Understanding Cryptography

by Christof Paar and Jan Pelzl

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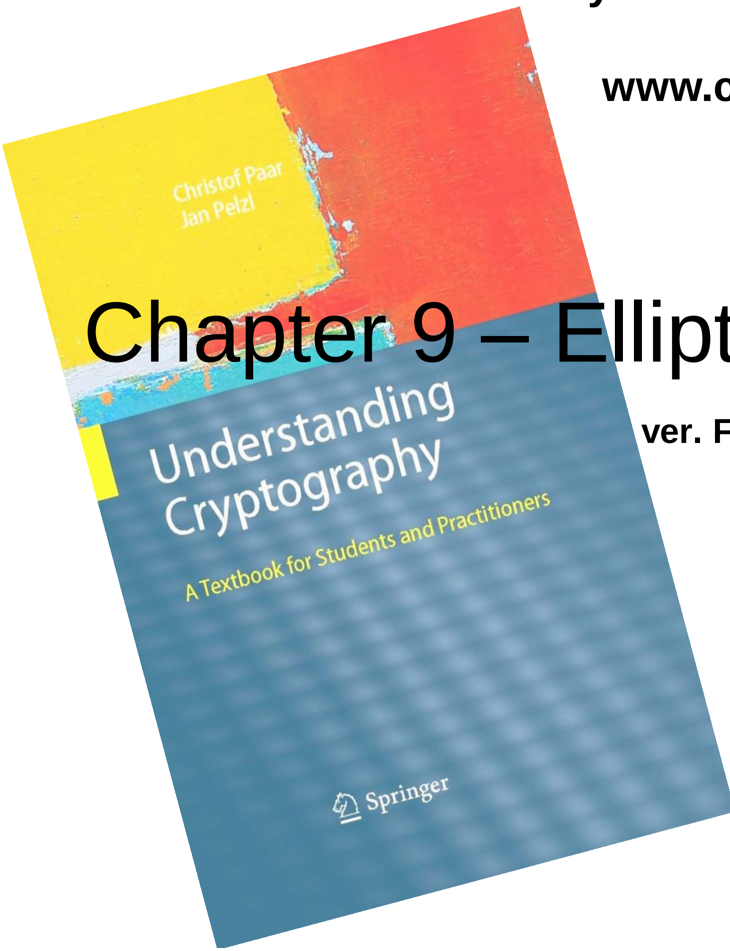

Chapter 9 – Elliptic Curve Cryptography

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These slides were prepared by Tim Güneysu, Christof Paar and Jan Pelzl

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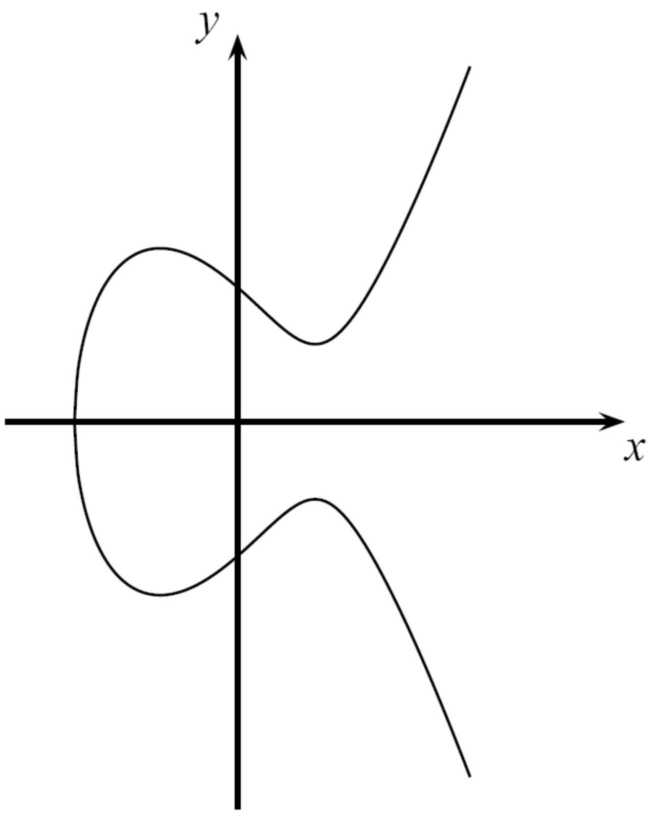
Content of this Chapter

- Introduction
- Computations on Elliptic Curves
- The Elliptic Curve Diffie-Hellman Protocol
- Security Aspects
- Implementation in Software and Hardware



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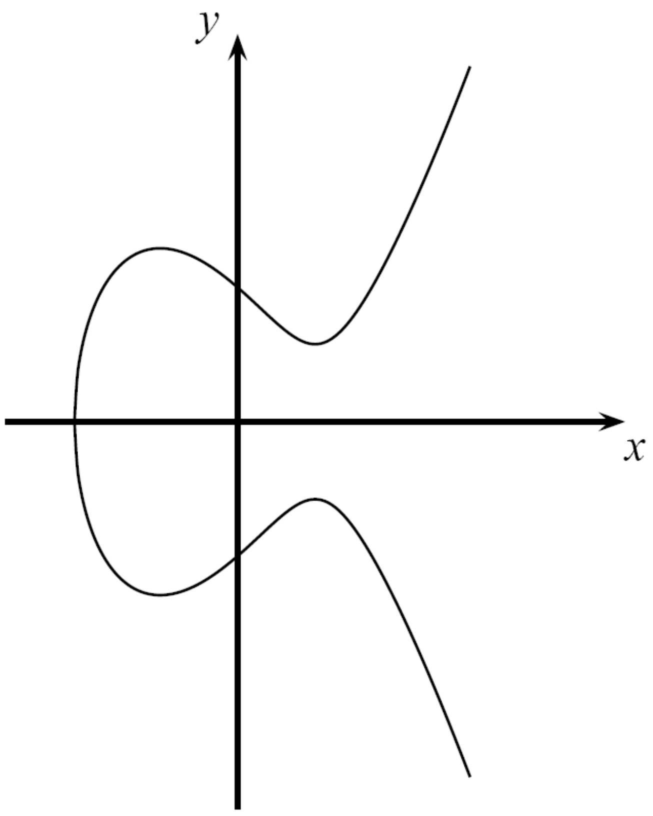

Motivation

- **Problem:**
  Asymmetric schemes like RSA and Elgamal require exponentiations in integer rings and fields with parameters of more than 1000 bits.
  - High computational effort on CPUs with 32-bit or 64-bit arithmetic
  - Large parameter sizes critical for storage on small and embedded
- **Motivation:**
  Smaller field sizes providing equivalent security are desirable
- **Solution:**
  Elliptic Curve Cryptography uses a group of points (instead of integers) for cryptographic schemes with coefficient sizes of 160-256 bits, reducing significantly the computational effort.

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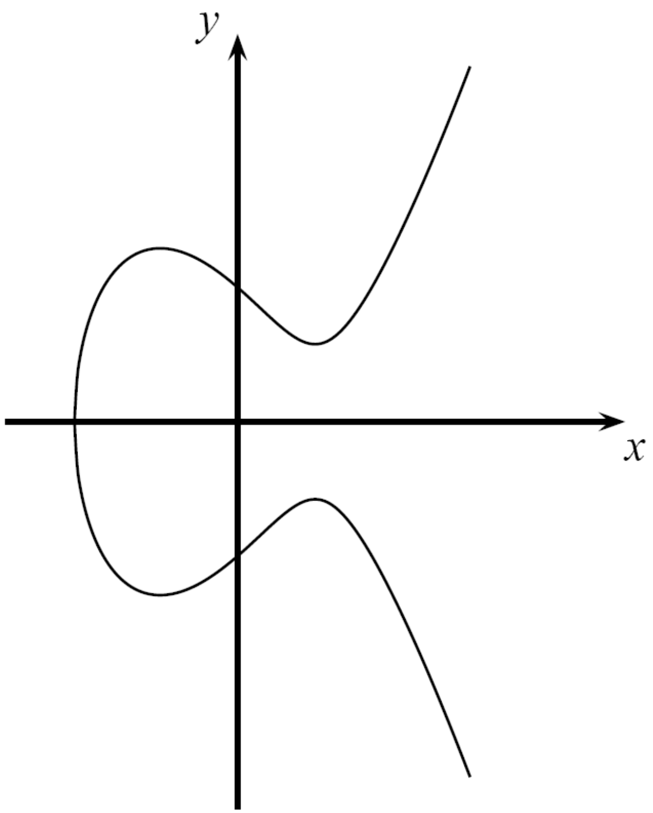


Computations on Elliptic Curves

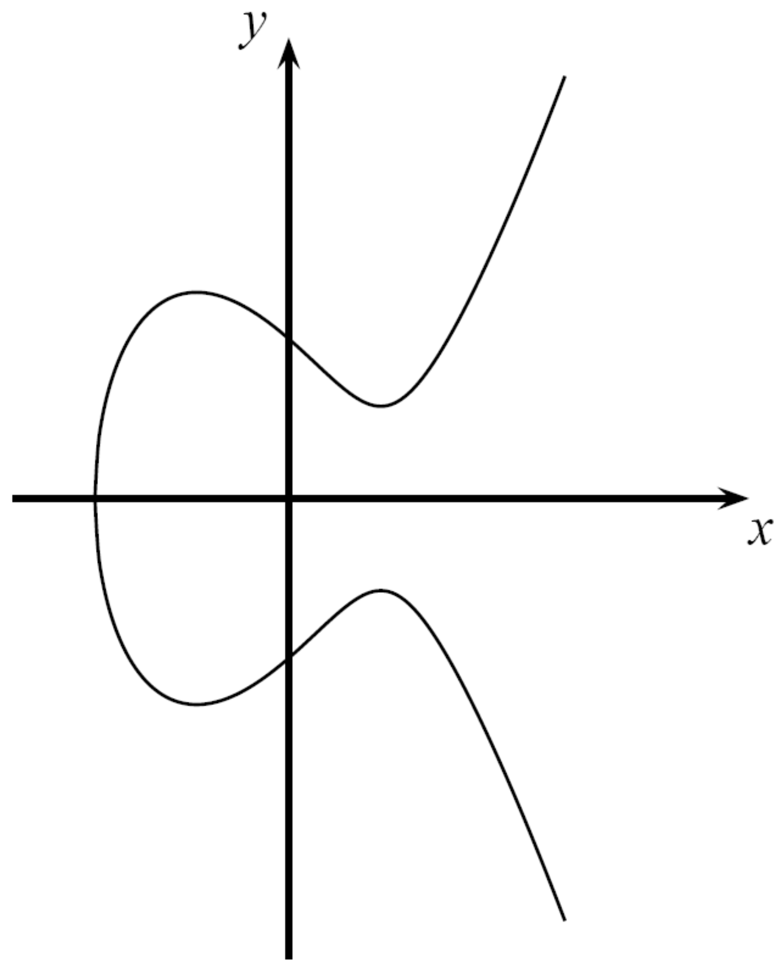
- Elliptic curves are polynomials that define points based on the (simplified) Weierstraß equation:

$$y^2 = x^3 + ax + b$$

for parameters a,b that specify the exact shape of the curve

- On the real numbers and with parameters a, b $\in$ R, an elliptic curve looks like this →

- Elliptic curves can not just be defined over the real numbers *R* but over many other types of finite fields.



Example: $y^2 = x^3 - 3x + 3$ over *R*

Computations on Elliptic Curves (ctd.)

- In cryptography, we are interested in elliptic curves module a prime *p*:

∈

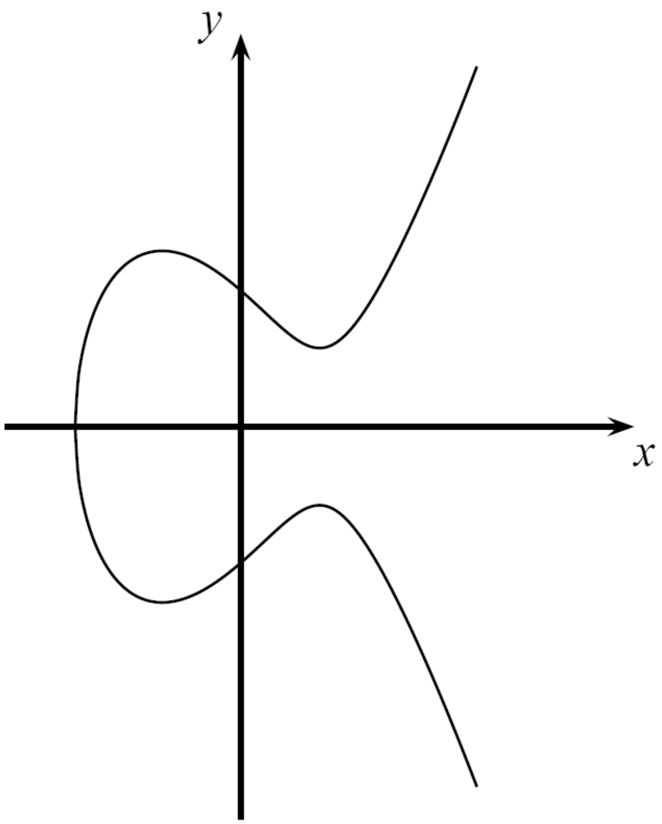
Definition: Elliptic Curves over prime fields

The elliptic curve over Z_p, $p>3$ is the set of all pairs $(x,y) \in Z_p$ which fulfill

$$\boldsymbol{y^2 = x^3 + ax + b \bmod p}$$

together with an imaginary point of infinity θ, where $a,b \in Z_p$ and the condition

$$\boldsymbol{4a^3+27b^2 \neq 0 \bmod p.}$$

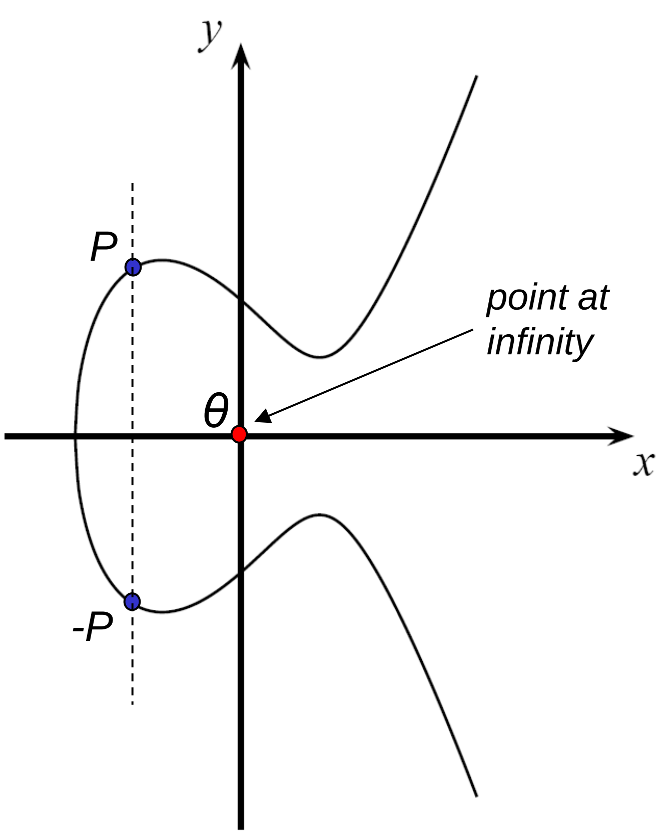


- Note that $Z_p = \{0,1,\ldots, p-1\}$ is a set of integers with modulo p arithmetic

Computations on Elliptic Curves (ctd.)

- Some special considerations are required to convert elliptic curves into a group of points
  - *In any group, a special element is required to allow for the identity operation, i.e., given $P \in E$: $P + \theta = P = \theta + P$*
  - *This identity point (which is not on the curve) is additionally added to the group definition*
  - *This (infinite) identity point is denoted by θ*

∈



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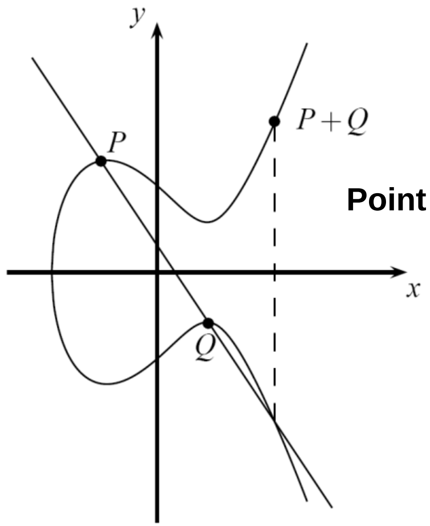
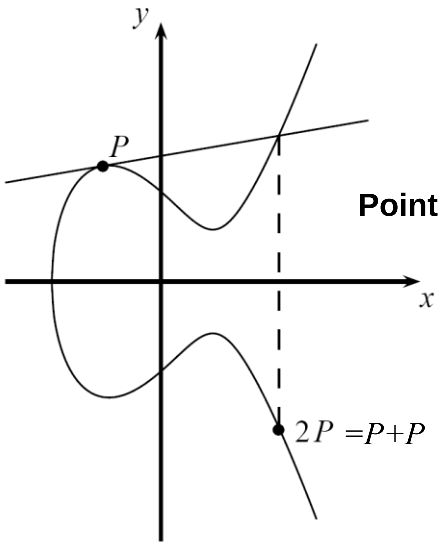
- Elliptic Curve are symmetric along the *x*-axis
  - Up to two solutions *y and -y* exist for each quadratic residue *x* of the elliptic curve
  - For each point $P = (x,y)$, the inverse or negative point is defined as $-P = (x,-y)$

Computations on Elliptic Curves (ctd.)

- Generating a *group of points* on elliptic curves based on point addition operation *P+Q = R, i.e.,* $(x_P,y_P)+(x_Q,y_Q) = (x_R,y_R)$
- Geometric Interpretation of point addition operation
  - *Draw straight line through P and Q; if P=Q use tangent line instead*
  - *Mirror third intersection point of drawn line with the elliptic curve along the x-axis*
- Elliptic Curve Point Addition and Doubling Formulas





$x_3 = s^2 - x_1 - x_2 \bmod p$ and $y_3 = s(x_1 - x_3) - y_1 \bmod p$

where

$$s = \begin{cases} \dfrac{y_2 - y_1}{x_2 - x_1} \bmod p & \text{; if P} \neq \text{Q (point addition)} \\ \dfrac{3x_1^2 + a}{2y_1} \bmod p & \text{; if P = Q (point doubling)} \end{cases}$$

Computations on Elliptic Curves (ctd.)

- **Example**: Given $E: y^2 = x^3+2x+2 \bmod 17$ and point $P=(5,1)$

  Goal: Compute $2P = P+P = (5,1)+(5,1)= (x_3,y_3)$

$$s = \frac{3x_1^2 + a}{2y_1} = (2 \cdot 1)^{-1}(3 \cdot 5^2 + 2) = 2^{-1} \cdot 9 \equiv 9 \cdot 9 \equiv 13 \bmod 17$$

$$x_3 = s^2 - x_1 - x_2 = 13^2 - 5 - 5 = 159 \equiv 6 \bmod 17$$

$$y_3 = s(x_1 - x_3) - y_1 = 13(5 - 6) - 1 = -14 \equiv 3 \bmod 17$$

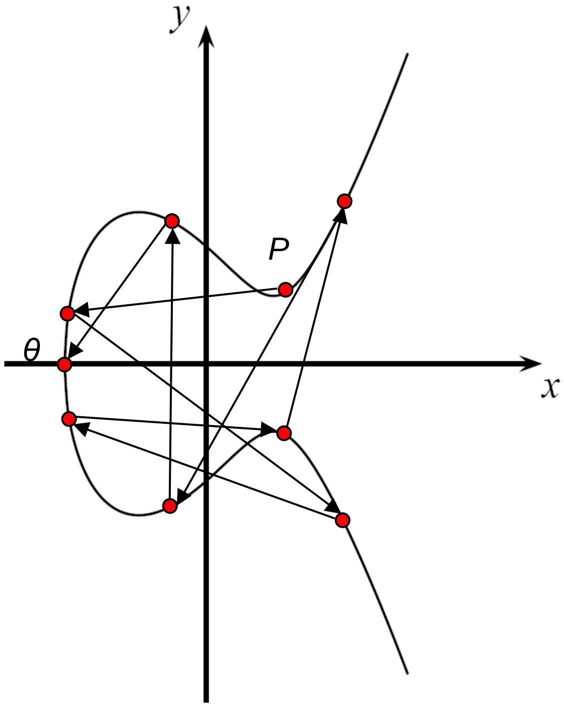
Finally $2P = (5,1) + (5,1) = (6,3)$

Computations on Elliptic Curves (ctd.)

- The points on an elliptic curve and the point at infinity θ form cyclic subgroups

$2P = (5,1)+(5,1) = (6,3)$	$11P = (13,10)$
$3P = 2P+P = (10,6)$	$12P = (0,11)$
$4P = (3,1)$	$13P = (16,4)$
$5P = (9,16)$	$14P = (9,1)$
$6P = (16,13)$	$15P = (3,16)$
$7P = (0,6)$	$16P = (10,11)$
$8P = (13,7)$	$17P = (6,14)$
$9P = (7,6)$	$18P = (5,16)$
$10P = (7,11)$	$19P = \theta$

This elliptic curve has order #E = |E| = 19 since it contains 19 points in its cyclic group.



Number of Points on an Elliptic Curve

- How many points can be on an arbitrary elliptic curve?
  - Consider previous example: *E:* $y^2 = x^3+2x+2 \bmod 17$ has 19 points
  - However, determining the point count on elliptic curves in general is hard
- But Hasse‘s theorem bounds the number of points to a restricted interval

> ***Definition: Hasse‘s Theorem:***
>
> *Given an elliptic curve module p, the number of points on the curve is denoted by #E and is bounded by*
>
> $$p+1-2\sqrt{p} \leq \#E \leq p+1+2\sqrt{p}$$

- **Interpretation:** The number of points is „close to“ the prime p
- **Example:** To generate a curve with about 2^{160} points, a prime with a length of about 160 bits is required

Elliptic Curve Discrete Logarithm Problem

- Cryptosystems rely on the hardness of the Elliptic Curve Discrete Logarithm Problem (ECDLP)

> ***Definition: Elliptic Curve Discrete Logarithm Problem (ECDLP)***
>
> *Given a primitive element P and another element T on an elliptic curve E. The ECDL problem is finding the integer d, where $1 \leq d \leq \#E$ such that*
>
> $$\underbrace{P + P + \ldots + P}_{d\ \text{times}} = dP = T.$$

- Cryptosystems are based on the idea that *d* is large and kept secret and attackers cannot compute it easily
- If *d* is known, an efficient method to compute the point multiplication *dP* is required to create a reasonable cryptosystem
  - Known Square-and-Multiply Method can be adapted to Elliptic Curves
  - The method for efficient point multiplication on elliptic curves: Double-and-Add Algorithm

Double-and-Add Algorithm for Point Multiplication

Double-and-Add Algorithm

Input: Elliptic curve *E,* an elliptic curve point *P* and *a* scalar *d* with bits d_i

Output: $T = d\,P$

Initialization:

$T = P$

Algorithm:

FOR $i = t - 1$ DOWNTO 0

$T = T + T \bmod n$

IF $d_i = 1$

$T = T + P \bmod n$

RETURN (T)

Example: $26P = (11010_2)P = (d_4 d_3 d_2 d_1 d_0)_2\ P$.

Step		
#0	$P = \mathbf{1}_2P$	inital setting
#1*a*	$P+P = 2P = \mathbf{10}_2P$	DOUBLE (bit d_3)
#1*b*	$2P+P = 3P = 10^2\ P+1_2P = \mathbf{11}_2P$	ADD (bit d_3=1)
#2*a*	$3P+3P = 6P = 2(11_2P) = \mathbf{110_2}P$	DOUBLE (bit d_2)
#2*b*		no ADD ($d_2 = 0$)
#3*a*	$6P+6P = 12P = 2(110_2P) = \mathbf{1100_2}P$	DOUBLE (bit d_1)
#3*b*	$12P+P = 13P = 1100_2P+1_2\ P = \mathbf{1101}_2P$	ADD (bit d_1=1)
#4*a*	$13P+13P = 26P = 2(1101_2P) = \mathbf{11010_2}P$	DOUBLE (bit d_0)
#4*b*		no ADD ($d_0 = 0$)

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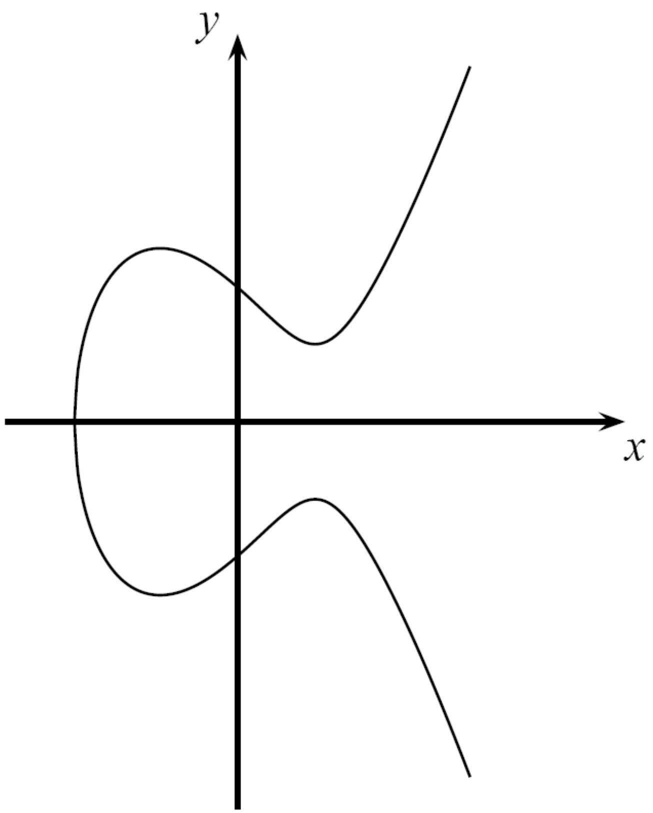

The Elliptic Curve Diffie-Hellman Key Exchange (ECDH)

- Given a prime *p*, a suitable elliptic curve *E* and a point $P=(x_P,y_P)$
- The Elliptic Curve Diffie-Hellman Key Exchange is defined by the following protocol:

Alice

Choose $k_{PrA} = a \in \{2, 3, \ldots, \#E-1\}$
Compute $k_{PubA} = A = aP = (x_A, y_A)$

Bob

Choose $k_{PrB} = b \in \{2, 3, \ldots, \#E-1\}$
Compute $k_{PubB} = B = bP = (x_B, y_B)$

$$\text{Alice} \xrightarrow{A} \text{Bob}$$

$$\text{Alice} \xleftarrow{B} \text{Bob}$$

Alice: Compute $aB = T_{ab}$

Bob: Compute $bA = T_{ab}$

- Joint secret between Alice and Bob: $T_{AB} = (x_{AB}, y_{AB})$
- Proof for correctness:
  - Alice computes aB=a(bP)=abP
  - Bob computes bA=b(aP)=abP since group is associative
- One of the coordinates of the point T_{AB} (usually the x-coordinate) can be used as session key (often after applying a hash function)

The Elliptic Curve Diffie-Hellman Key Exchange (ECDH) (ctd.)

- The ECDH is often used to derive session keys for (symmetric) encryption
- One of the coordinates of the point T_{AB} (usually the x-coordinate) is taken as session key

Alice		**Bob**	
Choose $k_{PrA} = a \in \{2, 3, \ldots, \#E-1\}$ Compute $k_{PubA} = A = aP = (x_A, y_A)$		Choose $k_{PrB} = b \in \{2, 3, \ldots, \#E-1\}$ Compute $k_{PubB} = B = bP = (x_B, y_B)$	**ECDH**
	$\xrightarrow{A}$		
	$\xleftarrow{B}$		
Compute $aB = T_{ab} = (x_T, y_T)$		Compute $bA = T_{ab} = (x_T, y_T)$	
Define key $k_{AES} = x_T$		Define key $k_{AES} = x_T$	**Symmetric encryption/decryption**
Given a message m: Encrypt $c = AES_{kAES}(m)$	$\xrightarrow{c}$		
		Received ciphertext c: Decrypt $m = AES^{-1}_{kAES}(c)$	

- In some cases, a hash function (see next chapters) is used to derive the session key

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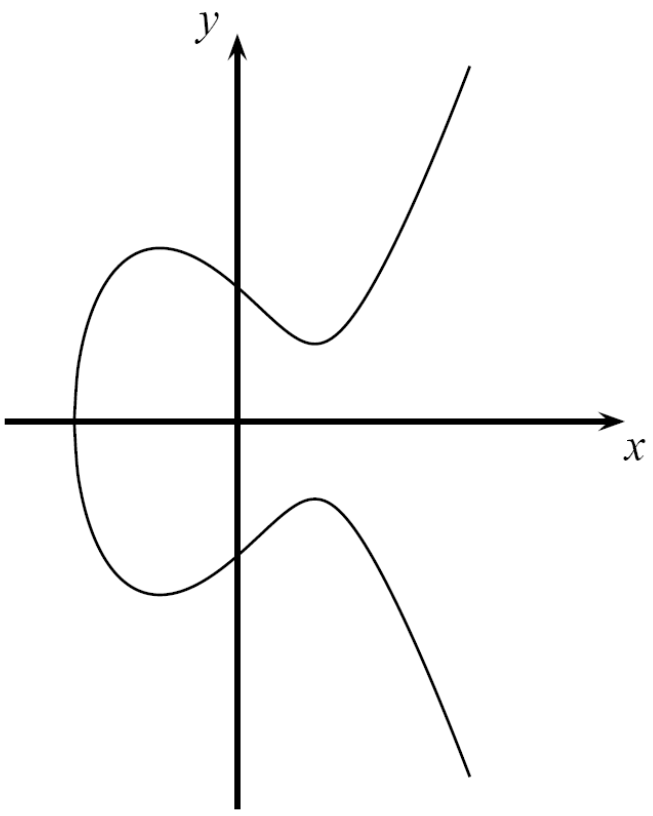

Security Aspects

- Why are parameters signficantly smaller for elliptic curves (160-256 bit) than for RSA (1024-3076 bit)?
  - Attacks on groups of elliptic curves are weaker than available factoring algorithms or integer DL attacks
  - Best known attacks on elliptic curves (chosen according to cryptographic criterions) are the Baby-Step Giant-Step and Pollard-Rho method
  - Complexity of these methods: on average, roughly $\sqrt{p}$ steps are required before the ECDLP can be successfully solved

- Implications to practical parameter sizes for elliptic curves:
  - An elliptic curve using a prime p with 160 bit (and roughly 2^{160} points) provides a security of 2^{80} steps that required by an attacker (on average)
  - An elliptic curve using a prime p with 256 bit (roughly 2^{256} points) provides a security of 2^{128} steps on average

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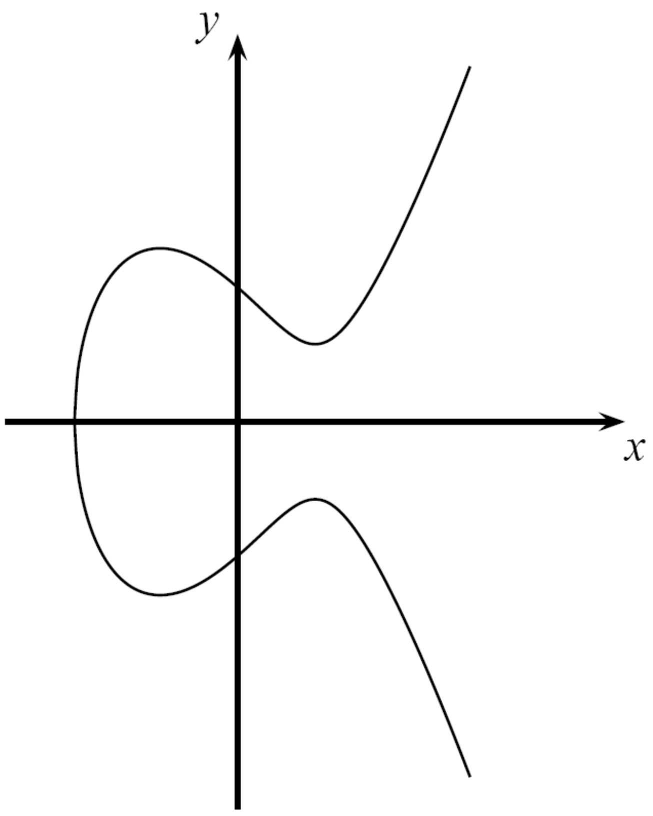
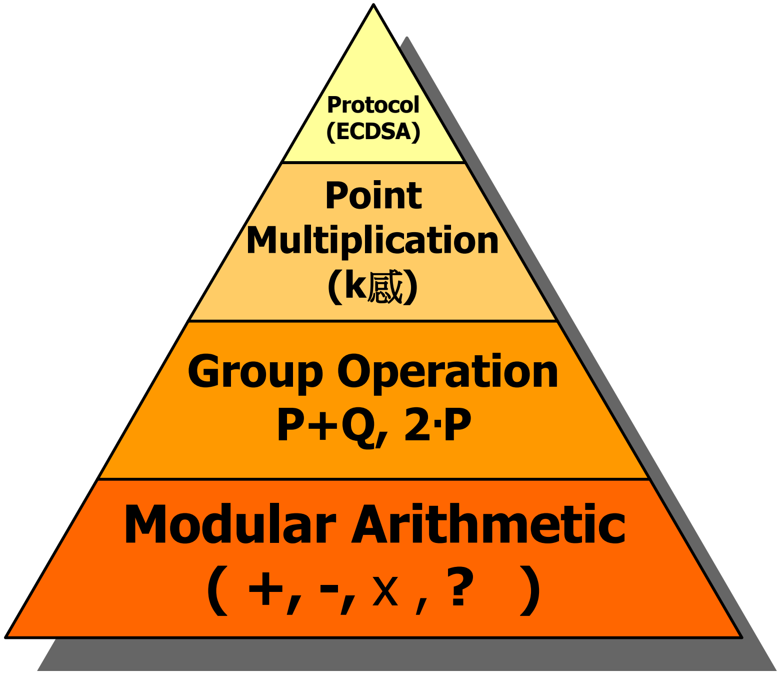


Implementations in Hardware and Software

- Elliptic curve computations usually regarded as consisting of four layers:
  - Basic modular arithmetic operations are computationally most expensive
  - Group operation implements point doubling and point addition
  - Point multiplication can be implemented using the Double-and-Add method
  - Upper layer protocols like ECDH and ECDSA

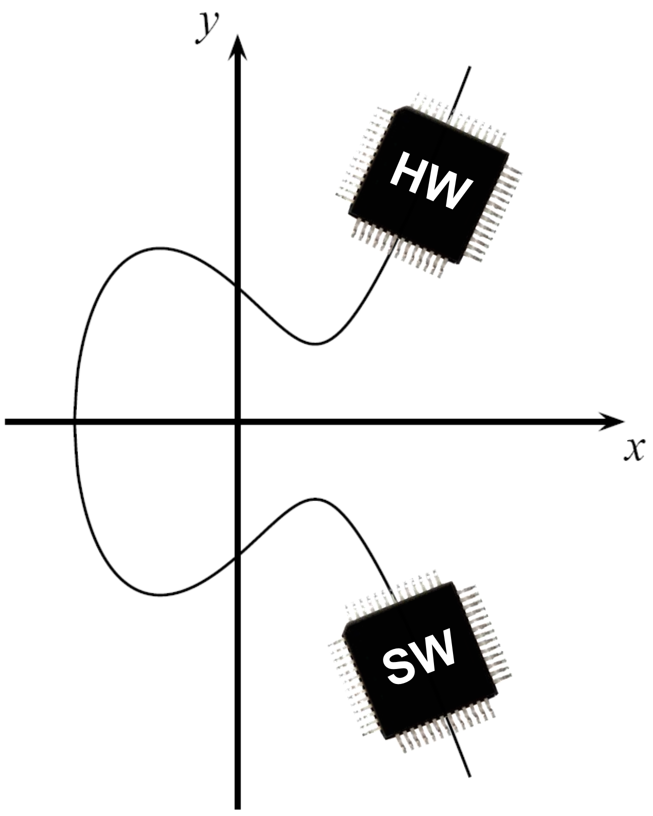
- Most efforts should go in optimizations of the modular arithmetic operations, such as
  - Modular addition and subtraction
  - Modular multiplication
  - Modular inversion



Implementations in Hardware and Software

- Software implementations
  - Optimized 256-bit ECC implementation on 3GHz 64-bit CPU requires about *2 ms* per point multiplication
  - Less powerful microprocessors (e.g, on SmartCards or cell phones) even take significantly longer (*>10 ms*)

- Hardware implementations
  - High-performance implementations with 256-bit special primes can compute a point multiplication in a few hundred microseconds on reconfigurable hardware
  - Dedicated chips for ECC can compute a point multiplication even in a few ten microseconds



Lessons Learned

- Elliptic Curve Cryptography (ECC) is based on the discrete logarithm problem. It requires, for instance, arithmetic modulo a prime.
- ECC can be used for key exchange, for digital signatures and for encryption.
- ECC provides the same level of security as RSA or discrete logarithm systems over Z_p with considerably shorter operands (approximately 160–256 bit vs. 1024–3072 bit), which results in shorter ciphertexts and signatures.
- In many cases ECC has performance advantages over other public-key algorithms.
- ECC is slowly gaining popularity in applications, compared to other public-key schemes, i.e., many new applications, especially on embedded platforms, make use of elliptic curve cryptography.