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Soft Computing

Fuzzy Propositions

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Fuzzy Propositions

- Two-valued logic vs. Multi-valued logic
- Examples of Fuzzy proposition
- Fuzzy proposition vs. Crisp proposition
- Canonical representation of Fuzzy proposition
- Graphical interpretation of Fuzzy proposition

Two-valued logic vs. Multi-valued logic

- The basic assumption upon which crisp logic is based - that every proposition is either TRUE or FALSE.

- The classical two-valued logic can be extended to multi-valued logic.

- As an example, three valued logic to denote true(1), false(0) and indeterminacy (1/2).

Two-valued logic vs. Multi-valued logic

Different operations with three-valued logic can be extended as shown in the truth table:

a	b					
0	0	0	0	1	1	1
0	½	0	½	1	1	½
0	1	0	1	1	1	0
½	0	0	½	½	½	½
½	½	½	½	½	½	1
½	1	½	1	½	1	½
1	0	0	1	0	0	0
1	½	½	1	0	½	½
1	1	1	1	0	1	1

Fuzzy connectives used in the above table are:

- AND ($\wedge$)
- OR ($\vee$)
- NOT ($\neg$)
- IMPLICATION ($\Longrightarrow$) and
- EQUAL ($=$)

Three-valued logic

Fuzzy connectives defined for such a three-valued logic better can be stated as follows:

Symbol	Connective	Usage	Definition
	NOT		
	OR		
	AND		
	IMPLICATION		
	EQUALITY		

Fuzzy proposition: Example 1

P: Ram is honest

T(P) = 0.0	: Absolutely false
T(P) = 0.2	: Partially false
T(P) = 0.4	: May be false or not false
T(P) = 0.6	: May be true or not true
T(P) = 0.8	: Partially true
T(P) = 1.0	: Absolutely true.

Fuzzy proposition: Example 2

P : Mary is efficient ; T(P) = 0.8
Q : Ram is efficient ; T(Q) = 0.6

- **Mary is not efficient.**

$$T(\neg P) = 1 - T(P) = 0.2$$

- **Mary is efficient and so is Ram.**

$$T(P \wedge Q) = min\{T(P), T(Q)\} = 0.6$$

Fuzzy proposition: Example 2

P : Mary is efficient ; T(P) = 0.8

Q : Ram is efficient ; T(Q) = 0.6

- **Either Mary or Ram is efficient**

$$T(P \vee Q) = max\{T(P), T(Q)\} = 0.8$$

- **If Mary is efficient then so is Ram**

$$T(P \Longrightarrow Q) = \max\{1 - T(P), T(Q)\} = 0.6$$

Fuzzy proposition vs. Crisp proposition

- The fundamental difference between crisp (classical) proposition and fuzzy propositions is in the range of their truth values.
- While each classical proposition is required to be either true or false, the truth or falsity of fuzzy proposition is a matter of degree.
- The degree of truth of each fuzzy proposition is expressed by a value in the interval [0,1] both inclusive.

Canonical representation of Fuzzy proposition

- Suppose, X is a universe of discourse of five persons. Intelligent of $x \in X$ is a fuzzy set as defined below.

  Intelligent: $\{(x_1, 0.3), (x_2, 0.4), (x_3, 0.1), (x_4, 0.6), (x_5, 0.9)\}$

- We define a fuzzy proposition as follows:
  P : x is Intelligent

- The canonical form of fuzzy proposition of this type, P is expressed by the sentence $P : v \; is \; F$.

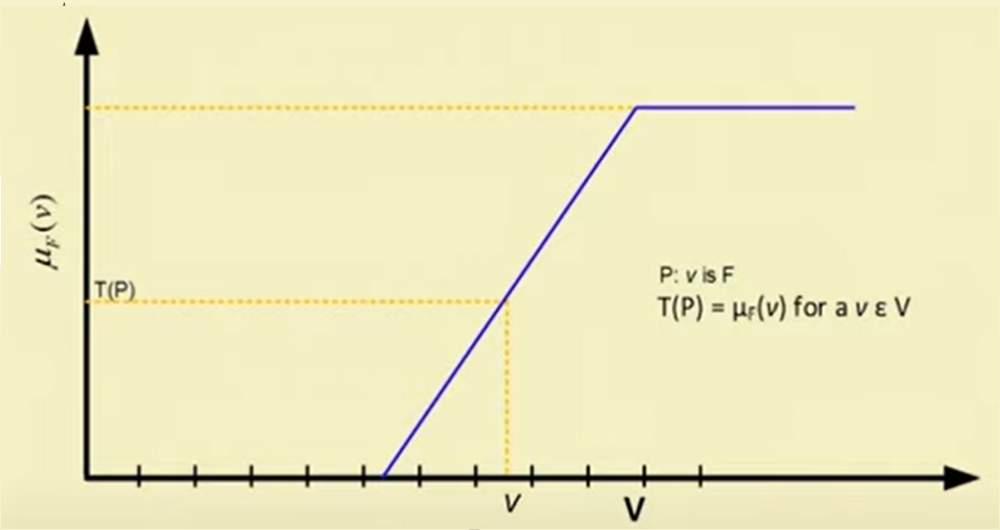
Canonical representation of Fuzzy proposition

- Predicate in terms of fuzzy set.

$P : v\ is\ F$; where v is an element that takes values v from some universal set V and F is a fuzzy set on V that represents a fuzzy predicate.

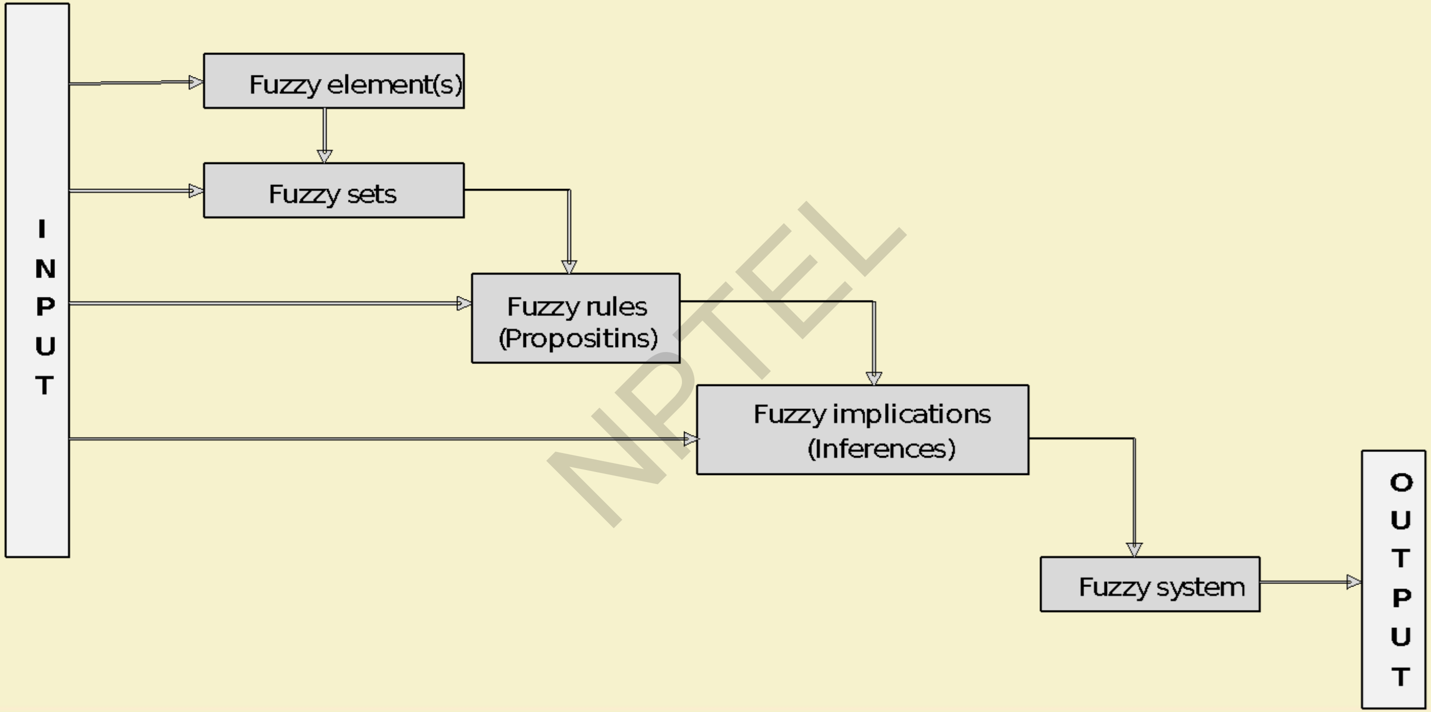
- In other words, given, a particular element v, this element belongs to F with membership grade $\mu_F(v)$.

Graphical interpretation of fuzzy proposition



- ✓ For a given value v of variable V in proposition P, $T(P)$ denotes the degree of truth of proposition P.

Fuzzy system



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Fuzzy Implication

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Fuzzy implications

- Fuzzy rule
- Examples of fuzzy implications
- Interpretation of fuzzy rules
- Product operators
- Zadeh’s Max-Min rule and some examples

Fuzzy rule

- A fuzzy implication (also known as fuzzy If-then rule, fuzzy rule, or fuzzy conditional statement) assumes the form :

  $$\textbf{If } x \textbf{ is } A \textbf{ then } y \textbf{ is } B$$

  where, A and B are two linguistic variables defined by fuzzy sets A and B on the universe of discourses X and Y, respectively.

- Often, x ***is*** A is called the **antecedent** or premise, while y ***is*** B is called the **consequence** or conclusion.

Fuzzy implication : Example 1

- If pressure is High then temperature is Low
- If mango is Yellow then mango is Sweet else mango is Sour
- If road is Good then driving is Smooth else traffic is High
- The fuzzy implication is denoted as $R : A \rightarrow B$
- In essence, it represents a binary fuzzy relation R on the (Cartesian) product of $A \times B$

Fuzzy implication : Example 2

- Suppose, P and T are two universes of discourses representing pressure and temperature, respectively as follows.

$$P = \{1,2,3,4\} \text{ and } T = \{10, 15, 20, 25, 30, 35, 40, 45, 50\}$$

- Let the linguistic variable High temperature and Low pressure are given as

$$T_{HIGH} = \{(20, 0.2), (25, 0.4), (30, 0.6), (35, 0.6), (40, 0.7), (45, 0.8), (50, 0.8)\}$$

$$P_{LOW} = \{(1, 0.8), (2, 0.8), (3, 0.6), (4, 0.4)\}$$

Fuzzy implication : Example 2

Then the fuzzy implication If temperature is High then pressure is Low can be defined as

$$R : T_{HIGH} \rightarrow P_{LOW}$$

where, $R = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 20 & 0.2 & 0.2 & 0.2 & 0.2 \\ 25 & 0.4 & 0.4 & 0.4 & 0.4 \\ 30 & 0.6 & 0.6 & 0.6 & 0.4 \\ 35 & 0.6 & 0.6 & 0.6 & 0.4 \\ 40 & 0.7 & 0.7 & 0.6 & 0.4 \\ 45 & 0.8 & 0.8 & 0.6 & 0.4 \\ 50 & 0.8 & 0.8 & 0.6 & 0.4 \end{array}$

Note : If temperature is 40 then what about low pressure?

Interpretation of fuzzy rules

In general, there are two ways to compute the fuzzy rule $A \rightarrow B$ as

- A coupled with B
- A entails B

Interpretation as A coupled with B

$R: A \rightarrow B = A \times B = \int_{X \times Y} \mu_A(x) * \mu_B(y) \,|(x, y)$; where $*$ is called a T-norm operator.

The most frequently used T-norm operators are:

- **Minimum** : $T_{min}(a, b) = min(a, b) = a \wedge b$
- **Algebraic product** : $T_{ap}(a, b) = ab$
- **Bounded product** : $T_{bp}(a, b) = 0 \vee (a + b - 1)$
- **Drastic product** : $T_{dp} = \begin{cases} a \; if \; b = 1 \\ b \; if \; a = 1 \\ 0 \; if \; a, b < 1 \end{cases}$

Here, $a = \mu_A(x)$ and $b = \mu_B(y)$. T is called the function of T-norm operator.

Interpretation as A coupled with B

Based on the T-norm operator as defined, we can automatically define the fuzzy rule $R : A \rightarrow B$ as a fuzzy set with two-dimensional MF:

$\mu_R(x, y) = f\big(\mu_A(x), \mu_B(y)\big) = f(a, b)$ with a $= \mu_A(x)$, $b = \mu_B(y)$ and f is the fuzzy implication function.

Interpretation as A coupled with B

In the following, few implications of $R : A \rightarrow B$

Min operator:

$$R_m = A \times B = \int_{X\times Y} \mu_A(x) \wedge \mu_B(y) \,|(x,y) \quad \text{or} \quad f_{min}(a,b) = a \wedge b$$

[Mamdani rule]

Algebraic product operator

$$R_{ap} = A \times B = \int_{X\times Y} \mu_A(x) . \mu_B(y) \,|(x,y) \quad \text{or} \quad f_{ap}(a,b) = ab$$

[Larsen rule]

Product Operators

Bounded product operator

$$R_{bp} = A \times B = \int_{X\times Y} \mu_A(x) \odot \mu_B(y) \,|(x,y)$$

$$= \int_{X\times Y} 0 \vee (\mu_A(x) + \mu_B(y) - 1) \,|(x,y) \text{ or } f_{bp}(a,b) = 0 \vee (a + b - 1)$$

Drastic product operator

$$R_{dp} = A \times B = \int_{X\times Y} \mu_A(x) \hat{\bullet} \mu_B(y) \,|(x,y) \text{ or } f_{dp}(a,b) = \begin{cases} a \; if \; b = 1 \\ b \; if \; a = 1 \\ 0 \; if \; otherwise \end{cases}$$

Interpretation of A entails B

There are three main ways to interpret such implication:

Material implication :

$$R : A \rightarrow B = \bar{A} \cup B$$

Propositional calculus :

$$R : A \rightarrow B = \bar{A} \cup (A \cup B)$$

Extended propositional calculus :

$$R : A \rightarrow B = (\bar{A} \cap \bar{B}) \cup B$$

Interpretation of A entails B

With the above mentioned implications, there are a number of fuzzy implication functions that are popularly followed in fuzzy rule-based system.

Zadeh's arithmetic rule :

$$R_{za} = \bar{A} \cup B = \int_{X\times Y} 1\wedge(1-\mu_A(x)+\mu_B(y))\,|(x,y) \quad \text{or}$$

$$f_{za}(a,b) = 1\wedge(1-a+b)$$

Zadeh's max-min rule :

$$R_{mm} = \bar{A} \cup (A \cap B) = \int_{X\times Y} (1-\mu_A(x)) \vee (\mu_A(x) \wedge \mu_B(y))\,|(x,y) \quad \text{or}$$

$$f_{mm}(a,b) = (1-a)\vee(a\wedge b)$$

Interpretation of A entails B

Boolean fuzzy rule:

$$R_{bf} = \bar{A} \cup B = \int_{X \times Y} (1 - \mu_A(x)) \vee \mu_B(x) \,|\, (x, y) \quad \text{or}$$

$$f_{bf}(a, b) = (1 - a) \vee b)$$

Goguen's fuzzy rule:

$$R_{gf} = \int_{X \times Y} \mu_A(x) * \mu_B(y) \,|\, (x, y) \quad \text{where} \quad a * b = \begin{cases} 1 \; if \; a \le b \\ \frac{b}{a} \; if \; a > b \end{cases}$$

Example 3: Zadeh's Max-Min rule

If x is A then y is B with the implication of Zadeh's max-min rule can be written equivalently as :

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y)$$

Here, Y is the universe of discourse with membership values for all $y \in Y$ is 1,

that is , $\mu_Y(y) = 1 \forall y \in Y$:

Suppose $X = \{a, b, c, d\}$ and $Y = \{1,2,3,4\}$ and

$A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$,

$B = \{(1, 0.2), (2, 1.0), (3, 0.8), (4, 0.0)\}$ are two fuzzy sets.

We are to determine $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$

Example 3: Zadeh's Max-Min rule

The computation of $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$ is as follows

$$A \times B = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline a & 0 & 0 & 0 & 0 \\ b & 0.2 & 0.8 & 0.8 & 0 \\ c & 0.2 & 0.6 & 0.6 & 0 \\ d & 0.2 & 1.0 & 0.8 & 0 \end{array}$$

and

$$\bar{A} \times Y = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline a & 1 & 1 & 1 & 1 \\ b & 0.2 & 0.2 & 0.2 & 0.2 \\ c & 0.4 & 0.4 & 0.4 & 0.4 \\ d & 0 & 0 & 0 & 0 \end{array}$$

Example 3: Zadeh's Max-Min rule

Therefore, $R_{mm} = (A \times B) \cup (\bar{A} \times Y)$

$$R_{mm} = (A \times B) \cup (\bar{A} \times Y) = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.4 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Example 4:

IF x is A THEN y is B ELSE y is C. The relation R is equivalent to

$$R = (A \times B) \cup (\bar{A} \times C)$$

The membership function of R is given by

$$\mu_R(x, y) = \max[\min\{\mu_A(x), \mu_B(y)\}, \min\{\mu_{\bar{A}}(x), \mu_C(y)\}]$$

Example 4:

$$X = \{a, b, c, d\}$$

$$Y = \{1, 2, 3, 4\}$$

$$A = \{(a, 0.0), (b, 0.8), (c, 0.6), (d, 1.0)\}$$

$$B = \{(1,0.2), (2,1.0), (3,0.8), (4,0.0)\}$$

$$C = \{(1,0.0), (2,0.4), (3,1.0), (4,0.8)\}$$

Determine the implication relation :

If x **is A then** y **is B else** y **is C**

Example 4:

Here, $A \times B =$

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.8 & 0.8 & 0 \\ 0.2 & 0.6 & 0.6 & 0 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

and $\bar{A} \times C =$

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{bmatrix} 0 & 0.4 & 1.0 & 0.8 \\ 0 & 0.2 & 0.2 & 0.2 \\ 0 & 0.4 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Example 4:

Here, $A \times B =$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline a & 0 & 0 & 0 & 0 \\ b & 0.2 & 0.8 & 0.8 & 0 \\ c & 0.2 & 0.6 & 0.6 & 0 \\ d & 0.2 & 1.0 & 0.8 & 0 \end{array}$$

and $\bar{A} \times C =$

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline a & 0 & 0.4 & 1.0 & 0.8 \\ b & 0 & 0.2 & 0.2 & 0.2 \\ c & 0 & 0.4 & 0.4 & 0.4 \\ d & 0 & 0 & 0 & 0 \end{array}$$

Example 4:

$$R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0.4 & 1.0 & 0.8 \\ 0.2 & 0.8 & 0.8 & 0.2 \\ 0.2 & 0.6 & 0.6 & 0.4 \\ 0.2 & 1.0 & 0.8 & 0 \end{bmatrix} \end{array}$$

Thank You!

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Fuzzy Inferences

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Fuzzy inferences

Let's start with propositional logic. We know the following in propositional logic.

1. **Modus Ponens** . $P, P \implies Q, \quad \Leftrightarrow Q$
2. **Modus Tollens** . $P \implies Q, \neg Q, \quad \Leftrightarrow, \neg Q$
3. **Chain rule** . $P \implies Q, Q \implies R, \quad \Leftrightarrow, P \implies R$

An example from propositional logic

Given.

1) $C \vee D$
2) $\sim H \implies (A \wedge \sim B)$
3) $C \vee D \implies \sim H$
4) $(A \wedge \sim B) \implies (R \vee S)$

From the above can we infer $R \vee S$?

Similar concept is also followed in fuzzy logic to infer a fuzzy rule from a set of given fuzzy rules (also called fuzzy rule base).

Inferring procedures in Fuzzy logic

Two important inferring procedures are used in fuzzy systems.

- **Generalized Modus Ponens (GMP)**

  If x is A Then y is B

  x is A'

  ____

  y is B'

- **Generalized Modus Tollens (GMT)**

  If x is A Then y is B

  y is B'

  ____

  x is A'

Fuzzy inferring procedures

- Here, A, B, A' and B' are fuzzy sets.
- To compute the membership function A' and B' the max-min composition of fuzzy sets B' and A', respectively with $R(x, y)$ (which is the known implication relation) is used.
- Thus,

$$B' = A' \circ R(x, y) \qquad \mu_{B`}(y) = \max[\min(\mu_{A'}(x), \mu_R(x, y))]$$

$$A' = B' \circ R(x, y) \qquad \mu_{A`}(x) = \max[\min(\mu_{B'}(y), \mu_R(x, y))]$$

Generalized Modus Ponens : Example

$$P: \textbf{\textit{If}}\ x\ \textbf{is}\ A\ \textbf{then}\ y\ \textbf{is}\ B$$

Let us consider two sets of variables x and y be

$X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$

Also, let us consider the following.

$$A = \{(x_1, 0.5), (x_2, 1), (x_3, 0.6)\}$$
$$B = \{(y_1, 1), (y_2, 0.4)\}$$

Example: Generalized Modus Ponens

Generalized Modus Ponens (GMP)

If x is A Then y is B

x is A'

____

y is B'

Generalized Modus Ponens

$$P: \textbf{\textit{If}}\ x\ \textbf{is}\ A\ \textbf{\textit{then}}\ y\ \textbf{is}\ B$$

Suppose, given a fact expressed by the proposition x is A',

where $A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$

We are to derive a conclusion in the form y **is** B'

Here, we should use generalized modus ponens (GMP).

Example. Generalized Modus Ponens

If x is A Then y is B
x is A'

____

y is B'

We are to find $B' = A' \circ R(x, y)$, where $R(x, y) = \max\{\mathrm{A} \times B, \bar{A} \times Y\}$

$$\mathrm{A} \times B = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.4 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array} \quad \text{and} \quad \bar{A} \times Y = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \\ 0.4 & 0.4 \end{bmatrix} \end{array}$$

Note. For $\mathrm{A} \times B$, $\mu_{A \times B}(x, y) = \min(\mu_A(x), \mu_B(y))$

Example. Generalized Modus Ponens

$$R(x,y) = (A \times B) \cup (\bar{A} \times Y) = \begin{array}{c c} & \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

Now $A' = \{(x_1, 0.6), (x_2, 0.9), (x_3, 0.7)\}$

Therefore $B' = A' \circ R(x,y) = [0.6 \quad 0.9 \quad 0.7] \circ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} = [0.9 \quad 0.5]$

Thus we derive that y is B' where $B' = \{(y_1, 0.9), (y_2, 0.5)\}$

Example. Generalized Modus Tollens

Generalized Modus Tollens (GMT)

If x is A Then y is B

y is B'

____

x is A'

Example. Generalized Modus Tollens

- Let the universe of discourses be $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2\}$, respectively.

- Assume that a proposition ***If*** x ***is*** A ***Then*** y ***is*** B given where
  $A = \{(x_1, 0.5), (x_2, 1), (x_3, 0.6)\}$ And $B = \{(y_1, 1), (y_2, 0.4)\}$

- Assume now that a fact expressed by a proposition $\boldsymbol{y}$ **is** $\boldsymbol{B'}$ is given
  where $B' = \{(y_1, 0.9), (y_2, 0.7)\}$

- From the above, we are to conclude that x ***is*** A'. That is, we are to determine A'

Example. Generalized Modus Tollens

1. We first calculate $R(x,y) = (A \times B) \cup (\bar{A} \times Y)$

$$R(x,y) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{cc} y_1 & y_2 \\ \left[\begin{array}{cc} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{array}\right] \end{array}$$

2. Next, we calculate $A' = B' \circ R(x,y)$

$$A' = [0.9 \quad 0.7] \circ \begin{bmatrix} 0.5 & 0.5 \\ 1 & 0.4 \\ 0.6 & 0.4 \end{bmatrix} = [0.5 \quad 0.9 \quad 0.6]$$

3. Hence, we calculate that x is A' where

$$A' = [(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)]$$

Practical example

Apply the fuzzy GMP rule to deduce **Rotation is quite slow**
Given that .

1. If temperature is High then rotation is Slow.
2. temperature is Very High

Let,

$X = \{30, 40, 50, 60, 70, 80, 90, 100\}$ be the set of temperatures.
$Y = \{10, 20, 30, 40, 50, 60\}$ be the set of rotations per minute.

Practice

The fuzzy set High(H), Very High (VH), Slow(S) and Quite Slow (QS) are given below.

$H \ = \{(70, 1), (80, 1), (90, 0.3)\}$

$VH \ = \{(80,0.6), (90, 0.9), (100, 1)\}$

$S \ = \{(30, 0.8), (40, 1.0), (50, 0.6)\}$

$QS \ = \{(10, 1), (20, 0.8), (30, 0.5)\}$

1. If temperature is High then rotation is Slow.

$$R \ = \ (H \times \ S) \cup (\bar{H} \times \ Y)$$

2. temperature is Very High

   Thus, to deduce "rotation is Quite Slow", we make use the composition rule

$$QS \ = \ VH \ \circ \ R(x, y)$$

Thank You!

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Soft Computing

Defuzzyfication Techniques-I

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What is defuzzification?

- Defuzzification means the fuzzy to crisp conversion.

Example 1.

Suppose, T_{HIGH} denotes a fuzzy set representing **temperature is High.**

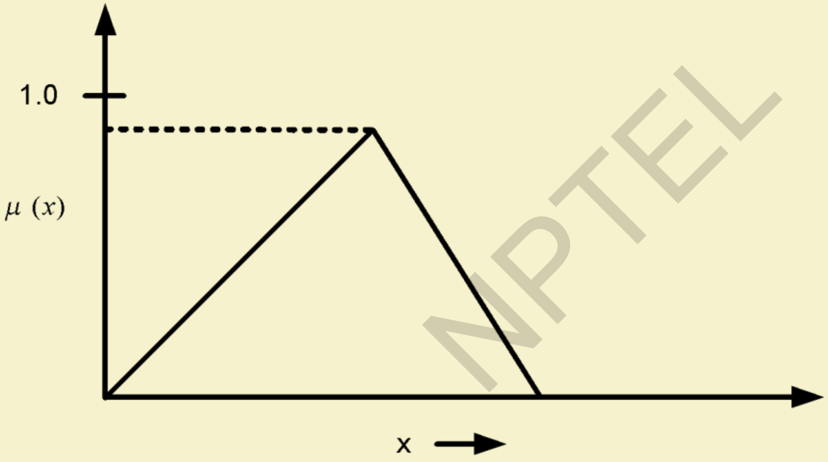
T_{HIGH} is given as follows.

$$T_{HIGH} = \{(15,0.1), (20, 0.4), (25,0.45), (30,0.55), (35,0.65), (40,0.7), (45,0.85), (50,0.9)\}$$

- What is the crisp value that implies the high temperature?

Example 2. Fuzzy to crisp

As an another example, let us consider a fuzzy set whose membership function is shown in the following figure.



What is the crisp value of the fuzzy set in this case?

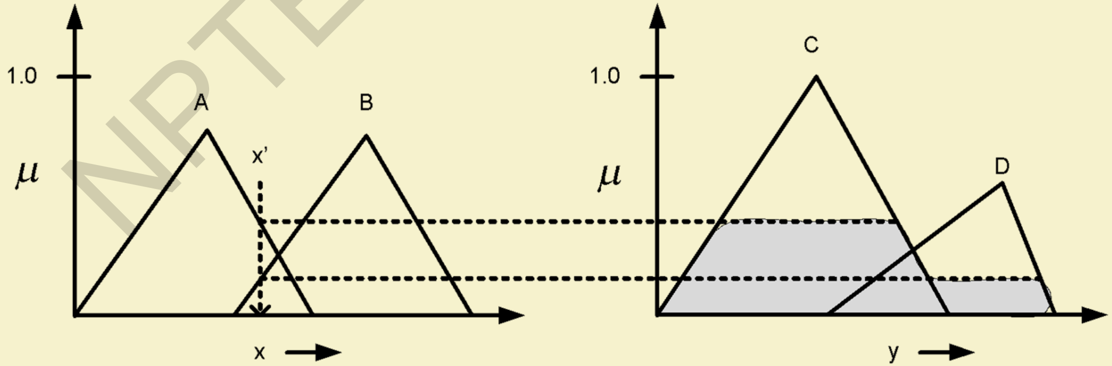
Example 3. Fuzzy to crisp

Now, consider the following two rules in the fuzzy rule base.

$$R1. If\ x\ is\ A\ then\ y\ is\ C$$
$$R2. If\ x\ is\ B\ then\ y\ is\ D$$

A pictorial representation of the above rule base is shown in the following figures.

What is the crisp value that can be inferred from the above rules given an input say x'?



Why defuzzification?

The fuzzy results generated can not be used in an application, where decision has to be taken only on crisp values.

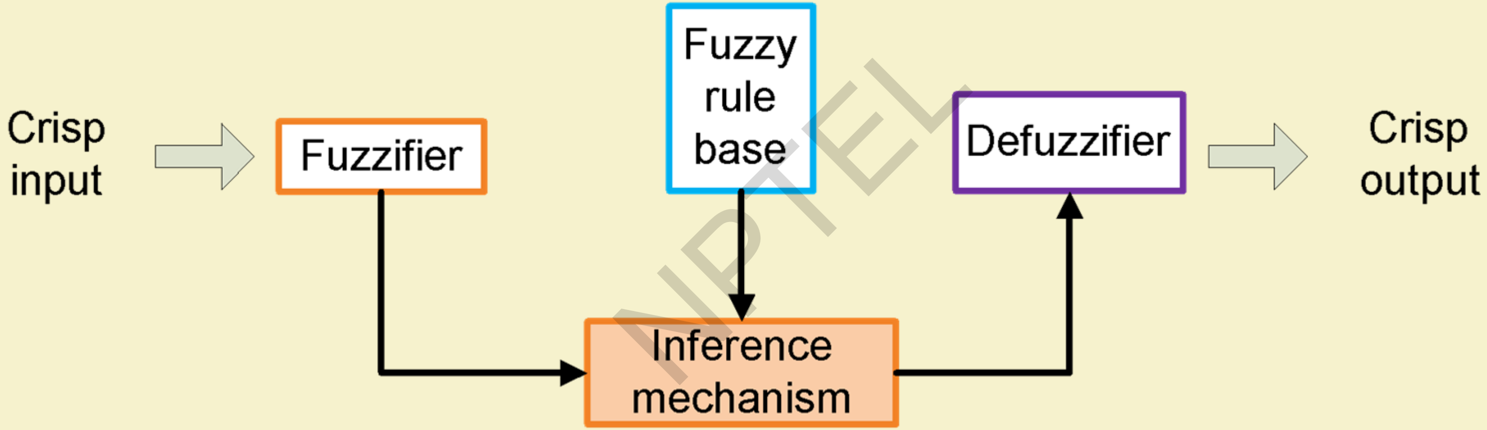
Example.

$$If\ temperature \text{ is } T_{HIGH} \text{ Then } rotation \text{ is } R_{FAST}.$$

Here, may be input T_{HIGH} is fuzzy, but action rotation should be based on the crisp value of R_{FAST} .

Generic structure of a Fuzzy system

Following figure shows a general framework of a fuzzy system.



Defuzzification Techniques

Defuzzification methods

A number of defuzzification methods are known. Such as

1) **Lambda-cut method**
2) **Weighted average method**
3) **Maxima methods**
4) **Centroid methods**

Lambda-cut method

Lambda-cut method

Lambda-cut method is applicable to derive crisp value of a fuzzy set or relation.

- Lambda-cut method for fuzzy relation

The same has been applied to Fuzzy set

- Lambda-cut method for fuzzy set

In many literature, Lambda-cut method is also alternatively termed as Alpha-cut method.

Lamda-cut method for fuzzy set

1) In this method a fuzzy set A is transformed into a crisp set A_λ for a given value of $\lambda (0 \leq \lambda \leq 1)$

2) In other-words, $A_\lambda = \{x | \mu_A(x) \geq \lambda\}$

3) That is, the value of Lambda-cut set A_λ is x, when the membership value corresponding to x is greater than or equal to the specified λ.

4) This Lambda-cut set A_λ is also called alpha-cut set.

Lambda-cut for a fuzzy set : Example

$A_1 = \{(x_1, 0.9), (x_2, 0.5), (x_3, 0.2), (x_4, 0.3)\}$

$\lambda = 0.6$

$A_{0.6} = \{(x_1, 1), (x_2, 0), (x_3, 0), (x_4, 0)\} = \{x_1\}$

$A_2 = \{(x_1, 0.1), (x_2, 0.5), (x_3, 0.8), (x_4, 0.7)\}$

$\lambda = 0.2$

$A_{0.2} = \{(x_1, 0), (x_2, 1), (x_3, 1), (x_4, 1)\} = \{x_2, x_3, x_4\}$

Lambda-cut sets : Example

Two fuzzy sets P and Q are defined on *x* as follows.

P	0.1	0.2	0.7	0.5	0.4
Q	0.9	0.6	0.3	0.2	0.8

Find the following :

a) $P_{0.2}, Q_{0.3}$

b) $(P \cup Q)_{0.6}$

c) $(P \cup \bar{P})_{0.8}$

d) $(P \cap Q)_{0.4}$

Lambda-cut for a fuzzy relation

The Lambda-cut method for a fuzzy set can also be extended to fuzzy relation also.

Example: For a fuzzy relation R

$$R = \begin{bmatrix} 1 & 0.2 & 0.3 \\ 0.5 & 0.9 & 0.6 \\ 0.4 & 0.8 & 0.7 \end{bmatrix}$$

We are to find λ-cut relations for the following values of

$$\lambda = 0, 0.2, 0.9, 0.5$$

$R_0 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $R_{0.2} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and

$R_{0.9} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $R_{0.5} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Some properties of λ -cut sets

If A and B are two fuzzy sets, defined with the same universe of discourse, then

1) $(A \cup B)_\lambda = A_\lambda \cup B_\lambda$

2) $(A \cap B)_\lambda = A_\lambda \cap B_\lambda$

3) $\overline{(A)}_\lambda \neq \overline{A_\lambda}$ except for value of $\lambda = 0.5$

4) For any $\lambda \leq \alpha$, where α varies between 0 and 1, it is true that $A_\alpha \subseteq A_\lambda$, where the value of A_0 is the universe of discourse.

Some properties of λ-cut relations

If R and S are two fuzzy relations, defined with the same fuzzy sets over the same universe of discourses, then

1) $(R \cup S)_\lambda = R_\lambda \cup S_\lambda$

2) $(R \cap S)_\lambda = R_\lambda \cap S_\lambda$

3) $\overline{(R)}_\lambda \neq \overline{R_\lambda}$

4) For $\lambda \leq \alpha$, where α between 0 and 1, then $R_\alpha \subseteq R_\lambda$

Summary: Lambda-cut methods

Lambda-cut method converts a fuzzy set (or a fuzzy relation) into a crisp set (or relation).

Output of a Fuzzy System

Output of a fuzzy System

The output of a fuzzy system can be a single fuzzy set or union of two or more fuzzy sets.

To understand the second concept, let us consider a fuzzy system with n-rules.

$$R_1 : \; If \;\; x \; is \; A_1 \; then \; y \; is \; B_1$$
$$R_2 : \; If \;\; x \; is \; A_2 \; then \; y \; is \; B_2$$

....

....

$$R_n : \; If \;\; x \; is \; A_n \; then \; y \; is \; B_n$$

In this case, the output y for a given input $x = x_1$ is possibly

$$B = B_1 \cup B_2 \cup \cdots \ldots \ldots \ldots B_n$$

Output fuzzy set : Illustration

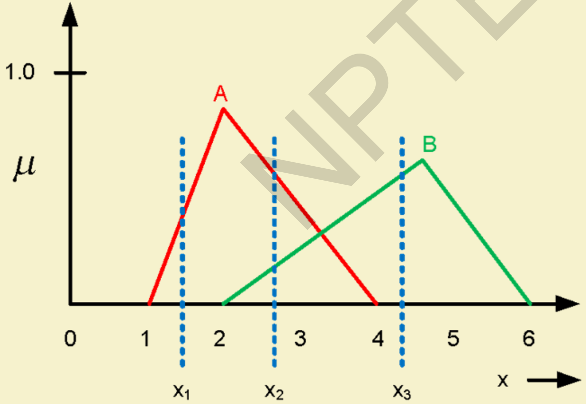
Suppose, two rules R_1 and R_2 are given as follows:

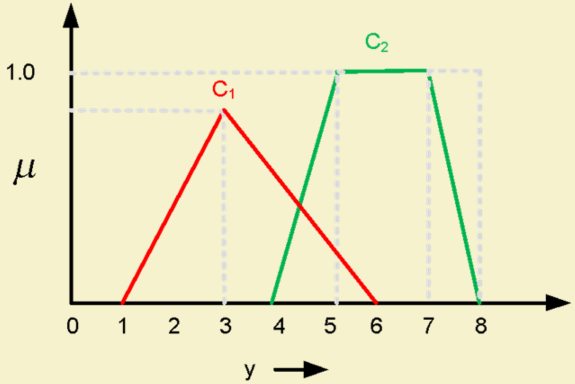
$R_1: if\ \ x\ is\ A_1 then\ y\ is\ C_1$

$R_2: if\ \ x\ is\ A_2 then\ y\ is\ C_2$

Here, the output fuzzy set $C = C_1 \cup C_2$

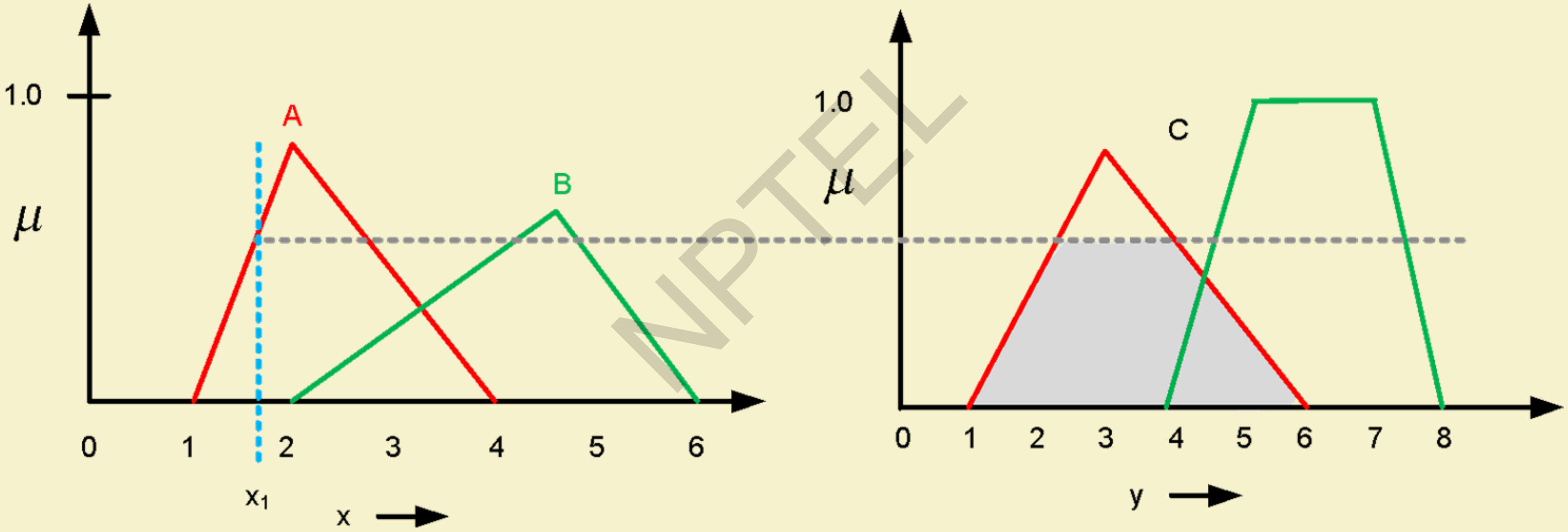
For instance, let us consider the following:





Output fuzzy set : Illustration

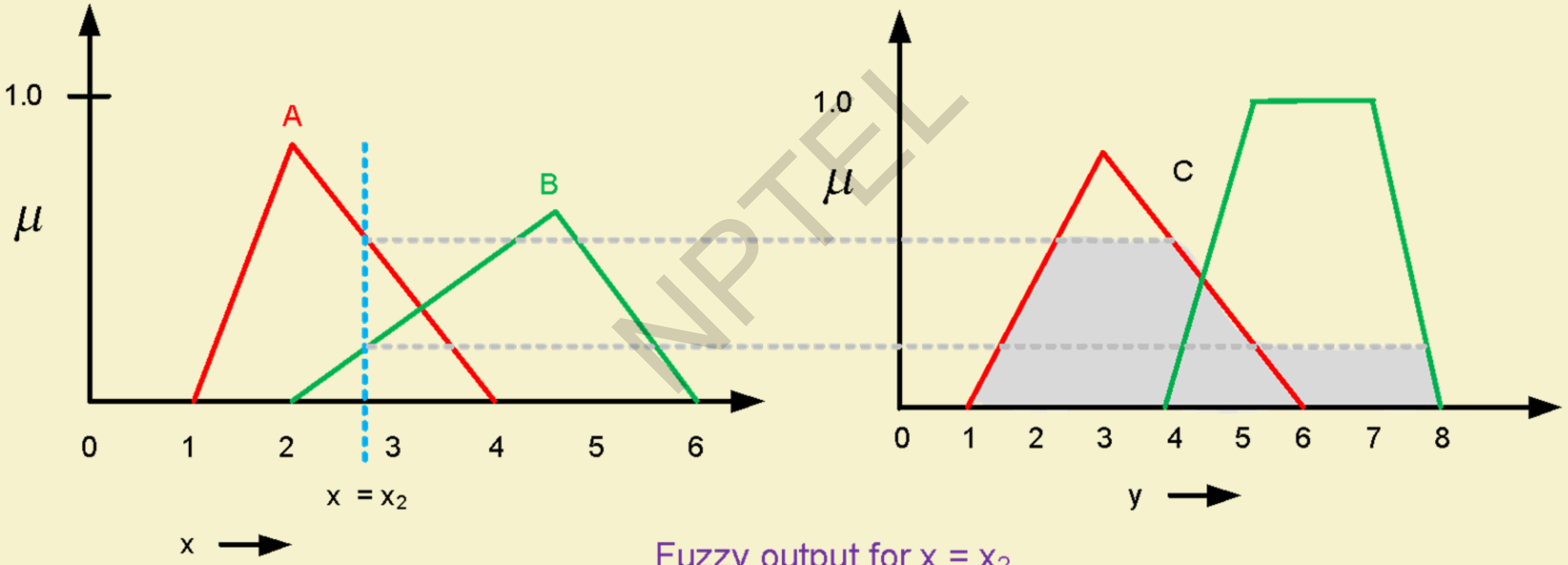
The fuzzy output for $x = x_1$ is shown below.



Fuzzy output for $x = x_1$

Output fuzzy set : Illustration

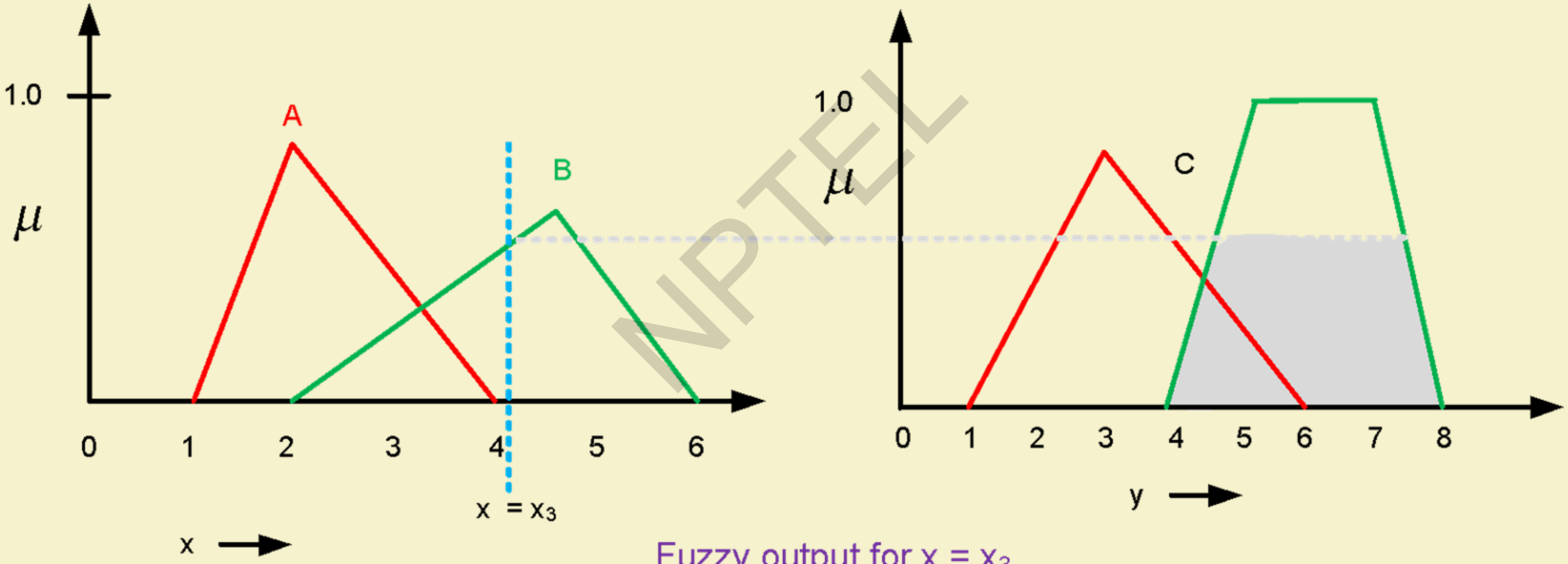
The fuzzy output for $x = x_2$ is shown below.



Fuzzy output for $x = x_2$

Output fuzzy set : Illustration

The fuzzy output for $x = x_3$ is shown below.



Fuzzy output for x = x_3

Thank You!!

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Soft Computing

Defuzzyfication Techniques-II

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Defuzzification Methods

Following defuzzification methods are known to calculate crisp output in the situations as discussed in the last lecture.

1. **Maxima Methods**
    a) Height method
    b) First of maxima (FoM)
    c) Last of maxima (LoM)
    d) Mean of maxima(MoM)
2. **Centroid methods**
    a) Centre of gravity method (CoG)
    b) Centre of sum method (CoS)
    c) Centre of area method (CoA)
3. **Weighted average method**

Maxima methods

1. **Maxima Methods**

    a) Height method

    a) First of maxima (FoM)

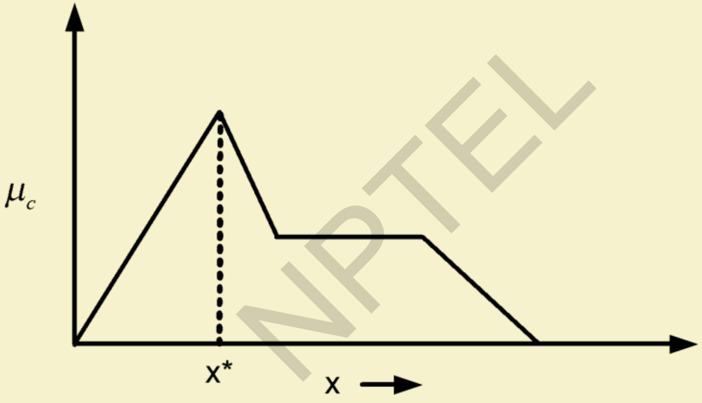
    a) Last of maxima (LoM)

    a) Mean of maxima(MoM)

Maxima method : Height method

This method is based on Max-membership principle, and defined as follows.

$$\mu_c(x^*) \geq \mu_c(x) \text{for all } x \in X$$

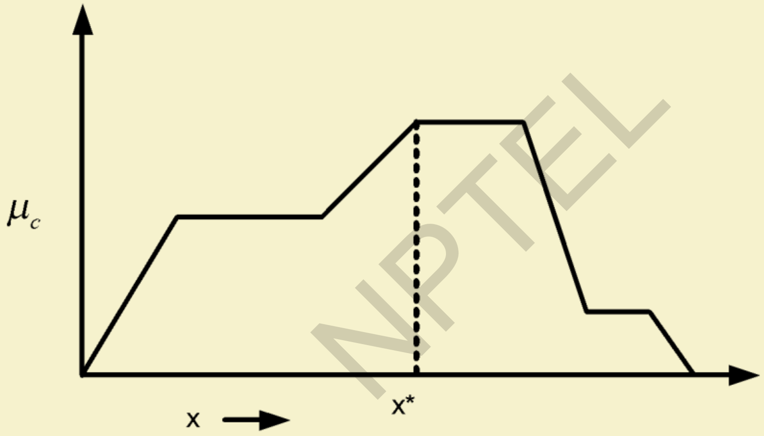


Note:

1. Here, x^* is the height of the output fuzzy set C.
2. This method is applicable when height is unique.

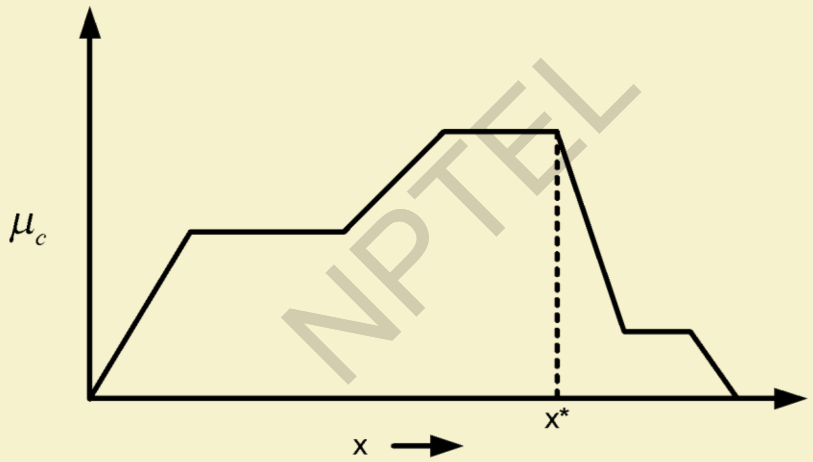
Maxima method : FoM

FoM: First of Maxima : $x^* = \min(x|C(x) = max_w C\{w\})$



Maxima method : LoM

LoM: Last of Maxima : $x^* = max(x|C(x) = max_w C\{w\})$



Maxima method : MoM

$$x^* = \frac{\sum_{x_i \in M}(x_i)}{|M|}$$

where, $M = \{x_i | \mu(xi) = h(C)\}$ where $h(C)$ is the height of the fuzzy set C

MoM : Example 1

Suppose, a fuzzy set **Young** is defined as follows:

$$Young = \{(15,0.5),(20,0.8),(25,0.8),(30,0.5),(35,0.3)\}$$

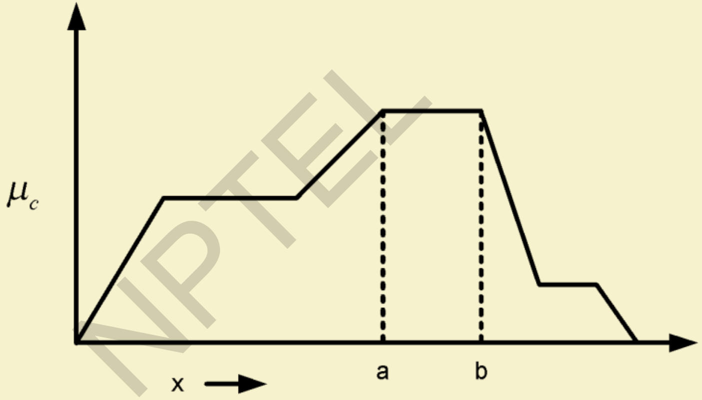
Then the crisp value of **Young** using MoM method is

$$x^* = \frac{20+25}{2} = 22.5$$

Thus, a person of 22.5 years old is treated as young!

MoM : Example 2

What is the crisp value of the fuzzy set using MoM in the following case?



$$x^* = \frac{a+b}{2}$$

Note:

- Thus, MoM is also synonymous to middle of maxima.
- MoM is also a general method of Height.

Centroid methods

2. **Centroid methods**

   a) Centre of gravity method (CoG)

   a) Centre of sum method (CoS)

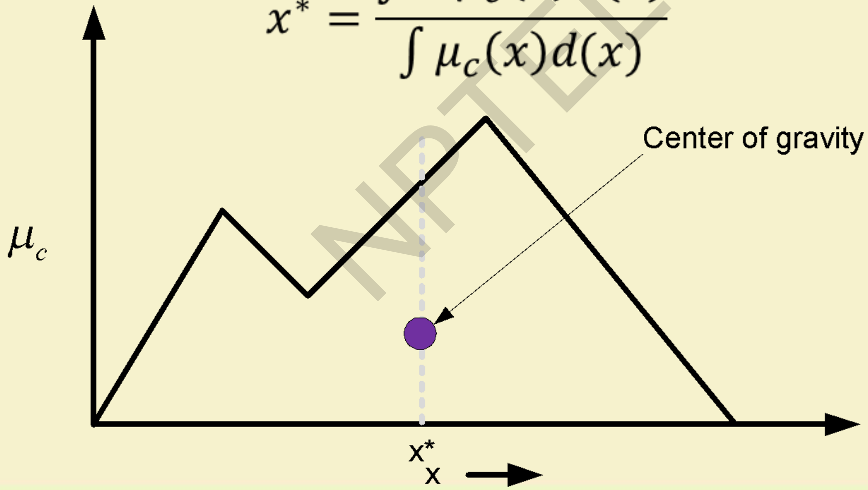
   a) Centre of area method (CoA)

Centroid method : CoG

1) The basic principle in CoG method is to find the point x where a vertical line would slice the aggregate into two equal masses.
2) Mathematically, the CoG can be expressed as follows :

$$x^* = \frac{\int x.\mu_c(x)d(x)}{\int \mu_c(x)d(x)}$$

3) Graphically,



Centroid method : CoG

Note:

1) x^* is the x-coordinate of centre of gravity.

2) $\int \mu_c(x)d(x)$ denotes the area of the region bounded by the curve μ_c

3) If μ_c is defined with a discrete membership function, then CoG can be stated as :

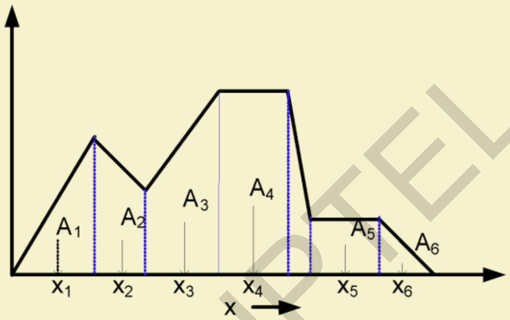
$$x^* = \frac{\sum x_i.\mu_c(x_i)}{\sum \mu_c(x_i)} \quad \text{for } i = 1 \text{ to } n$$

4. Here, x_i is a sample element and n represents the number of samples in fuzzy set C.

CoG : A geometrical method of calculation

Steps:

1) Divide the entire region into a number of small regular regions (e.g. triangles, trapezoid, etc.)

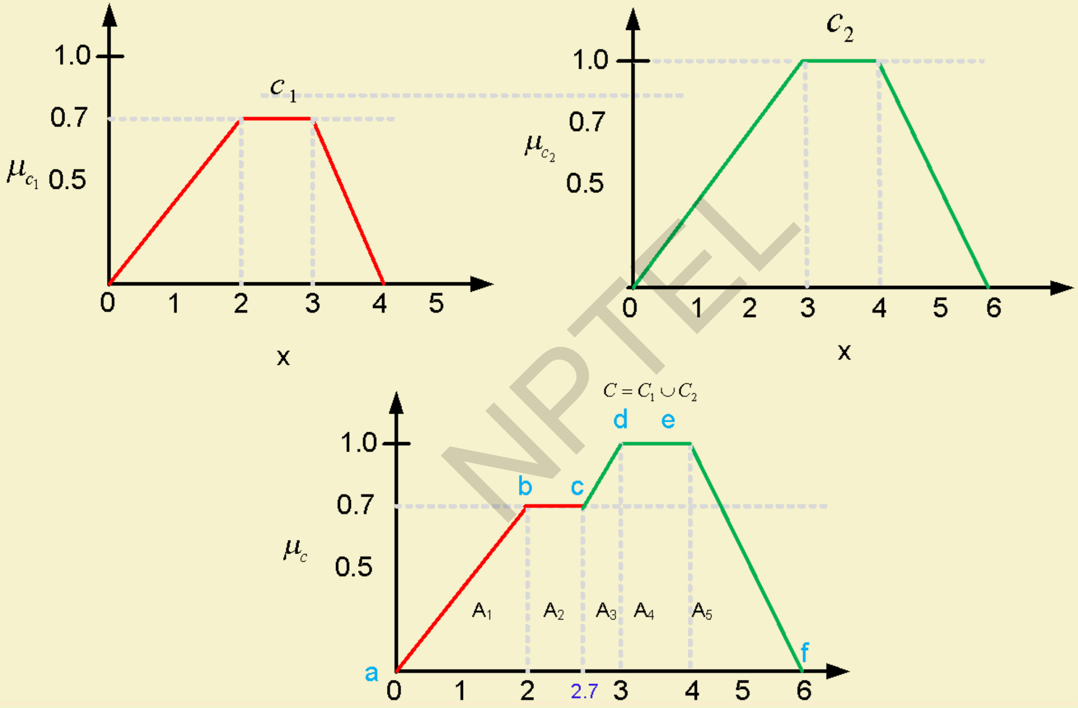


2) Let A_i and x_i denotes the area and $c.g.$ of the i^{th} portion.
3) Then x^* according to CoG is

$$x^* = \frac{\sum_{i=1}^{n} x_i.(A_i)}{\sum_{i=1}^{n} A_i}$$

where n is the number of smaller geometrical components.

CoG: An example of integral method of calculation



CoG: An example of integral method of calculation

$$\mu_c(x) = \begin{cases} 0.35x & 0 \le x < 2 \\ 0.7 & 2 \le x < 2.7 \\ x - 2 & 2.7 \le x < 3 \\ 1 & 3 \le x < 4 \\ (-.5x + 3) & 4 \le x \le 6 \end{cases}$$

For A_1: $y - 0 = \frac{0.7}{2}(x - 0)$, or $y = 0.35x$

For A_2: $y = 0.7$

For A_3: $y - 0 = \frac{1-0}{3-2}(x - 2), or\ y = x - 2$

For A_4: $y = 1$

For A_5: $y - 1 = \frac{0-1}{6-4}(x - 4), or\ y = -0.5x + 3$

CoG: An example of integral method of calculation

Thus, $x^* = \frac{\int x.\mu_c(x)d(x)}{\int \mu_c(x)d(x)} = \frac{N}{D}$

$$N = \int_0^2 0.35x^2dx + \int_2^{2.7} 0.7x^2dx + \int_{2.7}^3 (x^2-2x)dx + \int_3^4 xdx + \int_4^6 (-0.5x^2 + 3x)dx$$
$$= 10.98$$

$$D = \int_0^2 0.35xdx + \int_2^{2.7} 0.7xdx + \int_{2.7}^3 (x-2)dx + \int_3^4 dx + \int_4^6 (-0.5x + 3)dx$$
$$= 3.445$$

Thus, $x^* = \frac{10.98}{3.445} = 3.187$

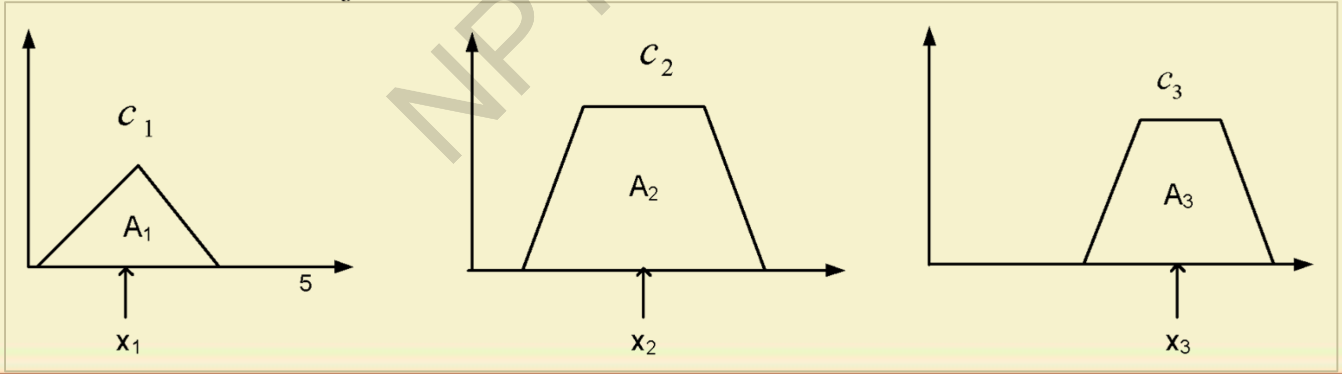
Centroid method : CoS

If the output fuzzy set $C = C_1 \cup C_2 \cup \cdots \ldots \ldots C_n$, then the crisp value according to CoS is defined as

$$x^* = \frac{\sum_{i=1}^{n} x_i A_{c_i}}{\sum_{i=1}^{n} A_{c_i}}$$

Here, A_{c_i} denotes the area of the region bounded by the fuzzy set c_i and x_i is the geometric centre of the area A_{c_i} .

Graphically,



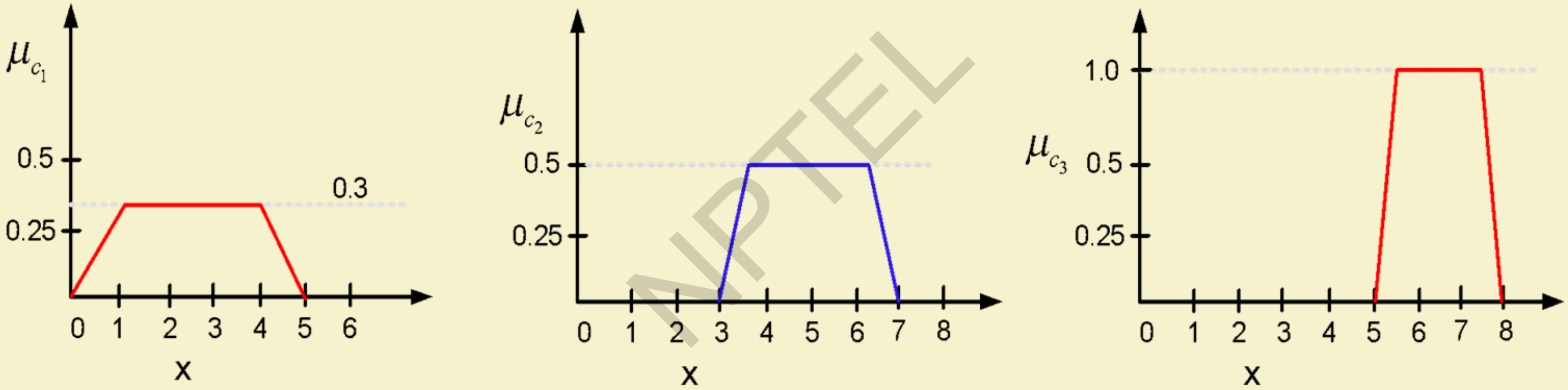
Centroid method : CoS

Note:

- In CoG method, the overlapping area is counted once, whereas, in CoS , the overlapping is counted twice or so.
- In CoS, we use the centre of area and hence, its name instead of centre of gravity as in CoG.

CoS: Example

Consider the three output fuzzy sets as shown in the following plots:



CoS: Example

In this case, we have

$$A_{c_1} = \frac{1}{2} \times 0.3 \times (3+5), x_1 = 2.5$$

$$A_{c_2} = \frac{1}{2} \times 0.5 \times (4+2), x_2 = 5$$

$$A_{c_3} = \frac{1}{2} \times 1.0 \times (3+1), x_3 = 6.5$$

Thus, $x^* = \frac{\frac{1}{2}\times0.3\times(3+5)\times2.5+\frac{1}{2}\times0.5\times(4+2)\times5+\frac{1}{2}\times1.0\times(3+1)\times6.5}{\frac{1}{2}\times0.3\times(3+5)+\frac{1}{2}\times0.5\times(4+2)+\frac{1}{2}\times1.0\times(3+1)}$

Note:

The crisp value of $C = C_1 \cup C_2 \cup C_3$ using CoG method can be found to be calculated as $x^* = 4.9$

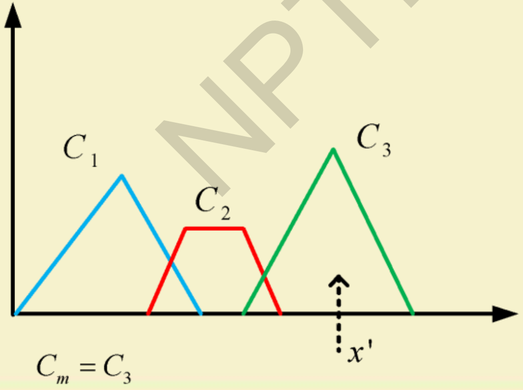
Centroid method: Centre of largest area

If the fuzzy set has two sub regions, then the centre of gravity of the sub region with the largest area can be used to calculate the defuzzified value.

Mathematically, $x^* = \dfrac{\int \mu_{c_m}(x).x' d(x)}{\int \mu_{c_m}(x)\, d(x)}$;

Here, C_m is the region with largest area, x' is the centre of gravity of C_m.

Graphically,



Weighted average methods

1. **Maxima Methods**
   a) Height method
   b) First of maxima (FoM)
   c) Last of maxima (LoM)
   d) Mean of maxima(MoM)
2. **Centroid methods**
   a) Centre of gravity method (CoG)
   b) Centre of sum method (CoS)
   c) Centre of area method (CoA)
3. **Weighted average method**

Weighted average method

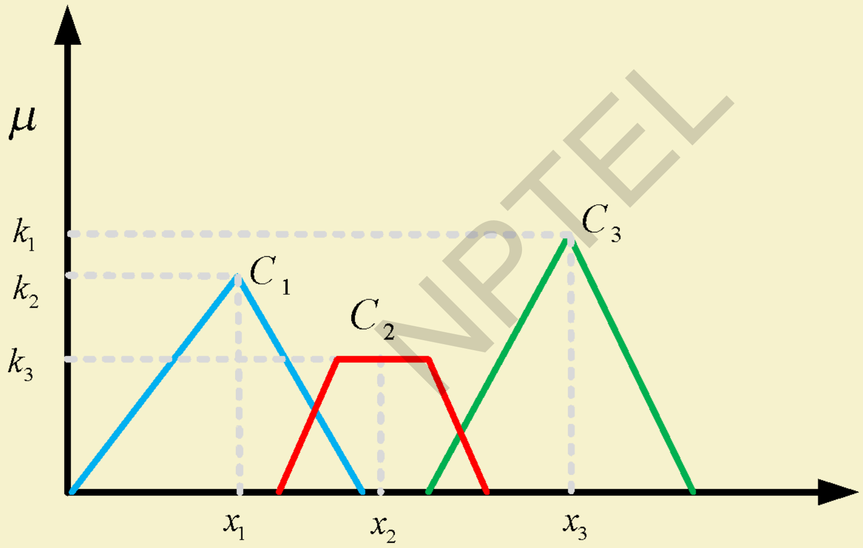
1) This method is also alternatively called "Sugeno defuzzification" method.
2) The method can be used only for symmetrical output membership functions.
3) The crisp value according to this method is

$$x^* = \frac{\sum_{i=1}^{n} \mu_{c_i}(x_i).x_i}{\sum_{i=1}^{n} \mu_{c_i}(x_i)}$$

where, $C_1, C_2, \ldots \ldots \ldots C_n$ are the output fuzzy sets and (x_i) is the value where middle of the fuzzy set C_i is observed.

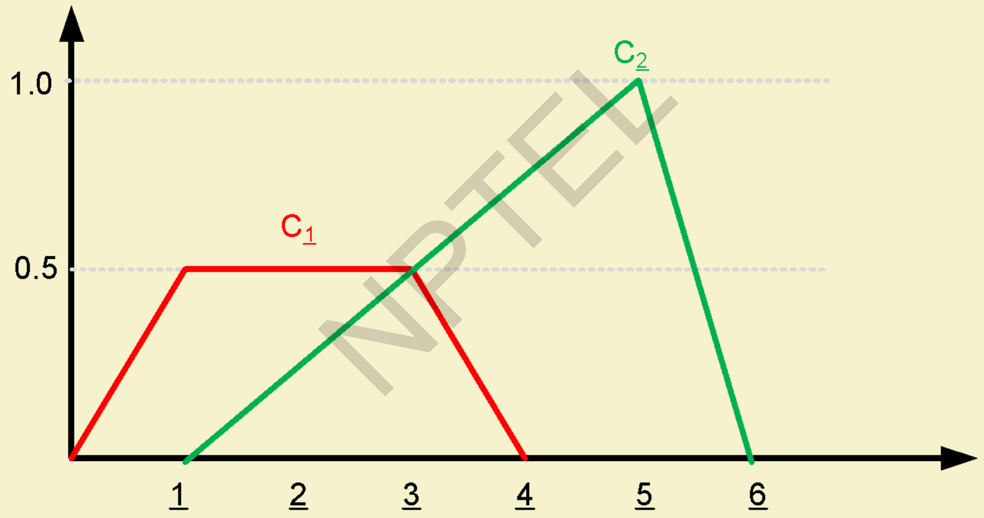
Weighted average method

Graphically ,



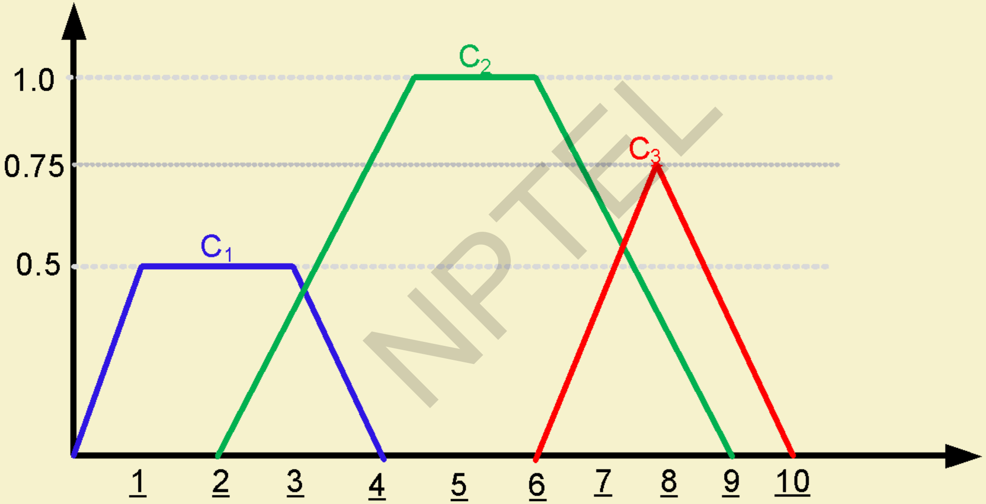
Exercise 1

Find the crisp value of the following using all defuzzified methods.



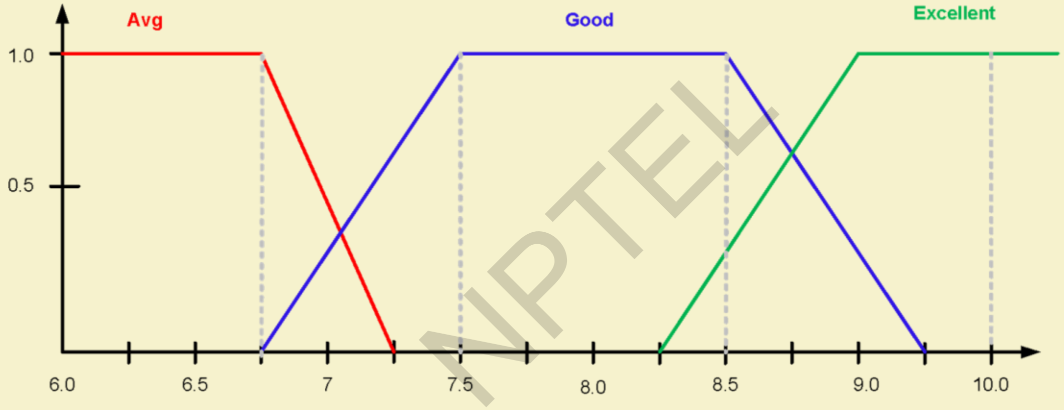
Exercise 2

Find the crisp value of the following using all defuzzified methods.



Exercise 3

- The membership function defining a student as *Average*, *Good*, and *Excellent* denoted by respective membership functions are as shown below.

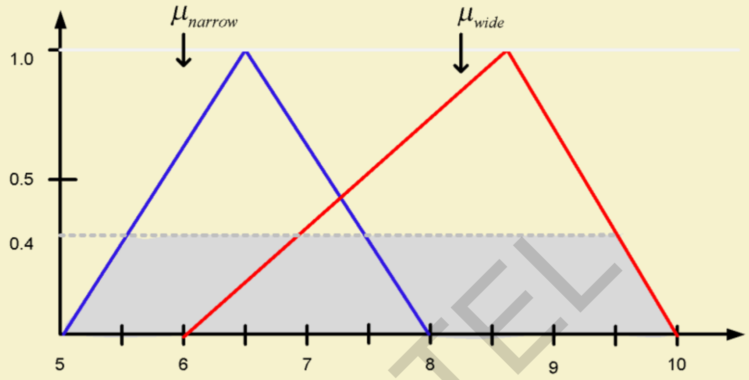


- Find the crisp value of **"Good Student"**

Hint:

Use CoG method to the portion *Good* to calculate it.

Exercise 4



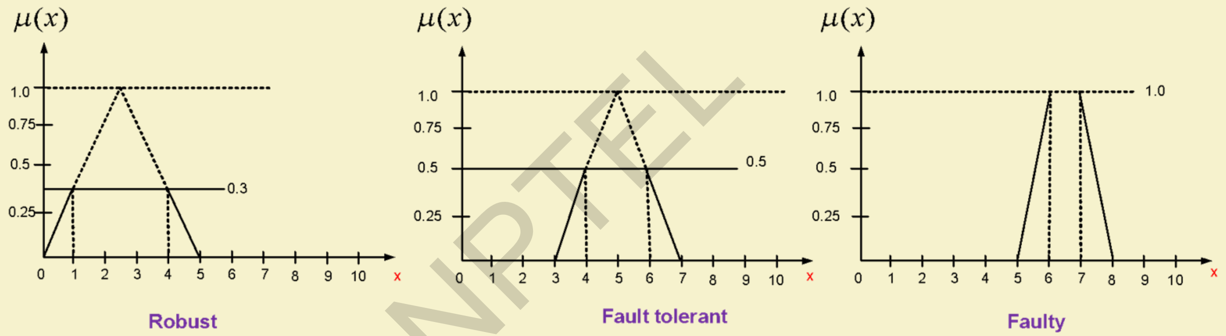
- The width of a road as narrow and wide is defined by two fuzzy sets, whose membership functions are plotted as shown above.
- If a road with its degree of membership value is 0.4 then what will be its width (in crisp) measure.

Hint:

Use CoG method for the shaded region

Exercise 5

- The faulty measure of a circuit is defined fuzzily by three fuzzy sets namely $Robust(R)$, $Fault\ tolerant\ (FT)\ and\ Faulty(F)$, defined by three membership functions with number of faults occur as universe of discourses and is shown below.



- Reliability is measured as $R^* = F \cup FT \cup R$ With a certain observation in testing $(x, 0.3) \in R, (x, 0.5) \in FT, (x, 0.8) \in F$.
- Calculate the reliability measure in crisp value.
- Calculate with 1) CoS 2) CoG .

Thank You!!