Byte Substitution Layer

In the field $GF(2^8)$,

which is used in AES, each element $A \in GF(2^8)$ is thus represented as:

$$A(x) = a_7x^7 + \cdots + a_1x + a_0, \quad a_i \in GF(2) = \{0,1\}.$$

Note that there are exactly $256 = 2^8$ such polynomials. The set of these 256 polynomials is the finite field $GF(2^8)$. It is also important to observe that every polynomial can simply be stored in digital form as an 8-bit vector

$$A = (a_7, a_6, a_5, a_4, a_3, a_2, a_1, a_0).$$

Definition 4.3.4 Extension field multiplication
Let $A(x), B(x) \in GF(2^m)$ *and let*

$$P(x) \equiv \sum_{i=0}^{m} p_i x^i, \quad p_i \in GF(2)$$

be an irreducible polynomial. Multiplication of the two elements $A(x), B(x)$ *is performed as*

$$C(x) \equiv A(x) \cdot B(x) \bmod P(x).$$

For AES, the irreducible polynomial

$$P(x) = x^8 + x^4 + x^3 + x + 1$$

is used. It is part of the AES specification.

Inversion in $GF(2^8)$ is the core operation of the Byte Substitution transformation, which contains the AES S-Boxes. For a given finite field $GF(2^m)$ and the corresponding irreducible reduction polynomial $P(x)$, the inverse A^{-1} of a nonzero element $A \in GF(2^m)$ is defined as:

$$A^{-1}(x) \cdot A(x) = 1 \bmod P(x).$$

Table 4.2 Multiplicative inverse table in $GF(2^8)$ for bytes xy used within the AES S-Box

X \ Y	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	00	01	8D	F6	CB	52	7B	D1	E8	4F	29	C0	B0	E1	E5	C7
1	74	B4	AA	4B	99	2B	60	5F	58	3F	FD	CC	FF	40	EE	B2
2	3A	6E	5A	F1	55	4D	A8	C9	C1	0A	98	15	30	44	A2	C2
3	2C	45	92	6C	F3	39	66	42	F2	35	20	6F	77	BB	59	19
4	1D	FE	37	67	2D	31	F5	69	A7	64	AB	13	54	25	E9	09
5	ED	5C	05	CA	4C	24	87	BF	18	3E	22	F0	51	EC	61	17
6	16	5E	AF	D3	49	A6	36	43	F4	47	91	DF	33	93	21	3B
7	79	B7	97	85	10	B5	BA	3C	B6	70	D0	06	A1	FA	81	82
8	83	7E	7F	80	96	73	BE	56	9B	9E	95	D9	F7	02	B9	A4
9	DE	6A	32	6D	D8	8A	84	72	2A	14	9F	88	F9	DC	89	9A
A	FB	7C	2E	C3	8F	B8	65	48	26	C8	12	4A	CE	E7	D2	62
B	0C	E0	1F	EF	11	75	78	71	A5	8E	76	3D	BD	BC	86	57
C	0B	28	2F	A3	DA	D4	E4	0F	A9	27	53	04	1B	FC	AC	E6
D	7A	07	AE	63	C5	DB	E2	EA	94	8B	C4	D5	9D	F8	90	6B
E	B1	0D	D6	EB	C6	0E	CF	AD	08	4E	D7	E3	5D	50	1E	B3
F	5B	23	38	34	68	46	03	8C	DD	9C	7D	A0	CD	1A	41	1C

Example 4.7. From Table 4.2 the inverse of

$$x^7 + x^6 + x = (1100\,0010)_2 = (C2)_{hex} = (xy)$$

is given by the element in row C, column 2:

$$(2F)_{hex} = (0010\,1111)_2 = x^5 + x^3 + x^2 + x + 1.$$

This can be verified by multiplication:

$$(x^7 + x^6 + x) \cdot (x^5 + x^3 + x^2 + x + 1) \equiv 1 \mod P(x).$$

4.4.1 Byte Substitution Layer

A_0	A_4	A_8	A_{12}
A_1	A_5	A_9	A_{13}
A_2	A_6	A_{10}	A_{14}
A_3	A_7	A_{11}	A_{15}

$$S(A_i) = B_i$$

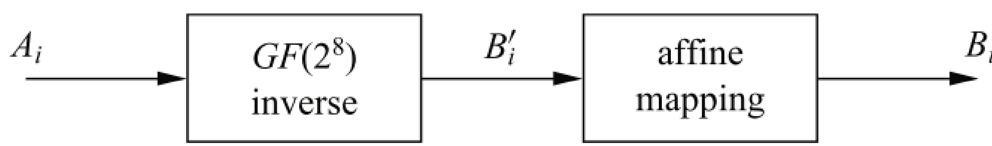

Fig. 4.4 The two operations within the AES S-Box which computes the function $B_i = S(A_i)$

$$\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \\ b_7 \end{pmatrix} \equiv \begin{pmatrix} 1&0&0&0&1&1&1&1 \\ 1&1&0&0&0&1&1&1 \\ 1&1&1&0&0&0&1&1 \\ 1&1&1&1&0&0&0&1 \\ 1&1&1&1&1&0&0&0 \\ 0&1&1&1&1&1&0&0 \\ 0&0&1&1&1&1&1&0 \\ 0&0&0&1&1&1&1&1 \end{pmatrix} \begin{pmatrix} b'_0 \\ b'_1 \\ b'_2 \\ b'_3 \\ b'_4 \\ b'_5 \\ b'_6 \\ b'_7 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \bmod 2.$$

Note that $B' = (b'_7, \ldots, b'_0)$ is the bitwise vector representation of $B'_i(x) = A_i^{-1}(x)$. This second step is referred to as *affine mapping*. Let's look at an example of how the S-Box computations work.

Table 4.3 AES S-Box: Substitution values in hexadecimal notation for input byte (xy)

									y								
		0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
	0	63	7C	77	7B	F2	6B	6F	C5	30	01	67	2B	FE	D7	AB	76
	1	CA	82	C9	7D	FA	59	47	F0	AD	D4	A2	AF	9C	A4	72	C0
	2	B7	FD	93	26	36	3F	F7	CC	34	A5	E5	F1	71	D8	31	15
	3	04	C7	23	C3	18	96	05	9A	07	12	80	E2	EB	27	B2	75
	4	09	83	2C	1A	1B	6E	5A	A0	52	3B	D6	B3	29	E3	2F	84
	5	53	D1	00	ED	20	FC	B1	5B	6A	CB	BE	39	4A	4C	58	CF
	6	D0	EF	AA	FB	43	4D	33	85	45	F9	02	7F	50	3C	9F	A8
	7	51	A3	40	8F	92	9D	38	F5	BC	B6	DA	21	10	FF	F3	D2
x	8	CD	0C	13	EC	5F	97	44	17	C4	A7	7E	3D	64	5D	19	73
	9	60	81	4F	DC	22	2A	90	88	46	EE	B8	14	DE	5E	0B	DB
	A	E0	32	3A	0A	49	06	24	5C	C2	D3	AC	62	91	95	E4	79
	B	E7	C8	37	6D	8D	D5	4E	A9	6C	56	F4	EA	65	7A	AE	08
	C	BA	78	25	2E	1C	A6	B4	C6	E8	DD	74	1F	4B	BD	8B	8A
	D	70	3E	B5	66	48	03	F6	0E	61	35	57	B9	86	C1	1D	9E
	E	E1	F8	98	11	69	D9	8E	94	9B	1E	87	E9	CE	55	28	DF
	F	8C	A1	89	0D	BF	E6	42	68	41	99	2D	0F	B0	54	BB	16

Example 4.8. Let's assume the input byte to the S-Box is $A_i = (C2)_{hex}$, then the substituted value is

$$S((C2)_{hex}) = (25)_{hex}.$$

On a bit level — and remember, the only thing that is ultimate of interest in encryption is the manipulation of bits — this substitution can be described as:

$$S(11000010) = (00100101).$$

Example 4.10. We assume the S-Box input $A_i = (11000010)_2 = (C2)_{hex}$. From Table 4.2 we can see that the inverse is:

$$A_i^{-1} = B_i' = (2F)_{hex} = (00101111)_2.$$

We now apply the B_i' bit vector as input to the affine transformation. Note that the least significant bit (lsb) b_0' of B_i' is at the rightmost position.

$$B_i = (00100101) = (25)_{hex}$$

Thus, $S((C2)_{hex}) = (25)_{hex}$, which is exactly the result that is also given in the S-Box Table 4.3.