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Soft Computing

Introduction to Soft Computing

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INTRODUCTION TO SOFT COMPUTING

- Concept of computation
- Hard computing
- Soft computing
- How soft computing?
- Hard computing vs. Soft computing
- Hybrid computing

CONCEPT OF COMPUTATION

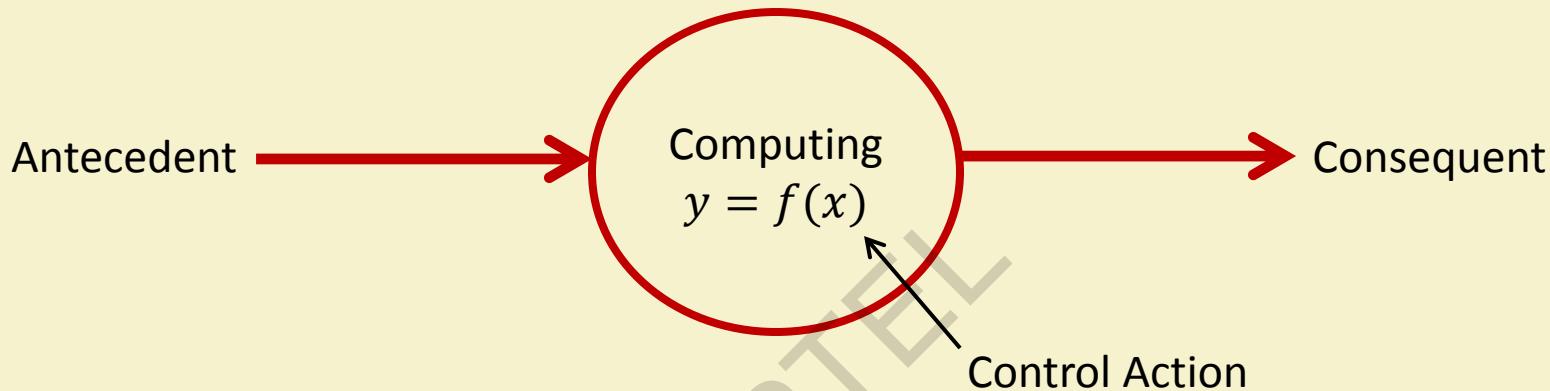


Figure: Basic of computing

$y = f(x)$, f is a mapping function.

f is also called a formal method or an **algorithm** to solve a problem.

Important characteristics of computing

- Should provide **precise solution**.
- Control action should be **unambiguous** and **accurate**.
- Suitable for problem, which is easy to **model mathematically**.

Hard computing

- In 1996, L. A. Zade (LAZ) introduced the term **hard computing**.
- According to LAZ: We term a computing as **Hard computing**, if
 - ✓ Precise result is guaranteed.
 - ✓ Control action is **unambiguous**.
 - ✓ Control action is **formally defined** (i.e., with mathematical model or algorithm).

Examples of hard computing

- Solving numerical problems (e.g., roots of polynomials, integration, etc.).
- Searching and sorting techniques.
- Solving computational geometry problems (e.g., shortest tour in a graph, finding closest pair of points given a set of points, etc.).
- many more...

Soft computing

- The term soft computing was proposed by the inventor of fuzzy logic, Lotfi A. Zadeh. He describes it as follows.

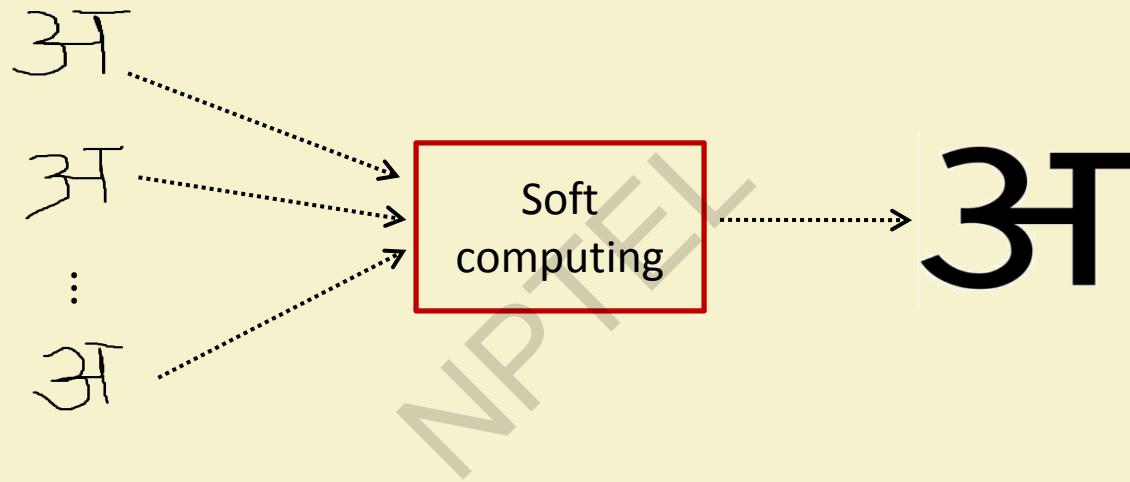
Definition 1: Soft computing

Soft computing is a collection of methodologies that aim to exploit the tolerance for imprecision and uncertainty to achieve tractability, robustness, and low solution cost. Its principal constituents are fuzzy logic, neuro-computing, and probabilistic reasoning. The role model for soft computing is the human mind.

Characteristics of soft computing

- It **does not require** any mathematical modeling of problem solving.
- It **may not yield** the precise solution.
- Algorithms are **adaptive** (i.e., it can adjust to the change of dynamic environment).
- Use some biological inspired methodologies such as genetics, evolution, Ant's behaviors, particles swarming, human nervous system, etc.).

Examples of soft computing



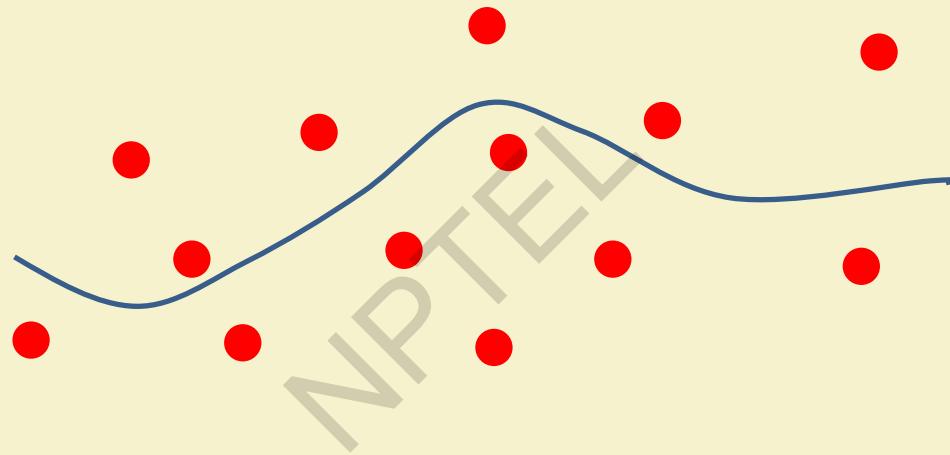
Example: Hand written character recognition
(Artificial Neural Networks)

Examples of soft computing



Example: Money allocation problem
(Evolutionary Computing)

Examples of soft computing



Example: Robot movement
(Fuzzy Logic)

How soft computing?

- How a **student** learns from his **teacher**?
 - Teacher asks questions and tell the answers then.
 - Teacher puts questions and hints answers and asks whether the answers are correct or not.
 - Student thus learn a topic and store in his memory.
 - Based on the knowledge he solves new problems.
- This is the way how human brain works.
- Based on this concept **Artificial Neural Network** is used to solve problems.

How soft computing?

- How **world** selects the best?
 - It starts with a population (random).
 - Reproduces another population (next generation).
 - Rank the population and selects the superior individuals.
- **Genetic algorithm** is based on this natural phenomena.
 - Population is synonymous to solutions.
 - Selection of superior solution is synonymous to exploring the optimal solution.

How soft computing?

- How a **doctor** treats his **patient**?
 - Doctor asks the patient about suffering.
 - Doctor find the symptoms of diseases.
 - Doctor prescribed tests and medicines.
- This is exactly the way **Fuzzy Logic** works.
 - Symptoms are correlated with diseases with uncertainty .
 - Doctor prescribes tests/medicines **fuzzily**.

Hard computing vs. Soft computing

Hard computing	Soft computing
<ul style="list-style-type: none">▪ It requires a precisely stated analytical model and often a lot of computation time.▪ It is based on binary logic, crisp systems, numerical analysis and crisp software.▪ It has the characteristics of precision and categoricity.	<ul style="list-style-type: none">▪ It is tolerant of imprecision, uncertainty, partial truth, and approximation.▪ It is based on fuzzy logic, neural nets and probabilistic reasoning.▪ It has the characteristics of approximation and dispositionality.

Hard computing vs. Soft computing

Hard computing	Soft computing
<ul style="list-style-type: none">▪ It is deterministic.▪ It requires exact input data.▪ It is strictly sequential.▪ It produces precise answers.	<ul style="list-style-type: none">▪ It incorporates stochasticity.▪ It can deal with ambiguous and noisy data.▪ It allows parallel computations.▪ It can yield approximate answers

Hybrid computing

- It is a combination of the conventional hard computing and emerging soft computing.

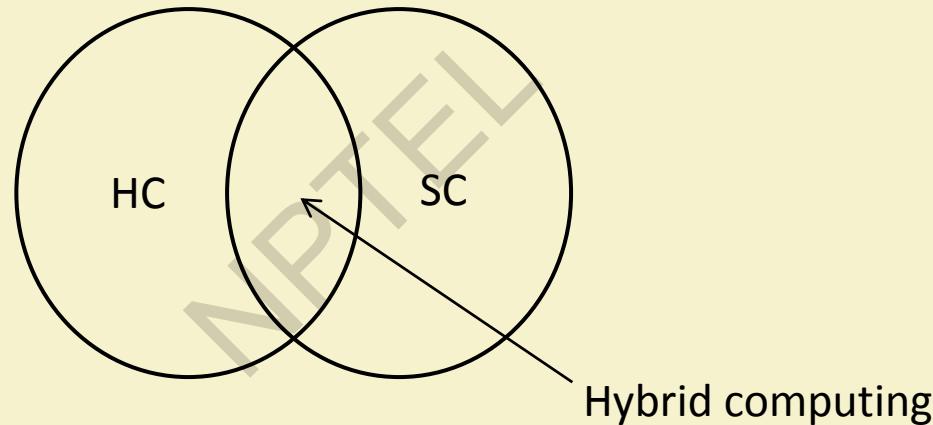


Figure: Concept of Hybrid Computing

In this course...

- You will be able to learn
 - Basic concepts of Fuzzy algebra and then how to solve problems using Fuzzy logic.
 - The framework of Genetic algorithm and solving varieties of optimization problems.
 - How to build an artificial neural network and train it with input data to solve a number of problems, which are not possible to solve with hard computing.

Thank You !!

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Soft Computing

Introduction to Fuzzy Logic

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What is Fuzzy logic?

- Fuzzy logic is a **mathematical language** to **express** something.
 - This means it has grammar, syntax, semantic like a language for communication.
- There are some other mathematical languages also known
 - **Relational algebra** (operations on sets)
 - **Boolean algebra** (operations on Boolean variables)
 - **Predicate algebra** (operations on well formed formulae (wff), also called predicate propositions)
- **Fuzzy logic deals with Fuzzy set or Fuzzy algebra.**

What is fuzzy?

- Dictionary meaning of **fuzzy** is **not clear, noisy**, etc.

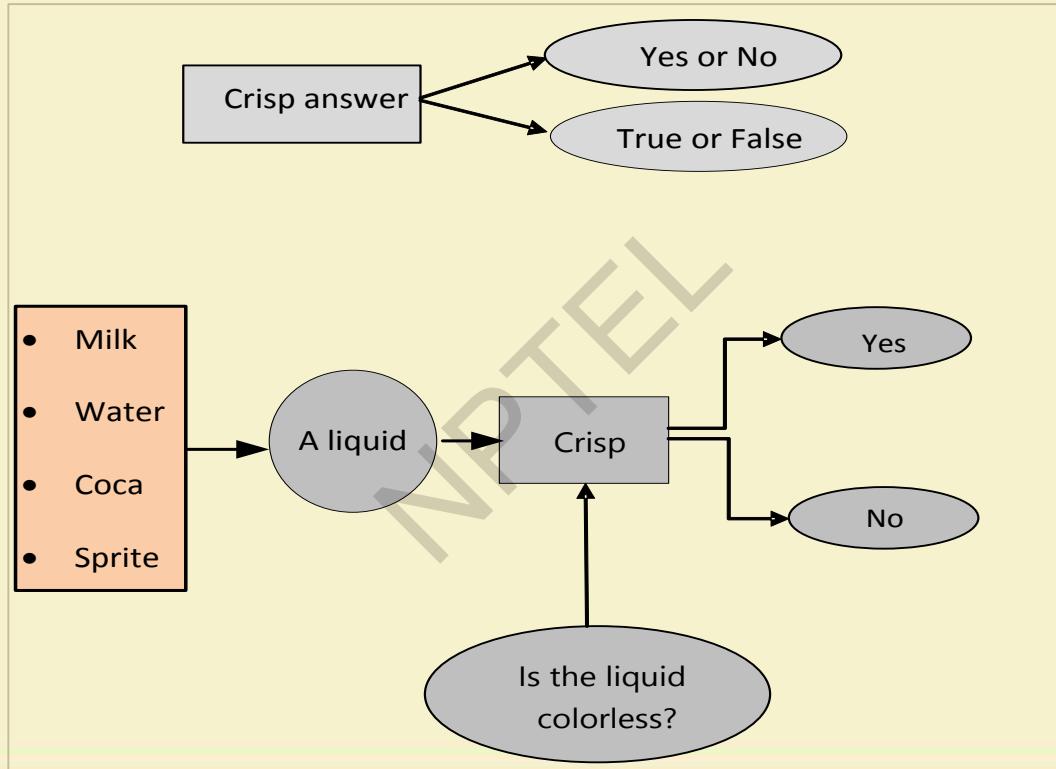
Example: Is the picture on this slide is fuzzy?

- Antonym of fuzzy is **crisp**

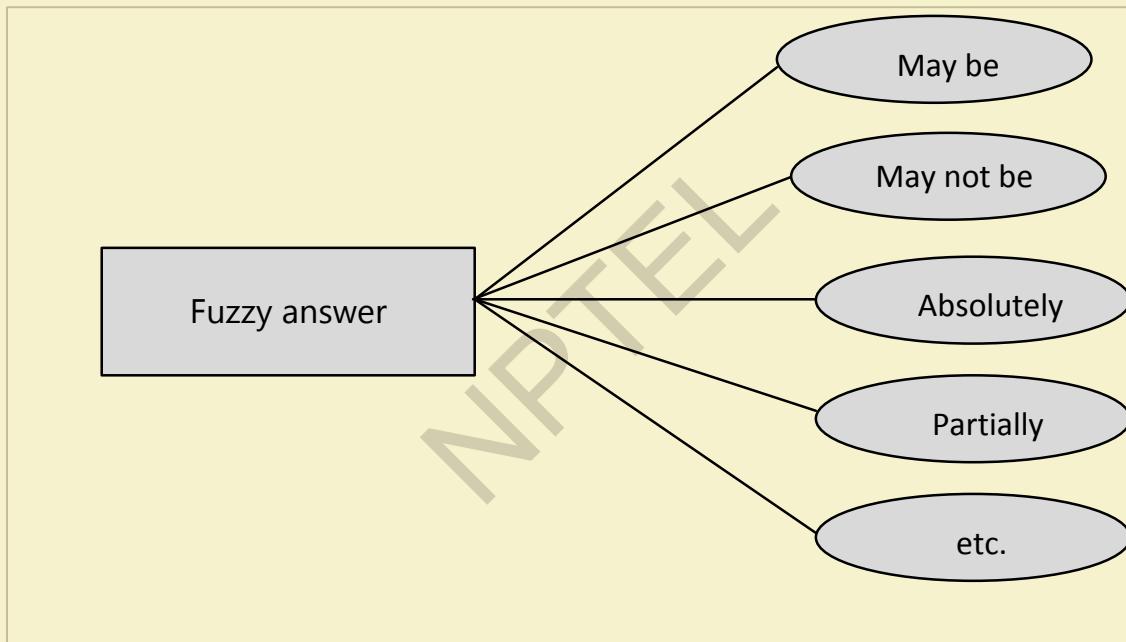
Example: Are the chips crisp?



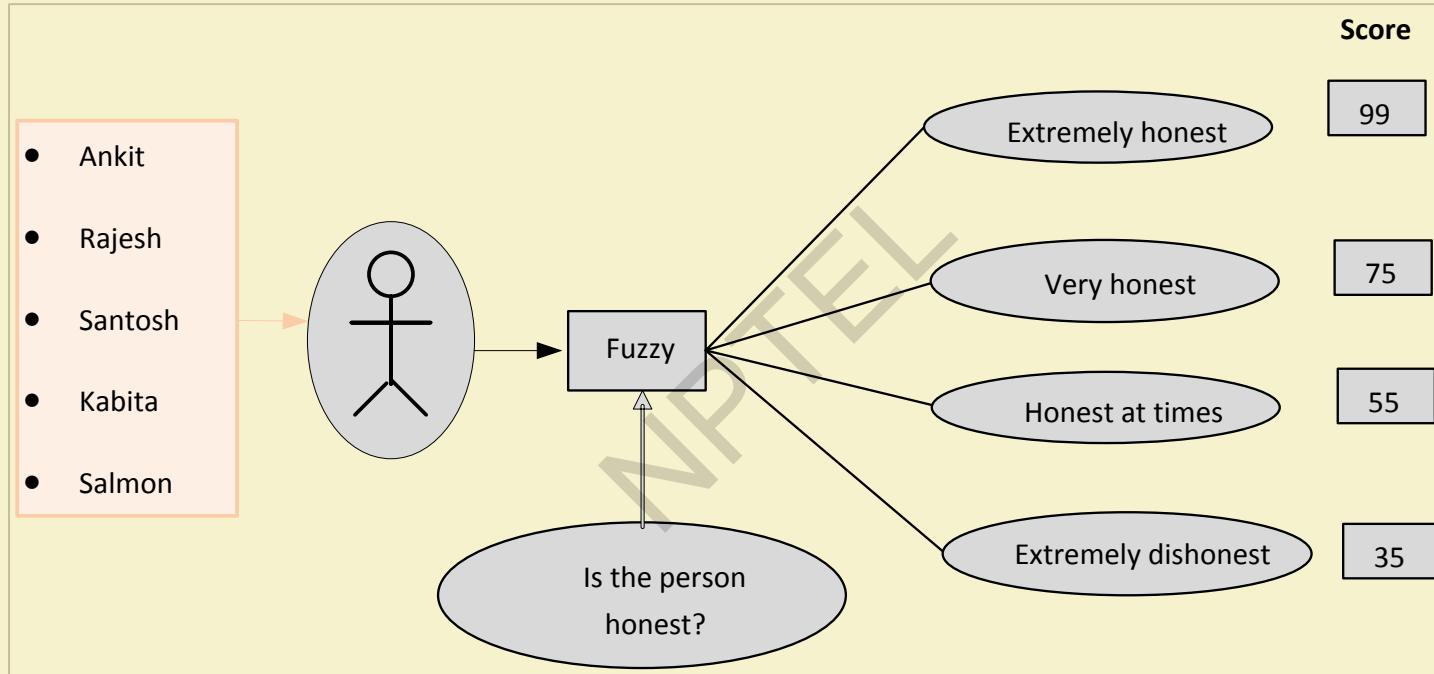
Example : Fuzzy logic vs. Crisp logic



Example : Fuzzy logic vs. Crisp logic



Example : Fuzzy logic vs. Crisp logic



World is fuzzy!



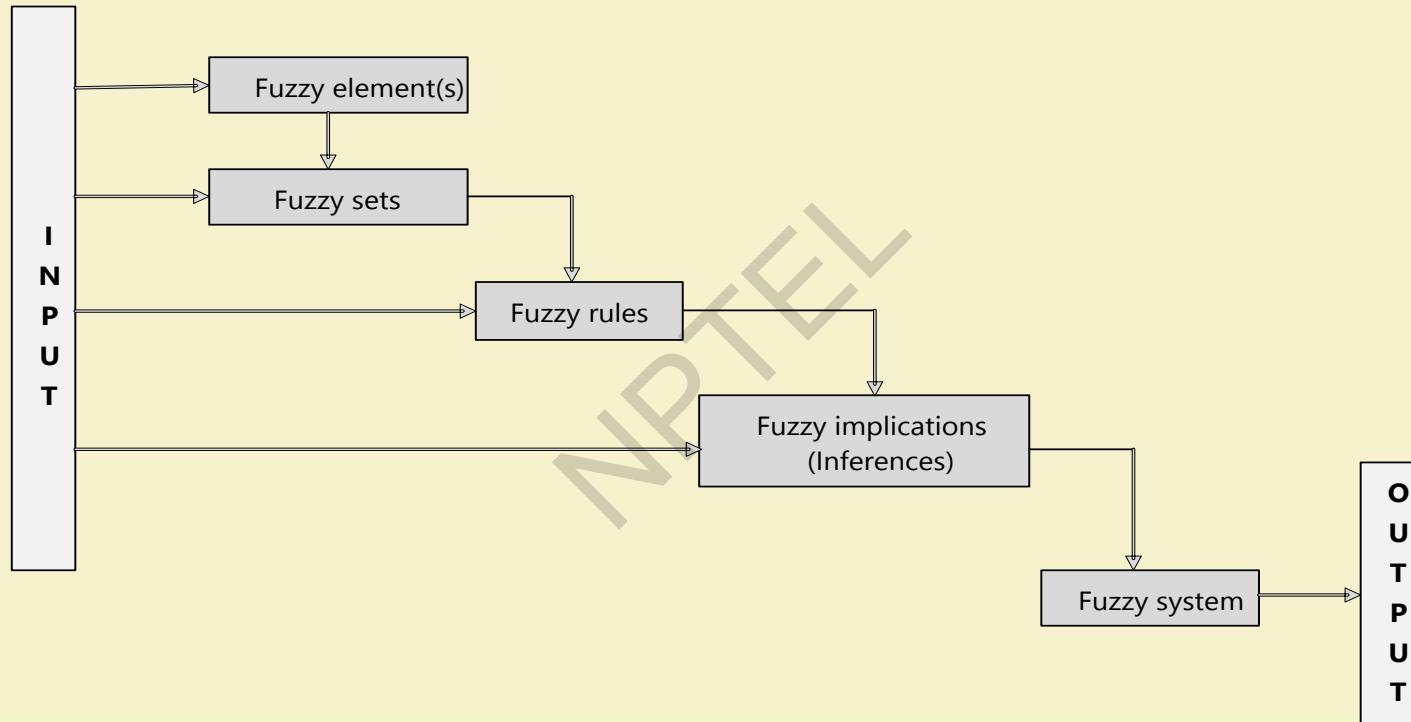
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Concept of fuzzy system



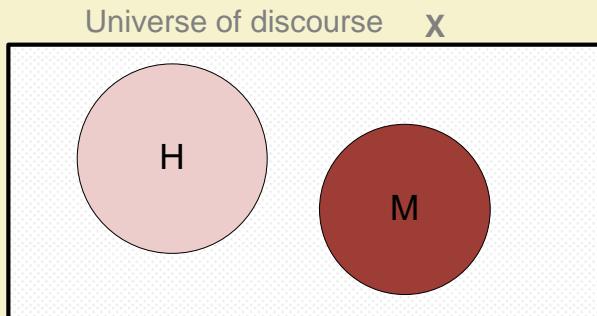
Concept of fuzzy set

To understand the concept of **fuzzy set** it is better, if we first clear our idea of **crisp set**.

X = The entire population of India.

H = All Hindu population = $\{h_1, h_2, h_3, \dots, h_L\}$

M = All Muslim population = $\{m_1, m_2, m_3, \dots, m_N\}$



Here, All are the sets of finite numbers of individuals.
Such a set is called **crisp set**.

Example of fuzzy set

Let us discuss about fuzzy set.

X = All students in NPTEL.

S = All **Good** students.

$S = \{(s, g(s)) \mid s \in X\}$ and $g(s)$ is a measurement of goodness of the student s .

Example:

$S = \{(Rajat, 0.8), (Kabita, 0.7), (Salman, 0.1), (Ankit, 0.9)\}$, etc.

Fuzzy set vs. Crisp set

Crisp set	Fuzzy set
<ul style="list-style-type: none">■ $S = \{s s \in X\}$■ It is a collection of elements.■ Inclusion of an element $s \in X$ into S is crisp, that is, has strict boundary yes or no.	<ul style="list-style-type: none">■ $F = (s, \mu(s)) s \in X$ and $\mu(s)$ is the degree of s.■ It is a collection of ordered pairs.■ Inclusion of an element $s \in X$ into F is fuzzy, that is, if present, then with a degree of membership.

Fuzzy set vs. Crisp set

Note: A crisp set is a fuzzy set, but, a fuzzy set is not necessarily a crisp set.

Example:

$$H = \{(h_1, 1), (h_2, 1) \dots \dots \dots, (h_L, 1)\}$$

$$\text{Person} = \{(p_1, 0), (p_2, 0) \dots \dots \dots, (p_N, 0)\}$$

In case of a crisp set, the elements are with extreme values of degree of membership namely either 1 or 0.

Degree of membership

How to decide the degree of memberships of elements in a fuzzy set?

City	Bangalore	Bombay	Hyderabad	Kharagpur	Madras	Delhi
μ	0.95	0.90	0.80	0.01	0.65	0.75

How the cities of **comfort** can be judged?

Example: Course evaluation in a crisp way

EX : Marks ≥ 90

A : $80 \leq \text{Marks} < 90$

B : $70 \leq \text{Marks} < 80$

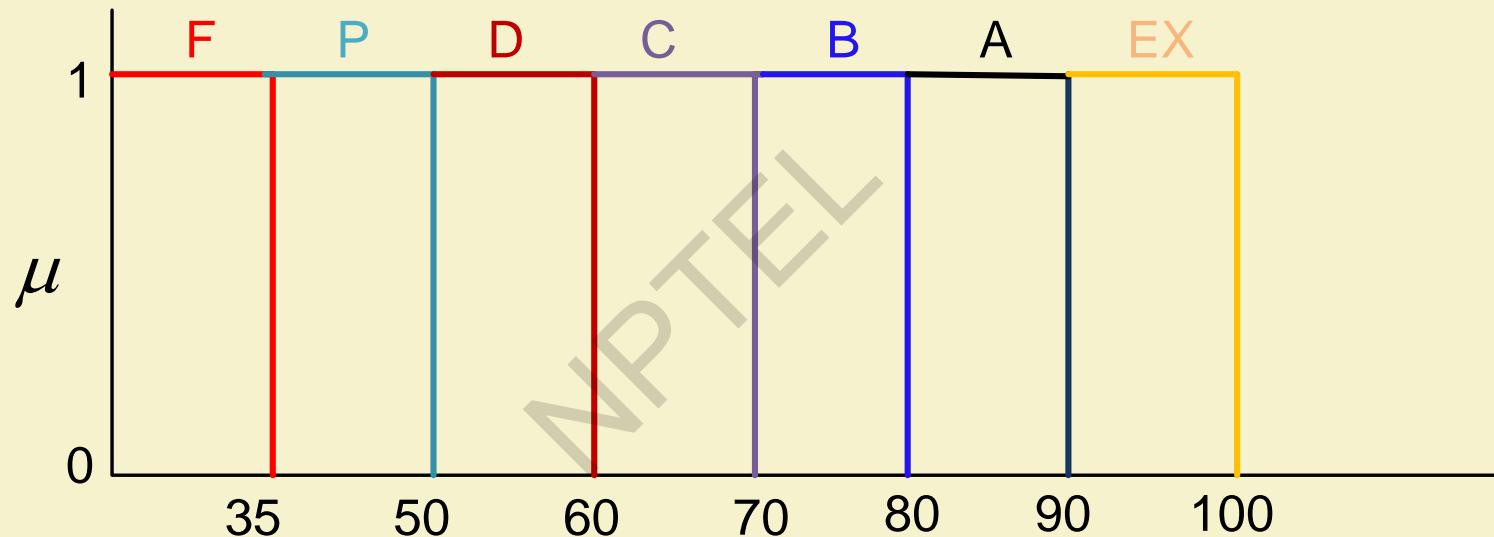
C : $60 \leq \text{Marks} < 70$

D : $50 \leq \text{Marks} < 60$

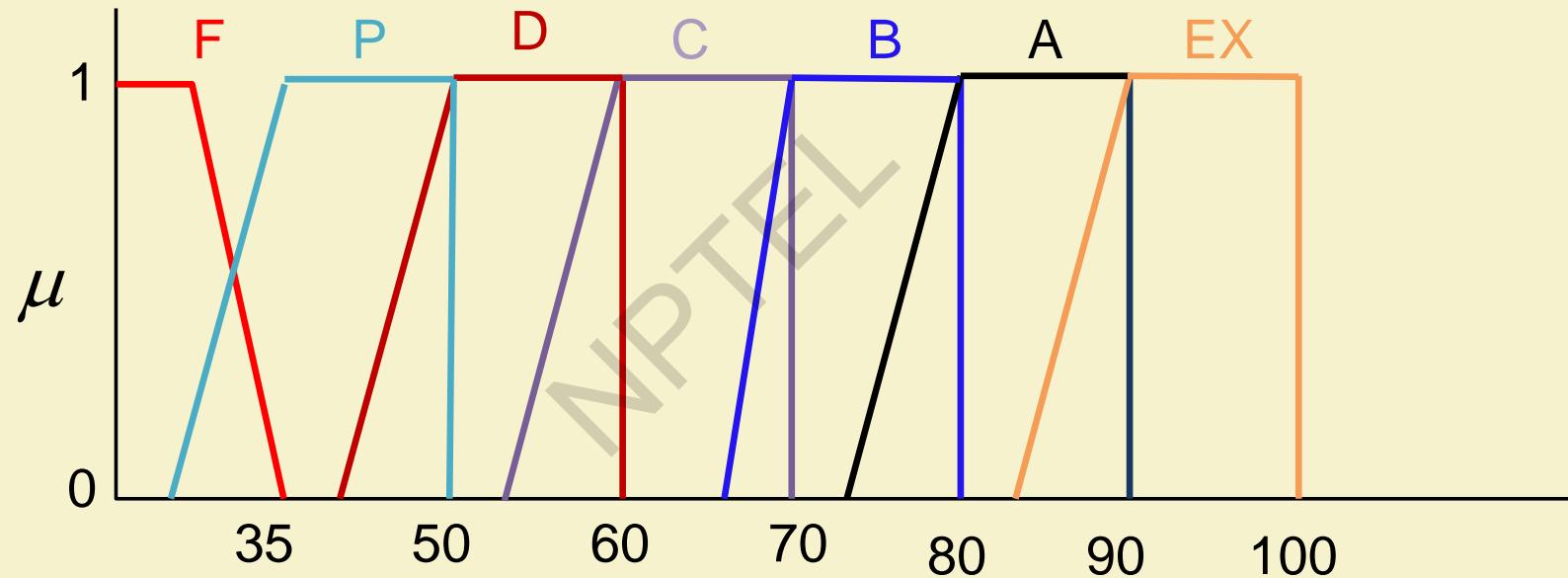
P : $35 \leq \text{Marks} < 50$

F : Marks ≤ 35

Example: Course evaluation in a crisp way



Example: Course evaluation in a fuzzy way



Few examples of fuzzy set

- High Temperature
- Low Pressure
- Colour of Apple
- Sweetness of Orange
- Weight of Mango

Note: Degree of membership values lie in the range [0...1].



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Some basic terminologies and notations

Definition 1: Membership function (and Fuzzy set)

If X is a universe of discourse and $x \in X$, then a fuzzy set A in X is defined as a set of ordered pairs, that is

$A = \{(x, \mu_A(x)) | x \in X\}$ where $\mu_A(x)$ is called the **membership function** for the fuzzy set A .

Note: $\mu_A(x)$ map each element of X onto a membership grade (or membership value) between 0 and 1 (both inclusive).

Question: How (and who) decides $\mu_A(x)$ for a fuzzy set A in X ?

Some basic terminologies and notations

Example:

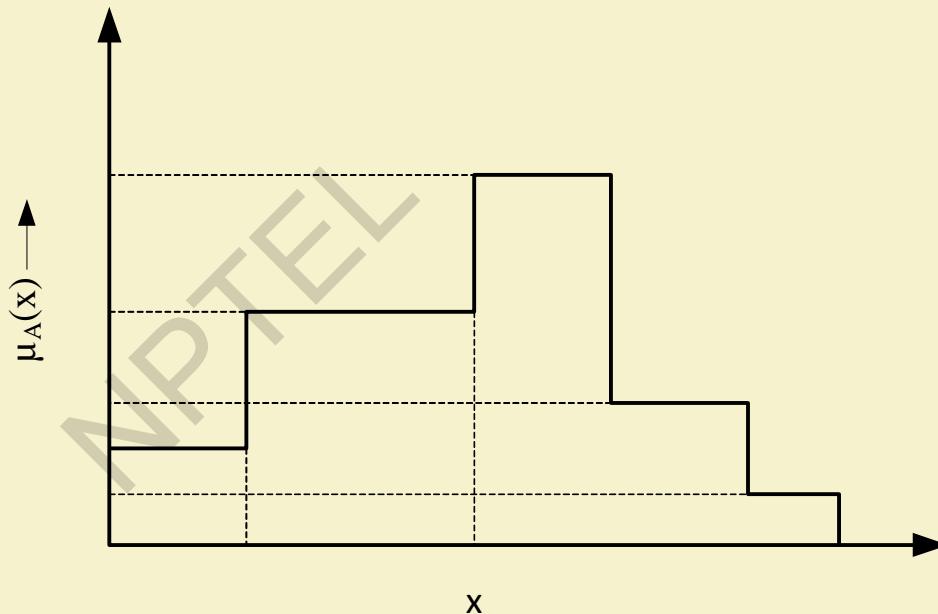
X = All cities in India

A = City of comfort

$A = \{(New\ Delhi, 0.7), (Bangalore, 0.9), (Chennai, 0.8), (Hyderabad, 0.6), (Kolkata, 0.3), (Kharagpur, 0)\}$

Membership function with discrete membership values

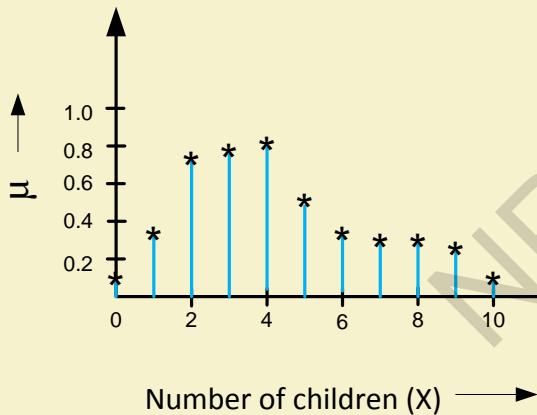
The membership values may be of discrete values.



A fuzzy set with discrete values of μ

Membership function with discrete membership values

Either elements or their membership values (or both) also may be of discrete values.



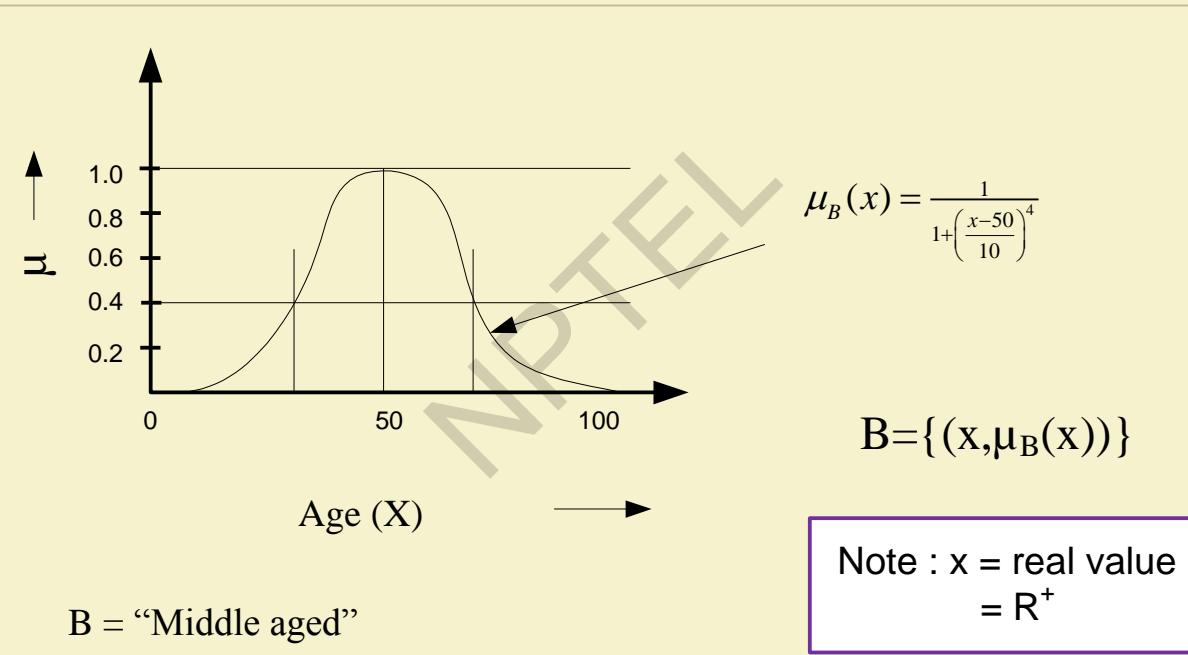
$$A = \{(0, 0.1), (1, 0.30), (2, 0.78), \dots, (10, 0.1)\}$$

Note : X = discrete value

How you measure happiness ??

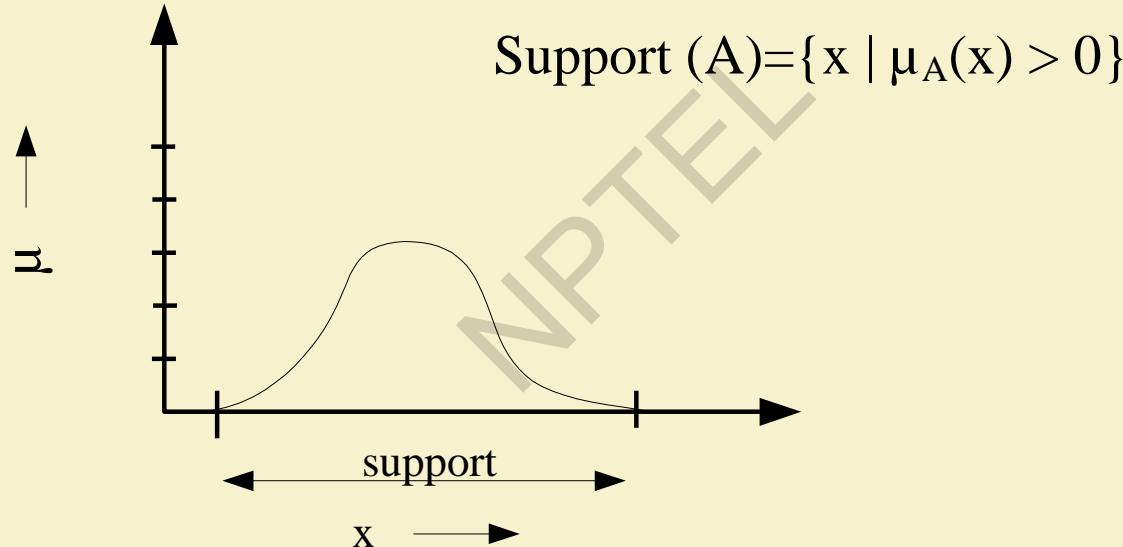
A = "Happy family"

Membership function with continuous membership values



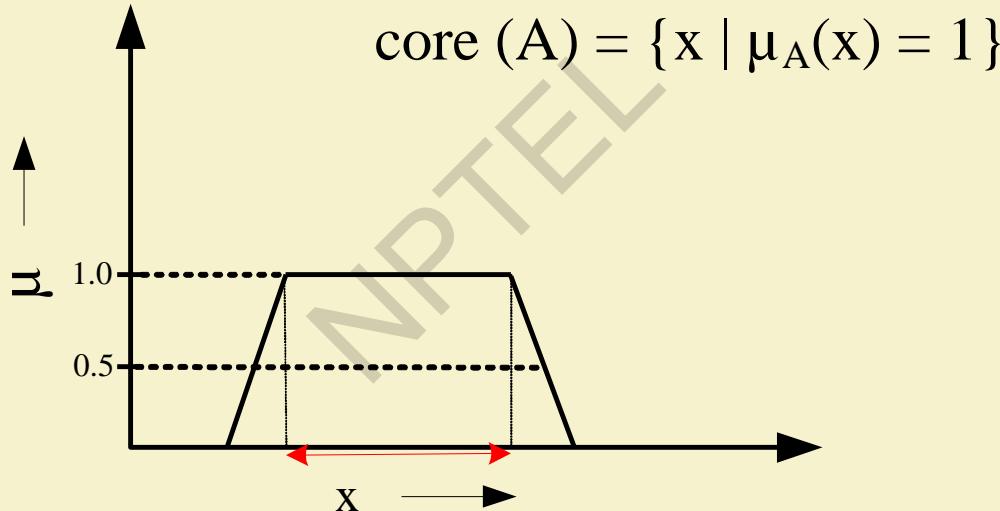
Fuzzy terminologies: Support

Support: The support of a fuzzy set A is the set of all points $x \in X$ such that $\mu_A(x) > 0$



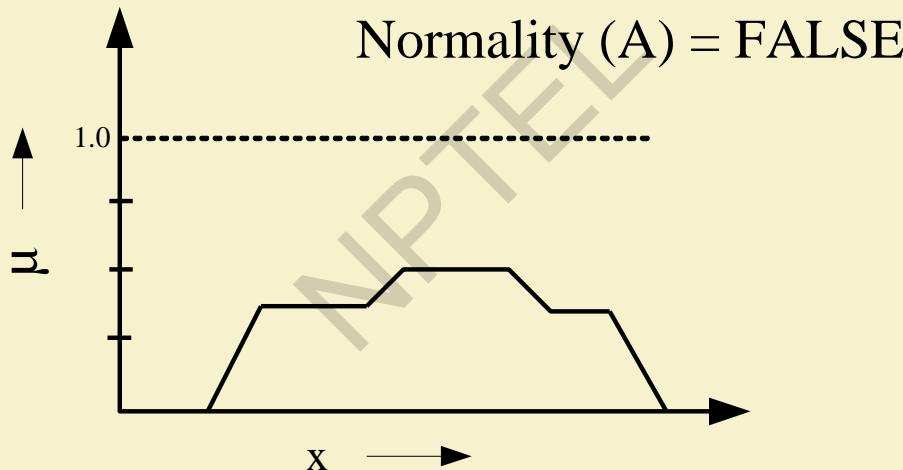
Fuzzy terminologies: Core

Core: The core of a fuzzy set A is the set of all points x in X such that $\mu_A(x) = 1$



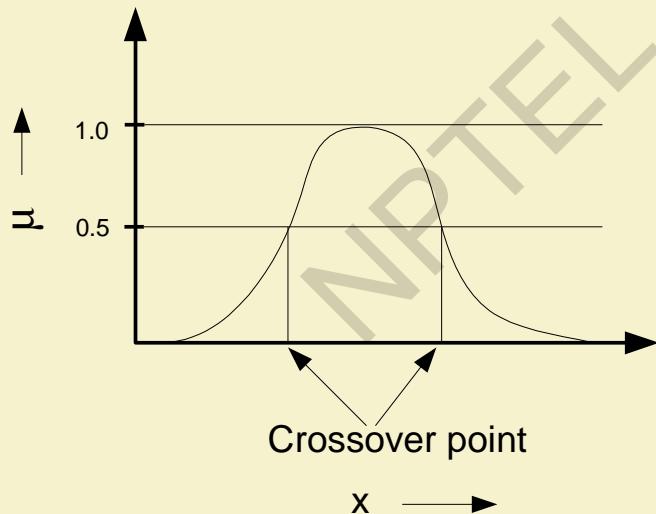
Fuzzy terminologies: Normality

Normality : A fuzzy set A is normal if its core is non-empty. In other words, we can always find a point $x \in X$ such that $\mu_A(x) = 1$



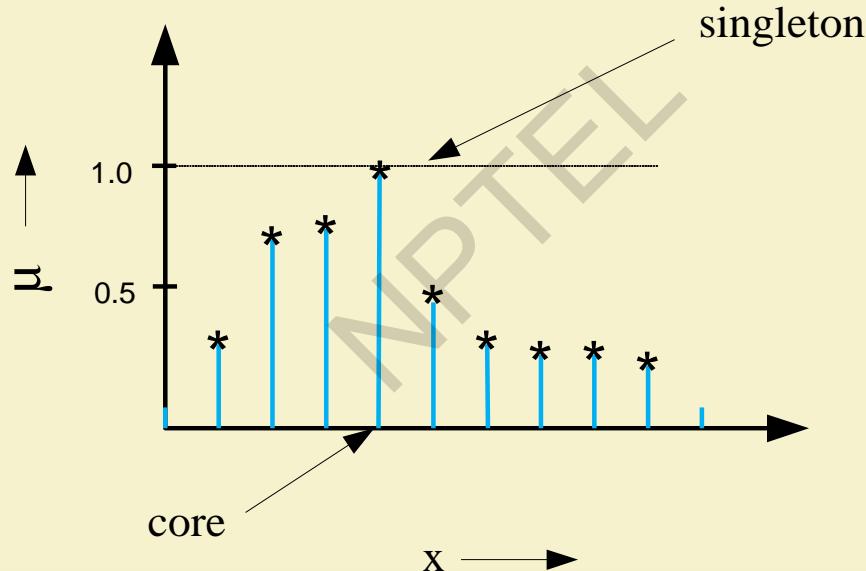
Fuzzy terminologies: Crossover points

Crossover point : A crossover point of a fuzzy set A is a point $x \in X$ at which $\mu_A(x) = 0.5$. That is $\text{Crossover}(A) = \{x | \mu_A(x) = 0.5\}$



Fuzzy terminologies: Fuzzy Singleton

Fuzzy Singleton : A fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called a fuzzy singleton. That is $|A| = \{x | \mu_A(x) = 1\}$



Fuzzy terminologies: α -cut and strong α -cut

α -cut and strong α -cut :

- ✓ The α -cut of a fuzzy set A is a crisp set defined by

$$A_\alpha = \{x | \mu_A(x) \geq \alpha\}$$

- ✓ Strong α -cut is defined similarly :

$$A'_\alpha = \{x | \mu_A(x) > \alpha\}$$

Note : Support (A) = A_0' and Core (A) = A_1 .

Fuzzy terminologies: Bandwidth

Bandwidth :

For a fuzzy set, the bandwidth (or width) is defined as the distance between the two unique crossover points:

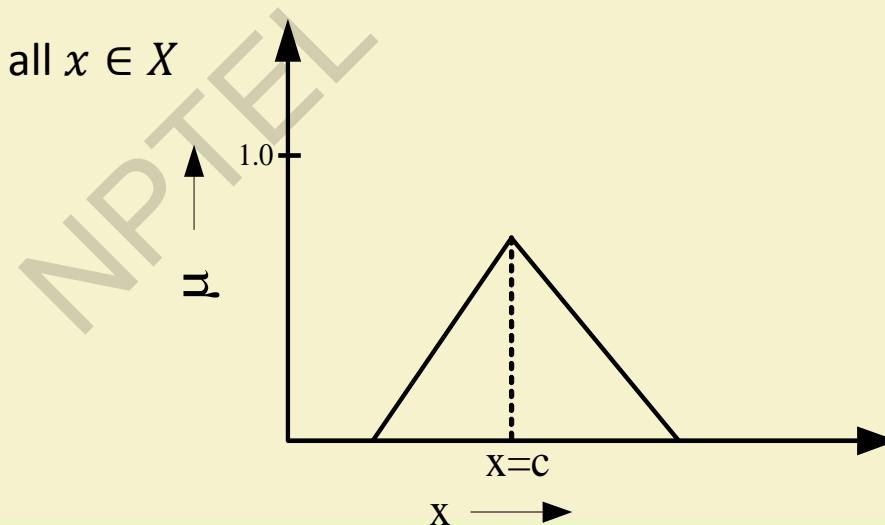
$$\text{Bandwidth } (A) = |x_1 - x_2|$$

where $\mu_A(x_1) = \mu_A(x_2) = 0.5$

Fuzzy terminologies: Symmetry

Symmetry :

A fuzzy set A is symmetric if its membership function around a certain point $x = c$, namely $\mu_A(x + c) = \mu_A(x - c)$ for all $x \in X$



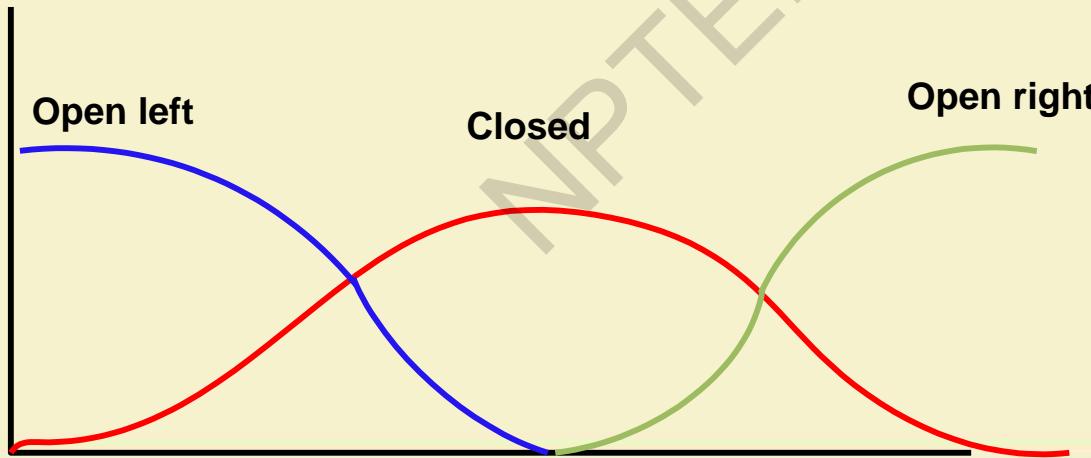
Fuzzy terminologies: Open and Closed

A fuzzy set A is

Open left : If $\lim_{x \rightarrow -\infty} \mu_A(x) = 1$ and $\lim_{x \rightarrow +\infty} \mu_A(x) = 0$

Open right: If $\lim_{x \rightarrow -\infty} \mu_A(x) = 0$ and $\lim_{x \rightarrow +\infty} \mu_A(x) = 1$

Closed: If $\lim_{x \rightarrow -\infty} \mu_A(x) = \lim_{x \rightarrow +\infty} \mu_A(x) = 0$



Fuzzy vs. Probability

Fuzzy : When we say about certainty of a thing

Example: A patient come to the doctor and he has to diagnose so that medicine can be prescribed.

Doctor prescribed a medicine with certainty 60% that the patient is suffering from flue. So, the disease will be cured with certainty of 60% and uncertainty 40%. Here, in stead of flue, other diseases with some other certainties may be.

Probability: When we say about the chance of an event to occur

Example: India will win the T20 tournament with a chance 60% means that out of 100 matches, India own 60 matches.

Prediction vs. Forecasting

The Fuzzy vs. Probability is analogical to Prediction vs. Forecasting

Prediction : When you start guessing about things.

Forecasting : When you take the information from the past job and apply it to new job.

The main difference:

Prediction is based on the **best guess** from experiences.

Forecasting is based on **data** you have actually recorded and packed from previous job.

Thank You !!

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Introduction to Soft Computing

Fuzzy membership functions

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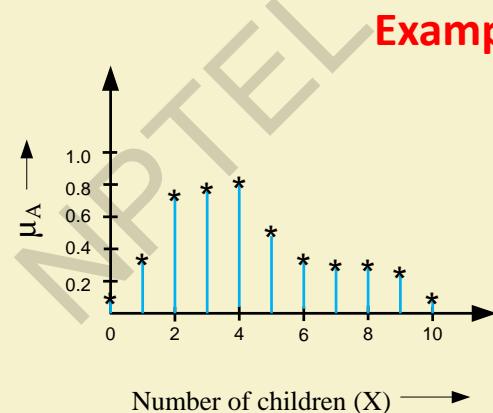
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Fuzzy membership functions

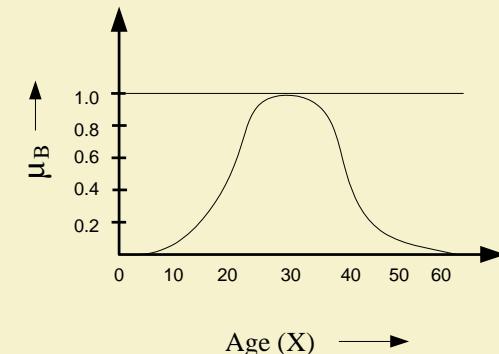
A fuzzy set is completely characterized by its membership function (sometimes abbreviated as *MF* and denoted as μ). So, it would be important to learn how a membership function can be expressed (mathematically or otherwise).

Note: A membership function can be on

- a) a discrete universe of discourse and
- b) a continuous universe of discourse.



A = Fuzzy set of “Happy family”

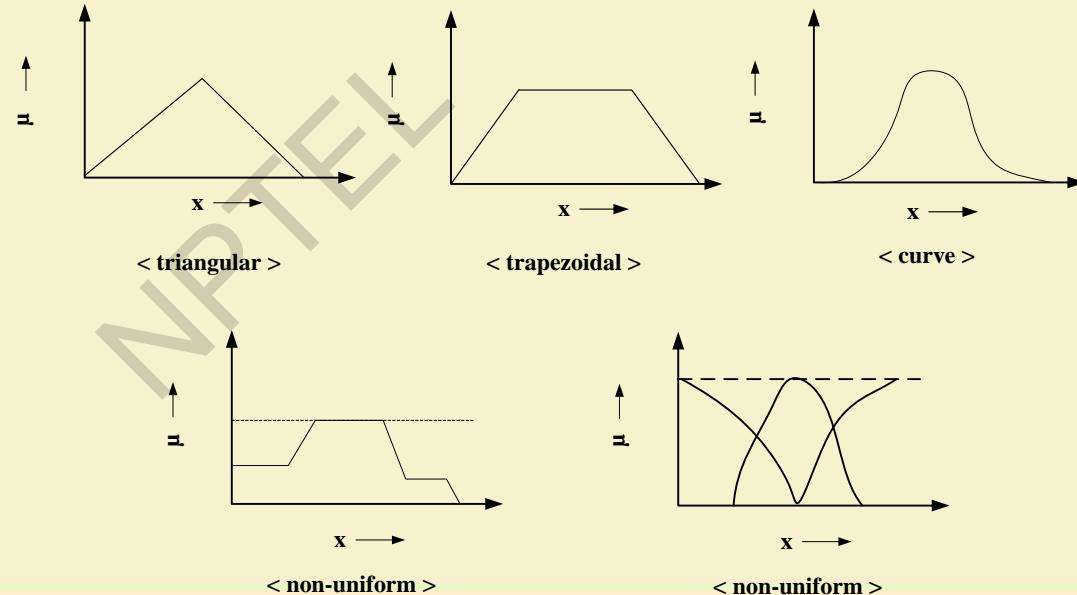


B = “Young age”

Fuzzy membership functions

So, membership function on a discrete universe of course is trivial. However, a membership function on a continuous universe of discourse needs a special attention.

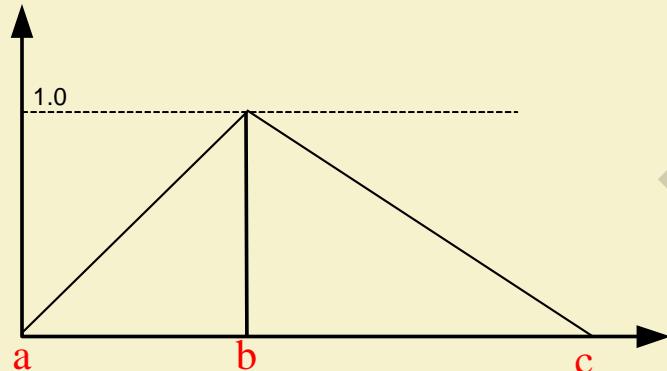
Following figures shows typical examples of membership functions.



Fuzzy MFs : Formulation and parameterization

In the following, we try to parameterize the different MFs on a continuous universe of discourse.

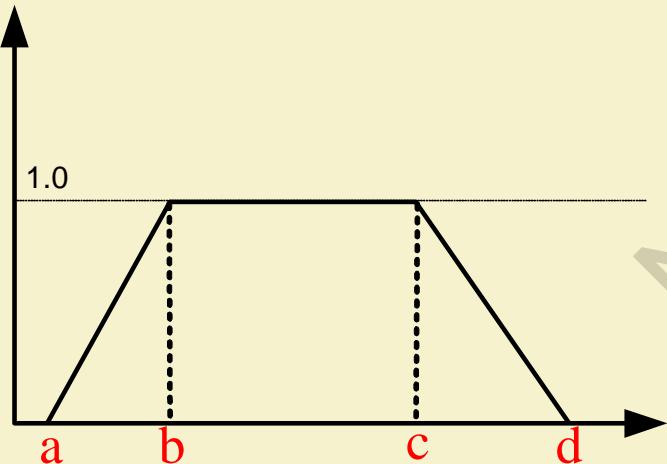
Triangular MFs : A triangular MF is specified by three parameters $\{a, b, c\}$ and can be formulated as follows.



$$\text{triangle}(x; a, b, c) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{a-b} & \text{if } a \leq x \leq b \\ \frac{c-x}{c-b} & \text{if } b \leq x \leq c \\ 0 & \text{if } c \leq x \end{cases}$$

Fuzzy MFs: Trapezoidal

A **trapezoidal MF** is specified by four parameters $\{a, b, c, d\}$ and can be defined as follows:

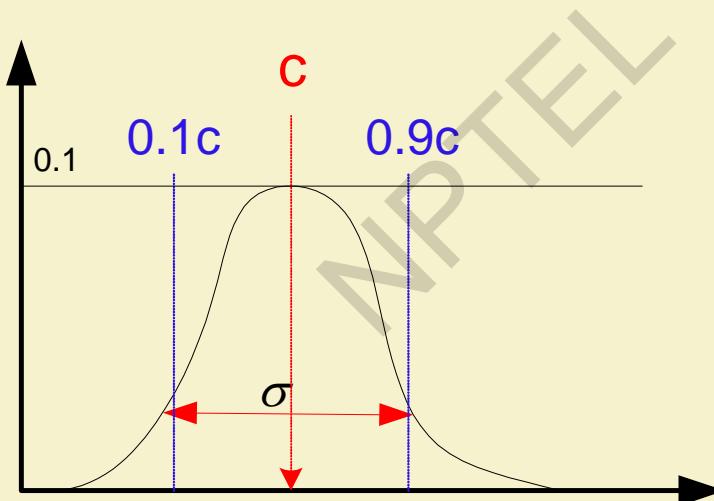


$$\text{trapozoid}(x; a, b, c, d) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c \leq x \leq d \\ 0 & \text{if } d \leq x \end{cases}$$

Fuzzy MFs: Gaussian

A Gaussian MF is specified by two parameters $\{c, \sigma\}$ and can be defined as below:

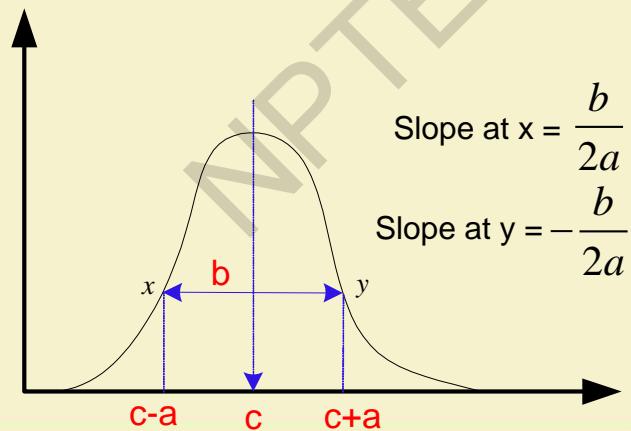
$$\text{gaussian } (x; c, \sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$



Fuzzy MFs: Generalized bell

It is also called **Cauchy MF**. A generalized bell MF is specified by three parameters $\{a, b, c\}$ and is defined as:

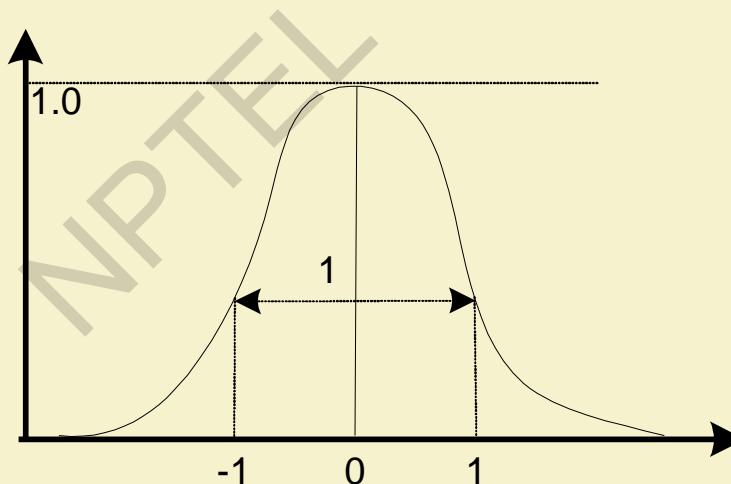
$$\text{bell}(x; a, b, c) = \frac{1}{1 + \left| \frac{x - c}{a} \right|^{2b}}$$



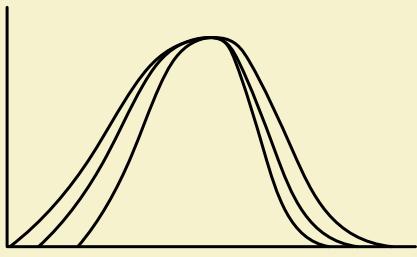
Example: Generalized bell MFs

Example: $\mu(x) = \frac{1}{1+|x|^2}$;

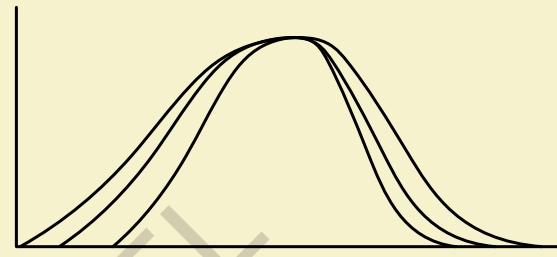
a = b = 1 and c = 0;



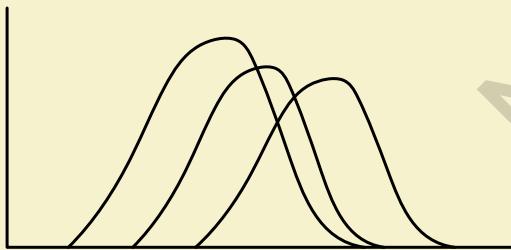
Generalized bell MFs: Different shapes



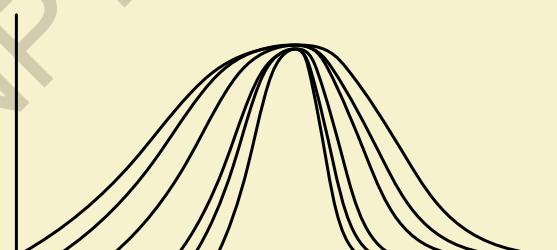
Changing a



Changing b



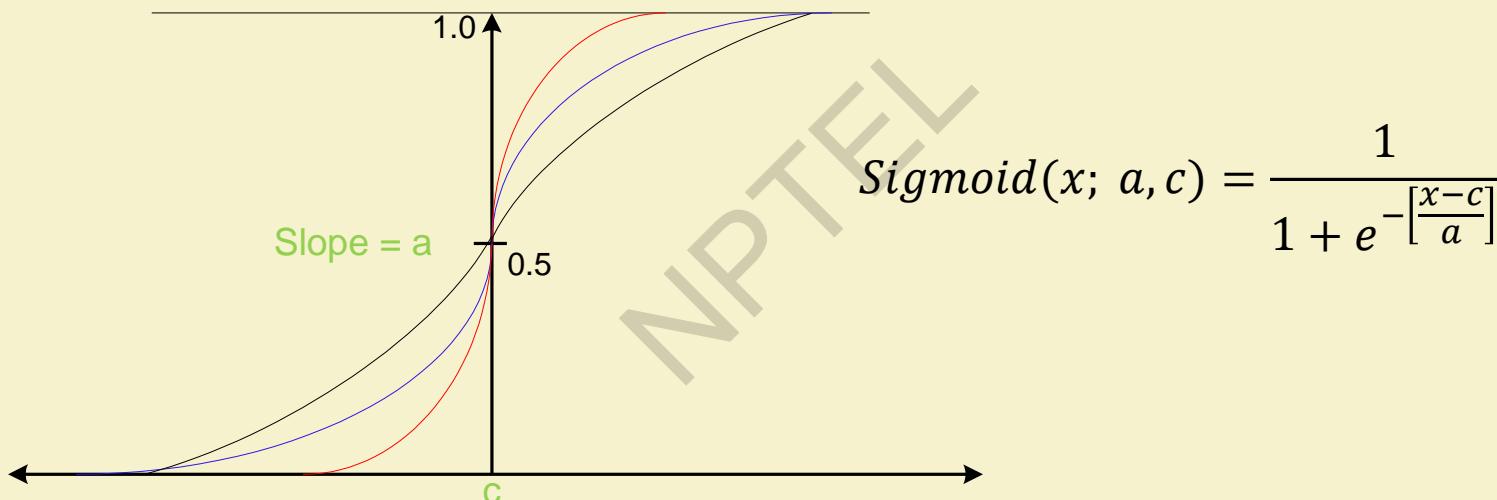
Changing a



Changing a and b

Fuzzy MFs: Sigmoidal MFs

Parameters: $\{a, c\}$; where c = crossover point and a = slope at c ;



Fuzzy MFs : Example

Example : Consider the following grading system for a course.

Excellent = Marks ≤ 90

Very good = $75 \leq \text{Marks} \leq 90$

Good = $60 \leq \text{Marks} \leq 75$

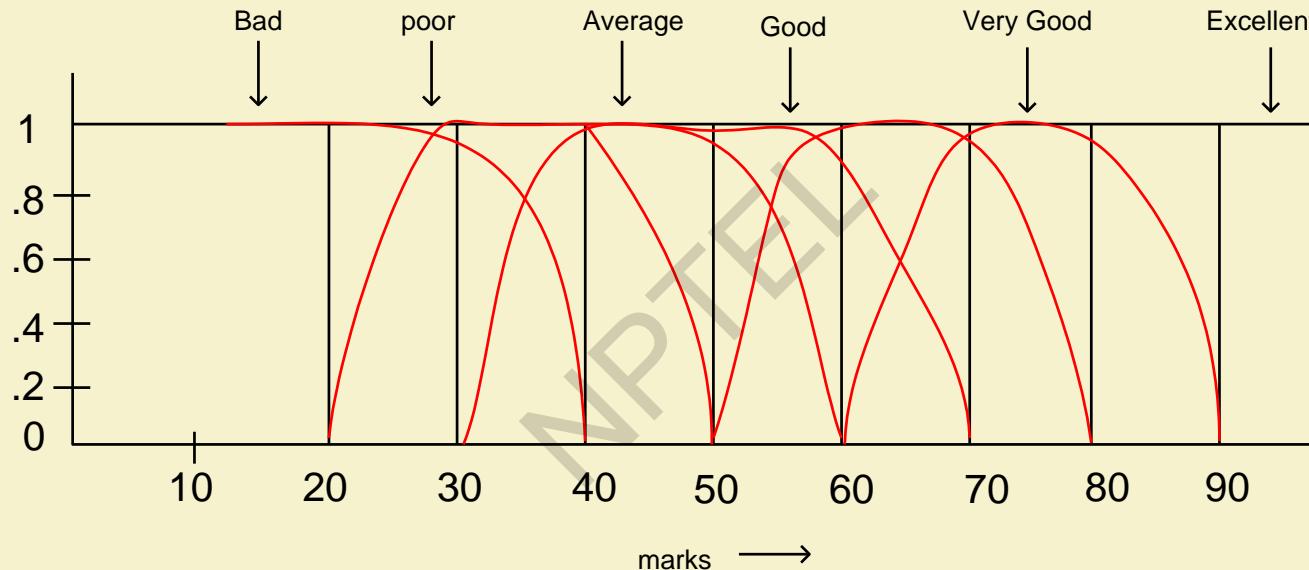
Average = $50 \leq \text{Marks} \leq 60$

Poor = $35 \leq \text{Marks} \leq 50$

Bad= Marks ≤ 35

Grading System

A fuzzy implementation will look like the following.



You can decide a standard fuzzy MF for each of the [fuzzy grade](#).

Few More on Membership Functions



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Generation of MFs

Given a membership function of a fuzzy set representing a **linguistic hedge**, we can derive many more MFs representing several other linguistic hedges using the concept of **Concentration** and **Dilation**.

1. **Concentration:** $A^k = [\mu_A(x)]^k; k > 1$
2. **Dilation:** $A^k = [\mu_A(x)]^k; k < 1$

Example : Age = { Young, Middle-aged, Old }

Thus, corresponding to Young, we have : **Not young, Very young, Not very young** and so on.
Similarly, with Old we can have : **Not old, Very old, Very very old, Extremely old**, etc.

Thus, $\mu_{Extremely\ old}(x) = (((\mu_{Old}(x))^2)^2)^2$ and so on

Or, $\mu_{More\ or\ less\ old}(x) = A^{0.5} = (\mu_{Old}(x))^{0.5}$

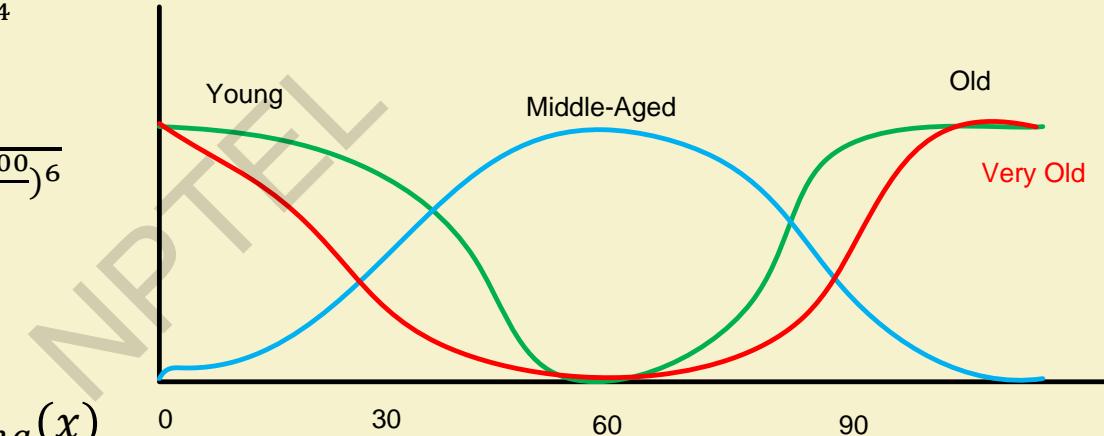
Linguistic variables and values

$$\mu_{young}(x) = \text{bell}(x, 20, 2, 0) = \frac{1}{1 + (\frac{x}{20})^4}$$

$$\mu_{old}(x) = \text{bell}(x, 30, 3, 100) = \frac{1}{1 + (\frac{x-100}{30})^6}$$

$$\mu_{middle-aged}(x) = \text{bell}(x, 30, 60, 50)$$

$$\text{Not young} = \overline{\mu_{young}(x)} = 1 - \mu_{young}(x)$$



$$\text{Young but not too young} = \mu_{young}(x) \cap \overline{\mu_{young}(x)}$$

Thank You !!

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Introduction to Soft Computing

Operations on Fuzzy sets

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Basic fuzzy set operations: Union

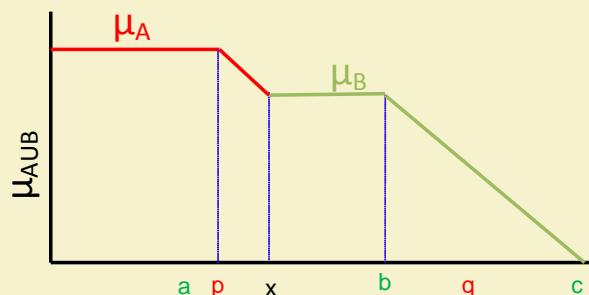
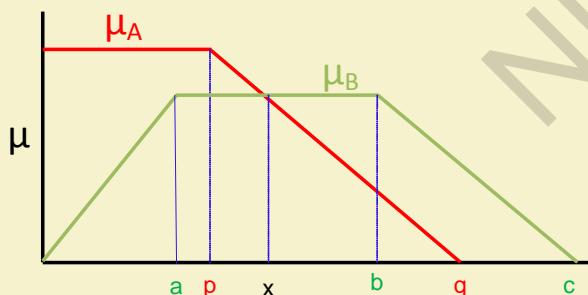
Union ($A \cup B$): $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$

Example:

$$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\} \text{ and}$$

$$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\};$$

$$C = A \cup B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.5)\}$$



Basic fuzzy set operations: Intersection

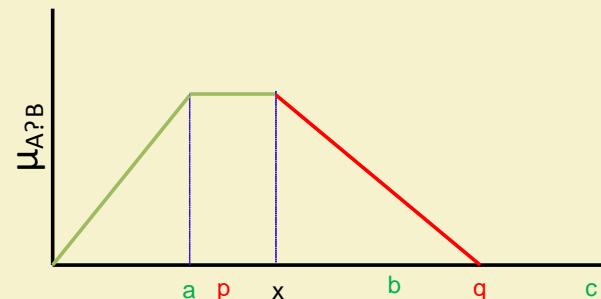
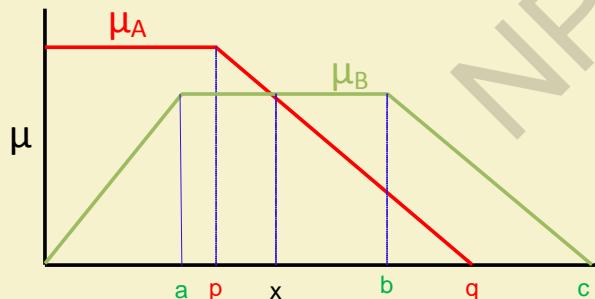
Intersection ($A \cap B$): $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$

Example:

$$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\} \text{ and}$$

$$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\};$$

$$C = A \cap B = \{(x_1, 0.2), (x_2, 0.1), (x_3, 0.4)\}$$



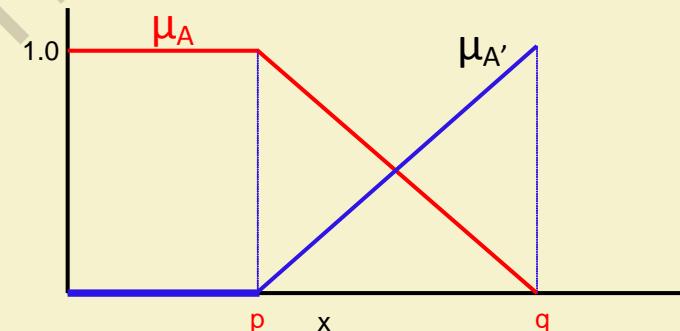
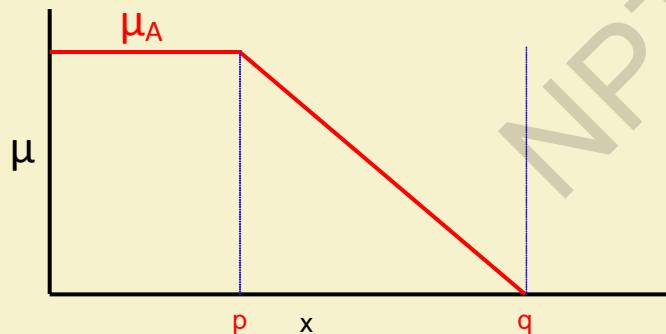
Basic fuzzy set operations: Complement

Complement (A^c): $\mu_{A^c}(x) = 1 - \mu_A(x)$

Example:

$$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$$

$$C = A^c = \{(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)\}$$



Basic fuzzy set operations: Products

Algebraic product or Vector product ($A \cdot B$):

$$\mu_{A \cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$$

Scalar product ($\alpha \times A$):

$$\mu_{\alpha A}(x) = \alpha \times \mu_A(x)$$

Basic fuzzy set operations: Sum and Difference

Sum ($A + B$):

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$$

Difference ($A - B = A \cap B^C$):

$$\mu_{A-B}(x) = \mu_{A \cap B^C}(x)$$

Disjunctive sum:

$$A \oplus B = (A^C \cap B) \cup (A \cap B^C)$$

Bounded Sum:

$$|A(x) \oplus B(x)| = \mu_{|A(x) \oplus B(x)|} = \min\{1, \mu_A(x) + \mu_B(x)\}$$

Bounded Difference:

$$|A(x) \ominus B(x)| = \mu_{|A(x) \ominus B(x)|} = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$$

Basic fuzzy set operations: Equality and Power

Equality ($A = B$):

$$\mu_A(x) = \mu_B(x)$$

Power of a fuzzy set A^α :

$$\mu_{A^\alpha}(x) = (\mu_A(x))^\alpha$$

- ✓ If $\alpha < 1$, then it is called **dilation**
- ✓ If $\alpha > 1$, then it is called **concentration**

Basic fuzzy set operations: Cartesian product

Caretian Product ($A \times B$): $\mu_{A \times B}(x, y) = \min(\mu_A(x), \mu_B(y))$

Example:

$$A(x) = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5), (x_4, 0.6)\}$$

$$B(y) = \{(y_1, 0.8), (y_2, 0.6), (y_3, 0.3)\}$$

$$A \times B = \min(\mu_A(x), \mu_B(y)) = \begin{matrix} & y_1 & y_2 & y_3 \\ x_1 & 0.2 & 0.2 & 0.2 \\ x_2 & 0.3 & 0.3 & 0.3 \\ x_3 & 0.5 & 0.5 & 0.3 \\ x_4 & 0.6 & 0.6 & 0.3 \end{matrix}$$

Properties of fuzzy sets

Commutativity :

$$A \cap B = B \cap A$$

$$A \cup B = B \cup A$$

Associativity :

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributivity :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$



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Properties of fuzzy sets

Idempotence :

$$A \cup A = A$$

$$A \cap A = \emptyset;$$

$$A \cup \emptyset; = A$$

$$A \cap \emptyset; = \emptyset;$$

Transitivity :

$$\text{If } A \subseteq B; B \subseteq C \text{ then } A \subseteq C$$

Involution :

$$(A^c)^c = A$$

De Morgan's law :

$$(A \cap B)^c = A^c \cup B^c$$

$$(A \cup B)^c = A^c \cap B^c$$



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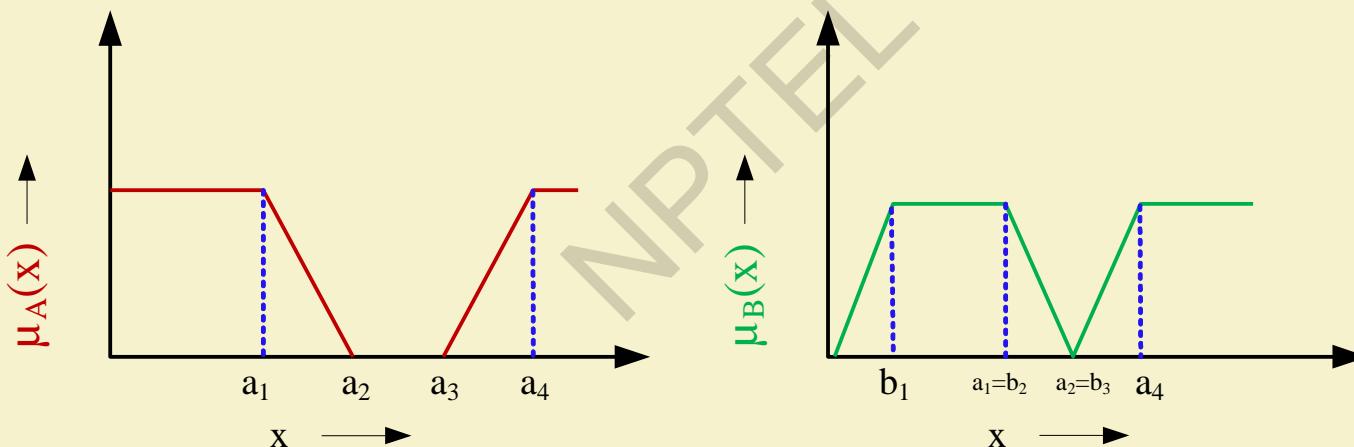


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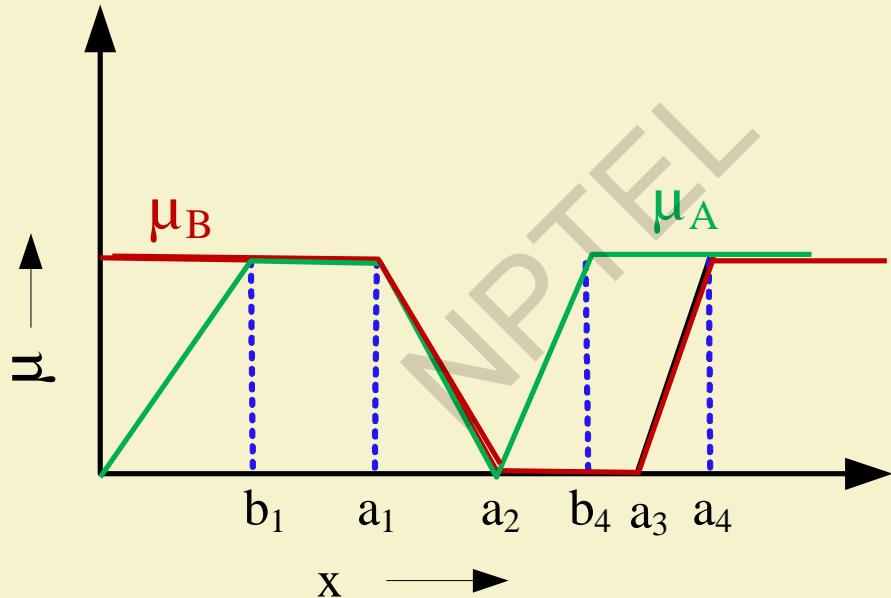
Example 1: Fuzzy Set Operations

Let A and B are two fuzzy sets defined over a universe of discourse X with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively. Two MFs $\mu_A(x)$ and $\mu_B(x)$ are shown graphically.



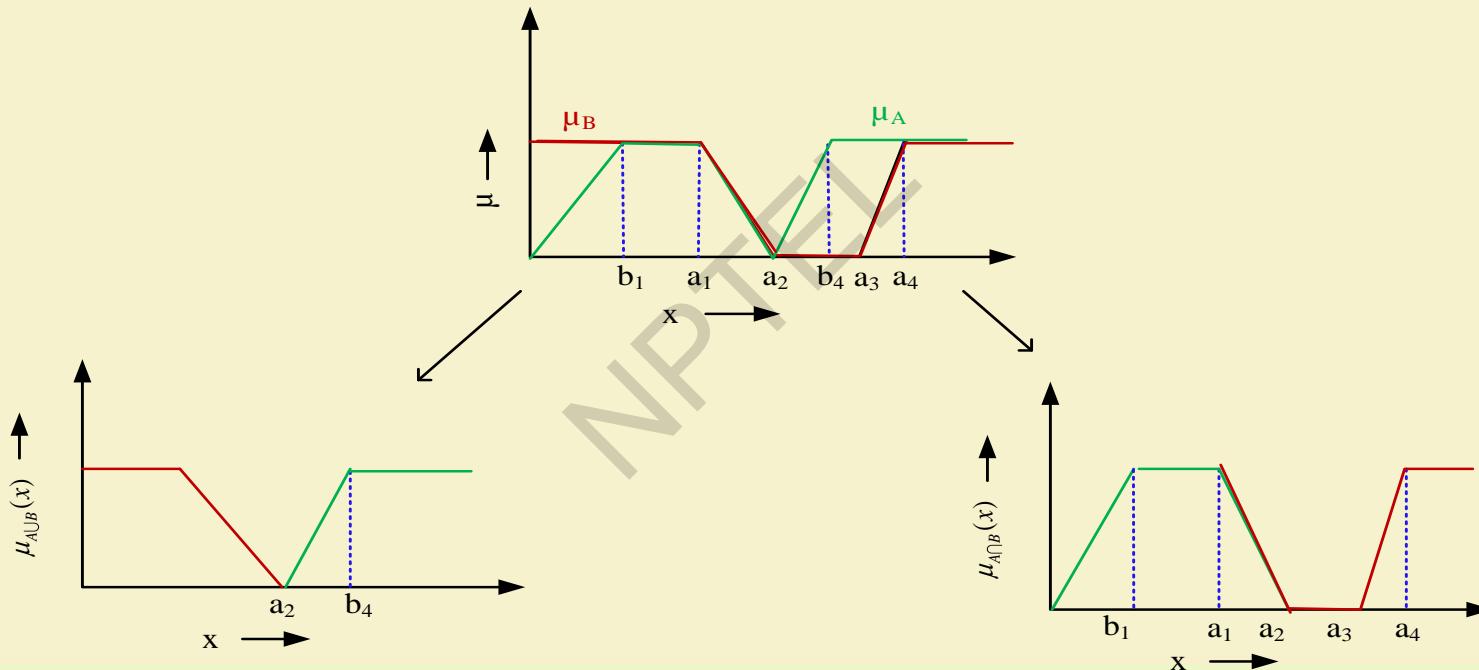
Example 1: Plotting two sets on the same graph

Let's plot the two membership functions on the same graph



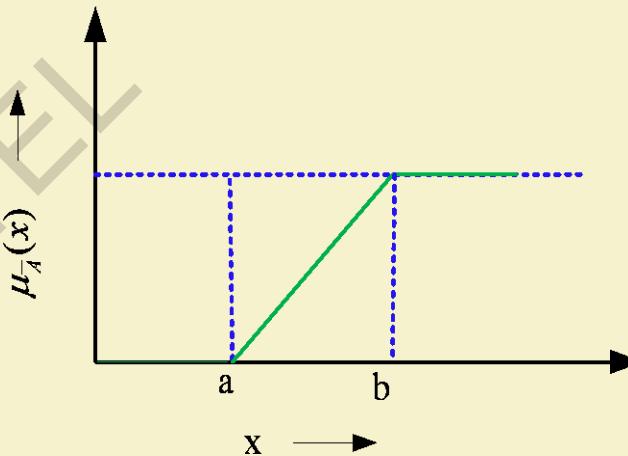
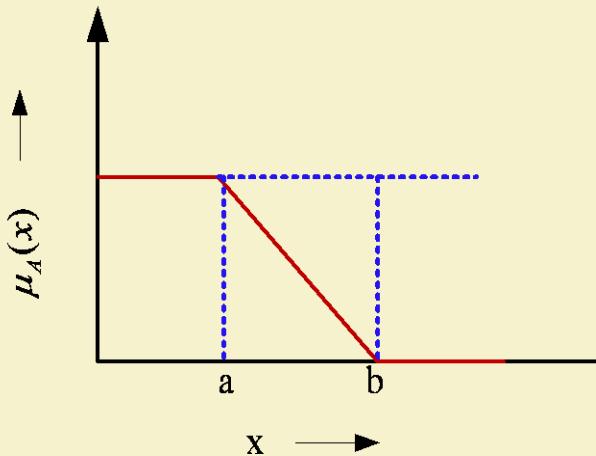
Example 1: Union and Intersection

The plots of union $A \cup B$ and intersection $A \cap B$ are shown in the following.



Example 1: Complementation

The plots of union $\mu_{\bar{A}}(x)$ of the fuzzy set A is shown in the following.



Fuzzy set operations: Practice

Consider the following two fuzzy sets A and B defined over a universe of discourse [0,5] of real numbers with their membership functions

$$\mu_A(x) = \frac{x}{1+x} \text{ and } \mu_B(x) = 2^{-x}$$

Determine the membership functions of the following and draw them graphically.

- I. \bar{A}, \bar{B}
- II. $A \cup B$
- III. $A \cap B$
- IV. $(A \cup B)^c$

[Hint: Use De' Morgan law]

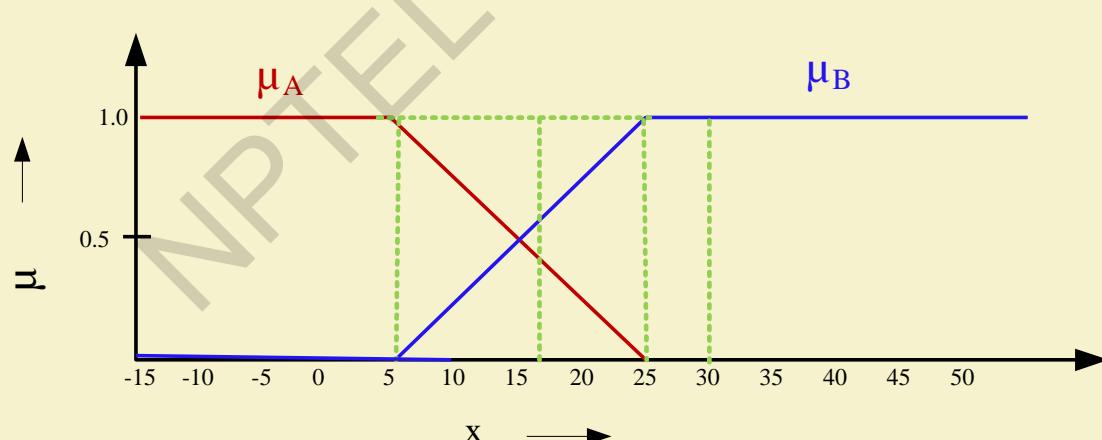
Example 2: A real-life example

Two fuzzy sets A and B with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively defined as below.

A = **Cold climate** with $\mu_A(x)$ as the MF.

B = **Hot climate** with $\mu_B(x)$ as the M.F.

Here, X being the universe of discourse representing entire range of temperatures.



Example 2: A real-life example

What are the fuzzy sets representing the following?

- 1. Not cold climate**
- 2. Not hot climate**
- 3. Extreme climate**
- 4. Pleasant climate**

Note: Note that "Not cold climate" \neq "Hot climate" and vice-versa.

Example 2: A real-life example

Answer would be the following.

- ✓ **Not cold climate**

\bar{A} with $1 - \mu_A(x)$ as the MF.

- ✓ **Not hot climate**

\bar{B} with $1 - \mu_B(x)$ as the MF.

- ✓ **Extreme climate**

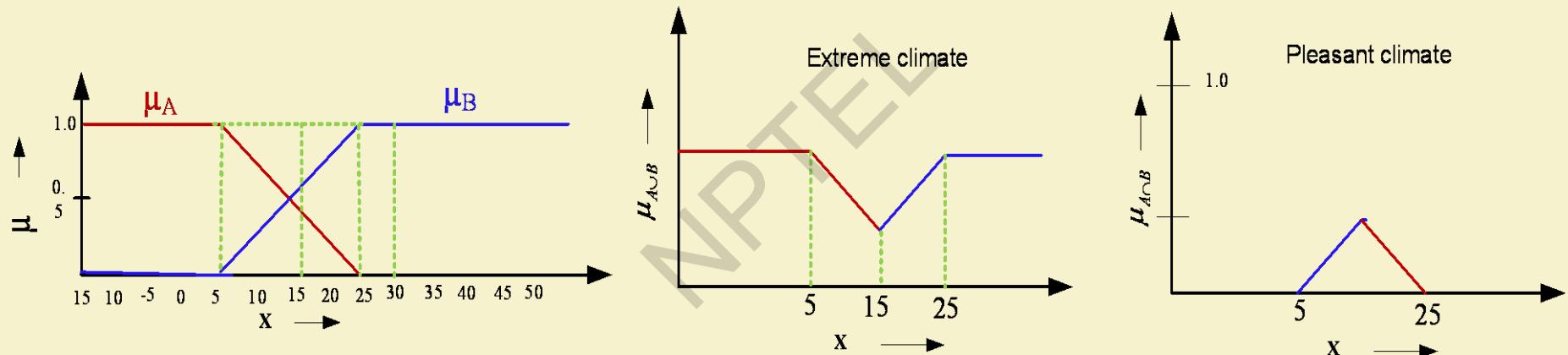
$A \cup B$ with $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$ as the MF.

- ✓ **Pleasant climate**

$A \cap B$ with $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$ as the MF.

Example 2: A real-life example

The plot of the MFs of $A \cup B$ and $A \cap B$ are shown in the following.



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Soft Computing

Fuzzy Relations

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Fuzzy Relations

- Crisp relations
- Operations on crisp relations
- Examples on crisp relations
- Fuzzy relations
- Operations on fuzzy relations
- Examples on fuzzy relations

Crisp relations

- **Order pairs:**

Suppose, A and B are two (crisp) sets. Then Cartesian product denoted as $A \times B$ is a collection of order pairs, such that

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

Note :

(1) $A \times B \neq B \times A$

(2) $|A \times B| = |A| \times |B|$

(3) $A \times B$ provides a mapping from $a \in A$ to $b \in B$.

A particular mapping so mentioned is called a **relation**.

Crisp relations

Example:

Consider the two crisp sets A and B as given below.

$$A = \{1, 2, 3, 4\} \quad B = \{3, 5, 7\}.$$

Then, $A \times B = \{(1, 3), (1, 5), (1, 7), (2, 3), (2, 5), (2, 7), (3, 3), (3, 5), (3, 7), (4, 3), (4, 5), (4, 7)\}$

Let us define a relation as $R = \{(a, b) | b = a + 1, (a, b) \in A \times B\}$

Then, $R = \{(2, 3), (4, 5)\}$ in this case.

Crisp relations

We can represent the relation R in a matrix form as follows.

$$R = \begin{matrix} & & 3 & 5 & 7 \\ 1 & \left[\begin{matrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \right] \\ 2 \\ 3 \\ 4 \end{matrix}$$

Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations defined over two crisp sets $x \in A$ and $y \in B$

- **Union:** $R(x, y) \cup S(x, y) = \max(R(x, y), S(x, y));$
- **Intersection:** $R(x, y) \cap S(x, y) = \min(R(x, y), S(x, y));$
- **Complement:** $\overline{R(x, y)} = 1 - R(x, y)$

Example: Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations defined over two crisp sets $x \in A$ and $y \in B$

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find the following

- $R \cup S$
- $R \cap S$
- \bar{R}

Composition of two crisp relations

Given R is a relation on X, Y and S is another relation on Y, Z . Then, $R \circ S$ is called a composition of relation on X and Z which is defined as follows.

$$R \circ S = \{(x, z) | (x, y) \in R \text{ and } (y, z) \in S \text{ and } \forall y \in Y\}$$

Max-Min Composition

Given the two relation matrices R and S , the **max-min composition** is defined as
 $T = R \circ S$;

$$T(x, z) = \max\{\min\{R(x, y), S(y, z) \text{ and } \forall y \in Y\}\}$$

Composition: Composition

Example : Given $X = \{1, 3, 5\}$; $Y = \{1, 3, 5\}$; $R = \{(x, y) | y = x + 2\}$;
 $S = \{(x, y) | x < y\}$

Here, R and S is on $X \times Y$.

Thus, we have $R = \{(1, 3), (3, 5)\}$, $S = \{(1, 3), (1, 5), (3, 5)\}$

$$R = \begin{matrix} & \begin{matrix} 1 & 3 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 3 \\ 5 \end{matrix} & \left[\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} \right] \end{matrix} \quad \text{and} \quad S = \begin{matrix} & \begin{matrix} 1 & 3 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 3 \\ 5 \end{matrix} & \left[\begin{matrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{matrix} \right] \end{matrix}$$

Using max-min composition

$$R \circ S = \begin{matrix} & \begin{matrix} 1 & 3 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 3 \\ 5 \end{matrix} & \left[\begin{matrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \right] \end{matrix}$$

Fuzzy relations

- Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp set X_1, X_2, \dots, X_n
- Here, n-tuples (x_1, x_2, \dots, x_n) may have varying degree of memberships within the relationship.
- The membership values indicate the strength of the relation between the tuples.

Fuzzy relations

Example:

$$X = \{ \text{typhoid}, \text{viral}, \text{cold} \}, Y = \{ \text{running nose}, \text{high temp}, \text{shivering} \}$$

The fuzzy relation R is defined as

	<i>running nose</i>	<i>high temperature</i>	<i>shivering</i>
<i>typhoid</i>	0.1	0.9	0.8
<i>viral</i>	0.2	0.9	0.7
<i>cold</i>	0.9	0.4	0.6

Fuzzy Cartesian product

Suppose

- A is a fuzzy set on the universe of discourse X with $\mu_A(x)|x \in X$
- B is a fuzzy set on the universe of discourse Y with $\mu_B(y)|y \in Y$

Then $R = A \times B \subset X \times Y$; where R has its membership function given

by $\mu_R(x, y) = \mu_{AxB}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$

Fuzzy Cartesian product

Example :

$$A = \{(a_1, 0.2), (a_2, 0.7), (a_3, 0.4)\} \text{ and } B = \{(b_1, 0.5), (b_2, 0.6)\}$$

$$R = A \times B = \begin{matrix} & b_1 & b_2 \\ a_1 & [0.2 & 0.2] \\ a_2 & [0.5 & 0.6] \\ a_3 & [0.4 & 0.4] \end{matrix}$$

Operations on Fuzzy relations

Let R and S be two fuzzy relations on $A \times B$.

- **Union:** $\mu_{R \cup S}(a, b) = \max\{\mu_R(a, b), \mu_S(a, b)\}$
- **Intersection:** $\mu_{R \cap S}(a, b) = \min\{\mu_R(a, b), \mu_S(a, b)\}$
- **Complement:** $\mu_{\bar{R}}(a, b) = 1 - \mu_R(a, b)$
- **Composition:** $T = R \circ S$

$$\mu_{R \circ S} = \max_{y \in Y} \{\min(\mu_R(x, y), \mu_S(y, z))\}$$

Operations on Fuzzy relations: Example

Example : $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2)$, $Z = (z_1, z_2, z_3)$,

$$R = \begin{matrix} & y_1 & y_2 \\ x_1 & 0.5 & 0.1 \\ x_2 & 0.2 & 0.9 \\ x_3 & 0.8 & 0.6 \end{matrix}$$

and $S = \begin{matrix} & z_1 & z_2 & z_3 \\ y_1 & 0.6 & 0.4 & 0.7 \\ y_2 & 0.5 & 0.8 & 0.9 \end{matrix}$

$$R \circ S = \begin{matrix} & z_1 & z_2 & z_3 \\ x_1 & 0.5 & 0.4 & 0.5 \\ x_2 & 0.5 & 0.8 & 0.9 \\ x_3 & 0.6 & 0.6 & 0.7 \end{matrix}$$

$$\begin{aligned}\mu_{R \circ S}(x_1, y_1) &= \max\{\min(\mu_R(x_1, y_1), \mu_S(y_1, z_1)), \min(\mu_R(x_1, y_2), \mu_S(y_2, z_1))\} \\ &= \max\{\min(0.5, 0.6), \min(0.1, 0.5)\} = \max\{0.5, 0.1\} = 0.5 \text{ and so on.}\end{aligned}$$

Fuzzy relation : An example

Consider the following two sets P and D , which represent a set of paddy plants and a set of plant diseases. More precisely

$P = \{P_1, P_2, P_3, P_4\}$ a set of four varieties of paddy plants

$D = \{D_1, D_2, D_3, D_4\}$ of the four various diseases affecting the plants.

In addition to these, also consider another set $S = \{S_1, S_2, S_3, S_4\}$ be the common symptoms of the diseases.

Let, R be a relation on $P \times D$, representing which plant is susceptible to which diseases, which is stated as

$$R = \begin{matrix} & \begin{matrix} D_1 & D_2 & D_3 & D_4 \end{matrix} \\ \begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{matrix} & \begin{bmatrix} 0.6 & 0.6 & 0.9 & 0.8 \\ 0.1 & 0.2 & 0.9 & 0.8 \\ 0.9 & 0.3 & 0.4 & 0.8 \\ 0.9 & 0.8 & 0.4 & 0.2 \end{bmatrix} \end{matrix}$$

Fuzzy relation : An example

Also, consider T be the another relation on $D \times S$, which is given by

$$S = \begin{bmatrix} S_1 & S_2 & S_3 & S_4 \\ D_1 & [0.1 & 0.2 & 0.7 & 0.9] \\ D_2 & [1.0 & 1.0 & 1.4 & 0.6] \\ D_3 & [0.0 & 0.0 & 0.5 & 0.9] \\ D_4 & [0.9 & 1.0 & 0.8 & 0.2] \end{bmatrix}$$

Obtain the association of plants with the different symptoms of the disease using **max-min composition**.

Hint: Find $R \circ T$, and verify that

$$R \circ S = \begin{bmatrix} P_1 & S_1 & S_2 & S_3 & S_4 \\ P_2 & [0.8 & 0.8 & 0.8 & 0.9] \\ P_3 & [0.8 & 0.8 & 0.8 & 0.9] \\ P_4 & [0.8 & 0.8 & 0.7 & 0.9] \end{bmatrix}$$

Fuzzy relation : Another example

Let, $R = x$ is relevant to y

and $S = y$ is relevant to z

be two fuzzy relations defined on $X \times Y$ and $Y \times Z$, respectively, where $X = \{1,2,3\}$, $Y = \{\alpha, \beta, \gamma, \delta\}$ and $Z = \{a, b\}$. Assume that R and S can be expressed with the following relation matrices :

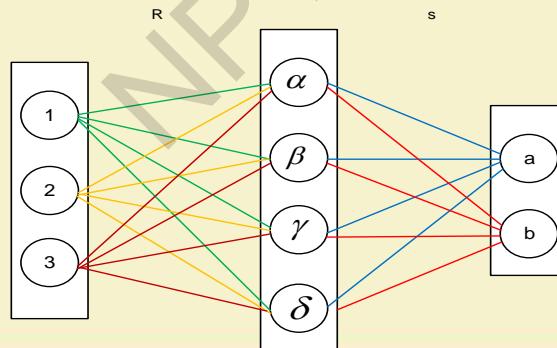
$$R = \begin{matrix} & \alpha & \beta & \gamma & \delta \\ 1 & [0.1 & 0.3 & 0.5 & 0.7] \\ 2 & [0.4 & 0.2 & 0.8 & 0.9] \\ 3 & [0.6 & 0.8 & 0.3 & 0.2] \end{matrix} \text{ and } S = \begin{matrix} & a & b \\ \alpha & [0.9 & 0.1] \\ \beta & [0.2 & 0.3] \\ \gamma & [0.5 & 0.6] \\ \delta & [0.7 & 0.2] \end{matrix}$$

Fuzzy relation : Another example

Now, we want to find $R \circ S$, which can be interpreted as a derived fuzzy relation **x is relevant to z** .

Suppose, we are only interested in the degree of relevance between $2 \in X$ and $a \in Z$. Then, using max-min composition,

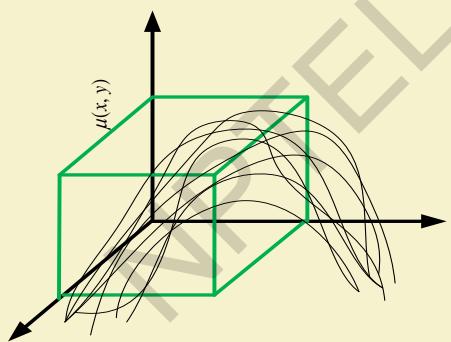
$$\begin{aligned}\mu_{R \circ S}(2, a) &= \max\{(0.4 \wedge 0.9), (0.2 \wedge 0.2), (0.8 \wedge 0.5), (0.9 \wedge 0.7)\} \\ &= \max\{0.4, 0.2, 0.5, 0.7\} = 0.7\end{aligned}$$



2D Membership functions : Binary fuzzy relations

(Binary) fuzzy relations are fuzzy sets $A \times B$ which map each element in $A \times B$ to a membership grade between 0 and 1 (both inclusive).

Note that a membership function of a binary fuzzy relation can be depicted with a 3D plot.



Important: Binary fuzzy relations are fuzzy sets with two dimensional MFs and so on.

2D membership function : An example

Let, $X = R^+ = y$ (the positive real line) and

$R = X \times Y = "y \text{ is much greater than } x"$

The membership function of $\mu_R(x, y)$ is defined as

$$\mu_R(x, y) = \begin{cases} \frac{(y - x)}{4} & \text{if } y > x \\ 0 & \text{if } y \leq x \end{cases}$$

Suppose, $X = \{3, 4, 5\}$ and $Y = \{3, 4, 5, 6, 7\}$, then

$$R = \begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 3 & 0 & 0.25 & 0.5 & 0.75 & 1.0 \\ 4 & 0 & 0 & 0.25 & 0.5 & 0.75 \\ 5 & 0 & 0 & 0 & 0.25 & 0.5 \end{bmatrix}$$

Example:

How you can derive the following?

If x is A or y is B then z is C ;

Given that

- R1: If x is A then z is C $[R1 \in A \times C]$
- R2: If y is B then z is C $[R2 \in B \times C]$

Hint:

- ✓ You have given two relations R1 and R2.
- ✓ Then, the required can be derived using the **union operation** of R1 and R2

Thank You !!

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