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NPTEL

Soft Computing

Introduction to Soft Computing

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INTRODUCTION TO SOFT COMPUTING

- **Concept of computation**
- **Hard computing**
- **Soft computing**
- **How soft computing?**
- **Hard computing vs. Soft computing**
- **Hybrid computing**

CONCEPT OF COMPUTATION

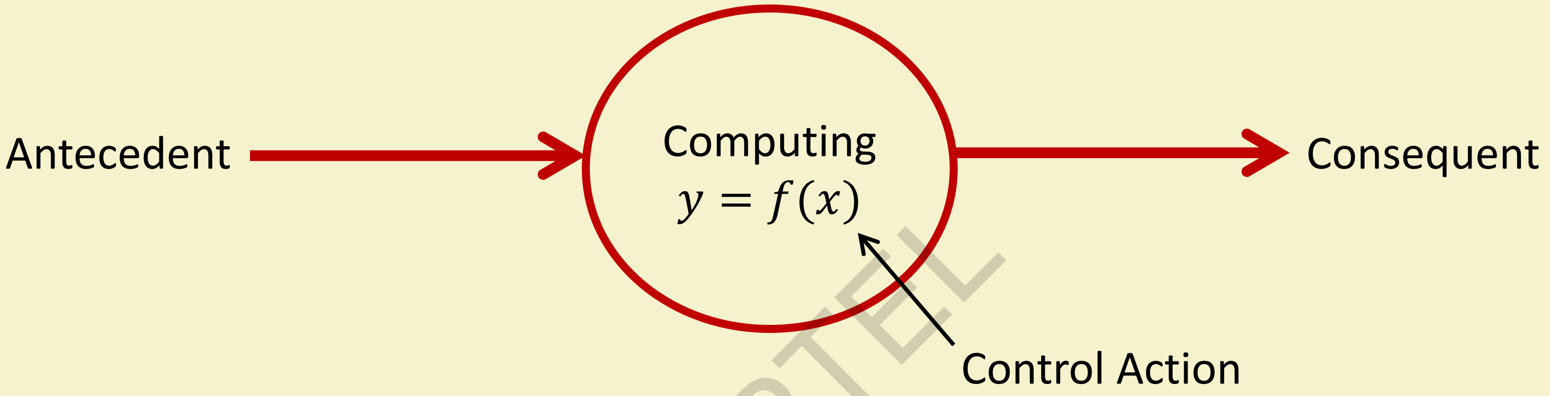


Figure: Basic of computing

$y = f(x), f$ is a mapping function.

f is also called a formal method or an algorithm to solve a problem.

Important characteristics of computing

- **Should provide precise solution.**
- **Control action should ne unambiguous and accurate.**
- **Suitable for problem, which is easy to model mathematically.**

Hard computing

- In 1996, L. A. Zade (LAZ) introduced the term hard computing.
- According to LAZ: We term a computing as Hard computing, if
  - ✓ Precise result is guaranteed.
  - ✓ Control action is unambiguous.
  - ✓ Control action is formally defined (i.e., with mathematical model or algorithm).

Examples of hard computing

- Solving numerical problems (e.g., roots of polynomials, integration, etc.).
- Searching and sorting techniques.
- Solving computational geometry problems (e.g., shortest tour in a graph, finding closet pair of points given a set of points, etc.).
- many more…

Soft computing

- **The term soft computing was proposed by the inventor of fuzzy logic, Lotfi A. Zadeh. He describes it as follows.**

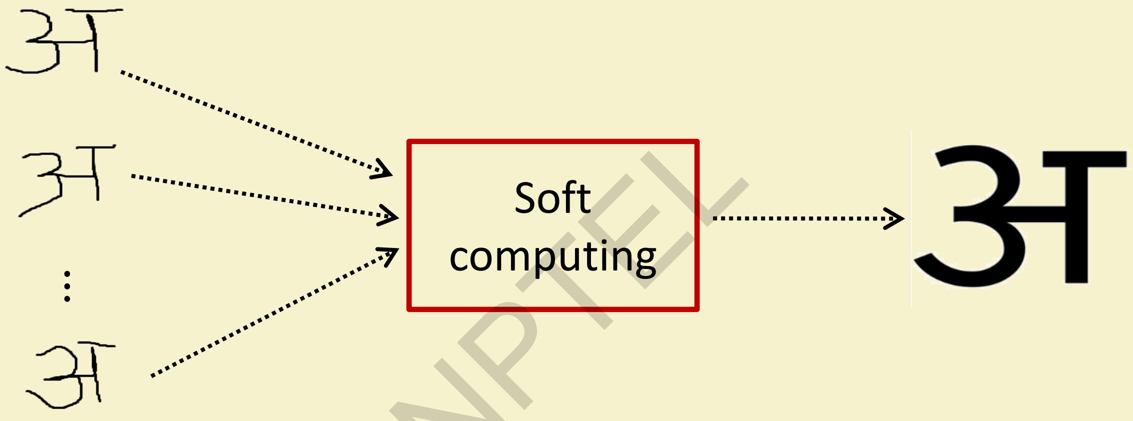
Definition 1: Soft computing

Soft computing is a collection of methodologies that aim to exploit the tolerance for imprecision and uncertainty to achieve tractability, robustness, and low solution cost. Its principal constituents are fuzzy logic, neuro-computing, and probabilistic reasoning. The role model for soft computing is the human mind.

Characteristics of soft computing

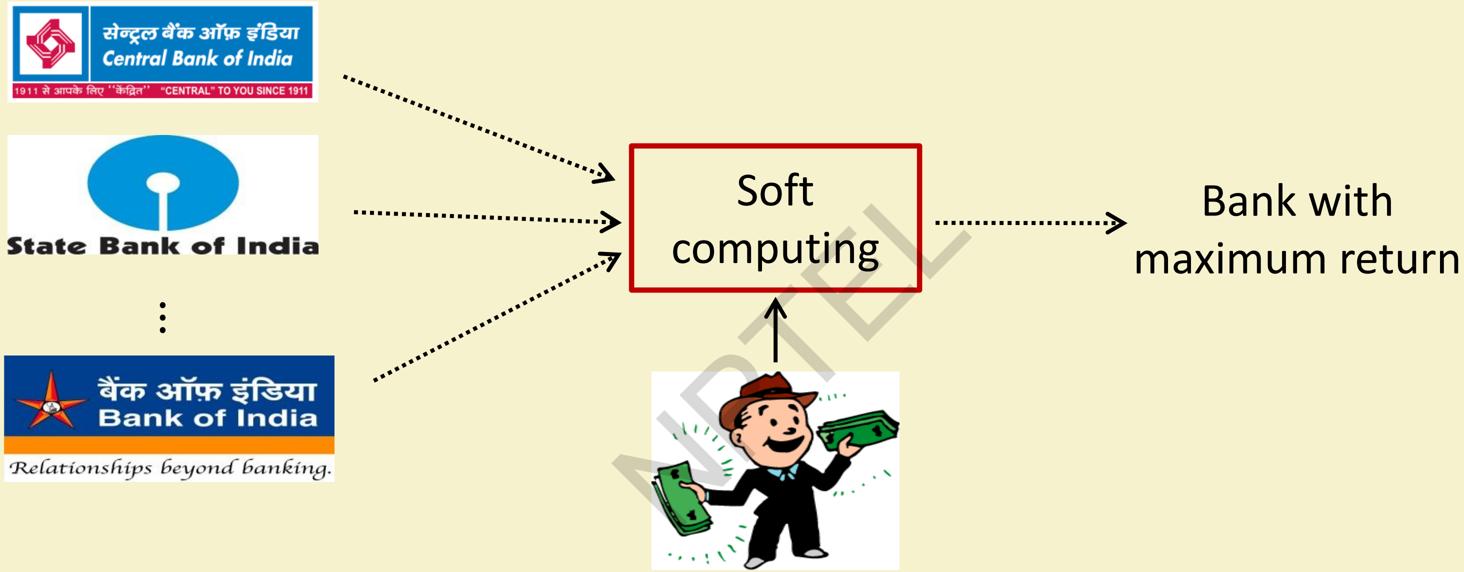
- It does not require any mathematical modeling of problem solving.
- It may not yield the precise solution.
- Algorithms are adaptive (i.e., it can adjust to the change of dynamic environment).
- Use some biological inspired methodologies such as genetics, evolution, Ant's behaviors, particles swarming, human nervous system, etc.).

Examples of soft computing



Example: Hand written character recognition
(Artificial Neural Networks)

Examples of soft computing



Example: Money allocation problem
(Evolutionary Computing)

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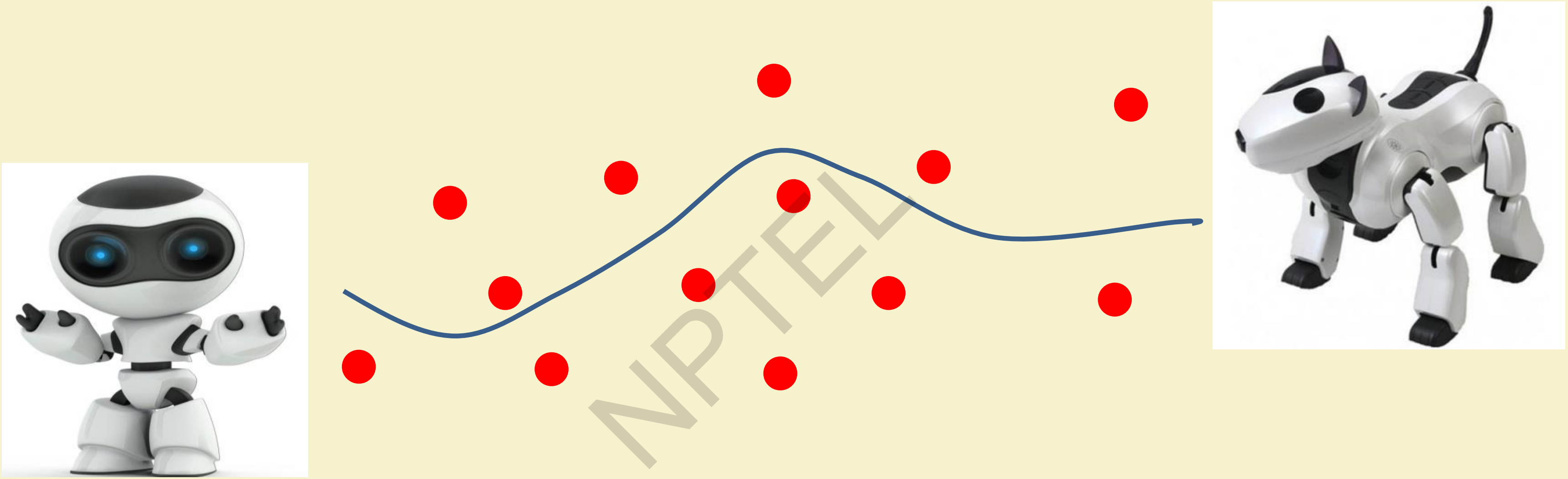
Examples of soft computing



Example: Robot movement
(Fuzzy Logic)

How soft computing?

- How a student learns from his teacher?
  - Teacher asks questions and tell the answers then.
  - Teacher puts questions and hints answers and asks whether the answers are correct or not.
  - Student thus learn a topic and store in his memory.
  - Based on the knowledge he solves new problems.
- This is the way how human brain works.
- Based on this concept Artificial Neural Network is used to solve problems.

How soft computing?

- How world selects the best?
  - It starts with a population (random).
  - Reproduces another population (next generation).
  - Rank the population and selects the superior individuals.
- Genetic algorithm is based on this natural phenomena.
  - Population is synonymous to solutions.
  - Selection of superior solution is synonymous to exploring the optimal solution.

How soft computing?

- How a doctor treats his patient?
  - Doctor asks the patient about suffering.
  - Doctor find the symptoms of diseases.
  - Doctor prescribed tests and medicines.
- This is exactly the way Fuzzy Logic works.
  - Symptoms are correlated with diseases with uncertainty .
  - Doctor prescribes tests/medicines fuzzily.

Hard computing vs. Soft computing

Hard computing	Soft computing
▪ It requires a precisely stated analytical model and often a lot of computation time.	▪ It is tolerant of imprecision, uncertainty, partial truth, and approximation.
▪ It is based on binary logic, crisp systems, numerical analysis and crisp software.	▪ It is based on fuzzy logic, neural nets and probabilistic reasoning.
▪ It has the characteristics of precision and categoricity.	▪ It has the characteristics of approximation and dispositionality.

Hard computing vs. Soft computing

Hard computing	Soft computing
▪ It is deterministic.	▪ It incorporates stochasticity.
▪ It requires exact input data.	▪ It can deal with ambiguous and noisy data.
▪ It is strictly sequential.	▪ It allows parallel computations.
▪ It produces precise answers.	▪ It can yield approximate answers

Hybrid computing

- It is a combination of the conventional hard computing and emerging soft computing.

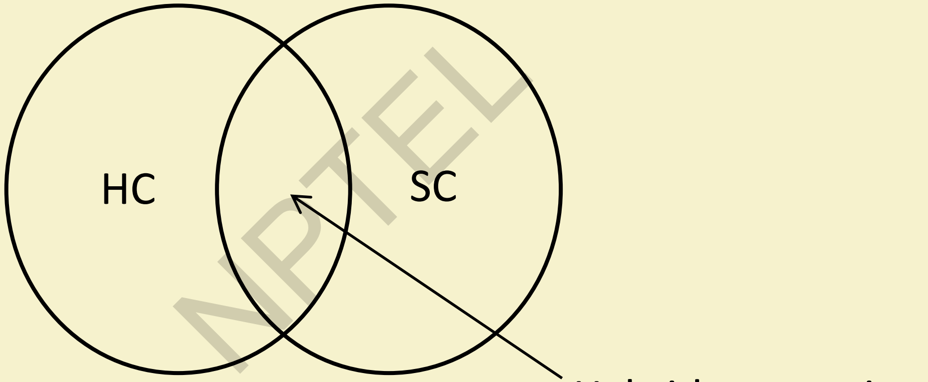


Figure: Concept of Hybrid Computing

In this course…

- You will be able to learn
  - Basic concepts of Fuzzy algebra and then how to solve problems using Fuzzy logic.
  - The framework of Genetic algorithm and solving varieties of optimization problems.
  - How to build an artificial neural network and train it with input data to solve a number of problems, which are not possible to solve with hard computing.

Thank You!!

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Soft Computing

Introduction to Fuzzy Logic

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What is Fuzzy logic?

- Fuzzy logic is a mathematical language to express something.
  - This means it has grammar, syntax, semantic like a language for communication.

- There are some other mathematical languages also known
  - **Relational algebra** (operations on sets)
  - **Boolean algebra** (operations on Boolean variables)
  - **Predicate algebra** (operations on well formed formulae (wff), also called predicate propositions)

- **Fuzzy logic deals with Fuzzy set or Fuzzy algebra.**

What is fuzzy?

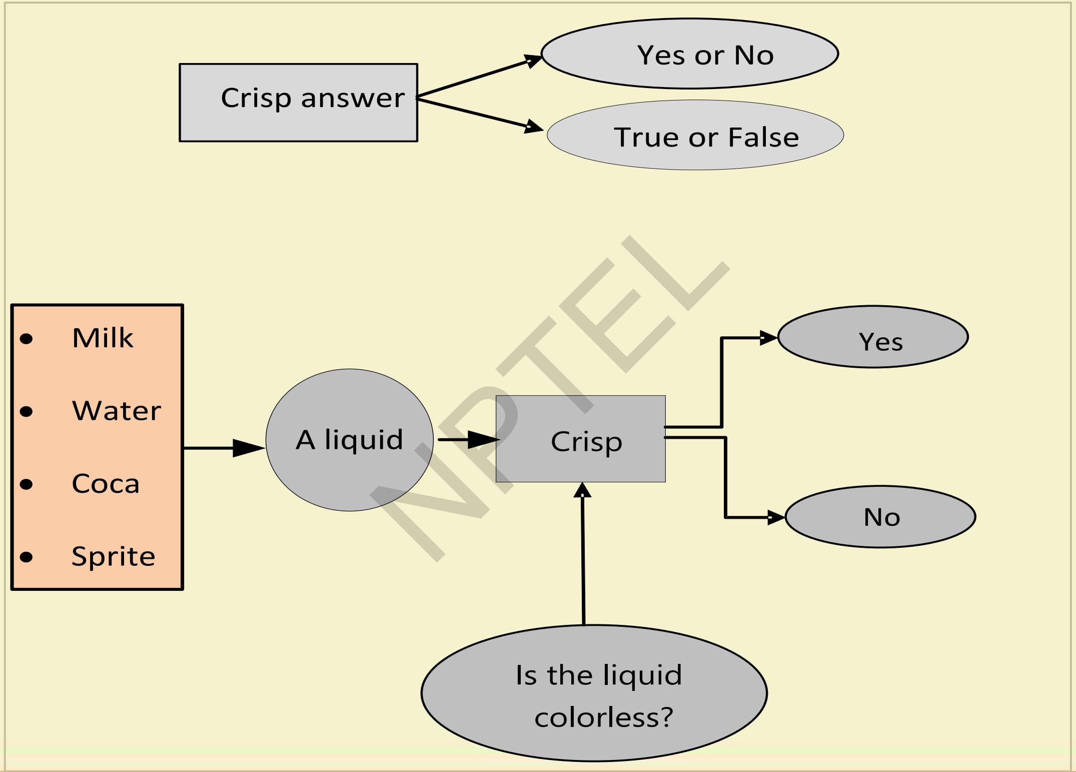
- Dictionary meaning of **fuzzy** is not clear, noisy, etc.

  Example: Is the picture on this slide is fuzzy?

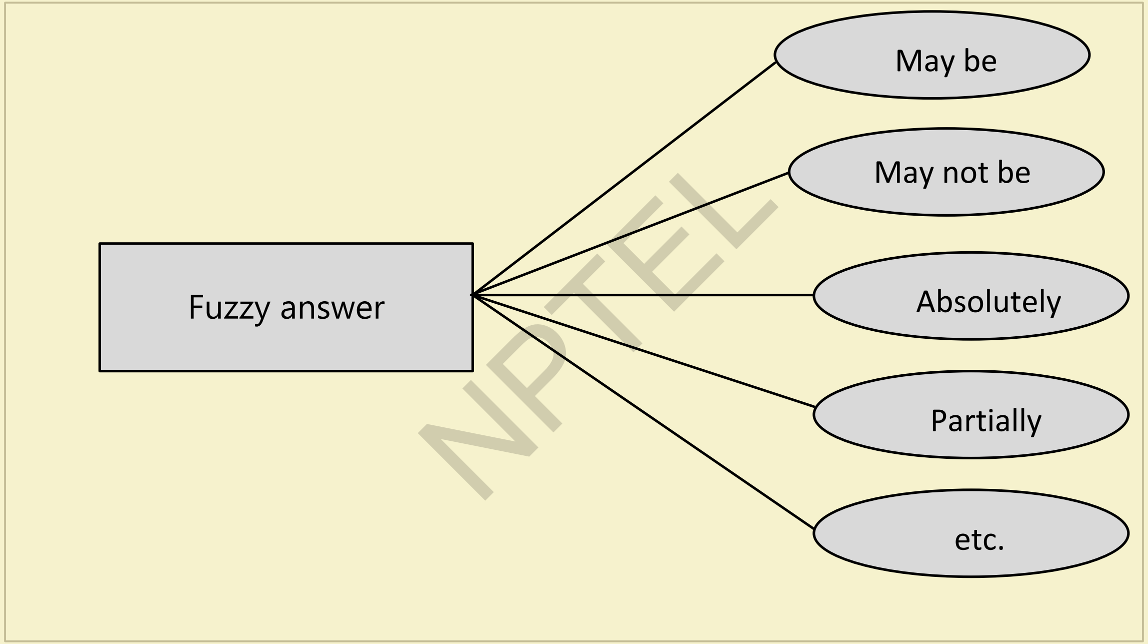
- Antonym of fuzzy is **crisp**

  Example: Are the chips crisp?

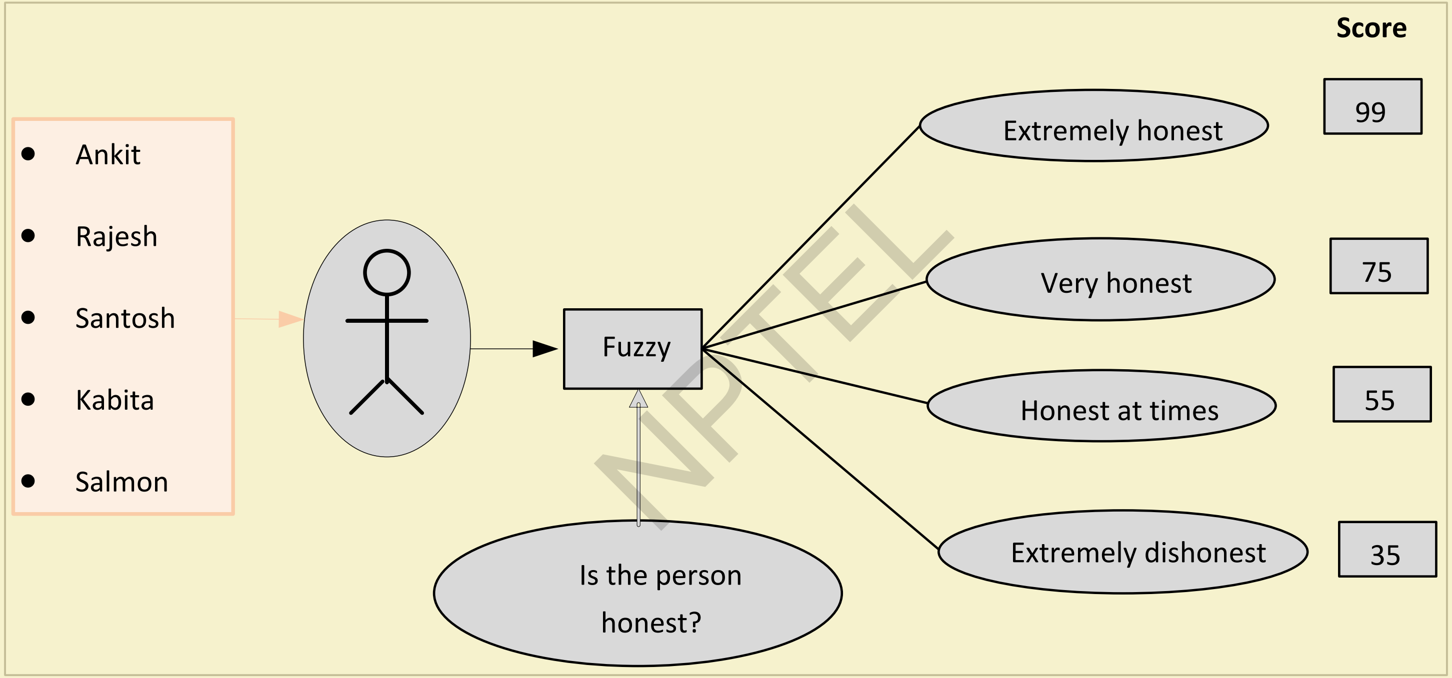
Example : Fuzzy logic vs. Crisp logic



Example : Fuzzy logic vs. Crisp logic

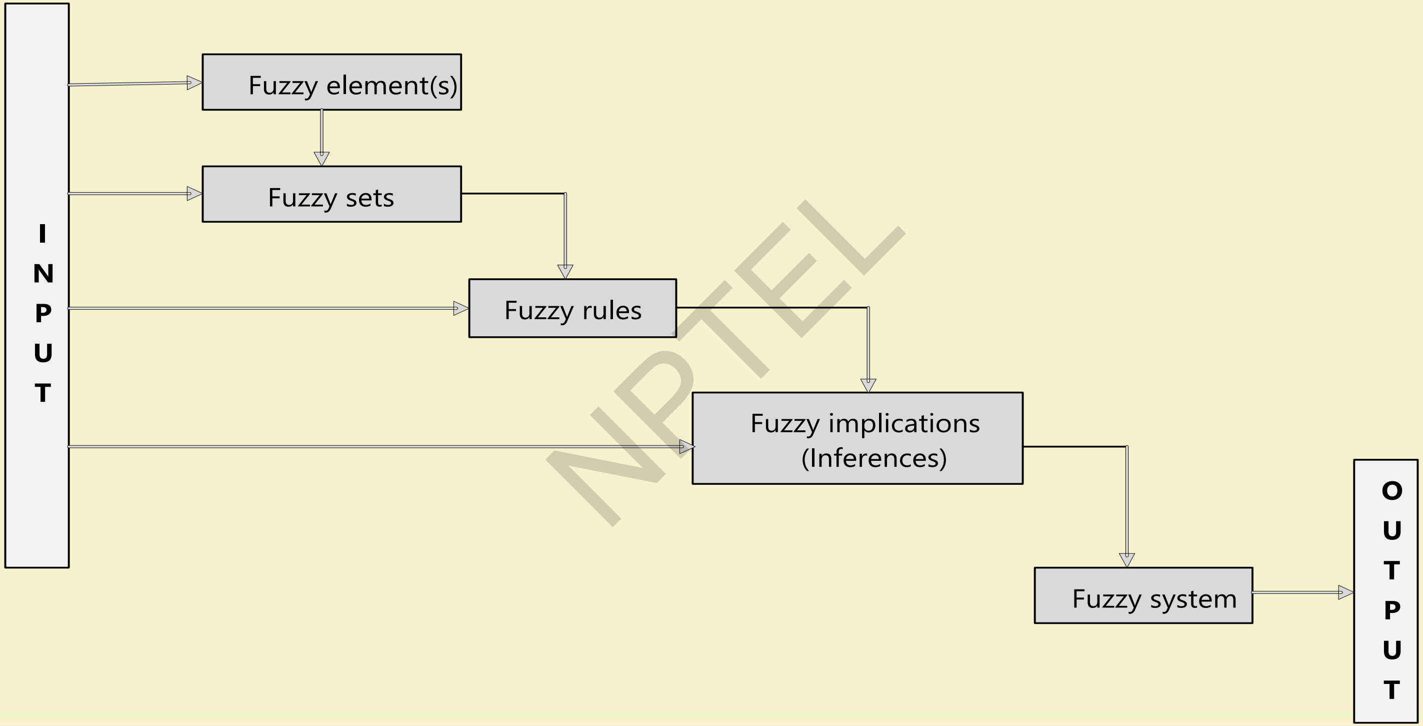


Example : Fuzzy logic vs. Crisp logic



World is fuzzy!

Concept of fuzzy system



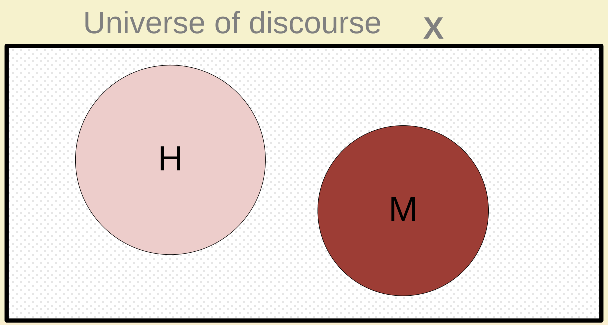
Concept of fuzzy set

To understand the concept of **fuzzy set** it is better, if we first clear our idea of **crisp set**.

X = The entire population of India.

H = All Hindu population = $\{h_1, h_2, h_3, \dots \dots, h_L,\}$

M = All Muslim population = $\{m_1, m_2, m_3, \dots \dots, m_N,\}$



Here, All are the sets of finite numbers of individuals. Such a set is called crisp set.

Example of fuzzy set

Let us discuss about fuzzy set.

X = All students in NPTEL.

S = All **Good students**.

$S = \{(s, g(s)) \mid s \in X\}$ and $g(s)$ is a measurement of goodness of the student s.

Example:

$S = \{(Rajat, 0.8), (Kabita, 0.7), (Salman, 0.1), (Ankit, 0.9)\}$, etc.

Fuzzy set vs. Crisp set

Crisp set	Fuzzy set
▪ $S = \{s \vert s \in X\}$	▪ $F = (s, \mu(s)) \vert s \in X$ and $\mu(s)$ is the degree of s.
▪ It is a collection of elements.	▪ It is a collection of ordered pairs.
▪ Inclusion of an element $s \in X$ into S is crisp, that is, has strict boundary yes or no.	▪ Inclusion of an element $s \in X$ into F is fuzzy, that is, if present, then with a degree of membership.

Fuzzy set vs. Crisp set

Note: A crisp set is a fuzzy set, but, a fuzzy set is not necessarily a crisp set.

Example:

$$\text{H} = \{(h_1, 1), (h_2, 1) \ldots \ldots \ldots, (h_L, 1)\}$$

$$\text{Person} = \{(p_1, 0), (p_2, 0) \ldots \ldots \ldots, (p_N, 0)\}$$

In case of a crisp set, the elements are with extreme values of degree of membership namely either 1 or 0.

Degree of membership

How to decide the degree of memberships of elements in a fuzzy set?

City	Bangalore	Bombay	Hyderabad	Kharagpur	Madras	Delhi
μ	0.95	0.90	0.80	0.01	0.65	0.75

How the cities of comfort can be judged?

Example: Course evaluation in a crisp way

EX : Marks ≥ 90

A : $80 \leq$ Marks < 90

B : $70 \leq$ Marks < 80

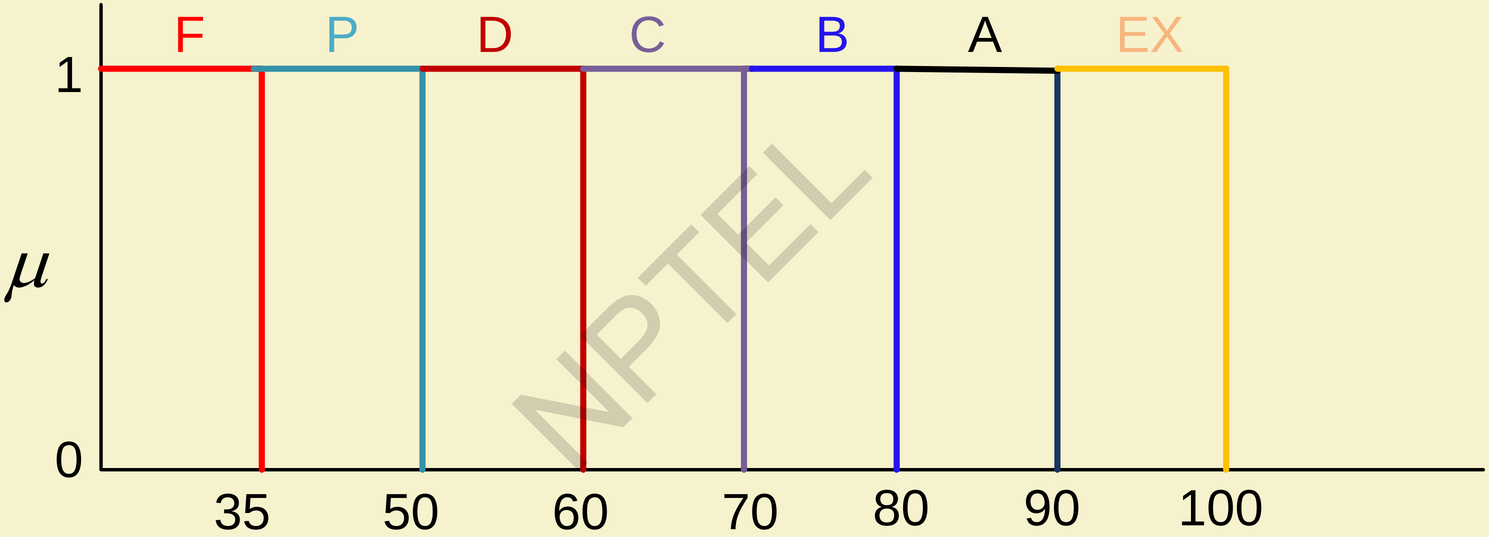
C : $60 \leq$ Marks < 70

D : $50 \leq$ Marks < 60

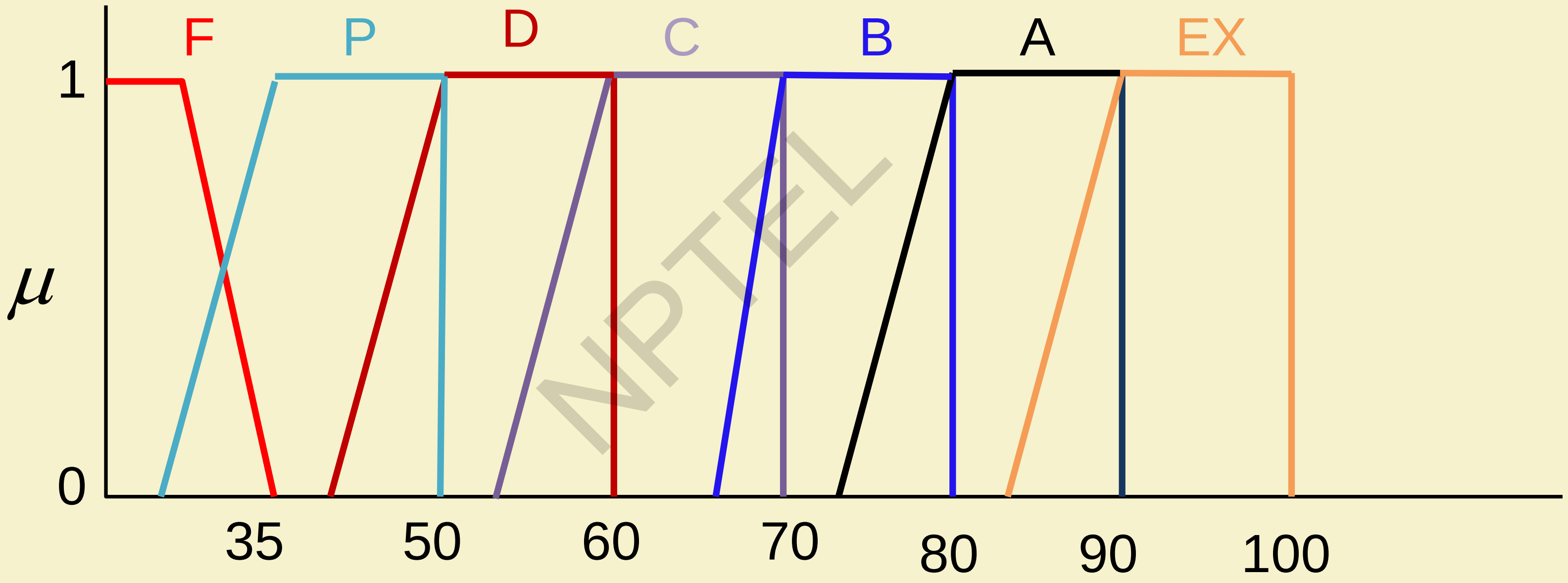
P : $35 \leq$ Marks < 50

F : Marks ≤ 35

Example: Course evaluation in a crisp way



Example: Course evaluation in a fuzzy way



Few examples of fuzzy set

- High Temperature
- Low Pressure
- Colour of Apple
- Sweetness of Orange
- Weight of Mango

Note: Degree of membership values lie in the range [0...1].

Some basic terminologies and notations

Definition 1: Membership function (and Fuzzy set)

If X is a universe of discourse and $x \in X$, then a fuzzy set A in X is defined as a set of ordered pairs, that is
$A = \{(x, \mu_A(x)) | x \in X\}$ where $\mu_A(x)$ is called the membership function for the fuzzy set A.

Note: $\mu_A(x)$ map each element of X onto a membership grade (or membership value) between 0 and 1 (both inclusive).

Question: How (and who) decides $\mu_A(x)$ for a fuzzy set A in X?

Some basic terminologies and notations

Example:

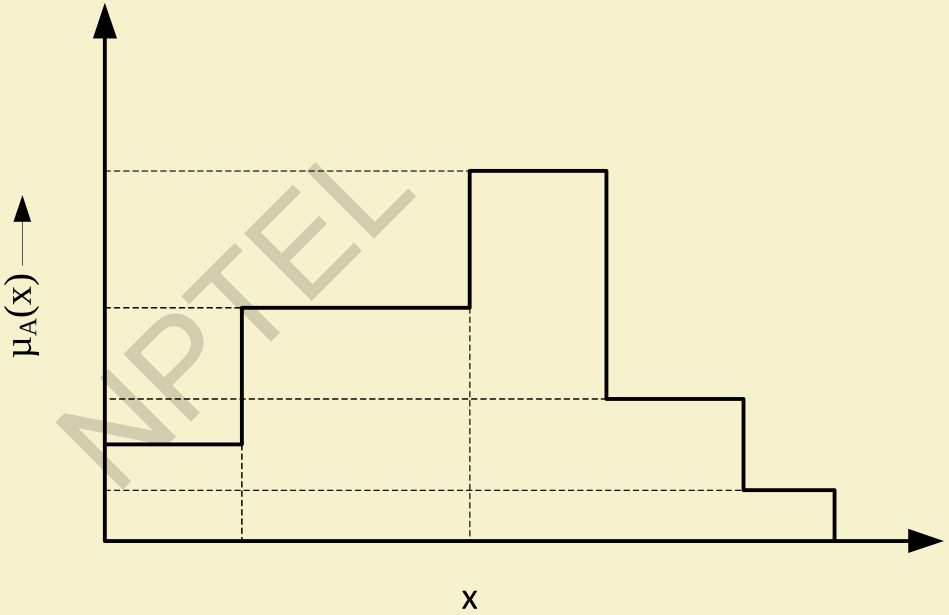
X = All cities in India

A = City of comfort

A={(New Delhi, 0.7), (Bangalore, 0.9), (Chennai, 0.8), (Hyderabad, 0.6), (Kolkata, 0.3), (Kharagpur, 0)}

Membership function with discrete membership values

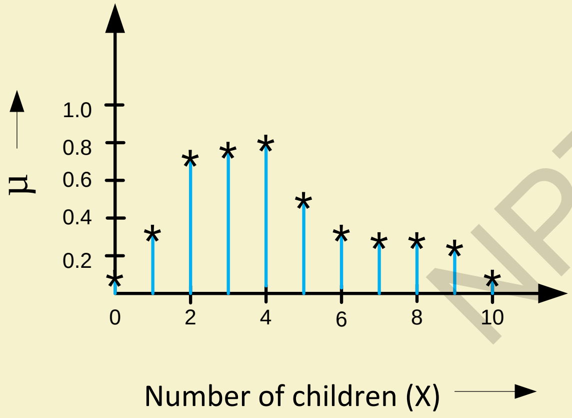
The membership values may be of discrete values.



A fuzzy set with discrete values of μ

Membership function with discrete membership values

Either elements or their membership values (or both) also may be of discrete values.



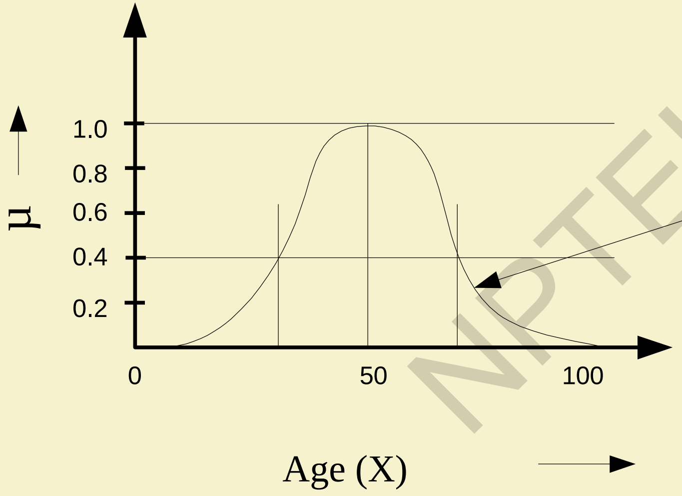
A = "Happy family"

A ={(0,0.1),(1,0.30),(2,0.78)……(10,0.1)}

Note : X = discrete value

How you measure happiness ??

Membership function with continuous membership values



$$\mu_B(x) = \frac{1}{1+\left(\frac{x-50}{10}\right)^4}$$

$$B=\{(x,\mu_B(x))\}$$

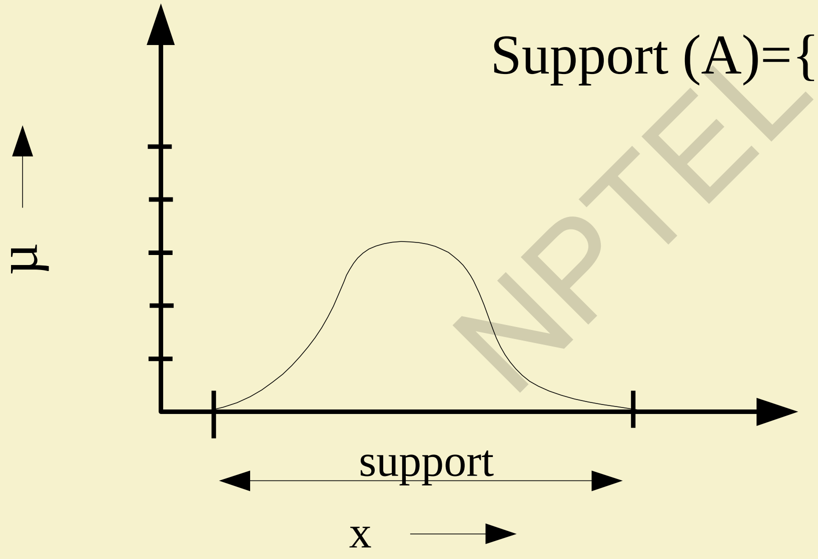
B = “Middle aged”

Note : x = real value = R^+

Fuzzy terminologies: Support

Support: The support of a fuzzy set A is the set of all points $x \in X$ such that $\mu_A(X) > 0$

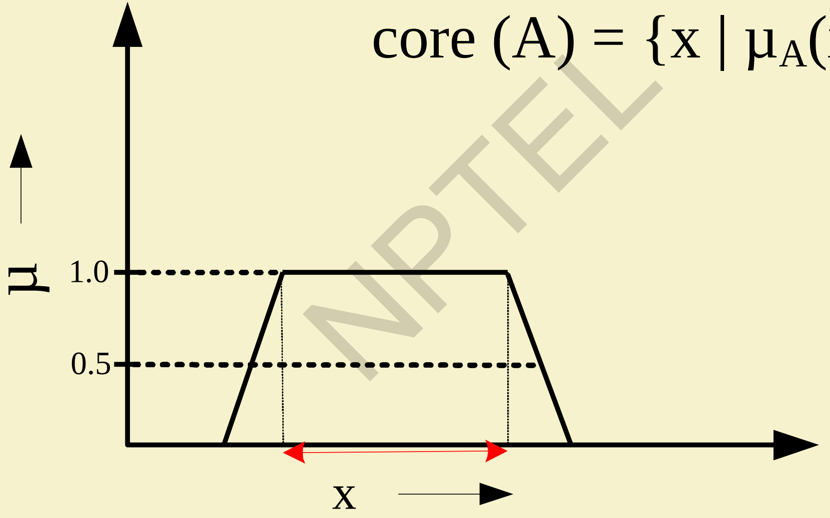
$$\text{Support (A)}=\{x \mid \mu_A(x) > 0\}$$



Fuzzy terminologies: Core

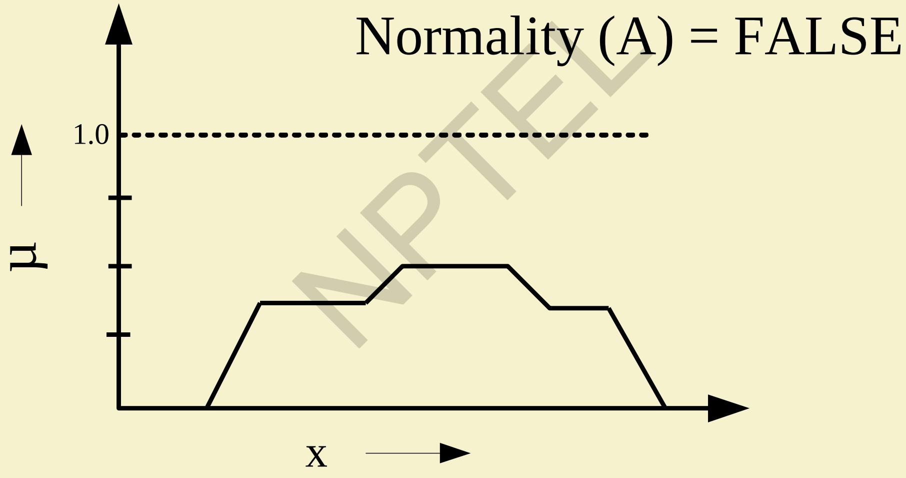
Core: The core of a fuzzy set A is the set of all points x in X such that $\mu_A(X) = 1$

$$\text{core}(A) = \{x \mid \mu_A(x) = 1\}$$



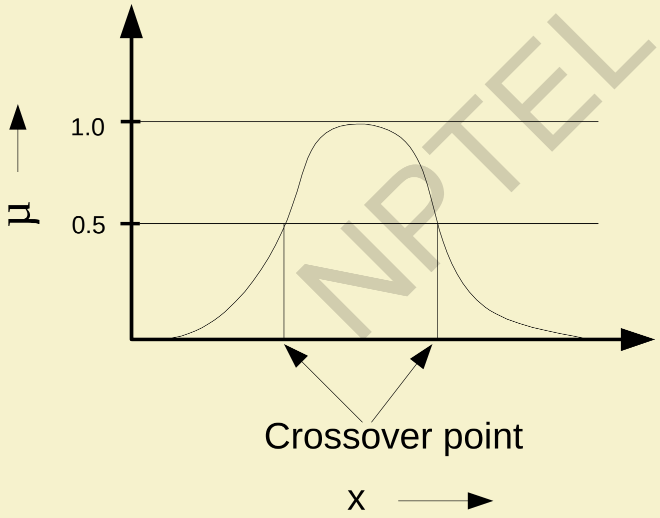
Fuzzy terminologies: Normality

Normality : A fuzzy set A is a normal if its core is non-empty. In other words, we can always find a point $x \in X$ such that $\mu_A(X) = 1$



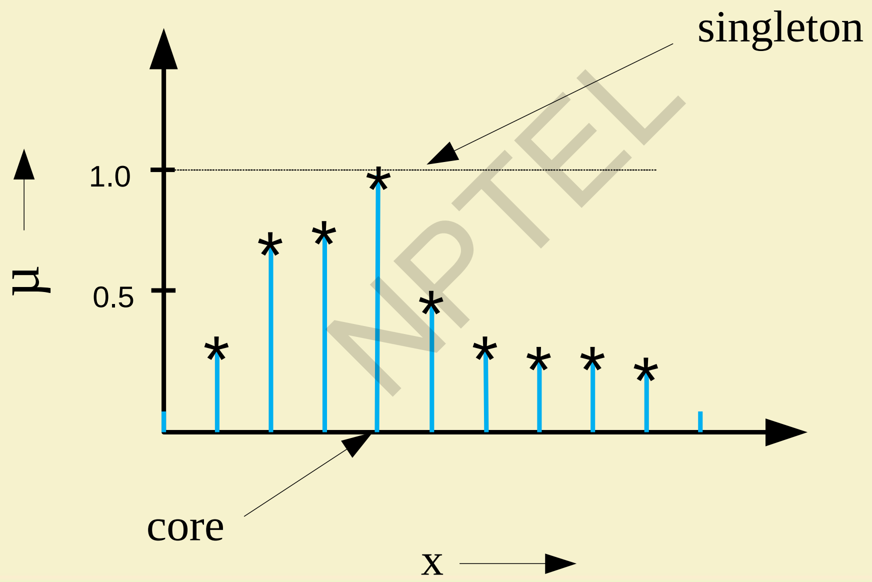
Fuzzy terminologies: Crossover points

Crossover point : A crossover point of a fuzzy set A is a point $x \in X$ at which $\mu_A(X) =$ 0.5. That is Crossover (A) =$\{x|\mu_A(x) = 0.5\}$



Fuzzy terminologies: Fuzzy Singleton

Fuzzy Singleton : A fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called a fuzzy singleton. That is $|A| = \{x|\mu_A(x) = 1\}$



Fuzzy terminologies: α-cut and strong α-cut

α-cut and strong α-cut :

- ✓ The α-cut of a fuzzy set A is a crisp set defined by

$$A_\alpha = \{x|\mu_A(x) \geq \alpha\}$$

- ✓ Strong α-cut is defined similarly :

$$A'_\alpha = \{x|\mu_A(x) > \alpha\}$$

Note : Support (A) = A_0’ and Core (A) = A_1.

Fuzzy terminologies: Bandwidth

Bandwidth :
For a fuzzy set, the bandwidth (or width) is defined as the distance between the two unique crossover points:

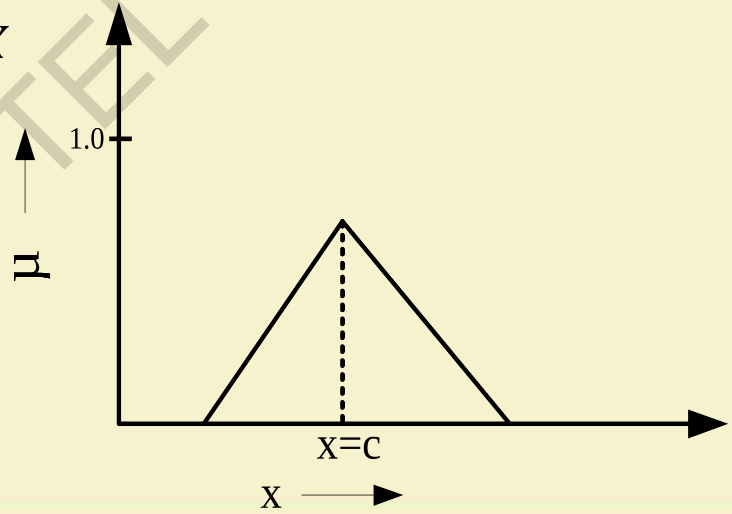
$$\text{Bandwidth } (A) = |x_1 - x_2|$$

where $\mu_A(x_1) = \mu_A(x_2) = 0.5$

Fuzzy terminologies: Symmetry

Symmetry :

A fuzzy set A is symmetric if its membership function around a certain point $x = c$, namely $\mu_A(x + c) = \mu_A(x - c)$ for all $x \in X$



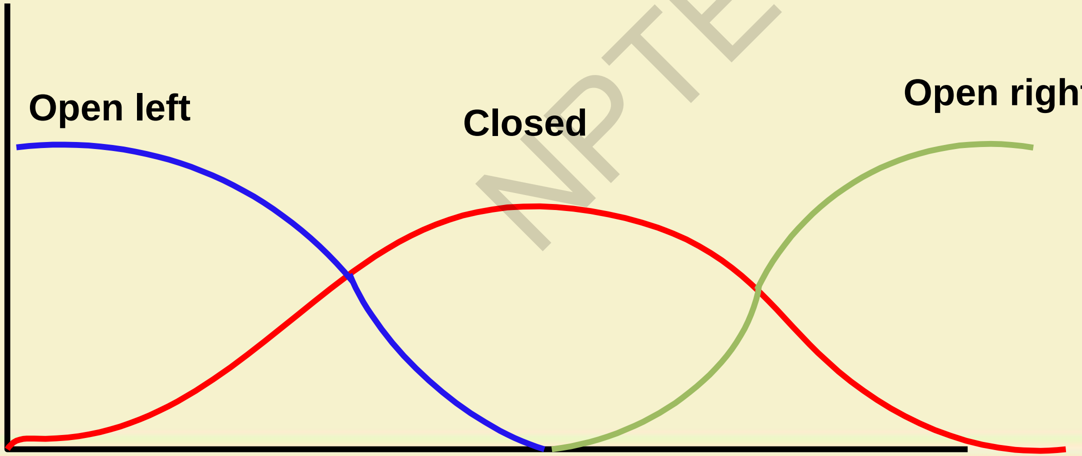
Fuzzy terminologies: Open and Closed

A fuzzy set A is

Open left : If $\lim x \rightarrow -\infty \; \mu_A(x) = 1$ and $\lim x \rightarrow +\infty \; \mu_A(x) = 0$

Open right: If $\lim x \rightarrow -\infty \; \mu_A(x) = 0$ and $\lim x \rightarrow +\infty \; \mu_A(x) = 1$

Closed: If $\lim x \rightarrow -\infty \; \mu_A(x) = \lim x \rightarrow +\infty \; \mu_A(x) = 0$



Fuzzy vs. Probability

Fuzzy : When we say about certainty of a thing

Example: A patient come to the doctor and he has to diagnose so that medicine can be prescribed.

Doctor prescribed a medicine with certainty 60% that the patient is suffering from flue. So, the disease will be cured with certainty of 60% and uncertainty 40%. Here, in stead of flue, other diseases with some other certainties may be.

Probability: When we say about the chance of an event to occur

Example: India will win the T20 tournament with a chance 60% means that out of 100 matches, India own 60 matches.

Prediction vs. Forecasting

The Fuzzy vs. Probability is analogical to Prediction vs. Forecasting

Prediction : When you start guessing about things.

Forecasting : When you take the information from the past job and apply it to new job.

The main difference:

Prediction is based on the best guess from experiences.

Forecasting is based on data you have actually recorded and packed from previous job.

Thank You!!

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Introduction to Soft Computing

Fuzzy membership functions

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Fuzzy membership functions

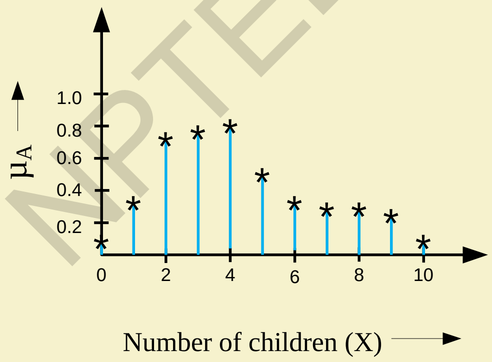
A fuzzy set is completely characterized by its membership function (sometimes abbreviated as MF and denoted as μ). So, it would be important to learn how a membership function can be expressed (mathematically or otherwise).

Note: A membership function can be on

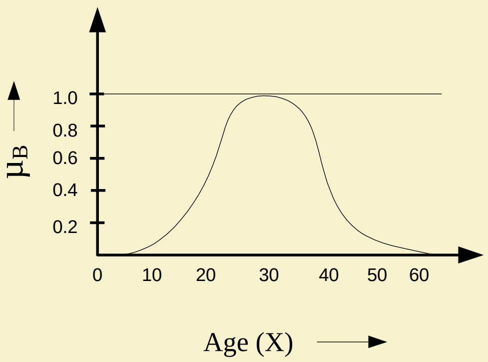
a) a discrete universe of discourse and

b) a continuous universe of discourse.

Example:



A = Fuzzy set of "Happy family"

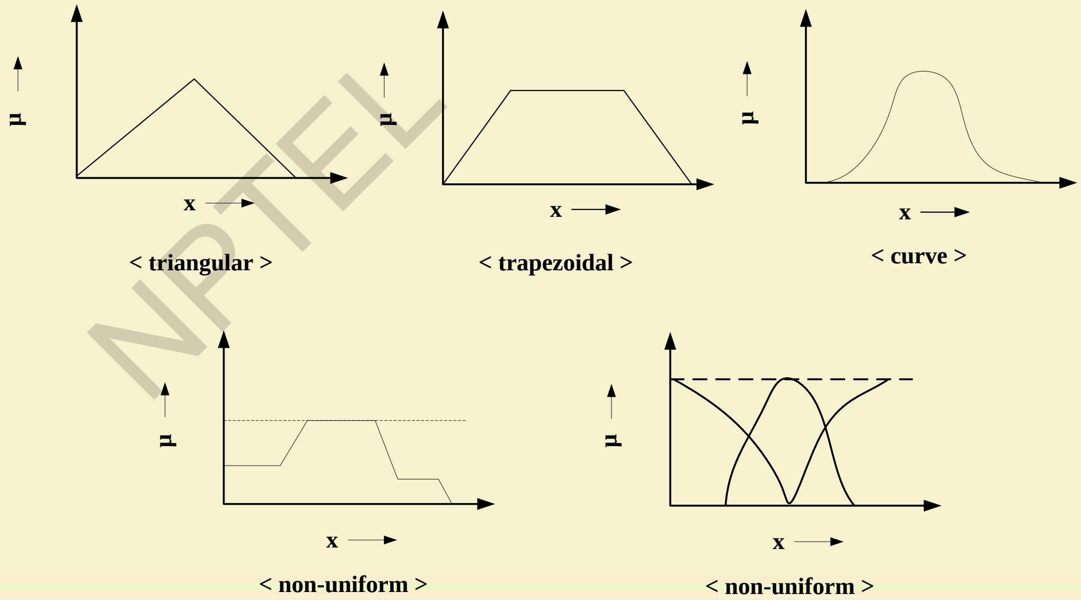


B = "Young age"

Fuzzy membership functions

So, membership function on a discrete universe of course is trivial. However, a membership function on a continuous universe of discourse needs a special attention.

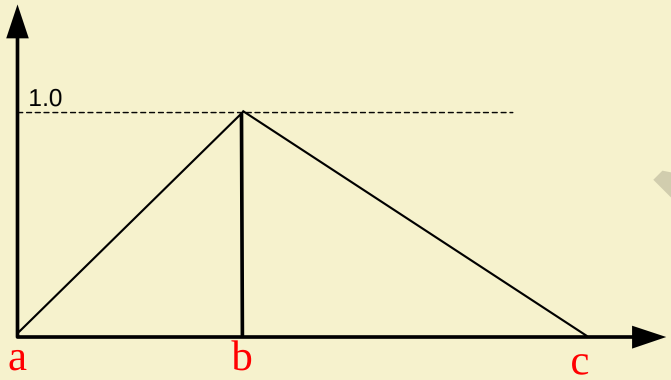
Following figures shows typical examples of membership functions.



Fuzzy MFs : Formulation and parameterization

In the following, we try to parameterize the different MFs on a continuous universe of discourse.

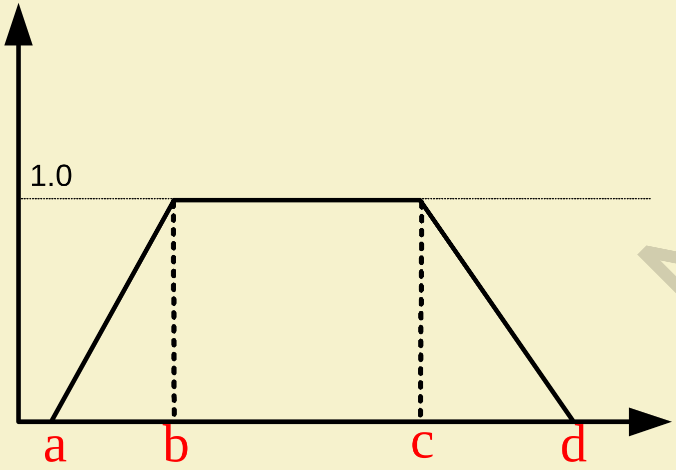
Triangular MFs : A triangular MF is specified by three parameters $\{a, b, c\}$ and can be formulated as follows.



$$\text{triangle}\,(x; a, b, c) = \begin{Bmatrix} 0 & if\ x \le a \\ \dfrac{x-a}{a-b} & if\ a \le x \le b \\ \dfrac{c-x}{c-b} & if\ b \le x \le c \\ 0 & if\ c \le x \end{Bmatrix}$$

Fuzzy MFs: Trapezoidal

A **trapezoidal MF** is specified by four parameters $\{a, b, c, d\}$ and can be defined as follows:

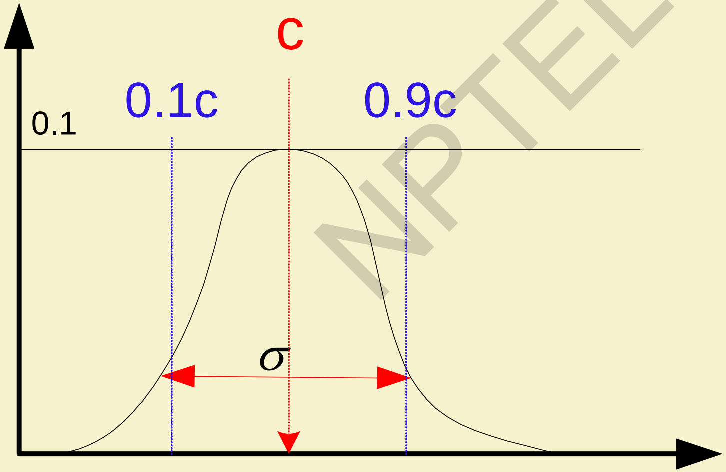


$$\text{trapozoid}\,(x; a, b, c, d) = \begin{Bmatrix} 0 & if\ x \leq a \\ \dfrac{x-a}{b-a} & if\ a \leq x \leq b \\ 1 & if\ b \leq x \leq c \\ \dfrac{d-x}{d-c} & if\ c \leq x \leq d \\ 0 & if\ d \leq x \end{Bmatrix}$$

Fuzzy MFs: Gaussian

A **Gaussian MF** is specified by two parameters $\{c, \sigma\}$ and can be defined as below:

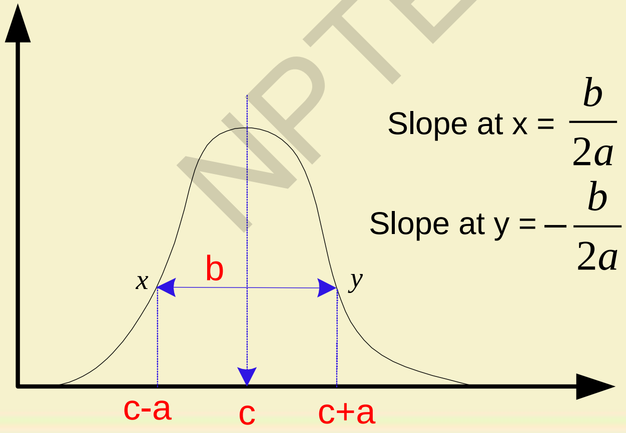
$$gaussian\,(x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$



Fuzzy MFs: Generalized bell

It is also called Cauchy MF. A generalized bell MF is specified by three parameters $\{a, b, c\}$ and is defined as:

$$bell(x;\ a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$



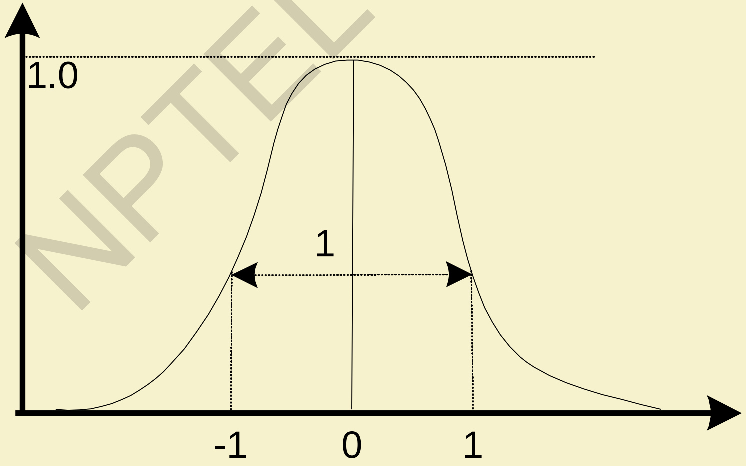
Slope at x = $\frac{b}{2a}$

Slope at y = $-\frac{b}{2a}$

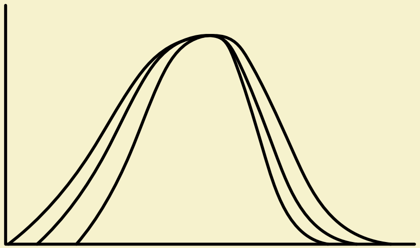
Example: Generalized bell MFs

Example: $\mu(x) = \frac{1}{1+|x|^2}$;

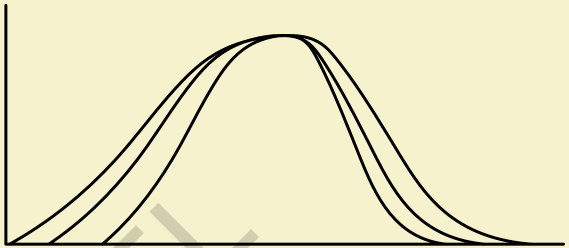
a = b = 1 and c = 0;



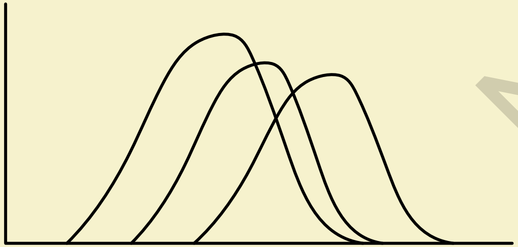
Generalized bell MFs: Different shapes



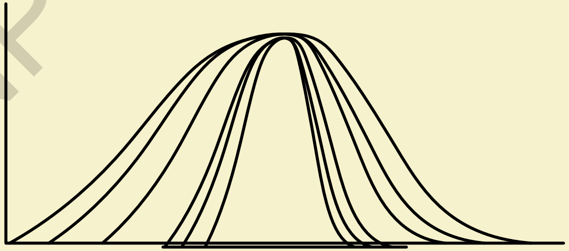
Changing a



Changing b



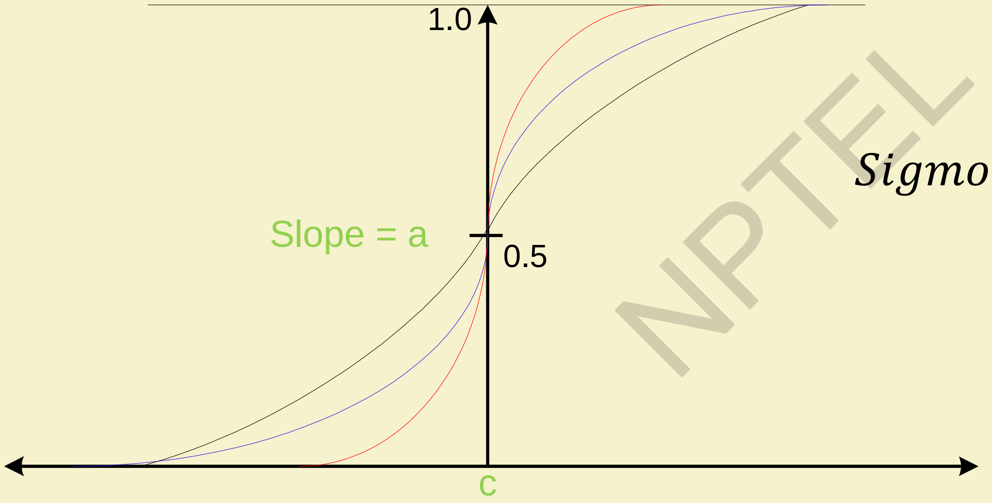
Changing a



Changing a and b

Fuzzy MFs: Sigmoidal MFs

Parameters: $\{a, c\}$; where c = crossover point and a = slope at c;



$$Sigmoid(x;\ a, c) = \frac{1}{1 + e^{-\left[\frac{x-c}{a}\right]}}$$

Fuzzy MFs : Example

Example : Consider the following grading system for a course.

Excellent = Marks $\leq$ 90

Very good = 75 $\leq$ Marks $\leq$ 90

Good = 60 $\leq$ Marks $\leq$ 75

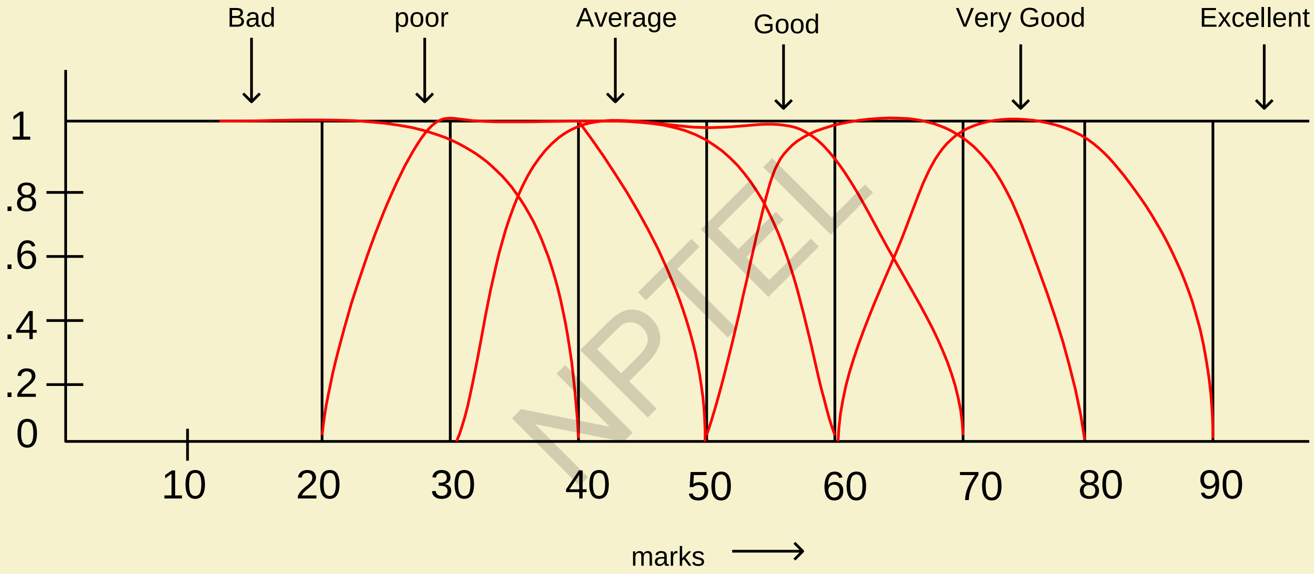
Average = 50 $\leq$ Marks $\leq$ 60

Poor = 35 $\leq$ Marks $\leq$ 50

Bad= Marks $\leq$ 35

Grading System

A fuzzy implementation will look like the following.



You can decide a standard fuzzy MF for each of the fuzzy grade.

Few More on Membership Functions

Generation of MFs

Given a membership function of a fuzzy set representing a linguistic hedge, we can derive many more MFs representing several other linguistic hedges using the concept of Concentration and Dilation.

1. **Concentration:** $A^k = [\mu_A(x)]^k; k > 1$
2. **Dilation:** $A^k = [\mu_A(x)]^k; k < 1$

Example : Age = { Young, Middle-aged, Old }

Thus, corresponding to Young, we have : Not young, Very young, Not very young and so on.
Similarly, with Old we can have : Not old, Very old, Very very old, Extremely old, etc.

Thus, $\mu_{Extremely\ old}(x) = (((\mu_{Old}(x))^2)^2)^2$ and so on

Or, $\mu_{More\ or\ less\ old}(x) = A^{0.5} = (\mu_{Old}(x))^{0.5}$

Linguistic variables and values

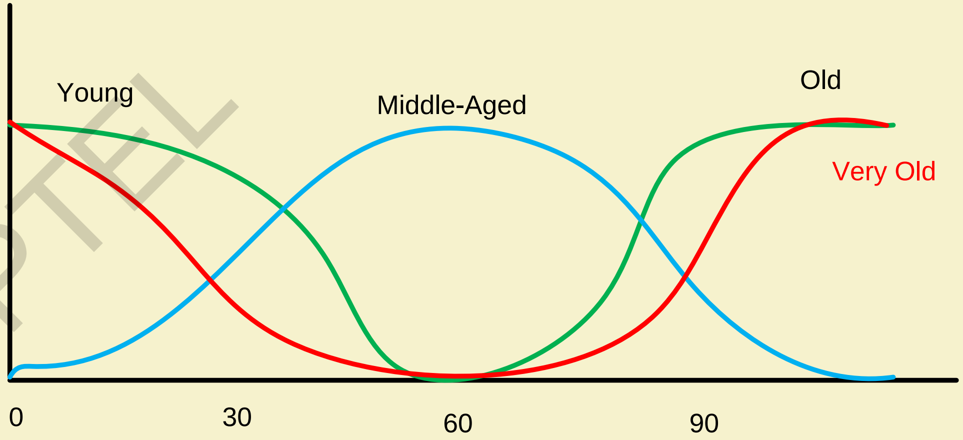
$\mu_{young}(x)$ =bell(x,20,2,0) $= \frac{1}{1+(\frac{x}{20})^4}$

$\mu_{old}(x)$ =bell(x,30,3,100) $= \frac{1}{1+(\frac{x-100}{30})^6}$

$\mu_{middle-aged}(x)$ =bell(x,30,60,50)

Not young=$\overline{\mu_{young}(x)} = 1 - \mu_{young}(x)$

Young but not too young = $\mu_{young}(x) \cap \overline{\mu_{young}(x)}$



Thank You!!

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Introduction to Soft Computing

Operations on Fuzzy sets

Prof. Debasis Samanta
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Basic fuzzy set operations: Union

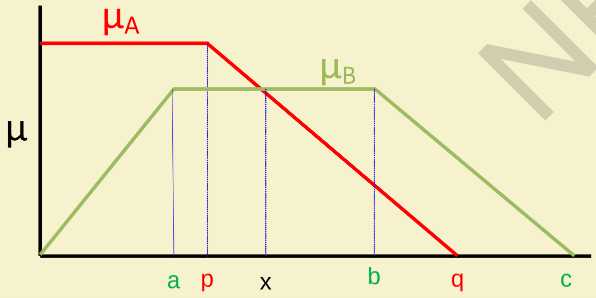
Union (A ∪ B): $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$

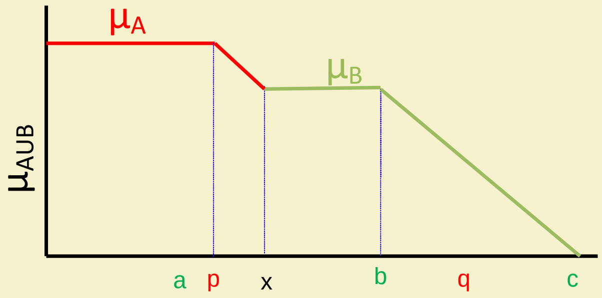
Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and

$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;

$C = A \cup B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.5)\}$





Basic fuzzy set operations: Intersection

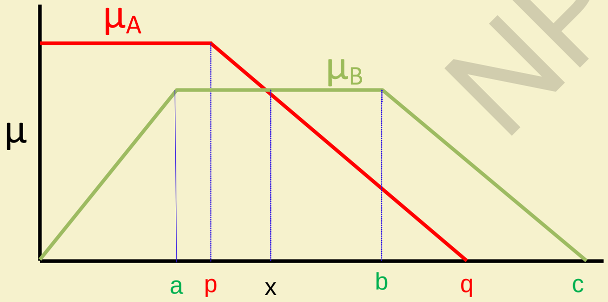
Intersection (A ∩ B): $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$

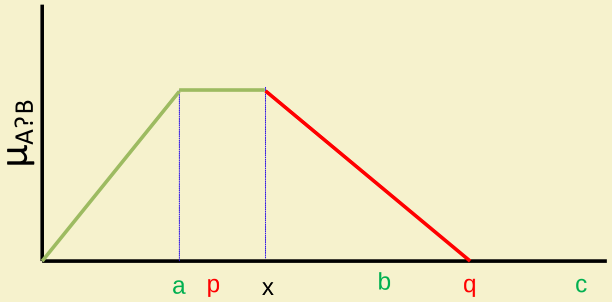
Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and

$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;

$C = A \cap B = \{(x_1, 0.2), (x_2, 0.1), (x_3, 0.4)\}$





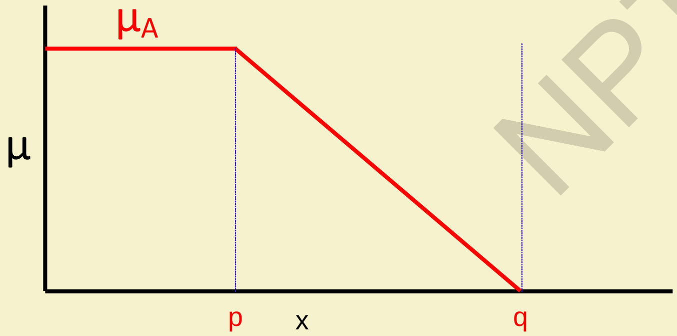
Basic fuzzy set operations: Complement

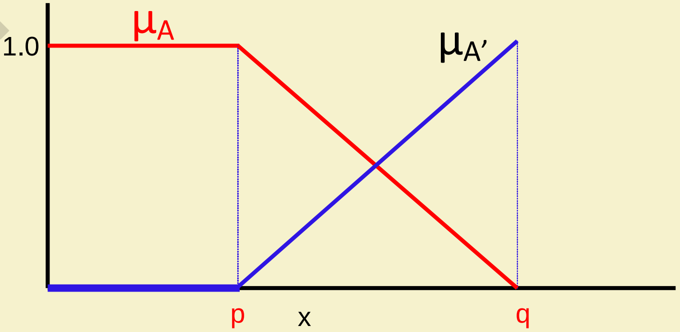
Complement (A^c): $\mu_{A^c}(x) = 1 - \mu_A(x)$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$

$C = A^c = \{(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)\}$





Basic fuzzy set operations: Products

Algebric product or Vector product ($A \cdot B$):

$$\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$$

Scalar product ($\alpha \times A$):

$$\mu_{\alpha A}(x) = \alpha \times \mu_A(x)$$

Basic fuzzy set operations: Sum and Difference

Sum (A + B**):**

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x).\mu_B(x)$$

Difference ($A - B \ = \ A \cap B^C$**) :**

$$\mu_{A-B}(x) = \mu_{A \cap B^C}(x)$$

Disjunctive sum:

$$A \oplus B \ = \ (A^C \cap B) \ \cup \ (A \cap B^C)$$

Bounded Sum:

$$| A(x) \oplus B(x) | = \mu_{|A(x) \oplus B(x)|} = \min\{1, \mu_A(x) + \mu_B(x)\}$$

Bounded Difference:

$$| A(x) \ominus B(x) | = \mu_{|A(x) \ominus B(x)|} = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$$

Basic fuzzy set operations: Equality and Power

Equality ($A = B$):

$$\mu_A(x) = \mu_B(x)$$

Power of a fuzzy set A^α:

$$\mu_{A^\alpha}(x) = (\mu_A(x))^\alpha$$

- ✓ If $\alpha < 1$, then it is called dilation
- ✓ If $\alpha > 1$, then it is called concentration

Basic fuzzy set operations: Cartesian product

Caretsian Product ($A \times B$): $\mu_{A\times B}(x, y) = \min(\mu_A(x), \mu_B(y))$

Example:

$A(x) = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5), (x_4, 0.6)\}$
$B(y) = \{(y_1, 0.8), (y_2, 0.6), (y_3, 0.3)\}$

$$\mathrm{A} \times B = \min\big(\mu_A(x), \mu_B(y)\big) = \begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.3 \end{bmatrix} \end{array}$$

Properties of fuzzy sets

Commutativity :

$$A \cap B = B \cap A$$
$$A \cup \mathrm{B} = B \cup \mathrm{A}$$

Associativity :

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributivity :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Properties of fuzzy sets

Idempotence :

$$A \cup A = A$$
$$A \cap A = \emptyset\,;$$
$$A \cup \emptyset; = A$$
$$A \cap \emptyset\,; = \emptyset;$$

Transitivity :

If $A \subseteq B; B \subseteq C$ then $A \subseteq C$

Involution :

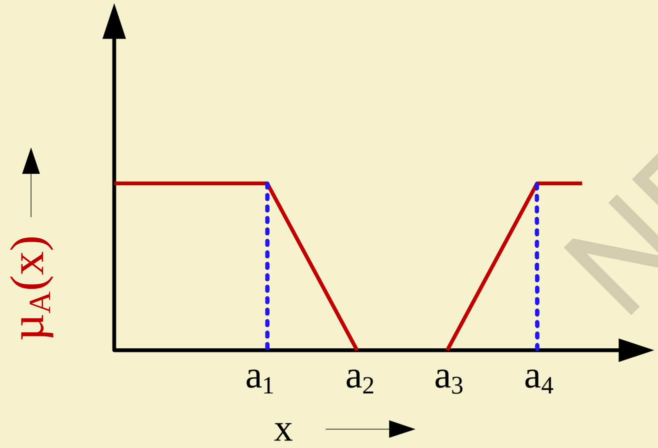
$$(A^c)^c = A$$

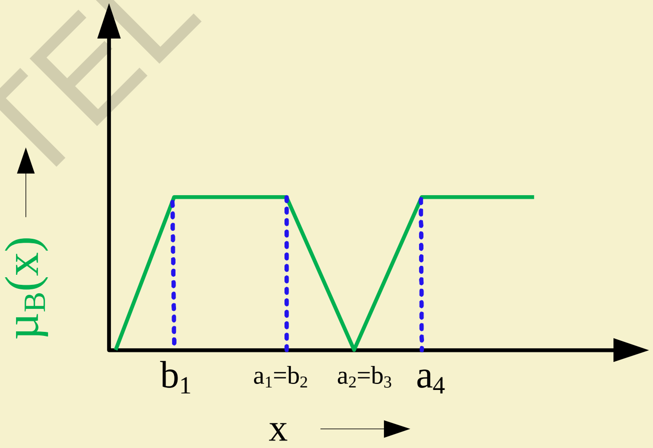
De Morgan's law :

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

Example 1: Fuzzy Set Operations

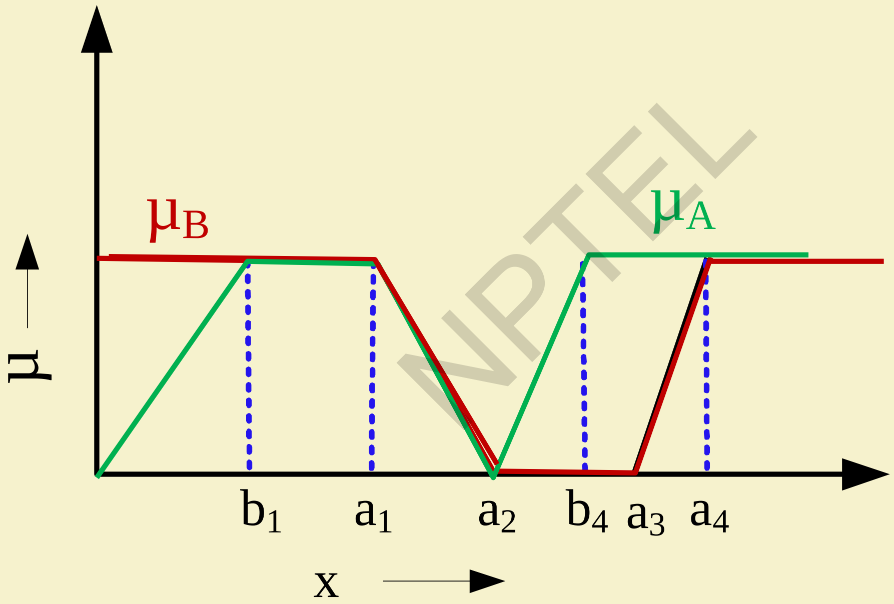
Let A and B are two fuzzy sets defined over a universe of discourse X with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively. Two MFs $\mu_A(x)$ and $\mu_B(x)$ are shown graphically.





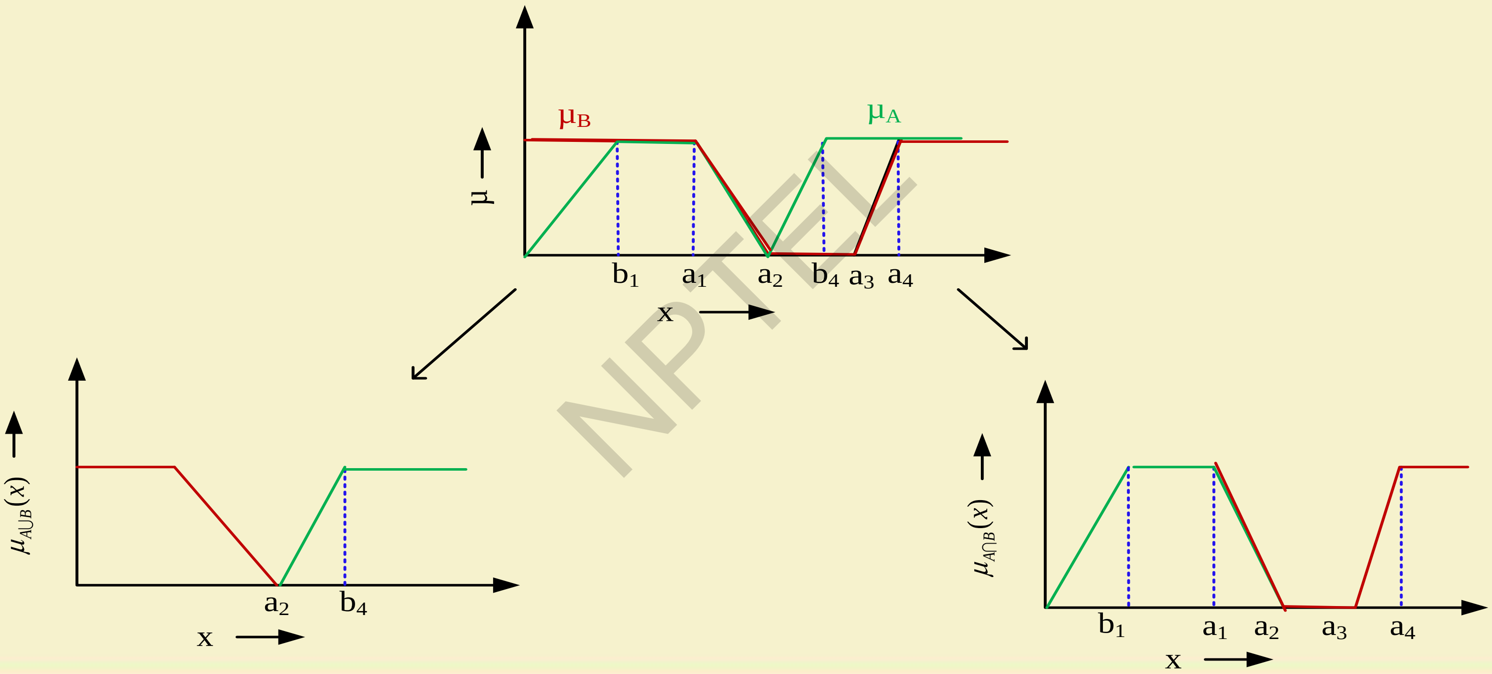
Example 1: Plotting two sets on the same graph

Let's plot the two membership functions on the same graph



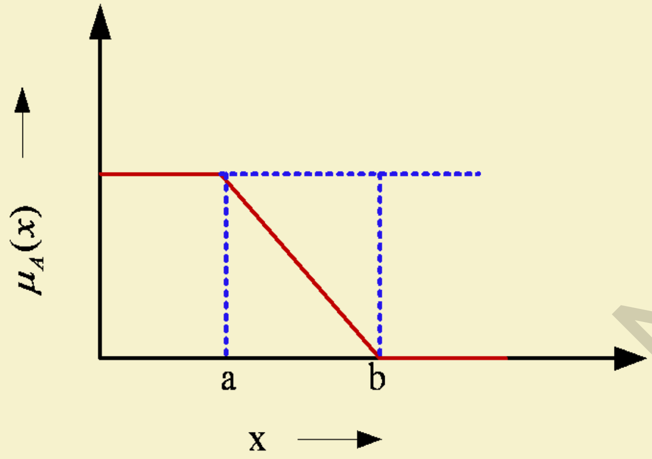
Example 1: Union and Intersection

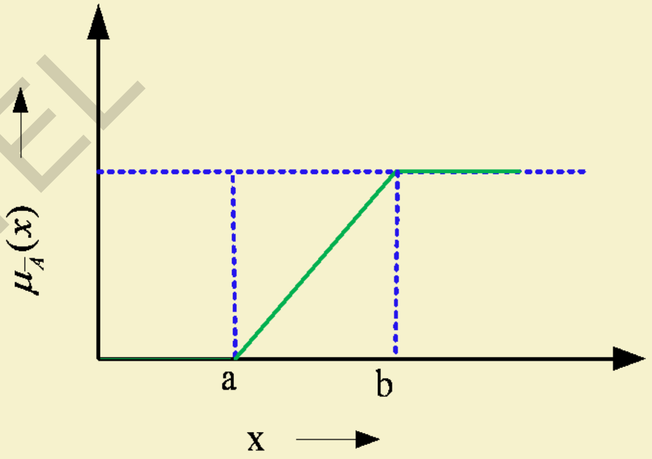
The plots of union $A \cup B$ and intersection $A \cap B$ are shown in the following.



Example 1: Complementation

The plots of union $\mu_{\bar{A}}(x)$ of the fuzzy set A is shown in the following.





Fuzzy set operations: Practice

Consider the following two fuzzy sets A and B defined over a universe of discourse [0,5] of real numbers with their membership functions

$$\mu_A(x) = \frac{x}{1+x} \text{ and } \mu_B(x) = 2^{-x}$$

Determine the membership functions of the following and draw them graphically.

I. $\bar{A}\,,\bar{B}$

II. $A \cup B$

III. $A \cap B$

IV. $(A \cup B)^c$

[Hint: Use De' Morgan law]

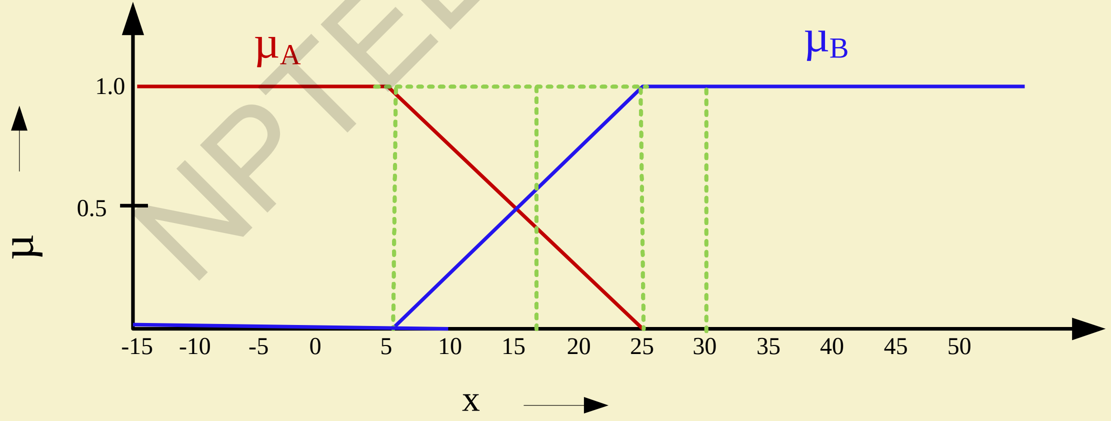
Example 2: A real-life example

Two fuzzy sets A and B with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively defined as below.

A = **Cold climate** with $\mu_A(x)$ as the MF.

B = **Hot climate** with $\mu_B(x)$ as the M.F.

Here, X being the universe of discourse representing entire range of temperatures.



Example 2: A real-life example

What are the fuzzy sets representing the following?

1. **Not cold climate**
2. **Not hot climate**
3. **Extreme climate**
4. **Pleasant climate**

Note: Note that ”Not cold climate” ≠ ”Hot climate” and vice-versa.

Example 2: A real-life example

Answer would be the following.

- ✓ **Not cold climate**

  $\bar{A}$ with $1 - \mu_A(x)$ as the MF.

- ✓ **Not hot climate**

  $\bar{B}$ with $1 - \mu_B(x)$ as the MF.

- ✓ **Extreme climate**

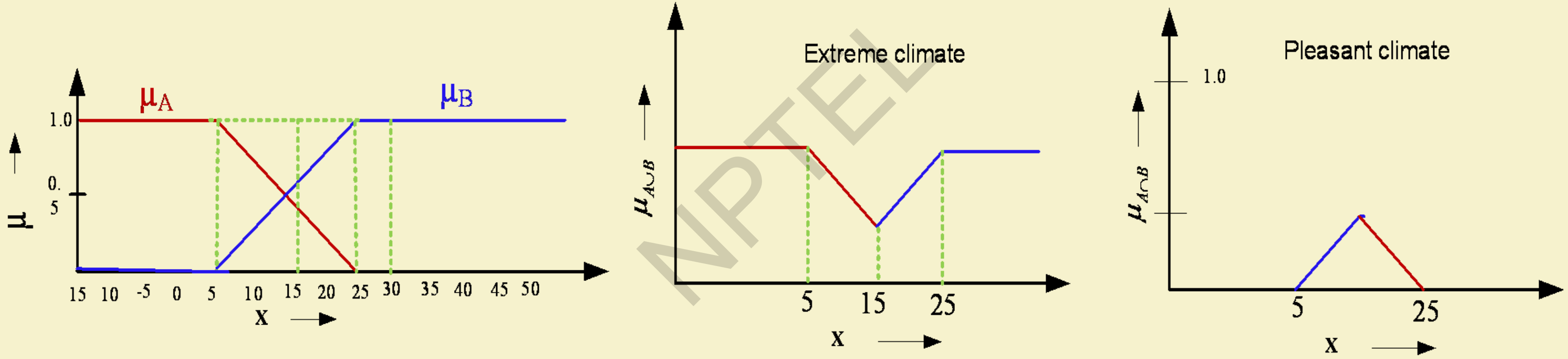
  $A \cup B$ with $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$ as the MF.

- ✓ **Pleasant climate**

  $A \cap B$ with $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$ as the MF.

Example 2: A real-life example

The plot of the MFs of $A \cup B$ and $A \cap B$ are shown in the following.



Thank You!!

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Soft Computing

Fuzzy Relations

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Fuzzy Relations

- Crisp relations
- Operations on crisp relations
- Examples on crisp relations
- Fuzzy relations
- Operations on fuzzy relations
- Examples on fuzzy relations

Crisp relations

- **Order pairs:**

  Suppose, A and B are two (crisp) sets. Then Cartesian product denoted as $A \times B$ is a collection of order pairs, such that

$$A \times B = \{(a, b) | a \in A \; and \; b \in B\}$$

Note :

(1) $A \times B \neq B \times A$ (2) $|A \times B| = |A| \times |B|$

(3) $A \times B$ provides a mapping from $a \in A$ to $b \in B$.

A particular mapping so mentioned is called a relation.

Crisp relations

Example:

Consider the two crisp sets A and B as given below.
$A = \{1, 2, 3, 4\}$ $B = \{3, 5, 7\}$.

Then, $A \times B = \{(1, 3), (1, 5), (1, 7), (2, 3), (2, 5), (2, 7), (3, 3), (3, 5), (3, 7), (4, 3), (4, 5), (4, 7)\}$

Let us define a relation as $R = \{(a, b) | b = a + 1, (a, b) \in A \times B\}$

Then, $R = \{(2, 3), (4, 5)\}$ in this case.

Crisp relations

We can represent the relation R in a matrix form as follows.

$$R = \begin{array}{c c} & \begin{array}{ccc} 3 & 5 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations defined over two crisp sets $x \in A$ and $y \in B$

- **Union:** $R(x, y) \cup S(x, y) = max(R(x, y), S(x, y));$
- **Intersection:** $R(x, y) \cap S(x, y) = min(R(x, y), S(x, y));$
- **Complement:** $\overline{R(x, y)} = 1 - R(x, y)$

Example: Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations defined over two crisp sets $x \in A$ and $y \in B$

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \quad S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find the following

- $R \cup S$
- $R \cap S$
- $\bar{R}$

Composition of two crisp relations

Given R is a relation on X, Y and S is another relation on Y, Z. Then, $R \circ S$ is called a composition of relation on X and Z which is defined as follows.

$$R \circ S = \{(x,z) | (x,y) \in R \; and \; (y,z) \in S \; and \; \forall y \in Y\}$$

Max-Min Composition

Given the two relation matrices R and S, the max-min composition is defined as $T = R \circ S$;

$$T(x,z) = max\{min\{R(x,y), S(y,z) \; and \; \forall y \in Y\}\}$$

Composition: Composition

Example : Given $X = \{1,3,5\}$; $Y = \{1,3,5\}$; $R = \{(x,y) | y = x + 2\}$;
$S = \{(x,y) | x < y\}$

Here, R and S is on $X \times Y$.

Thus, we have $R = \{(1,3),(3,5)\}$, $S = \{(1,3),(1,5),(3,5)\}$

$$R = \begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad and \quad S = \begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Using max-min composition $$R \circ S = \begin{array}{c c} & \begin{array}{ccc} 1 & 3 & 5 \end{array} \\ \begin{array}{c} 1 \\ 3 \\ 5 \end{array} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

Fuzzy relations

- Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp set $X_1, X_2, \ldots, X_n$
- Here, n-tuples $(x_1, x_2, \ldots, x_n)$ may have varying degree of memberships within the relationship.
- The membership values indicate the strength of the relation between the tuples.

Fuzzy relations

Example:

$$X = \{typhoid, viral, cold\}, Y = \{running\ nose, high\ temp, shivering\}$$

The fuzzy relation R is defined as

$$R = \begin{array}{c c} & \begin{array}{ccc} running\ nose & high\ temperature & shivering \end{array} \\ \begin{array}{c} typhoid \\ viral \\ cold \end{array} & \begin{bmatrix} 0.1 & 0.9 & 0.8 \\ 0.2 & 0.9 & 0.7 \\ 0.9 & 0.4 & 0.6 \end{bmatrix} \end{array}$$

Fuzzy Cartesian product

Suppose

- A is a fuzzy set on the universe of discourse X with $\mu_A(x) | x \in X$
- B is a fuzzy set on the universe of discourse Y with $\mu_B(y) | y \in Y$

Then $R = A \times B \subset X \times Y$; where R has its membership function given by $\mu_R(x, y) = \mu_{AxB}(x, y) = min\{\mu_A(x), \mu_B(y)\}$

Fuzzy Cartesian product

Example :

$A = \{(a1, 0.2), (a2, 0.7), (a3, 0.4)\}$ and $B = \{(b1, 0.5), (b2, 0.6)\}$

$$R = A \times B = \begin{array}{c c} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{bmatrix} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Operations on Fuzzy relations

Let R and S be two fuzzy relations on $A \times B$.

- **Union:** $\mu_{R \cup S}(a,b) = max\{\mu_R(a,b), \mu_S(a,b)\}$
- **Intersection:** $\mu_{R \cap S}(a,b) = min\{\mu_R(a,b), \mu_S(a,b)\}$
- **Complement:** $\mu_{\bar{R}}(a,b) = 1 - \mu_R(a,b)$
- **Composition:** $T = R \circ S$

$$\mu_{R \circ S} = max_{y \in Y}\{min\left(\mu_R(x,y), \mu_S(y,z)\right)\}$$

Operations on Fuzzy relations: Example

Example : $X = (x_1, x_2, x_3), \; Y = (y_1, y_2), \; Z = (z_1, z_2, z_3),$

$$R = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \end{array} \quad and \quad S = \begin{array}{c} \\ y_1 \\ y_2 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{bmatrix} \end{array}$$

$$R \circ S = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{bmatrix} \end{array}$$

$$\mu_{R \circ S}(x_1, y_1) = max\{min(\mu_R(x_1, y_1), \mu_S(y_1, z_1)), min(\mu_R(x_1, y_2), \mu_S(y_2, z_1))\}$$

$$= max\{min(0.5, 0.6), min(0.1, 0.5)\} = max\{0.5, 0.1\} = 0.5 \; and \; so \; on.$$

Fuzzy relation : An example

Consider the following two sets P and D, which represent a set of paddy plants and a set of plant diseases. More precisely

$P = \{P_1, P_2, P_3, P_4\}$ a set of four varieties of paddy plants

$D = \{D_1, D_2, D_3, D_4\}$ of the four various diseases affecting the plants.

In addition to these, also consider another set $S = \{S_1, S_2, S_3, S_4\}$ be the common symptoms of the diseases.

Let, R be a relation on $P \times D$, representing which plant is susceptible to which diseases, which is stated as

$$R = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \end{array} \\ \begin{bmatrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.4 & 0.2 \end{bmatrix} \end{array}$$

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Fuzzy relation : An example

Also, consider T be the another relation on $D \times S$, which is given by

$$S = \begin{array}{c} \\ D_1 \\ D_2 \\ D_3 \\ D_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.1 & 0.2 & 0.7 & 0.9 \\ 1.0 & 1.0 & 1.4 & 0.6 \\ 0.0 & 0.0 & 0.5 & 0.9 \\ 0.9 & 1.0 & 0.8 & 0.2 \end{bmatrix} \end{array}$$

Obtain the association of plants with the different symptoms of the disease using **max-min composition**.

Hint: Find $R \circ T$, and verify that

$$R \circ S = \begin{array}{c} \\ P_1 \\ P_2 \\ P_3 \\ P_4 \end{array} \begin{array}{c} \begin{array}{cccc} S_1 & S_2 & S_3 & S_4 \end{array} \\ \begin{bmatrix} 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.8 & 0.9 \\ 0.8 & 0.8 & 0.7 & 0.9 \end{bmatrix} \end{array}$$

Fuzzy relation : Another example

Let, $R = x$ is relevant to y
and $S = y$ is relevant to z
be two fuzzy relations defined on $X \times Y$ and $Y \times Z$, respectively, where $X = \{1,2,3\}$, $Y = \{\alpha, \beta, \gamma, \delta\}$ and $Z = \{a, b\}$. Assume that R and S can be expressed with the following relation matrices :

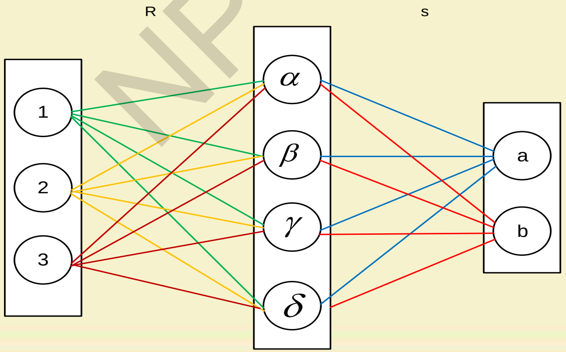
$$R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} \alpha & \beta & \gamma & \delta \end{array} \\ \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix} \end{array} \quad and \quad S = \begin{array}{c} \\ \alpha \\ \beta \\ \gamma \\ \delta \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix} \end{array}$$

Fuzzy relation : Another example

Now, we want to find $R \circ S$, which can be interpreted as a derived fuzzy relation x **is relevant to** z.

Suppose, we are only interested in the degree of relevance between $2 \in X$ and $a \in Z$. Then, using max-min composition,

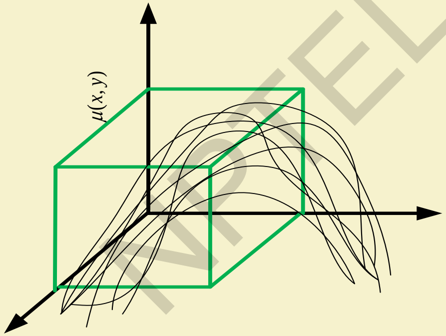
$$\mu_{R \circ S}(2, a) = \max\{(0.4 \wedge 0.9), (0.2 \wedge 0.2), (0.8 \wedge 0.5), (0.9 \wedge 0.7)\}$$
$$= \max\{0.4, 0.2, 0.5, 0.7\} = 0.7$$



2D Membership functions : Binary fuzzy relations

(Binary) fuzzy relations are fuzzy sets $A \times B$ which map each element in $A \times B$ to a membership grade between 0 and 1 (both inclusive).

Note that a membership function of a binary fuzzy relation can be depicted with a 3D plot.



Important: Binary fuzzy relations are fuzzy sets with two dimensional MFs and so on.

2D membership function : An example

Let, $X = R^{+} = y$ (the positive real line) and
$R = X \times Y =$ "y is much greater than x"

The membership function of $\mu_R(x,y)$ is defined as

$$\mu_R(x,y) = \begin{cases} \frac{(y-x)}{4} & if\ y > x \\ 0 & if\ y \le x \end{cases}$$

Suppose, $X = \{3,4,5\}$ and $Y = \{3,4,5,6,7\}$, then

$$R = \begin{array}{c} \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0.25 & 0.5 & 0.75 & 1.0 \\ 0 & 0 & 0.25 & 0.5 & 0.75 \\ 0 & 0 & 0 & 0.25 & 0.5 \end{bmatrix} \end{array}$$

Example:

How you can derive the following?

If x is A <u>or</u> y is B then z is C;

Given that

- R1: If x is A then z is C $\quad [R1 \in A \times C]$
- R2: If y is B then z is C $\quad [R2 \in B \times C]$

Hint:

✓ You have given two relations R1 and R2.

✓ Then, the required can be derived using the union operation of R1 and R2

Thank You!!