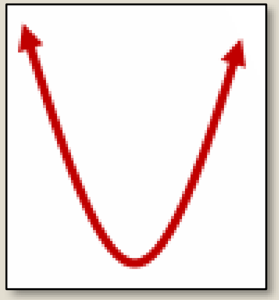
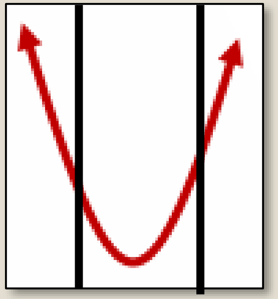
DATA SCIENCE FOR ENGINEERS

Week 5

Session Co-Ordinator : Abhijit Bhakte

Constraints Optimization

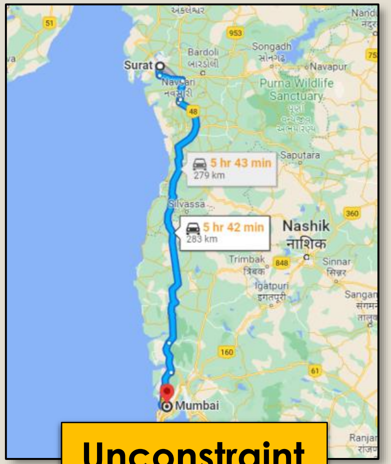




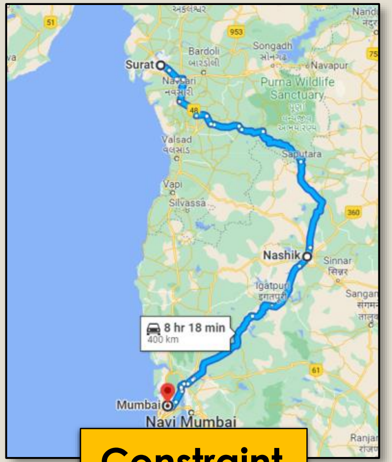
- Unconstraint optimization: In the Last week, the functions we examined were unconstrained, meaning they either had ***no boundaries, or the boundaries were soft.***
- Constraint optimization: In the week, we will be examining the functions with constraint. A constraint is a ***hard limit placed on the value of a variable***, which ***prevents us from going forever in certain directions.***

- Example: Travel car from Surat to Mumbai

Constraint → must visit the Nashik city

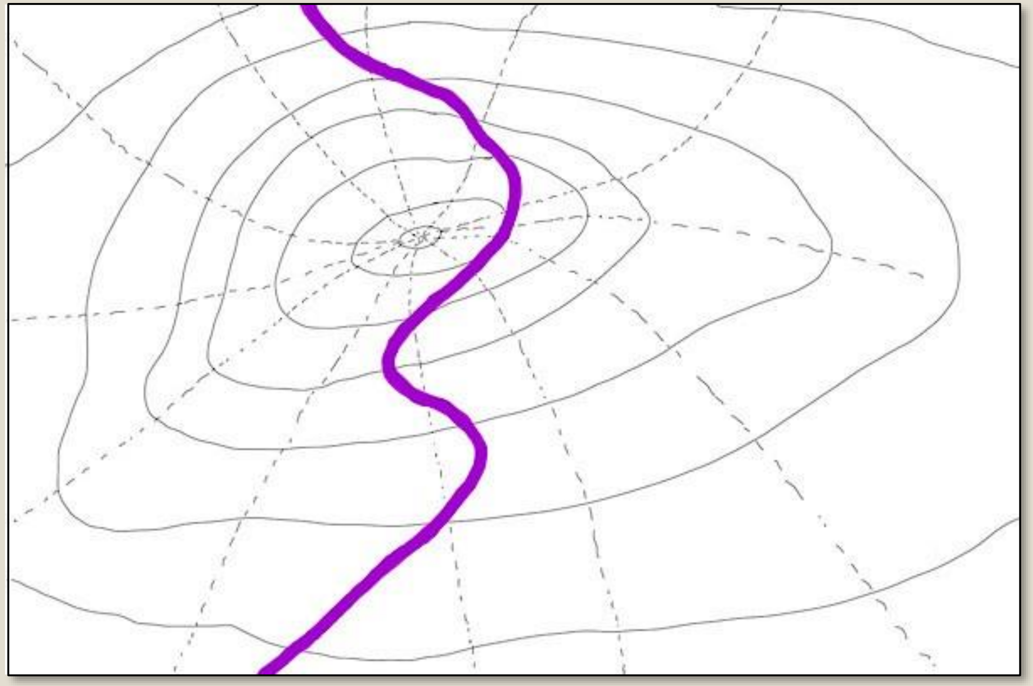


Unconstraint



Constraint

- *Task/Problem:* Climb as high as possible on the mountain to have a better view of the moon.
- *Constraint*: The car must be on the road



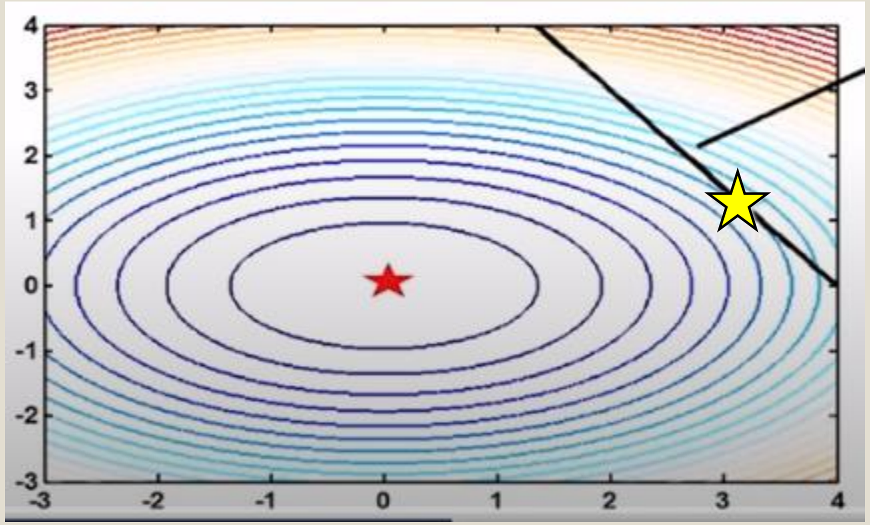
Equality constraint optimization

$$\min_{x_1, x_2} \; 2x_1^2 + 4x_2^2$$

$$st \quad 3x_1 + 2x_2 = 12$$

Constraint Minimum

Unconstraint Minimum



$3x_1 + 2x_2 = 12$

Lagrange Multipliers

- The Lagrangian multiplier is a technique **used in optimization problems that involve constraints.**

- It helps us **find the optimal solution of an objective function** while satisfying one or more constraints

- For the objective function f(x,y) with constraint function g(x,y)

$$\boxed{\nabla f(x, y) = -\lambda \nabla g(x, y)}$$

Que: find Minima for $f(x_1, x_2) = 2x_1^2 + 4x_2^2$ with constraint $g(x_1, x_2) \rightarrow 3x_1 + 2x_2 = 12$

- Applying 1st order condition

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = -\lambda \begin{bmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \end{bmatrix} \rightarrow \begin{bmatrix} 4x_1 \\ 8x_2 \end{bmatrix} = -\lambda \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$4x_1 = -3\lambda \quad (1) \qquad 8x_2 = -2\lambda \quad (2) \qquad 3x_1 + 2x_2 = 12 \quad (3)$$

- Solving equation (1) and (2) we get $x_1 = 3x_2$

- Substituting above variables in the constraint equation (3)

$$3x_1 + 2x_2 = 12 \rightarrow \qquad 3(3x_2) + 2x_2 = 12 \rightarrow \qquad x_2 = 1.09$$

- Calculating other point by putting x_2 in the above equations gives $x_1 = 3.27$

$$(x_1, x_2) = (3.27, 1.09)$$

Que: For a rectangle whose perimeter is 20m, find the dimensions that will maximize the area?

- Let consider breadth = b, length = l , Area = A and Perimeter = P

$$\text{Maximize } f(x) = A = l \times b$$

$$st \;\; g(x) = 2l + 2b = 20$$

- Applying 1st order condition

$$\underline{\boldsymbol{\nabla f(l, b) = -\lambda \, \nabla g(l, b)}}$$

- $b = -2\lambda \quad (1)$
- $l = -2\lambda \quad (2)$
- $2l + 2b = 20 \quad (3)$

- Solving above equation (1) and (2) **gives $\boldsymbol{l = b}$** . Putting these value in equation (3) gives

$$\boldsymbol{l = 5m} \text{ and } \boldsymbol{b = 5m}$$

- Calculating the dimensions, the Area of the rectangle is ***A = 5*5 = 25***

Que: Find the pt. on the circle $x^2 + y^2 = 80$ which are closest to & farthest from the point (1,2)?

- Distance d from any point (x,y) to the point (1,2) is $d = \sqrt{(x-1)^2 + (y-2)^2}$

Maximize $f(x,y) = (x-1)^2 + (y-2)^2$

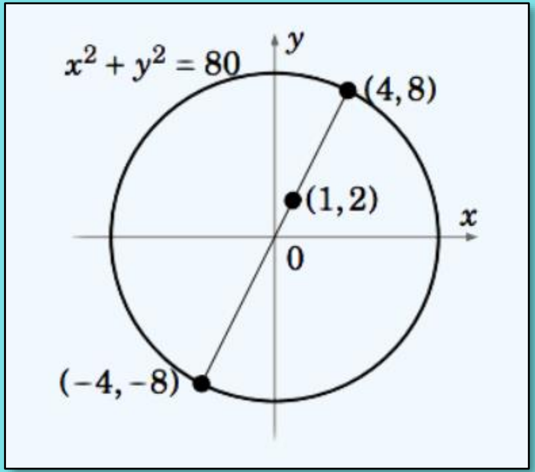
$st \;\; g(x) = x^2 + y^2 = 80$

- Applying 1st order condition

$$\underline{\boldsymbol{\nabla f(x,y) = -\lambda\, \nabla g(x,y)}}$$

$$\frac{\partial f}{\partial x} = 2(x-1) \qquad \frac{\partial g}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = 2(y-2) \qquad \frac{\partial g}{\partial y} = 2y$$



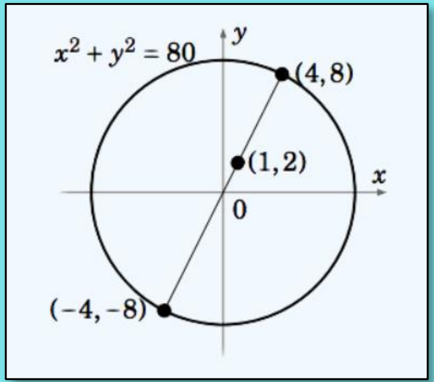
- Putting all derivatives into the main 1st order equation gives

$$\frac{x-1}{x} = -\lambda \quad (1) \qquad \frac{y-2}{y} = -\lambda \quad (2) \qquad x^2 + y^2 = 80 \quad (3)$$

- Solving equation (1) and (2) gives **y = 2x.** This value substituted in equation (3)

$$x^2 + y^2 = 80 \;\rightarrow\; x^2 + (2x)^2 = 80 \;\rightarrow\; x^2 + 4x^2 = 80 \;\rightarrow\; \boldsymbol{x = \pm\, 4}$$

- Putting x values in equation (3) gives $y = \pm\, \mathbf{8}$
- So we obtained the points where (4,8) is the nearest and (-4,-8) is the farthest point from (1,2)



Que: $\mathrm{f(x,y,z)} = x + z$

$\mathrm{g(x,y,z)} = x^2 + y^2 + z^2 = 1$

- Applying 1st order condition

$$\underline{\boldsymbol{\nabla f(x,y,z) = -\lambda\, \nabla g(x,y,z)}}$$

$$1 = -\,2\lambda x \quad (1) \qquad\qquad 0 = -\,2\lambda y \quad (2) \qquad\qquad 1 = -\,2\lambda z \quad (3)$$

- Solving equation (1) and (3) we get **x = z** . And with equ. (2) we get **y = 0**
- Substituting above variables in the constraint equation

$$x^2 + y^2 + z^2 = 1 \;\rightarrow \qquad x^2 + x^2 = 1 \;\rightarrow \qquad 2x^2 = 1 \;\rightarrow \qquad x = \pm\sqrt{\frac{1}{2}}$$

- Calculating other point by putting x in the above equations gives

Pt.1$= (\sqrt{\frac{1}{2}}, 0, \sqrt{\frac{1}{2}})$ **pt.2** $=(-\sqrt{\frac{1}{2}}, 0, -\sqrt{\frac{1}{2}})$

Q) Which of the following methods is used to solve optimization problems with equality constraints?

a) Simplex Method
b) Gradient Descent
c) Lagrange Multipliers
d) Genetic Algorithm

Solution

Lagrange Multipliers is a technique commonly used to solve optimization problems with equality constraints.

Q) In an optimization problem with equality constraints, the objective is to:

a) Minimize the objective function only.
b) Maximize the objective function only.
c) Satisfy the equality constraints only.
d) Optimize the objective function while satisfying the equality constraints.

Solution

The objective is to optimize the objective function while satisfying the equality constraints.

Q) Which of the following represents an equality constraint in a linear programming problem?

a) $3x+2y\leq10$

b) $2x-5y=8$

c) $4x+6y\geq20$

d) $7x+3y>15$

Solution

An equality constraint is represented by an equation with an equal sign, where the left-hand side is equal to the right-hand side.

Q) Consider the optimization problem: Minimize: **f(x, y) = x^2 + y^2** Subject to: **x + y = 5**
What are the coordinates (x, y) of the optimal solution?

a) (2, 3)
b) (3, 2)
c) (2.5, 2.5)
d) (1, 4)

Solution

- Applying 1st order condition f(x,y) and g(x,y)

$$\underline{\boldsymbol{\nabla f(x,y) = -\lambda\, \nabla g(x,y)}}$$

$$2x = -\lambda \quad (1) \qquad\qquad 2y = -\lambda \quad (2) \qquad\qquad x + y = 5 \quad (3)$$

- Solving equation (1) and (2) we get **x = y**
- Substituting above variables in the constraint equation

$$x + y = 5 \;\rightarrow \qquad x + x = 5 \;\rightarrow \qquad x = \frac{5}{2} = 2.5$$

- As we know x=y; hence optimal solution is (x,y)=(2.5,2.5)

Inequality constraint optimization

$$\min_{x_1,x_2} \; 2x_1^2 + 4x_2^2$$

$$st \quad 3x_1 + 2x_2 \leq 12$$

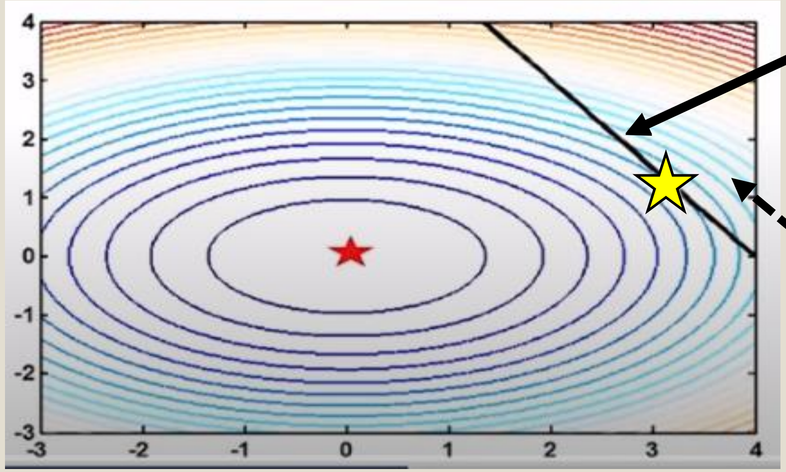
★ Constraint Minimum

★ Unconstraint Minimum

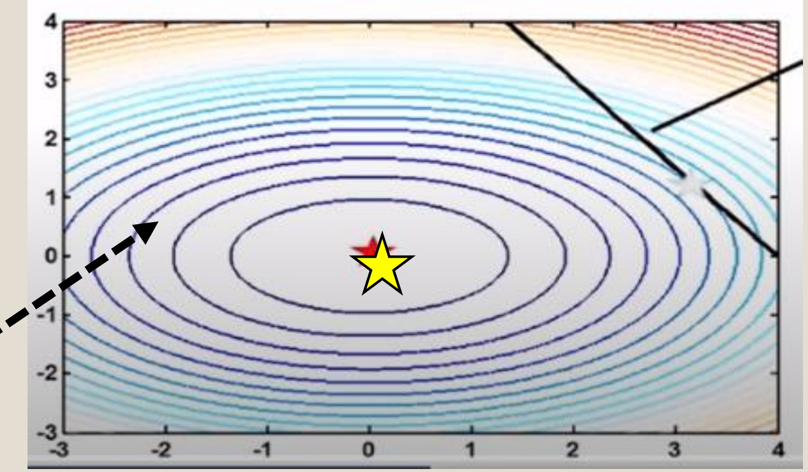
$$\min_{x_1,x_2} \; 2x_1^2 + 4x_2^2$$

$$st \quad 3x_1 + 2x_2 \geq 12$$

$$3x_1 + 2x_2 = 12$$



Feasible region



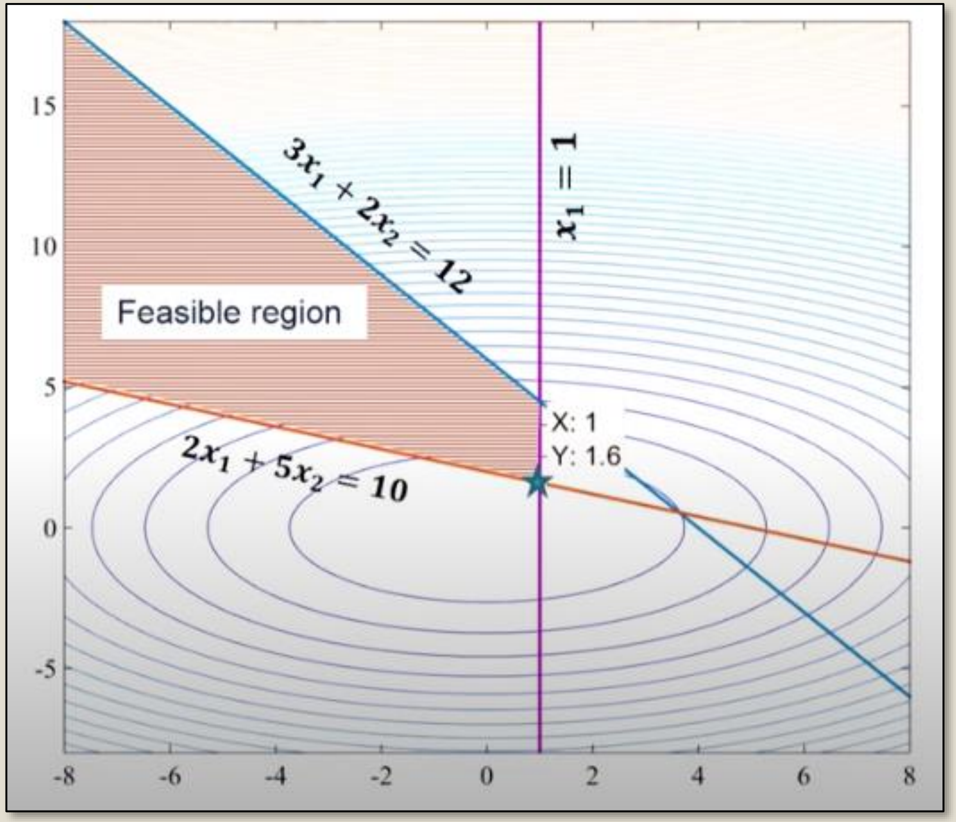
Multivariate inequality constraint

$$\min_{x_1,x_2} \; 2x_1^2 + 4x_2^2$$

$$st \;\; 3x_1 + 2x_2 \leq 12$$

$$2x_1 + 5x_2 \geq 10$$

$$x_1 \leq 1$$



$$\min_{x_1, x_2} \; 2x_1^2 + 4x_2^2$$

$$st$$

$$3x_1 + 2x_2 \leq 12 \Rightarrow (a)$$

$$2x_1 + 5x_2 \geq 10 \Rightarrow (b)$$

$$x_1 \leq 1 \Rightarrow (c)$$

- Lagrangian

$$L(x_1, x_2, \mu_1, \mu_2, \mu_3) = 2x_1^2 + 4x_2^2 + \mu_1(3x_1 + 2x_2 - 12) + \mu_2(10 - 2x_1 - 5x_2) + \mu_3(x_1 - 1)$$

- First order KKT conditions

$$4x_1 + 3\mu_1 - 2\mu_2 + \mu_3 = 0$$

$$8x_2 + 2\mu_1 - 5\mu_2 = 0$$

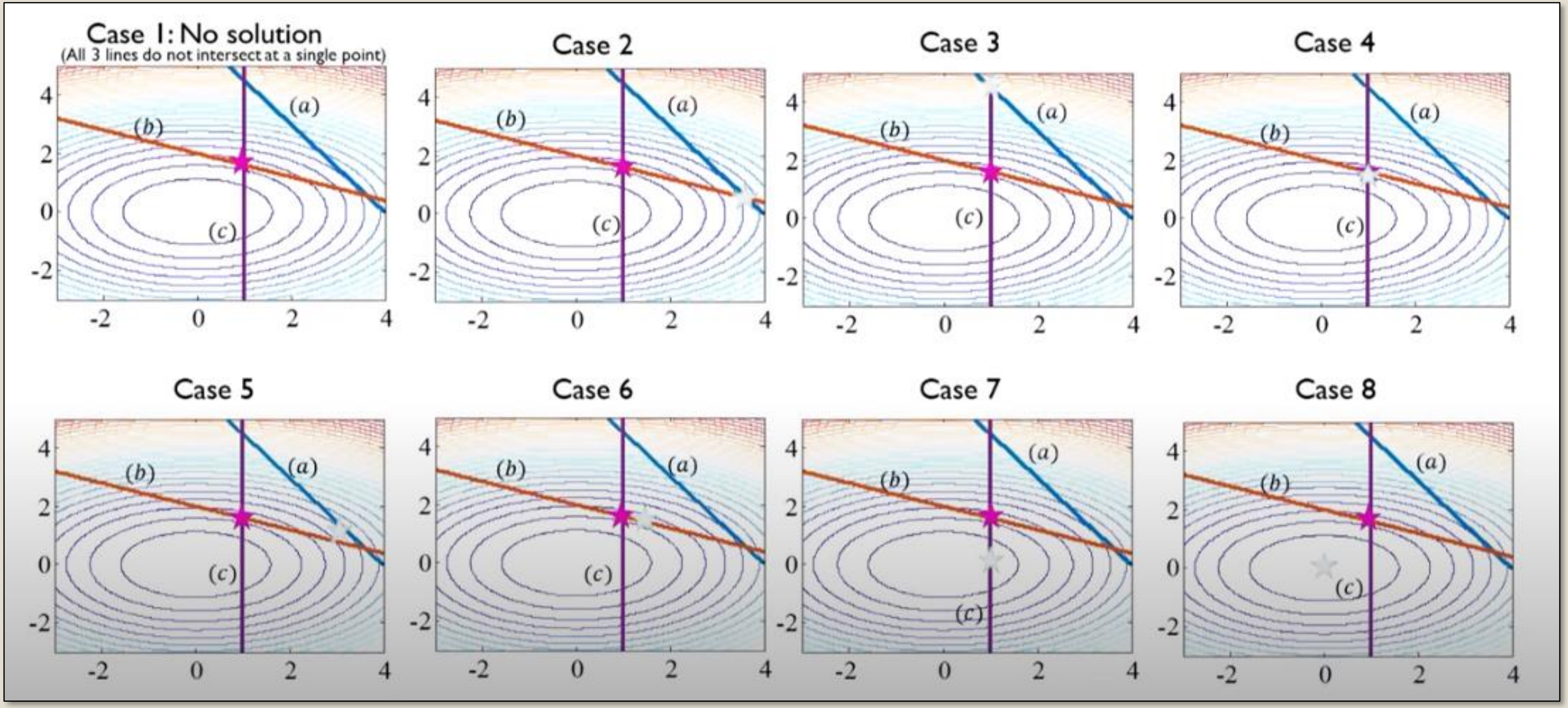
$$\mu_1(3x_1 + 2x_2 - 12) = 0$$

$$\mu_2(10 - 2x_1 - 5x_2) = 0$$

$$\mu_3(x_3 - 1) = 0$$

$$\mu_i \geq 0$$

Sl.no	Active (A) /Inactive (I) constraints (a)	(b)	(c)	Solution (x, μ)	Possible optima (Y/N)	Remark
1	A	A	A	Infeasible	N	Equations do not have a valid solution.
2	A	A	I	$x = [3.6364 \quad 0.5455]$ $\mu = [-5.2 \quad -1.45 \quad 0]$	N	$x_1 \leq 1$ is not satisfied, $\mu_1 < 0, \mu_2 < 0$
3	A	I	A	$x = [1 \quad 4.5]$ $\mu = [-18 \quad 0 \quad 50]$	N	$\mu_1 < 0$
4	I	A	A	$x = [1 \quad 1.6]$ $\mu = [0 \quad 2.56 \quad 1.12]$	Y	All constraints and KKT conditions satisfied
5	A	I	I	$x = [3.27 \quad 1.09]$ $\mu = [-4.36 \quad 0 \quad 0]$	N	$x_1 \leq 1$ is not satisfied
6	I	A	I	$x = [1.21 \quad 1.51]$ $\mu = [0 \quad 2.45 \quad 0]$	N	$x_1 \leq 1$ is not satisfied
7	I	I	A	$x = [1 \quad 0]$ $\mu = [0 \quad 0 \quad -4]$	N	$2x_1 + 5x_2 \geq 10$ is not satisfied
8	I	I	I	$x = [0 \quad 0]$ $\mu = [0 \quad 0 \quad 0]$	N	$2x_1 + 5x_2 \geq 10$ is not satisfied


Case 1: No solution
(All 3 lines do not intersect at a single point)
Case 2
Case 3
Case 4
Case 5
Case 6
Case 7
Case 8
(a)
(b)
(c)
-2
0
2
4

Q) Which of the following represents an inequality constraint in optimization?
a) $x+y = 10$
b) $2x-3y \leq 5$
c) $4x-3y = 25$
d) $3x+2y = 12$

Q) For an optimization problem, what does the feasible region represent?
a) The region where the objective function is minimized
b) The region where the objective function is maximized
c) The region satisfying all the constraints
d) The region outside the constraints

Q) If the feasible region of an optimization problem is empty, it means that:

a) The problem has no solution
b) The problem has multiple solutions
c) The problem is unbounded
d) The problem is well-posed

Solution

An empty feasible region indicates that there are no valid points that satisfy all the given constraints.

Q) In linear programming, a feasible solution that also satisfies all the constraints with equality is called:

a) An degenerate solution
b) A bounded solution
c) An infeasible solution
d) A optimal solution

Solution

An optimal solution is one that not only satisfies all the constraints but also optimizes the objective function according to the given optimization criteria.

Q) For given function $z = 4x^2 - 5xy + 6y$ subject to the constraint $x + y = 9$; $2x - 9y \leq 1$ and $xy + x \geq 3$ find lagrangian function

a) $L(x, y, \lambda, \mu_1, \mu_2) = 4x^2 - 5xy + 6y + \lambda(x + y - 9) + \mu_1(1 - 2x + 9y) + \mu_2(3 - xy - x)$

b) $\mathbf{L(x, y, \lambda, \mu_1, \mu_2) = 4x^2 - 5xy + 6y + \lambda(x + y - 9) + \mu_1(2x - 9y - 1) + \mu_2(3 - xy - x)}$

c) $L(x, y, \lambda, \mu_1, \mu_2) = 4x^2 - 5xy + 6y + \lambda(x + y - 9) + \mu_1(1 - 2x + 9y) + \mu_2(xy + x - 3)$

d) $L(x, y, \lambda, \mu_1, \mu_2) = 4x^2 - 5xy + 6y + \lambda(x + y - 9) + \mu_1(2x - 9y - 1) + \mu_2(xy + x - 3)$

Thank you