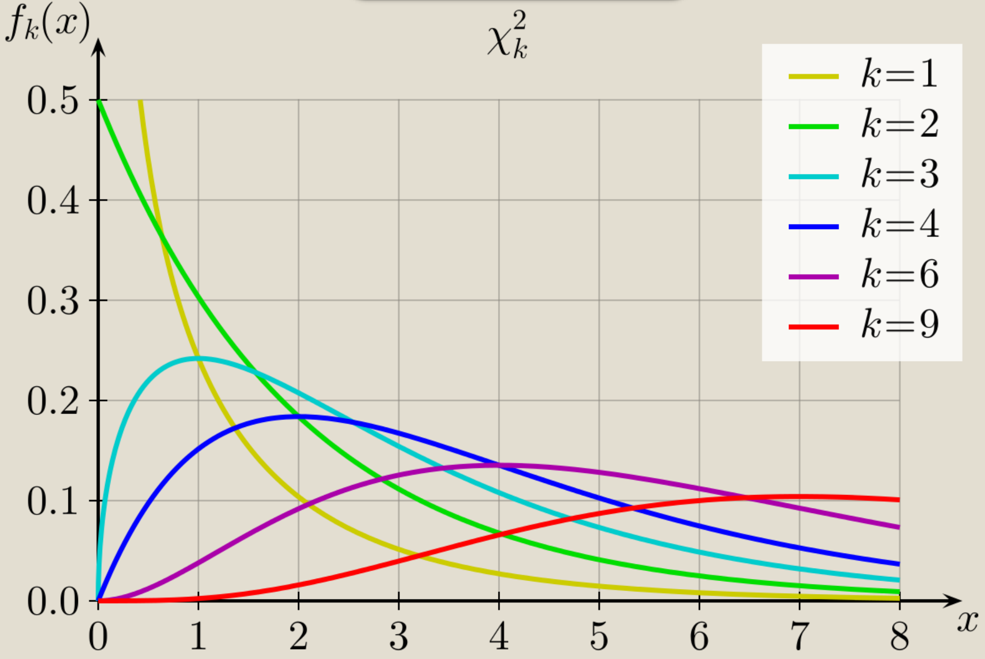
where
n = the number of trials (or the number being sampled)
x = the number of successes desired
p = probability of getting a success in one trial
$q = 1 - p$ = the probability of getting a failure in one trial

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where σ is the standard deviation and μ the mean

Distributions

Chi-square



$$f(x) = \frac{1}{\Gamma\left(\frac{k}{2}\right) 2^{k/2}} x^{\frac{k}{2}-1} . e^{-x/2}$$

- The function is characterize by one parameter i.e., degree of freedom (k)
- The range of distribution is 0 to Inf
- Mean=k; variance= 2k
- This help in hypothesis testing (discuss next)

Q) What type of events does the binomial distribution model?

A) Continuous events

B) Discrete events

C) Events with a normal distribution

D) Events with a uniform distribution

Explanation:

The binomial distribution models the number of successes in a fixed number of independent trials, where each trial has two possible outcomes (success or failure). Therefore it is used for discrete events.

Q) What is the shape of the Gaussian distribution?

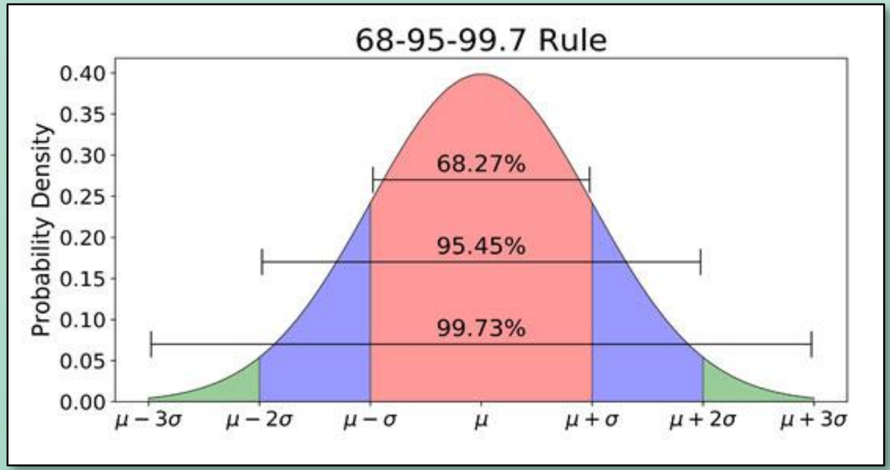
A) U-shaped

B) Skewed to the left

C) Bell-shaped

D) Skewed to the right

Explanation:



Q) What are the parameters that define a chi square distribution?

A) Mean and standard deviation

B) Probability of success and number of trials

C) Degree of freedom

D) Mode and median

Explanation:

As shown in density function (k=dof) is the only parameter use to generate distribution

$$f(x) = \frac{1}{\Gamma\left(\frac{k}{2}\right) 2^{k/2}} x^{\frac{k}{2}-1} . e^{-x/2}$$

Q) What percentage of data falls within one standard deviation of the mean in a normal distribution?

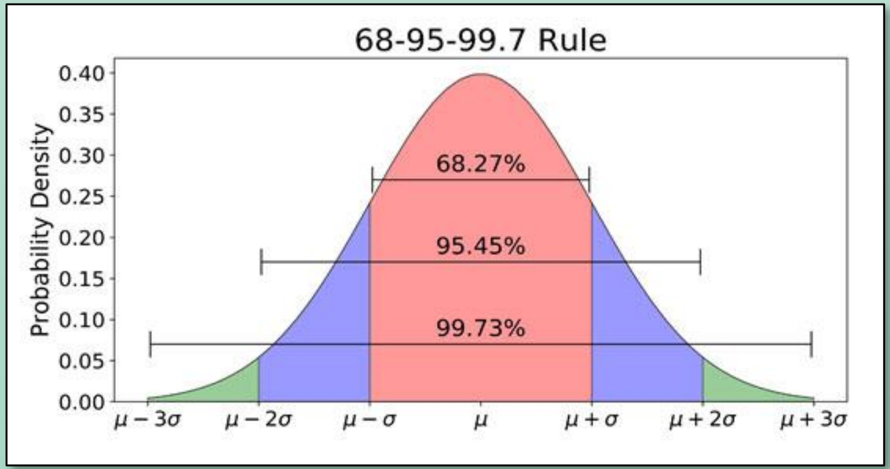
A) 25%

B) 50%

C) 68%

D) 95%

Explanation:



Expected Value

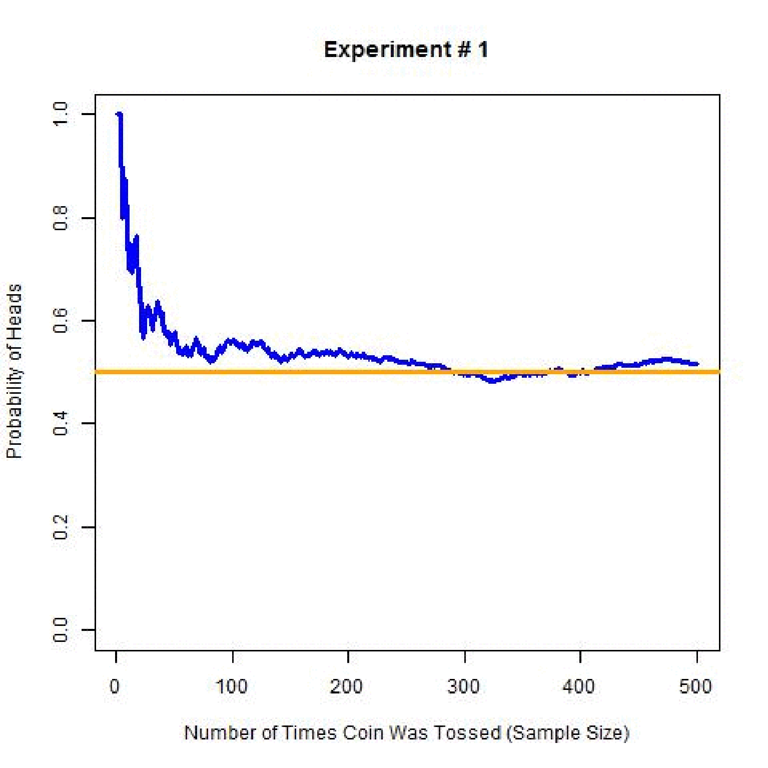
- It's a way to understand what might happen on average if you repeat something many times.
- Ex → if coin tossed, probability of head
- For discrete distribution:

$$E[X] = \sum_{i=1}^{n} x_i . p(x_i)$$

- For continuous distribution:

$$E[X] = \int_{-\infty}^{\infty} x_i . f(x_i)\, dx$$

- Mean: $\mu = E[X]$
- Variance: $\sigma^2 = E[(x-\mu)^2] = E[X^2] - \mu^2$



Q) If you roll a fair six-sided die, what is the expected value of a single roll?

A) 3.5

B) 6

C) 4.5

D) 2.5

Explanation:

x	1	2	3	4	5	6
P(x)	1/6	1/6	1/6	1/6	1/6	1/6

Using formulae $\boldsymbol{E}[\boldsymbol{X}] = \sum_{i=1}^{n} \boldsymbol{x_i} . \boldsymbol{p}(\boldsymbol{x_i})$

$$\boldsymbol{E}[\boldsymbol{X}] = \mathbf{1}.\frac{\mathbf{1}}{\mathbf{6}} + \mathbf{2}.\frac{\mathbf{1}}{\mathbf{6}} + \mathbf{3}.\frac{\mathbf{1}}{\mathbf{6}} + \mathbf{4}.\frac{\mathbf{1}}{\mathbf{6}} + \mathbf{5}.\frac{\mathbf{1}}{\mathbf{6}} + \mathbf{6}.\frac{\mathbf{1}}{\mathbf{6}} = \frac{\mathbf{7}}{\mathbf{2}} = \mathbf{3.5}$$

Q) In a normal (Gaussian) distribution, where is the expected value located?

A) At the mode of the distribution

B) At the median of the distribution

C) At the mean of the distribution

Explanation:

Expected value generally represents the mean of the distribution

Properties of expectation and variance

➢ **Expectation**

- $E(ax_1 + b) = a\, E(x_1) + b$
- $E(ax_1 + bx_2) = a\, E(x_1) + b\, E(x_1)$

➢ **Variance**

- $V(ax_1 + b) = a^2\, V(x_1)$
- $V(ax_1 + bx_2) = a^2\, V(x_1) + b^2\, V(x_2) + 2ab\, cov(x_1, x_2)$

Q) If Var(X) = 9 and Var(Y) = 16, what is the variance of 2X - 3Y, if X and Y are independent?

A) 120

B) 150

C) 180

D) 100

Explanation:

- Using formulae

$$V(aX + bY) = a^2\, V(X) + b^2\, V(Y) + 2ab\, cov(X, Y)$$

$$V(2X - 3Y) = 2^2\, V(X) + -3^2\, V(Y) + 0$$

$$V(2X - 3Y) = 4 * 9 + 9 * 16 + 0 = 180$$

Q) Let X and Y be two independent random variables with expectations E(X) = 5 and E(Y) = 3. What is the expectation of the random variable Z = 2X - 3Y?

A) 4

B) 1

C) -4

D) -6

Explanation:

- Using formulae

$$E(aX + bY) = a\, E(X) + b\, E(Y))$$

$$\mathrm{E}(2X - 3Y) = 2\, E(X) - 3\, E(Y)$$

$$E(2X - 3Y) = 2 * 5 - 3 * 3 = 1$$

Covariance & Correlation

Covariance

- Covariance indicates the direction of the linear relationship between variables
- Covariance values are not standardized.
- Value can be anything

$$\operatorname{cov}(X, Y) = \sum_{i=1}^{N} \frac{(x_i - \bar{x})(y_i - \bar{y})}{N}.$$

Correlation

- Correlation measures both the strength and direction of the linear relationship between two variables
- Correlation values are standardized
- Value lie between -1 and +1

$$Correlation = \frac{Cov\ (x, y)}{\sigma x * \sigma y}$$

Properties of joint pdf

- Joint pdf of two random variables x and y: f(x,y)

$$\boldsymbol{P(x \le a, y \le b) = \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dxdy}$$

- **Covariance between x and y**

$$\boxed{\boldsymbol{\sigma}_{x,y} = E\left[(x - \boldsymbol{\mu}_x)(y - \boldsymbol{\mu}_y)\right]}$$

- **Correlation between x and y:**

$$\boxed{\rho_{x,y} = \frac{\boldsymbol{\sigma}_{x,y}}{\boldsymbol{\sigma}_x\,\boldsymbol{\sigma}_y}}$$

Q) If two random variables are independent, what can we say about their covariance and correlation?

a) Covariance is zero, and correlation is zero.

b) Covariance is zero, and correlation can be any value.

c) Covariance can be any value, and correlation is zero.

d) Covariance can be any value, and correlation is one.

Explanation:

- When two random variables are independent, they have no linear relationship, so both the covariance and correlation will be zero.

Q) The correlation coefficient between X and Y is 0.6. The covariance between them is 25. Then find the product between variance of both the variables?

a) 39.33

b) 41.66

c) 40.11

d) 36.99

Explanation:

- Using formulae

$$\rho_{x,y} = \frac{\boldsymbol{\sigma}_{x,y}}{\boldsymbol{\sigma}_x\,\boldsymbol{\sigma}_y}$$

$$0.6 = \frac{25}{\boldsymbol{\sigma}_x\,\boldsymbol{\sigma}_y}$$

$$\boldsymbol{\sigma}_x\,\boldsymbol{\sigma}_y = \frac{25}{0.6} = 41.667$$

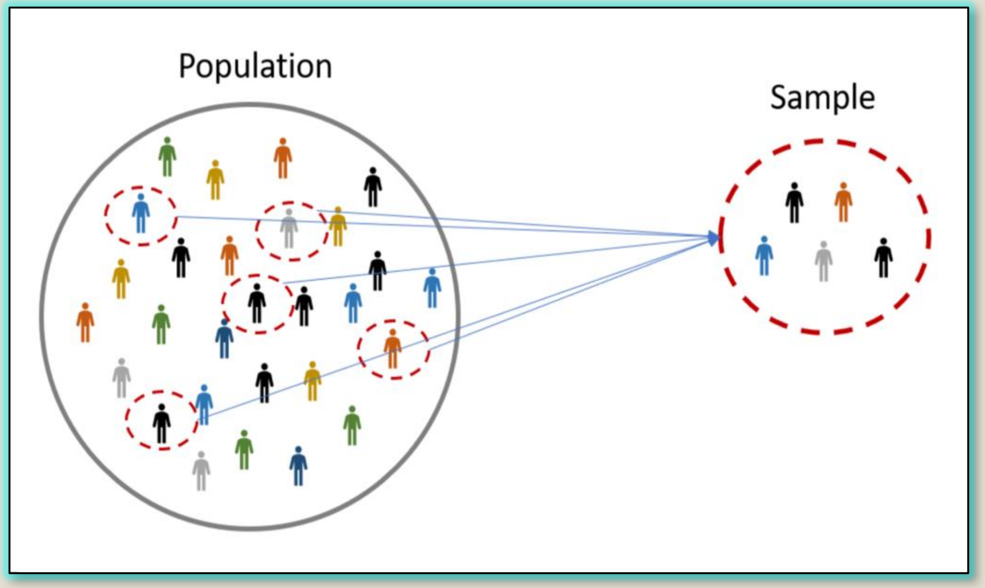
R studio to calculate probability

- ***pnorm(X, mean,std,'lower tail'=T/F)*** : it gives the probability of the PDF of normal distribution for provided X random variable
- **pchisq:** chi-square distribution (parameter is DOF)
- ***pbinom****: binomial distribution (parameters are trials and success probability)*
- ***punif****: uniform distribution (min and max value)*

- ***qnorm(p, mean,std,'lower tail'=T/F)*** : it gives the random variable of the PDF of normal distributionfor provided probability p
- **qchisq:** chi-square distribution (parameter is DOF)
- ***qbinom****: binomial distribution (parameters are trials and success probability)*
- ***qunif****: uniform distribution (min and max value)*

Statistical sampling

- **Population:** Set of all possible outcome of random experiment
- **Sample set:** Finite set of observation obtained from experiment
- Sampling **help to make inferences** about the population
- The inference may be uncertain because samples might be uncertain

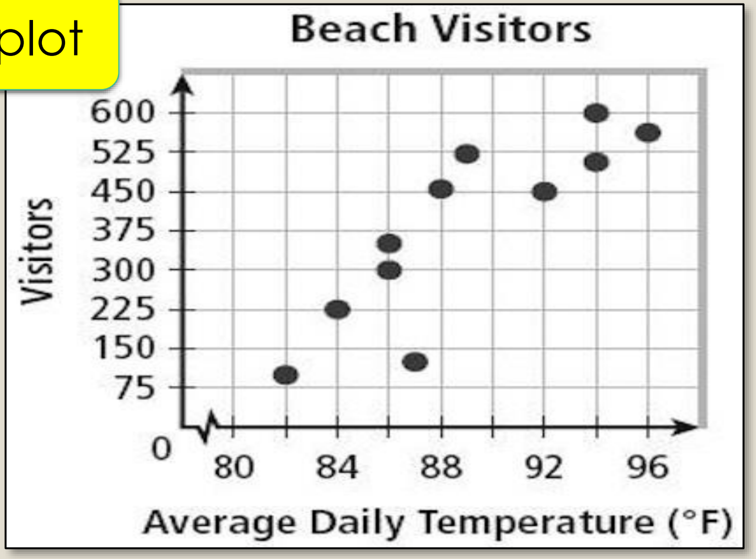


Statistical Analysis

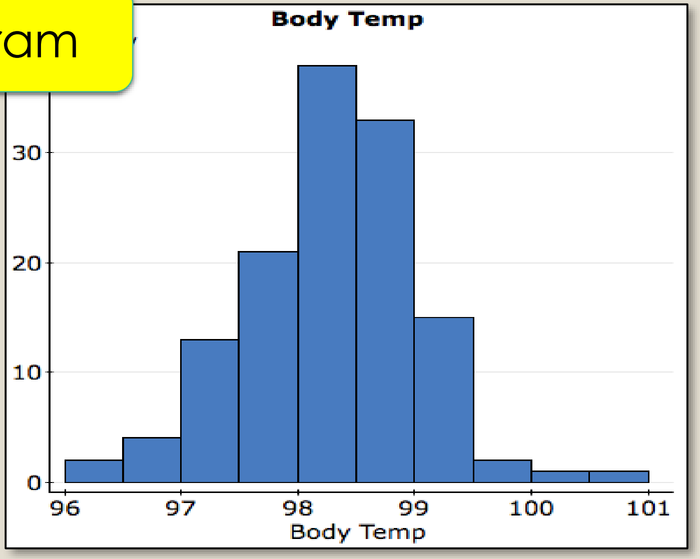
- Statistical Analysis
  - Descriptive
    - **Graphical**
      presenting the data in plot
    - **Numerical**
      Summarizing sample set
  - Inferential
    - **Estimation**
      Estimating parameters of PDF
    - **Hypothesis Testing**
      Making judgement about f(x)

Graphical statistics

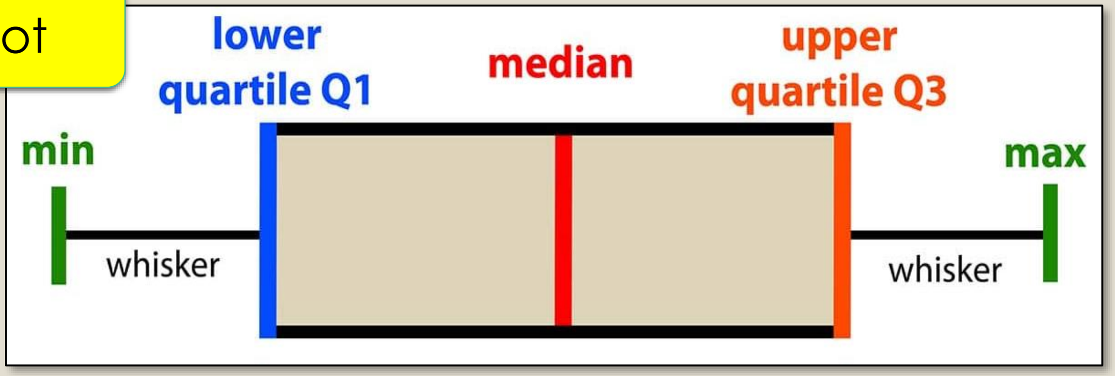
Scatter plot



Histogram



Box plot



Numerical Statistics

Mean: Mean is average or norm ➔ $\frac{1+3+4+6+6+7+8}{7} = 5$

Median: Median is middle value ➔ 1 3 4 **6** 6 7 8

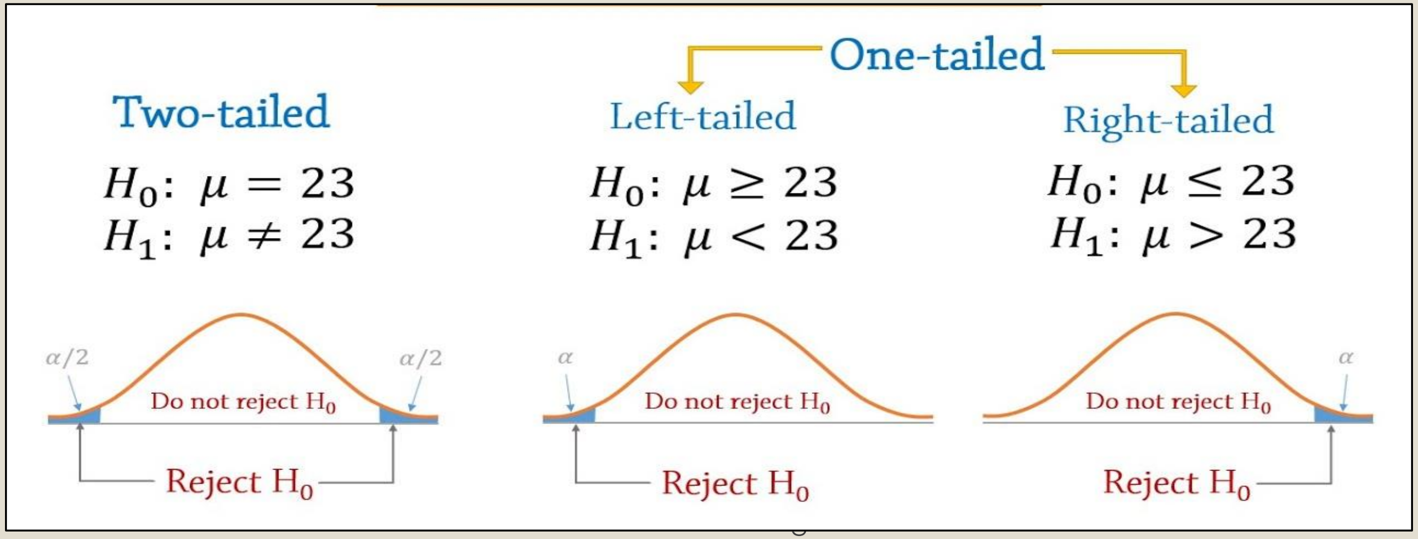
Mode: Mode is most frequent value ➔ 1 3 4 6 6 7 8

Range: Difference between lowest and highest value ➔ 8 -1= 7

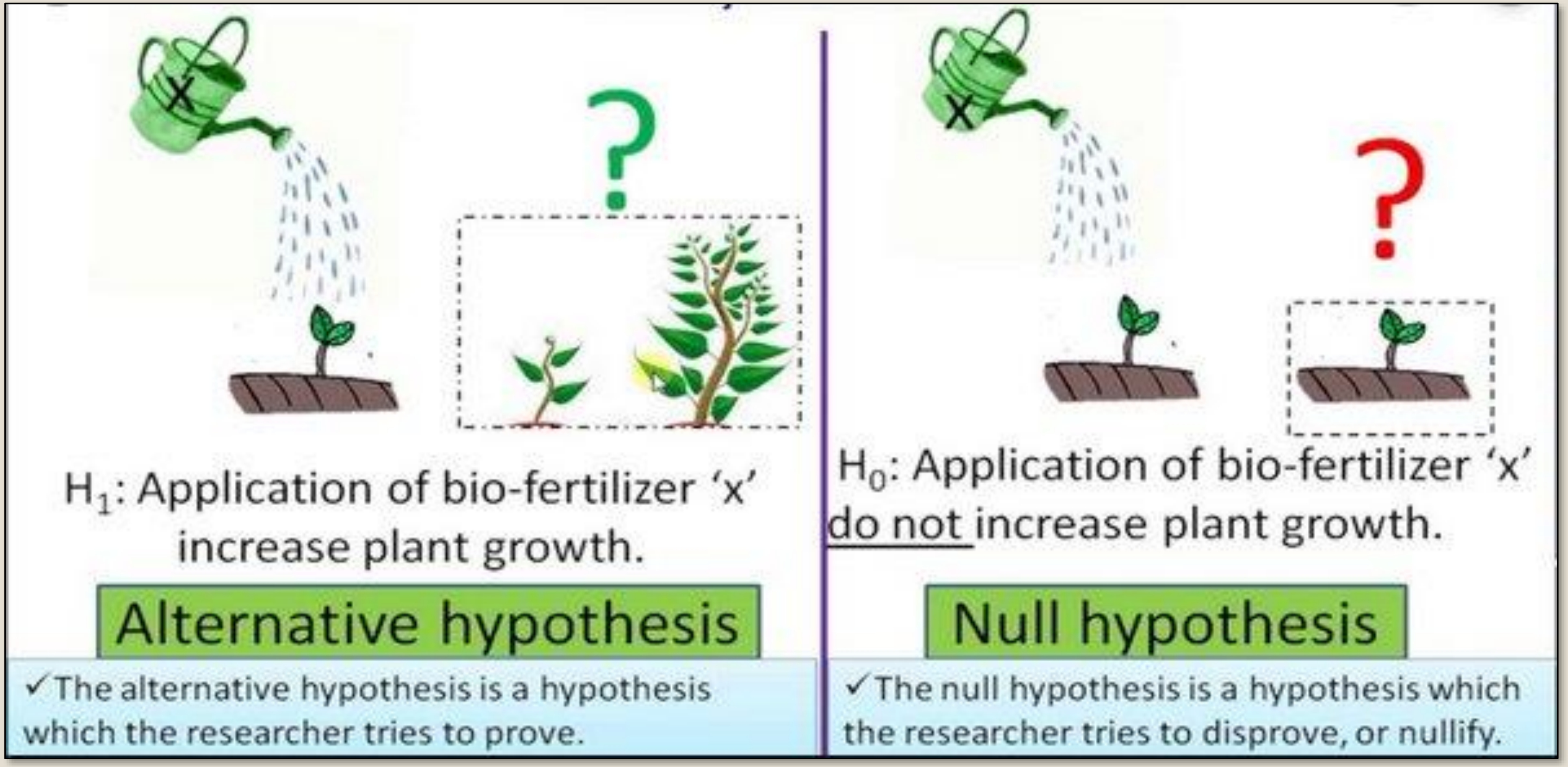
Goals scored in seven matched

Hypothesis testing

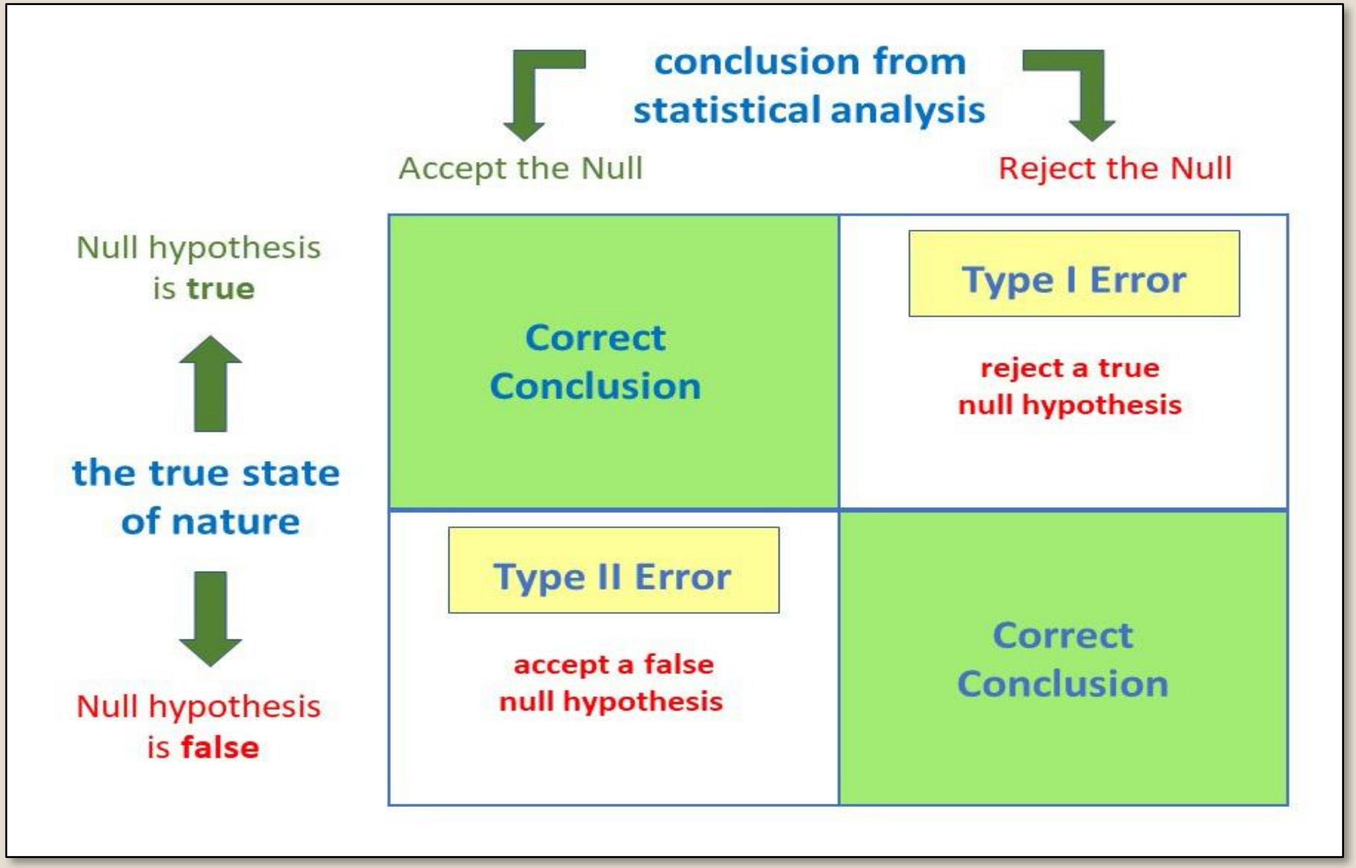
- Hypothesis testing is **used to make decisions**
- Ex: Whether company stock yield profit that desired value
- Ex: Whether the effect of drug A similar to drug B
- Hypothesis testing is generally converted to a test of mean and variance parameter of population

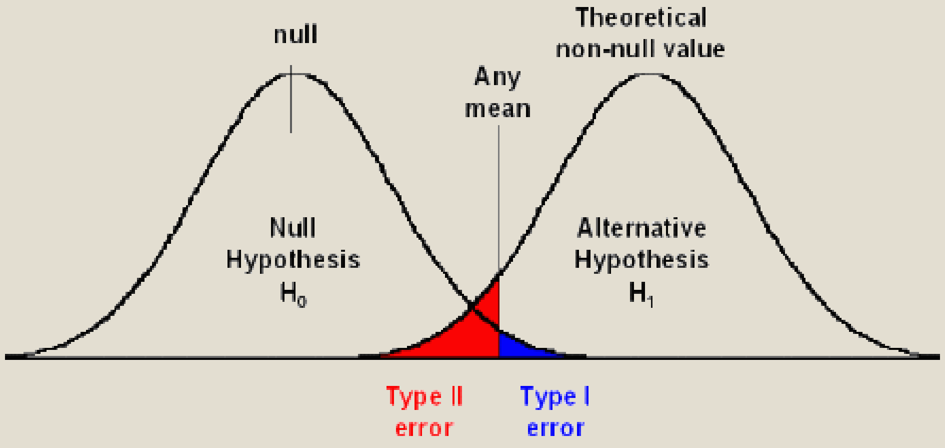


Null & Alternative hypothesis



Errors in hypothesis testing





z-test vs t-test

z-test

- Used when population variance in known
- Used for sample size greater than 30
- Based on normal distribution

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

$$\sigma = known\ std.dev.$$

T-test

- Used when variance is not known
- Use for sample size less then 30
- Based on student-t distribution

$$z = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}}$$

$$s = sample\ std.dev$$

Que: A teacher claims that the mean score of students in his class is greater than 82 with a standard deviation of 20. If a sample of 81 students was selected with a mean score of 90 then check if there is enough evidence to support this claim at a 0.05 significance level

- Number of students = n = 81
- Sample mean = $\bar{x} = 90$
- Population Std. deviation = $\sigma = 20$

$$\mathbf{H_0: \mu = 82}$$
$$\mathbf{H_1: \mu > 82}$$

From z table critical value of α is 1.645 (this is calculated using Rstudio with *pnorm* & *qnorm)*

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = 3.6$$

As 3.6 > 1.645 thus, the **null hypothesis is rejected**

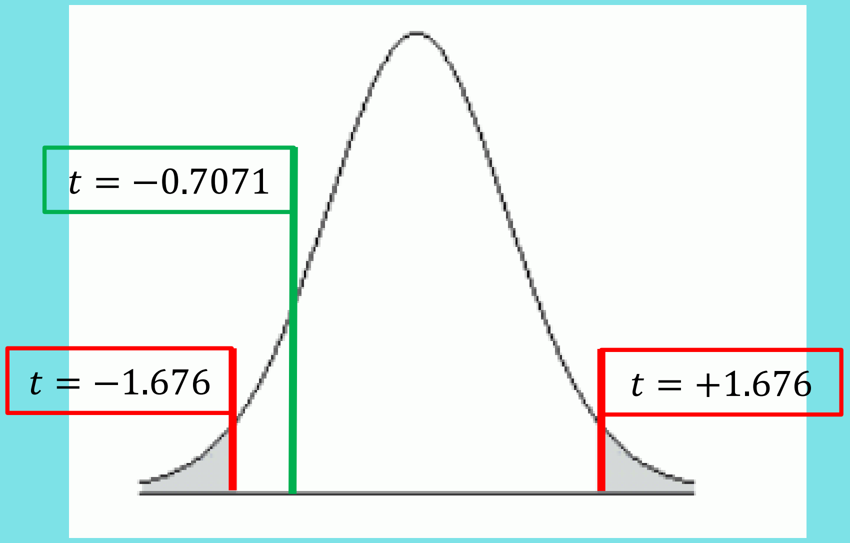
Que: A researcher wants to test whether the mean weight of a sample of 50 apples is significantly different from a claimed population mean weight of 150 grams. The sample mean weight is 149 grams, and the sample standard deviation is 10 grams. Calculate the t-statistic for this test.

- Number of apples = n = 50
- DOF=n-1=49
- Population mean $= \mu = 150$
- Sample mean = $\bar{x} = 149$
- Sample Std. deviation = $s = 10$

$$\mathbf{H_0: \mu = 150}$$
$$\mathbf{H_1: \mu \neq 150}$$

From t table critical value of α is ± 1.676 (95%)
(this is calculated using Rstudio with *pt* & *qt)*

$$t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{149 - 150}{\frac{10}{\sqrt{50}}} = -0.7071$$



As -0.7071 > -1.676 thus, the **null hypothesis is accepted**