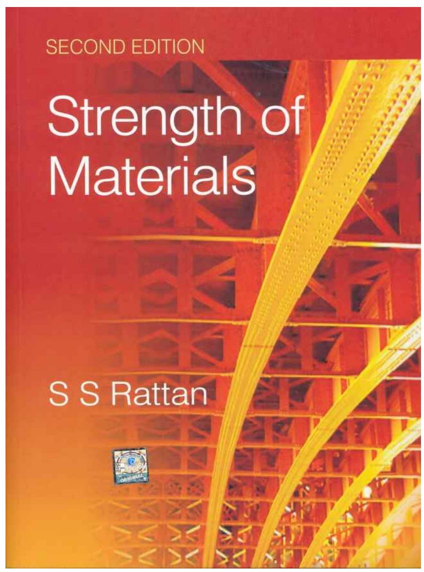
SECOND EDITION
Strength of Materials
S S Rattan

Strength of Materials

Secend Editien

About the Author

S S Rattan is currently serving as Professor and Head in the Department of Mechanical Engineering at the National Institute of Technology, Kurukshetra. He did his postgraduation in Mechanical Engineering from Punjab Engineering College, Chandigarh, in 1981. Since then he has been engaged in teaching and research. He obtained his PhD degree from Kurukshetra University in 1995. Dr Rattan has authored several research papers, which have been published in national and international journals as well as in the proceedings of conferences in India and abroad. He is also the recipient of Sir Rajendra Nath Mookerjee Memorial Prize for a paper published in *Journal of Institute of Engineers*. Besides these, Dr Rattan is the author of two other books titled *Theory of Machines* and *Fluid Mechanics and Fluid Machines*. He is also a member of Indian Society of Theoretical and Applied Mechanics.

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To the Memory of
My Parents

Contents

Preface

An engineer always endeavours to design structural or machine members that are safe, durable and economical. To accomplish this, he has to evaluate the load-carrying capacity of the members so that they are able to withstand the various forces acting on them. The subject *Strength of Materials* deals with the strength, stability and rigidity of various structural or machine members such as beams, columns, shafts, springs, cylinders, etc. These days, a number of books on the subject are available in the market. It is observed that although most of the books are having a few good features in them, their overall ranking drops when considered on composite parameters like coverage of a topic, lucidity of writing, variety of solved and unsolved problems, quality of diagrams etc. Usually, the students have to supplement one book with a few others to comprehend the subject. The present book aims to provide most of the good features in a single book.

The book aims to be useful to degree-level students of mechanical and civil engineering as well as those preparing for AMIE and various other competitive examinations. However, diploma-level students will also find the book to be of immense use. The book will also benefit postgraduate students to some extent as it contains some advanced topics like bending of curved bars, rotating discs and cylinders, plastic bending and circular plates, etc.

The salient features of the book are

- Moderately concise and compact text covering all major topics
- Simple language aimed to benefit average and weak students
- Logical and evolutionary approach in descriptions for better imagination and visualisation
- Derivation of physical concepts from simple and readily comprehensible principles
- Summary at the end of each chapter
- An appendix containing important relations and results
- Rich pedagogy including
  - Over 540 illustrations
  - 440 Solved Examples
  - 220 Objective Type Questions
  - 175 Review Questions
  - 210 Problems

It is necessary for students using this book to have completed a course in applied mechanics. **Chapters 1 and 2** introduce the concept of simple and compound stresses at a point. The theory shows that an axial load may produce shear stresses along with normal stresses depending upon the section considered. The latter chapter also discusses the utility of Mohr's circle in transformation of stress at a point. **Chapter 3** explains the concept of strain energy that forms the basis of analysis in many cases. **Chapters 4 to 8** are related to beams which may be simply supported, fixed at one or both ends or continuous having more than two supports. The analysis includes the computations of bending moment, shear force, bending and shear stresses under transverse loads and deflection of beams. The concept of plastic deformation of beams beyond the elastic limit, being an advanced topic is taken up later in **Chapter 17**.

Sometimes, curved members such as rings and hooks are also loaded. **Chapter 9** discusses the stresses developed in such members. **Chapter 10** takes up the theory of torsion, which also includes its application to shafts transmitting power. **Chapter 11** discusses springs based on the same theory.

Columns are important members of structures. **Chapter 12** discusses the equilibrium of columns and struts. However, the computation of stress in plane frame structures, which is mostly included in the civil engineering curriculum, is discussed later in **Chapter 18**. Some other important machine members include cylinders and spheres under internal or external pressures; flywheels, discs and cylinders, which rotate while performing the required function, are covered in **chapters 13 and 14**. Design of mechanical members is mostly based on certain criteria of failure, and various theories based on the same are taken up in **Chapter 15.**

Chapter 16 illustrates how circular plates are stressed under concentrated and uniform loads. The properties of materials as well as the methods to determine the same are discussed in **Chapter 19**.

The first edition of the book aimed at providing the rudiments of the subject in a simple manner for easy comprehension by students. Simple mathematical derivations were favoured instead of more elegant but perplexed ones so that those with limited mathematical skills could easily grasp the essence. However, to make the book more purposeful and acceptable to a wider section of users, the present edition aims at making it more exhaustive. Many new sections under various chapters have been added apart from rewriting of some of the previous sections. Many more worked-out examples as well as unsolved problems have been added. The objective type questions, which had been contained in an appendix in the previous edition, are now provided at the end of each chapter as was suggested by most of the readers. Effort has been made to eliminate all sorts of errors and misprints as far as possible. In spite of the addition of a large amount of material, care has been taken to let the book remain concise and compact.

Though students are expected to exert and solve the numerical problems given at the end of each chapter, hints to most of these are available at the publisher's website of the book for the benefit of average and weak students. However, full solutions of the unsolved problems are available to the faculty members at the same site. The facility can be availed by logging on to https://www.mhhe.com/rattan/som2.

In preparing the script, I relied heavily on the works of renowned authors whose writings are considered classics in the field. I am indeed indebted to them. I sincerely acknowledge the help of my many colleagues, who helped me in one form or the other in preparing this treatise. I also acknowledge the efforts of the editorial and production staff of the McGraw-Hill Education for taking pains in bringing out this edition in an excellent format.

I am also grateful to the following reviewers of this book who took out time from their busy schedules and sent me useful comments and suggestions:

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Finally, I am also indebted to my wife, Neena, and my children, Ravneet and Jasmeet, for being patient with me while I went about the arduous task of preparing the manuscript. But for their sacrifice, I would not have been able to complete it in the most satisfying way.

A creation by a human being can never be perfect. A number of mistakes might have crept in even in this revised text. I shall be highly grateful to the readers and the users of the book for their uninhibited comments and for pointing out the errors do feel free to contact me at ss_rattan@hotmail.com

S S Rattan

Symbols

a	area
A	area
b	width
B	width
d	diameter, height, depth
D	diameter
e	eccentricity
E	modulus of elasticity, Young's modulus
F	force
g	acceleration due to gravity
G	shear modulus, modulus of rigidity
h	height, distance
H	height
I	moment of inertia
l	length
j	number of joints
J	polar moment of inertia
k	torsional stiffness, stiffness of spring
K	Bulk modulus
l	length
L	length, load factor
m	mass, modular ratio, number of members
M	moment, bending moment, mass
n	number of coils
p	pressure, compressive stress
P	force, load
q	shear flow
r	radius
R	radius, reaction
s	length
S	shape factor
t	thickness, time, temperature
T	torque
u	strain energy density
U	strain energy, resilience
V	volume
w	rate of loading
W	force, weight, load
x,y,z	rectangular coordinates, distances
Z	section modulus
σ	direct stress
θ	angle
ϕ	angle, shear strain
α	angle, coefficient of thermal expansion
δ	increment of quantity, deflection, extension
τ	shear stress
Δ	elongation
ε	direct strain
π	3.1416
ν	Poisson's ratio
ψ	angle
γ	angle
ω	angular velocity
ρ	density

Chapter 1

Simple Stress and Strain

External forces acting on individual structural or machine members of an engineering design are common. An engineer always endeavours to have such designs that are safe, durable and economical. Thus, load-carrying capacity of the members being designed is of paramount importance to know their dimensions to minimise the cost. The subject *Strength of Materials* deals with the strength or the load-carrying capacity of various members such as beams and columns. It also considers their stability and rigidity. *Theory of Structures* involves the application of these principles to structures made up of beams, columns, slabs and arches.

The force acting on a body is termed as *load*. A *concentrated load* is also known as a *point load*, and a distributed load over a length is known as *distributed load*. A distributed load of constant value is called *uniformly distributed load*. If a structure as a whole is in equilibrium, its members are also in equilibrium individually which implies that the resultant of all the forces acting on a member must be zero. However, the forces acting on a body tend to deform or tear the body. For example, a load P acting on a body tends to pull it apart (Fig. 1.1a). This type of pull may also be applied if one end of the body is fixed (Fig. 1.1b). In this case, the balancing force is provided by the reaction of the fixed end. Such type of a pulling force is known as *tension* or *tensile force*. A tensile force tends to increase the length and decrease the cross-section of the body.

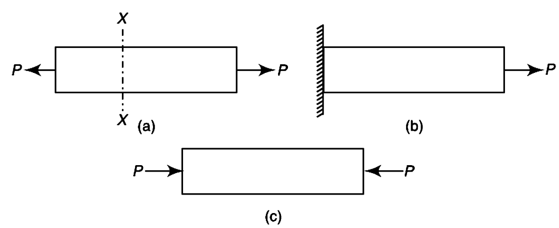


Fig. 1.1

In a similar way, a force tending to push or compress a body is known as *compression* or *compressive force* which tends to shorten the length (Fig. 1.1c).

Usually, the forces acting on a body along the longitudinal axis are known as *direct* or *axial forces*, and the forces acting normal to the longitudinal axis of a body are known as *transverse* or *normal* forces.

In the elementary theory of analysis, a material subjected to external forces is assumed to be perfectly elastic, i.e., the deformations caused to the body totally disappear as soon as the load or forces are removed. Other assumptions are that the materials are *isotropic* (same properties in all directions) and *homogeneous* (same properties anywhere in the body).

1.1 STRESS

The applied external forces on a body are transmitted to the supports through the material of the body. This phenomenon tends to deform the body and causes it to develop equal and opposite internal forces. These *internal forces* by virtue of cohesion between particles of the material tend to resist the deformation. The magnitude of the internal resisting forces is equal to the applied forces but the direction is opposite.

Let the member shown in Fig. 1.1a be cut through the section X-X as shown in Fig. 1.2. Now, each segment of the member is in equilibrium under the action of the force P and the internal resisting force. The resisting force per unit area of the surface is known as *intensity of stress* or simply *stress* and is denoted by σ. Thus, if the load P is assumed as uniformly distributed over a sectional area A, then the stress σ is given by

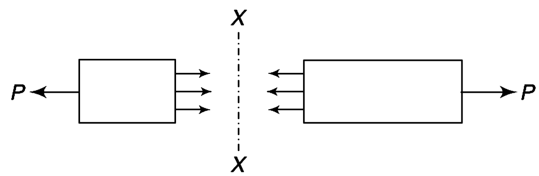


Fig. 1.2

$$\sigma = P/A \tag{1.1}$$

However, if the intensity of stress is not uniform throughout the body, then the stress at any point is defined as

$$\sigma = \delta P/\delta A$$

where δA = infinitesimal area of cross-section
and δP = load applied on area δA

The stress may be tensile or compressive depending upon the nature of forces applied on the body.

Stress at the elastic limit is usually referred as *proof stress*.

Units

The unit of stress is N/m^2 or Pascal (Pa). However, this is a very small unit, almost the stress due to placing an apple on an area of 1 m^2. Thus, it is preferable to express stress in units of MN/m^2 or MPa.

$$1 \text{ MN/m}^2 = 1 \text{ MPa} = 1 \times 10^6 \text{ N/m}^2 = 1 \text{ N/mm}^2$$

Also $$1 \text{ GPa} = 1000 \text{ MPa} = 1000 \text{ N/mm}^2 = 1 \text{ kN/mm}^2$$

In numerical problems, it is always convenient to express the units of stress mentioned in MPa and GPa in the form of N/mm^2.

1.2 SHEAR STRESS

When two equal and opposite parallel forces not in the same line act on two parts of a body, then one part tends to slide over or shear from the other across any section and the stress developed is termed as *shear stress*. In Figs. 1.3 a and b, the material is sheared along any section X-X whereas in a riveted joint (Fig.1.3c), the shearing is across the rivet diameter.

If P is the force applied and A is the area being sheared, then the intensity of shear stress is given by

$$\tau = P/A \tag{1.2}$$

and if the intensity of shear stress varies over an area,

$$\tau = \delta P/\delta A$$

Remember that shear stress is always tangential to the area over which it acts.

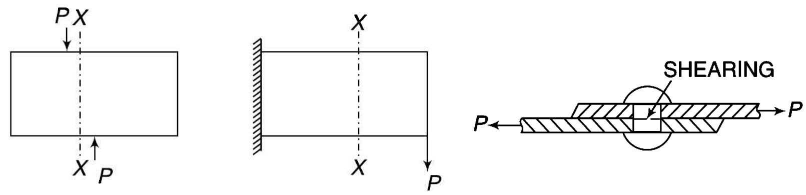


Fig. 1.3

Complimentary Shear Stress

Consider an infinitely small rectangular element *ABCD* under shear stress of intensity τ acting on planes *AD* and *BC* as shown in Fig. 1.4a. It is clear from the figure that the shear stress acting on the element will tend to rotate the block in the clockwise direction. As there is no other force acting on the element, static equilibrium of the element can only be attained if another couple of the same magnitude is applied in the counter-clockwise direction. This can be achieved by having shear stress of intensity τ' on the faces *AB* and *CD* (Fig. 1.4b).

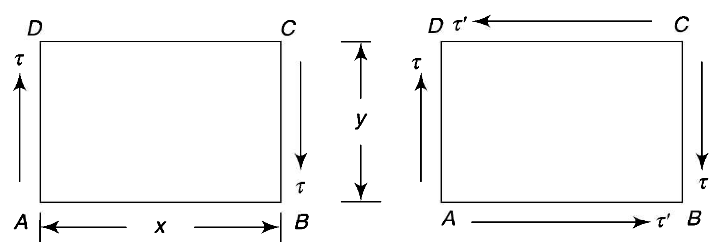


Fig. 1.4

Assuming x and y to be the lengths of the sides *AB* and *BC* of the rectangular element and a unit thickness perpendicular to the figure,

The force of the given couple $= \tau \cdot (y \cdot 1)$

The moment of the given couple $= (\tau \cdot y) \cdot x$

Similarly,

The force of balancing couple $= \tau' \cdot (x \cdot 1)$

The moment of balancing couple $= (\tau' \cdot x) \cdot y$

For equilibrium, equating the two,

$$(\tau \cdot y) \cdot x = (\tau' \cdot x) \cdot y$$

or

$$\tau = \tau'$$

which shows that the magnitude of the balancing shear stresses is the same as of the applied stresses. The shear stresses on the transverse pair of faces are known as *complimentary shear stresses*. Thus every shear stress is always accompanied by an equal complimentary shear stress on perpendicular planes.

The presence of complimentary shear stress may cause an early failure of *anisotropic materials* such as timber which is weaker in shear along the grain than normal to the grain.

- *Owing to the characteristic of complimentary shear stresses for the equilibrium of members subjected to shear stresses, near a free boundary on which no external force acts, the shear stress must follow a direction parallel to the boundary. This is because any component of the shear force perpendicular to the surface will find no complimentary shear stress on the boundary plane.*

1.3 SAINT-VENANT'S PRINCIPLE

According to this principle, the distribution of internal stresses or strains on sections of a body, which are at sufficient distance from the surfaces of the load applications, is not affected by the nature of actual application of load over the surface. Thus, in case of a rod under simple tension, the load applied may be centrally concentrated (Fig. 1.5a), distributed over the outer circumference (Fig. 1.5b), distributed in a circular way on the end surface (Fig. 1.5c) or uniformly distributed over the end surface (Fig. 1.5d), the distribution of stresses is found to be uniformly distributed in all types of loading for points distant more than about two to three times the diameter of the rod.

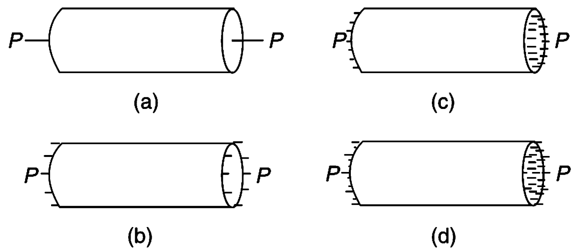


Fig. 1.5

Figure 1.6a shows a member having two rigid plates attached at the two ends. When the loads are applied at the centre of each plate, the plates move towards each other causing the member to be shorter while increasing the lateral dimensions of the member. It can safely be assumed that the plane sections will remain plane and all elements of the member deform in the same way. In such a case, the distribution of strains throughout the member is uniform, i.e., the axial and lateral strains are constant. However, if the plates are removed and the concentrated loads are applied directly as shown in Fig. 1.6b, the elements near the points of applications of loads deform appreciably while no deformation takes place at the corners resulting in very large stresses in the vicinity of points of application of loads and almost no deformation near the ends. But if the elements farther and farther from the ends are observed, a more nearly uniform distribution of the strains and stresses across a section is obtained.

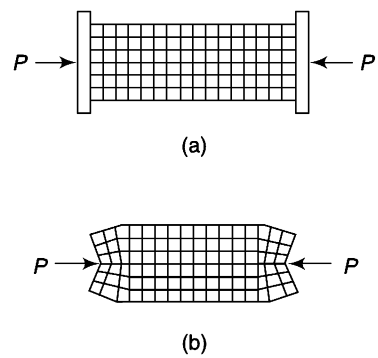


Fig. 1.6

1.4 STRAIN

The deformation of a body under a load is proportional to its length. To study the behaviour of a material, it is convenient to study the deformation per unit length of a body than its total deformation. The elongation per unit length of a body is known as *longitudinal strain* or simply *strain* and is denoted by Greek symbol ε. If Δ is the elongation of a body of length L, the strain ε is given by

$$\varepsilon = \Delta/L \tag{1.3}$$

As it is a ratio of similar quantities, it is dimensionless.

Shear Strain

A rectangular element of a body is distorted by shear stresses as shown in Fig. 1.7. If the lower surface is assumed to be fixed, the upper surface slides relative to the lower surface and the corner angles are altered by angle φ. *Shear strain* is defined as the change in the right angle of the element measured in radians and is dimensionless.

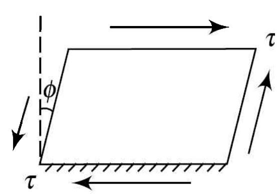


Fig. 1.7

1.5 MODULUS OF ELASTICITY AND MODULUS OF RIGIDITY

For elastic bodies, the ratio of stress to strain is constant and is known as *Young's modulus* or the *modulus of elasticity* and is denoted by E, i.e.,

$$E = \sigma/\varepsilon \tag{1.4}$$

Strain has no units as it is a ratio. Thus, E has the same units as stress.

The materials that maintain this ratio are said to obey *Hooke's law* which states that within elastic limits, strain is proportional to the stress producing it. The elastic limit of a material is determined by plotting a tensile test diagram (Refer Section 1.15).

Young's modulus is the stress required to cause a unit strain. As a unit strain means elongation of a body equal to its original length (since $\varepsilon = \Delta/L$), this implies that Young's modulus is the stress or the force required per unit area to elongate the body by its original size or to cause the length to be doubled. However, for most of the engineering materials, the strain does not exceed 1/1000. Obviously, mild steel has a much higher value of Young's modulus E as compared to rubber.

Similarly, for elastic materials, the shear strain is found to be proportional to the applied shear stress within the elastic limit. *Modulus of rigidity* or *shear modulus* denoted by G is the ratio of shear stress to shear strain, i.e.,

$$G = \tau/\varphi \tag{1.5}$$

1.6 ELONGATION OF A BAR

An expression for the elongation of a bar of length L and cross-sectional area A under the action of a force P is obtained below:

$$\text{As} \quad E = \frac{\sigma}{\varepsilon} \qquad \therefore \qquad \varepsilon = \frac{\sigma}{E} \quad \text{or} \quad \frac{\Delta}{L} = \frac{\sigma}{E}$$

Thus, elongation of a bar of length L, $\Delta = \dfrac{\sigma \cdot L}{E} = \dfrac{PL}{AE}$ (1.6)

1.7 PRINCIPLE OF SUPERPOSITION

The principle of superposition states that if a body is acted upon by a number of loads on various segments of a body, then the net effect on the body is the sum of the effects caused by each of the loads acting independently on the respective segment of the body. Thus, each segment can be considered for its equilibrium. This is done making a diagram of the segment along with various forces acting on it. This diagram is generally referred as *free-body diagram*. The principle of superposition is applicable to all parameters like stress, strain and deflection. However, it is not applicable to materials with nonlinear stress–strain characteristics which do not follow Hooke's law.

Example 1.1 A circular steel bar of various cross-sections is subjected to a pull of 800 kN as shown in Fig. 1.8. Determine the extension of the bar. E = 204 GPa.

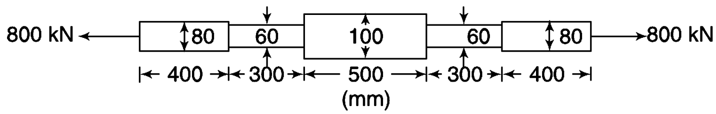


Fig. 1.8

Solution

Given A circular steel bar of following cross-sections:

— Two lengths of 400 mm each of 80-mm diameter

— Two lengths of 300 mm each of 60-mm diameter

— Single length of 500 mm of 100-mm diameter

P = 800 kN $\quad\quad$ E = 204 GPa

To find Extension of the bar

$$\text{Total extension, } \Delta = \frac{PL}{AE} \quad \text{...(Eq. 1.6)}$$

$$= \frac{800\,000}{(\pi/4) \times 204\,000}\left(\frac{400 \times 2}{80^2} + \frac{300 \times 2}{60^2} + \frac{500}{100^2}\right)$$

$$= 4.993\ (0.125 + 0.167 + 0.05) = 1.708 \text{ mm}$$

Example 1.2 A bar made up of two square sections, one of steel and the other of aluminium is shown in Fig. 1.9. The bar is acted upon by a compressive force P. Determine the value of P if the total decrease in length of the bar is 0.3 mm. Take E_s = 205 GPa and E_{al} = 75 GPa.

Solution

Given A bar with following two square cross-sections:

— Steel section of 600-mm length with 30 mm × 30 mm cross-section

— Aluminium section of 800-mm length with 60 mm × 60 mm cross-section

Δ = 0.3 mm $\quad$ E_s = 205 GPa $\quad$ E_{al} = 75 GPa.

To find Applied force P

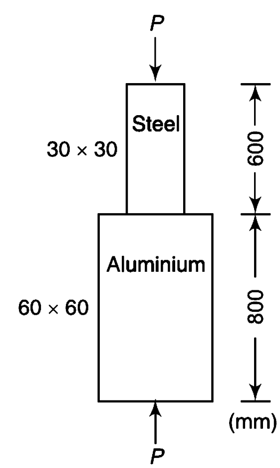


Fig. 1.9

Force P

Force applied can be found by considering the total decrease.

$$\text{Total decrease, } \Delta = \frac{PL}{AE} \quad \text{...(Eq. 1.6)}$$

$$0.3 = P\left[\frac{600}{30^2 \times 205\,000} + \frac{800}{60^2 \times 75\,000}\right]$$

or $\quad \dfrac{P}{10^6}(3.252 + 2.963) = 0.3$

or $\quad 6.215\,P = 0.3 \times 10^6$ or P = 48 270 N or 48.27 kN

Example 1.3 A steel bar of 25-mm diameter is acted upon by forces as shown in Fig. 1.10. What is the total elongation of the bar?

Take E = 190 GPa.

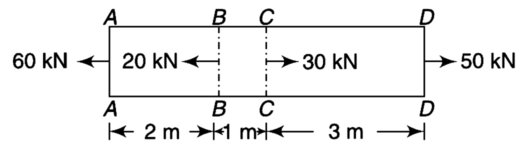


Fig. 1.10

Solution

Given A steel bar having three segments with different forces as shown in the figure.

$$d = 25 \text{ mm} \qquad E = 190 \text{ GPa}$$

To find Elongation of bar

$$\text{Area of the section} = \frac{\pi}{4}(25)^2 = 490.87 \text{ mm}^2$$

Calculation of forces in various segments

Forces in various segments are considered by taking the free-body diagram of each segment as follows (Fig. 1.11):

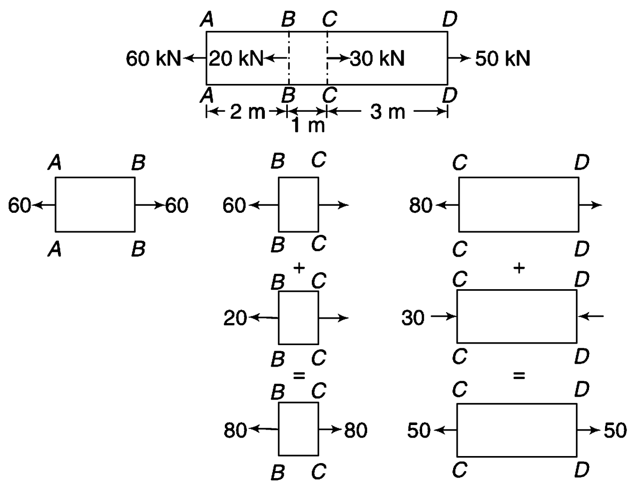


Fig. 1.11

Segment AB At Section *AA*, the force is 60 kN tensile and for force equilibrium of this segment, it is to be 60 kN tensile at *BB* also.

Segment BC

Force at Section *BB* = 60 kN tensile (as above) + 20 kN (applied tensile force at *B*)
= 80 kN (tensile)
= Force at Section *CC*

Segment CD

Force at Section *CC* = 80 kN tensile (as above) – 30 kN (applied comp. force at *CC*)
= 50 kN (tensile)
= Force at Section *DD*

Determination of total elongation

$$\Delta = \frac{PL}{AE} \qquad \text{...(Eq. 1.6)}$$

$$= \frac{1}{490.87 \times 190\,000}(60\,000 \times 2000 + 80\,000 \times 1000 + 50\,000 \times 3000) = 3.75 \text{ mm}$$

Example 1.4 A steel circular bar has three segments as shown in Fig. 1.12. Determine

(i) the total elongation of the bar
(ii) the length of the middle segment to have zero elongation of the bar
(iii) the diameter of the last segment to have zero elongation of the bar
Take E = 205 GPa.

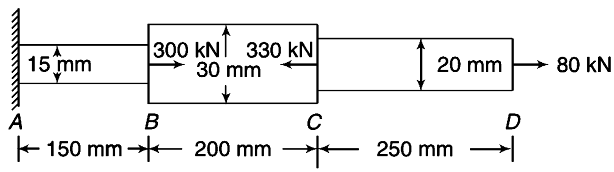


Fig. 1.12

Solution

Given A steel circular bar having three segments with different forces
E = 205 GPa.

To find

— Total elongation
— Length of middle segment for zero elongation of bar
— Diameter of last segment for zero elongation of bar

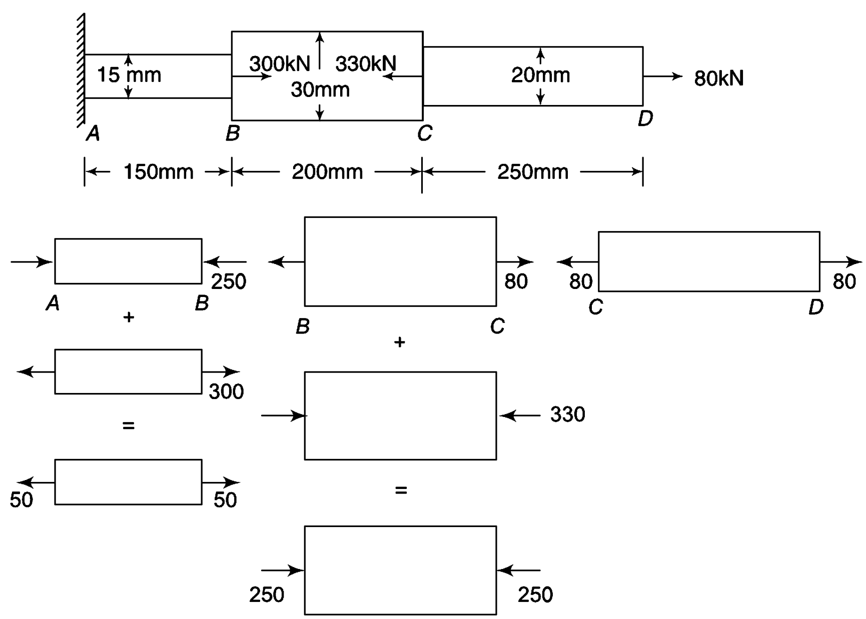


Fig. 1.13

Calculation of forces in various segments

(i) *Segment CD* At Section D, the force is 80 kN tensile and for force equilibrium of this segment, at C also it is to be 80 kN tensile.

Segment BC

Force at Section C = 80 kN tensile (as above) – 330 kN (comp. Force at Section C)
= –250 kN (compressive)
= Force at Section B

Segment AB

Force at Section B = –250 kN (comp. as above) + 300 kN (tensile force at section B)
= 50 kN (tensile)
= Force at section A

Total elongation

Total elongation,

$$\Delta = \frac{1}{(\pi/4)\times 205\,000}\left(\frac{80\,000\times 250}{20^2} - \frac{250\,000\times 200}{30^2} + \frac{50\,000\times 150}{15^2}\right)$$

$$= \frac{1}{161\,007}(50\,000 - 55\,555.5 + 33\,333.3) = 0.173 \text{ mm}$$

Length of middle segment for zero elongation of bar

Let the length of the middle segment be L to have zero elongation of the bar.

Then $$\Delta = \frac{1}{161\,007}\left(50\,000 - \frac{250\,000\times L}{30^2} + 33\,333.3\right) = 0$$

or $$L = \frac{30^2}{250\,000} \times 83\,333.3 = 300 \text{ mm}$$

Diameter of last segment for zero elongation of bar

Let the diameter of the last segment be d to have zero elongation of the bar.

$\therefore$ $$\delta = \frac{1}{161\,007}\left(\frac{80\,000\times 250}{d^2} - 55\,555.5 + 33\,333.3\right) = 0$$

$$d^2 = \frac{80\,000\times 250}{22\,222.2} = 900 \quad \text{or} \quad d = 30 \text{ mm}$$

Example 1.5 A circular steel bar having three segments is subjected to various forces at different cross-sections as shown in Fig. 1.14. Determine the necessary force to be applied at Section C for the equilibrium of the bar. Also, find the total elongation of the bar. Take E = 202 GPa.

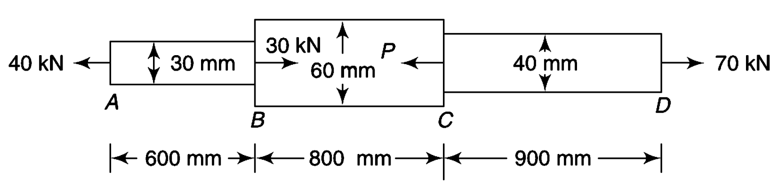


Fig. 1.14

Solution

Given A steel circular bar having three segments with different forces.
E = 202 GPa.

To find Total elongation of bar

Calculation of forces in various segments

Forces in various segments are considered by taking the free-body diagram of each segment (Fig. 1.15).

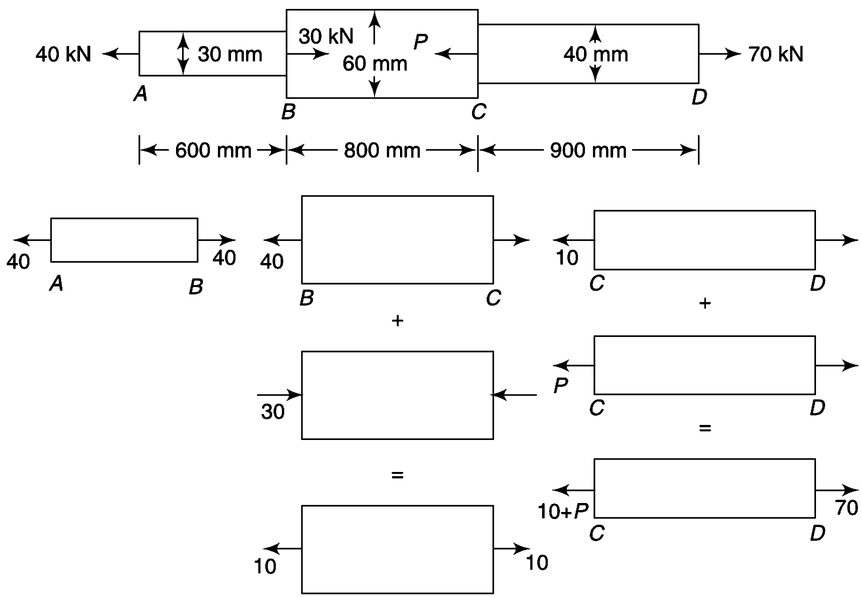


Fig. 1.15

Segment AB At Section *A,* the force is 40 kN tensile and for force equilibrium of this segment, it is to be 40 kN tensile at *B* also.

Segment BC

Force at Section B = 40 kN tensile (as above) – 30 kN (compressive at Section B)
= 10 kN (tensile)
= Force at Section C

Segment CD

Force at Section C = 10 kN tensile (as above) + P (tensile force at Section C)
= Force at Section D (= 70 kN tensile)

Thus, $10 + P = 70$ or $P = 60$ kN (tensile)

P can also be found directly by equating leftward and rightward forces, i.e.,

$$P + 40 = 30 + 70 \text{ or } P = 60 \text{ kN}$$

Determination of elongation of the bar

$$\Delta = \frac{PL}{AE}$$

$$= \frac{1}{(\pi/4)\times 202\,000}\left(\frac{40\,000\times 600}{30^2}+\frac{10\,000\times 800}{60^2}+\frac{70\,000\times 900}{40^2}\right)$$

$$= \frac{1}{158\,650}(26\,667 + 2222 + 39\,375) = 0.4303 \text{ mm}$$

Example 1.6 A vertical circular steel bar of length $3l$ fixed at both of its ends is loaded at intermediate sections by forces W and $2W$ as shown in Fig. 1.16. Determine the end reactions if W = 1.5 kN.

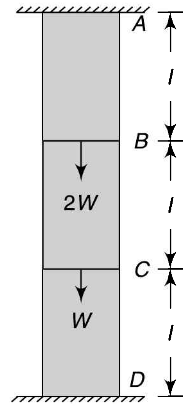


Fig. 1.16

Solution

Given A circular fixed end steel bar having three segments with different forces

$$W = 1500 \text{ N}$$

To find End reactions

Forces in various segments

Let A be the area of cross-section of each section and E, the modulus of elasticity.

Let the reaction at the upper fixed end be R (tensile). Forces in various segments are considered by taking the free-body diagram of each segment (Fig. 1.17).

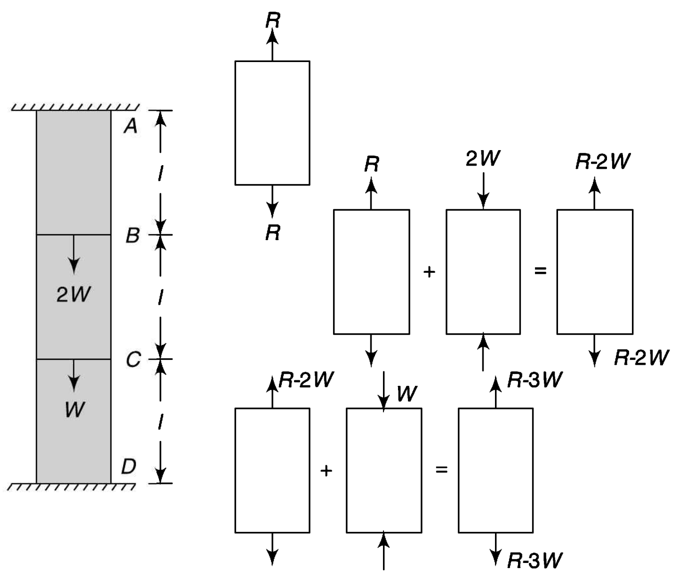


Fig. 1.17

Segment AB At Section A, the force is R tensile and for force equilibrium of this segment, it is to be R tensile at B also.

Segment BC

Force at Section B = R tensile (as above) – $2W$ (compressive at Section B)
= $R - 2W$ (tensile)
= Force at Section C

Segment CD

Force at Section C = $R - 2W$ tensile (as above) – W (compressive at Section C)
= $R - 3W$ (tensile)
= Force at D

End reactions

As the total extension is to be zero since both the ends are fixed,

$$\frac{Rl}{AE} + \frac{(R-2W)l}{AE} + \frac{(R-3W)l}{AE} = 0$$

or $$R + R - 2W + R - 3W = 0 \quad \text{or} \quad 3R = 5W \text{ or } R = 5W/3$$

$\therefore$ reaction at the lower end $= R - 3W = \dfrac{5W}{3} - 3W = -\dfrac{4W}{3}$, i.e., the reaction is upwards.

Numerical calculations

Reaction at the upper end $= R = \dfrac{5W}{3} = \dfrac{5 \times 1.5}{3} = 2.5$ kN upwards

Reaction at the lower end $= -\dfrac{4W}{3} = -\dfrac{4 \times 1.5}{3} = -2$ kN upwards

1.8 BARS OF TAPERING SECTION

Bars of tapering section can be of conical section or of trapezoidal section with uniform thickness.

Conical Section

Consider a bar of conical section under the action of an axial force P as shown in Fig. 1.18.

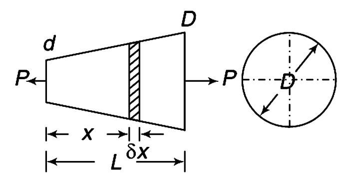


Fig. 1.18

Let

D = diameter at the larger end

d = diameter at the smaller end

L = length of the bar

E = Young's modulus of the bar material

Consider a very small length δx at a distance x from the small end.

The diameter at a distance x from the small end $= d + \dfrac{D - d}{L} \cdot x$

The extension of the small length $= \dfrac{P \cdot \delta x}{\dfrac{\pi}{4}\left(d + \dfrac{D - d}{L} x\right)^2 \cdot E}$ $\quad \left(\Delta = \dfrac{PL}{AE}\right)$

Extension of the whole rod $= \displaystyle\int_0^L \frac{4P}{\pi(d + (D - d)x/L)^2 \cdot E} \cdot dx$

$$= \frac{4P}{\pi E}\int_0^L \left(d + \frac{D - d}{L} \cdot x\right)^{-2} \cdot dx = -\frac{4P}{\pi E} \cdot \frac{L}{(D - d)}\left(\frac{1}{(d + (D - d)x/L)}\right)_0^L$$

$$= \frac{4PL}{\pi E(D - d)}\left(\frac{1}{d} - \frac{1}{D}\right) = \frac{4PL}{\pi E(D - d)}\left(\frac{D - d}{dD}\right) = \frac{4PL}{\pi EdD} \qquad (1.7)$$

Trapezoidal Section of Uniform Thickness

Let

B = width at the larger end

b = width at the smaller end

t = thickness of the section

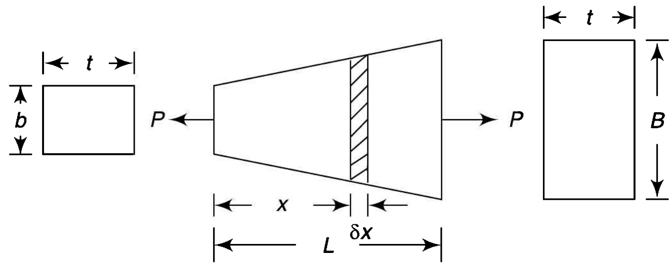


Fig. 1.19

L = length of the bar

E = Young's modulus of the bar material

Consider a very small length δx at a distance x from the small end of the rod (Fig.1.19).

The width at a distance x from the small end $= b + \dfrac{B-b}{L} \cdot x = b + kx$ ....[Taking $k = (B - b)/L$]

The area of cross-section at this distance $= (b + kx) \cdot t$

The extension of the small length $= \dfrac{P \cdot \delta x}{(b+kx)t \cdot E}$

$$\text{Extension of the whole rod} = \int_0^L \frac{P}{(b+kx)t \cdot E} \cdot dx = \frac{P}{tE}\int_0^L \frac{1}{(b+kx)} \cdot dx$$

$$= \frac{P}{tE}\frac{1}{k}[\log_e(b+kx)]_0^L = \frac{P}{ktE}\left(\log_e \frac{b+kL}{b}\right) = \frac{P}{ktE}\log_e \frac{B}{b} \qquad (1.8)$$

$$\ldots\ldots\left(b + kL = b + \frac{B-b}{L} \cdot L = B\right)$$

Example 1.7 A tapering conical bar of 1-m length has diameters of 20 mm and 50 mm at the two ends. Find the elongation of the bar under an axial tensile load of 250 kN. E = 205 GPa.

Solution

Given A tapering conical bar

D = 50 mm	L = 1000 mm
d = 20 mm	P = 250 kN
E = 205 GPa	

To find Elongation of bar

$$\text{Elongation of the bar} = \frac{4PL}{\pi EdD} = \frac{4 \times 250 \times 10^3 \times 1000}{\pi \times 205 \times 10^3 \times 20 \times 50} = 1.55 \text{ mm}$$

Example 1.8 A flat steel plate of trapezoidal section has a thickness of 20 mm and tapers uniformly from a width of 80 mm to 30 mm in a length of 500 mm. What will be the elongation of the plate under an axial load of 200 kN? E = 205 Gpa.

Solution

Given A flat steel plate of trapezoidal section

$B = 80$ mm $\quad L = 500$ mm

$b = 30$ mm $\quad t = 20$ mm

$P = 200$ kN $\quad E = 205$ GPa

To find Elongation of plate

$$k = \frac{B-b}{L} = \frac{80-30}{500} = 0.1$$

$$\text{Elongation of the rod} = \frac{P}{ktE}\log_e \frac{B}{b}$$

$$= \frac{200\times 10^3}{0.1\times 20\times 205\times 10^3}\log_e \frac{80}{30}$$

$$= 0.488 \text{ mm}$$

Example 1.9 A tension bar tapers from $(d + a)$ diameter to $(d - a)$ diameter. Prove that the error involved in using the mean diameter to calculate the Young's modulus is $(10a/d)^2$ percent.

Solution

Given A tapering conical bar with diameters $(d + a)$ and $(d - a)$ at the two ends.

To find Error involved if mean diameter is used to calculate E

Let P be the load applied and the L the length of the tapered bar.

Calculation of E when tapered bar relation is applied

$$\Delta = \frac{4PL}{\pi E(d-a)(d+a)} \quad \text{or} \quad E = \frac{4PL}{\pi(d^2-a^2)\Delta}$$

Calculation of E when tapered mean diameter is used

The mean diameter is d.

$$\Delta = \frac{PL}{AE} \quad \text{or} \quad E = \frac{PL}{(\pi d^2/4)\Delta} = \frac{4PL}{\pi d^2 \Delta}$$

Calculation of Error involved

$$\text{Error} = \left(\frac{4PL}{\pi(d^2-a^2)\Delta} - \frac{4PL}{\pi d^2\Delta}\right)\Bigg/\frac{4PL}{\pi(d^2-a^2)\Delta}$$

$$= 1 - \frac{d^2-a^2}{d^2} = 1 - \left[1-\left(\frac{a}{d}\right)^2\right] = \left(\frac{a}{d}\right)^2$$

$$\%\text{error} = \left(\frac{a}{d}\right)^2 \times 100 = \left(\frac{10a}{d}\right)^2$$

Example 1.10 A steel plate, 22 mm thick and 220 mm wide at one end, tapers uniformly to 12 mm thick and 180 mm wide at the other end. Determine the elongation under a pull of 20 kN when the length of the plate is 2.4 m. E = 205 GPa.

Solution

Given A steel plate of varying cross-section of dimensions as shown in Fig. 1.20.

$E = 205$ GPa $\quad P = 20$ kN

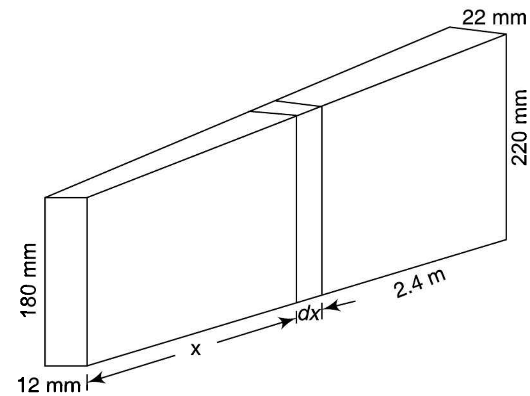


Fig. 1.20

To find Elongation of plate

Width at the smaller end = 180 mm
Width at the larger end = 220 mm
Thickness at the smaller end = 12 mm
Thickness at the larger end = 22 mm

Consider an elementary strip of length *dx* of the plate at a distance from the small end

Area of elementary strip

$$\text{Width of the strip} = 180 + \frac{220 - 180}{2400}x = 180 + \frac{x}{60}$$

$$\text{Thickness of the strip} = 12 + \frac{22 - 12}{2400}x = 12 + \frac{x}{240}$$

$$\text{Area of the strip} = \text{width} \times \text{thickness} = \left(180 + \frac{x}{60}\right)\left(12 + \frac{x}{240}\right)$$

Elongation of elementary strip

$$\text{Elongation of the strip} = \frac{PL}{AE} = \frac{P}{\left(180 + \frac{x}{60}\right)\left(12 + \frac{x}{240}\right)} \cdot \frac{dx}{E}$$

Elongation of whole plate

$$\text{Total elongation} = \frac{P}{E}\int_0^{2400} \frac{dx}{\left(180 + \frac{x}{60}\right)\left(12 + \frac{x}{240}\right)}$$

$$= \frac{P}{E}\int_0^{2400} \frac{dx}{4\left(45 + \frac{x}{240}\right)\left(12 + \frac{x}{240}\right)}$$

$$= \frac{P}{4E}\int_0^{2400}\left(-\frac{1}{33}\right)\left(\frac{1}{45 + \frac{x}{240}} - \frac{1}{12 + \frac{x}{240}}\right)$$

$$= -\frac{P}{132E}\left[240\ln\left(45+\frac{x}{240}\right)-240\ln\left(12+\frac{x}{240}\right)\right]_0^{2400}$$

$$= -\frac{20\,000\times 240}{132\times 205\times 10^3}\,[\ln 55-\ln 45-\ln 22+\ln 10]$$

$$= (-0.1774)\,(-0.5878)$$

$$= 0.1043 \text{ mm}$$

1.9 ELONGATION DUE TO SELF-WEIGHT

The elongation due to self-weight of bars of rectangular and conical sections may be considered as follows:

Rectangular Section

Consider a bar hanging freely under its own weight as shown in Fig. 1.21.
Consider a small length δx of the bar at a distance x from the free end.

Let A = area of cross-section of the bar
w = weight per unit volume of the bar
W = weight of the whole bar = wAL
W_x = weight of the bar below the small section = wAx

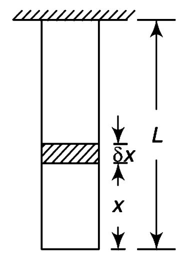


Fig. 1.21

The extension of the small length = $\dfrac{W_x \cdot \delta x}{A \cdot E} = \dfrac{wAx \cdot \delta x}{A \cdot E}$

$$\text{Extension of the whole rod} = \int_0^L \frac{wx}{E}\cdot dx = \frac{w}{E}\left(\frac{x^2}{2}\right)_0^L = \frac{wL^2}{2E} = \frac{wAL\cdot L}{2AE} = \frac{WL}{2AE} \qquad (1.9)$$

= deformation due to a weight W at the lower end/2

Thus the deformation of the bar under its own weight is equal to half the deformation due to a direct load equal to the weight of the body applied at the lower end.

Conical Section

Consider a small length δx of the bar at a distance x from the free end (Fig.1.22).

Let A = area of cross-section at the small length
w = weight per unit volume of the bar
W = weight of the whole bar = wAL
W_x = weight of the bar below the section = $wAx/3$

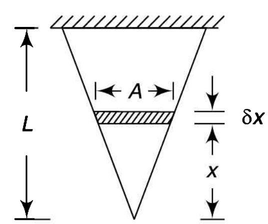


Fig. 1.22

The extension of a small length = $\dfrac{W_x \cdot \delta x}{A \cdot E} = \dfrac{wAx \cdot \delta x}{3A \cdot E}$

$$\text{Extension of the whole rod} = \int_0^L \frac{wx}{3E}\cdot dx = \frac{w}{3E}\int_0^L x\cdot dx = \frac{w}{3E}\left(\frac{x^2}{2}\right)_0^L = \frac{wL^2}{6E} = \frac{wAL\cdot L}{6AE} = \frac{WL}{6AE} \qquad (1.10)$$

Comparing it with Eq. 1.9, this elongation is one-third that of the rectangular section of the same length under own weight of the bar.

Example 1.11 A steel wire of 8-mm diameter is used to lift a weight of 1.5 kN at its lowest end. The density of the wire material is 8000 kg/m^3. Determine the elongation of the wire if the length of the wire is 100 m. E = 205 GPa.

Solution

Given A steel wire

d = 8 mm　　ρ = 8000 kg/m^3

L = 100 m　　E = 205 GPa

P = 1500 N

To find Elongation of wire

Weight of wire

Specific weight of wire, $w = 8000 \times 9.81 \text{ N/m}^3 = 8 \times 9.81 \times 10^{-6} \text{ N/mm}^3$

$$\text{Weight of wire, } W = \frac{\pi}{4} \times (0.008)^2 \times 100 \times 8 \times 9.81 = 0.3945 \text{ N}$$

Elongation of wire

$$\text{Elongation of wire due to load} = \frac{PL}{AE} = \frac{1500 \times 100\,000}{(\pi/4) \times 8^2 \times 205\,000} = 14.56 \text{ mm}$$

$$\text{Elongation of wire due to self-weight} = \frac{wL^2}{2E} \qquad \text{...(Eq.1.9)}$$

$$= \frac{8 \times 9.81 \times 10^{-6} \times (100 \times 10^3)^2}{2 \times 205\,000} = 1.91$$

Total elongation = 14.56 + 1.91 = 16.47 mm

1.10 COLUMN OF UNIFORM STRENGTH

Let a bar of varying cross-sectional area be acted upon by a load P as shown in Fig. 1.23. It is desired to know the cross-section of the bar so that it has a constant uniform compressive stress σ throughout when the weight of the bar is also taken into account.

Consider a small length dx at a distance x from the top.

Let A = area at distance x

$A + dA$ = area at distance $x + dx$

w = weight per unit volume of the bar

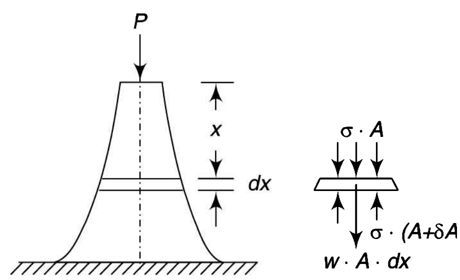


Fig. 1.23

Considering the balance of forces acting on the small length,

$$\sigma(A + dA) = \sigma A + \text{weight of the small length } dx \text{ of the bar}$$

or $$\sigma(A + dA) = \sigma A + wAdx$$

or $$\sigma \cdot dA = wAdx$$

or $$\frac{dA}{A} = \frac{w}{\sigma}dx$$

Integrating both sides, $\log_e A = \dfrac{w}{\sigma}x + C$

At the top, where $x = 0$, let Area $A = a$

Then, $\log_e a = 0 + C$ or $C = \log_e a$

Thus $$\log_e A = \frac{w}{\sigma}x + \log_e a \quad \text{or} \quad \log_e \frac{A}{a} = \frac{w}{\sigma}x \quad \text{or} \quad \frac{A}{a} = e^{wx/\sigma} \quad \text{or} \quad A = ae^{wx/\sigma} \tag{1.11}$$

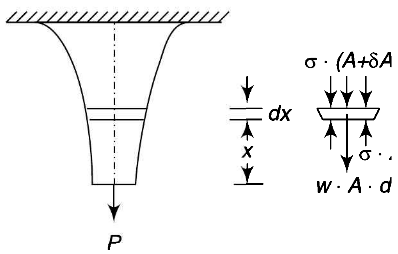


Fig. 1.24

- The bar may also have been considered suspending from top (tie bar) as shown in Fig. 1.24. The force balance will be (σ is tensile in this case),

  $\sigma (A + dA) = \sigma A$ + weight of the small length dx of the bar

  i.e., the same equation as above. Thus the final expression will be the same if x is taken from bottom and a is the area at the bottom.

Example 1.12 A vertical tie bar is used to withstand a uniform tensile stress of 30 MPa throughout its length. Determine the diameter of the section at a point 6 m above the section where the diameter is 30 mm. Density of the bar material is 7500 kg/m³.

Solution

Given A tie bar of uniform stress of 30 MPa

$\rho = 7500$ kg/m³

To find Diameter of a section at 6 m above the section of 30-mm diameter

Specific weight, $w = 7500 \times 9.81$ N/m³ $= 7.5 \times 9.81 \times 10^{-6}$ N/mm³

Calculation of diameter

We have, $A = ae^{wx/\sigma}$ ...(Eq. 1.11)

or $$D^2 = d^2 e^{wx/\sigma} = 30^2 \cdot e^{7.5\times 9.81\times 10^{-6}\times 3000/30}$$

$$= 900 \times 1.0148 = 913.34$$

or $$D = 30.22 \text{ mm}$$

1.11 STATICALLY INDETERMINATE SYSTEMS

When a system comprises two or more members of different materials, the forces in various members cannot be determined by the principle of statics alone. Such systems are known as *statically indeterminate systems*. In such systems, additional equations are required to supplement the equations of statics to determine the unknown forces. Usually, these equations are obtained from deformation conditions of the system and are known as *compatibility equations*. A compound bar is a case of an indeterminate system and is discussed below:

Compound Bar

A bar consisting of two or more bars of different materials in parallel is known as a *composite* or *compound bar*. In such a bar, the sharing of load by each can be found by applying equilibrium and the compatibility equations.

Consider the case of a solid bar enclosed in a hollow tube as shown in Fig. 1.25. Let the subscripts 1 and 2 denote the solid bar and the hollow tube respectively.

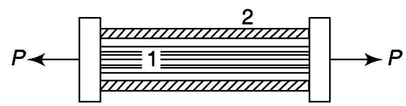


Fig. 1.25

Equilibrium Equation As the total load must be equal to the load taken by individual members,

$$P = P_1 + P_2 \tag{i}$$

Compatibility Equation The deformation of the bar must be equal to the tube.

$$\frac{P_1 L}{A_1 E_1} = \frac{P_2 L}{A_2 E_2} \quad \text{or} \quad P_1 = \frac{P_2 A_1 E_1}{A_2 E_2} \tag{ii}$$

Inserting (ii) in (i),

$$P = \frac{P_2 A_1 E_1}{A_2 E_2} + P_2 = \frac{P_2 A_1 E_1 + P_2 A_2 E_2}{A_2 E_2} = \frac{P_2 (A_1 E_1 + A_2 E_2)}{A_2 E_2}$$

or
$$P_2 = \frac{P \cdot A_2 E_2}{A_1 E_1 + A_2 E_2} \tag{1.12}$$

Similarly,
$$P_1 = \frac{P \cdot A_1 E_1}{A_1 E_1 + A_2 E_2} \tag{1.13}$$

Example 1.13 Three equally spaced rods in the same vertical plane support a rigid bar *AB*. Two outer rods are of brass, each 600 mm long and of 25 mm in diameter. The central rod is of steel that is 800 mm long and 30 mm in diameter. Determine the forces in the rods due to an applied load of 120 kN through the midpoint of the bar. The bar remains horizontal after the application of load. Take $E_s/E_b = 2$.

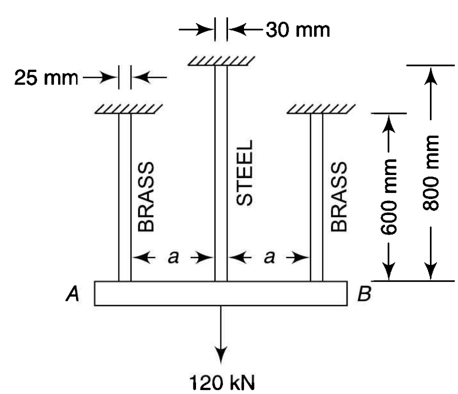


Fig. 1.26

Solution

Given A rigid bar system as shown in Fig. 1.26.

$$E_s/E_b = 2.$$

To find Forces in brass and steel rods

Applying compatibility equation

As the bar remains horizontal after the application of load, the elongation of each of the brass bars and of the steel bar are the same, $\Delta_b = \Delta_s$

or $$\frac{P_b L_b}{A_b E_b} = \frac{P_s L_s}{A_s E_s}$$

or $$P_b = \frac{L_s}{L_b} \cdot \frac{E_b}{E_s}\left(\frac{d_b}{d_s}\right)^2 P_s = \frac{800}{600} \cdot \frac{1}{2}\left(\frac{25}{30}\right)^2 P_s$$

or $$P_b = 0.463 P_s$$

Applying equilibrium equation

$$2P_b + P_s = P$$

or $$2 \times 0.463\, P_s + P_s = 120$$

or $$1.926\, P_s = 120$$

or $$P_s = 62.3 \text{ kN and } P_b = 28.\ 84 \text{ kN}$$

Example 1.14 Three equidistant vertical rods, each of 20 mm diameter, support a load of 25 kN in the same plane as shown in Fig. 1.27. Initially, all the rods are adjusted to share the load equally. Neglecting any chance of buckling, and taking E_s= 205 GPa and E_c = 100 GPa, determine the final stresses when a further load of 20 kN is added.

Solution

Given Three equidistant vertical rods supporting a load of 25 kN as shown in Fig. 1.27.

$L_s = 3.6$ m $E_s = 205$ Gpa

$L_c = 2.8$ m $E_c = 100$ Gpa

$d = 20$ mm

Initial load = 25 kN

To find Final stresses when a further load of 20 kN is added

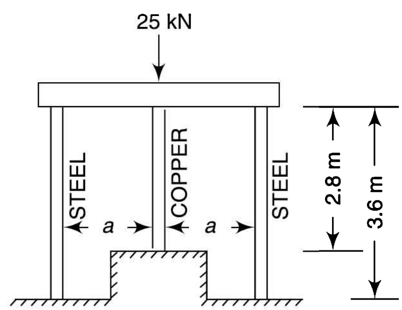


Fig. 1.27

$$A = (\pi/4)\, 20^2 = 100\,\pi \text{ mm}^2$$

Finding of initial stress in each rod

$$\sigma_i = \frac{25\,000}{100\pi \times 3} = 26.53 \text{ MPa}$$

Finding of additional stresses in each rod

On adding a further load of 20 kN, let the increase of stress in the steel rod be σ_s and in the copper rod be σ_c.

From equilibrium equation, the additional load P is

$$(2\sigma_s + \sigma_c)A = P \quad \text{or} \quad (2\sigma_s + \sigma_c) \times 100\pi = 20\,000 \qquad \text{(i)}$$

From the compatibility equation, $\delta_c = \delta_s$

$$\frac{\sigma_c L_c}{E_c} = \frac{\sigma_s L_s}{E_s}$$

or $$\sigma_c = \frac{L_s}{L_c} \cdot \frac{E_c}{E_s}\sigma_s = \frac{3.6}{2.8} \times \frac{100\,000}{205\,000}\sigma_s \quad \text{or} \quad \sigma_c = 0.627\,\sigma_s$$

Inserting this value of σ_c in (i)

$$(2\sigma_s + 0.627\sigma_s) \times 100\,\pi = 20\,000$$

or $$2.627\,\sigma_s = 63.662$$

or $$\sigma_s = 24.23 \text{ MPa and } \sigma_c = 15.19 \text{ Mpa}$$

Finding of total stresses in each rod

Final stress in steel rod = 24.23 + 26.53 = 50.76 MPa

Final stress in copper rod = 15.19 + 26.53 = 41.72 MPa

Example 1.15 A steel rod of 16-mm diameter passes through a copper tube of 20-mm internal diameter and of 32-mm external diameter. The steel rod is fitted with nuts and washers at each end. The nuts are tightened till a stress of 24 MPa is developed in the steel rod. A cut is then made in the copper tube for half the length to remove 2 mm from its thickness. Assuming the Young's modulus of steel to be twice that of copper, determine

(i) the stress existing in the steel rod

(ii) the stress in the steel rod if a compressive load of 6 kN is applied to the ends of the steel rod

Solution

Given

— A steel rod in a hollow copper tube, fitted with nuts at ends as shown in Fig. 1.28.

— The nuts are tightened till a stress of 24 MPa is developed in steel rod

— A cut is made in copper tube for half the length and 2 mm is removed from its thickness.

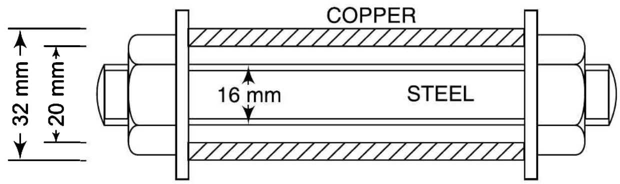


Fig. 1.28

To find

— Stress in steel rod

— Stress in steel rod when a compressive load of 6 kN is applied to ends of steel rod.

$$A_s = \frac{\pi}{4} \times 16^2 = 64\pi \text{ mm}^2 \quad \text{and} \quad A_c = \frac{\pi}{4}(32^2 - 20^2) = 156\pi \text{ mm}$$

On tightening the nut, the steel rod is elongated and the stress induced is tensile whereas the tube is shortened and the stress is compressive.

Let σ_{s1} = stress in the steel rod = 24 MPa

σ_{c1} = stress in the copper tube

Determination of initial stress in copper tube

From equilibrium equation

Push on copper tube = Pull on steel rod

$$\sigma_{c1} \times A_c = \sigma_{s1} \times A_s$$

or
$$\sigma_{c1} \times 156\pi = 24 \times 64\pi$$

or
$$\sigma_{c1} = 9.846 \text{ MPa} \quad \text{(compressive)}$$

When the copper tube is reduced in diameter

A_c' = reduced area of cross-section of the tube = $\frac{\pi}{4}(28^2 - 20^2) = 96\pi$

Let σ_{s2} = stress in the steel rod

σ_{c2}' = stress in the reduced section of tube

and σ_{c2} = stress in the remaining section of tube

From equilibrium equation,

Forces in each section of copper tube as well as in the steel rod are to be equal

i.e., $$\sigma_{c2} \times 156\,\pi = \sigma'_{c2} \times 96\,\pi = \sigma_{s2} \times 64\,\pi \qquad \text{(i)}$$

$$\sigma_{c2} = 0.4103\,\sigma_{s2} \quad \text{and} \quad \sigma'_{c2} = 0.6667\,\sigma_{s2}$$

Compatibility equation

When the cross-section of the tube is reduced, the change in length of the rod as well as of the tube is to be of the same nature, i.e., either the length of both is increased or decreased. Let us assume that the length of each is reduced which means a reduction of tensile stress in the rod and increase of compressive stress in the tube.

Thus reduction in length of steel rod = reduction in length of copper tube

$$\frac{\sigma_{s1} - \sigma_{s2}}{E_s} \cdot L = \frac{\sigma_{c2} - \sigma_{c1}}{E_c} \cdot \frac{L}{2} + \frac{\sigma'_{c2} - \sigma_{c1}}{E_c} \cdot \frac{L}{2}$$

or $$\sigma_{s1} - \sigma_{s2} = \sigma_{c2} + \sigma'_{c2} - 2\sigma_{c1} \;\ldots\ldots\; (E_s = 2E_c) \qquad \text{(ii)}$$

or $$24 - \sigma_{s2} = 0.4103\sigma_{s2} + 0.6667\sigma_{s2} - 2 \times 9.846$$

or $$2.077\sigma_{s2} = 43.692 \quad \text{or} \quad \sigma_{s2} = 21.036 \text{ MPa}$$

As the stress in the steel rod is decreased from 24 MPa to 21.036 MPa, the assumption of reduction of the length of the two is correct. In case, the lengths are assumed to be increased, the stress in the steel rod is increased and in the copper tube decreased. The equation formed would have been

$$\frac{\sigma_{s2} - \sigma_{s1}}{E_s} \cdot L = \frac{\sigma_{c1} - \sigma_{c2}}{E_s} \cdot \frac{L}{2} + \frac{\sigma_{c1} - \sigma'_{c2}}{E_s} \cdot \frac{L}{2}$$

and the result would have been the same, i.e., $\sigma_{s2} = 21.036$ MPa which would have indicated that the length actually would be reduced due to decrease in the stress of steel rod.

When a compressive load of 6 kN is applied to the ends of the steel rod

When a compressive load of 6 kN is applied to the ends of the steel rod, the length of the rod is further reduced.

From equilibrium equation,

$$\sigma_{c3} \times 156\,\pi = \sigma'_{c3} \times 96\,\pi = \sigma_{s3} \times 64\,\pi + 6000 \qquad \text{[as in (i)]}$$

or $$\sigma_{c3} \times 156 = \sigma'_{c3} \times 96 = \sigma_{s3} \times 64 + 1909.9$$

or $$\sigma_{c3} = 0.4103\,\sigma_{s3} + 12.243 \quad \text{and} \quad \sigma'_{c3} = 0.6667\,\sigma_{s3} + 19.895$$

Applying compatibility equation,

$$\sigma_{s1} - \sigma_{s3} = \sigma_{c3} + \sigma'_{c3} - 2\sigma_{c1} \qquad \text{[as in (ii)]}$$

or $$24 - \sigma_{s3} = 0.4103\sigma_{s3} + 12.243 + 0.6667\sigma_{s3} + 19.895 - 2 \times 9.846$$

or $$2.077\sigma_{s3} = 11.554$$

or $$\sigma_{s3} = 5.56 \text{ MPa}$$

Example 1.16 A round steel rod supported in a recess is surrounded by a co-axial brass tube as shown in Fig. 1.29. The level of the upper end of the rod is 0.08 mm below that of the tube. Determine

(i) the magnitude of the maximum permissible axial load which can be applied to a rigid plate resting on the top of the tube, the permissible values of the compressive stresses are 105 MPa for steel and 75 MPa for brass

(ii) the amount by which the tube is shortened by a load if the compressive stresses in the steel and the brass are the same

Take E_s = 210 GPa and E_b = 105 GPa.

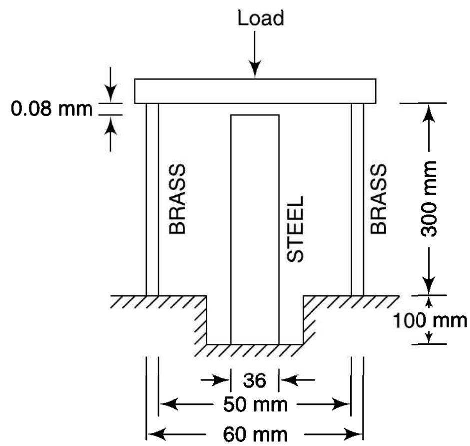


Fig. 1.29

Solution

Given A round steel rod supported in a recess and surrounded by a co-axial brass tube as shown in Fig. 1.29.

$\sigma_{smax} = 105$ MPa $\qquad \sigma_{bmax} = 75$ MPa

$E_s = 210$ GPa $\qquad E_b = 105$ GPa

To find

— Maximum axial load to be applied to rigid plate

— Shortening of tube, if compressive stresses in steel and brass are same

$$A_s = \frac{\pi}{4} \times 36^2 = 324\pi \text{ mm}^2 \quad \text{and} \quad A_b = \frac{\pi}{4} \times (60^2 - 50^2) = 275\pi \text{ mm}^2$$

Determination of load before the compression of the rod

Let W_b be the load applied for the initial compression of the tube before the compression of the rod starts. Then as

$$\delta_b = \frac{\sigma_b L}{E} \quad \text{or} \quad 0.08 = \frac{\sigma_b \times 300}{105\,000} \quad \text{or} \quad \sigma_b = 28 \text{ MPa}$$

$$\therefore \qquad W_b = 28 \times 275\pi = 24\,190 \text{ N}$$

Determination of additional load

Limiting value of stress in the brass = 75 MPa

$\therefore$ Maximum value of stress due to additional load can be = 75 – 28 = 47 MPa

Let W be the additional load to compress both, the tube and the bar. Let σ_s be the stress induced in the steel rod and σ_b the additional stress in the brass tube.

From equilibrium equation, $\sigma_s . A_s + \sigma_b A_b = W$

From compatibility equation, $\Delta_s = \Delta_b$

$$\text{or} \qquad \frac{\sigma_s L_s}{E_s} = \frac{\sigma_b L_b}{E_b} \quad \text{or} \quad \sigma_s = \frac{L_b}{L_s} \cdot \frac{E_s}{E_b} \sigma_b = \frac{300}{400} \cdot \frac{210\,000}{105\,000} \sigma_b$$

$$\text{or} \qquad \sigma_s = 1.5\sigma_b$$

$\therefore$ stress induced in the steel rod = 1.5 × 47 = 70.5 MPa

It is less than the permissible value of stress for steel.

Thus $W = p_s . A_s + p_b A_b = 70.5 \times 324\pi + 47 \times 275\pi = 112\ 365$ N

Determination of total load

Total maximum load = 112 365 + 24 190 = 136 555 N or 136.555 kN

Determination shortening of tube

Let Δ be the shortening of the steel rod. This will also be the additional shortening of the brass tube. Then

$$\Delta_b + \Delta = \frac{\sigma_b \cdot L_b}{E_b}$$

or $$\sigma_b = \frac{105\ 000}{300}(0.08 + \Delta) \quad \text{and} \quad \sigma_s = \frac{210\ 000}{400} \cdot \Delta$$

Equating the stresses in the steel and the brass,

$$= \frac{105\ 000}{300}(0.08 + \Delta) = \frac{210\ 000}{400}\Delta \quad \text{or} \quad 0.08 + \Delta = 1.5\ \Delta$$

or $$0.5\ \Delta = 0.08 \text{ or } \Delta = 0.16 \text{ mm}$$

Total shortening = 0.08 + 0.16 = 0.24 mm

Example 1.17 Three wires of the same material and cross-section support a rigid bar which further supports a weight of 5 kN. The length of the wires is 5 m, 8 m and 6 m in order. The spacing between the wires is 2 m and the weight acts at 1.6 m from the first wire. Determine the load carried by each wire.

Solution

Given Three wires of same material and cross-section and of different lengths supporting a weight as shown in Fig. 1.30.

To find Load carried by each wire

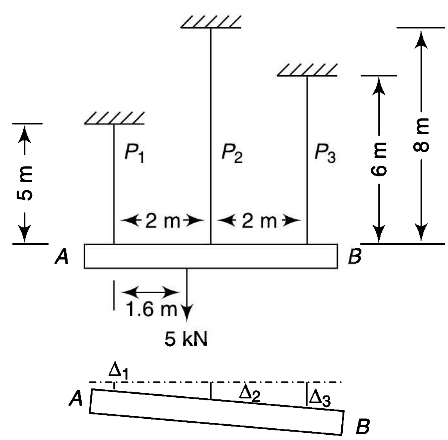


Fig. 1.30

As the wires are of different lengths and the weight suspended is unsymmetrical, the bar will not remain horizontal but will be tilted. Let it take the shape as shown in Fig. 1.30.

Let P_1, P_2 and P_3 be the loads taken by the first, second and the third wire respectively.

Equilibrium equation

$$P_1 + P_2 + P_3 = P = 5000 \qquad \text{(i)}$$

Moment equation

Taking moments about the first wire,

$$2P_2 + 4\ P_3 = 1.6 \times 5000 = 8000$$

or $$P_2 = 4000 - 2\ P_3 \qquad \text{(ii)}$$

Compatibility equation

From symmetry, $$\Delta_2 = \frac{\Delta_1 + \Delta_3}{2} \quad \text{or} \quad 2\left(\frac{P_2 L_2}{AE}\right) = \frac{P_1 L_1}{AE} + \frac{P_3 L_3}{AE}$$

or $$2\ P_2 L_2 = P_1 L_1 + P_3 L_3$$

or $$2\ P_2 \times 8 = P_1 \times 5 + P_3 \times 6$$

or $$16\ P_2 = 5\ P_1 + 6\ P_3$$

or $$16\ (4000 - 2\ P_3) = 5P_1 + 6\ P_3$$

or $$64\ 000 - 32\ P_3 = 5P_1 + 6\ P_3$$

or $$5P_1 = 64\,000 - 38\,P_3$$

or $$P_1 = 12\,800 - 7.6\,P_3 \tag{iii}$$

Inserting the values of P_1 and P_2 from (ii) and (iii) in (i),

$$12\,800 - 7.6\,P_3 + 4000 - 2\,P_3 + P_3 = 5000$$

or $$8.6\,P_3 = 11\,800 \quad \text{or} \quad P_3 = 1372 \text{ N} \quad \text{or} \quad 1.372 \text{ kN}$$

$$P_2 = 4000 - 2\,P_3 = 4000 - 2 \times 1372 = 1256 \text{ N or } 1.256 \text{ kN}$$

$$P_1 = 12\,800 - 7.6 \times 1372 = 2373 \text{ N or } 2.373 \text{ kN}$$

Example 1.18 A system of three bars supports a vertical load P as shown in Fig. 1.31. The outer bars are identical and of the same material whereas the inner bar is of different material. Determine the forces in the bars of the system.

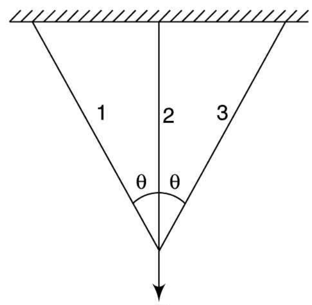


Fig. 1.31

Solution

Given A system of three bars supporting a vertical load. Bars 1 and 3 are identical and of same material.

To find Force in each bar

Owing to symmetry, forces in the outer bars 1 and 3 will be equal. Let it be P_1 and the force in the inner bar P_2. The dotted lines show the deformed shape of the system under the load P (Fig. 1.32).

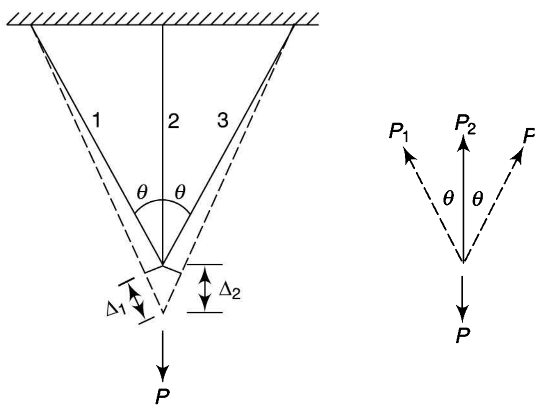


Fig. 1.32

Equilibrium equation

$$2\,P_1 \cos\theta + P_2 = P \quad \text{....(assuming negligible change in } \theta) \tag{i}$$

Compatibility equation

$$\Delta_1 = \Delta_2 \cos\theta \text{ or } \frac{P_1 L_1}{A_1 E_1} = \frac{P_2 L_2}{A_2 E_2} \cos\theta$$

or $$P_1 = \frac{A_1 E_1 P_2 L_2}{A_2 E_2 L_1} \cos\theta = \frac{A_1 E_1 P_2 (L_1 \cos\theta)}{A_2 E_2 L_1} \cos\theta = \frac{A_1 E_1 P_2}{A_2 E_2} \cos^2\theta \tag{ii}$$

Substituting this value of P_1 in (i),

$$2\,\frac{A_1 E_1 P_2}{A_2 E_2} \cos^3\theta + P_2 = P$$

or
$$P_2 = \frac{P}{1 + \dfrac{2A_1E_1}{A_2E_2}\cos^3\theta}$$

From (ii),
$$P_1 = \frac{A_1E_1}{A_2E_2} \cdot \frac{P}{1 + \dfrac{2A_1E_1}{A_2E_2}\cos^3\theta} \cdot \cos^2\theta$$

$$= P\cos^2\theta \frac{1}{\left(\dfrac{A_2E_2}{A_1E_1}\right)} \cdot \frac{1}{\left(1 + \dfrac{2A_1E_1}{A_2E_2}\cos^3\theta\right)} = \frac{P\cos^2\theta}{\left(\dfrac{A_2E_2}{A_1E_1} + 2\cos^3\theta\right)}$$

Example 1.19 A horizontal bar supported by two suspended vertical wires 240 mm apart fixed to a rigid support. A load W is attached to the bar. The left-hand side wire is of copper with a diameter of 5 mm and the right-hand side wire is of steel of 3-mm diameter. The length of both the wires is 4 m initially. Find the position of the weight on the bar relative to the copper wire so that both the wires extend by the same amount.

Also, calculate the load, stresses and the elongation of each wire if W = 1 kN. Neglect the weight of the bar and take E_s = 210 GPa and E_c = 120 GPa.

Solution

Given A horizontal bar system as shown in Fig. 1.33. Both wires extend by the same amount.

To find

— Position of weight on bar
— Load on each wire
— Stress in each wire
— Elongation of each wire

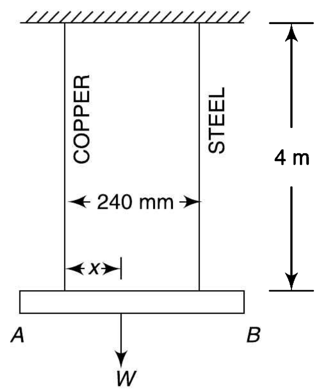


Fig. 1.33

Let the load W be placed at a distance x from the copper wire and P_s and P_c the forces in steel and copper wires respectively (Fig. 1.33).

$$A_c = \frac{\pi}{4}(5)^2 = 6.25\pi \text{ mm}^2 \text{ and } A_s = \frac{\pi}{4}(3)^2 = 2.25x \text{ mm}^2$$

Moments equations

$$\text{Taking moments about } A,\ 240\,P_s = W \cdot x \text{ or } P_s = \frac{W \cdot x}{240} \quad \text{(i)}$$

$$\text{Taking moments about } B,\ 240\,P_c = W(240 - x)$$

or
$$P_c = \frac{W \cdot (240 - x)}{240} \quad \text{(ii)}$$

Dividing (ii) by (i),
$$\frac{P_c}{p_s} = \frac{240 - x}{x} \quad \text{(iii)}$$

Compatibility equation

As both the wires extend by the same amount, $\Delta_c = \Delta_s$

or
$$\frac{P_cL_c}{A_cE_c} = \frac{P_sL_s}{A_sE_s} \quad \text{or} \quad \frac{P_c}{p_s} = \frac{A_c}{A_s} \cdot \frac{E_c}{E_s} \quad \ldots\ldots(\because L_c = L_s)$$

$$= \frac{6.25\pi}{2.25\pi} \cdot \frac{120\ 000}{210\ 000} = 1.587 \qquad \text{(iv)}$$

From (iii) and (iv), $\dfrac{240 - x}{x} = 1.587$ or $x = 92.77$ mm

Load on each wire

$$P_c = \frac{W(240 - x)}{240} = \frac{1000 \times (240 - 92.77)}{240} = 613.46 \text{ N}$$

$$P_s = \frac{Wx}{240} = \frac{1000 \times 92.77}{240} = 386.54 \text{ N}$$

Stress in each wire

$$\sigma_c = \frac{P_c}{A_c} = \frac{613.46}{6.25\pi} = 31.24 \text{ MPa}$$

$$\sigma_s = \frac{P_s}{A_s} = \frac{386.54}{2.25\pi} = 54.68 \text{ MPa}$$

Elongation of each wire

$$\Delta = \frac{\sigma_c \cdot L}{E_c} = \frac{31.24 \times 4000}{120\ 000} = 1.041 \text{ mm}$$

Example 1.20 Three identical pin-connected bars support a load P as shown in Fig. 1.34. All the bars are of the same area of cross-section and same length. Determine

(i) the force in each bar
(ii) the vertical displacement of the point where the load is applied
Neglect the possibility of lateral buckling of the bars.

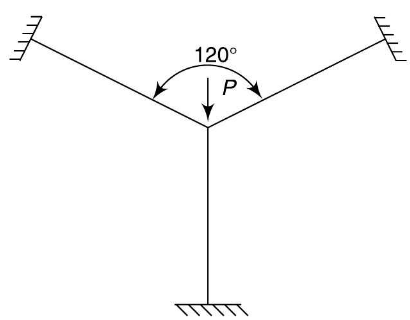


Fig. 1.34

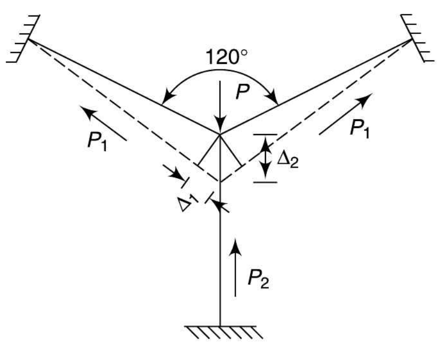


Fig. 1.35

Solution

Given Three identical pin-connected bars of equal cross-section and length supporting a load P.

To find

— Force in each bar
— Vertical displacement of load point

Figure 1.35 shows the deformed figure in dotted lines after the application of load. Assuming that there is negligible change in the angles after the deforming of the bars,

Equilibrium equation

$$2P_1 \cos 60° + P_2 = P$$

or
$$P_1 = P - P_2 \quad \text{(i)}$$

Compatibility equation

$$\Delta_1 = \Delta_2 \cos 60°$$

or
$$\frac{P_1 L}{AE} = \frac{P_2 L}{AE} \cos 60° \quad \text{or} \quad P_1 = \frac{P_2}{2} \quad \text{(ii)}$$

From (i) and (ii), $\frac{P_2}{2} = P - P_2$ or $P_2 = \frac{2P}{3}$

and
$$P_1 = P/3$$

Vertical displacement of the joint

$$\Delta_2 = \frac{P_2 L}{AE} = \frac{2PL}{3AE}$$

Example 1.21 A suspended bar system consists of two cross-sections as shown in Fig. 1.36. Initially its lower surface is 0.8 mm above the ground surface. Determine the reaction of the lower support and the stresses in each section when a load of 40 kN is applied as shown in the figure. Take E = 205 GPa

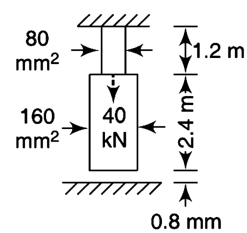


Fig. 1.36

Solution

Given A suspended bar system consisting of two cross-sections; lower surface is 0.8 mm above the ground surface.

To find Reaction of lower support and stresses in each section when a load of 40 kN is applied as shown in the figure.

Let R_1 = reaction of the upper support
R_2 = reaction of the lower support
Then $R_1 + R_2 = 40\,000$ or $R_1 = 40\,000 - R_2$

The free-body diagrams of the two portions of the bar system is shown in Fig. 1.37. It is clear that the upper portion is in tension whereas the lower portion is in compression.

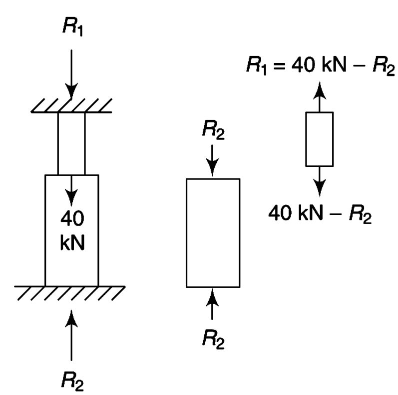


Fig. 1.37

Variation of lengths of two portions

Elongation of the upper portion, $\Delta_1 = \frac{P_1 L_1}{A_1 E} = \frac{(40\,000 - R_2) \times 1200}{80 \times 205\,000}$

Shortening of the lower portion, $\Delta_2 = \frac{P_2 L_2}{A_2 E} = \frac{R_2 \times 2400}{160 \times 205\,000}$

Compatibility equation

Elongation of upper portion – shortening of lower portion = net elongation = 0.8 mm

or
$$\frac{(40\,000 - R_2) \times 1200}{80 \times 205\,000} - \frac{R_2 \times 2400}{160 \times 205\,000} = 0.8$$

or
$$(40\,000 - R_2) \times 15 - 15R_2 = 0.8 \times 205\,000$$

or
$$40\,000 - 2R_2 = 10\,933 \text{ or } R_2 = 14\,533 \text{ N}$$

and
$$R_1 = 40\,000 - 14\,533 = 25\,467 \text{ N}$$

$\sigma_1 = 25\ 467/80 = 318.3$ MPa (tensile)
$\sigma_2 = 14\ 533/160 = 90.8$ MPa (compressive)

Example 1.22 A rigid horizontal bar *AB* hinged at *A* is supported by a 1.2-m long steel rod and a 2.4 m long bronze rod, both rigidly fixed at the upper ends (Fig.1.38). A load of 48 kN is applied at a point that is 3.2 m from the hinge point *A*. The areas of cross-section of the steel and bronze rods are 850 mm² and 650 mm² respectively. Find

(i) the stress in each rod
(ii) the reaction at the pivot point
E_s = 205 GPa and E_b = 82 GPa

Solution

Given A loaded bar system as shown in Fig. 1.38.

$A_s = 850$ mm² $\quad A_b = 650$ mm²
$E_s = 205$ GPa $\quad E_b = 82$ GPa

To find

— Stress in each rod
— Reaction at pivot point

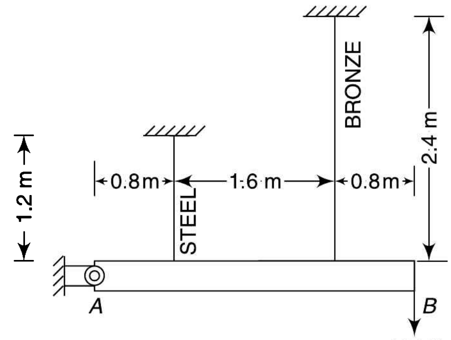


Fig. 1.38

Refer Fig. 1.39.

Let P_s and P_b be the forces in the steel and bronze wires respectively as the load is applied.

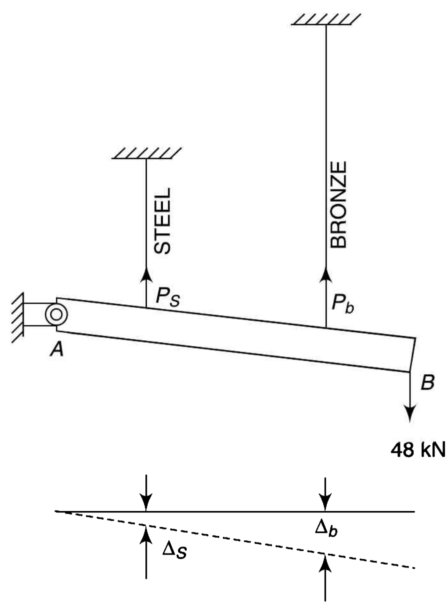


Fig. 1.39

Moments equation

Taking moments about the pivot point,

$$P_s \times 800 + P_b \times 2400 - 48\ 000 \times 3200 = 0$$

$$P_s + 3\,P_b = 192\ 000 \qquad \text{(i)}$$

Compatibility equation

$$\frac{\Delta_s}{\Delta_b} = \frac{800}{2400} = \frac{1}{3} \text{ or } \frac{\Delta_s}{1} = \frac{\Delta_b}{3}$$

or
$$\frac{P_s L_s}{A_s E_s} = \frac{1}{3}\frac{P_b L_b}{A_b E_b} \quad \text{or} \quad \frac{P_s \times 1200}{850 \times 205\,000} = \frac{1}{3} \times \frac{P_b \times 2400}{650 \times 82\,000}$$

$$P_s = 2.179\, P_b \qquad \text{(ii)}$$

Calculation of stresses

From (i) and (ii), $2.179\, P_b + 3\, P_b = 192\,000$

or $$P_b = 37\,073 \text{ N}$$

and $$P_s = 192\,000 - 3 \times 37\,073 = 80\,781 \text{ N}$$

$$\sigma_b = \frac{37\,073}{650} = 57.04 \text{ MPa and} \qquad \sigma_s = \frac{80\,781}{850} = 95.04 \text{ MPa}$$

Reaction at pivot point

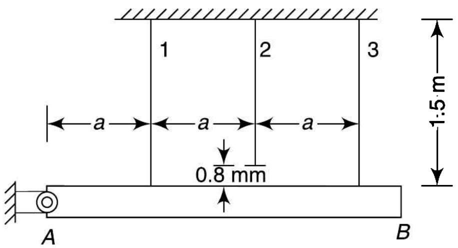
The reaction at the pivot can be found from force equation, let it be downwards,

$$P_s + P_b - R_a = 48\,000$$

$$R_a = 80\,781 + 37\,073 - 48\,000 = 69\,854 \text{ N or } 69.854 \text{ kN}$$

Thus the assumed direction is correct.

Example 1.23 A rigid bar *AB* is to be suspended from three steel rods as shown in Fig. 1.40. The lengths of the outer rods are 1.5 m each, whereas the length of the middle rod is shortened than these by an amount of 0.8 mm. The area of cross-section of all the rods is the same and is equal to 1600 mm². Determine the stresses in the rods after the assembly of the structure. E_s = 205 GPa.



(a)

Fig. 1.40

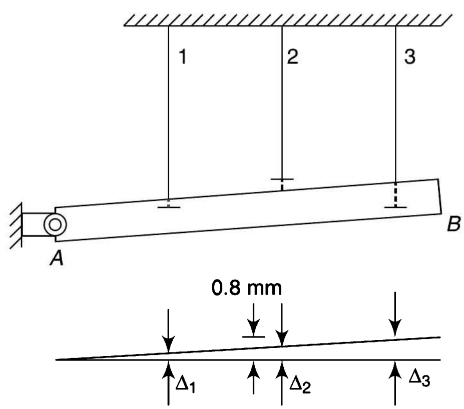


Fig. 1.41

Solution

Given A rigid bar to be supported by three steel rods of unequal lengths as shown in Fig. 1.40.

E_s = 205 GPa

To find Stress in each rod

The position of the rigid bar after the assembly is shown in Fig. 1.41. It is raised upward by amounts Δ_1, Δ_2 and Δ_3 at the rod positions 1, 2 and 3 respectively. Thus, the rods 1 and 3 are shortened by amounts Δ_1 and Δ_3 respectively whereas the rod 2 is elongated by an amount $(0.8 - \Delta_2)$.

Compatibility equations

$$\frac{\Delta_3}{\Delta_1} = \frac{3a}{a} \quad \text{or} \quad \frac{P_3L_3/A_3E_3}{P_1L_1/A_1E_1} = 3$$

As $L_3 = L_1, A_3 = A_1$ and $E_3 = E_1$

$\therefore$ $$P_3 = 3\,P_1 \tag{i}$$

Also, $$\frac{\Delta_2}{\Delta_1} = \frac{2a}{a} \quad \text{or} \quad \frac{0.8 - P_2L_2/A_2E_2}{P_1L_1/A_1E_1} = 2$$

The length of the rod 2 is shorter by 0.8 mm. However, to find the elongation of the rod, this may be ignored as its effect will be negligible and the length of rod 2 can be taken equal to that of rod 1.

Thus $L_2 = L_1$

Also, $A_2 = A_1$ and $E_2 = E_1$

$\therefore$ $$\frac{0.8 - P_2L_1/A_1E_1}{P_1L_1/A_1E_1} = 2 \quad \text{or} \quad \frac{0.8A_1E_1}{P_1L_1} - \frac{P_2}{P_1} = 2$$

or $$\frac{0.8 \times 1600 \times 205\,000}{P_1 \times 1500} - \frac{P_2}{P_1} = 2 \quad \text{or} \quad 174\,933 - P_2 = 2\,P_1$$

or $$2\,P_1 + P_2 = 174\,933 \tag{ii}$$

Moments equation

Taking moments about A, $P_1.a + P_3.3a = P_2.2a$

or $$P_1 + 3P_3 = 2P_2 \tag{iii}$$

Calculating forces and stresses in each rod

Solving (i), (ii) and (iii),

From (i) and (iii), $P_1 + 3 \times 3P_1 = 2P_2$

or $$P_2 = 5P_1 \tag{iv}$$

From (ii) and (iv), $2P_1 + 5P_1 = 174\,933$ or $P_1 = 24\,990$ N

$$P_2 = 24\,990 \times 5 = 124\,952 \text{ N}$$

From (i), $P_3 = 24\,990 \times 3 = 74\,971$ N

$$\sigma_1 = 24\,990/1600 = 15.62 \text{ MPa (comp)}$$

$$\sigma_2 = 15.62 \times 5 = 78.1 \text{ MPa (tensile)}$$

$$\sigma_3 = 15.62 \times 3 = 46.86 \text{ MPa (compressive)}$$

1.12 TEMPERATURE STRESSES

The length of a material which undergoes a change in temperature also changes and if the material is free to do so, no stresses are developed in the material. However, if the material is constrained, stresses are developed in the material which are known as *temperature stresses.*

Consider a bar of length L. If its temperature is increased through t^o, its length is increased by an amount $L.\alpha.t$, where α is the coefficient of thermal expansion (Fig. 1.42a). But if the bar is constrained and is prevented from expansion, the temperature stress is induced in the material which is given by (Fig. 1.42b)

$$\Delta = L\,\alpha\,t = \frac{\sigma L}{E}$$

or $$\sigma = \alpha\,t\,E \tag{1.14}$$

and $$\text{temperature strain, } \varepsilon = \sigma/E = \alpha.t \tag{1.15}$$

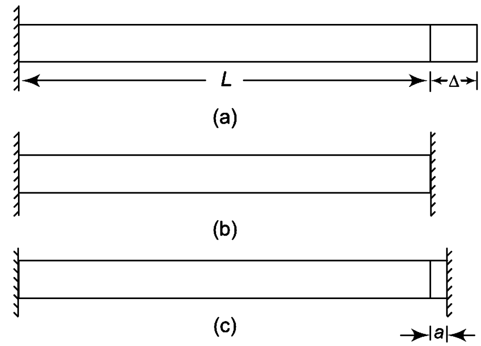


Fig. 1.42

- Temperature stresses are developed if a material is prevented from expansion which is $L\,\alpha\,t$. In case a support yields or is unable to prevent the expansion completely, then if the yield is a (Fig. 1.42c),

$$\Delta = (L\alpha t - a) = \frac{\sigma L}{E}$$

or
$$\sigma = \frac{(L\alpha t - a)E}{L}$$

- If a bar is of tapering section (Fig. 1.43)

$$\text{The elongation, } \Delta = \frac{4PL}{\pi EdD} \qquad \text{(Eq. 1.7)}$$

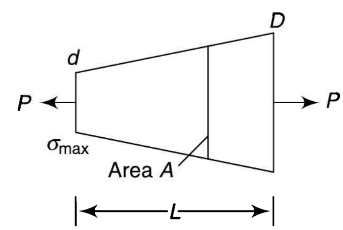


Fig. 1.43

$$\therefore \qquad \Delta = L\,\alpha\,t = \frac{4PL}{\pi EdD} \quad \text{or} \quad P = \frac{\alpha t \cdot \pi EdD}{4}$$

Stress at any cross-section $= \dfrac{P}{A} = \dfrac{\alpha t.\pi EdD}{4A}$ where A is the area of the cross-section.

Maximum stress will be at a section with minimum diameter,

Thus, $\sigma_{max} = \dfrac{\alpha t.\pi EdD}{4(\pi d^2/4)} = \dfrac{\alpha tED}{d}$

For bars of uniform cross-section, $d = D$ and thus $\sigma = \alpha tE$ as before.

Compound Sections

Consider a copper rod enclosed in a steel tube as shown in Fig. 1.44 rigidly joined at each end. Now, if the temperature is increased by t^o, the copper rod would tend to expand more as compared to the steel tube. As the two are joined together, the copper is prevented its full expansion and is put in compression. The final position of the compound bar will be as shown in the figure.

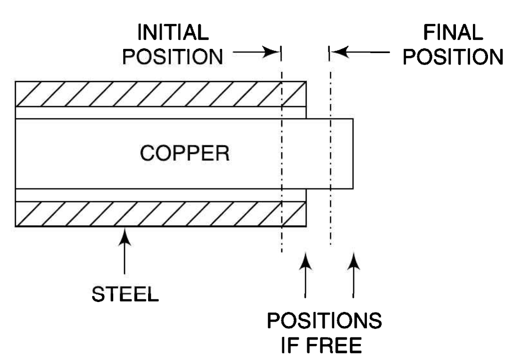


Fig. 1.44

Let

σ_s = tensile stress in steel

σ_c = tensile stress in copper

A_s = cross-sectional area of steel
A_c = cross-sectional area of copper

From Equilibrium Equation

Tensile force in steel = compressive force in copper

$$\sigma_s \cdot A_s = \sigma_c \cdot A_c \tag{1.16}$$

or

$$\sigma_s \cdot E_s \cdot A_s = \varepsilon_c E_c \cdot A_c$$

Compatibility Equation

Let α_s = coefficient of thermal expansion in steel
α_c = coefficient of thermal expansion in copper

Now Elongation of steel tube (due to temperature + due to tensile stress)
= Elongation of copper rod (due to temperature – due to compressive stress)

or Temperature strain of steel + tensile strain
= Temperature strain of copper – compressive strain

$$\alpha_s t + \sigma_s/E_s = \alpha_c t - \sigma_c/E_c$$

or

$$\alpha_s t + \varepsilon_s = \alpha_c t - \varepsilon_c$$

or

$$\varepsilon_s + \varepsilon_c = (\alpha_c - \alpha_s)\, t \tag{1.17}$$

Equations (1.16) and (1.17) are sufficient to solve the problems.

Example 1.24 Two parallel walls, 8 m apart, are to be stayed together by a steel rod of 30 mm diameter with the help of washers and nuts at the ends. The steel rod is passed through the metal plates and is heated. When its temperature is raised to 90°C, the nuts are tightened. Determine the pull in the bar when it is cooled to 24°C if

(i) the ends do not yield
(ii) the total yielding at the ends is 2 mm

E = 205 GPa and coefficient of thermal expansion of steel, $\alpha_s = 11 \times 10^{-6}$/°C.

Solution

Given Two parallel walls 8 m apart

d = 30 mm $\quad E$ = 205 GPa.
E = 205 GPa $\quad \alpha_s = 11 \times 10^{-6}$ /°C.

To find Pull in bar when temperature varies from 90°C to 24°C if

— ends do not yield
— total yielding at ends is 2 mm

$$A = \frac{\pi}{4}(30)^2 = 225\pi \text{ mm}^2$$

Pull when the ends do not yield

$$P = \sigma \cdot \mathrm{A} = \alpha tEA$$
$$= 11 \times 10^{-6} \times (90 - 24) \times 205\,000 \times 225\,\pi = 105\,202 \text{ N}$$

Pull when the ends yield by 2 mm

$$\Delta = (\alpha Lt - 2) = \frac{PL}{AE}$$

or

$$P = \alpha\, t\, A\, E - \frac{2AE}{L} = 105\,202 - \frac{2 \times 225\pi \times 205\,000}{8000}$$
$$= 105\,202 - 36\,227$$
$$= 68\,975 \text{ N} \quad \text{or} \quad 68.975 \text{ kN}$$

Example 1.25 A composite bar made up of copper, steel and brass is rigidly attached to the end supports as shown in Fig. 1.45. Determine the stresses in the three portions of the bar when the temperature of the composite system is raised by 70°C if

(i) the supports are rigid

(ii) the supports yield by 0.6 mm

$$E_c = 100 \text{ GPa}; E_s = 205 \text{ GPa}; E_b = 95 \text{ GPa}$$
$$\alpha_c = 18 \times 10^{-6}/°\text{C}; \alpha_s = 11 \times 10^{-6}/°\text{C}; \alpha_b = 19 \times 10^{-6}/°\text{C}$$

Solution

Given A composite bar as shown in Fig. 1.45.

$t = 70°\text{C}$

$E_c = 100$ GPa; $E_s = 205$ GPa; $E_b = 95$ GPa

$\alpha_c = 18 \times 10^{-6}/°\text{C}$; $\alpha_s = 11 \times 10^{-6}/°\text{C}$;

$\alpha_b = 19 \times 10^{-6}/°\text{C}$

To find Stresses in three portions if

— supports are rigid

— supports yield by 0.6 mm.

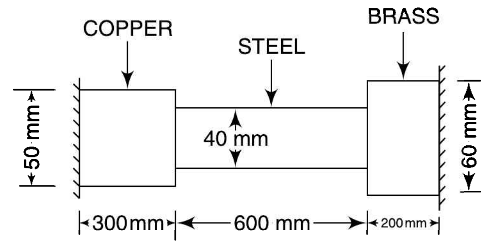


Fig. 1.45

$$A_c = (\pi/4)\ 50^2 = 625\ \pi \text{ mm}^2$$
$$A_s = (\pi/4)\ 40^2 = 400\ \pi \text{ mm}^2$$
$$A_b = (\pi/4)\ 60^2 = 900\ \pi \text{ mm}^2$$

From equilibrium equations

When the temperature is raised, each portion tends to elongate which is resisted by the rigid supports and the compressive stresses are developed in each portion. However, the forces so developed in each portion are equal,

i.e.,
$$\sigma_c \cdot A_c = \sigma_s \cdot A_s = \sigma_b \cdot A_b$$

or
$$\sigma_c = \frac{A_b}{A_c}\sigma_b = \frac{900\pi}{625\pi} \times \sigma_b = 1.44\sigma_b$$

and
$$\sigma_s = \frac{A_b}{A_s}\sigma_b = \frac{900\pi}{400\pi} \times \sigma_b = 2.25\sigma_b$$

Elongation in the absence of supports

$$\Delta = \Delta_c + \Delta_s + \Delta_b$$
$$= \alpha_c L_c t_c + \alpha_s L_s t_s + \alpha_b L_b t_b$$
$$= 18 \times 10^{-6} \times 300 \times 70 + 11 \times 10^{-6} \times 600 \times 70 + 19 \times 10^{-6} \times 200 \times 70$$
$$= 70 \times 10^{-6}\ (5400 + 6600 + 3800)$$
$$= 1.106 \text{ mm}$$

Calculation of stresses

Due to supports, this elongation is prevented and stresses are developed in the materials.

$$\Delta = \frac{\sigma_c L_c}{E_c} + \frac{\sigma_s L_s}{E_s} + \frac{\sigma_b L_b}{E_b}$$

Thus,
$$\frac{1.44\sigma_b \times 300}{100\,000} + \frac{2.25\sigma_b \times 600}{205\,000} + \frac{\sigma_b \times 200}{95\,000} = 1.106$$

or
$$(0.004\,32 + 0.006\,59 + 0.002\,11)\ \sigma_b = 1.106$$
$$0.013\,02\ \sigma_b = 1.106$$
$$\sigma_b = 84.95 \text{ MPa}$$

$$\sigma_c = 84.95 \times 1.44 = 122.33 \text{ MPa}$$
$$\sigma_s = 84.95 \times 2.25 = 191.13 \text{ MPa}$$

When the supports yield by 0.6 mm

$$0.0132\,\sigma_b = 1.106 - 0.6 = 0.506$$
$$\sigma_b = 38.33 \text{ MPa}$$
$$\sigma_c = 38.33 \times 1.44 = 55.20 \text{ MPa}$$
$$\sigma_s = 38.33 \times 2.25 = 86.24 \text{ MPa}$$

Example 1.26 A steel tube of 35-mm outer diameter and 30-mm inner diameter encloses a gunmetal rod of 25-mm diameter and is rigidly joined at each end. If at a temperature of 40°C there is no longitudinal stress, determine the stresses developed in the rod and the tube when the temperature of the assembly is raised to 240°C.

Coefficient of thermal expansion of steel = 11×10^{-6}/°C.
Coefficient of thermal expansion of gun metal = 18×10^{-6}/°C.
Young's modulus for steel = 205 GPa
Young's modulus for gun metal = 91.5 GPa
Also find the increase in length if the original length of the assembly is 1 m.

Solution

Given A composite system at 40°C as shown in Fig. 1.46.

$E_c = 91.5$ GPa $E_s = 205$ GPa;
$\alpha_c = 18 \times 10^{-6}$/°C $\alpha_s = 11 \times 10^{-6}$/°C
$L = 1$ m $t = 240°$

To find

— Stresses in rod and tube
— Increase in length

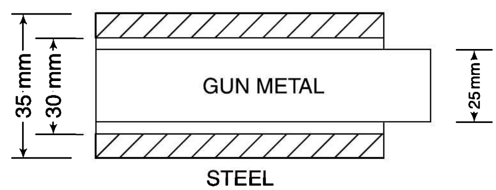


Fig. 1.46

$$A_s = \frac{\pi}{4}(35^2 - 30^2) = 255.25 \text{ mm}^2$$

$$A_g = \frac{\pi}{4} \times 25^2 = 490.87 \text{ mm}^2$$

As the coefficient of expansion of the gunmetal is more as compared to that of steel, the final expansion will be less than the free expansion of gunmetal due to temperature rise and thus compressive stresses will be developed in the gunmetal rod. In a similar way, as the coefficient of expansion of the steel is less, the final expansion will be more than the free expansion of steel due to temperature rise and thus it will have tensile stresses.

Compatibility equation

Temperature strain of steel + tensile strain = Temperature strain of copper – compressive strain

i.e., $$\alpha_s t + \frac{\sigma_s}{E_s} = \alpha_g t - \frac{\sigma_g}{E_g} \quad \text{or} \quad \alpha_s t + \frac{P}{A_s E_s} = \alpha_g t - \frac{P}{A_g E_g}$$

or $$P\left(\frac{1}{A_s E_s} + \frac{1}{A_g E_g}\right) = t(\alpha_g - \alpha_s)$$

or $$P = \frac{t(\alpha_g - \alpha_s)}{\dfrac{1}{A_s E_s} + \dfrac{1}{A_g E_g}} = \frac{(240 - 40)(18 - 11) \times 10^{-6}}{\dfrac{1}{255.25 \times 205\,000} + \dfrac{1}{490.87 \times 91\,500}}$$

$$= \frac{1400 \times 10^{-6}}{19.11 \times 10^{-9} + 22.26 \times 10^{-9}} = 33\,841\text{ N}$$

$$\sigma_s = \frac{33\,841}{255.25} = 132.6\text{ MPa and } \sigma_g = \frac{33\,841}{490.87} = 68.94\text{ MPa}$$

Increase in length

Increase in length of assembly

= Elongation of steel tube (due to temperature + due to tensile stress)

= Elongation of copper rod (due to temperature – due to compressive stress)

Using the first equation, increase in length

$$= \alpha_s Lt + \frac{\sigma_s L}{E_s} = L\left(\alpha_s t + \frac{\sigma_s}{E_s}\right) = 1000\left(11 \times 10^{-6} \times 200 + \frac{132.6}{205\,000}\right) = 2.847\text{ mm}$$

Example 1.27 Rails are laid such that there is no stress in them at 24°C. If the rails are 32 m long. Determine

(i) the stress in the rails at 80°C, when there is no allowance for expansion
(ii) the stress in the rails at 80°C, when there is an expansion allowance of 8 mm per rail
(iii) the expansion allowance for no stress in the rails at 80°C
(iv) the maximum temperature for no stress in the rails when expansion allowance is 8 mm

Coefficient of linear expansion, $\alpha = 11 \times 10^{-6}$/°C and E = 205 GPa

Solution

Given Rail having no stress at 24°C

$$L = 32\text{ m}$$

$$\alpha = 11 \times 10^{-6}/°\text{C} \qquad E = 205\text{ GPa}$$

To find

— Stress at 80°C, no allowance for expansion
— Stress at 80°C, expansion allowance 8 mm per rail
— Expansion allowance for no stress in rails at 80°C
— Maximum temperature for no stress, expansion allowance 8 mm

Change in temperature = 80° – 24° = 56°

Stress at 80°C, no allowance for expansion

$$\sigma = \alpha\, t\, E = 11 \times 10^{-6} \times 56 \times 205\,000 = 126.28\text{ MPa}$$

When expansion allowance is 8 mm

$$\Delta = \alpha Lt - 8 = \frac{\sigma L}{E}$$

or $$11 \times 10^{-6} \times 32\,000 \times 56 - 8 = \frac{\sigma \times 32\,000}{205\,000}$$

or $$19.712 - 8 = 0.1561\,\sigma \text{ or } \sigma = 75.03\text{ MPa}$$

If stresses are to be zero

The expansion allowance

$$\Delta = \alpha Lt = 11 \times 10^{-6} \times 32\,000 \times 56 = 19.71\text{ mm}$$

No stress in the rails, expansion allowance 8 mm

$$8 = \alpha Lt$$

or $$8 = 11 \times 10^{-6} \times 32\,000 \times t \quad \text{or} \quad t = 22.73°\text{C}$$

Example 1.28 A steel rod of 16-mm diameter and 3-m length passes through a copper tube of 50-mm external and 40-mm internal diameter and of the same length. The tube is closed at each end with the help of 30-mm thick steel plates which are tightened by nuts till the length of the copper tube is reduced by 0.6 mm. The temperature of the whole assembly is then raised by 56°C. Determine the stresses in the steel and copper before and after the rise of temperature. Assume that the thickness of the steel plates at the ends do not change during tightening of the nuts.

$$E_s = 210 \text{ GPa}; E_c = 100 \text{ GPa}; \alpha_s = 12 \times 10^{-6}/°\text{C}; \alpha_c = 17 \times 10^{-6}/°\text{C}$$

Solution

Given An assembly of steel rod and copper tube as shown in Fig. 1.47.

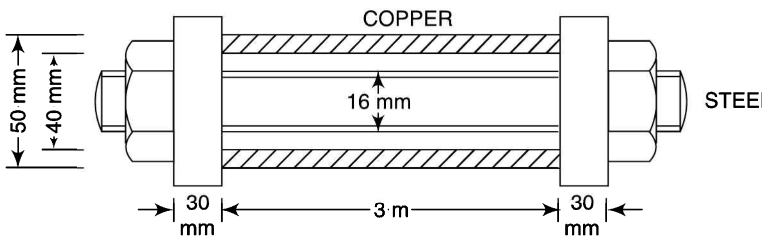


Fig. 1.47

$t = 56°\text{C}$

$E_s = 210$ GPa; $E_c = 100$ GPa; $\alpha_s = 12 \times 10^{-6}/°\text{C}$; $\alpha_c = 17 \times 10^{-6}/°\text{C}$

To find Stresses in the steel and copper

$$A_s = (\pi/4)\, 16^2 = 64\,\pi \text{ mm}^2;$$
$$A_c = (\pi/4)\,[50^2 - 40^2] = 225\,\pi \text{ mm}^2$$

Stresses due to tightening of nuts

As $\Delta = \dfrac{\sigma_c L}{E_c}$ $\quad\therefore\quad 0.6 = \dfrac{\sigma_c \times 3000}{100\,000}$ or $\sigma_c = 20$ MPa (compressive)

The force in the rod and the tube is the same, $\sigma_s \cdot A_s = \sigma_c \cdot A_c$

or $\sigma_s \times 64\,\pi = 20 \times 225\,\pi$

or $\sigma_s = 70.3$ MPa (tensile)

Stresses due to temperature rise

As the coefficient of expansion of copper is more than that of steel, it expands more. Thus, compressive stress is induced in the copper tube and tensile in the steel rod.

As $\sigma_s \cdot A_s = \sigma_c \cdot A_c$

$\therefore$ $\sigma_s = (A_c/A_s)\,\sigma_c = (225/64)\,\sigma_c = 3.516\,\sigma_c$

Compatibility equation

Temperature strain of steel + tensile strain of steel
= Temperature strain of copper – compressive strain of copper

i.e.,
$$\alpha_s L_s t + \frac{\sigma_s L_s}{E_s} = \alpha_c L_c t - \frac{\sigma_c L_c}{E_c}$$

$$12 \times 10^{-6} \times (3000 + 60) \times 56 + \frac{3.516\sigma_c \times 3060}{210\,000} = 17 \times 10^{-6} \times 3000 \times 56 - \frac{\sigma_c \times 3000}{100\,000}$$

or $2.056 + 0.051\sigma_c = 2.856 - 0.03\,\sigma_c$

or $0.081\,\sigma_c = 0.8$ or $\sigma_c = 9.87$ MPa

and $\sigma_s = 3.516 \times \sigma_c = 3.516 \times 9.87 = 34.7$ MPa

Final stresses

$$\sigma_c = 20 + 9.87 = 29.87 \text{ MPa} \quad \text{(compressive)}$$

and

$$\sigma_s = 70.3 + 34.6 = 104.9 \text{ MPa} \quad \text{(tensile)}$$

Example 1.29 A steel rod of 30-mm diameter is enclosed in a brass tube of 42-mm external diameter and 32-mm internal diameter. Each is 360 mm long and the assembly is rigidly held between two stops 360 mm apart. The temperature of the assembly is then raised by 50°C. Determine

(i) stresses in the tube and the rod
(ii) stresses in the tube and the rod if the stops yield by 0.15 mm
(iii) yield of the stops if the force at the stops is limited to 60 kN

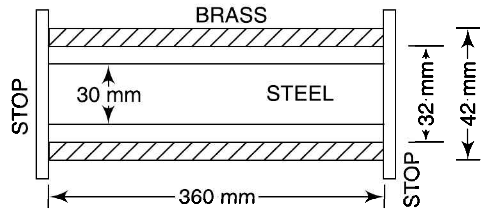


Fig. 1.48

E_s = 205 GPa; E_b = 90 GPa; $\alpha_s = 11 \times 10^{-6}$/°C; $\alpha_b = 19 \times 10^{-6}$/°C

Solution

Given An assembly of steel rod and brass tube as shown in Fig. 1.48.

$$t = 50°\text{C}$$

$$E_s = 205 \text{ GPa};\ E_b = 90 \text{ GPa};\ \alpha_s = 11 \times 10^{-6}/°\text{C};\ \alpha_b = 19 \times 10^{-6}/°\text{C}$$

To find

— Stresses in rod and tube
— stresses in rod and tube, stops yield by 0.15 mm
— yield of stops, force at stops 60 kN

$$A_s = (\pi/4)\, 30^2 = 225\, \pi \text{ mm}^2$$

$$A_b = (\pi/4)\, (42^2 - 32^2) = 185\, \pi \text{ mm}^2$$

When temperature rises by 50°C

$$\text{Stress in the steel rod} = \alpha_s\, t\, E_s = 11 \times 10^{-6} \times 50 \times 205\,000$$
$$= 112.75 \text{ MPa (compressive)}$$

$$\text{Stress in the brass tube} = \alpha_b\, t\, E_b = 19 \times 10^{-6} \times 50 \times 90\,000$$
$$= 85.5 \text{ MPa (compressive)}$$

When stops yields by 0.15 mm

$$\Delta_s = (\alpha_s Lt - 0.15) = \frac{\sigma_s L}{E_s}$$

or

$$\sigma_s = \alpha_s\, t\, E_s - \frac{0.15 E_s}{L} = 112.75 - \frac{0.15 \times 205\,000}{360}$$
$$= 112.75 - 85.42 = 27.33 \text{ MPa (compressive)}$$

and

$$\Delta_b = (\alpha_b Lt - 0.15) = \frac{\sigma_b L}{E_b}$$

or

$$\sigma_b = \alpha_b\, t\, E_b - \frac{0.15 E_b}{L} = 85.5 - \frac{0.15 \times 90\,000}{360}$$
$$= 85.5 - 37.5 = 48 \text{ MPa (compressive)}$$

Force at stops is 60 kN

When the force at the stops is limited to 60 kN, let the yield of the stops be δ,

Then

$$\Delta_s = (\alpha_s Lt - \delta) = \frac{\sigma_s L}{E_s}$$

or $$\sigma_s = \alpha_s\, t\, E_s - \frac{\delta E_s}{L} = 112.75 - \frac{\delta \times 205\,000}{360} = 112.75 - 569.44\,\delta$$

and $$\Delta_b = (\alpha_b L t - \delta) = \frac{\sigma_b L}{E}$$

or $$\sigma_b = \alpha_b\, t\, E_b - \frac{\delta E_b}{L} = 85.5 - \frac{\delta \times 90\,000}{360}$$
$$= 85.5 - 250\,\delta$$

From equilibrium equation

Force exerted by steel rod + Force exerted by brass tube = total force on the stops

$$\sigma_s . A_s + \sigma_b . A_b = P$$

$$(112.75 - 569.44\,\delta).225\,\pi + (85.5 - 250\,\delta) \cdot 185\,\pi = 60\,000$$
$$79\,698 - 402\,513\,\delta + 49\,692 - 145\,299\,\delta = 60\,000$$
$$547\,812\,\delta = 69\,390$$
$$\delta = 0.127 \text{ mm}$$

Example 1.30 A rigid block *AB* weighing 180 kN is supported by three rods symmetrically placed as shown in Fig. 1.49. Before attaching the weight, the lower ends of the rods are set at the same level. The areas of cross-section of the steel and copper rods are 800 mm^2 and 1350 mm^2 respectively. Determine

(i) the stresses in the rods, if the temperature is raised by 25°C
(ii) the stresses in the rods, if the temperature is raised by 50°C
(iii) the temperature rise for no stress in the copper rod

E_c = 95 GPa; α_c = 18 × 10^{-6}/°C; E_s = 205 GPa; α_s = 11 × 10^{-6}/°C

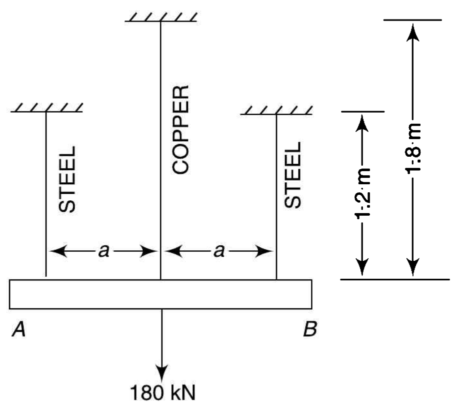


Fig. 1.49

Solution

Given An rigid block assembly as shown in Fig. 1.49.

A_s = 800 mm^2 A_c = 1350 mm^2
E_c = 95 GPa α_c = 18 × 10^{-6}/°C
E_s = 205 GPa α_s = 11 × 10^{-6}/°C

To find

— Stresses in rods, temperature rise 25°C
— Stresses in rods, temperature rise 50°C
— Temperature rise for no stress in copper rod

(i) When temperature is raised by 25°C

Considering the increase in temperature alone (neglecting the weight of the block), the elongation of a copper rod is more as compared to steel rods. On the other hand, if the temperature does not change, there is elongation of all rods and there is tensile stress in all the rods.

Total elongation of each rod is the sum of elongations due to temperature and due to weight. As the block is rigid, it will remain horizontal under all conditions. Thus, the total elongation of each rod is the same.

Compatibility equation

Assume the stress in the copper rod to be compressive, i.e., the force acting upwards.

Then $$\alpha_s L_s t + \frac{P_s L_s}{A_s E_s} = \alpha_c L_c t - \frac{P_c L_c}{A_c E_c}$$

$$11 \times 10^{-6} \times 1200 \times 25 + \frac{P_s \times 1200}{800 \times 205\,000} = 18 \times 10^{-6} \times 1800 \times 25 - \frac{P_c \times 1800}{1350 \times 95\,000}$$

$$330\,000 + 7.317\, P_s = 810\,000 - 14.035 P_c$$

$$P_s = 65\,601 - 1.918\, P_c \qquad \text{(i)}$$

Equilibrium equation

$$2\, P_s - P_c = 180\,000$$

or $$P_s - 0.5P_c = 90\,000$$

$$65601 - 1.918\, P_c - 0.5P_c = 90\,000 \qquad \text{[from (i)]}$$

or $$2.418\, P_c = -24\,399$$

or $$P_c = -10\,090 \text{ N (compressive)}$$

and $$P_s = 90\,000 + 0.5 \times (-10090)$$
$$= 84\,955 \text{ N (tensile)}$$

$$\sigma_c = -\frac{10\,090}{1350} = -7.474 \text{ MPa}$$

which shows that the stress in the copper rod is opposite of what was assumed, i.e., tensile and not compressive.

$$\sigma_s = \frac{84\,955}{800} = 106.19 \text{ MPa (tensile)}$$

(ii) When temperature is raised by 50°C

Compatibility equation

$$11 \times 10^{-6} \times 1200 \times 50 + 7.317 \times 10^{-6}\, P_s = 18 \times 10^{-6} \times 1800 \times 50 - 14.035 \times 10^{-6}\, P_c$$

$$660\,000 + 7.317\, P_s = 1\,620\,000 - 14.035\, P_c$$

$$P_s = 131\,201 - 1.918\, P_c$$

Equilibrium equation

$$2\, P_s - P_c = 180\,000$$

or $$P_s - 0.5P_c = 90\,000$$

or $$131\,201 - 1.918\, P_c = 90\,000 \quad \text{or} \quad P_c = 17\,039 \text{ N (compressive)}$$

$$\sigma_c = 17\,039/1350 = 12.62 \text{ MPa (compressive)}$$

$$P_s = 90\,000 + 0.5 \times 17\,039 = 98\,520 \text{ N}$$

$$\sigma_s = 98\,520/800 = 123.1 \text{ MPa (tensile)}$$

(iii) As there is to be no stress and hence no load on the copper rod, $\sigma_c = 0$

Hence load in each rod = 180 000/2 = 90 000 N

Compatibility equation

$$11 \times 10^{-6} \times 1200 \times t + 7.317 \times 10^{-6}\, P_s = 18 \times 10^{-6}\, P_s = 18 \times 10^{-6} \times 1800 \times t - 0$$

$$13\,200\, t + 7.317\, P_s = 32\,400\, t$$

$$19\,200\, t = 7.317 \times 90\,000$$

$$t = 34.3°$$

Example 1.31 A rigid rod *ABC* is hinged at *A* and attached to brass bars *BD* and *CE* as shown in Fig. 1.50. The temperature of the steel bar is raised by 40°C and that of the brass bar is lowered by 40°C. Determine the normal stresses in the steel and the brass bars.

$$E_b = 92 \text{ GPa}; \alpha_b = 20 \times 10^{-6}/°\text{C}; E_s = 202 \text{ GPa}; \alpha_s = 11.5 \times 10^{-6}/°\text{C}$$

Solution

Given A rigid block assembly as shown in Fig. 1.50.

$A_s = 200 \text{ mm}^2$ $\qquad A_b = 300 \text{ mm}^2$

$E_b = 92$ GPa $\qquad \alpha_b = 20 \times 10^{-6}/°\text{C}$

$E_s = 202$ GPa $\qquad \alpha_s = 11.5 \times 10^{-6}/°\text{C}$

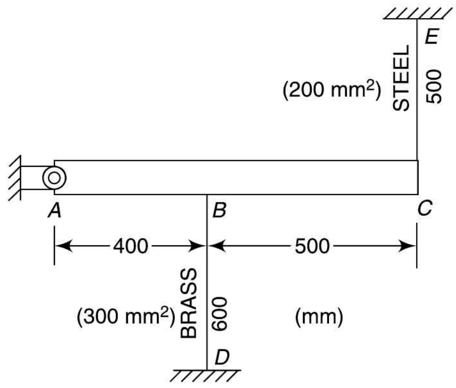


Fig. 1.50

Rise of temperature of steel bar = 40°C
Decrease of temperature of brass bar = 40°C
To find Stresses in steel and brass bars

Free extension of the steel bar = $L\,\alpha\,t = 500 \times 11.5 \times 10^{-6} \times 40$
$= 0.23$ mm

Free contraction of the brass bar = $L\,\alpha\,t = 600 \times 20 \times 10^{-6} \times 40$
$= 0.48$ mm

From geometric consideration

As the rod *ABC* is rigid, and the steel bar expands whereas the brass bar contracts, no temperature stresses are developed if

$$\frac{\Delta_s}{\Delta_b} = \frac{900}{400} = 2.25$$

$$\Delta_s = 2.25\Delta_b = 2.25 \times 0.48 = 1.08 \text{ mm}$$

This means that if the contraction of the brass bar is 0.48 mm, the extension of the steel bar must be 1.08 mm to have no temperature stresses. However, the extension of the steel bar is only 0.23 mm which indicates that the final position of the rigid bar will be such that the contraction of the brass bar is lesser than 0.48 mm and the extension of the steel bar is more than 0.23 mm. Thus, tensile stress is developed in the steel bar and compressive stress in the bronze bar as shown in Fig. 1.51.

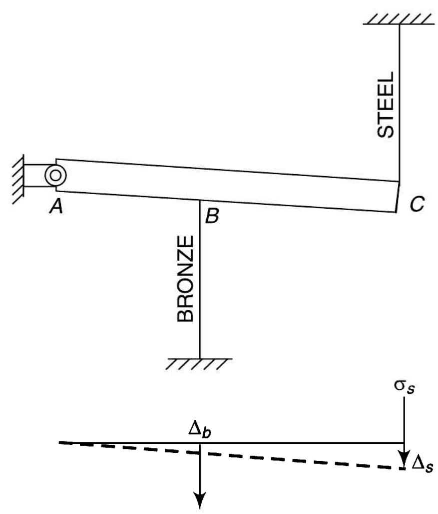


Fig. 1.51

Let the net extension of the steel bar = Δ_s'

and the net contraction of the brass bar = Δ_b'

Compatibility equation

$$\Delta_s' = L_s\alpha_s\, t_s + \frac{\sigma_s L_s}{E_s} = 0.23 + \frac{\sigma_s \times 500}{202\,000}$$

$$\Delta_b' = L_b\alpha_b\, t_b + \frac{\sigma_b L_b}{E_b} = 0.48 - \frac{\sigma_b \times 600}{92\,000}$$

$$\therefore \quad 0.23 + \frac{\sigma_s \times 500}{202\,000} = 2.25\left(0.48 - \frac{\sigma_b \times 600}{92\,000}\right)$$

or $$0.23 + 0.002\,475\,\sigma_s = 1.08 - 0.014\,674\,\sigma_b$$

or $$\sigma_s + 5.929\,\sigma_b - 343.4 = 0 \quad \text{(i)}$$

Moment equation

Taking moments about A, $400\sigma_b \cdot A_b - 900\sigma_s \cdot A_s = 0$

or $$400 \times 300\sigma_b - 900 \times 200\sigma_s = 0 \quad \text{or} \quad \sigma_b = 1.5\sigma_s$$

Stresses

From (i), $$\sigma_s + 5.929 \times 1.5\,\sigma_s - 343.4 = 0 \quad \text{or} \quad \sigma_s = 34.7 \text{ MPa}$$

and $$\sigma_b = 1.5\sigma_s = 1.5 \times 34.7 = 52.1 \text{ MPa}$$

1.13 SHRINKING ON

A thin tyre of steel or of any other metal can be shrunk on to wheels of slightly smaller diameter by heating the tyre to a certain degree which increases its diameter. When the tyre has been mounted and the temperature falls to the normal temperature, the steel tyre tends to come to its original diameter and thus tensile (hoop) stress is set up in the tangential direction.

As shown in Fig. 1.52, let d and D be the diameters of the steel tyre and of the wheel on which the steel tyre is to be mounted, then

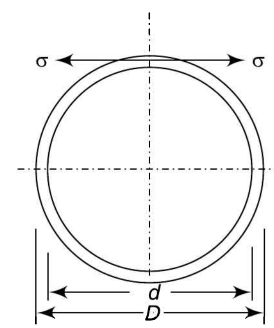


Fig. 1.52

The strain, $$\varepsilon = \frac{\pi D - \pi d}{\pi d} = \frac{D-d}{d}$$

Circumferential tensile stress or hoop stress $= \varepsilon.E = \left(\dfrac{D-d}{d}\right)E$ (1.18)

Example 1.32 A thin tyre of steel is to be mounted on to a rigid wheel of 1.2-m diameter. Determine the internal diameter of the tyre if the hoop stress is limited to 120 MPa.

Also determine the least temperature to which the tyre should be heated so that it can be slipped on to the wheel.

$$E_s = 210 \text{ GPa and } \alpha_s = 11 \times 10^{-6}/°\text{C}$$

Solution

Given A thin tyre of steel to be mounted on to a rigid wheel

$d = 1.2$ m $\sigma = 120$ MPa

$E_s = 210$ GPa $\alpha_s = 11 \times 10^{-6}/°\text{C}$

To find

— Internal diameter of tyre
— Least temperature rise

Determination of internal diameter

$$\text{Tensile strain, } \varepsilon = \frac{D-d}{d} = \left(\frac{\sigma}{E}\right)$$

or $$\frac{D}{d} = \left(\frac{\sigma}{E}\right) + 1 = \frac{\sigma + E}{E} \quad \text{or} \quad \frac{d}{D} = \frac{E}{\sigma + E}$$

or $$d = \frac{DE}{\sigma + E} = \frac{1200 \times 210\,000}{120 + 210\,000} = 1199.31 \text{ mm or } 1.19931 \text{ m}$$

Determination of least temperature

Increase in the circumferential length $= \pi\,(D - d)$

Thus $$\alpha\, L\, t = \pi\,(D - d)$$

or $$11 \times 10^{-6} \times (\pi \times 1199.31) \times t = \pi\,(1200 - 1199.31)$$

$$t = 52.3° \text{ C}$$

1.14 STRAIN ANALYSIS

So far, the effect of an axial force on the length of a bar or rod has been considered. In case of a tensile force, the length increases, and in a compressive force, it decreases. However, this axial increase or decrease takes place at the cost of a change in the lateral dimensions of the bar or rod. If an axial tensile force is applied to a bar, its length is increased and its lateral dimensions, i.e., the width and breadth or the diameter are decreased (Fig. 1.53). Therefore, any direct stress produces a strain in its own direction as well as an opposite kind of strain in all directions at right angles to its own direction known as *lateral strain.*

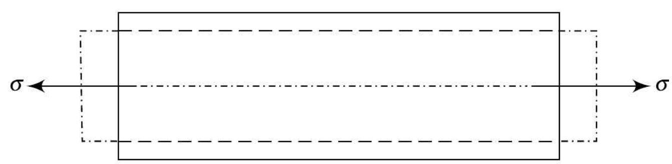


Fig. 1.53

Poisson's Ratio

The ratio of the lateral strain to the longitudinal strain of a material, when it is subjected to a longitudinal stress, is known as *Poisson's ratio* and is denoted by ν. It is found that for elastic materials, the lateral strain is proportional to the longitudinal strain, i.e., the ratio of the lateral strain to the longitudinal strain is constant. Thus

$$\frac{\text{Lateral strain}}{\text{Longitudinal strain}} = \text{constant} = \nu \tag{1.19}$$

The value of ν lies between 0.25 and 0.34 for most of the metals.

Lateral strain $= -\,\nu \times$ Longitudinal strain $= -\,\nu \cdot \sigma/E$

(negative sign indicates that it is opposite to the longitudinal strain)

Two-dimensional Stress System

Consider a system with two pure normal stresses σ_1 and σ_2 as shown in Fig. 1.54.

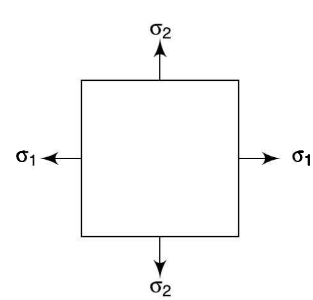


Fig. 1.54

Strain due to σ_1 in its own direction $= \sigma_1/E$

Strain due to σ_2 in the direction of $\sigma_1 = -\,\nu\sigma_2/E$

Thus, net strain in the direction of σ_1, $$\varepsilon_1 = \sigma_1/E - \nu\sigma_2/E \tag{1.20}$$

In a similar way,

Net strain in the direction of σ_2, $$\varepsilon_2 = \sigma_2/E - \nu\sigma_1/E \tag{1.21}$$

Remember that a tensile stress is taken positive whereas a compressive stress negative.

Three-dimensional Stress System

Let there be a system with three pure normal stresses σ_1, σ_2 and σ_3 as shown in Fig. 1.55.

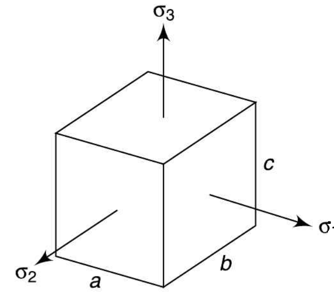


Fig. 1.55

Strain due to σ_1 in its own direction $= \sigma_1/E$

Strain due to σ_2 in the direction of $\sigma_1 = -\nu\sigma_2/E$

Strain due to σ_3 in the direction of $\sigma_1 = -\nu\sigma_3/E$

Thus, the net strain in the direction of σ_1, $\varepsilon_1 = \sigma_1/E - \nu\sigma_2/E - \nu\sigma_3/E$

In a similar way, $\varepsilon_2 = \sigma_2/E - \nu\sigma_3/E - \nu\sigma_1/E$

and $\varepsilon_3 = \sigma_3/E - \nu\sigma_1/E - \nu\sigma_2/E$

Volumetric Strain

When a body is acted upon by three mutually perpendicular forces, there is change in the volume of the body which is referred as the *dilatation* or *dilation* of the material. Volumetric strain is defined as the ratio of increase in volume of a body to its original volume when it is acted upon by three mutually perpendicular stresses σ_1, σ_2, and σ_3. For a rectangular solid body of sides a, b and c (Fig. 1.34), let ε_1, ε_2 and ε_3 be the corresponding strains.

Initial volume $= a \cdot b \cdot c$

Final volume $= (a + a\varepsilon_1)(b + b\varepsilon_2)(c + c\varepsilon_3) = abc\,(1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3)$

$$\text{Volumetric strain, } \varepsilon_v = \frac{\text{Increase in volume}}{\text{Original volume}}$$

$$= \frac{abc\,(1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3) - abc}{abc}$$

$$= (1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3) - 1$$

$$= 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 + \varepsilon_1\varepsilon_2\varepsilon_3 - 1$$

$$\approx \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \tag{1.22}$$

Thus if the products of very small quantities are neglected, the volumetric strain is the algebraic sum of the three mutually perpendicular strains.

- Volumetric strain can also be obtained by differentiating the expression for volume, $V = abc$

$$dV = bc(da) + ca(db) + ab(dc)$$

Dividing by abc (or V) throughout,

$$\frac{dV}{V} = \frac{da}{a} + \frac{db}{b} + \frac{dc}{c}$$

or $$\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

In terms of stresses the volumetric strain can be expressed by substituting the values of ε_1, ε_2 and ε_3 from above.

Volumetric strain,

$$\frac{\Delta V}{V} = (\sigma_1/E - \nu\sigma_2/E - \nu\sigma_3/E) + (\sigma_2/E - \nu\sigma_3/E - \nu\sigma_1/E) + (\sigma_3/E - \nu\sigma_1/E - \nu\sigma_2/E)$$

$$= \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\nu)}{E} \tag{1.23}$$

and Change in volume $= V \times \dfrac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\nu)}{E}$ (1.24)

If force is unidirectional, i.e., σ_2 and σ_3 are zero,

$$\text{Volumetric strain} = \frac{\sigma(1 - 2\nu)}{E} = \varepsilon(1 - 2\nu)$$

$$\text{Change in volume} = V \times \frac{\sigma(1 - 2\nu)}{E} = V\varepsilon(1 - 2\nu) \quad (1.25)$$

Example 1.33 What will be the percentage change in the volume of a steel bar of 20-mm diameter and 600-mm length when a tensile stress of 180 MPa is applied to it along its longitudinal axis?

E_s = 205 GPa, ν = 0.3

Solution

Given A steel bar

d = 20 mm $\quad$ L = 600 mm

E_s = 205 GPa $\quad$ ν = 0.3

σ = 180 MPa

To find Change in volume

$$\text{Volume of the bar, } V = \frac{\pi}{4} \times 20^2 \times 600 = 60\,000\,\pi \text{ mm}^3$$

$$\text{Change in volume} = V \times \frac{\sigma(1 - 2\nu)}{E} = 60\,000\,\pi \times \frac{180(1 - 2 \times 0.3)}{205\,000} = 66.2 \text{ mm}^3$$

$$\text{Percentage change in volume} = \frac{66.2}{60\,000\pi} \times 100 = 0.035$$

Example 1.34 The tangential (hoop) and longitudinal stresses in the plates of a cylindrical boiler of 2.2 m diameter and 3.5 m in length are 90 MPa and 45 MPa respectively. Determine the increase in its internal capacity. Neglect compressive stress due to steam on the inner surface.

E = 205 GPa; ν = 0.3

Solution

Given A cylindrical boiler

d = 2.2 m $\quad$ l = 3.5 m

σ_c = 90 MPa $\quad$ σ_l = 45 MPa

E = 205 GPa $\quad$ ν = 0.3

To find Increase in capacity of boiler

$$V = (\pi/4)\, 2.2^2 \times 3.5 = 4.235\,\pi \text{ m}^3$$

Determination of volumetric strain

As compressive stress due to steam on the inner surface is neglected, $\sigma_z = 0$

$$\varepsilon_x = \frac{\sigma_c}{E} - \frac{\nu\sigma_l}{E} = \frac{1}{E}(90 - 0.3\ X\ 45) = \frac{76.5}{E}$$

$$\varepsilon_y = \frac{\sigma_l}{E} - \frac{\nu\sigma_c}{E} = \frac{1}{E}(45 - 0.3 \times 90) = \frac{18}{E}$$

The diameter of a boiler is directly proportional to its circumference. Thus ε_x also is the diametrical strain along any two perpendicular radii.

$$\text{Volumetric strain, } \varepsilon = \varepsilon_y + \varepsilon_x + \varepsilon_x$$
$$= 18/E + 76.5/E + 76.5/E = 171/205\,000 = 834.1 \times 10^{-6}$$

As strain is a ratio, change in volume can be found directly in m^3.

Change in capacity

$$\text{Change in volume} = \varepsilon \times V = (834.1 \times 10^{-6}) \times 4.235\,\pi$$
$$= 0.011\,098 \text{ m}^3$$
$$= (0.011\,098 \times 1000)\, l = 11.098\, l \qquad \text{(in litres)}$$

Example 1.35 A steel bar 35 mm × 35 mm in section and 100 mm in length is acted upon by a tensile load of 180 kN along its longitudinal axis and 400 kN and 300 kN along the axes of the lateral surfaces. Determine

(i) change in the dimensions of the bar
(ii) change in volume
(iii) longitudinal axial load acting alone to produce the same longitudinal strain as in (i)

E = 205 GPa; ν = 0.3

Solution

Given A steel bar 35 mm × 35 mm in section and 100 mm in length as shown in Fig. 1.56. A tensile load of 180 kN along longitudinal axis and 400 kN and 300 kN along the axes of the lateral surfaces act.

(Loads are shown only on the visible faces. Equal and opposite loads also act on the invisible faces).

E = 205 GPa $\qquad$ ν = 0.3

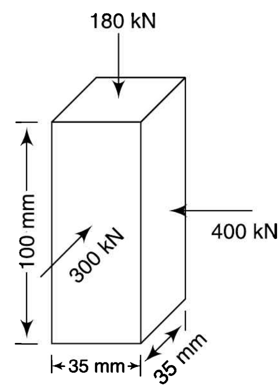


Fig. 1.56

To find

— Change in dimensions
— change in volume
— axial load to produce same longitudinal strain as above

Let σ_1, σ_2 and σ_3 be the stresses along longitudinal and two transverse axes respectively.

Determination of change in dimensions

$$\sigma_1 = \frac{180\,000}{35 \times 35} = 146.9 \text{ MPa}$$

$$\sigma_2 = \frac{400\,000}{100 \times 35} = 114.3 \text{ MPa}$$

$$\sigma_3 = \frac{300\,000}{100 \times 35} = 85.7 \text{ MPa}$$

In longitudinal direction,

$$\Delta L = \frac{L}{E}(\sigma_1 - \nu\sigma_2 - \nu\sigma_3)$$

$$= \frac{100}{205\,000}(146.9 - 0.3 \times 114.3 - 0.3 \times 85.7)$$

$$= 0.04239 \text{ mm}$$

In the direction of 400-kN load,

$$\Delta L = \frac{L}{E}(\sigma_2 - \nu\sigma_1 - \nu\sigma_3)$$

$$= \frac{35}{205\,000}(114.3 - 0.3 \times 146.9 - 0.3 \times 85.7) = 0.0076 \text{ mm}$$

In the direction of 300-kN load,

$$\Delta L = \frac{L}{E}(\sigma_3 - \nu\sigma_1 - \nu\sigma_2)$$

$$= \frac{35}{205\,000}(85.7 - 0.3 \times 146.9 - 0.3 \times 114.3) = 0.001\,25 \text{ mm}$$

Change in volume

$$\text{Change in volume} = (100 + 0.042\,39)\,(35 + 0.0076)(35 + 0.001\,25) - 100 \times 35 \times 35 = 82.92 \text{ mm}^3$$

$$\text{or directly change in volume} = V \times \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\nu)}{E}$$

$$= (100 \times 35 \times 35)\,\frac{(146.9 + 114.3 + 85.7)(1 - 2 \times 0.3)}{205\,000}$$

$$= 82.92 \text{ mm}^3$$

Longitudinal axial load

Let σ be the longitudinal stress to have the same strain,

$$\varepsilon = \frac{\Delta L}{L} = \frac{0.042\,39}{100} = \frac{\sigma}{205\,000},\ \sigma = 86.9 \text{ MPa}$$

Longitudinal load = $\sigma \times A = 86.9 \times 35 \times 35 = 106\,452$ N or 106.452 kN

Example 1.36 A square steel bar of dimensions 50 mm × 50 mm × 150 mm is subjected to an axial load of 250 kN. Determine the decrease in length of the bar if

(i) the lateral strain is fully prevented by applying external uniform pressure on the rectangular surfaces

(ii) only one-third of the lateral strain is prevented by the external pressure

E = 205 GPa and ν = 0.3

Solution

Given A square steel bar of dimensions 50 mm × 50 mm × 150 mm as shown in Fig. 1.57. An axial load of 250 kN is applied.

E = 205 GPa $\qquad$ ν = 0.3

To find

Decrease in length if

— lateral strain is fully prevented

— one-third of lateral strain is prevented

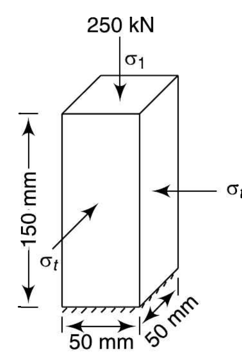


Fig. 1.57

Lateral strain is fully prevented

$$\sigma_1 = \frac{250\,000}{50 \times 50} = 100 \text{ MPa}$$

Let the compressive stresses applied on each of the similar lateral sides be σ_t to prevent the lateral strain (Fig. 1.57). Then

$$\frac{1}{E}(\sigma_t - v\sigma_t - v\sigma_1) = 0$$

or $$(\sigma_t - 0.3 \times \sigma_t - 0.3 \times 100) = 0$$

or $$0.7\,\sigma_t = 30$$

$$\sigma_t = 42.857 \text{ MPa}$$

$$\text{Decrease in length} = \frac{L}{E}(\sigma_1 - v\sigma_t - v\sigma_t)$$

$$= \frac{150}{205\,000}(100 - 0.3 \times 2 \times 42.857)$$

$$= 0.054\,36 \text{ mm}$$

Only one-third of the lateral strain is prevented

In the absence of compressive stresses on the sides to prevent the lateral strain, The lateral strain = $v\sigma_1/E$ (tensile)

Now, one-third of this is to be prevented, i.e., $v\sigma_1/3E$ and leaving $2v\sigma_1/3E$ as such.

Let the compressive stresses applied on the sides be σ_t.

Then

$$\frac{1}{E}(\sigma_t - v\sigma_t - v\sigma_1) = -\frac{2v\sigma_1}{3E}$$

The two strains are of opposite directions.

or $$(\sigma_t - 0.3 \times \sigma_t - 0.3 \times 100) = -2 \times 0.3 \times 100/3 \qquad (\sigma_2 = \sigma_3)$$

or $$0.7\,\sigma_t = 30 - 20$$

$$\sigma_t = 14.286 \text{ MPa}$$

$$\text{Decrease in length} = \frac{L}{E}(\sigma_1 - v\sigma_t - v\sigma_t)$$

$$= \frac{150}{205\,000}(100 - 0.3 \times 2 \times 14.286) \qquad (\sigma_2 = \sigma_3)$$

$$= 0.0669 \text{ mm}$$

1.15 TENSILE TEST DIAGRAM

The behaviour of a material subjected to an increased tensile load is studied by testing a specimen in a tensile testing machine and plotting the stress–strain diagram. Stress–strain diagrams of different materials vary widely. However, it is possible to distinguish some common characteristics among various stress–strain diagrams of various groups of materials. It is observed that broadly the materials can be divided into two categories on the basis of these characteristics: ductile materials and brittle materials.

Ductile materials such as steel and many alloys of other metals have the ability to yield at normal temperatures. The plot between strain and the corresponding stress of a ductile material is represented graphically by a tensile test diagram. Figure 1.58 shows a stress vs. strain diagram for steel in which the stress is calculated on the basis of original area of a steel bar. Most of other engineering materials show a similar pattern to a varying degree. The following are the salient features of the diagram:

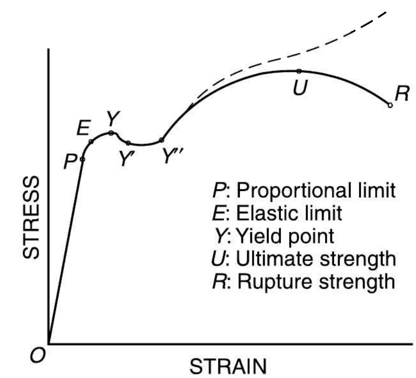


Fig. 1.58

- When the load is increased gradually, the strain is proportional to load or stress upto a certain value. Line OP indicates this range and is known as the *line of proportionality*. Hooke's law is applicable in this range. The stress at the end point P is known as the *proportional limit.*
- If the load is increased beyond the limit of proportionality, the elongation is found to be more rapid, though the material may still be in the elastic state, i.e., on removing the load, the strain vanishes. This elongation with a relatively small increase in load is caused by slippage of the material along oblique surfaces and is mainly due to shear stresses. The point E depicts the elastic limit. Hooke's law cannot be applied in this range as the strain is not proportional to stress. Usually, this point is very near to P and many times the difference between P and E is ignored and therefore elastic limit is taken as the limit of proportionality.
- When the load is further increased, plastic deformation occurs, i.e., on removing the load, the strain is not fully recoverable. At point Y, metal shows an appreciable strain even without further increasing the load. Actually, the curve drops slightly at this point to Y' and the yielding goes up to the point Y''. The points Y' and Y'' are known as the *upper* and *lower yield points* respectively. The stress–strain curve between Y and Y'' is not steady.
- After the yield point, further straining is possible only by increasing the load. The stress–strain curve rises up to the point U, the strain in the region Y to U is about 100 times that from O to Y'. The stress value at U is known as the *ultimate stress* and is mostly plastic which is not recoverable.
- If the bar is stressed further, it begins to form a *neck*, or a local reduction in cross-section occurs. After this, somewhat lower loads are sufficient to keep the specimen elongating further. Ultimately, the specimen fractures at point R. It is noted that fracture occurs along a cone-shaped surface at about 45° with the original surface of the specimen indicating that shear is primarily responsible for the failure of ductile materials.
- If the load is divided by the original area of the cross-section, the stress is known as the *nominal stress*. This is lesser at the rupture load than at the maximum load. However, the stress obtained by dividing with the reduced area of cross-section is known as the *actual* or *true* stress and is greater at the maximum load. It is shown in the figure by the dotted line.

In brittle materials such as cast iron, glass and stone, etc., rupture occurs without any appreciable change in the rate of elongation and there is no difference between the ultimate strength and the rupture strength. The strain at the time of rupture is much smaller for brittle materials as compared to ductile materials. There is no neck formation of the specimen of a brittle material and the rupture occurs along a perpendicular surface to the load indicating that normal stresses are primarily responsible for the failure of brittle materials.

For detailed information, refer Section 18.3.

1.16 FACTOR OF SAFETY

A machine component must be designed so that the load carried by it under normal conditions of utilisation is considerably below its ultimate load. This smaller load is referred as the *allowable load* or the *design load* or the *working load.* Usually, the allowable load is only a fraction of the ultimate load or the load-carrying capacity of a component. This is done to ensure safe working of the component against uncertainties of various factors during the operation of a machine, e.g., homogeneousness of the material, number of loadings during the life of the component, type of loading (static or sudden), method of analysis used, natural causes, etc. Thus, a large portion of the load-carrying capacity of the component is kept as reserve for safe performance of the component. The ratio of the ultimate load to the allowable or working load is known factor of safety. Thus,

$$\text{Factor of safety} = \frac{\text{ultimate load}}{\text{allowable load}}$$

- As the stress is the load per unit area, *allowable stress* is the allowable load per unit area. Also, allowable stress is also known as *design stress* or *working stress*. Thus factor of safety is also defined as,

$$\text{Factor of safety} = \frac{\text{ultimate stress}}{\text{allowable stress}}$$

- Refer Section 18.3 also.

1.17 ELASTIC CONSTANTS

The factors to determine the deformations produced by a stress system acting on a material within elastic limits are constant and termed as *elastic constants*. Two elastic constants, *modulus of elasticity* and *modulus of rigidity*, have already been defined in Section 1.4. A third elastic constant is being defined in this section.

If three mutually perpendicular stresses of equal intensity are applied to a body of initial volume V as shown in Fig. 1.59, then the ratio of the direct stress to the volumetric strain is known as the *bulk modulus* (K) of the body.

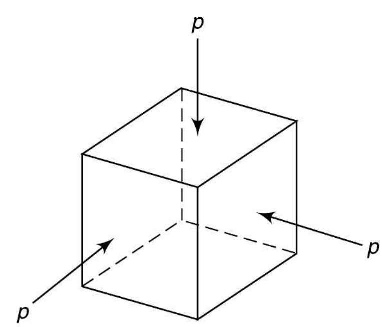


Fig. 1.59

Usually, bulk modulus is applicable mainly to fluid problems with pressure intensity p in all directions and thus

$$K = \frac{\text{Direct stress}}{\text{Volumetric strain}} = \frac{-p}{\varepsilon_v} \tag{1.26}$$

Volumetric strain, $$\varepsilon_v = \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2\nu)}{E} \quad \text{(Eq. 1.23)}$$

For three perpendicular stresses of equal intensity p (compressive),

$$\varepsilon_v = \frac{3(1 - 2\nu)(-p)}{E}$$

Therefore,

$$K = \frac{-p}{-3(1 - 2\nu)p/E}$$

or $$E = 3K(1 - 2\nu) \tag{1.27}$$

1.18 RELATION BETWEEN ELASTIC CONSTANTS

Consider a square element *ABCD* under the action of a simple shear stress τ (Fig.1.60a). The resultant distortion of the element is shown in Fig. 1.60b The total change in the corner angles is $\pm\ \varphi$. However, for convenience sake, the side *AB* may be considered to be fixed as shown in Fig. 1.60c. As angle φ is extremely small, *CC′* and *DD′* can be assumed to be arcs. Let *CE* be a perpendicular on the diagonal *AC′*.

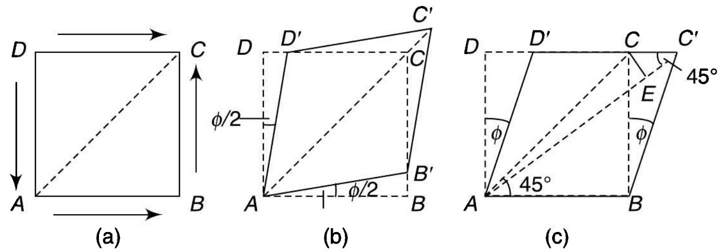


Fig. 1.60

Linear strain of the diagonal *AC* can approximately be taken as

$$\varepsilon = \frac{AC' - AC}{AC}$$

$$= \frac{EC'}{AC}$$

$$= \frac{CC' \cos 45°}{AB/\cos 45°}$$

$$= \frac{\varphi \cdot BC \cos^2 45°}{BC} \qquad (CC' = \varphi \cdot BC \text{ and } AB = BC)$$

$$= \frac{\varphi}{2}$$

But modulus of rigidity, $G = \tau/\varphi$ or $\varphi = \tau/G$ (Eq.1.5)

$$\therefore \qquad \varepsilon = \frac{\tau}{2G} \qquad \text{(i)}$$

It will be shown in Section 2.1 that in a state of simple shear on two perpendicular planes, the planes at 45° are subjected to a tensile stress (magnitude equal to that of the shear stress) while the planes at 135° are subjected to a compressive stress of the same magnitude with no shear stress on these planes. Thus, planes *AC* and *BD* are subjected to tensile and compressive stresses respectively each equal to τ in magnitude as shown in Fig. 1.61.

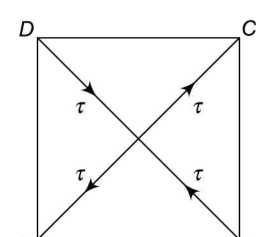


Fig. 1.61

Hence linear strain of diagonal *AC* is

$$\varepsilon = \frac{\tau}{E} - \left(-\frac{\nu\tau}{E}\right) = \frac{\tau}{E}(1+\nu) \qquad \text{(ii)}$$

From (i) and (ii),

$$\frac{\tau}{2G} = \frac{\tau}{E}(1+\nu)$$

or $$E = 2G(1+ \nu)$$
As $$E = 3K(1 - 2\nu) \qquad \text{(Eq. 1.27)}$$
$\therefore$ $$E = 2G(1 + \nu) = 3K(1 - 2\nu) \qquad (1.28)$$

This equation relates the elastic constants.

Also from above, $1 + \nu = \dfrac{E}{2G}$, $\therefore\ 2 + 2\nu = \dfrac{E}{G}$ (i)

and $$1 - 2\nu = \frac{E}{3K} \qquad \text{(ii)}$$

Adding (i) and (ii), $3 = E\left(\dfrac{1}{G} + \dfrac{1}{3K}\right) = \dfrac{E}{3KG}(3K + G)$

or $$E = \frac{9KG}{3K + G} \qquad (1.29)$$

Example 1.37 A bar of 24-mm diameter and 400-mm length is acted upon by an axial load of 38 kN. The elongation of the bar and the change in diameter are measured as 0.165 mm and 0.0031 mm respectively. Determine

(i) the Poisson's ratio (ii) the values of the three moduli

Solution

Given A circular bar

$d = 24$ mm $L = 400$ mm
$P = 38\ 000$ N $\delta L = 0.165$ mm
$\delta d = 0.0031$ mm

To find

— Poisson's ratio
— E, G and K

$$A = \frac{\pi}{4} \times 24^2 = 144\,\pi \text{ mm}^2$$

$$\sigma = \frac{38\ 000}{144\pi} = 84 \text{ MPa}$$

Determination of Poisson's ratio

$$\text{Lateral strain} = \nu \cdot \text{Linear strain}$$

$$\frac{\delta d}{d} = \nu \frac{\delta L}{L} \quad \text{or} \quad \frac{0.0031}{24} = \nu \frac{0.165}{400} \quad \text{or} \quad \nu = 0.313$$

Determination of three moduli

$$E = \frac{\sigma}{\varepsilon} = \frac{84}{0.165/400} = 203\ 636 \text{ MPa}$$

Also, $$E = 2G\,(1 + \nu) = 3K\,(1 - 2\nu)$$

$\therefore$ $$G = \frac{E}{2(1 + \nu)} = \frac{203\,636}{2(1 + 0.313)} = 77\ 546 \text{ MPa}$$

and $$K = \frac{E}{3(1 - 2\nu)} = \frac{203\,636}{3(1 - 2 \times 0.313)} = 181\ 494 \text{ MPa}$$

Example 1.38 A bar, 12 mm in diameter, is acted upon by an axial load of 20 kN. The change in diameter is measured as 0.003 mm. Determine

(i) the Poisson's ratio
(ii) the modulus of elasticity and the bulk modulus
The value of the modulus of rigidity is 80 GPa.

Solution

Given A circular bar

$d = 12$ mm $\qquad P = 20\,000$ N

$\delta d = 0.003$ mm $\qquad G = 80$ GPa

To find

— Poisson's ratio
— E and K

$$A = (\pi/4)\ 12^2 = 36\ \pi\ \text{mm}^2$$
$$\sigma = 20\,000/36\pi = 176.84\ \text{MPa}$$

Determination of Poisson's ratio

$$\text{Lateral strain} = \nu \cdot \text{Linear strain}$$

$$\frac{\delta d}{d} = \nu\varepsilon \quad \text{or} \quad \frac{0.003}{12} = \nu\varepsilon \quad \text{or} \quad \varepsilon = 0.000\,25/\nu \tag{i}$$

Now,
$$E = 2G\,(1 + \nu) = 2 \times 80\,000\,(1 + \nu) = 160\,000 + 160\,000\,\nu$$

Also,
$$E = \frac{\sigma}{\varepsilon} = \frac{176.84}{0.000\,25/\nu} = 707\,360\,\nu \qquad \text{[using (i)]}$$

$\therefore$
$$707\,360\,\nu = 160\,000 + 160\,000\,\nu$$
$$\nu = 0.2923$$

Determination of E and K

Thus
$$E = 707\,360 \times 0.2923 = 206\,771\ \text{MPa}$$

and
$$K = \frac{E}{3(1-2\nu)} = \frac{206\,771}{3(1-2\times 0.2923)} = 165\,921\ \text{MPa}$$

1.19 THREE-DIMENSIONAL STRESS SYSTEMS

To visualize the stress condition at a point Q within a body, consider a small cube of side a with centre at Q as shown in Fig. 1.62. The stresses act on each of the six faces of the cube. However, for clarity sake, stresses are shown only on the three visible sides. The stress components shown are σ_x, σ_y and σ_z, the normal stresses on faces perpendicular to the x-, y- and z-axes respectively. Similarly, the shear stress components τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx}, τ_{zy} are also shown. In defining these components, the first subscript indicates the axis perpendicular to the plane upon which the stress is exerted and the second subscript identifies the direction of the component. For example, τ_{xy} represents the y component of the shear stress acting on a surface perpendicular to the x-axis whereas τ_{yx} is the x component of the shear stress on a surface perpendicular to the y-axis. In reality, the stresses acting on two parallel surfaces of the cube varies slightly from the stresses at Q, the error disappears as the side a of the cube approaches zero.

The normal and shear forces acting on different faces of the cube are obtained by multiplying the stress component by the corresponding surface area of the face (Fig. 1.63). Equal and opposite forces to the forces shown in the figure act on the hidden faces of the cube. If a projection on the xy plane is considered, it is seen that the moments about the z-axis with non-zero values are the shear forces (Fig. 1.64)

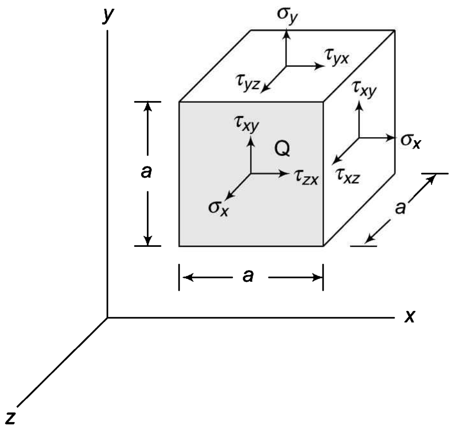


Fig. 1.62

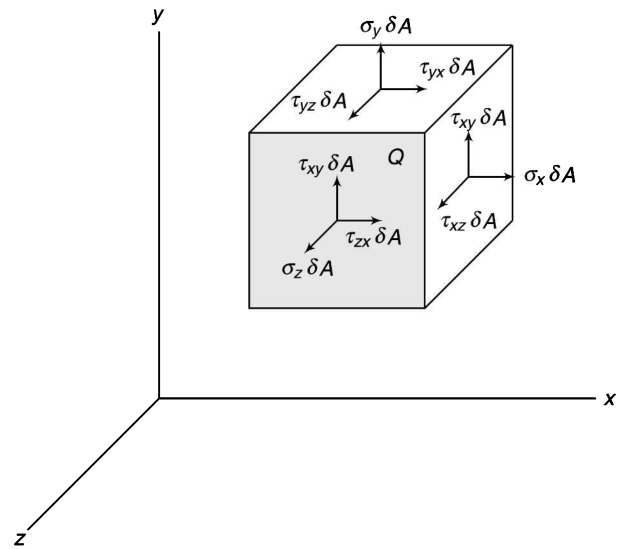


Fig. 1.63

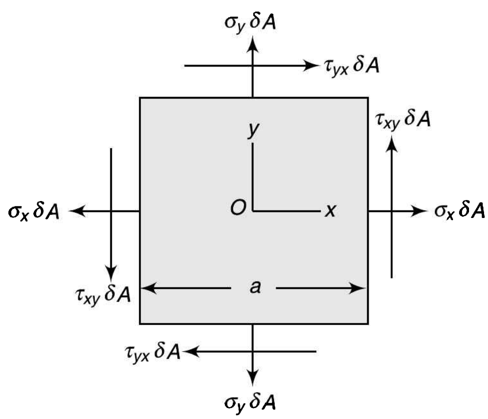


Fig. 1.64

i.e., $$\tau_{xy}\delta A \cdot a - \tau_{yx}\delta A \cdot a = 0 \quad \text{or} \quad \tau_{xy} = \tau_{yx}$$

which indicates that the y component of the shear stress on a face perpendicular to the x-axis is equal to x component of the shear stress on a face perpendicular to the y-axis. Similarly, it can be shown that $\tau_{yz} = \tau_{zx}$ and $\tau_{zx} = \tau_{xz}$

In order to determine the stress components on a plane of arbitrary orientation, consider a tetrahedron as shown in Fig. 1.65. It has three of its faces parallel to the coordinate planes whereas the fourth face ABC is perpendicular to the line QN. In the figure, the face QAB is shown parallel to yOz plane or the plane perpendicular to x-axis. Similarly, QBC is parallel to xOy and the plane QCA parallel to plane xOz.

Let l_x, l_y and l_z be the direction cosines of the line QN with respect to x-, y- and z-directions. Then if δA is the area of the face ABC, areas of the faces perpendicular to the x-, y- and z-axis will be $\delta A \cdot l_x$, $\delta A \cdot l_y$ and $\delta A \cdot l_z$ respectively.

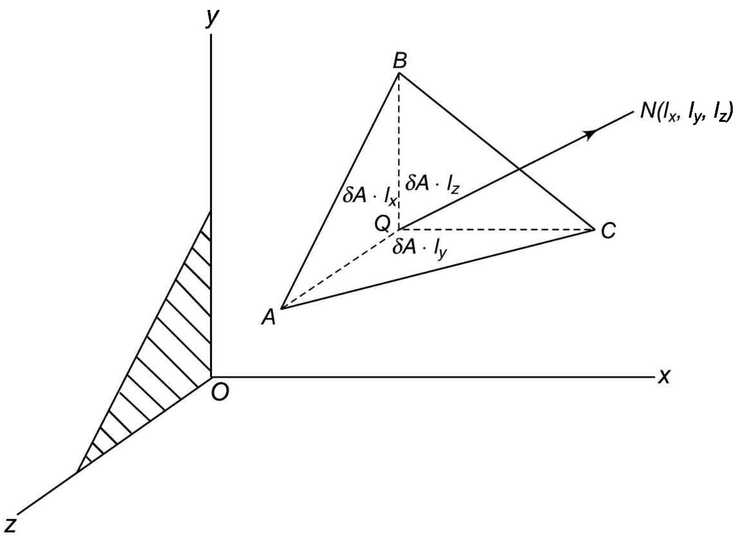


Fig. 1.65

Let also the state of stress at point Q be defined by the stress components σ_x, σ_y, σ_z, σ_{xy} σ_{yz} and σ_{zx}. The forces exerted on the faces are obtained by multiplying each stress component by the corresponding area of that face. On the face ABC, the normal force will be of magnitude $\sigma \cdot \delta A$ along QN and a shear force of magnitude $\tau \cdot \delta A$ perpendicular to QN but of unknown direction on the face ABC.

Now, for equilibrium of the tetrahedron, the sum of components of all the forces along QN must be zero, i.e., $F_n = 0$

Normal force on face $QAB = \sigma_x \cdot \delta A \cdot l_x$ (Fig. 1.66)

Component of this force along $QN = (\sigma_x \cdot \delta A \cdot l_x)l_x$

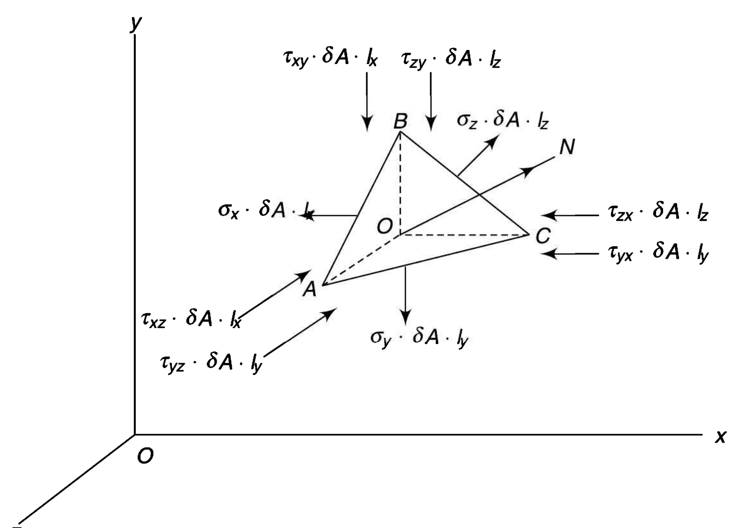


Fig. 1.66

Similarly, components of normal forces on other faces along QN can be determined.
Shear stress on face QAB along y-axis = τ_{xy}
Shear force due to this force = $\tau_{xy} \cdot \delta A \cdot l_x$
Component of this shear force along $QN = (\tau_{xy} \cdot \delta A \cdot l_x)l_y$
Similarly, component of shear force $\tau_{xz} \cdot \delta A \cdot l_x$ on face QAB along $QN = (\tau_{xz} \cdot \delta A \cdot l_x)\, l_z$
Similarly, components of shear forces on other faces along QN can be determined.
Therefore,

$$\sigma \cdot \delta A - (\sigma_x \cdot \delta A \cdot l_x)l_x - (\tau_{xy} \cdot \delta A \cdot l_x)l_y - (\tau_{xz} \cdot \delta A \cdot l_x)l_z - (\sigma_y \cdot \delta A \cdot l_y)l_y - (\tau_{yz} \cdot \delta A \cdot l_y)l_z$$
$$-(\tau_{yx} \cdot \delta A \cdot l_y)l_x - (\sigma_z \cdot \delta A \cdot l_z)l_z - (\tau_{zx} \cdot \delta A \cdot l_z)l_x - (\tau_{zy} \cdot \delta A \cdot l_z)l_y = 0$$

Dividing throughout by δA and remembering that $\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zx}$ and $\tau_{zx} = \tau_{xz}$,

$$\sigma = \sigma_x \cdot l_x^2 + \sigma_y \cdot l_y^2 + \sigma_z \cdot l_z^2 - 2\tau_{xy} \cdot l_x l_y - 2\tau_{yz} \cdot l_y l_z + 2\tau_{zx} \cdot l_z l_x \tag{1.30}$$

Let shear force of magnitude $\tau \cdot \delta A$ acts along QS perpendicular to QN. Let l_{sx}, l_{sy} and l_{sz} be the direction cosines of the line QS with respect to x, y and z-directions. As normal-stress direction and the shear-stress direction are at right angles to each other, the following equation must be satisfies,

$$l_x l_{sx} + l_y l_{sy} + l_z l_{sz} = 0 \tag{1.31}$$

Then

$$\tau \cdot \delta A - (\sigma_x \cdot \delta A \cdot l_x)l_{sx} - (\tau_{xy} \cdot \delta A \cdot l_x)l_{sy} - (\tau_{xz} \cdot \delta A \cdot l_x)l_{sz} - (\sigma_y \cdot \delta A \cdot l_y)l_{sy} - (\tau_{yz} \cdot \delta A \cdot l_y)l_{sz}$$
$$-(\tau_{yx} \cdot \delta A \cdot l_y)l_{sx} - (\sigma_z \cdot \delta A \cdot l_z)l_{sz} - (\tau_{zx} \cdot \delta A \cdot l_z)l_{sx} - (\tau_{zy} \cdot \delta A \cdot l_z)l_{sy} = 0$$

or

$$\tau = \sigma_x l_x l_{sx} + \tau_{xy} l_x l_{sy} + \tau_{xz} l_x l_{sz} + \sigma_y l_y l_{sy} + \tau_{yz} l_y l_{sz} + \tau_{yx} l_y l_{sx} + \sigma_z l_z l_{sz} + \tau_{zx} l_z l_{sx} + \tau_{zy} l_z l_{sy}$$
$$= \sigma_x l_x l_{sx} + \sigma_y l_y l_{sy} + \sigma_z l_z l_{sz} + \tau_{xy}(l_x l_{sy} + l_y l_{sx}) + \tau_{xz}(l_x l_{sz} + l_z l_{sx}) + \tau_{yz}(l_y l_{sz} + l_z l_{sy})$$

Principal Stress

Let the inclined plane be a principal plane so that shear stress on the face ABC is zero. Let the direction cosines of the normal to the surface with the coordinate axes x, y and z be l_{nx}, l_{ny} and l_{nz} respectively. Then by Newton's law in x-direction (neglecting gravity and inertia effects),

Normal force on face $ABC = \sigma \cdot \delta A$
Component of this force in x-direction = $\sigma \cdot \delta A \cdot l_{nx}$
Area of face $QAB = \delta A \cdot l_{nx}$
Force on face QAB in x-direction = $\sigma_x \delta A \cdot l_{nx}$
Similarly, components of forces on other faces in the x-direction can be determined.

Then, $\quad \sigma \cdot \delta A \cdot l_{nx} - \sigma_x \cdot \delta A \cdot l_{nx} - \tau_{yx} \cdot \delta A \cdot l_{ny} - \tau_{zx} \cdot \delta A \cdot l_{nz} = 0$

or $\quad (\sigma_x - \sigma)l_{nx} + \tau_{yx} \cdot l_{ny} + \tau_{zx} \cdot l_{nz} = 0$ or $(\sigma_x - \sigma)l_{nx} + \tau_{yx} \cdot l_{ny} + \tau_{zx} \cdot l_{nz} = 0$

Similarly, $\quad \tau_{xy} \cdot l_{nx} + (\sigma y - \sigma) \cdot l_{ny} + \tau_{zy} \cdot l_{nz} = 0$ or $\tau_{xy} \cdot l_{nx} + (\sigma_y - \sigma) \cdot l_{ny} + \tau_{yz} \cdot l_{nz} = 0$

$\quad \tau_{xz} \cdot l_{nx} + \tau_{yz} \cdot l_{ny} + (\sigma_z - \sigma) \cdot l_{nz} = 0$ or $\tau_{zx} \cdot l_{nz} + \tau_{yz} \cdot l_{ny} + (\sigma_z - \sigma) \cdot l_{nz} = 0$

From these equations, it is required to find three direction cosines l_{nx}, l_{ny} and l_{nz} and the magnitude of the principal stress. By Cramer's rule,

$$l_{nx} = \frac{\begin{vmatrix} 0 & \tau_{xy} & \tau_{zx} \\ 0 & \sigma_y - \sigma & \tau_{yz} \\ 0 & \tau_{yz} & \sigma_z - \sigma \end{vmatrix}}{\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix}}$$

As the numerator is zero, l_{nx} is zero and similarly l_{ny} and l_{nz} will also be zero unless the denominator is also zero. As the direction cosines cannot be zero because

$$l_{nx}^2 + l_{ny}^2 + l_{nz}^2 = 0$$

Therefore, the necessary condition required for the solution is that the denominator is zero i.e.

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{yz} \\ \tau_{zx} & \tau_{yz} & \sigma_z - \sigma \end{vmatrix} = 0$$

On expanding, the following cubic equation is obtained,

$$\begin{aligned} &\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma \\ &-(\sigma_x\sigma_y\sigma_z - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 + 2\tau_{xy}\tau_{yz}\tau_{zx}) = 0 \end{aligned} \tag{1.32}$$

On solving, three real roots of the equation are obtained which indicates that there are always three real values that satisfy the equation.

Summary

1. The resisting force per unit area of cross-section of a body is known as *intensity of stress* or simply *stress*.
2. *Shear stress* exists on two parts of a body when two equal and opposite parallel forces, not in the same line, act and one part tends to slide over or shear from the other across any section.
3. The elongation per unit length of a body is known as *strain*. It is dimensionless.
4. *Shear strain* is the change in the right angle of a rectangular element measured in radians and is dimensionless.
5. The ratio of stress to strain is constant within elastic limits and is known as *Young's modulus* or the *modulus of elasticity*. It is denoted by E and $E = \sigma/\varepsilon$.
6. *Modulus of rigidity* or *shear modulus* denoted by G is the ratio of shear stress to shear strain, i.e. $G = \tau/\varphi$.
7. The materials in which strain is proportional to stress are said to obey the *Hooke's law*.
8. Elongation of a bar of length L is given by, $\Delta = PL/AE$
9. The principle of superposition states that if a body is acted upon by a number of loads, then the net effect on the body is the sum of the effects caused by each load acting independently.
10. Increase in length due to temperature rise $= L \cdot \alpha \cdot t$
11. Temperature stress is given by $\sigma = \alpha\, t\, E$

12. For elastic materials, the ratio of lateral strain to longitudinal strain is constant and is known as *Poisson's ratio* (v).
13. Volumetric strain is the ratio of increase in volume of a body to its original volume when it is acted upon by three mutually perpendicular stresses.
14. Volumetric strain = $\dfrac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2v)}{E} \approx \varepsilon_1 + \varepsilon_2 + \varepsilon_3.$
15. Factor of safety = $\dfrac{\text{ultimate load}}{\text{allowable load}}$ or $\dfrac{\text{ultimate stress}}{\text{allowable stress}}$
16. Relation between various elasticity constants of a material is
$$E = 2G(1 + v) = 3K(1 - 2v)$$

Objective Type Questions

1. A material with identical properties in all directions is known as
(a) homogeneous (b) isotropic (c) elastic (d) none of these

2. The units of stress in the SI system are
(a) kg/m^2 (b) N/mm^2 (c) MPa (d) any one of these

3. In a lap-riveted joint, the rivets are mainly subjected to _____ stress.
(a) shear (b) tensile (c) bending (d) compressive

4. The resistance to deformation of a body per unit area is known as
(a) stress (b) strain
(c) modulus of elasticity (d) modulus of rigidity

5. Stress developed due to external force in an elastic material
(a) depends on elastic constants (b) does not depend on elastic constants
(c) depends partially on elastic constants

6. Strain is defined as deformation per unit
(a) area (b) length (c) load (c) volume

7. Units of strain are
(a) mm/m (b) mm/mm (c) m/mm (d) no units

8. Hooke's law is valid up to the
(a) elastic limit (b) yield point
(c) limit of proportionality (d) ultimate point

9. The ratio of linear stress to linear strain is known as
(a) bulk modulus (b) modulus of rigidity
(c) Young's modulus (d) modulus of elasticity

10. The units of modulus of elasticity are the same as of
(a) stress (b) modulus of rigidity (c) pressure (d) any one of these

11. The change in length due to a tensile force on body is given by
(a) PL/AE (b) PLA/E (c) PLE/A (d) AE/PL

12. Approximate Value of Young's modulus for mild steel is
(a) 100 GPa (b) 205 MPa (c) 205 GPa (d) 100 MPa

13. 1MPa is equal to
(a) 1N/m^2 (b) 1N/mm^2 (c) 1kN/m^2 (d) 1kN/mm^2

14. Deformation of a bar under its own weight is …… the deformation due to a direct load equal to the weight of the body applied at the lower end.
(a) double (b) four times (c) half (d) equal to

15. Elongation of a conical bar under its own weight is ……that of a rectangular section of the same length.
(a) two-third (b) one-third (c) half (d) equal to

16. If a bar of length L m extends by l mm under load W, the strain is
(a) $1000\ l/L$ (b) l/L (c) $0.001l/L$ (d) none of these

17. Four wires of same material are applied the same load. In which of the following cases, the elongation will be maximum (length, diameter)?
(a) 2 m, 1 mm (b) 4 m, 2 mm (c) 8 m, 4 mm (d) 12 m, 6 mm

18. If $E_s = 3E_a$, the stress in steel of a composite bar made of aluminium and steel strips each having a cross-sectional area of 300 mm^2 and subjected to an axial load of 12 kN is
(a) 10 MPa (b) 20 MPa (c) 30 MPa (d) 40 MPa

19. The ratio of modulus of rigidity to modulus of elasticity for most of the materials is
(a) 0.5 (b) > 0.5 (c) < 0.5 (d) > 1

20. Temperature stress is a function of
(a) modulus of elasticity (b) coefficient of linear expansion
(c) change in temperature (d) all of these

21. Factor of safety is defined as ratio of
(a) ultimate stress to allowable stress (b) ultimate stress to yield stress
(c) allowable stress to ultimate stress (d) allowable stress to yield stress

22. The modulus of rigidity in terms of modulus of elasticity is given by

(a) $\dfrac{E}{2(1-\nu)}$ (b) $\dfrac{E}{2(1+\nu)}$ (c) $\dfrac{2E}{(1-\nu)}$ (d) $\dfrac{2E}{(1+\nu)}$

23. The modulus of elasticity in terms of bulk modulus is given by
(a) $3K/(1+2\nu)$ (b) $3K(1+2\nu)$ (c) $3K/(1-2\nu)$ (d) $3K(1-2\nu)$

24. The ratio of Young's modulus and bulk modulus is given by
(a) $3/(1+2\nu)$ (b) $3(1-2\nu)$ (c) $3(1-\nu)$ (d) none of these

25. The modulus of elasticity in terms of bulk modulus and modulus of rigidity is

(a) $\dfrac{9KG}{3K+G}$ (b) $\dfrac{9KG}{K+3G}$ (c) $\dfrac{3K+G}{9KG}$ (d) $\dfrac{K+3G}{9KG}$

26. The bulk modulus of a material having $E = 200$ GPa and $G = 80$ GPa is
(a) 233.3 GPa (b) 133.3 GPa (c) 250 GPa (d) 160 GPa

27. For a Poisson's ratio 0.4 for a material, the ratio of the shear modulus of elasticity to modulus of elasticity is
(a) 14/5 (b) 5/7 (c) 7/5 (d) 5/14

28. The ratio of lateral strain to linear strain is known as
(a) modulus of rigidity (b) elastic limit
(c) Poisson's ratio (d) modulus of elasticity

29. The stress at which elongation of a material is quite large as compared to the increase in load is known as
(a) ultimate point (b) yield point (c) elastic limit (d) rupture point

30. The limit up to which the stress is linearly proportional to strain is _____ limit.
(a) elastic (b) plastic (c) ultimate (d) rupture

31. Poisson's ratio ν is defined as ratio of
(a) axial strain to transverse strain (b) transverse strain to axial strain
(c) shear strain to axial strain (d) axial strain to shear strain

Answers

1. (b)	2. (b and c)	3. (a)	4. (a)	5. (b)	6. (b)
7. (d)	8. (c)	9. (c and d)	10. (d)	11. (a)	12. (c)
13. (b)	14. (c)	15. (b)	16. (c)	17. (a)	18. (c)
19. (c)	20. (d)	21. (a)	22. (b)	23. (d)	24. (b)
25. (a)	26. (b)	27. (d)	28. (c)	29. (b)	30. (a)
31. (b)					

Review Questions

1.1 What do you mean by tensile, compressive and shear forces? Give examples.
1.2 What is stress? In what way does the shear stress differ from direct stress? Explain.
1.3 What is complimentary shear stress? What is its significance?
1.4 Explain the terms: strain, shear strain, Young's modulus and modulus of rigidity.
1.5 Define the principle of superposition. What is its utility?
1.6 Deduce expressions to determine the elongation of
(i) a bar of tapering section, and
(ii) a trapezoidal section of uniform thickness.
1.7 Find an expression for elongation of a bar of rectangular section and a conical section due to self-weight.
1.8 What is meant by a column of uniform strength? How is its area of cross-section along its length related to that at the top?
1.9 What are compound bars? What are equilibrium and compatibility equations?
1.10 What do you mean by temperature stresses? Explain.
1.11 Define the term Poisson's ratio. Write the expressions for strains in the three principal directions.
1.12 What is volumetric strain? Show that it is the algebraic sum of three mutually perpendicular strains.
1.13 Plot a tensile test diagram for steel. Explain its salient features.
1.14 Define the term *factor of safety* and its importance.
1.15 Define bulk modulus. Deduce the relation $E = 3K(1 - 2\nu)$.
1.16 Derive a relation between Young's modulus, modulus of rigidity and the poisson's ratio.
1.17 Deduce an expression among three elastic constants of a material.

Numerical Problems

1.1 A rod made up of a number of circular cross-sections as shown in Fig. 1.67 is subjected to a tensile force of 125 kN. Determine the elongation of the rod. $E = 205$ GPa. (0.7 mm)
1.2 The loading on a steel bar of 30-mm diameter is as shown in Fig. 1.68. Find the elongation of the bar. $E_s = 205$ GPa (0.224 mm)
1.3 Determine the force P necessary for the equilibrium of a steel bar shown in Fig. 1.69. The diameters of the first, middle and the last segments of the bar are 30 mm, 25 mm and 30 mm respectively. Also, find the elongation of the bar. $E = 200$ GPa. (250 kN, 0.536 mm)

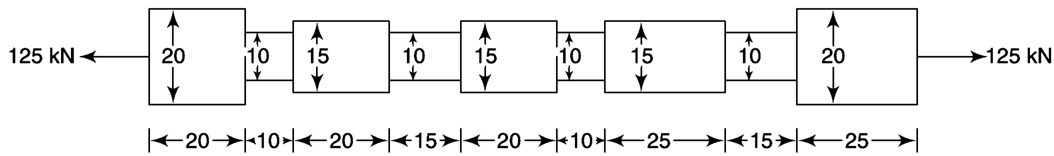


Fig. 1.67

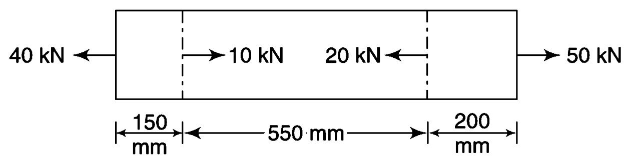


Fig. 1.68

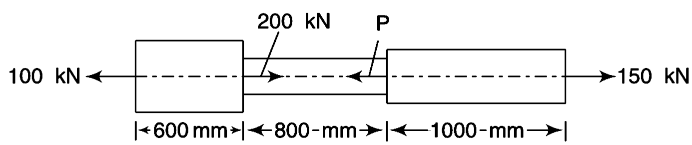


Fig. 1.69

1.4 Determine the total compression of a bar made up of three circular cross-sections as shown in Fig. 1.70, the diameters being 10 mm, 20 mm and 30 mm of the top, middle and the bottom portions respectively. Take $E_s = 210$ GPa, $E_b = 105$ GPa and $E_c = 100$ GPa. (0.15 mm)

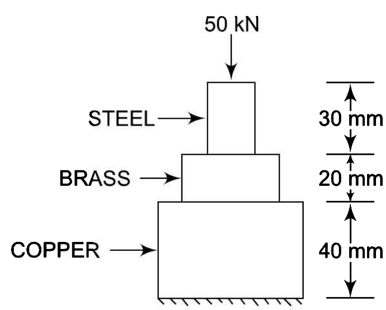


Fig. 1.70

1.5 A flange coupling is used to join two shafts which are to transmit 300 kW at 800 rpm. 8 bolts are used at a pitch diameter of 120 mm. Assuming a mean shear stress of 80 MPa, determine the diameter of the bolts. (9 mm)

1.6 Determine the contraction of a 10-mm thick flat aluminium plate of trapezoidal section which tapers uniformly from a width of 50 mm to 30 mm in a length of 400 mm when an axial compressive load of 80 kN is applied? $E = 78$ GPa (1.048 mm)

1.7 A 20-mm thick and 200-mm wide steel plate tapers uniformly to 10-mm thickness and 150-mm width over a length of 2 m. Determine the increase in length when a pull of 18 kN is applied. $E = 200$ GPa. (0.073 mm)

1.8 An assembly of a steel bar of 60-mm diameter enclosed in an aluminium tube of 70-mm internal diameter and 110-m external diameter is compressed between two rigid parallel plates by a force of 300 kN. The length of the assembly is 1 m. Determine the stresses in the tube and the bar if $E_s = 200$ GPa and $E_a = 70$ GPa (21.8 MPa, 62.3 MPa)

1.9 A 6-m high column is designed to withstand a uniform compressive stress of 20 MPa throughout its height. Find the diameter at the base and at the midsection of the column if the diameter at the top is 24 mm. Density of the column material is 7400 kg/m^3. (24.26 mm, 24.13 mm)

1.10 A steel rod of 24-mm diameter passes centrally through a 1-m long brass tube of 36-mm external diameter and 30-mm internal diameter. The tube is closed at each end by rigid washers and nuts screwed to the rod. The nuts are tightened till the compressive force in the brass tube is 25 kN. Determine the stresses in the rod and the tube. (55.24 MPa tensile; 80.38 MPa compressive)

1.11 A load of 800 kN is applied to a reinforced concrete column of 560-mm diameter which has four steel rods of 36-mm diameter embedded in it. Determine the stress in the concrete and the steel. Take E for steel = 210 GPa and E for concrete = 15 GPa.

Also, find the adhesive force between the concrete and the steel.

[2.67 MPa (concrete); 37.43 MPa (steel); 139.3 kN]

1.12 In a framed structure shown in Fig. 1.71, all the three rods are of the same material and with the same area of cross-section of 120 mm^2. The central rod is 240 mm long but is longer than its requirement by 2 mm. Determine the forces in the rods if their lower ends are welded together. E = 205 GPa.

(115.83 kN in rod 2 and 66.9 kN in rods 1 and 3)

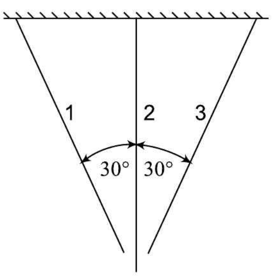


Fig. 1.71

1.13 In the framed structure of Fig. 1.71, the outer rods are of steel and of 260 mm^2 area of cross-section whereas the central rod is of brass and of 420 mm^2 area of cross-section. The length of the central rod is 1200 mm. Initially all the rods are of required length. However, while assembling, the central rod is heated through 40°C. Determine the stresses developed in the rods. E for steel = 205 GPa, E for brass = 85 GPa and α for brass = 19×10^{-6}/°C.

(17.9 kN in 2, 10.34 kN in 1 and 3)

1.14 A steel sleeve of 24-mm internal diameter and 36-mm external diameter encloses an aluminum rod of 22-mm diameter. The length of the rod is 0.4 mm longer than that of the sleeve which is 400 mm long as shown in Fig. 1.72. Determine

(i) the compressive load upto which only the rod is stressed
(ii) the maximum load on the assembly if the permissible stresses in aluminium and steel are 130 MPa and 175 MPa respectively
(iii) the deformation of the assembly under maximum load

E_a = 75 GPa and E_s = 205 GPa

(28.5 kN; 105.9 kN; 0.293 mm)

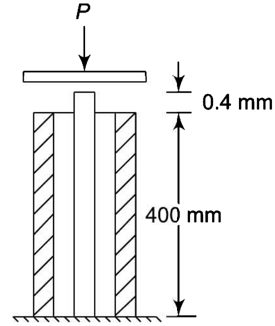


Fig. 1.72

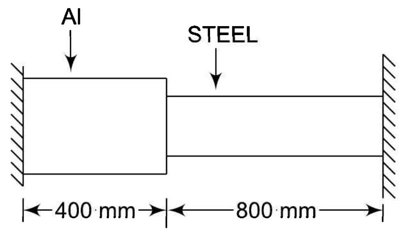


Fig. 1.73

1.15 A composite bar made up of aluminium and steel is rigidly attached to the end supports at 60°C as shown in Fig. 1.73. Find the stresses in the two portions of the bar when the temperature of the composite system falls to 20°C if (i) the ends do not yield (ii) the ends yield by 0.25 mm.

$$E_s = 200 \text{ GPa}; E_a = 70 \text{ GPa}$$
$$\alpha_s = 11.7 \times 10^{-6}/°\text{C}; \quad \alpha_a = 23.4 \times 10^{-6}/°\text{C}$$

Cross-sectional areas, $A_s = 250 \text{ mm}^2$; $A_a = 375 \text{ mm}^2$

(63.9 MPa, 95.9MPa; 42.6 MPa, 63.9 MPa)

1.16 A composite bar of 20 mm × 20 mm cross-section is made up of three flat bars as shown in Fig. 1.74. All the three bars are rigidly connected at the ends when the temperature is 20°C. Determine

(i) the stresses developed in each bar when the temperature of the composite bar is raised to 60°C

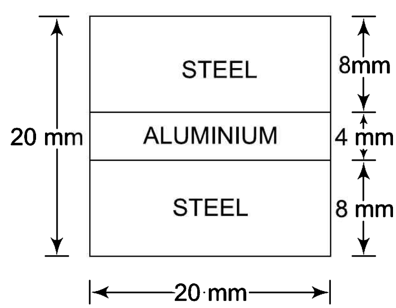


Fig. 1.74

(ii) the final stresses in each bar when a load of 17.6 kN is applied to the composite bar

$$E_a = 80 \text{ GPa} \qquad \alpha_a = 11 \times 10^{-6}/°\text{C}$$
$$E_s = 200 \text{ GPa} \qquad \alpha_s = 22 \times 10^{-6}/°\text{C}$$

($\sigma_s = 8$ MPa; $\sigma_a = 32$ MPa; $\sigma_s = 42$ MPa; $\sigma_a = 52$ MPa)

1.17 A load of 120 kN is applied to bar of 20-mm diameter. The bar which is 400 mm long is elongated by 0.7 mm. Determine the modulus of elasticity of the bar material. If the Poisson's ratio is 0.3, find the change in diameter. (218 GPa; 0.0105 mm)

1.18 A metallic prismatic specimen is subjected to an axial stress of σ_x and on one pair of sides no constraint is exerted whereas on the other, the lateral strain is restricted to one half of the strain which would be there in the absence of any constraint. Show that the modified modulus of elasticity will be $[2E/(2 - \nu^2)]$.

1.19 A prismatic bar is stretched in such a way that all the lateral strain is prevented. Find the value of the modified modulus of elasticity. What will be the modified Poisson's ratio?

$[E(1 - \nu)/(1 + \nu)(1 - 2\nu)$; zero)]

1.20 A steel bar of 10-mm diameter is subjected to an axial load of 12 kN. If the change in diameter is found to be 0.0022 mm, determine the Poisson's ratio and the modulus of elasticity and the bulk modulus. Take $G = 78$ GPa. (0.29; 201.4 GPa, 159.8 GPa)

1.21 An axial load of 56 kN is applied to a bar of 36-mm diameter and 1-m length. The extension of the bar is measured to be 0.265 mm whereas the reduction in diameter is 0.003 mm. Calculate the Poisson's ratio and the values of the three moduli. (0.314, $E = 207.6$ GPa, $G = 79$ GPa, $K = 186$ GPa)

Chapter 2

Compound Stress and Strain

In the first chapter, direct and shear forces were assumed to act independently. Also, the stresses were determined on planes in the normal or tangential directions. However, in most of the cases, direct and shear forces act simultaneously on a body and the maximum value of the resultant stress may act in some other direction than of the load application. It is, therefore, necessary to find out stresses on planes other than those of load application.

In practical problems, the stress varies from point to point in a loaded member. Therefore, the equilibrium of an element at a point is to be considered by taking the element of infinitesimal dimensions so that the stresses approach the conditions at the point. Such an infinitesimal element may be considered of any convenient shape and the stresses may be assumed uniformly distributed over the surface of the element. In a two-dimensional analysis, the thickness of the element does not affect the results and for convenience sake, may be taken as unity.

2.1 STRESS ANALYSIS

While analysing a stress system, the general conventions have been taken as follows:

- A tensile stress is positive and compressive stress, negative.
- A pair of shear stresses on parallel planes forming a clockwise couple is positive and a pair with counter-clockwise couple, negative.
- Clockwise angle is taken as positive and counter-clockwise, negative.

The following cases are being considered:

(i) Direct stress condition
(ii) Bi-axial stress condition
(iii) Pure shear stress condition
(iv) Bi-axial and shear stresses condition

(i) Direct Stress Condition

Let a bar be acted upon by an external force P resulting in a direct tensile stress σ_x along its length (Fig. 2.1a). The stress on any transverse section such as $BCDE$ will have a pure normal stress acting on it. The stress acting on an arbitrary plane $ACDF$ inclined at an angle θ with the vertical plane will have two components:

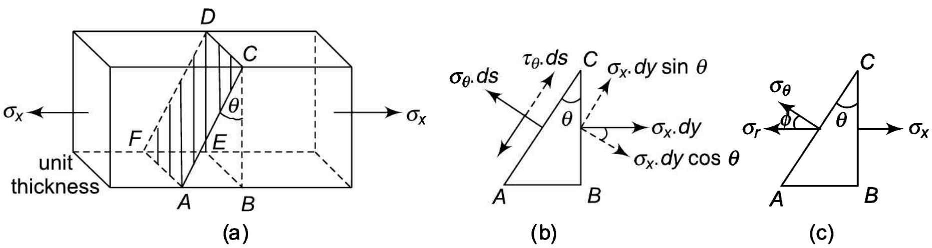


Fig. 2.1

— normal component known as *direct stress component*
— tangential component known as *shear stress component.*

These stress components can be determined from the consideration of force balance.

If the bar is imagined cut through the section $ACDF$, each portion of the bar is also in equilibrium under the action of forces due to the stresses developed. For convenience, a triangular prismatic element $ABCDEF$ containing the plane $ACDF$ can be taken for the force analysis.

Figure 2.1b shows the forces acting on the triangular element.

Let

dy = the length of the side BC
ds = the length of the side AC
σ_x = normal stress acting on the plane $BCDE$
σ_θ = normal stress acting on the plane $ACDF$
τ_θ = tangential or shear stress acting on the plane $ACDF$

Assume a unit thickness of the prism and equate the forces along normal and tangential directions to the plane $ACDF$ of the prism for its equilibrium, i.e.,

$$\sigma_\theta \cdot ds - \sigma_x . dy . \cos\theta = 0$$

or
$$\sigma_\theta = \frac{\sigma_x \, dy \cos\theta}{ds} = \frac{\sigma_x \, dy \cos\theta}{dy/\cos\theta} = \sigma_x \cos^2\theta \tag{2.1}$$

and
$$\tau_\theta \cdot ds + \sigma_x . dy . \sin\theta = 0 \qquad \text{(assuming } \tau_\theta \text{ clockwise as positive)}$$

or
$$\tau_\theta = -\frac{\sigma_x \, dy \sin\theta}{ds} = -\frac{\sigma_x \, dy \sin\theta}{dy/\cos\theta} = -\sigma_x \sin\theta\cos\theta = -\frac{1}{2}\sigma_x \sin 2\theta \tag{2.2}$$

The negative sign shows that τ_θ is counter-clockwise and not clockwise on the inclined plane.

- When $\theta = 0°$, $\sigma_\theta = \sigma_x$ and $\tau_\theta = 0$
- When $\theta = 45°$, $\sigma_\theta = \sigma_x/2$ and $\tau_\theta = -\sigma_x/2$ (maximum, counter-clockwise)
- When $\theta = 90°$, $\sigma_\theta = 0$ and $\tau_\theta = 0$
- When $\theta = 135°$ $\sigma_\theta = \sigma_x/2$ and $\tau_\theta = \sigma_x/2$ (maximum, clockwise)

Figures 2.2 a and b show the planes inclined at different angles to the vertical alongwith the stresses acting on them. The following can be noted from these figures along with the above observations:

- A plane at angle θ with the vertical also is the plane with angle $(180°+\theta)$. Thus, a plane at angle 45° clockwise with the vertical can also be mentioned as the plane at 225° clockwise or 135° counter-clockwise. Similarly, a plane at angle –45° with the vertical would also mean a plane at angle 45° counter-clockwise or angle 225° counter-clockwise or angle 135° clockwise.
- The normal stress on the inclined plane decreases with the increase in angle θ, from maximum on the vertical plane to zero on the horizontal plane.

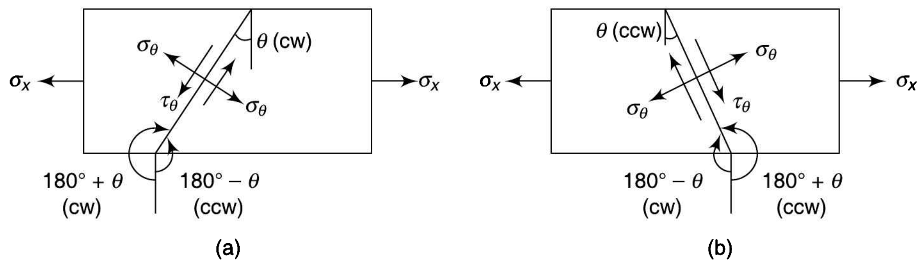


Fig. 2.2

- The shear stress is negative (counter-clockwise) between 0° and 90° and positive (clockwise) between 0° and –90°. Remember that a plane at 135° to the vertical also means a plane at –45° as described above.
- The maximum shear stress is equal to one half the applied stress.

The resultant stress on the plane *ACDF* (Fig. 2.1c),

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2}$$
$$= \sigma_x\sqrt{\cos^4\theta + \sin^2\theta\cos^2\theta}$$
$$= \sigma_x\cos\theta\sqrt{\cos^2\theta + \sin^2\theta}$$
$$= \sigma_x \cos\theta \tag{2.3}$$

Inclination with the normal stress, $\tan\varphi = \dfrac{\sigma_x \sin\theta\cos\theta}{\sigma_x\cos^2\theta} = \tan\theta$

or $$\varphi = \theta \tag{2.4}$$

i.e., it is always in the direction of the applied stress.

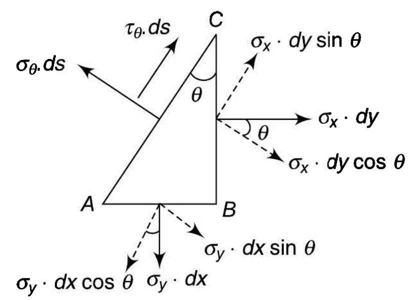


Fig. 2.3

(ii) Bi-axial Stress Condition

Let an element of a body be acted upon by two tensile stresses acting on two perpendicular planes of the body as shown in Fig. 2.3. Let *dx*, *dy* and *ds* be the lengths of the sides *AB*, *BC* and *AC* respectively. Considering unit thickness of the body and resolving the forces in the direction of σ_θ,

$$\sigma_\theta \cdot ds - \sigma_x \cdot dy \cdot \cos\theta - \sigma_y \cdot dx \cdot \sin\theta = 0$$

or $$\sigma_\theta = \frac{\sigma_x\, dy\cos\theta}{ds} + \frac{\sigma_y\, dx\sin\theta}{ds} = \frac{\sigma_x\, dy\cos\theta}{dy/\cos\theta} + \frac{\sigma_y\, dx\sin\theta}{dx/\sin\theta}$$

$$= \sigma_x\cos^2\theta + \sigma_y\sin^2\theta \tag{2.5}$$

The expression may be put in the following form,

$$\sigma_\theta = \sigma_x\left(\frac{1+\cos 2\theta}{2}\right) + \sigma_y\left(\frac{1-\cos 2\theta}{2}\right) = \frac{1}{2}(\sigma_x+\sigma_y) + \frac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta \tag{2.6}$$

Resolving the forces in the direction of τ_θ,

$$\tau_\theta \cdot \text{ds} + \sigma_x.\text{dy}\cdot\sin\theta - \sigma_y.\text{dx}\cdot\cos\theta = 0$$

or $$\tau_\theta = -\frac{\sigma_x\, dy \sin\theta}{ds} + \frac{\sigma_y\, dx \cos\theta}{ds} = -\frac{\sigma_x\, dy \sin}{dy/\cos\theta} + \frac{\sigma_y\, dx\cos\theta}{dx/\sin\theta}$$

$$= -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta$$

$$= -(\sigma_x - \sigma_y)\sin\theta\cos\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta \tag{2.7}$$

which indicates that it is counter-clockwise if σ_x is more than σ_y.
Resultant stress,

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2}$$

$$= \left[\left\{\frac{1}{2}(\sigma_x+\sigma_y) + \frac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta\right\}^2 + \left\{-\frac{1}{2}(\sigma_x-\sigma_y)\sin 2\theta\right\}^2\right]^{1/2}$$

$$= \left[\frac{1}{4}(\sigma_x+\sigma_y)^2 + \frac{1}{4}(\sigma_x-\sigma_y)^2\cos^2 2\theta + \frac{1}{2}(\sigma_x+\sigma_y)(\sigma_x-\sigma_y)\cos 2\theta + \frac{1}{4}(\sigma_x-\sigma_y)^2\sin^2 2\theta\right]^{1/2}$$

$$= \left[\frac{1}{4}(\sigma_x+\sigma_y)^2 + \frac{1}{4}(\sigma_x-\sigma_y)^2 + \frac{1}{2}(\sigma_x^2-\sigma_y^2)\cos 2\theta\right]^{1/2}$$

$$= \left[\frac{1}{4}(\sigma_x^2+\sigma_y^2+2\sigma_x\sigma_y+\sigma_x^2+\sigma_y^2-2\sigma_x\sigma_y) + \frac{1}{2}(\sigma_x^2-\sigma_y^2)\cos 2\theta\right]^{1/2}$$

$$= \left[\frac{1}{2}(\sigma_x^2+\sigma_y^2) + \frac{1}{2}(\sigma_x^2-\sigma_y^2)\cos 2\theta\right]^{1/2}$$

$$= \left[\frac{1}{2}\sigma_x^2(1+\cos 2\theta) + \frac{1}{2}\sigma_y^2(1-\cos 2\theta)\right]^{1/2}$$

$$= \left[\frac{1}{2}\sigma_x^2\cdot 2\cos^2\theta + \frac{1}{2}\sigma_y^2\cdot 2\sin^2\theta\right]^{1/2}$$

$$= \sqrt{\sigma_x^2\cos^2\theta + \sigma_y^2\sin^2\theta} \tag{2.8}$$

and the angle of inclination of the resultant with σ_θ.

$$\tan\varphi = \frac{\tau_\theta}{\sigma_\theta} = \frac{-(\sigma_x-\sigma_y)\sin\theta\cos\theta}{\sigma_x\cos^2\theta + \sigma_y\sin^2\theta} = \frac{\sigma_y - \sigma_x}{\sigma_x\cot\theta + \sigma_y\tan\theta} \tag{2.9}$$

For greatest obliquity or inclination of the resultant with the normal stress,

$$\frac{d(\tan\varphi)}{d\theta} = 0$$

or $$-\sigma_x \operatorname{cosec}^2\theta + \sigma_y\sec^2\theta = 0 \quad \text{or} \quad \sigma_x\operatorname{cosec}^2\theta = \sigma_y\sec^2\theta$$

$$\tan^2\theta = \frac{\sigma_x}{\sigma_y} \quad \text{or} \quad \tan\theta = \sqrt{\frac{\sigma_x}{\sigma_y}} \tag{2.10}$$

$\therefore$
$$\tan\varphi_{max} = \frac{\sigma_y - \sigma_x}{\sigma_x\sqrt{\sigma_y/\sigma_x} + \sigma_y\sqrt{\sigma_x/\sigma_y}} = \frac{\sigma_y - \sigma_x}{2\sqrt{\sigma_x\sigma_y}} \tag{2.10a}$$

The angle of inclination of the resultant with σ_x,

$$\tan\alpha = \frac{\sigma_y \cdot dx}{\sigma_x \cdot dy} = \frac{\sigma_y \cdot dy \cdot \tan\theta}{\sigma_x \cdot dy} = \frac{\sigma_y}{\sigma_x}\tan\theta \tag{2.11}$$

The above results show the following:

- The normal stress on the inclined plane varies between the values of σ_x and σ_y as the angle θ is increased from 0° to 90°. For equal values of the two axial stresses ($\sigma_x = \sigma_y$), σ_θ is always equal to σ_x or σ_y.
- The shear stress is zero on planes with angles 0° and 90°, i.e., on horizontal and vertical planes. It has maximum value numerically equal to one half the difference between given normal stresses which occurs on planes at ± 45° to the given planes.

$$\tau_{max} = \pm\frac{1}{2}(\sigma_x - \sigma_y) \tag{2.12}$$

and the normal stress across the same plane,

$$\sigma_{45°} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 90° = \frac{1}{2}(\sigma_x + \sigma_y) \tag{2.13}$$

- Shear stress in a body subjected to two equal perpendicular stresses is zero (Refer Eq. 2.7).
- If any of the given stresses is compressive, the stress can be replaced by a negative sign in the above derived expressions, i.e., σ_x with $-\sigma_x$ and σ_y with $-\sigma_x$.
- If σ_y is compressive, the maximum value of shear stress across a plane at 45° plane is

$$\tau_{max} = \frac{1}{2}[(\sigma_x - (-\sigma_y)] = \frac{1}{2}(\sigma_x + \sigma_y)$$

and if σ_x is numerically equal to σ_y,

$$\tau_{max} = \sigma_x = \sigma_y \tag{2.14}$$

(iii) Pure Shear Stress Condition

Let an element of a body be acted upon by shear stresses on its two perpendicular faces as shown in Fig. 2.4. Let dx, dy and ds be the lengths of the sides AB, BC and AC respectively.

Considering unit thickness of the body and resolving the forces in the direction of σ_θ,

$$\sigma_\theta \cdot ds - \tau \cdot dx \cdot \cos\theta - \tau \cdot dy \cdot \sin\theta = 0$$

or
$$\sigma_\theta = \frac{\tau\, dx\cos\theta}{ds} + \frac{\tau\, dy\sin\theta}{ds} = \frac{\tau\, dx\cos\theta}{dx/\sin\theta} + \frac{\tau\, dy\sin\theta}{dy/\cos\theta}$$

$$= \tau\sin\theta\cos\theta + \tau\sin\theta\cos\theta = \tau\cdot\sin 2\theta \tag{2.15}$$

Resolving the forces in the direction of τ_θ,

$$\tau_\theta \cdot \mathrm{ds} - \tau \cdot \mathrm{dy} \cdot \cos\theta + \tau \cdot \mathrm{dx} \cdot \sin\theta = 0$$

or
$$\tau_\theta = \frac{\tau\, dy\cos\theta}{ds} - \frac{\tau\, dx\sin\theta}{ds} = \frac{\tau\, dy\cos\theta}{dy/\cos\theta} - \frac{\tau\, dx\sin}{dx/\sin\theta}$$

$$= \tau\cos^2\theta - \tau\sin^2\theta = \tau\left[\left(\frac{1+\cos 2\theta}{2}\right) - \left(\frac{1-\cos 2\theta}{2}\right)\right] = \tau\cos 2\theta \tag{2.16}$$

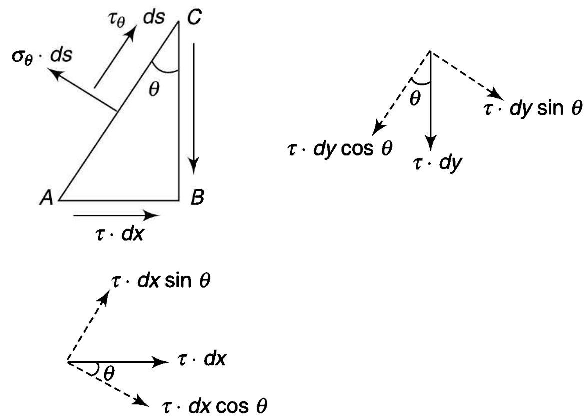


Fig. 2.4

which shows that it is up the plane for $\theta < 45°$ and down the plane for $\theta > 45°$.

The resultant stress on the plane AC, $\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \tau\sqrt{(\sin 2\theta)^2 + (\cos 2\theta)^2}$

$$= \tau \tag{2.17}$$

Inclination with the direction of shear stress planes, $\tan\varphi = \dfrac{\sin 2\theta}{\cos 2\theta} = \tan 2\theta$

or
$$\varphi = 2\theta \tag{2.18}$$

The above equations show that

- The normal stress is positive (tensile) when θ is between 0° and 90° and negative (compressive) between 90° and 180°. Maximum values being at 45° ($= \tau$) and 135° ($= -\tau$)
- The shear stress is positive (clockwise) for $\theta < 45°$ and negative (counter-clockwise) for $\theta > 45°$ and $< 135°$ and again positive between $\theta > 135°$ and $< 180°$.
- The shear stress is zero at 45° and 135° where the normal stress is maximum.

These conclusions indicate that when a body is acted upon by pure shear stresses on two perpendicular planes, the planes inclined at 45° are subjected to a tensile stress of magnitude equal to that of the shear stress while the planes inclined at 135° are subjected to a compressive stress of the same magnitude with no shear stress on these planes.

Compare this result with Eq. 2.12.

(iv) Bi-axial and Shear Stresses Condition

Let an element of a body be acted upon by two tensile stresses along with shear stresses acting on two perpendicular planes of the body as shown in Fig. 2.5. Let dx, dy and ds be the lengths of the sides AB, BC and AC respectively.

Considering unit thickness of the body and resolving the forces in the direction of σ_θ,

$$\sigma_\theta \cdot ds = \sigma_x \cdot dy \cdot \cos\theta + \sigma_y \cdot dx \cdot \sin\theta + \tau \cdot dy \cdot \sin\theta + \tau \cdot dx \cdot \cos\theta$$

or
$$\sigma_\theta = \frac{\sigma_x\, dy \cos\theta}{ds} + \frac{\sigma_y\, dx \sin\theta}{ds} + \frac{\tau \cdot dy \sin\theta}{ds} + \frac{\tau \cdot dx \cos\theta}{ds}$$

$$= \frac{\sigma_x\, dy \cos\theta}{dy/\cos\theta} + \frac{\sigma_y\, dx \sin\theta}{dx/\sin\theta} + \frac{\tau\, dy \sin\theta}{dy/\cos\theta} + \frac{\tau\, dx \cos\theta}{dx/\sin\theta}$$

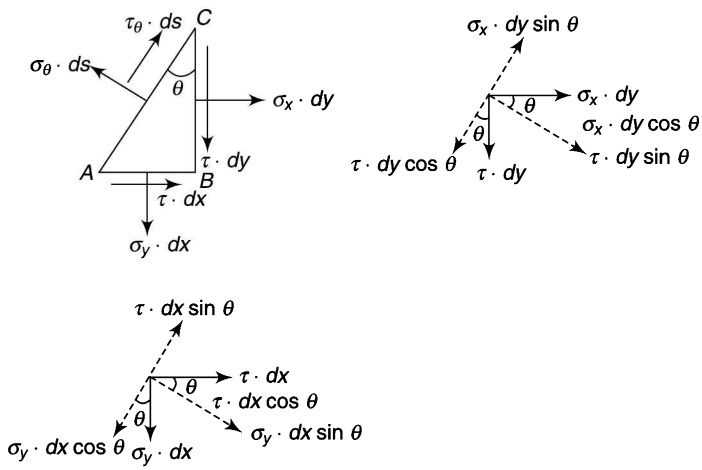


Fig. 2.5

$$= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau \sin\theta\cos\theta + \tau\sin\theta\cos\theta$$

$$= \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau \sin 2\theta \tag{2.19}$$

$$= \sigma_x\left(\frac{1+\cos 2\theta}{2}\right) + \sigma_y\left(\frac{1-\cos 2\theta}{2}\right) + \tau\cdot\sin 2\theta$$

$$= \frac{1}{2}(\sigma_x+\sigma_y) + \frac{1}{2}(\sigma_x-\sigma_y)\cos 2\theta + \tau\cdot\sin 2\theta \tag{2.20}$$

Resolving the forces in the direction of τ,

$$\tau\theta\cdot ds + \sigma_x\cdot dy\cdot\sin\theta - \sigma_y\cdot dx\cdot\cos\theta - \tau\cdot dy\cdot\cos\theta + \tau\cdot dx\cdot\sin\theta = 0$$

or
$$\tau_\theta = -\frac{\sigma_x\, dy\sin\theta}{ds} + \frac{\sigma_y\, dx\cos\theta}{ds} + \frac{\tau\cdot dy\cos\theta}{ds} - \frac{\tau\cdot dx\sin\theta}{ds} -$$

$$= -\frac{\sigma_x\, dy \sin}{dy/\cos\theta} + \frac{\sigma_y\, dx\cos\theta}{dx/\sin\theta} + \frac{\tau\, dy\cos\theta}{dy/\cos\theta} - \frac{\tau\, dx\sin}{dx/\sin\theta}$$

$$= -\sigma_x\sin\theta\cos\theta + \sigma_y\sin\theta\cos\theta + \tau\cos^2\theta - \tau\sin^2\theta$$

$$= -\frac{1}{2}(\sigma_x-\sigma_y)\sin 2\theta + \tau\left[\left(\frac{1+\cos 2\theta}{2}\right) - \left(\frac{1-\cos 2\theta}{2}\right)\right]$$

$$= -\frac{1}{2}(\sigma_x-\sigma_y)\sin 2\theta + \tau\cos 2\theta \tag{2.21}$$

Equations 2.19, 2.20 and 2.21 can be used to determine the stresses on any inclined plane in a material under a general state of stress.

To determine the planes having maximum and minimum values of direct stress, differentiate Eq. 2.20 with respect to θ and equate to zero, i.e.,

$$\frac{d\sigma_\theta}{d\theta} = 0 - \frac{1}{2}(\sigma_x-\sigma_y)\,2\sin 2\theta + 2\tau\cdot\cos 2\theta = 0$$

or
$$\frac{1}{2}(\sigma_x-\sigma_y)2\sin 2\theta = 2\tau\cdot\cos 2\theta$$

or $$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} \tag{2.22}$$

This equation provides two values of 2θ differing by 180° or θ by 90°, the planes along which the direct stresses have the maximum and minimum values.

Note that the same values of θ are also obtained by equating τ to zero, which indicates that shear stress is zero or does not exist on these planes. Thus, it is concluded that shear stresses are zero on the planes with maximum or minimum values of direct stress (they are known as *principal planes* and the corresponding stresses as *principal stresses*, to be discussed in the next section).

If σ_x and σ_y are not alike, i.e., if one of them is compressive (say σ_y is compressive), corresponding expressions can be obtained by replacing σ_y with $-\sigma_y$.

Note the following in general:

- As the material as a whole is in equilibrium under the action of external forces and internal resistances, an element of any shape at any point in a material will also be in equilibrium under the internal or external forces.
- An element of any shape may be considered for force analysis. Usually, the choice is made depending upon the requirements. For example, if the stresses on longitudinal and transverse axes are required, a rectangular element is a suitable choice whereas if the stresses on some inclined plane are to be found, then a triangular element has to be preferred.
- Relations derived above for various cases are valid when
  - — inclination is measured in the clockwise direction with the vertical plane
  - — compressive stresses are taken negative, and
  - — the direction of the shear stresses is clockwise on the vertical planes and counter-clockwise on the horizontal planes.

In case, these parameters are chosen differently, relations have to be modified as below:

- If angle is measured counter-clockwise with the vertical planes (Fig. 2.2b), θ is replaced by $-\theta$ or $(180° - \theta)$.
- If angle is taken counter-clockwise with the horizontal plane, find out θ with the vertical and use the relations as usual.
- If the direction of the shear stresses is counter-clockwise on the vertical planes and clockwise on the horizontal planes replace τ with $-\tau$.

Thus, to use the derived relations directly, it must be ensured that the parameters are taken in the proper way.

2.2 SUM OF DIRECT STRESSES ON TWO MUTUALLY PERPENDICULAR PLANES

Direct stress on an inclined plane at angle θ is given by,

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau \cdot \sin 2\theta \qquad \text{(Eq. 2.20)} \qquad \text{(i)}$$

Direct stress on an inclined plane at angle $(\theta+90°)$ will be,

$$\sigma_{(\theta+90°)} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2(\theta + 90°) + \tau \cdot \sin 2(\theta + 90°)$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta - \tau \cdot \sin 2\theta \qquad \text{(ii)}$$

Adding (i) and (ii),

$$\sigma_\theta + \sigma_{(\theta+90^\circ)} = \sigma_x + \sigma_y$$

Since the sum of direct stresses σ_x and σ_y is constant, the sum of direct stresses on two mutually perpendicular planes at a point at any angle θ and $(\theta + 90°)$ remains constant and equal to $\sigma_x + \sigma_y$.

2.3 PRINCIPAL STRESSES

In general, a body may be acted upon by direct stresses and shear stresses. However, it will be seen that even in such complex systems of loading, there exist three mutually perpendicular planes, on each of which the resultant stress is wholly normal. These are known as *principal planes* and the normal stress across these planes, as *principal stresses*. The larger of the two stresses σ_1 is called the *major principal stress,* and the smaller one σ_2, as the *minor principal stress*. The corresponding planes are known as *major* and *minor principal planes*. In two-dimensional problems, the third principal stress is taken to be zero.

As shear stress is zero in principal planes,

$$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau \cos 2\theta = 0 \quad \text{(Eq. 2.21)}$$

or

$$\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta = \tau \cos 2\theta$$

or

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} \quad (2.23)$$

which provides two values of 2θ differing by 180° or two values of θ differing by 90°. Thus, the two principal planes are perpendicular to each other (Also, refer Eq. 2.22).

From Fig. 2.6,

$$\sin 2\theta = \pm \frac{2\tau}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$\cos 2\theta = \pm \frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

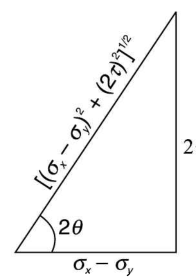


Fig. 2.6

Right-hand sides of both the above equations should have the same signs, positive or negative while using them. Substituting these values of $\sin 2\theta$ and $\cos 2\theta$ in Eq. 2.20, two values of the direct stresses, i.e., of principal stresses corresponding to two values of 2θ are obtained.

$$\sigma_{1,2} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau \cdot \sin 2\theta$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\frac{(\sigma_x - \sigma_y)^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}} \pm \tau \cdot \frac{2\tau^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\frac{(\sigma_x - \sigma_y)^2 + 4\tau^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2} \quad (2.24)$$

- Unless it is possible to know by inspection which of the two principal planes corresponds to major principal stress, it is necessary to substitute one of the values of θ (inclination of principal planes) into the equation,

$$\sigma = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau \sin 2\theta$$

to find which of the two values of angles corresponds to major principal stress.

2.4 MAXIMUM (PRINCIPAL) SHEAR STRESSES

In any complex system of loading, the maximum and the minimum normal stresses are the principal stresses and the shear stress is zero in their planes. To find the maximum value of shear stress and its plane in such a system, consider the equation for shear stress in a plane, i.e.,

$$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau \cos 2\theta \qquad \text{(Eq. 2.21)}$$

For maximum value of τ_θ, differentiate it with respect to θ and equate to zero,

$$\frac{d\tau_\theta}{d\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau \sin 2\theta = 0$$

or

$$\tan 2\theta = -\frac{\sigma_x - \sigma_y}{2\tau} \qquad (2.25)$$

This indicates that there are two values of 2θ differing by 180° or two values θ differing by 90°. Thus, maximum shear stress planes lie at right angle to each other.

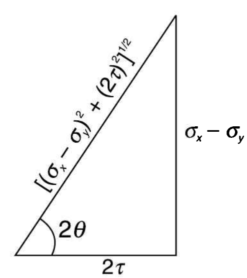


Fig. 2.7

Now, as $\tan 2\theta = -\dfrac{(\sigma_x - \sigma_y)}{2\tau}$ can be represented as shown in Fig. 2.7,

$$\sin 2\theta = \mp \frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}; \qquad \cos 2\theta = \pm \frac{2\tau}{\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2}}$$

Right-hand sides of both the above equations should have the opposite signs, if one positive the other negative while using them. Substituting these values of $\sin 2\theta$ and $\cos 2\theta$ in Eq. 2.21, two values of the shear stress are obtained.

$$\therefore \qquad \tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau \cos 2\theta$$

$$= -\left(\mp\frac{1}{2}(\sigma_x - \sigma_y)\frac{\sigma_x - \sigma_y}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}\right) \pm \tau \cdot \frac{2\tau}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$= \pm\frac{1}{2}\frac{(\sigma_x - \sigma_y)^2 + 4\tau^2}{\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}}$$

$$= \pm\frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

This provides maximum and minimum values of shear stress, both numerically equal. In fact, the negative or minimum value indicates that it is at right angles to the positive value as discussed above and the two are the complimentary shear stresses. Thus, magnitude of the maximum or principal shear stress is given by

$$\tau_{max} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

As maximum principal stress, $\sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$ (i)

and minimum principal stress, $\sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$ (ii)

Subtracting (ii) from (i), $\sigma_1 - \sigma_2 = \sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$

$\therefore\ \tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2)$

Thus, in general, $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$ (2.26)

Now, principal planes are given by, $\tan 2\theta_p = \dfrac{2\tau}{\sigma_x - \sigma_y}$

and planes of maximum shear stress, $\tan 2\theta_s = -\dfrac{\sigma_x - \sigma_y}{2\tau}$

Multiplying the two, $\tan 2\,\theta_p \cdot \tan 2\,\theta_s = -1$ which means $2\theta_s = 2\theta_p + 90°$ or $\theta_s = \theta_p + 45°$

This indicates that the planes of maximum shear stress lie at 45° to the planes of principal axes.

- The above conclusion can also be drawn from the fact that the case of biaxial stresses on a rectangular element discussed in the previous section is a case of principal stresses, as no shear stress acts on the horizontal or vertical planes. Thus, σ_x and σ_y also denote principal stresses in the element and as in case of biaxial stresses, the maximum value of shear stress lies in the planes at 45° to the principal planes and is

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

- Unless it is possible to know by inspection which of the two planes corresponds to a particular direction of principal shear stress, it is necessary to substitute one of the values of θ into the equation, $\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\ \sin 2\theta\ +\ \tau\ \cos 2\theta$ to find which of the two values of angles corresponds to a particular direction.

2.5 NORMAL STRESS ON THE PLANES OF MAXIMUM SHEAR STRESS

Direct stress on an inclined plane is given by,

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau \cdot \sin 2\theta \qquad \text{(Eq. 2.20)}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + \cos 2\theta\left[\frac{1}{2}(\sigma_x - \sigma_y) + \tau \cdot \tan 2\theta\right]$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + \cos 2\theta\left[\frac{1}{2}(\sigma_x - \sigma_y) - \tau \cdot \frac{\sigma_x - \sigma_y}{2\tau}\right] \qquad \text{(Using Eq. 2.25)}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) \qquad (2.27)$$

Thus normal stresses of magnitude $(\sigma_x + \sigma_y)/2$ act on each of the two planes of maximum shear stresses.

2.6 MOHR'S STRESS CIRCLE

The stress components on any inclined plane can easily be found with the help of a geometrical construction known as *Mohr's stress circle*.

Two Perpendicular Direct Stresses

Let the material of a body at a point be subjected to two like direct tensile stresses σ_x and σ_y ($\sigma_x > \sigma_y$), on two perpendicular planes *AD* and *AB* respectively (Fig. 2.8).

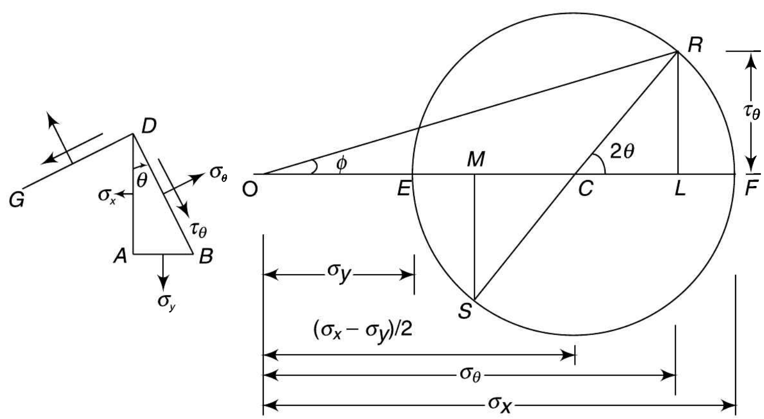


Fig. 2.8

Make the following constructions:

- On x-axis, take $OF = \sigma_x$ and $OE = \sigma_y$ to some scale. A stress is taken towards the right of the origin O (positive) if tensile and toward left (negative) if compressive.
- Bisect *EF* at *C*.
- With *C* as centre and *CE* (= *CF*) as radius, draw a circle.

  The radius *CF* represents the plane *AD* (of direct stress σ_x) and *CE*, the plane *AB* (of direct stress σ_y). Note that the two planes *AD* and *AB* which are at 90° are represented at 180° apart (or at double the angle) in the Mohr's circle. This indicates that any angular position of a plane can be located at double the angle from a particular plane.
- Locate an inclined plane in this circle by marking a radial line at double the angle at which the required plane is inclined with a given plane, e.g., if the plane *BD* is inclined at angle θ with plane *AD* in the counter-clockwise direction, then mark radius *CR* at an angle 2θ with *CF* in the counter-clockwise direction.
- Draw $LR \perp x$-axis. Join *OR*.

Now, it can be shown that *OL* and *LR* represent the normal and the shear stress components on the inclined plane *BD*.

From the geometry of the figure,

$$OC = \frac{1}{2}(OC + OC) = \frac{1}{2}(OF - CF) + (OE + CE)$$

$$= \frac{1}{2}(OF - CF) + (OE + CF) \qquad \text{......}(CE = CF)$$

$$= \frac{1}{2}(OF + OE) = \frac{1}{2}(\sigma_x + \sigma_y)$$

$$CL = CR \cos 2\theta = CF \cos 2\theta = \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta \qquad (CR = CF)$$

Thus $OL = OC + CL = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta = \sigma_\theta$ ...(Refer Eq.2.23)

And $LR = CR \sin 2\theta = CF \sin 2\theta = \frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta = \tau_\theta$ ...(Refer Eq.2.24)

A positive expression of τ_θ indicates it is clockwise for counter-clockwise angle θ.

The resultant of *OL* and *LR* is represented by *OR* at an angle φ with the *OL,* i.e., with the direction of σ_θ. Thus the components *OL* and *RL* represent the normal and shear stress components on the plane *BD.*

Note the following:

- Direct stress component on the inclined plane *BD* represented by *OR* is on the right side of the origin, it is positive or tensile.
- Shear stresses giving a clockwise rotation are assumed positive and are above the *x*-axis. In the present case, the shear component *LR* represents a clockwise direction.
- The stress components on a plane *DG* perpendicular to *BD* are obtained by rotating the radial line *CR* through double the angle, i.e., 180° in clockwise or counter-clockwise direction. Thus, *CS* represents the plane *DG. OM* indicates the tensile component and the *SM* the shear component.

Two Perpendicular Direct Stresses with Simple Shear

In the above-discussed case, *CR* and *CS* represent two perpendicular planes having direct tensile stresses *OL* and *OM* and shear stresses *LR* (clockwise) and *MS* (= *LR*, counter-clockwise) respectively. Now, if these happen to be the known stresses on two perpendicular planes, then stresses on any other inclined plane can easily be found by locating that plane relative to any of these planes.

Let *CR* and *CS* represent two perpendicular planes *BD* and *AB* respectively so that $OL = \sigma_x$, $OM = \sigma_y$ and *LR* and *MS* each equal to τ in the clockwise and counter-clockwise directions respectively (Fig. 2.9). Now if it is desired to find stresses on an inclined plane at angle θ clockwise with plane *BD*, a radial line *CP* may be drawn at angle 2θ in the clockwise direction with *CR.* Then *ON* and *NP* will represent the direct and shear components respectively on the plane *AD* and the resultant is given by *OP.*

Thus the procedure may be summarised as follows:

- Take *OL* and *OM* as the direct components of the two perpendicular stresses σ_x and σ_y.
- At *L* and *M* draw $\perp$s *LR* and *MS* on the *x*-axis each equal to τ using the same scale as for the direct stresses. For the stress system shown in Fig. 2.8, *LR* is taken upwards as the direction on plane *BD* is clockwise and *MS* downwards as the direction on plane *AB* is counter-clockwise.
- Bisect *LM* at *C* and draw a circle with *C* as centre and radius equal to *CR* (= *CS*).

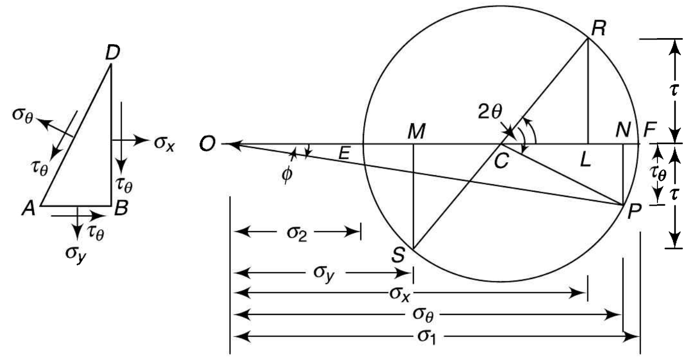


Fig. 2.9

- Rotate the radial line CR through angle 2θ in the clockwise direction if θ is taken clockwise and let it take the position CP.
- Draw $NP \perp$ on x-axis. Join OP.

It can be proved that ON and NP represent the normal and the shear stress components on the inclined plane AD.

From the geometry of the figure,

$$OC = \frac{1}{2}(\sigma_x + \sigma_y) \qquad \text{as before.}$$

$$\begin{aligned} CN &= CP \cos (2\theta - \beta) \\ &= CR \cos (2\theta - \beta) && \ldots(CP = CR) \\ &= CR (\cos 2\theta \cos \beta + \sin 2\theta \sin \beta) \\ &= (CR \cos \beta) \cos 2\theta + (CR \sin \beta) \sin 2\theta \\ &= CL \cos 2\theta + LR \sin 2\theta \\ &= \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta + \tau \cdot \sin 2\theta && \ldots(CL = OL - OM) \end{aligned}$$

Thus $ON = OC + CN = \frac{1}{2}(\sigma_y + \sigma_x) + \frac{1}{2}(\sigma_x - \sigma_y) \cos 2\theta + \tau \cdot \sin 2\theta = \sigma_\theta$ ...(Eq.2.20)

and

$$\begin{aligned} NP &= CP \sin (2\theta - \beta) \\ &= CR \sin (2\theta - \beta) \\ &= CR (\sin 2\theta \cos \beta - \cos 2\theta \sin \beta) \\ &= (CR \cos \beta) \sin 2\theta - (CR \sin \beta) \cos 2\theta \\ &= CL \sin 2\theta - LR \cos 2\theta \\ &= \frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta - \tau \cos 2\theta = -\tau_\theta && \ldots(\text{Eq.2.21}) \end{aligned}$$

As NP is below the x-axis, therefore, the shear stress is negative or counter-clockwise.

Mathematically, $NP = -\left[\frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta - \tau \cos 2\theta\right] = -\frac{1}{2}(\sigma_x - \sigma_y) \sin 2\theta + \tau \cos 2\theta]$

Principal Stresses

As shear stress is zero on the principal planes, OF represents the major principal plane with maximum normal stress. In a similar way, OE represents the minor principal plane.

$$OF = OC + CF = OC + CR = OC + \sqrt{CL^2 + LR^2}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + \sqrt{\left\{\frac{1}{2}(\sigma_x - \sigma_y)\right\}^2 + \tau^2}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

= Major principal stress

$$OE = OC - CE = OC - CR = OC - \sqrt{CL^2 + LR^2}$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

= Minor principal stress

The angles of inclination of planes of major and minor principal stresses are $\beta/2$ and $(90° + \beta/2)$ respectively clockwise with the plane of stress σ_x.

Example 2.1 Two pieces of wood of section 50 mm × 30 mm are joined together along a plane at 60° with the x-axis. If the required strength of the joint is to be 7.5 MPa in tension and 4 MPa in shear, determine the maximum force which the member can sustain.

Solution

Given Two pieces of wood joined together at an inclined plane as shown in Fig. 2.10.

$\sigma_\theta = 7.5$ MPa $\qquad \tau_\theta = 4$ MPa

To find Maximum force P along x-axis

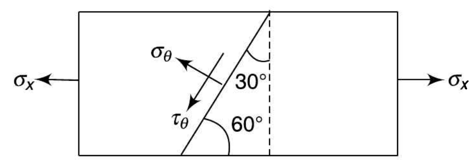


Fig. 2.10

Let the maximum stress along the x-axis be σ_x.
Angle with the plane of $\sigma_x = 90° - 60° = 30°$

Strength of joint in tension

$$\sigma_\theta = \sigma_x \cos^2\theta = \sigma_x \cos^2 30° = \frac{3}{4}\sigma_x$$

or $$7.5 = \frac{3}{4}\cdot\sigma_x \quad \text{or} \quad \sigma_x = 10 \text{ MPa} \qquad \text{(i)}$$

Strength of joint in shear

$$\tau_\theta = -\frac{1}{2}\sigma_x \sin 2\theta = -\frac{1}{2}\sigma_x \sin 60° = -\frac{\sqrt{3}}{4}\sigma_x.$$

or $$-4 = -\frac{\sqrt{3}}{4}\sigma_x. \quad \text{or} \quad \sigma_x = 9.24 \text{ MPa} \qquad \text{(ii)}$$

(shear stress at the joint is assumed counter-clockwise to have positive axial tensile stress σ_x)

From (i) and (ii), for safety of the joint, the maximum axial stress to be taken by the member is 9.24 MPa.

Maximum force, $P = (50 \times 30) \times 9.24 = 13\,860$ N or 13.86 kN

Example 2.2 A rectangular block is subjected to two perpendicular stresses of 10 MPa tension and 10 MPa compression. Determine the stresses on planes inclined at (i) 30°, (ii) 45°, and (iii) 60° with the plane of compressive stress.

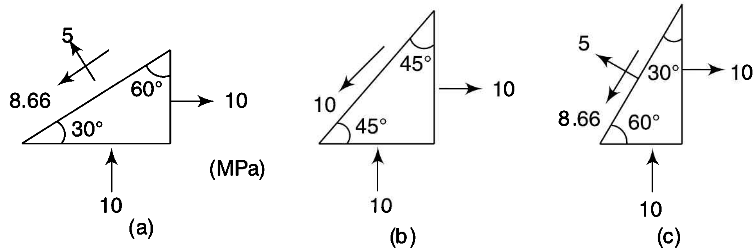


Fig. 2.11

Solution

Given

A biaxial stress system,

$$\sigma_x = 10 \text{ MPa} \qquad \sigma_y = -10 \text{ MPa}$$

To find Stresses on planes inclined at (i) 30°, (ii) 45°, and (iii) 60° with the plane of compressive stress

***Plane at* 30° *with the plane of* σ_y**

Inclination with the vertical plane or plane of σ_x = 90° – 30° = 60° (Fig. 2.11a)

$$\sigma_{60} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta \qquad \text{(Eq. 2.5)}$$

$$= 10\,(\cos^2 60° - \sin^2 60°) = 10\left(\frac{1}{4} - \frac{3}{4}\right) = -5 \text{ MPa (compression)}$$

or using the expression,

$$\sigma_{60} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta \qquad \text{(Eq. 2.6)}$$

$$= \frac{1}{2}(10 + (-10)) + \frac{1}{2}[10 - (-10)]\cos 120° = -5 \text{ MPa (compression)}$$

$$\tau_{60} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta$$

$$= -\frac{1}{2}[10 - (-10)]\sin 120° = -8.66 \text{ MPa (ccw)}$$

Solution by Mohr's circle

Solution by Mohr's circle is shown in Fig. 2.12. Adopt the following procedure:

- Take *OF* equal to 10 MPa to some suitable scale to the right of *O* for tensile stress. Similarly, take *OE* to the left for compressive stress.
- The centre of *EF* is the centre of the Mohr's circle. Draw the circle passing through *E* and *F* points. Now *OF* and *OE* represent the planes of tensile and compressive stresses respectively.
- Make angle *FCR* =120° i.e double the angle of the inclined plane with *OF* in the clockwise direction.

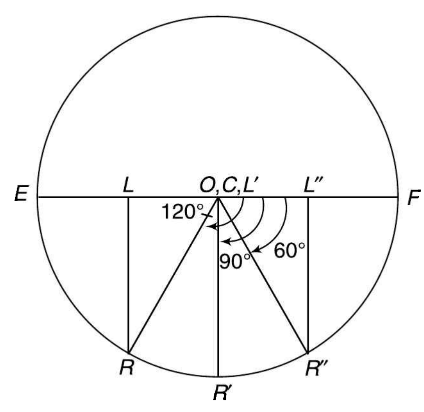


Fig. 2.12

Then CR represents the inclined plane.

$\sigma_{60} = OL = 5$ MPa (compressive being on left of point O)

$\tau_{60} = LR = 8.66$ MPa (ccw being below the x-axis)

Plane at 45° with the plane of σ_y

(ii) Inclination with the vertical plane = 90° – 45° = 45° (Fig. 2.11b)

$$\sigma_{45} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta$$

$$= 10(\cos^2 45° - \sin^2 45°) = 10\left(\frac{1}{2} - \frac{1}{2}\right) = 0$$

and

$$\tau_{45} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta$$

$$= -\frac{1}{2}[10 - (-10)]\sin 90° = -10 \text{ MPa (ccw)}$$

Solution by Mohr's circle is self-explanatory (Fig. 2.12).

Plane at 60° with the plane of σ_y

Inclination with the vertical plane = 90° – 60° = 30° (Fig. 2.11c)

$$\sigma_{60} = 10(\cos^2 30° - \sin^2 30°) = 10\left(\frac{3}{4} - \frac{1}{4}\right) = 5 \text{ MPa}$$

and

$$\tau_{60} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta$$

$$= -\frac{1}{2}[10 - (-10)]\sin 60° \text{ (ccw)}$$

Solution by Mohr's circle is self-explanatory (Fig. 2.12).

Example 2.3 A piece of material is subjected to two perpendicular stresses as follows:

(a) tensile stresses of 100 MPa and 60 Mpa
(b) tensile stress of 100 MPa and compressive stress of 60 Mpa
(c) compressive stress of 100 MPa and tensile stress of 60 Mpa
(d) compressive stresses of 100 MPa and 60 MPa

Determine normal and tangential stresses on a plane inclined at 30° to the plane of 100 MPa stress. Also find the resultant and its inclination with the normal stress.

Solution

Given Four biaxial stress systems,

(a) $\sigma_x = 100$ MPa $\sigma_y = 60$ Mpa
(b) $\sigma_x = 100$ MPa $\sigma_y = -60$ Mpa
(c) $\sigma_x = -100$ MPa $\sigma_y = 60$ Mpa
(d) $\sigma_x = -100$ MPa $\sigma_y = -60$ MPa

To find Stresses on planes inclined at 30° with the plane of σ_x.

(a) Biaxial stress system, both stresses tensile

Inclination with the vertical plane = 30°

$$\sigma_{30} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta$$

$$= 100\cos^2 30° + 60\sin^2 30°$$

$$= 100 \times \frac{3}{4} + 60 \times \frac{1}{4} = 90 \text{ MPa (tensile)}$$

$$\tau_{30} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta$$

$$= -\frac{1}{2}(100 - 60)\sin 60° = -17.32 \text{ MPa (ccw)}$$

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sqrt{90^2 + (-17.32)^2} = 91.65 \text{ MPa}$$

Its inclination with normal stress σ_{30}

$$\tan\varphi = \frac{\tau_{30}}{\sigma_{30}} = \frac{17.32}{90} = 0.1924 \qquad \text{or} \qquad \varphi = 10.9°$$

The inclination of the resultant with σ_x can also be found,

$$\tan\alpha = \frac{\sigma_y}{\sigma_x}\tan 30° = \frac{60}{100}\tan 30° = 0.346 \qquad \text{or} \qquad \alpha = 19.1°$$

Thus $\alpha + \varphi = 19.1° + 10.9° = 30° = \theta$

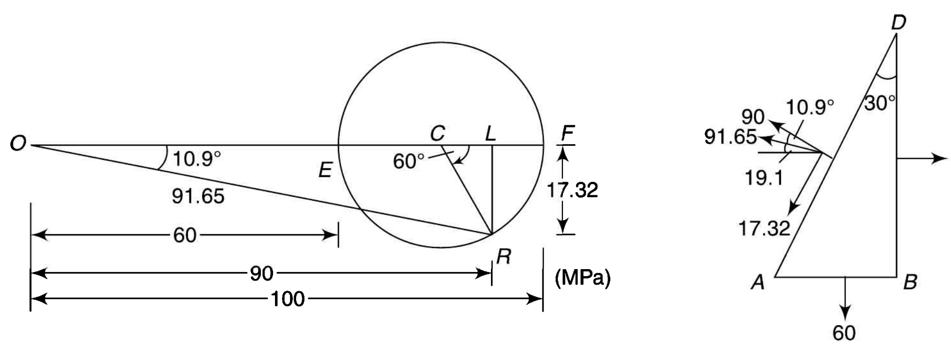


Fig. 2.13

Solution by Mohr's circle

Solution by Mohr's circle is shown in Fig. 2.13 (a). The procedure is as follows:

- Take *OF* and *OE* equal to 100 MPa and 60 MPa respectively to some suitable scale to the right of *O* for tensile stresses.
- The centre of *EF* is the centre of the Mohr' circle. Draw the circle passing through *E* and *F* points. Now *OF* and *OE* represents the planes of tensile stresses 100 MPa and 60 MPa respectively.
- Make angle *FCR* = 60°, i.e., double the angle of the inclined plane with *OF* in the clockwise direction.

Then *CR* represents the inclined plane.

$$\sigma_{30} = OL = 90 \text{ MPa (tensile)}$$
$$\tau_{30} = LR = 17.32 \text{ MPa (counter-clockwise)}$$
$$\sigma_r = OR = 91.65 \text{ MPa}$$

Inclination of the resultant with *OL* or σ_{30}, $\quad \varphi = 10.9°$

The results are shown in Fig. 2.13 (b).

(b) Biaxial stress system, tensile along x-axis and compressive along y-axis

$$\sigma_{30} = 100\cos^2 30° - 60\sin^2 30° = 100 \times \frac{3}{4} - 60 \times \frac{1}{4} = 60 \text{ MPa (tensile)}$$

$$\tau_{30} = -\frac{1}{2}[100-(-60)]\sin 60° = -80\times 0.866 = -69.28 \text{ MPa (ccw)}$$

$$\sigma_r = \sqrt{\sigma_\theta^2+\tau_\theta^2} = \sqrt{60^2+(-69.28)^2} = 91.65 \text{ MPa}$$

inclination with σ_{30}, $\tan\varphi = \dfrac{\tau_{30}}{\sigma_{30}} = \dfrac{69.28}{60} = 1.155$

or $\varphi = 49.1°$

α can be found to be $-19.1°$

Solution by Mohr's circle is shown in Fig. 2.14a which is self-explanatory.

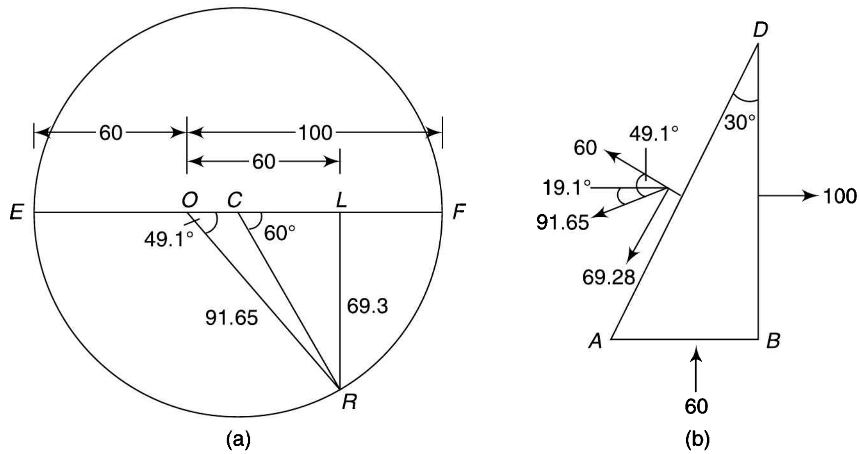


Fig. 2.14

$\sigma_{30} = OL = 60$ MPa (tensile)

$\tau_{30} = LR = 69.3$ MPa (counter-clockwise)

$\sigma_r = OR = 91.65$ MPa

Inclination of the resultant with OL or σ_{30}, $\varphi = 49.1°$

The results are shown in Fig. 2.14b.

(c) Biaxial stress system, compressive along x-axis and tensile along y-axis

$$\sigma_{30} = -100\cos^2 30° + 60\sin^2 30° = -100\times\frac{3}{4} + 60\times\frac{1}{4} = -60 \text{ MPa (comp.)}$$

$$\tau_{30} = -\frac{1}{2}(-100-60)\sin 60° = 80\times 0.866 = 69.28 \text{ MPa (cw)}$$

$$\sigma_r = \sqrt{\sigma_\theta^2+\tau_\theta^2} = \sqrt{(-60)^2+(69.28)^2} = 91.65 \text{ MPa}$$

Inclination with σ_{30}, $\tan\varphi = \dfrac{\tau_{30}}{\sigma_{30}} = \dfrac{69.28}{60} = 1.155$

or $\varphi = 49.1°$

α can be found to be $-19.1°$. The results are shown in Fig. 2.15a.

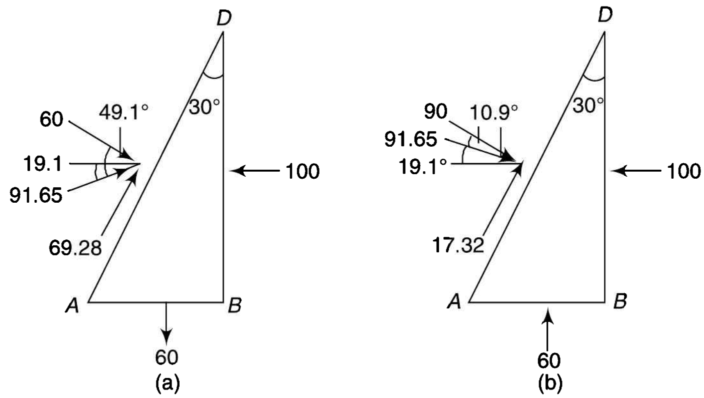


Fig. 2.15

(d) Biaxial stress system, both stresses compressive

$$\sigma_{30} = -100\cos^2 30° - 60\sin^2 30° = -100 \times \frac{3}{4} - 60 \times \frac{1}{4} = -90 \text{ MPa (comp.)}$$

$$\tau_{30} = -\frac{1}{2}[-100 - (-60)]\sin 60° = 20 \times 0.866 = 17.32 \text{ MPa (cw)}$$

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sqrt{(-90)^2 + (17.32)^2} = 91.65 \text{ MPa}$$

Inclination with σ_{30},
$$\tan\varphi = \frac{\tau_{30}}{\sigma_{30}} = \frac{17.32}{90} = 0.1924$$

or
$$\varphi = 10.9°$$

α can be found to be –19.1°. The results are shown in Fig. 2.15b.

Example 2.4 A piece of material is subjected to two perpendicular tensile stresses of 100 MPa and 60 MPa. Determine the plane on which the resultant stress has maximum obliquity with the normal. Also, find the resultant stress on this plane.

Solution

Given A biaxial stress system,

$$\sigma_x = 100 \text{ MPa} \qquad \sigma_y = 60 \text{ MPa}$$

To find

— Plane on which resultant stress has maximum obliquity with normal
— Resultant stress on that plane

Angle for maximum obliquity

For maximum obliquity of the resultant with the normal to a plane is given by

$$\tan\theta = \sqrt{\frac{\sigma_x}{\sigma_y}} = \sqrt{\frac{100}{60}} = 1.29 \quad \text{or} \quad \theta = 52.24° \qquad \text{...(Eq. 2.10)}$$

Determination of stresses

Direct stress,

$$\sigma_{52.24} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta$$
$$= 100\cos^2 52.24° + 60\sin^2 52.24°$$

$$= 100 \times 0.375 + 60 \times 0.735$$
$$= 37.5 + 37.5$$
$$= 75 \text{ MPa}$$

Shear stress,

$$\tau_{52.24} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta$$
$$= -\frac{1}{2}(100 - 60)\sin 104.48 = -19.365 \text{ MPa (ccw)}$$

Resultant stress,

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sqrt{75^2 + 19.365^2} = 77.46 \text{ MPa}$$

$$\tan\varphi = \frac{19.365}{75} = 0.2582 \quad \text{or} \quad \varphi = 14.48°$$

Solution by direct relations

The resultant and its inclination can also be found directly by using relations

$$\sigma_r = \sqrt{\sigma_x^2 \cos^2\theta + \sigma_x^2 \sin^2\theta}$$
$$= \sqrt{100^2\cos^2 52.24 + 60^2 \sin^2 52.24°}$$
$$= \sqrt{3749.8 + 2250}$$
$$= 77.46 \text{ MPa}$$

Inclination with σ_x,

$$\tan\alpha = \frac{60}{100}\tan 52.24°$$

or $$\alpha = 37.76°$$

(Note that $\alpha + \varphi = 37.76° + 14.48° = 52.24° = \theta$)

The results are shown in Fig. 2.16.

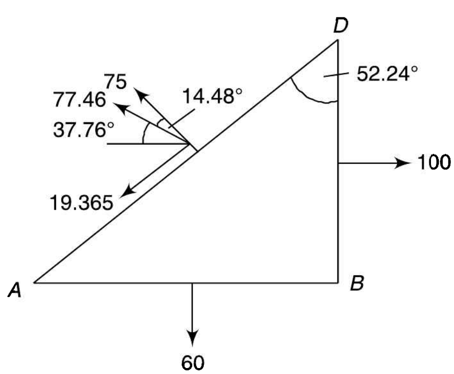


Fig. 2.16

Solution by Mohr's circle

Solution by Mohr's circle is shown in Fig. 2.17. *OR* is tangent to the circle for maximum obliquity.

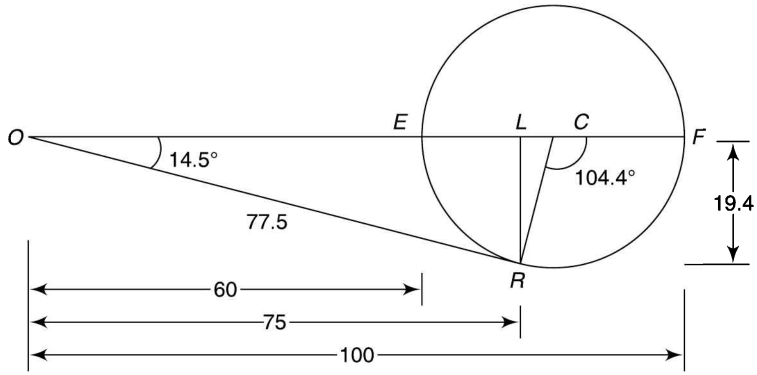


Fig. 2.17

$$\theta = 104.4/2 = 52.2°$$
$$\sigma_{52.2} = OL = 75 \text{ MPa (tensile)}$$
$$\tau_{52.2} = LR = 19.4 \text{ (counter-clockwise)}$$
$$\sigma_r = OR = 77.5 \text{ MPa}$$

Inclination of the resultant with OL or $\sigma_{52.2}$, $\varphi = 14.5°$

Example 2.5 The stresses on two perpendicular planes through a point in a body are 160 MPa and 100 MPa, both compressive along with a shear stress of 80 MPa. Determine

(i) The normal and the shear stresses on a plane inclined at 30° to the plane of 160 MPa stress. Find also the resultant stress and its direction
(ii) The normal stress on a plane at 90° to the inclined plane mentioned in (i)
(iii) Show the results diagrammatically

Solution

Given A biaxial and shear stress system,

$$\sigma_x = -160 \text{ MPa} \qquad \sigma_y = -100 \text{ MPa} \qquad \tau = 80 \text{ MPa}$$

To find

— Normal and shear stresses on a plane at 30° to plane of 160 MPa stress
— Resultant stress and its direction
— Normal stress on a plane at 90° to the above plane
— Diagrammatical presentation of results

Calculations of stresses

$$\sigma_{30} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau \cdot \sin 2\theta$$
$$= \frac{1}{2}(-160 - 100) + \frac{1}{2}(-160 + 100)\cos 60° + 80 \sin 60°$$
$$= -75.7 \text{ MPa}$$
$$\tau_{30} = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau\cos 2\theta = -\frac{1}{2}(-160 + 100)\sin 60° + 80\cos 60°$$
$$= 66 \text{ MPa (clockwise)}$$

$$\sigma_r = \sqrt{\sigma_\theta^2 + \tau_\theta^2} = \sqrt{(-75.7)^2 + (66)^2} = 100.4 \text{ MPa}$$

Inclination with σ_{30}, $\tan \varphi = \dfrac{\tau_{30}}{\sigma_{30}} = \dfrac{66}{75.7} = 0.872$

or $\varphi = 41.1°$

Calculation of normal stress at 120°

$$\sigma_{30°} + \sigma_{(30°+90°)} = \sigma_x + \sigma_y \qquad \text{(Eq. 2. 26a)}$$

$$\therefore \quad -75.7 + \sigma_{(30°+90°)} = -160 - 100$$

$$\sigma(30°+90°) = -184.3 \text{ MPa}$$

Diagrammatical presentation of results

The results are shown diagrammatically in Fig. 2.18.

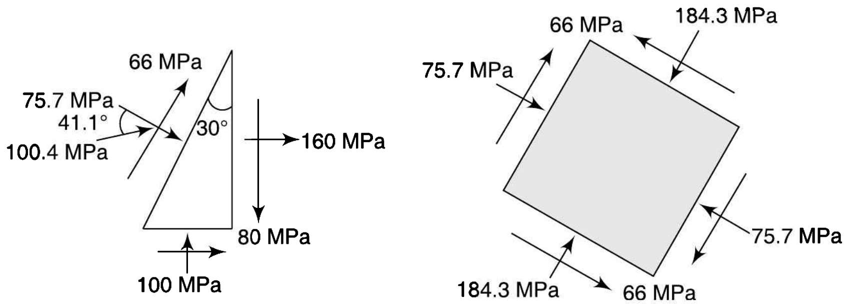


Fig. 2.18

Example 2.6 The stresses on two perpendicular planes through a point in a body are 30 MPa and 15 MPa both tensile along with a shear stress of 25 MPa. Find

(i) the magnitude and direction of principal stresses
(ii) the planes of maximum shear stress
(iii) the normal and shear stresses on the planes of maximum shearing stress

Solution

Given A biaxial and shear stress system,

$$\sigma_x = 30 \text{ MPa} \qquad \sigma_y = 15 \text{ MPa} \qquad \tau = 25 \text{ MPa}$$

To find

— Principal stresses
— Planes of maximum shear stresses
— Normal and shear stresses on the planes of maximum shear stress.

Refer Fig. 2.19.

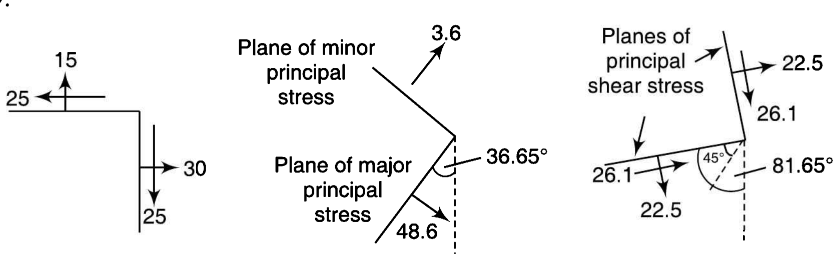


Fig. 2.19

Principal stresses

$$\sigma_1, \sigma_2 = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

$$= \frac{1}{2}(30 + 15) \pm \frac{1}{2}\sqrt{(30 - 15)^2 + 4(25)^2}$$

$$= 22.5 \pm \frac{1}{2}\sqrt{225 + 2500}$$

$$= 22.5 \pm 26.1$$

$$= 48.6 \text{ MPa (tensile)} \quad \text{and} \quad -3.6 \text{ MPa (compressive)}$$

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} = \frac{2 \times 25}{30 - 15} = \frac{50}{15} = 3.333$$

or $$2\theta = 73.3$$

or $$\theta_1 = 36.65° \quad \text{and} \quad \theta_2 = 36.65° + 90° = 126.65°$$

The major principal stress of 48.6 MPa occurs on the plane inclined at 36.65° with the plane of 30 MPa tensile stress (clockwise), and the minor principal stress occurs on plane at 126.65°. This can be verified from the following calculation:

$$\sigma = 30 \cos^2 36.65° + 15 \sin^2 36.65° + 25 \sin [2(36.65°)] = 48.6 \text{ MPa}$$

Planes of maximum shear stresses

Maximum shear stress occurs on planes at 45° to the principal planes, i.e., on planes at 36.65° + 45° = 81.65° and 126.65° + 45° = 171.65° (or at 36.65° – 45° = – 8.35° which is the same plane as 171.65°)

Stresses on planes of maximum shear stress

Normal stress on planes of maximum shear stress,

$$\sigma_{81.65} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau \sin 2\theta$$

$$= 30 \cos^2 81.65° + 15 \sin^2 81.65° + \tau \sin 2(81.65°)$$

$$= 0.632 + 14.684 + 7.184$$

$$= 22.5 \text{ MPa}$$

On the plane at –8.35°, the stress value will be found to be same.

The above result is also obtained from,

$$\sigma = \pm \frac{1}{2}(\sigma_x + \sigma_y) = \pm\frac{1}{2}(30 + 15) = \pm 22.5 \text{ MPa} \qquad \text{(Eq. 2.26a)}$$

Shear stress on planes of maximum shear stress,

$$\tau_{max} = \pm \frac{1}{2}(48.6 - (-3.6)) = \pm 26.1 \text{ MPa}$$

To correlate the direction of the maximum shear stress, it is necessary to calculate the shear stress at one of the angles,

$$\tau_{81.65} = -\frac{1}{2}(30 - 15) \sin 2(81.65°) + 25 \cos 2(81.65°)$$

$$= -2.15 - 23.95 = -26.1 \text{ MPa}$$

Thus, on a plane inclined at 81.65° with the plane of 30 MPa, the shear stress is 26.1 MPa counter-clockwise and on a plane inclined at 171.65°, the shear stress is 26.1 MPa clockwise.

Solution by Mohr's circle

Solution by Mohr's circle is shown in Fig. 2.20 which is self-explanatory.

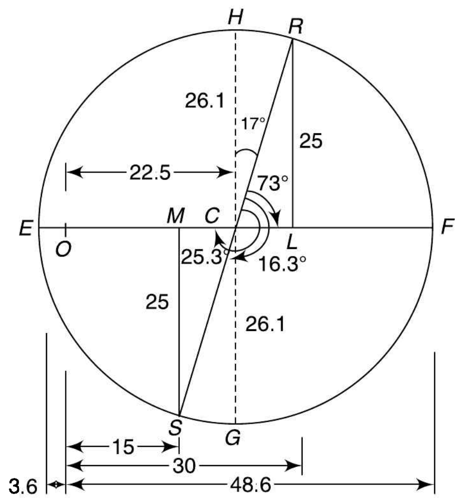


Fig. 2.20

In brief, OL = 30 MPa, OM = 15 MPa, LR = MS = 25 MPa. Mohr's circle is drawn with C, the midpoint of LM as centre and passing through R and S.

- Major principal stress = OF = 48.6 MPa (tensile) at angle 73/2 = 36.5° clockwise of CR or plane of 30 MPa tensile stress.
- Minor principal stress = OE = 3.6 MPa (compressive) at angle 253/2 = 126.5° clockwise of CR or plane of 30 MPa tensile stress.
  Maximum shear stress = CG = CH = 26.1 MPa
- Inclination of the planes of maximum shear stresses:
  163/2 = 81.5° clockwise of CR or plane of 30 MPa tensile stress and
  (343/2 = 171.5°) clockwise or 17/2 = 8.5° counter-clockwise of CR or plane of 30 MPa tensile stress.

$$\sigma_{81.65} = OC = 22.5 \text{ MPa}$$

Example 2.7 A circle of 200-mm diameter is drawn on a mild steel plate before it is subjected to two mutually perpendicular stresses of 160 MPa and 40 MPa along with a shear stress of 60 MPa. After stressing, the circle on the plate deforms to an ellipse. Determine the lengths of the major and minor axes of the ellipse and their directions. E = 205 GPa and Poisson's ratio = 0.3.

Solution

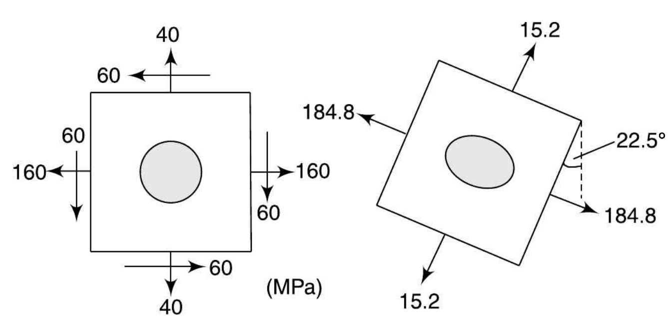


Fig. 2.21

Given A biaxial and shear stress system,

$\sigma_x = 140$ MPa $\quad \sigma_x = 40$ MPa $\quad \tau = 60$ MPa

$E = 205$ GPa $\quad \nu = 0.3 \quad d = 200$ mm

To find Major and minor axes of the ellipse formed

Calculations of principal stresses

The plate acted upon by stresses is shown in Fig. 2.21.

$$\text{Principal stress} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

$$= \frac{1}{2}(160 + 40) \pm \frac{1}{2}\sqrt{(160 - 40)^2 + 4(60)^2}$$

$$= 100 \pm \frac{1}{2}\sqrt{14400 + 14400}$$

$$= 100 \pm 84.8$$

$$= 184.8 \text{ MPa (tensile) and } 15.2 \text{ MPa (tensile)}$$

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} = \frac{2 \times 60}{160 - 40} = 1$$

or $\quad 2\theta = 45°$

or $\quad \theta_1 = 22.5°$ and $\theta_2 = 22.5° + 90° = 112.5°$

The major principal stress 184.8 MPa occurs on the plane inclined at 22.5° with the plane of 160 MPa tensile stress (clockwise) and the minor stress occurs on plane at 112.5° (may be verified).

Calculation of major and minor axes

Increase in diameter along major principal stress,

$$= \frac{L}{E}(\sigma_1 - \nu\sigma_2) = \frac{200}{205\,000}(184.8 - 0.3 \times 15.2) = 0.176$$

$\therefore$ Major axis of ellipse = 200+0.176 = 200.176 mm

Decrease in diameter along minor principal stress,

$$= \frac{L}{E}(\sigma_2 - \nu\sigma_1) = \frac{200}{205\,000}(15.2 - 0.3 \times 184.8) = -0.039$$

$\therefore$ minor axis of ellipse = 200–0.039 = 199.961 mm

Example 2.8 || The stresses on two mutually perpendicular planes through a point in a body are 120 MPa and 30 MPa both tensile alongwith a shear stress of 60 MPa. Determine

(i) the magnitude and direction of principal stresses stating whether the stress condition is uniaxial or biaxial

(ii) the planes of maximum shear stress

(iii) the normal and shear stresses on the planes of maximum shearing stress.

Solution

Given A biaxial and shear stress system,

$\sigma_x = 120$ MPa $\quad \sigma_x = 30$ MPa $\quad \tau = 60$ MPa

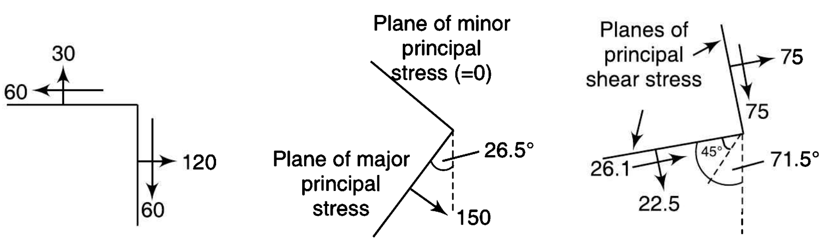


Fig. 2.22

To find

— Principal stresses
— Planes of maximum shear stresses
— Normal and shear stresses on planes of maximum shear stress

Calculations for principal stresses

Refer Fig. 2.22.

$$\text{Principal stress} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

$$= \frac{1}{2}(120 + 30) \pm \frac{1}{2}\sqrt{(120 - 30)^2 + 4(60)^2}$$

$$= 75 \pm \frac{1}{2}\sqrt{8100 + 14\,400}$$

$$= 75 \pm 75$$

$$= 150 \text{ MPa (tensile)} \quad \text{and } 0$$

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} = \frac{2 \times 60}{120 - 30} = \frac{120}{90} = 1.333$$

or $2\theta = 53°$

or $\theta_1 = 26.5°$ and $\theta_2 = 26.5° + 90° = 116.5°$

The major principal stress of 150 MPa occurs on the plane inclined at 26.5° with the plane of 120 MPa tensile stress and no stress occurs on plane at 116.5°.

(Verification: $\sigma = 120 \cos^2 26.5° + 30 \sin^2 26.5° + 60 \sin [2(26.5°)] = 150$ MPa)

Planes of maximum shear stresses

Maximum shear stress occurs on planes at 45° to the principal planes,

i.e., on planes at 26. 5° + 45° = 71. 5° and 116.5° + 45° = 161.5°

Stresses on planes of maximum shear stress

Normal stress on planes of maximum shear stress ,

$$\sigma_{71.5} = \frac{1}{2}(\sigma_x + \sigma_y) = \frac{1}{2}(120 + 30) = 75 \text{ MPa}$$

On the plane at 161.5°, the stress value will also be found to be same.

$$\tau_{max} = \pm \frac{1}{2}(150 - 0) = \pm 75 \text{ MPa}$$

On a plane at 71.5°, the direction of the shear stress is counter-clockwise.

$$\left[\text{Verification: } \tau_\theta = -\frac{1}{2}(120 - 30)\sin 2(71.5°) + 60\cos 2(71.5°) = -75 \text{ MPa}\right]$$

Example 2.9 The stresses on two mutually perpendicular planes through a point in a body are 80 MPa and 50 MPa both tensile. Determine the maximum value of the shear stress which can be applied so that the maximum value of the permissible principal stress is limited to 120 MPa. What will be the inclination of the principal stress and the magnitude of the maximum shear stress?

Solution

Given

$$\sigma_x = 80 \text{ MPa} \qquad \sigma_y = 50 \text{ MPa} \qquad \sigma_1 = 120 \text{ MPa}$$

To find

— Inclination of principal stress

— Maximum shear stresses

Calculation of shear stress

$$\text{Maximum principal stress} = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

$$120 = \frac{1}{2}(80 + 50) + \frac{1}{2}\sqrt{(80 - 50)^2 + 4\tau^2}$$

$$120 = 65 + \frac{1}{2}\sqrt{900 + 4\tau^2}$$

$$900 + 4\tau^2 = 12\ 100$$

$$\tau = 52.9 \text{ MPa}$$

Inclination of principal stresses

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} = \frac{2 \times 52.9}{80 - 50} = \frac{105.8}{30} = 3.527$$

or $$2\theta = 74.2$$

or $$\theta_1 = 37.1°$$

and $$\theta_2 = 37.1° + 90° = 127.1° \text{ (of minor principal stress)}$$

Minimum principal stress

$$\text{Minimum principal stress} = \frac{1}{2}(\sigma_x + \sigma_y) - \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

$$= \frac{1}{2}(80 + 50) - \frac{1}{2}\sqrt{(80 - 50)^2 + 4 \times 52.9^2}$$

$$= 10 \text{ MPa}$$

The major principal stress 120 MPa occurs on the plane inclined at 37.1° with the plane of 80 MPa tensile stress and the minor stress occurs on plane at 127.1°. (may be verified)

Maximum shear stress

$$\tau_{\max} = \pm \frac{1}{2}(120 - 10) = \pm 55 \text{ MPa}$$

Example 2.10 || The resultant stress on a plane at a point in a material under stress is 80 MPa inclined at 30° to the normal to the plane (Fig. 2.23). The normal component of stress on another plane at right angle to the first plane is 60 MPa. Determine

(i) the resultant stress on the second plane

(ii) the principal stresses and their planes

(iii) the maximum shear stresses and their planes

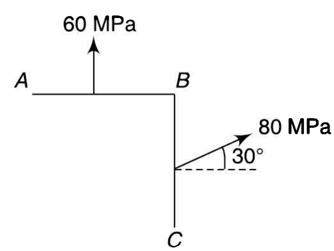


Fig. 2.23

Solution

Given A stress system as shown in Fig. 2.23.

To find

— Resultant stress on the second plane *AB*

— Principal stresses and planes

— Maximum shear stresses and planes

The resultant stress on plane BC can be resolved into two components (Fig. 2.24).

The normal stress = 80 cos 30° = 69.28 MPa

The tangential stress = 80 sin 30° = 40 MPa

Thus, on the plane BC, a shear stress of magnitude 40 MPa acts alongwith a normal stress of 69.28 MPa. On the plane AB, a complimentary shear stress of the same magnitude will act as shown in the figure.

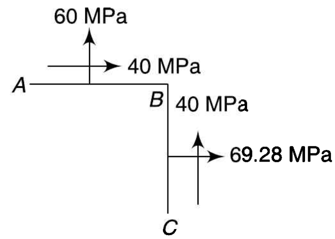


Fig. 2.24

Resultant stress

Resultant stress on plane $AB = \sqrt{60^2 + 40^2} = 72.1\text{ MPa}$

$$\text{Its inclination with the normal} = \tan^{-1}\frac{40}{60} = 33.7°$$

Principal stresses

$$\text{Major principal stress, } \sigma_1 = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

$$= \frac{1}{2}(69.28 + 60) + \frac{1}{2}\sqrt{(69.28 - 60)^2 + 4(-40)^2}$$

$$= 64.64 + 40.27$$

$$= 104.91 \text{ MPa (tensile)}$$

$$\text{Minor principal stress} = 64.64 - 40.27 = 24.37 \text{ MPa (tensile)}$$

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} = \frac{2 \times (-40)}{69.3 - 60} = -8.6$$

or $\quad 2\theta = -83.4°$

or $\quad \theta_1 = -41.7° \quad \text{and} \quad \theta_2 = -41.7° - 90° = -131.7°$

The major principal stress 105 MPa occurs on the plane inclined at 41.7° with the plane of 69.3 MPa tensile stress (may be verified).

Maximum shear stress

$$\text{Maximum shear stress} = \tau_{max} = \frac{1}{2}\sqrt{(69.3 - 60)^2 + 4 \times 40^2} = 40.3 \text{ MPa}$$

Maximum shear stress occurs on planes at 45° to the principal planes, i.e., on planes at 41.7° + 45° = 86.7° and 131.7° + 45° = 176.7°

On a plane at 86.7°, the direction of the shear stress is counter-clockwise (may be verified).

Example 2.11 In a 2-D stress system, stresses at a point in a material are 50 MPa compressive and 30 MPa shearing in one plane and 20 MPa tensile and a shearing stress in another plane at 60° to the first one. Determine the value of the shearing stress in the second plane and the principal stresses and position of their planes.

Solution

Given A stress system as shown in Fig. 2.25a.

To find

— Shear stress in plane at 60° to the vertical plane

— Principal stresses and planes

Let σ_y be the second tensile stress in a plane perpendicular to that of σ_x. Then shear stress in this plane is also 30 MPa, i.e., equal to the shear stress in the plane perpendicular to it.

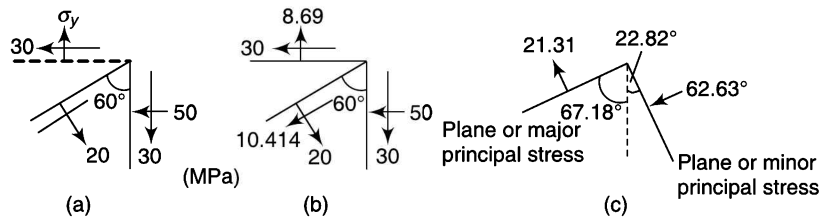


Fig. 2.25

Calculation of shear stress in second plane

Normal stress in a plane at an angle θ to the first one,

$$\sigma_\theta = \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau \cdot \sin 2\theta \qquad \text{(Eq. 2.20)}$$

Thus, in a plane at 60°,

$$20 = \frac{1}{2}(-50 + \sigma_y) + \frac{1}{2}(-50 - \sigma_y)\cos 120° + 30 \cdot \sin 120°$$

$$40 = -50 + \sigma_y + (-50 - \sigma_y)(-0.5) + 60 \times 0.866$$

$$40 = -50 + \sigma_y + 25 + 0.5\sigma_y + 51.96$$

$$1.5\,\sigma_y = 13.04 \quad \text{or} \quad \sigma_y = 8.69 \text{ MPa}$$

$\therefore$

$$\tau_\theta = -(\sigma_x - \sigma_y)\sin 2\theta + \tau \cdot \cos 2\theta$$

$$= -(-50 - 8.69)\sin 120° + 30\cos 120° = 10.414 \text{ MPa}$$

Principal stresses

$$\text{Principal stresses} = = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

$$= = \frac{1}{2}(-50 + 8.69) \pm \frac{1}{2}\sqrt{(-50 - 8.69)^2 + 4(30)^2}$$

$$= -20.66 \pm 41.97$$

$$= 21.31 \text{ and } -62.63 \text{ MPa}$$

$$\tan 2\theta = \frac{2\tau}{\sigma_x - \sigma_y} = \frac{2 \times 30}{-50 - 8.69} = -1.0223$$

or $\quad 2\,\theta = -45.63$ or $\theta_1 = -22.82°$ or 157.18°

and $\quad \theta_2 = -22.82° + 90° = 67.18°$

The major principal stress 21.31 MPa occurs on the plane inclined at 67.18° with the plane of 50 MPa stress.

- Verification: $\sigma_1 = -50\cos^2 67.18° + 8.69\sin^2 67.18° + 30\sin[2(67.18°)] = 21.31$ MPa

Example 2.12 || Figure 2.26 shows the resultant stresses on two planes at a certain point in a material. On a certain plane it is 80 MPa compressive at an angle of 30° to its normal and on another plane, it is 60 MPa tensile at an angle of 75° to its normal. Determine the angle between the planes. Also, find the principal stresses and their directions to the given plane.

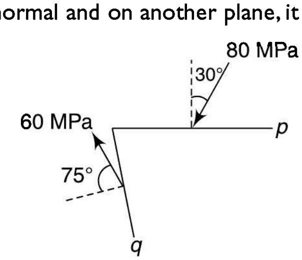


Fig. 2.26

Solution

Given A stress system as shown in Fig. 2.26.

To find

— Angle between the planes

— Principal stresses and their directions

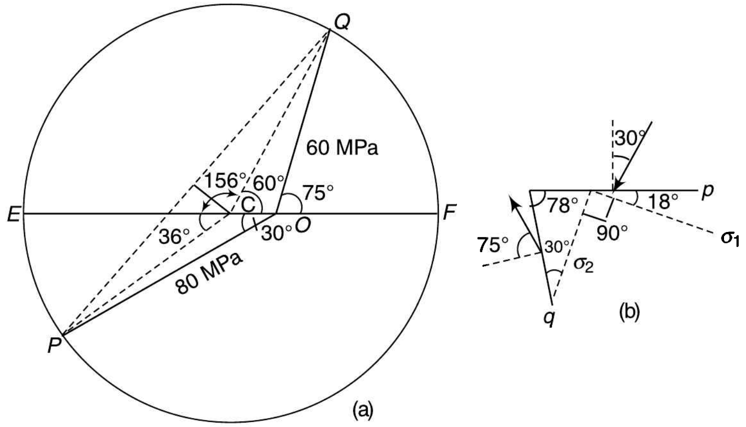


Fig. 2.27

Solution by Mohr's circle method

On the plane p, the resultant stress is 80 MPa compressive, its normal component will be compressive and the shear component counter-clockwise. Thus, take a radial line OP at an angle of 30° with the horizontal to the left of O downwards indicating compressive normal stress and counter-clockwise shear stress (Fig. 2.27a).

On the plane q, the resultant stress is 60 MPa tensile. Take a radial line OQ at an angle of 75° with the horizontal to the right of O upwards indicating tensile normal stress and clockwise shear stress.

Draw a circle passing through points P and Q and having centre on the horizontal line through O. The circle can be drawn by drawing a right bisector of PQ, the intersection of which with the horizontal line through O is the point C. Now, Mohr's circle is drawn with centre C and radius equal to CP or CQ.

Angle between the planes = $\angle PCQ/2$ = 156°/2 = 78°

Major principal stress = OE = 82.5 MPa (compressive) at an angle equal to half of $\angle ECP$ i.e 36/2 = 18° clockwise of plane p.

Minor principal stress = OF = 52.5 MPa (tensile) at an angle equal to half of $\angle FCQ$ i.e 60/2 = 30° clockwise of plane q.

The principal planes are shown in Fig. 2.27b.

Example 2.13 Figure 2.28 shows the stresses at a point in a material subjected to 2-D stresses. The stresses on a certain plane are 90 MPa tensile and 40 MPa shear whereas in another plane 60 MPa tensile and 30 MPa shear. Determine the angle between the planes. Also, find the magnitudes and directions of the principal stresses.

Solution

Given A stress system as shown in Fig. 2.28.

To find

— Angle between the planes
— Principal stresses and their directions

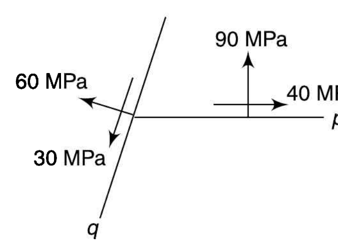


Fig. 2.28

Solution by Mohr's circle method

The Mohr's circle is self-explanatory as shown in Fig. 2.29a.

Take OL = 90 MPa ; LP = 40 MPa; OM = 60 MPa; MQ = 30 MPa

Join PQ. Draw right bisector of PQ intersecting OL in C. With C as centre draw a circle passing through P and Q.

Angle between the planes = $\angle PCQ/2$ = 146°/2 = 73°

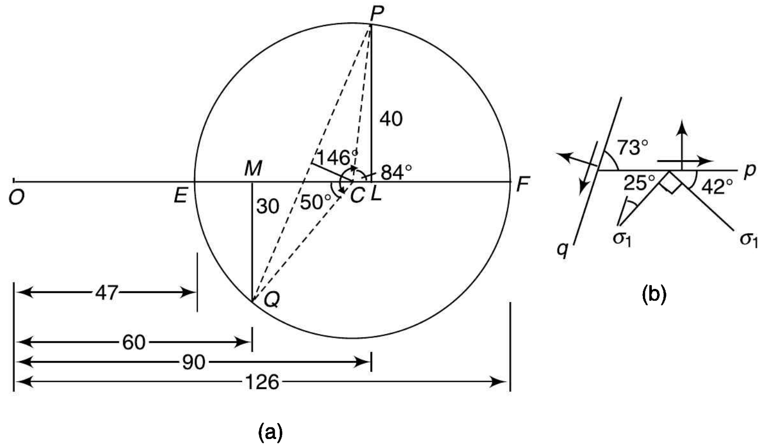


Fig. 2.29

Major principal stress = OF = 126 MPa (tensile) at 84°/2 or 42° clockwise of plane p
Minor principal stress = OE = 47 MPa (tensile) at 50°/2 clockwise of plane q
The principal planes are shown in Fig. 2.29b.

2.7 THREE COPLANAR STRESSES

Consider three planes p, q and r and their representation in a Mohr's circle as CP, CQ and CR as shown in Fig. 2.30a.

Draw vertical lines through P, Q and R. If O is the origin, then normal distances from O to vertical lines through p', q' and r' are the normal stresses on these planes.

Then

$\angle PCQ = 2\theta$ counter-clockwise

and $\angle PCR = 2\beta$ clockwise

Let the vertical through P intersect the Mohr's circle at P' as shown in the figure. Join QP' and RP'.

Then $\angle PP'Q = \theta$

(as the angle subtended by chord PQ on the circumference of circle is half of the angle subtended at the centre)

Similarly, $\angle PP'R = \beta$

Thus, the following procedure may be adopted to draw the Mohr's circle if three normal stresses on three planes are known:

- Mark a point O'. Draw a vertical line $O'O''$ through it (Fig. 2.30).
- Draw three lines p', q' and r' parallel to the vertical line through O' at distances representing the direct stresses in the directions p, q and r to a suitable scale. Tensile stresses are taken on the right side (positive) of $O'O''$ whereas the compressive stresses (negative) on the left. Assuming that the stresses in the directions p and q are tensile and in the direction r it is compressive, the lines p', q' are taken towards right, and r' towards left.
- Take a point at a convenient position on the middle vertical line. Assume that the middle line is p' and the point taken is P'.
- Draw a line through P' making an angle θ with the vertical through P' in the same direction sense as the plane q has with the plane p. In this case it is taken counter-clockwise. Let this line intersect with the vertical through q' at Q.
- In the same way, draw a line through P' making an angle β with the vertical through P' in the clockwise direction. Let it intersect with the vertical through r' at R.

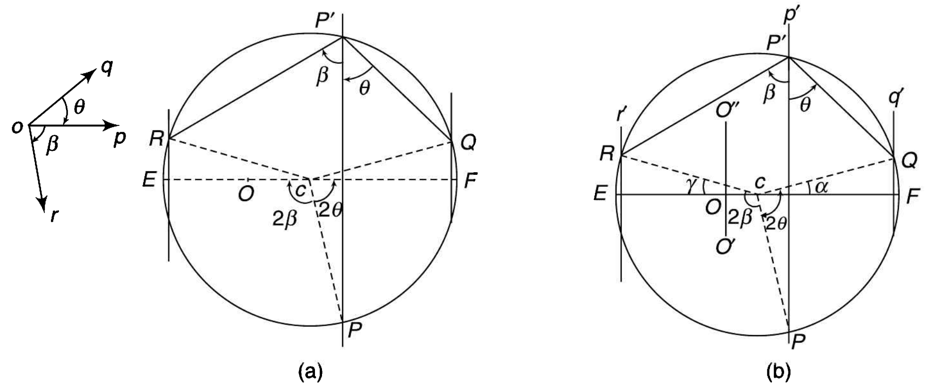


Fig. 2.30

- Draw a circle passing through P', Q and R by taking perpendicular bisectors of $P'Q$ and $P'R$ (not shown in the figure), the intersection locates the centre C. This is the Mohr's circle.
- Let the vertical line through P' have the second intersect point with the circumference of the circle at P as shown in the figure. Join CP, CQ and CR.

Now the radial lines CP, CQ and CR represent the planes of stresses p, q and r respectively because

— The angle made by plane CQ with the plane CP at the centre of the circle is the angle made by chord PQ at the centre which is 2θ, i.e., double of angle $PP'Q$ made by the chord PQ at the circumference of the circle

— The angle made by plane CR with the plane OP at the centre is 2β clockwise, i.e., twice of that made by chord PR at P' on the circumference of the circle

Let a horizontal line through C intersects the vertical through O' at O and the circle at E and F as shown in the above figure.

Principal stresses are given by OF and OE.

The angular position of major principal stress is $\alpha/2$ clockwise of plane CQ and of minor principal stress $\gamma/2$ counter-clockwise of plane CR.

Example 2.14 Figure 2.31 shows three direct stresses in three coplanar directions p, q and r at a particular point. Determine the magnitude and the direction of the principal stresses.

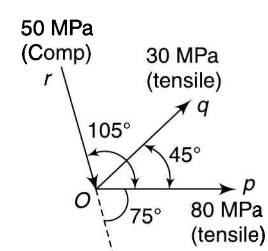


Fig. 2.31

Solution

Given A stress system as shown in Fig. 2.31.

To find Principal stresses and their directions

Solution by Mohr's circle method

Draw the Mohr's circle as follows (Fig. 2.32):

- Take a vertical line $O'O''$. Draw three more vertical lines p', q' and r' parallel to line $O'O''$ at distances representing stresses of 80 MPa, 30 MPa and 50 MPa respectively to a suitable scale. Lines of 80 MPa and 30 MPa are taken to the right being tensile and 50 MPa on the left being compressive.

- Take a convenient point on the middle line q', say point Q'. Draw a line making an angle 45° through this point in the clockwise direction as plane p is at 45° of plane q in the clockwise direction. Let this line intersect line p' at the point P.
- Similarly, draw a line through Q' at an angle (105° – 45°) i.e. 60° in the counter-clockwise direction as plane r' is at 60° in counter-clockwise direction of plane q. Let this line intersect line r' at R.
- Draw a circle passing through points P, Q' and R by taking right bisectors of PQ' and RQ'. Let their point of intersecting be C which is the centre of the circle.
- Let the other point of intersection of line q' with the circumference be Q. Then CP, CQ and CR represent the planes of p, q and r respectively.
- Draw a horizontal line through C intersecting the circle at E and F.

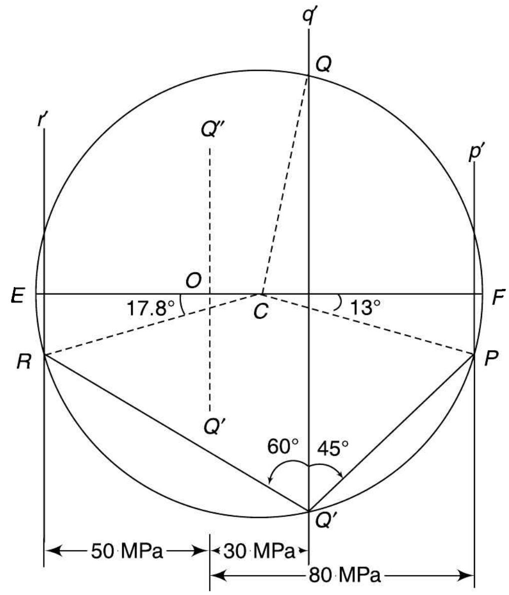


Fig. 2.32

Major principal stress = OF = 82.5 MPa (tensile) at an angle equal to half of $\angle FCQ$, i.e., 13/2 = 6.5° counter-clockwise of p.

Minor principal stress = OE = 52.5 MPa (compressive) at an angle equal to half of $\angle ECR$, i.e, 17.8 = 8.9° clockwise of plane r.

2.8 ELLIPSE OF STRESS

This is another graphical method to be used when a material is subjected to direct stresses σ_x and σ_y. The method is as follows:

- Draw two circles with O as centre and radii equal to σ_x and σ_y taken to a suitable scale (Fig. 2.33).
- Through O draw AB parallel to the inclined plane.
- Draw $OE \perp AB$ through O intersecting the inner circle at D and outer circle at E.

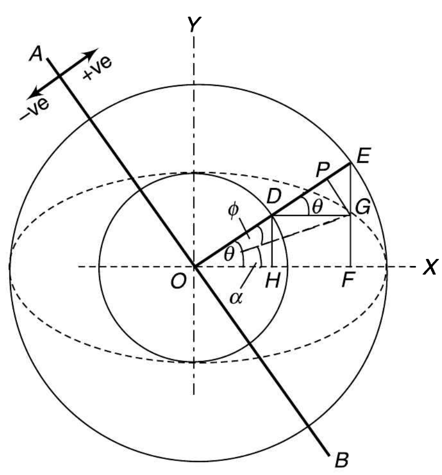


Fig. 2.33

- Draw EF and DH $\perp$ OX.
- Draw $DG \perp EF$ and $GP \perp OE$.

Now, $OP = OD + DP = OD + DG \cos\theta$

$= \sigma_y + (DE \cos\theta)\cos\theta = \sigma_y + (\sigma_x - \sigma_y)\cos^2\theta$

$= \sigma_x \cos^2\theta + \sigma_y(1 - \cos^2\theta) = \sigma_x\cos^2\theta + \sigma_y \sin^2\theta = \sigma_\theta$ (Refer Eq.2.5)

and $PG = DG \sin\theta = (DE \cos\theta)\sin\theta$

$= (\sigma_x - \sigma_y)\cos\theta \sin\theta = \dfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta = \tau_\theta$ (Refer Eq.2.24 with τ =0)

Also, $OF = OE\cos\theta = \sigma_x \cos\theta$

and $FG = HD = \sigma_y \sin\theta$

$\therefore$ $OG = \sqrt{\sigma_x^2 \cos^2\theta + \sigma_y^2 \sin^2} = \sigma_r$ (Refer Eq. 2.8)

$$\tan\alpha = \frac{\sigma_y \sin\theta}{\sigma_x\cos\theta} = \frac{\sigma_y}{\sigma_x}\tan\theta \qquad \text{(Refer Eq. 2.11)}$$

$$\tan\varphi = \frac{PG}{OP} = \frac{(\sigma_x - \sigma_y)\sin\theta\cos\theta}{\sigma_x \cos^2\theta + \sigma_y \sin^2\theta} \qquad \text{(Refer Eq. 2.9)}$$

For different values of θ, point G can be located, the locus of which is evidently an ellipse as shown in the figure. The diagram is thus known as ellipse of stress.

Example 2.15 A piece of material is subjected to two perpendicular stresses as follows:

(a) tensile stresses of 100 MPa and 60 MPa
(b) tensile stress of 100 MPa and compressive stress of 60 MPa

Determine normal and tangential stresses on a plane inclined at 30° to the plane of 100-MPa stress. Also, find the resultant and its inclination with the normal stress using ellipse stress method.

Solution

Given A biaxial stress system,

(i) σ_x = 100 MPa and σ_y = 60 MPa
(ii) σ_x = 100 MPa and σ_y = – 60 MPa

To find

$\sigma_{30°}$, $\tau_{30°}$, σ_r, φ

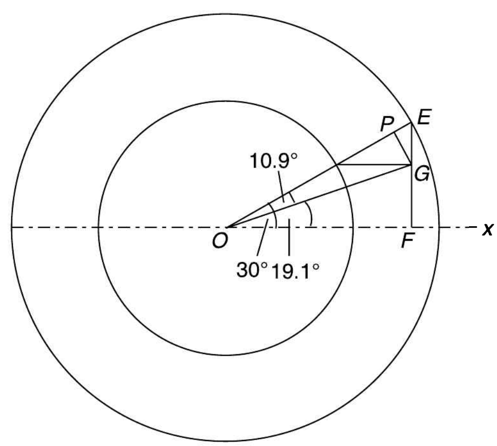


Fig. 2.34

Solution by Mohr's circle method

(a) In the statement, it is not specified whether the inclined plane is clockwise or counter-clockwise relative to the plane of 100-MPa stress. So, it can be taken in any sense relative to OX. This merely affects the sense of direction of shear stress on the inclined plane. The ellipse of stress for the given data is shown in Fig. 2.34 which is self-explanatory. The results are

$\sigma_\theta = OP$ = 90 MPa; $\tau_\theta = GP$ = 17.3 MPa; $\sigma_r = OG$ = 91.6 MPa;
$\varphi = \angle POG$ = 10.9°; $\alpha = \angle FOG$ = 19.1°

(b) The ellipse of stress for the given data is shown in Fig. 2.35. OE represents the tensile stress and OD the compressive stress. The results are

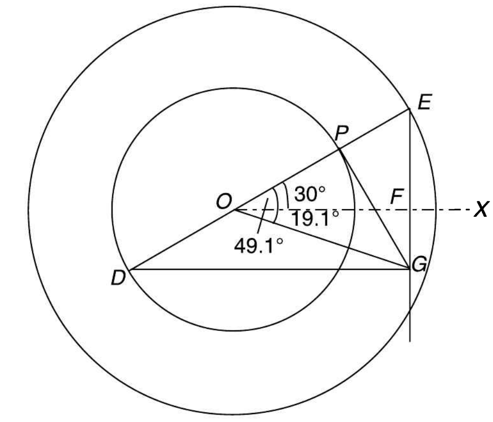


Fig. 2.35

$\sigma_\theta = OP = 60$ MPa; $\tau_\theta = GP = 69.3$ MPa; $\sigma_r = OG = 91.6$ MPa;
$\varphi = \angle POG = 49.1°$; $\alpha = \angle FOG = 19.1°$

2.9 STRAIN ANALYSIS

If direct and shear strains along x- and y-directions are known, normal strain (ε_θ) and the shear strain (φ_θ) in a direction at angle θ with the x-direction of a body can be found by the following method:

Normal Strain

Let a rectangular element $OACB$ with angle of the diagonal θ with the direction of ε_x or x-axis distorts to become a parallelogram $OA'C'B'$ under the action of linear strains ε_x, ε_y and shear strain φ as shown in Fig. 2.36. Point C moves to C'. Let r be the length of the diagonal OC.

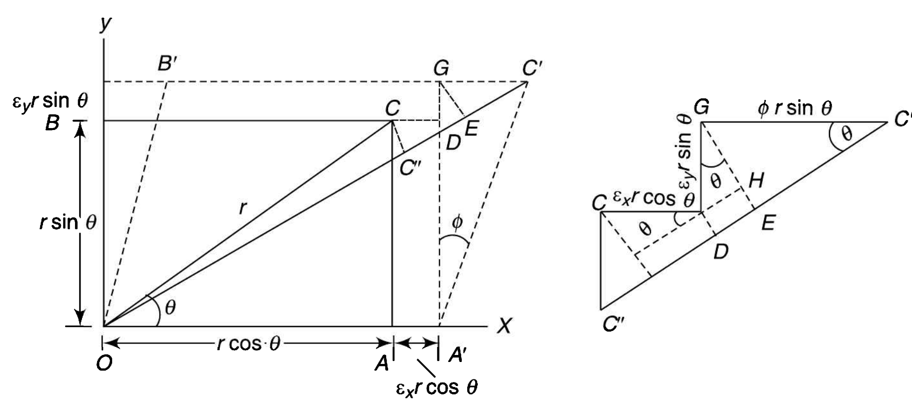


Fig. 2.36

Now, elongation of the diagonal $= C''C' = C''D + DE + EC'$

$$= (\varepsilon_x \cdot r \cdot \cos\theta)\cos\theta + (\varepsilon_y \cdot r \cdot \sin\theta)\sin\theta + (\varphi \cdot r \cdot \sin\theta)\cos\theta$$

$$= \varepsilon_x \cdot r \cdot \cos^2\theta + \varepsilon_y \cdot r \cdot \sin^2\theta \sin\theta + \varphi \cdot r \cdot \sin\theta \cdot \cos\theta$$

Since strain of the diagonal, $\varepsilon_\theta = C''C'/r$

$$\therefore \quad \varepsilon_\theta = \varepsilon_x \cdot \cos^2\theta + \varepsilon_y \cdot \sin^2\theta + \varphi \cdot \sin\theta \cdot \cos\theta \tag{2.28}$$

$$= \frac{1}{2}\,\varepsilon_x\,(1+\cos 2\theta) + \frac{1}{2}\,\varepsilon_y\,(1-\cos 2\theta) + \frac{1}{2}\,\varphi \sin 2\theta$$

$$= \frac{1}{2}\,(\varepsilon_x + \varepsilon_y) + \frac{1}{2}\,(\varepsilon_x - \varepsilon_y)\cos 2\theta + \frac{1}{2}\,\varphi \sin 2\theta \tag{2.28a}$$

Compare the results with bi-axial and shear stresses conditions (Eq. 2.25).

- In a linear strain system, $\varepsilon_\theta = \varepsilon_x \cdot \cos^2\theta$ or $\varepsilon_x\left(\dfrac{1+\cos 2\theta}{2}\right)$
- In a pure shear system and for $\theta = 45°, \varepsilon_{45}° = \varphi/2$.

Shear Strain

The shear strain at a point on a plane inclined at angle θ is the change in the angle between two straight lines perpendicular to each other. As shown in Fig. 2.37, if these lines are OC and OE before distortion, they become OC' and OE' after distortion. Let the angle between OC and OC' be α and between OE and OE' be γ.

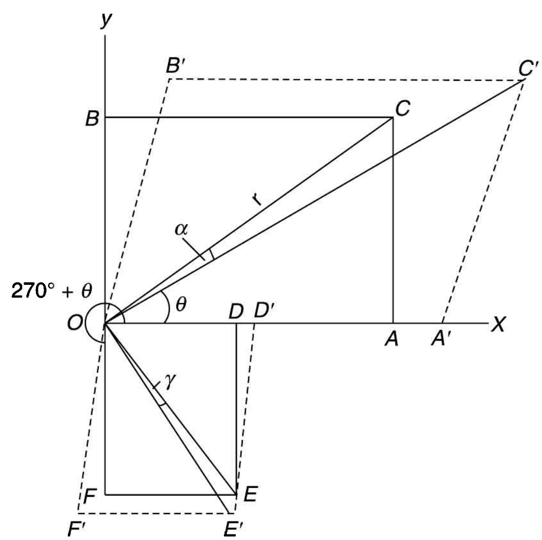


Fig. 2. 37

Thus, shear strain = φ_θ = change of angle of OC – change of angle of OE

$= \alpha - \gamma$

As the angle α is small, $\alpha \cong \tan \alpha = CC''/r$

$$CC' = CF + FC'' = CF + (GE - GH)$$

$$= (\varepsilon_x \cdot r \cdot \cos\theta)\sin\theta + [(\varphi \cdot r \cdot \sin\theta)\sin\theta - (\varepsilon_y \cdot r \cdot \sin\theta)\cos\theta]$$

$$= \varepsilon_x \cdot r \cdot \sin\theta \cdot \cos\theta + \varphi \cdot r \cdot \sin^2\theta - \varepsilon_y \cdot r \cdot \sin\theta \cdot \cos\theta$$

$$= \frac{1}{2}(\varepsilon_x - \varepsilon_y) r \cdot \sin 2\theta + \varphi \cdot r \cdot \sin^2\theta$$

$$\alpha = CC''/r = \frac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2\theta + \varphi \sin^2\theta$$

Angle γ can be found by inserting $\theta = -(90° - \theta) = 270° + \theta$, in the above equation.

$$\gamma = \frac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2(270° + \theta) + \varphi \sin^2(270° + \theta)$$

$$= \frac{1}{2}(\varepsilon_x - \varepsilon_y)\sin(180° + 2\theta) + \varphi \sin^2(270° + \theta)$$

$$= -\frac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2\theta + \varphi \cos^2\theta$$

$$\text{Shear strain} = \varphi_\theta = \alpha - \beta = (\varepsilon_x - \varepsilon_y)\sin 2\theta + \varphi(\sin^2\theta - \cos^2\theta)$$

$$= (\varepsilon_x - \varepsilon_y)\sin 2\theta - \varphi\cos 2\theta \qquad (2.29)$$

Compare the results with bi-axial and shear stress conditions (Eq. 2.26).

2.10 PRINCIPAL STRAINS

The maximum and the minimum values of strains on any plane at a point are known as the *principal strains* and the corresponding planes as the *principal planes for strains*.

To obtain the condition for principal strains, differentiating Eq. 2.40 with respect to θ and equating to zero,

$$\frac{d\varepsilon}{d\theta} = 0 - \frac{1}{2}(\varepsilon_x - \varepsilon_y)\, 2\sin 2\theta + \varphi \cdot \cos 2\theta$$

or
$$(\varepsilon_x - \varepsilon_y)\sin 2\theta = \varphi \cdot \cos 2\theta$$

or
$$\tan 2\theta = \frac{\varphi}{\sigma_x - \sigma_y} \qquad (2.30)$$

Values of principal strains can be obtained in a similar way as for principal stresses:

$$\text{Principal strain} = \frac{1}{2}(\varepsilon_x + \varepsilon_y) \pm \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \varphi^2}$$

$$\text{As } \tan 2\theta = \tan 2\theta = \frac{\varphi}{\sigma_x - \sigma_y}, \text{ From Fig. 2.38,}$$

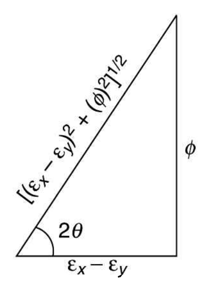


Fig. 2.38

$$\sin 2\theta = \pm\frac{\varphi}{\sqrt{(\varepsilon_y - \varepsilon_x)^2 + \varphi^2}}$$

$$\cos 2\theta = \pm\frac{\varepsilon_x - \varepsilon_y}{\sqrt{(\varepsilon_y - \varepsilon_x)^2 + \varphi^2}}$$

Right-hand sides of both the above equations should have the same signs, positive or negative while using them.

- In principal planes, $\varphi_\theta = (\varepsilon_x - \varepsilon_y)\sin 2\theta - \varphi\cos 2\theta$

$$= (\varepsilon_x - \varepsilon_y)\frac{\varphi}{\sqrt{(\sigma_x - \sigma_y)^2 + \varphi^2}} - \varphi\frac{\varepsilon_x - \varepsilon_y}{\sqrt{(\sigma_x - \sigma_y)^2 + \varphi^2}} = 0$$

- It can be shown that the planes of principal strains are the same as of principal stresses as follows:

$$\tan 2\theta = \frac{\varphi}{\varepsilon_x - \varepsilon_y} = \frac{\tau/G}{(1/E)[(\sigma_x - \nu\sigma_y - \nu\sigma_z) - (\sigma_y - \nu\sigma_z - \nu\sigma_x)]}$$

$$= \frac{\tau \cdot E}{G[(\sigma_x - \nu\sigma_y - \nu\sigma_z) - (\sigma_y - \nu\sigma_z - \nu\sigma_x)]}$$

$$= \frac{\tau \cdot 2G(1+\nu)}{G(\sigma_x - \sigma_y)(1+\nu)} = \frac{2\tau}{\sigma_x - \sigma_y}$$

which is the same equation as Eq. 2.31 indicating that the planes of principal strains are the same as of principal stresses and thus can simply be referred as *principal planes*.

2.11 PRINCIPAL SHEAR STRAINS

For shear strain to be maximum or minimum, differentiating Eq. 2.40 with respect to θ and equating to zero,

$$\frac{d\varphi}{d\theta} = (\varepsilon_x - \varepsilon_y)\, 2\cos 2\theta + \varphi \cdot 2\sin 2\theta = 0$$

or
$$\tan 2\theta = -\frac{\varepsilon_x - \varepsilon_y}{\varphi}$$

Comparing this with Eq. 2.33, it can be shown that the planes of maximum shear strain are inclined at 45° to the planes of maximum shear strain as in case of maximum shear stress.

2.12 SUM OF DIRECT STRAINS ON TWO MUTUALLY PERPENDICULAR PLANES

Direct strain on an inclined plane at angle θ is given by,

$$\varepsilon_\theta = \varepsilon_x \cdot \cos^2\theta + \varepsilon_y \cdot \sin^2\theta + \varphi \cdot \sin\theta \cdot \cos\theta \tag{i}$$

Direct strain on an inclined plane at angle $(90°+\theta)$ will be,

$$\varepsilon_{\theta+90°} = \varepsilon_x \cdot \cos^2(90°+\theta) + \varepsilon_y \cdot \sin^2(90°+\theta) + \varphi \cdot \sin(90°+\theta) \cdot \cos(90°+\theta)$$

$$= \varepsilon_x \cdot \sin^2\theta + \varepsilon_y \cdot \cos^2\theta - \varphi \cdot \cos\theta \sin\theta \tag{ii}$$

From (i) and (ii),

$$\varepsilon_\theta + \varepsilon_{\theta+90°} = \varepsilon_x + \varepsilon_y \tag{2.31}$$

Since the sum of direct strains ε_x and ε_y is constant, the sum of direct strains on two mutually perpendicular planes at a point at any angle θ and $(90°+\theta)$ remains constant and equal to $\varepsilon_x + \varepsilon_y$.

2.13 MOHR'S STRAIN CIRCLE

Comparing Eqs. 2.38 and 2.20, it may be observed that the Mohr's circle used for the stress analysis can also be used for strain analysis. The linear strains can be taken along horizontal axis and shear strain along the vertical axis, the magnitude of the shear strain taken to be half. Thus, in the strain circle (Fig. 2.39),

$$OC = \frac{1}{2}(\varepsilon_x + \varepsilon_y)$$

and
$$CR = \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \varphi^2}$$

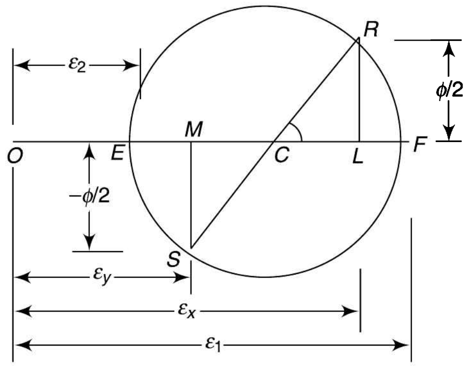


Fig. 2.39

The construction of the strain circle, when three coplanar linear strains in three directions at a point are known, is exactly similar to that for the case when three coplanar stresses are known (Section 2.5).

2.14 PRINCIPAL STRESSES FROM PRINCIPLE STRAINS

Two-dimensional System

In a two-dimensional system $\sigma_3 = 0$, and

$$\varepsilon_1 = \sigma_1/E - \nu\,\sigma_2/E$$

or $$\varepsilon_1 \cdot E = \sigma_1 - \nu\,\sigma_2 \quad \text{or} \quad \sigma_1 = \varepsilon_1 \cdot E + \nu\,\sigma_2 \qquad \text{(i)}$$

and $$\varepsilon_2 = \sigma_2/E - \nu\,\sigma_1/E$$

or $$\varepsilon_2 \cdot E = \sigma_2 - \nu\,\sigma_1 \qquad \text{(ii)}$$

Inserting the value of σ_1 from (i) in (ii),

$$\varepsilon_2 \cdot E = \sigma_2 - \nu(\varepsilon_1 \cdot E + \nu\,\sigma_2) = \sigma_2(1 - \nu^2) - \nu\,\varepsilon_1 \cdot E$$

or $$\sigma_2 = \frac{E(\nu\varepsilon_1 + \varepsilon_2)}{1 - \nu^2} \qquad (2.32)$$

Similarly, $$\sigma_1 = \frac{E(\nu\varepsilon_2 + \varepsilon_1)}{1 - \nu^2} \qquad (2.33)$$

Three-Dimensional System

We have

$$\varepsilon_1 = \sigma_1/E - \nu\,\sigma_2/E - \nu\,\sigma_3/E \qquad \text{(i)}$$
$$\varepsilon_2 = \sigma_2/E - \nu\,\sigma_3/E - \nu\,\sigma_1/E \qquad \text{(ii)}$$
$$\varepsilon_3 = \sigma_3/E - \nu\,\sigma_1/E - \nu\,\sigma_2/E \qquad \text{(iii)}$$

From (i),

$$\sigma_1 = E\varepsilon_1 + \nu(\sigma_2 + \sigma_3)$$

Inserting this value in (ii) and (iii),

$$E\varepsilon_2 = \sigma_2 - \nu\,\sigma_3 - \nu\,[E\varepsilon_1 + \nu\,(\sigma_2 + \sigma_3)]$$

or $$E(\varepsilon_2 + \nu\,\varepsilon_1) = \sigma_2\,(1 - \nu^2) - \sigma_3\,(1 + \nu)\nu \qquad \text{(iv)}$$

Similarly,

$$E(\varepsilon_3 + \nu\,\varepsilon_1) = \sigma_3\,(1 - \nu^2) - \sigma_2\,(1 + \nu)\nu \qquad \text{(v)}$$

Multiplying (iv) by (ν) and (v) by ($1-\nu$),

$$E(\varepsilon_2 + \nu\,\varepsilon_1)\,\nu = \sigma_2(1-\nu^2)\,\nu - \sigma_3(1+\nu)\nu^2 \quad \text{(vi)}$$

$$E(\varepsilon_3 + \nu\,\varepsilon_1)\,(1-\nu) = \sigma_3(1-\nu^2)\,(1-\nu) - \sigma_2(1-\nu^2)\,\nu \quad \text{(vii)}$$

Adding (vi) and (vii),

$$E[(\varepsilon_2 + \nu\,\varepsilon_1)\,\nu + (\varepsilon_3 + \nu\,\varepsilon_1)\,(1-\nu)] = \sigma_3(1-\nu^2)\,(1-\nu) - \sigma_3(1+\nu)\nu^2$$

$$E(\varepsilon_2\,\nu + \nu\,\varepsilon_1\,\nu + \varepsilon_3 + \nu\,\varepsilon_1 - \nu\,\varepsilon_3 - \nu\,\varepsilon_1\,\nu) = \sigma_3[(1-\nu^2)\,(1-\nu) - (1+\nu)\nu^2]$$

$$E(\varepsilon_2\,\nu + \varepsilon_3 + \nu\,\varepsilon_1 - \nu\,\varepsilon_3) = \sigma_3[1-\nu^2 - \nu + \nu^3 - \nu^2 - \nu^3]$$

$$E[(1-\nu)\varepsilon_3 + (\varepsilon_1 + \varepsilon_2)\,\nu] = \sigma_3[1-\nu^2 - \nu - \nu^2]$$

$$= \sigma_3(1+\nu)(1-2\nu)$$

Thus

$$\sigma_3 = \frac{E[(1-\nu)\varepsilon_3 + (\varepsilon_1 + \varepsilon_2)\nu]}{(1+\nu)(1-2\nu)} \quad (2.34)$$

Similarly,

$$\sigma_1 = \frac{E[(1-\nu)\varepsilon_2 + (\varepsilon_3 + \varepsilon_1)\nu]}{(1+\nu)(1-2\nu)} \quad (2.35)$$

and

$$\sigma_2 = \frac{E[(1-\nu)\varepsilon_2 + (\varepsilon_3 + \varepsilon_1)\nu]}{(1+\nu)(1-2\nu)} \quad (2.36)$$

Example 2.16 If the principal strains at a point in a material are in the ratio 4:3:2, determine the ratio of principal stresses at that point. Poisson's ratio is 0.3.

Solution

Given Ratio of principal strains: 4:3:2.

Poisson's ratio = 0.3

To find Ratio of principal stresses

Relations for stresses and strains

$$\sigma_1 = \frac{E[(1-\nu)\varepsilon_1 + (\varepsilon_2 + \varepsilon_3)\nu]}{(1+\nu)(1-2\nu)} \quad \text{...(Eq. 2.35)}$$

$$\sigma_2 = \frac{E[(1-\nu)\varepsilon_2 + (\varepsilon_3 + \varepsilon_1)\nu]}{(1+\nu)(1-2\nu)} \quad \text{...(Eq. 2.36)}$$

$$\sigma_3 = \frac{E[(1-\nu)\varepsilon_3 + (\varepsilon_1 + \varepsilon_2)\nu]}{(1+\nu)(1-2\nu)} \quad \text{...(Eq. 2.34)}$$

The given ratio of principal strains,

$$\varepsilon_1 : \varepsilon_2 : \varepsilon_3 = 4 : 3 : 2 \text{ i.e. } \varepsilon_1 = 2\varepsilon_3,\ \varepsilon_2 = 1.5\varepsilon_3$$

Since denominator is the same in all cases and E is common, these may be dropped to find the ratio.

Ratio of principal stresses

$$\sigma_1 : \sigma_2 : \sigma_3 = [(1-0.3)2\varepsilon_3 + (1.5\varepsilon_3 + \varepsilon_3)0.3]$$

$$:[(1-0.3)1.5\varepsilon_3 + (\varepsilon_3 + 2\varepsilon_3)0.3] : [(1-0.3)\varepsilon_3 + (2\varepsilon_3 + 1.5\varepsilon_3)0.3]$$

$$= 2.15 : 1.95 : 1.75$$

$$= 1.229 : 1.114 : 1$$

2.15 STRAIN ROSETTE

Strain in any direction can be measured by using an instrument known as strain gauge. In case, directions of principal strains are known (no shear strain), two strain gauges can be used to measure the strains in these directions and by using Eqs 2.35 and 2.36, the principal stresses can be calculated. However, many times the directions of the principal stresses are not known. In such cases, a set of three strain gauges, known as a strain rosette, can be used to find the strain in three known directions in order to determine stress condition at a point under consideration.

Let ε_x and ε_y be the linear strains in x- and y-directions and φ be the shear strain at the point under consideration. Then linear strains in any three arbitrary chosen directions at angles θ_1, θ_2 and θ_3 made with the x-axis will be (using Eq. 2.28)

$$\varepsilon_{\theta 1} = \varepsilon_x \cdot \cos^2\theta_1 + \varepsilon_y \cdot \sin^2\theta_1 + \varphi \cdot \sin\theta_1 \cdot \cos\theta_1$$
$$\varepsilon_{\theta 2} = \varepsilon_x \cdot \cos^2\theta_2 + \varepsilon_y \cdot \sin^2\theta_2 + \varphi \cdot \sin\theta_2 \cdot \cos\theta_2$$
$$\varepsilon_{\theta 3} = \varepsilon_x \cdot \cos^2\theta_3 + \varepsilon_y \cdot \sin^2\theta_3 + \varphi \cdot \sin\theta_3 \cdot \cos\theta_3$$

If three arbitrary directions are chosen in a set manner and $\varepsilon_{\theta 1}$, $\varepsilon_{\theta 2}$ and $\varepsilon_{\theta 3}$ are measured along these directions, then ε_x and ε_y, the linear strains in x- and y-directions, and φ, the shear strain at the point can be calculated by using the above equations. Principal strains and principal stresses can then be calculated.

Rectangular Strain Rosette

In case, the three strain gauges are set at 0°, 45° and 90° with the x-direction, it is known as a *rectangular strain rosette* or *45° strain rosette*. Thus in this case,

$$\theta_1 = 0°, \qquad \theta_2 = 45° \qquad \text{and} \qquad \theta_3 = 90°$$

The above equations can be written as

$$\varepsilon_{0°} = \varepsilon_x$$
$$\varepsilon_{45°} = \frac{1}{2}(\varepsilon_x + \varepsilon_y + \varphi)$$
$$\varepsilon_{90°} = \varepsilon_y$$

From which, $\varepsilon_x = \varepsilon_{0°}$, $\varepsilon_y = \varepsilon_{90°}$ and $\varphi = 2\varepsilon_{45°} - \varepsilon_x + \varepsilon_y$ (2.37)

On knowing these values, principal strains and principal stresses can be calculated.

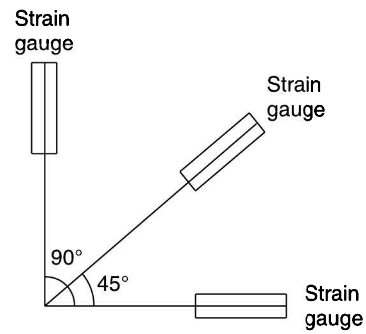


Fig. 2.40

Equiangular Strain Rosette

If the three strain gauges are set at 0°, 60° and 120° with the x-direction, it is known as a *equiangular strain rosette* or *delta strain rosette* or *60° strain rosette*. Thus in this case,

$$\theta_1 = 0°, \qquad \theta_2 = 60° \qquad \text{and} \qquad \theta_3 = 120°$$

The above equations can be written as

$$\varepsilon_{0°} = \varepsilon_x \qquad \text{(i)}$$

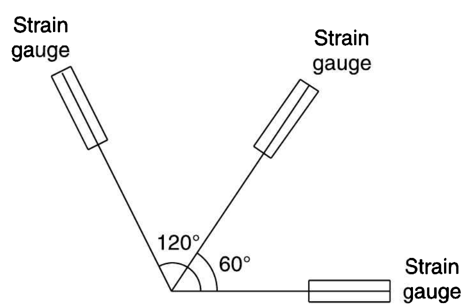


Fig. 2.41

$$\varepsilon_{60°} = \frac{1}{4}(\varepsilon_x + 3\varepsilon_y + \sqrt{3}\varphi) \qquad \text{(ii)}$$

$$\varepsilon_{120°} = \frac{1}{4}(\varepsilon_x + 3\varepsilon_y - \sqrt{3}\varphi) \qquad \text{(iii)}$$

From which, $\varepsilon_x = \varepsilon_{0°}$ (2.38)

Subtracting (iii) from (ii), $\varphi = \dfrac{2}{\sqrt{3}}(\varepsilon_{60°} - \varepsilon_{120°})$ (2.39)

From (ii), $3\varepsilon_y = 4\varepsilon_{60°} - \varepsilon_x - \sqrt{3}\varphi = 4\varepsilon_{60°} - \varepsilon_{0°} - \sqrt{3}\cdot\frac{2}{\sqrt{3}}(\varepsilon_{60°} - \varepsilon_{120°})$

or
$$\varepsilon_y = \frac{1}{3}(2\varepsilon_{60°} + 2\varepsilon_{120°} - \varepsilon_{0°}) \tag{2.40}$$

On knowing these values, principal strains and principal stresses can be calculated.

Example 2.17 The following readings are recorded by a rectangular strain rosette (the angles are with the *x*-axis):

$$\varepsilon_{0°} = 400 \times 10^{-6},\ \varepsilon_{45°} = 200 \times 10^{-6} \text{ and } \varepsilon_{90°} = -100 \times 10^{-6}$$

Determine the principal strains and stresses. $E = 210$ GPa and Poisson's ratio 0.3.

Solution

Given readings of a rectangular strain rosette,

$$\varepsilon_{0°} = 400 \times 10^{-6},\ \varepsilon_{45°} = 200 \times 10^{-6} \text{ and } \varepsilon_{90°} = -100 \times 10^{-6}$$
$$E = 210 \text{ GPa} \qquad \nu = 0.3$$

To find Principal strains and stresses

Principal strains

For a rectangular strain rosette (From Eq. 2.37),

$$\varepsilon_x = \varepsilon_{0°} = 400 \times 10^{-6},\ \varepsilon_y = \varepsilon_{90°} = -100 \times 10^{-6}$$

and $\varphi = 2\varepsilon_{45°} - \varepsilon_x + \varepsilon_y = 2 \times 200 \times 10^{-6} - 400 \times 10^{-6} - 100 \times 10^{-6} = -100 \times 10^{-6}$

$$\text{Principal strains, } \varepsilon_1, \varepsilon_2 = \frac{1}{2}(\varepsilon_x + \varepsilon_y) \pm \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \varphi^2}$$
$$= \frac{10^{-6}}{2}\left[(400 - 100) \pm \sqrt{(400 + 100)^2 + (-100)^2}\right]$$
$$= 404.95 \times 10^{-6} \text{ and } -104.95 \times 10^{-6}$$

Principal stresses

$$\sigma_1 = \frac{E(\nu\varepsilon_2 + \varepsilon_1)}{1 - \nu^2} = \frac{210\,000(-0.3 \times 104.95 + 404.95) \times 10^{-6}}{1 - 0.3^2} = 86.2 \text{ MPa}$$

$$\sigma_2 = \frac{E(\nu\varepsilon_1 + \varepsilon_2)}{1 - \nu^2} = \frac{210\,000\,(0.3 \times 404.95 - 104.95) \times 10^{-6}}{1 - 0.3^2} = 3.82 \text{ MPa}$$

Example 2.18 The strains measured on a strain rosette are as follows: (the angles are with the *x*-axis):

$$\varepsilon_{0°} = 450 \times 10^{-6}, \quad \varepsilon_{60°} = -600 \times 10^{-6} \quad \text{and} \quad \varepsilon_{120°} = 150 \times 10^{-6}$$

Determine the stresses and their directions. $E = 200$ GPa and Poisson's ratio = 0.3.

Solution

Given Readings of a strain rosette,

$$\varepsilon_{0°} = 450 \times 10^{-6}, \quad \varepsilon_{60°} = -600 \times 10^{-6} \quad \text{and} \quad \varepsilon_{120°} = 150 \times 10^{-6}$$
$$E = 200 \text{ GPa} \qquad \nu = 0.3$$

To find Stresses and their directions

Principal strains

From Eq. 2.38, $\varepsilon_x = \varepsilon_{0°} = 450 \times 10^{-6}$

From Eq. 2.39, $\varphi = \frac{2}{\sqrt{3}}(\varepsilon_{60°} - \varepsilon_{120°}) = \frac{2}{\sqrt{3}}(-600 \times 10^{-6} - 150 \times 10^{-6}) = -866 \times 10^{-6}$

From Eq. 2.40,

$$\varepsilon_y = \frac{1}{3}(2\varepsilon_{60°} + 2\varepsilon_{120°} - \varepsilon_{0°})$$

$$= \frac{1}{3}(-2 \times 600 \times 10^{-6} + 2 \times 150 \times 10^{-6} - 450 \times 10^{-6}) = -450 \times 10^{-6}$$

$$\text{Principal strains, } \varepsilon_1, \varepsilon_2 = \frac{1}{2}(\varepsilon_x + \varepsilon_y) \pm \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \varphi^2}$$

$$= \frac{10^{-6}}{2}[(450 - 450) \pm \sqrt{(450 + 450)^2 + (-866)^2}]$$

$$= 624.5 \times 10^{-6} \text{ and } -624.5 \times 10^{-6}$$

Principal stresses

Principal stresses,

$$\sigma_1 = \frac{E(\nu\varepsilon_1 + \varepsilon_2)}{1 - \nu^2} = \frac{200\,000\,(-0.3 \times 624.5 + 624.5) \times 10^{-6}}{1 - 0.3^2} = -96 \text{ MPa}$$

$$\sigma_2 = \frac{E(\nu\varepsilon_1 + \varepsilon_2)}{1 - \nu^2} = \frac{200\,000\,(-0.3 \times 624.5 + 624.5) \times 10^{-6}}{1 - 0.3^2} = -96 \text{ MPa}$$

Principal planes

$$\text{Principal planes, } \tan 2\theta = \frac{\varphi}{\varepsilon_x - \varepsilon_y} = \frac{-866 \times 10^{-6}}{450 \times 10^{-6} + 450\ X\ 10^{-6}} = 0.962$$

or $2\theta = 40°$ or $\theta = 22°$clockwise with x-axis

Example 2.19 Figure 2.42 shows the strains in three directions p, q and r in a plane, the magnitudes being 600×10^{-6}, -150×10^{-6} and 250×10^{-6}. Determine the magnitude and direction of the principal strains in this plane.

Assuming no stress in a plane perpendicular to this plane, find the principal stresses at the point. Take $E = 205$ GPa and $\nu = 0.3$.

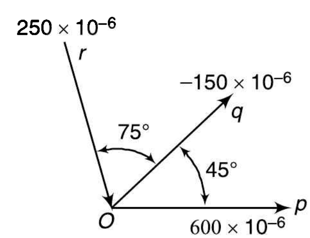


Fig. 2.42

Solution

Given Readings of a strain rosette,

$\varepsilon_{0°} = 600 \times 10^{-6}$, $\varepsilon_{45°} = -150 \times 10^{-6}$, and $\varepsilon_{120°} = 250 \times 10^{-6}$,

$E = 205$ GPa $\quad \nu = 0.3$

To find Principal stresses

Assuming x-axis along direction of $\varepsilon_{0°}(p)$

Calculation of linear and shear strains

$$\varepsilon_\theta = \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \frac{1}{2}\varphi \sin 2\theta \qquad \text{(Eq. 2.28)}$$

- When $\theta = 0°$

$$\varepsilon_0 = \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y) = 600 \times 10^{-6}$$

or
$$\varepsilon_x = 600 \times 10^{-6} \quad \text{(i)}$$

- When $\theta = 45°$

$$\varepsilon_{45} = \frac{1}{2}(600 \times 10^{-6} + \varepsilon_y) + \frac{1}{2}(600 \times 10^{-6} - \varepsilon_y)\cos 90° + \frac{1}{2}\varphi \sin 90°$$

$$-150 \times 10^{-6} = \frac{1}{2}(600 \times 10^{-6} + \varepsilon_y) + \frac{1}{2}\varphi$$

or
$$\varepsilon_y + \varphi = -900 \times 10^{-6} \quad \text{(ii)}$$

- When $\theta = 120°$

$$\varepsilon_{120} = \frac{1}{2}(600 \times 10^{-6} + \varepsilon_y) + \frac{1}{2}(600 \times 10^{-6} - \varepsilon_y)\cos 240° + \frac{1}{2}\varphi \sin 240°$$

$$250 \times 10^{-6} = \frac{1}{2}(600 \times 10^{-6} + \varepsilon_y) - \frac{1}{4}(600 \times 10^{-6} - \varepsilon_y) - \frac{\sqrt{3}}{4}\varphi$$

$$= \frac{1}{4}\,600 \times 10^{-6} + \frac{3}{4}\varepsilon_y - \frac{\sqrt{3}}{4}\varphi$$

$$\varepsilon_y - 0.577\,\varphi = 133.3 \times 10^{-6} \quad \text{(iii)}$$

Subtracting (iii) from (ii),

$$1.577\,\varphi = -1033 \times 10^{-6}$$
$$\varphi = -655 \times 10^{-6}$$
$$\varepsilon_y = -900 \times 10^{-6} - (-655 \times 10^{-6}) = -245 \times 10^{-6}$$

Principal strains

$$\text{Principal strains} = \frac{1}{2}(\varepsilon_x + \varepsilon_y) \pm \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \varphi^2}$$

$$= \frac{10^{-6}}{2}(600 - 245) \pm \frac{10^{-6}}{2}\sqrt{(600 + 245)^2 + 655^2}$$

$$= (177 \pm 535)10^{-6}$$

$$\varepsilon_1 = 712 \times 10^{-6}$$

and
$$\varepsilon_2 = -358 \times 10^{-6}$$

Direction of principal strains

$$\tan 2\theta = \frac{\varphi}{\sigma_x - \sigma_y} = \frac{655 \times 10^{-6}}{(600 + 245)10^{-6}} = 0.775 \quad \text{(Eq. 2.30)}$$

$$2\,\theta = 37.8° \quad \text{or} \quad \theta = 18.9° \text{ and } 108.9°$$

Principal stresses

$$\sigma_1 = \frac{E(\nu\varepsilon_2 + \varepsilon_1)}{1 - \nu^2} = \frac{205\,000 \times 10^{-6}\,[0.3 \times (-358) + 712]}{1 - 0.3^2} = 136.2 \text{ MPa}$$

$$\sigma_2 = \frac{E(\nu\varepsilon_1 + \varepsilon_2)}{1 - \nu^2} = \frac{205\,000 \times 10^{-6}\,[0.3 \times 712 - 358]}{1 - 0.3^2} = -32.5 \text{ MPa}$$

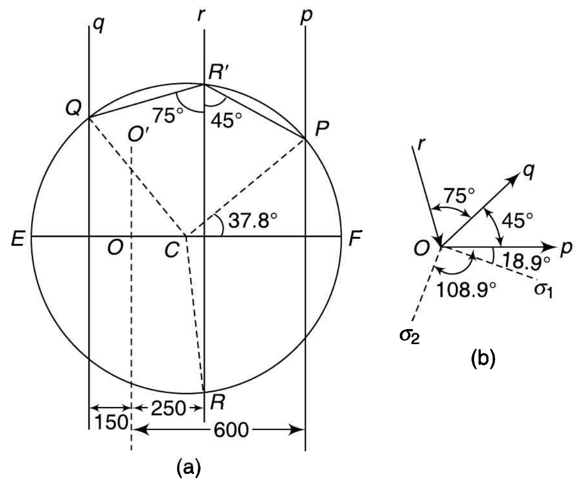


Fig. 2.43

Solution by Mohr's circle

Mohr's circle can be drawn as shown in Fig. 2.43a which is self-explanatory.

Major principal strain $\varepsilon_1 = OF = 712 \times 10^{-6}$ at 37.8/2 or 18.9° clockwise of plane p

Minor principal strain $\varepsilon_2 = OE = -358 \times 10^{-6}$ at 108.9° clockwise of plane p

The results are shown in Fig. 2.43b.

Summary

1. Stresses on an inclined plane of system with direct and shear stresses are

$$\sigma_\theta = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau \sin 2\theta$$

$$= \frac{1}{2}(\sigma_x + \sigma_y) + \frac{1}{2}(\sigma_x - \sigma_y)\cos 2\theta + \tau \cdot \sin 2\theta$$

$$\tau_\theta = -\frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta + \tau \cos 2\theta$$

2. Sum of direct stresses on two mutually perpendicular planes, $\sigma_\theta + \sigma_{(\theta + 90°)} = \sigma_x + \sigma_y$
3. In complex systems of loading, there exist three mutually perpendicular planes, on each of which the resultant stress is wholly normal. These are known as *principal planes* and the normal stresses across these planes as *principal stresses*.
4. The inclination of principal planes is given by $\tan 2\theta = \dfrac{2\tau}{\sigma_x - \sigma_y}$
5. The magnitude of major and minor principal stresses are given by

$$\sigma_{1,2} = \frac{1}{2}(\sigma_x + \sigma_y) \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$$

6. Maximum shear stress $= \dfrac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2} = \dfrac{1}{2}(\sigma_1 - \sigma_2)$

7. A material acted upon by pure shear stresses on two perpendicular planes will have a tensile stress equal to the magnitude of the shear stress on the planes at 45° and a compressive stress of the same magnitude on the planes at 135° with no shear stress on these planes.
8. Principal stresses from principal strains for 2-D,
$$\sigma_1 = \frac{E(\nu\varepsilon_2 + \varepsilon_1)}{1-\nu^2} \quad \text{and} \quad \sigma_2 = \frac{E(\nu\varepsilon_1 + \varepsilon_2)}{1-\nu^2}$$
9. Linear strain in an inclined plane,
$$\varepsilon_\theta = \frac{1}{2}(\varepsilon_x + \varepsilon_y) + \frac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\theta + \frac{1}{2}\varphi \sin 2\theta$$
10. Shear strain in an inclined plane,
$$\varphi_\theta = (\varepsilon_x - \varepsilon_y)\sin 2\theta - \varphi \cos 2\theta$$
11. Principal strains $= \frac{1}{2}(\varepsilon_x + \varepsilon_y) \pm \frac{1}{2}\sqrt{(\varepsilon_x - \varepsilon_y)^2 + \varphi^2}$
12. Sum of direct strains on two mutually perpendicular planes, $\varepsilon_\theta + \varepsilon_{(\theta + 90°)} = \varepsilon_x\, \varepsilon_y$

Objective Type Questions

1. Under uniaxial loading, the maximum shear stress is _____ times the uniaxial stress.
(a) half (b) two (c) 1.5 (d) three

2. The shear stress on the principal plane is
(a) $\frac{\sigma_x + \sigma_y}{2}$ (b) $\frac{\sigma_x - \sigma_y}{2}$ (c) $\sigma_x + \sigma_y$ (d) Zero

3. If a body is acted upon by pure shear stresses on two perpendicular planes, the planes inclined at 45° are subjected to no _____ stress.
(a) tensile (b) compressive (c) shear

4. Normal stress on a plane, the normal to which is inclined at angle θ with the line of action of uniaxial stress σ is given by
(a) $\sigma \sin^2\theta$ (b) $\sigma \cos^2\theta$ (c) $\sigma/\sin^2\theta$ (d) $\sigma/\cos^2\theta$

5. Shear stress on a plane the normal to which is inclined at angle θ with the line of action of uniaxial stress σ is given by
(a) $\frac{1}{2}\sigma \sin 2\theta$ (b) $\sigma \sin^2\theta$ (c) $\frac{1}{2}\sigma \cos 2\theta$ (d) $\sigma \cos^2\theta$

6. In case of biaxial stresses, the maximum value of shear stress is
(a) difference of normal stresses (b) half the difference of normal stresses
(c) sum of normal stresses (d) half the sum of normal stresses

7. In case of biaxial stresses, the shear stress is maximum on a plane at ___ degrees to the given planes.
(a) 45° (b) 90° (c) 135° (d) 180°

8. Normal stress on an oblique plane inclined at angle 45° to the axis of a bar of square cross-section of side s when acted upon by a tensile force P is
(a) $\frac{P}{s^2}$ (b) $\frac{2P}{s^2}$ (c) $\frac{P}{2s^2}$ (d) $\frac{P}{4s^2}$

9. Each normal stress on two planes of maximum shear stresses are
(a) difference of normal stresses (b) half the difference of normal stresses
(c) sum of normal stresses (d) half the sum of normal stresses

10. In a Mohr's circle, the radius gives the value of
(a) minimum shear stress (b) maximum normal stress
(c) minimum normal stress (d) maximum shear stress

11. Ellipse of stress can be drawn only when a body is acted upon by
(a) one normal stress (b) two normal stresses
(c) one shear stress (d) two normal stresses and one shear stress

12. The angle between planes of principal stresses and principal strains is
(a) zero (b) 45° (c) 90° (d) 135°

13. A strain rosette consists of at least ______strain gauges
(a) two (b) three (c) four (d) five

14. In a rectangular strain rosette, the strain gauges are set at
(a) 45° (b) 60° (c) 75° (d) 90°

15. In an equiangular strain rosette, the strain gauges are set at
(a) 45° (b) 60° (c) 75° (d) 90°

16. In a linear strain system, strain in a direction θ with x-direction is
(a) $\varepsilon_x \cdot \cos\theta$ (b) $\varepsilon_x \cdot \sin\theta$ (c) $\varepsilon_x \cdot \sin^2\theta$ (d) $\varepsilon_x \cdot \cos^2\theta$

Answers

1. (a)	2. (d)	3. (c)	4. (b)	5. (c)	6. (b)
7. (a)	8. (c)	9. (d)	10. (d)	11. (b)	12. (a)
13. (b)	14. (a)	15. (b)	16. (d)		

Review Questions

2.1 Show that in a direct stress system, the maximum shear stress in a body is half the magnitude of the applied stress.

2.2 Deduce expressions for stresses on an inclined plane in a body subjected to a bi-axial stress condition.

2.3 Show that shear stress in a body acted upon by two equal perpendicular stresses is zero.

2.4 Show that a body subjected to a pure shear is also acted upon by tensile and compressive stresses as well.

2.5 From first principles, show that a state of pure shear exists in a body subjected to equal perpendicular stresses of different nature.

2.6 What do you mean by principal planes and principal stresses? Derive the expression for principal stresses for a body subjected to direct and shear stresses.

2.7 What is Mohr's stress circle? How is it useful in the solution of stress-analysis problems?

2.8 Deduce the expression for the linear strain in a body in a direction inclined at angle θ with the x-axis when direct and shear strains along x- and y-directions are known.

Numerical Problems

2.1 A circular bar of steel is subjected to an axial pull of 50 kN. Find the diameter of the bar if the maximum intensity of shear stress on any oblique plane is not to exceed 60 MPa. (23 mm)

2.2 A piece of material is subjected to two perpendicular tensile stresses of 300 MPa and 150 MPa. Determine the normal and shear stress components on a plane the normal of which makes an angle of 40° with the 300 MPa stress. Also, find the resultant. (237.5 MPa; 74 MPa; 248.8 MPa; 22.8°)

2.3 The stresses on two perpendicular planes through a point are 120 MPa tensile, 80 MPa compression and 60 MPa shear. Determine the normal and shear stress components on a plane at 60° to that of the 120 MPa stress and also the resultant and its inclination with the normal component on the plane.
(22 MPa; 116.5 MPa; 118.6 MPa; 79.3°)

2.4 Determine the position of the plane on which the resultant stress is the most inclined to the normal in a system of two perpendicular compressive stresses of 120 MPa and 180 MPa. Also, find the value of the resultant stress. (39.23°; 147 MPa comp)

2.5 Two perpendicular tensile stresses of 300 MPa and 150 MPa act at a point in a material. Draw the Mohr's stress circle and find the normal and shear stress components on a plane the normal of which makes an angle of 40° with the 300 MPa stress. Also, find the resultant.
(237.5 MPa; 74 MPa ; 248.8 MPa ;22.8°)

2.6 Draw the Mohr's stress circle for a bi-axial stress system having two direct stresses of 30 MPa (tensile) and 20 MPa (compressive). Determine the magnitude and the direction of the resultant stresses on planes which make angles of (i) 25°, and (ii) 70° with the 30 MPa stress. Also, find the normal and shear stresses on these planes. (For 25° plane: 21 MPa (tensile), 19 MPa; 28.5 MPa; 42°;
For 70° plane: 14.2 MPa (comp.), 16 MPa; 21. MPa; 131.5°)

2.7 At a point in a steel bar the stresses on two mutually perpendicular planes are 10 MPa tensile and 5 MPa tensile whereas the shear stress across these planes is 2.5 MPa. Determine, using Mohr's circle, the normal as well as the shear stresses on a plane making an angle of 30° with the plane of the first stress. Also, find the magnitude and the direction of the resultant stress on the same plane.
(10.9 MPa; 0.9 MPa; 10.95 MPa; 5°)

2.8 The normal stresses at a point in an elastic material are 100 MPa and 60 MPa respectively at right angle to each other with a shearing stress of 50 MPa. Determine the principal stresses and the position of principal planes if (i) both normal stresses are tensile, and (ii) 100 MPa stress is tensile, and 60 MPa stress is compressive. Also determine the maximum shear stress and its plane in the two cases.
(133.8 MPa tensile, 26.2 MPa tensile, 34.1° and 124.1°, 53.8 MPa, 79.1°;
114.3 MPa tensile , 74.3 MPa comp., 16° and 106°, 94.3 MPa, 61°)

2.9 The principal stresses at a point in a bar are 50 MPa tensile and 30 MPa compressive. Calculate the normal stress, shear stress and the resultant stress on a plane inclined at 50° to the axis of major principal stress. Also, find the maximum shear stress at the point.
(16.95 MPa, 39.4 MPa, 42.88 MPa; 40 MPa)

2.10 The direct stresses in two mutually perpendicular directions are 300 MPa and 150 MPa both tensile accompanied by complimentary shear stresses of intensity 225 MPa. Determine the normal and tangential stresses on two planes equally inclined to the planes of the direct stresses.
(450 MPa, –75 MPa; zero, 75 MPa)

2.11 At a certain point in a strained material, direct stresses of 120 MPa tensile and 90 MPa compressive exist on two perpendicular planes. These stresses are also accompanied by a shear stress on the planes. The major principal stress at the point due to these is 150 MPa. Determine the magnitude of the shear stresses on the two planes. Also, find the maximum shear stress at the point. (84.85 MPa, 135 MPa)

2.12 The resultant stress on a plane BC at a point in a material is 240 MPa tensile inclined at 30° to the normal to the plane as shown in Fig. 2.44. On a plane AB perpendicular to plane BC, the normal component of stress is 180 MPa. Determine

(i) the resultant stress on plane AB
(ii) the principal stresses and principal planes
(iii) the maximum shear stress

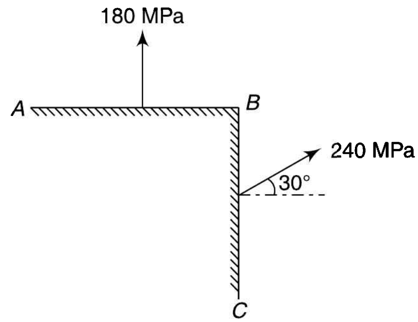


Fig. 2.44

(216.3 MPa; 56.5°; 314.6 MPa, 73 MPa; 41.7°, 131.7°; 120.8 MPa; 86.7°, 176.7°)

2.13 A piece of material is acted upon by tensile stresses of 50 MPa and 25 MPa at right angle to each other. Determine by ellipse of stress, the magnitude and direction of the resultant stress on a plane at 45° to the 50 MPa stress. (39.5 MPa, 18° with normal stress, 27° with 100 MPa stress.)

2.14 The stresses at a point in three coplanar directions are measured as σ_0 = 80 MPa (tensile), σ_{60} = 400 MPa (tensile) and σ_{120} = 200 MPa (compressive) where subscripts indicates the relative angular position of the planes in degrees. Determine the principal stresses and the planes.

[449 MPa (tensile) at 14° to 400 MPa and 251 MPa (compressive) at 16° to 200 MPa]

2.15 The readings of a strain gauge rosette inclined at 45° with each other are 4×10^{-6}, 3×10^{-6} and 1.6×10^{-6}, the first gauge being along x-axis. Determine the principal strains and the planes.

(4.04×10^{-6}, 1.58×10^{-6}; 5° and 95°)

Chapter 3

Strain Energy

When an elastic body is loaded within elastic limits, it deforms and some work is done which is stored within the body in the form of internal energy. This stored energy in the deformed body is known as *strain energy* or *potential energy of deformation* and is denoted by *U*. It is recoverable without loss as soon as the load is removed from the body. However, if the elastic limit is exceeded, there is permanent set of deformations and the particles of the material of the body slide one over another. The work done in doing so is spent in overcoming the cohesion of the particles and the energy spent appears as heat in the strained material of the body. The concept of strain energy is very important in strength of materials as it is associated with the deformation of the body. The deflection of a body depends upon the manner of application of the load, i.e., whether the applied load is gradual, sudden or impact. If a body is acted upon by sudden or impact load, the instantaneous deformation is much more as compared to when the load is gradually applied. In such cases, strain energy is a convenient tool to solve problems associated with deformations.

3.1 STRAIN ENERGY

When a *gradual* or *static* load is applied to a body of an elastic material, the body is strained and work is done on the body which is stored within the body in the form of internal energy. The work done by the load in straining the body is known as the *strain energy*. For most of the elastic materials, the elongation increases linearly with the increase in load and therefore, the load-elongation diagram is a straight line within elastic limits (Fig. 3.1). The work done in straining a material is equal to the area under the diagram at any instant.

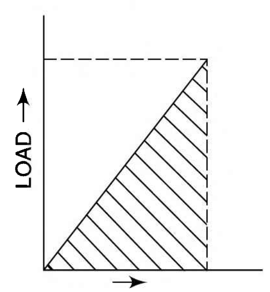


Fig. 3.1

Let P be the gradual load applied to a bar of length L and of uniform area A. Let Δ be the total elongation under the load.

Then strain energy, U = Average load × Elongation = $\frac{1}{2} \cdot P \cdot \Delta$ (3.1)

The strain energy may also be expressed in the following forms:

$$U = \frac{1}{2} \cdot (\sigma A) \cdot (\varepsilon \cdot L) = \frac{1}{2} \cdot \sigma \cdot \varepsilon \cdot AL = \frac{1}{2} \times \text{stress} \times \text{strain} \times \text{volume} \quad (3.2)$$

or $$U = \frac{1}{2} \cdot \sigma \cdot \frac{\sigma}{E} \cdot AL = \frac{\sigma^2}{2E} \cdot AL = \frac{\sigma^2}{2E} \times \text{volume} \tag{3.3}$$

or $$U = \frac{P^2}{A^2} \cdot \frac{1}{2E} . AL = \frac{P^2 L}{2AE} \tag{3.4}$$

Resilience

It is the ability of a material to regain its original shape on removal of the applied load. It is defined as the *strain energy per unit volume* in simple tension or compression and is equal to $\sigma^2/2E$. It is also referred as *strain-energy density* and denoted by u.

Distinction between *strain energy* and *resilience* is hardly followed by authors and each is referred in place of the other.

Proof Resilience

It is the value of resilience at the elastic limit or at proof stress. It is also known as the *modulus of resilience* or *resilience modulus.*

Strain-energy density, $u = \frac{1}{2} \times \text{stress} \times \text{strain}$ ... (From Eq. 3.2)

$$= \frac{1}{2} \cdot \sigma_e \cdot \varepsilon_e = \frac{1}{2} \cdot E\varepsilon_e \cdot \varepsilon_e = \frac{E}{2} \cdot \varepsilon_\varepsilon^2 \quad \text{(i)}$$

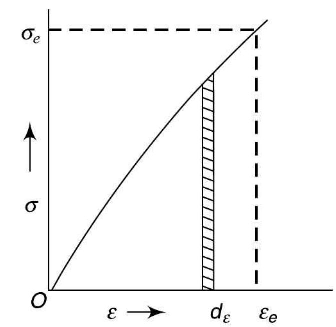


Fig. 3.2

It can be shown that the proof resilience is the area under the stress–strain curve. Figure 3.2 shows an element of area of width $d\varepsilon$ located under the stress–strain diagram.

Elemental area = $\sigma \cdot d\varepsilon = E\varepsilon \cdot d\varepsilon$

Let ε_e denote the strain corresponding to the elastic limit of stress σ_e.

Then, area under the curve up to elastic limit $= E\int_0^{\varepsilon_e} \varepsilon \cdot d\varepsilon = E\left(\frac{\varepsilon^2}{2}\right)_o^{\varepsilon_e} = \frac{E}{2} \cdot \varepsilon_e{}^2$ which is same as (i) above.

Thus proof resilience is the area under the stress–strain diagram up to the elastic limit.

Modulus of Toughness

It is the strain energy per unit volume required to cause the material to rupture. It is the area under the stress–strain diagram up to rupture point (Fig. 3.3). Thus, it is a measure of the ability of a material to absorb energy before fracture.

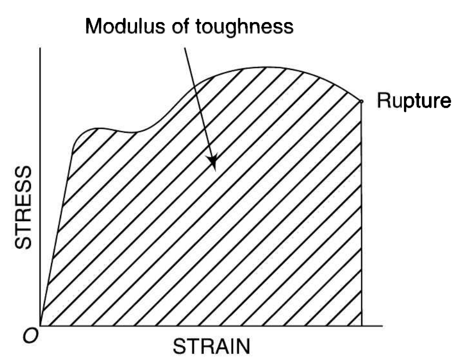


Fig. 3.3

3.2 STRAIN ENERGY (THREE-DIMENSIONAL STRESS SYSTEM)

Consider a unit cube acted upon by three principal stresses σ_1, σ_2 and σ_3 (Fig. 3.4). Let the corresponding strains be ε_1, ε_2 and ε_3. Then, for gradually applied stresses,

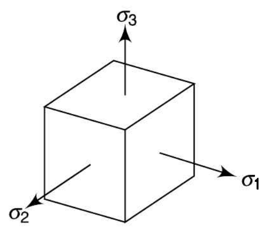


Fig. 3.4

Strain energy,

$$U = \frac{1}{2}\sigma_1\varepsilon_1 + \frac{1}{2}\sigma_2\varepsilon_2 + \frac{1}{2}\sigma_3\varepsilon_3$$

$$= \frac{1}{2}\sigma_1\left(\frac{\sigma_1 - \nu\sigma_2 - \nu\sigma_3}{E}\right) + \frac{1}{2}\sigma_2\left(\frac{\sigma_2 - \nu\sigma_3 - \nu\sigma_1}{E}\right) + \frac{1}{2}\sigma_3\left(\frac{\sigma_3 - \nu\sigma_1 - \nu\sigma_2}{E}\right)$$

$$= \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \text{ per unit volume} \tag{3.5}$$

- For a system with equal principal stresses $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$,

$$U = \frac{1}{2E}[\sigma^2 + \sigma^2 + \sigma^2 - 2\nu(\sigma\sigma + \sigma\sigma + \sigma\sigma)] = \frac{3\sigma^2}{2E}(1 - 2\nu) \tag{3.6}$$

As $$E = 3K(1 - 2\nu) \qquad \text{(From Eq. 1.27)}$$

Thus, $$U = \frac{3\sigma^2}{2 \times 3K(1 - 2\nu)}(1 - 2\nu) = \frac{\sigma^2}{2K} \tag{3.7}$$

- For a two-dimensional system

$$U = \frac{1}{2E}(\sigma_1^2 + \sigma_1^2 - 2\nu\sigma_1\sigma_2) \text{ per unit volume} \tag{3.8}$$

3.3 SHEAR STRAIN ENERGY

Consider a block with dimensions L, b and h as shown in Fig. 3.5. Assume it to be rigidly fixed to the ground. A shear force P is applied gradually along the top surface.

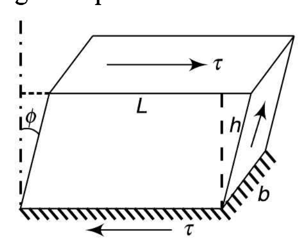


Fig. 3.5

Strain energy, U = Work done in straining = $\frac{1}{2}$ × Final couple × Angle turned

$$= \frac{1}{2} \times \text{Final force} \times h \times \varphi = \frac{1}{2} \times (\text{Shear stress} \times \text{Area}) \times h \times \varphi$$

$$= \frac{1}{2} \cdot \tau \cdot (L \times b) \times h \times \frac{\tau}{G} \quad \text{...... (As } G = \tau/\varphi \text{ or } \varphi = \tau/G)$$

$$= \frac{\tau^2}{2G} \cdot (L \cdot b \cdot h)$$

$$= \frac{\tau^2}{2G} \times \text{Volume} \tag{3.9}$$

or
$$\text{Shear strain energy per unit volume} = \frac{\tau^2}{2G} \tag{3.10}$$

It is similar to $\sigma^2/2E$ for direct stress.

3.4 SHEAR STRAIN ENERGY (THREE-DIMENSIONAL STRESS SYSTEM)

Volumetric strain of a cube is given by,

$$\varepsilon_v = \frac{(\sigma_1 + \sigma_2 + \sigma_3)(1 - 2v)}{E} \quad \text{(Eq. 1.23)}$$

Note that in the above equation,

- If $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$, i.e., the stresses are of equal value, there is only volumetric change and no distortion takes place
- If $\sigma_1 + \sigma_2 + \sigma_3 = 0$, there is no change in volume and thus there is only distortion of the cube

Thus the total strain energy given by Eq. 3.5 can be assumed to consist of two parts:

1. Volumetric strain energy due to change in volume
2. Shear strain energy due to distortion of the cube

Thus, Total strain energy = Volumetric strain energy + Shear strain energy

Let each principal stress consists of two components, one (each σ) causing change in volume only and the other $(\sigma_1', \sigma_2', \sigma_3')$ causing distortion only,

i.e.,
$$\sigma_1 = \sigma + \sigma_1'$$
$$\sigma_2 = \sigma + \sigma_2'$$
$$\sigma_3 = \sigma + \sigma_3'$$

Adding, $\sigma_1 + \sigma_2 + \sigma_3 = 3\sigma + (\sigma_1' + \sigma_2' + \sigma_3')$

Then $(\sigma_1' + \sigma_2' + \sigma_3')$ must be zero since this set causes only distortion.

or
$$\sigma = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$$

Now total strain energy,

$$U = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \text{ per unit volume} \quad \text{(Eq. 3.5)}$$

Volumetric strain energy,

$$U_v = \frac{1}{2} \times \text{Average stress} \times \text{Volumetric strain}$$

$$= \frac{1}{2}\left(\frac{\sigma_1+\sigma_2+\sigma_3}{3}\right)\times\frac{(\sigma_1+\sigma_1+\sigma_1)(1-2\nu)}{E} \qquad \text{(Using Eq. 1.23)}$$

$$= \frac{1}{6E}(\sigma_1+\sigma_2+\sigma_3)^2(1-2\nu) \tag{3.11}$$

Thus, shear strain energy

$$= \frac{1}{2E}[\sigma_1^2+\sigma_2^2+\sigma_3^2-2\nu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)]-\frac{1}{6E}(\sigma_1+\sigma_2+\sigma_3)^2(1-2\nu)$$

$$= \frac{1}{6E}\{3(\sigma_1^2+\sigma_2^2+\sigma_3^2)-6\nu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)\} - \frac{1}{6E}\{\sigma_1^2+\sigma_2^2+\sigma_3^2+2(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)(1-2\nu)\}$$

$$= \frac{1}{6E}[3(\sigma_1^2+\sigma_2^2+\sigma_3^2)-6\nu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)-(\sigma_1^2+\sigma_2^2+\sigma_3^2) - 2(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)+2\nu(\sigma_1^2+\sigma_2^2+\sigma_3^2)+4\nu(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)]$$

$$= \frac{1}{6E}[(\sigma_1^2+\sigma_2^2+\sigma_3^2)(3-1+2\nu)-(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)(6\nu+2-4\nu)]$$

$$= \frac{1+\nu}{6E}[2(\sigma_1^2+\sigma_2^2+\sigma_3^2)-2(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1)]$$

$$= \frac{1+\nu}{6\times 2G(1+\nu)}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]$$

$$= \frac{1}{12G}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2] \tag{3.12}$$

- For a 2-dimensional stress system, the relation for shear strain energy reduces to

$$\frac{1}{12G}[(\sigma_1-\sigma_2)^2+(\sigma_2)^2+(-\sigma_1)^2]=\frac{1}{6G}(\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2) \tag{3.13}$$

3.5 STRAIN ENERGY DUE TO BENDING AND TORSION

Strain energy due to bending in beams, $U = \int \frac{M^2 \cdot dx}{2EI}$ (3.14)

(Refer Section 7.7)

Strain energy due to torsion in shafts, $U = \frac{\tau^2}{4G} \times \text{Volume}$ (3.15)

(Refer Section 10.6)

3.6 STRAIN ENERGY OF BARS OF TAPERING SECTION

Bars of tapering section can be of conical section or of trapezoidal section with uniform thickness.

Conical Section

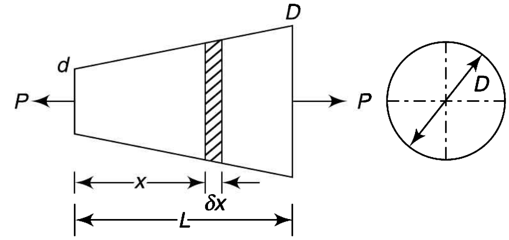


Fig. 3.6

Consider a bar of conical section under the action of axial force P as shown in Fig. 3.6

Let

D = diameter at the larger end

d = diameter at the smaller end

L = length of the bar

E = Young's modulus of the bar material

Consider a very small length δx at a distance x from the small end.

The diameter at a distance x from the small end $= d + \dfrac{D-d}{L} \cdot x$

Strain energy of the small length $= \dfrac{P^2 \cdot \delta x}{2A_x E}$

Strain energy of the whole rod $= \displaystyle\int_0^L \frac{P^2}{2\left(\frac{\pi}{4}\right)\left(d + \frac{D-d}{L}\cdot x\right)^2 \cdot E} \cdot dx$

$$= \frac{2P^2}{\pi E}\int_0^L \left(d + \frac{D-d}{L}\cdot x\right)^{-2} . dx = -\frac{2P^2}{\pi E}\cdot\frac{L}{(D-d)}\left(\frac{1}{d + \frac{D-d}{L}\cdot x}\right)_0^L$$

$$= \frac{2P^2 L}{\pi E(D-d)}\left(\frac{1}{d} - \frac{1}{D}\right) = \frac{2P^2 L}{\pi E(D-d)}\left(\frac{D-d}{dD}\right) = \frac{2P^2 L}{\pi EdD} \quad (3.16)$$

Trapezoidal Section of Uniform Thickness

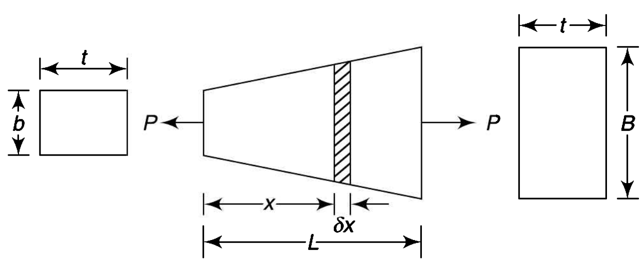


Fig. 3.7

Let

B = width at the larger end

b = width at the smaller end

t = thickness of the section
L = length of the bar
E = Young's modulus of the bar material

Consider a very small length δx at a distance x from the small end of the rod (Fig.3.7).

The width at a distance x from the small end $= b + \dfrac{B-b}{L} \cdot x = b + kx$ ....[Taking $k = (B-b)/L$]

The area of cross-section at this distance $= (b + kx) \cdot t$

$$\text{Strain energy of a small length} = \frac{P^2 \cdot \delta x}{2(b+kx)t \cdot E}$$

$$\text{Strain energy of the whole rod} = \int_0^L \frac{P^2}{2(b+kx)t \cdot E} \cdot dx = \frac{P}{2tE}\int_0^L \frac{P^2}{(b+kx)} \cdot dx$$

$$= \frac{P^2}{2tE}\frac{1}{k}[\log_e(b+kx)]_0^L = \frac{P^2}{2ktE}\left(\log_e \frac{b+kL}{b}\right) = \frac{P^2}{2ktE}\log_e \frac{B}{b} \tag{3.17}$$

$$\ldots\ldots\left(b + kL = b + \frac{B-b}{L} \cdot L = B\right)$$

Example 3.1 A 1.5-m long steel bar has a cross-sectional area of 800 mm^2. With an elastic limit of 180 MPa, determine its proof resilience. Take E = 205 GPa.

Solution

Given

σ = 180 MPa	L = 1.5 m = 1500 mm
a = 800 mm^2	E =205 GPa = 205 000 MPa

To find

— Proof resilience

As mentioned earlier, the distinction between *strain energy* and *resilience* is hardly followed and each is referred in place of the other, i.e., it may be expressed as total strain energy or strain energy per unit volume at the elastic limit.

Determination of proof resilience

$$\text{Proof resilience} = \frac{\sigma^2}{2E} \times \text{volume}$$

$$= \frac{180^2}{2 \times 205\,000} \times 800 \times 1500$$

$$= 94\,830 \text{ N.mm} \quad \text{or} \quad 94.83 \text{ N.m}$$

or $$\text{proof resilience} = \frac{180^2}{2 \times 205\,000} = 0.079 \text{ N.mm/mm}^3$$

Example 3.2 || A steel bar is acted upon by forces as shown in Fig. 3.8. Determine the strain energy stored in the bar if A is the area of cross-section of the bar and E is the modulus of elasticity.

Solution

Given A steel bar acted upon by forces as shown in Fig. 3.8.

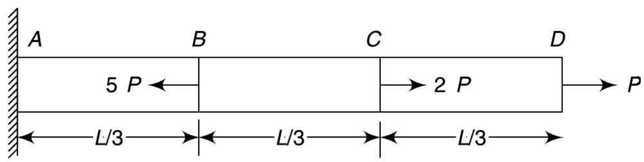


Fig. 3.8

To find Strain energy

Forces in various segments

Segment CD At Section D, the load is P tensile and for force equilibrium of this segment, at C also it is to be P tensile.

Segment BC

Force at Section $C = P$ tensile (from above) + 2 P (tensile)
= 3 P (tensile)
= Force at Section B

Segment AB

Force at Section B = 3 P tensile (from above) – 5 P (compressive)
= – 2 P (compressive)
= Force at Section A

Total strain energy

$$= \sum \frac{P^2 L}{2AE} = \frac{L/3}{2AE}[P^2 + (3P)^2 + (-2P)^2] = \frac{7P^2 L}{3AE}$$

Example 3.3 || Two elastic bars are of equal length and of the same material; one is of circular cross-section of 80-mm diameter and the other of square cross-section of 80 mm side. Both absorb the same amount of strain energy under axial forces. Compare the stresses in the two bars.

Solution

Given Two elastic bars of same length and material, i.e., L and E are same.
First bar: circular of 80-mm diameter; *Second:* Square cross-section of 80-mm side
Equal strain energy under axial forces
To find Ratio of stresses in the two cases

Strain energies of bars

Strain energy of the first bar $= \dfrac{\sigma_1{}^2}{2E} \cdot AL = \dfrac{\sigma_1{}^2}{2E} \cdot \dfrac{\pi}{4} \times 80^2 \times L$

Strain energy of the second bar $= \dfrac{\sigma_2^2}{2E} \times 80^2 \times L$

Equating the strain energies

$$\frac{\sigma_1^2}{2E}\cdot\frac{\pi}{4}\times 80^2\times L=\frac{\sigma_2^2}{2E}\times 80^2\times L$$

or
$$\frac{\sigma_1}{\sigma_2}=\sqrt{\frac{4}{\pi}}=1.128$$

Example 3.4 Derive an expression for strain energy of a prismatic bar under its own weight.

Solution

Given A prismatic bar
To find Strain energy under its own weight

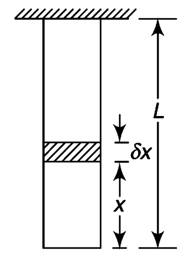


Fig. 3.9

Consider a bar hanging freely under its own weight as shown in Fig. 3.9. Consider a small length δx of the bar at a distance x from the free end.

Strain energy of elementary length

Let A = area of cross-section of the bar
w = weight per unit volume of the bar
W_x = weight of the bar below the small section = wAx

Force applied on the elementary length is the weight of the portion of the bar below it.

$$\text{Strain energy of the element} = \frac{(wAx)^2 dx}{2AE}$$

Total strain energy

$$\text{Total strain energy} = \int_0^L \frac{(wAx)^2 dx}{2AE}=\frac{w^2A}{2E}\int_0^L x^2 dx=\frac{w^2AL^3}{6E}$$

Example 3.5 Show that the ratio of strain energies of two circular bars of equal length, one made of aluminium and the other of steel subjected to same axial tensile load is given by $U_a/U_s = (d_s/d_a)^2 \cdot E_s/E_a$, where d_s and d_a are the diameters and E_s and E_a are the young's moduli of the steel and aluminium respectively.

Solution

Given Two circular bars of equal length but of different materials and subjected to same axial load
To find Ratio of strain energies of the two bars

Strain energies of bars

Let P be the load and L be the length of the two bars.

$$\text{Strain energy of aluminium bar, } U_a = \frac{P^2L}{2A_aE_a}$$

$$\text{Strain energy of steel bar, } U_s = \frac{P^2L}{2A_sE_s}$$

Ratio of strain energies

$$\therefore \quad \frac{U_a}{U_s}=\frac{A_sE_s}{A_aE_a}=\left(\frac{d_s}{d_a}\right)^2\cdot\frac{E_s}{E_a}$$

Example 3.6 Compare the strain energies of the bars shown in Fig. 3.10 (ii), (iii), (iv) and (v) with the strain energy of bar (i) for a constant load P in all cases. Smaller cross-sectional areas are half of the larger cross-sectional areas.

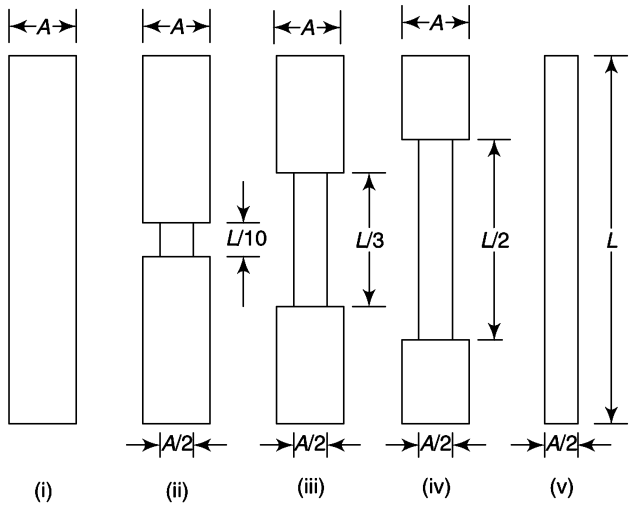


Fig. 3.10

Solution

Given Bars of uniform and non-uniform cross-sectional areas
To find Ratio of strain energies for a constant load

Strain energies of bars (i) and (ii)

Let A = area of larger cross-section
a = area of smaller cross-section
L = total length of bar
L_A = length of section with larger area
L_a = length of section with smaller area

Thus

As the load is the same in all cases,

Stress in the larger section, $\sigma = P/A$

Stress in the smaller section, $\sigma_1 = \frac{P}{a} = \frac{P/A}{a/A} = \frac{\sigma}{1/2} = 2\sigma$

Strain energy of bar (i), $U_1 = \frac{\sigma^2}{2E} \cdot AL$

Strain energy of non-uniform section of bar of Fig. 3.10 (ii),

U_2 = Strain energy of larger section + strain energy of smaller section

$$= \frac{\sigma^2}{2E} \cdot A \cdot L_A + \frac{(2\sigma)^2}{2E} \cdot a \cdot L_a$$

Ratio of strain energies

$$\text{Ratio of (ii) and (i)} = \frac{\frac{\sigma^2}{2E} \cdot A \cdot L_A + \frac{(2\sigma)^2}{2E} \cdot a \cdot L_a}{\frac{\sigma^2}{2E} \cdot AL} = \frac{2aL_A + 4a \cdot L_a}{2a \cdot L} = \frac{L_A + 2L_a}{L}$$

Thus, $U_2/U_1 = 0.9 + 2 \times 0.1 = 1.1$

Similarly,

$$U_3/U_1 = 0.667 + 2 \times 0.333 = 1.33$$
$$U_4/U_1 = 0.5 + 2 \times 00.5 = 1.5$$
$$U_5/U_1 = 0 + 2 \times 1 = 2$$

It can be observed that a smaller diameter of the bar gives higher strain energy than a larger diameter bar.

Example 3.7 Compare the strain energies of the bars shown in Fig. 3.10 (ii), (iii), (iv) and (v) with the strain energy of bar, (i) when the bars are subjected to maximum permissible stress.

Solution

Given Bars of uniform and non-uniform cross-sectional areas
To find Ratio of strain energies for same maximum permissible stress

Strain energies of bars (i) and (ii)

In this case, the maximum value of stress in all the cases is σ which means that in the first case the maximum stress is uniform, whereas in other cases the maximum stress is in the sections with smaller cross-sections.

Bar of Fig. 3.10 (*i*) Stress in the uniform section is σ.

Strain energy, $U_1 = \dfrac{\sigma^2}{2E} \times AL$

Bar of Fig. 3.10 (*ii*)

Stress in the smaller section = maximum permissible stress = σ

Stress in the larger section = $\sigma \times \dfrac{a}{A} = \dfrac{\sigma}{2}$

Strain energy of non-uniform section of Fig. 3.10 (ii),

U_2 = Strain energy of larger section + strain energy of smaller section

$$= \frac{(\sigma/2)^2}{2E} \cdot A \cdot L_A + \frac{\sigma^2}{2E} \cdot a \cdot L_a$$

Ratio of strain energies

$$\text{Ratio} = \frac{\dfrac{(\sigma/2)^2}{2E} \cdot AL_A + \dfrac{\sigma^2}{2E} \cdot a \cdot L_a}{\dfrac{\sigma^2}{2E} \cdot AL} = \frac{\dfrac{2aL_A}{4} + a \cdot L_a}{2aL}$$

$$= \frac{1}{4L}(L_A + 2L_a) = \frac{1}{4}(0.9 + 2 \times .1) = 0.275$$

Similarly for other bars,

$$\frac{U_3}{U_1} = \frac{1}{4}[0.667 + 2X0.333] = 0.333$$

$$\frac{U_4}{U_1} = \frac{1}{4}[0.5 + 2 \times 0.5] = 0.375$$

$$\frac{U_5}{U_1} = \frac{1}{4}[0 + 2 \times 1] = 0.5$$

Example 3.8 Compare the strain energies per unit volume of the bars shown in Fig. 3.10 (ii), (iii), (iv) and (v) with the strain energy per unit volume of bar (i) when the bars are subjected to maximum permissible stress.

Solution

Given Bars of uniform and non-uniform cross-sectional areas

To find Ratio of strain energies per unit volume for same maximum permissible stress

Strain energies of bars (i) and (ii)

Stress in the uniform section (i) = maximum permissible stress = σ

$$\text{Strain energy per unit volume, } U_1 = \frac{\sigma^2}{2E}$$

For non-uniform section of Fig. 3.10 (ii),

Stress in the smaller section = maximum permissible stress = σ

$$\text{Stress in the larger section} = \sigma \cdot \frac{a}{A} = \frac{\sigma}{2}$$

$$\text{Volume} = A \cdot L_A + aL_a$$

Strain energy of non-uniform section of Fig. 3.10 (ii)

$$= \frac{(\sigma/2)^2}{2E} \cdot A \cdot L_A + \frac{\sigma^2}{2E} \cdot a \cdot L_a$$

Strain energy per unit volume of non-uniform section,

$$U_2 = \frac{\frac{(\sigma/2)^2}{2E} \cdot A \cdot L_A + \frac{\sigma^2}{2E} \cdot a \cdot L_a}{A.L_A + aL_a} \frac{\sigma^2}{2E}\left(\frac{\frac{1}{4} \cdot 2a \cdot L_A + a \cdot L_a}{2a \cdot L_A + aL_a}\right) = \frac{\sigma^2}{2E}\left(\frac{L_A/2 + L_a}{2L_A + L_a}\right)$$

Ratio of strain energies

$$\text{Ratio, } \frac{U_2}{U_1} = \frac{\frac{\sigma^2}{2E}\left(\frac{L_A/2 + L_a}{2L_A + L_a}\right)}{\frac{\sigma^2}{2E}} = \frac{L_A/2 + L_a}{2L_A + L_a} = \frac{0.9/2 + 0.1}{2 \times 0.9 + 0.1} = 0.289$$

Similarly,

$$\frac{U_3}{U_1} = \frac{0.6667/2 + 0.3333}{2 \times 0.6667 + 0.3333} = 0.4$$

$$\frac{U_4}{U_1} = \frac{0.5/2 + 0.5}{2 \times 0.5 + 0.5} = 0.5$$

$$\frac{U_5}{U_1} = \frac{0/2 + 1}{2 \times 0 + 1} = 1$$

Example 3.9 The cross-sections of two bars *A* and *B* made up of the same material and each 320 mm long are as follows:

— *Bar A*: 24-mm diameter for a length of 80 mm and 48 mm for the remaining 240 mm

— *Bar B*: 24-mm diameter for a length of 240 mm and 48 mm for the remaining 80 mm

An axial blow to bar *A* produces a maximum instantaneous stress of 160 MPa. Determine

(i) the maximum instantaneous stress produced by the same blow to bar *B*

(ii) the ratio of energies stored by the two bars when subjected to maximum permissible stress
(iii) the ratio of energies per unit volume of the two bars when subjected to maximum permissible stress

Solution

(Refer Fig. 3.11)

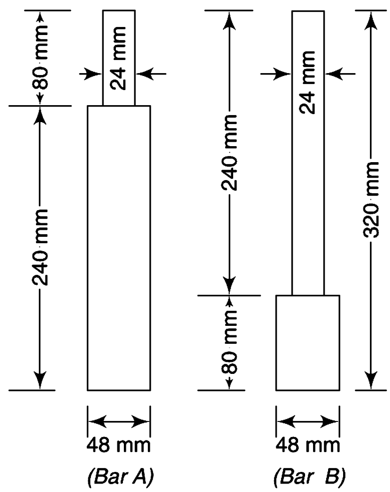


Fig. 3.11

Given Two bars of non-uniform cross-sectional areas; an axial blow to bar A produces a maximum instantaneous stress of 160 MPa.

To find

— Maximum instantaneous stress produced by the same blow to bar B
— Ratio of energies for maximum permissible stress
— Ratio of energies per unit volume for maximum permissible stress

The larger diameter of the bars = 2 × diameter of the bars

∴ larger areas of cross-sections = 4 × smaller area of cross-sections

For the same blow to Bar B, the strain energy produced by the blow should equal to that produced by the blow to the first bar.

In Bar A,

Maximum instantaneous stress in the smaller cross-section = 160 MPa

Maximum instantaneous stress in the larger cross-section = 160/4 = 40 MPa

In Bar B,

Let maximum instantaneous stress in the smaller cross-section = σ

Then maximum instantaneous stress in the larger cross-section = $\sigma/4$

Equating strain energies of bars

Strain energy of Bar A = strain energy of Bar B

$$\frac{40^2}{2E}\times\frac{\pi}{4}(48)^2\times 240+\frac{160^2}{2E}\times\frac{\pi}{4}(24)^2\times 80$$

$$=\frac{(\sigma/4)^2}{2E}\times\frac{\pi}{4}(48)^2\times 80+\frac{\sigma^2}{2E}\times\frac{\pi}{4}(24)^2\times 240$$

Dividing throughout by $\dfrac{\pi}{4} \times 24^2 \times \dfrac{80}{2E}$,

$$40^2(12+16) = \sigma^2\left(\frac{1}{4}+3\right) \quad \text{or} \quad \sigma = 117.4 \text{ MPa}$$

Ratio of strain energies

Let maximum stress in the smaller cross-section = σ

Then stress in the larger cross-section = $\sigma/4$

$$\frac{U_B}{U_A} = \frac{\dfrac{(\sigma/4)^2}{2E} \times \dfrac{\pi}{4}(48)^2 \times 80 + \dfrac{\sigma^2}{2E} \times \dfrac{\pi}{4}(24)^2 \times 240}{\dfrac{(\sigma/4)^2}{2E} \times \dfrac{\pi}{4}(48)^2 \times 240 + \dfrac{\sigma^2}{2E} \times \dfrac{\pi}{4}(24)^2 \times 80}$$

Dividing throughout by $\dfrac{\sigma^2}{2E} \times \dfrac{\pi}{4}(24)^2 \times 80$

$$\frac{U_B}{U_A} = \frac{(1/16) \times 2^2 + 3}{(1/16)2^2 \times 3 + 1} = 1.857$$

Ratio of strain energies per unit volume

The ratio of strain energies per unit volume of two bars for the same maximum stress in the smaller cross-sections,

$$\text{Volume of Bar } A = \frac{\pi}{4}(48)^2 \times 240 + \frac{\pi}{4}(24)^2 \times 80 \text{ mm}^3;$$

$$\text{Volume of Bar } B = \frac{\pi}{4}(48)^2 \times 80 + \frac{\pi}{4}(24)^2 \times 240 \text{ mm}^3;$$

$$\frac{U_B}{U_A} = \frac{\left[\dfrac{(\sigma/4)^2}{2E} \times \dfrac{\pi}{4}(48)^2 \times 80 + \dfrac{\sigma^2}{2E} \times \dfrac{\pi}{4}(24)^2 \times 240\right] \dfrac{4/\pi}{48^2 \times 80 + 24^2 \times 240}}{\left[\dfrac{(\sigma/4)^2}{2E} \times \dfrac{\pi}{4}(48)^2 \times 240 + \dfrac{\sigma^2}{2E} \times \dfrac{\pi}{4}(24)^2 \times 80\right] \dfrac{4/\pi}{48^2 \times 240 + 24^2 \times 80}}$$

$$= 1.857 \times \frac{24^2 \times 80(2^2 \times 3 + 1)}{24^2 \times 80(2^2 + 3)} = 1.857 \times 1.857 = 3.448$$

Example 3.10 || A steel rope lowers a load of 9.5 kN with a uniform velocity of 750 mm/s. When the length of the rope unwound is 8 m, it suddenly gets jammed and the load is brought to a halt. Determine the stress developed in the rope due to sudden stoppage and the maximum instantaneous elongation if the diameter of the rope is 20 mm. E_s = 205 GPa.

Solution

Given A steel rope lowering a load gets jammed

W = 9500 N $\quad d$ = 20 mm

v = 750 mm/s $\quad L$ = 8 m

E_s = 205 GPa.

To find

— Stress in the rope due to sudden stoppage

— Maximum instantaneous elongation

$$A = \frac{\pi}{4}.(20)^2 = 100\pi \text{ mm}^2$$

Kinetic energy

$$\text{K.E. of the load} = \frac{1}{2}mv^2 = \frac{Wv^2}{2g} = \frac{9\,500 \times 750^2}{2 \times 9810} = 272\,362 \text{ N.mm}$$

When the rope is suddenly gets jammed, its kinetic energy is converted into strain energy.

Strain energy

Let the stress developed in the rope due to sudden stoppage be σ.

$$\text{Then strain energy gained by the rope} = \frac{\sigma^2}{2E}AL$$

$$= \frac{\sigma^2}{2 \times 205\,000} \times 100\pi \times 8000 = 6.13\,\sigma^2 \text{ N.m}$$

Equating the energies

$$6.13\,\sigma^2 = 272\,362 \qquad \text{or} \qquad \sigma = 210.8 \text{ MPa}$$

$$\text{Maximum instantaneous elongation} = \frac{\sigma L}{E} = \frac{210.8 \times 8000}{205\,000} = 8.226 \text{ mm}$$

3.7 STRESSES DUE TO VARIOUS TYPES OF LOADING

A loading may be gradual, sudden or by impact (shock). The case of gradual loading has already been discussed in section 3.1.

Suddenly Applied Load

When a load is applied suddenly to a bar, it may be assumed that the load W is the same throughout the elongation.

Strain energy stored in the material = Work done by the load = $W \cdot \Delta$

Let P be the gradually applied load which produces the same elongation.

$$\text{Then, work done or the strain energy} = \frac{1}{2}P\Delta$$

$$\text{Equating the two, } W \cdot \Delta = \frac{1}{2} \cdot P \cdot \Delta \qquad \text{or} \qquad W = \frac{1}{2}P \tag{3.18}$$

Thus the magnitude of a suddenly applied load required to produce the same elongation is half that of gradually applied load.

Also, if σ is the stress produced by the gradually applied load,

$$W = \frac{1}{2} \cdot (\sigma A) \qquad \text{or} \qquad \sigma = \frac{2W}{A}$$

i.e., the maximum stress induced in the body is twice the stress induced by the load of the same magnitude applied gradually.

Impact or Shock Load

Let a load W drop through a height h before it commences to deform the body. After falling on the collar of the body as shown in Fig. 3.12, the collar oscillates for some time and finally the load settles at the same position as it would have by the gradually applied load, provided the limit of proportionality is not exceeded. Of course, the instantaneous extension and the stress are much greater than the steady state values.

Work done (potential energy loss) by the load = $W(h + \Delta)$

Let σ be the stress due to a gradually applied load P that causes the same deflection Δ.

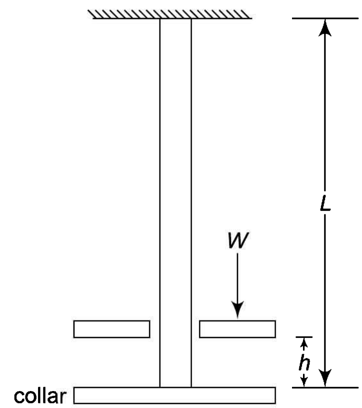


Fig. 3.12

Then, strain energy stored in the material = $\dfrac{\sigma^2}{2E} \cdot AL$

Equating the two, $W(h + \Delta) = \dfrac{\sigma^2}{2E} \cdot AL \cdot AL$ or $W\left(h + \dfrac{\sigma L}{E}\right) = \dfrac{\sigma^2}{2E} \cdot AL$

Multiplying throughout by $2E/AL$ and rearranging,

$$\sigma^2 - \left(\frac{2W}{A}\right)\sigma - \frac{2WEh}{AL} = 0$$

This is a quadratic in σ and the solution is given by,

$$\sigma = \frac{W}{A} \pm \frac{1}{2}\sqrt{\left(\frac{2W}{A}\right)^2 + 4 \times \frac{2WEh}{AL}} = \frac{W}{A}\left(1 + \sqrt{1 + \frac{2AEh}{WL}}\right) \tag{3.19}$$

If $h = 0$, $\sigma = 2W/A$, i.e., the case of a suddenly applied load.

Example 3.11 || A 1-m long bar of rectangular cross-section 50 mm × 80 mm is subjected to an axial load of 1.2 kN. Determine the maximum stress and the strain energy developed in the bar if the load applied (i) is gradual, (ii) is sudden, and (iii) falls through a height of 25 mm. E = 205 GPa.

Solution

Given A rectangular cross-sectional bar subjected to an axial load

P = 1200 N $\qquad$ h = 25 mm

L = 1 m $\qquad$ E = 205 GPa

To find Maximum stress and strain energy when load applied is sudden, gradual and falls through a height of 25 mm

$$A = 50 \times 80 = 4000 \text{ mm}^2$$

Load applied is gradual

$$\sigma = \frac{P}{A} = \frac{1200}{4000} = 0.3 \text{ MPa}$$

$$\text{Strain energy} = \frac{\sigma^2}{2E} \times AL = \frac{\sigma^2}{2 \times 205\,000} \times 4000 \times 1000$$

$$= 9.755\sigma^2 = 9.755 \times 0.3^2 = 0.878 \text{ N.mm}$$

Load applied is sudden

$$\sigma = \frac{2P}{A} = \frac{2 \times 1200}{4000} = 0.6 \text{ MPa}$$

Strain energy = 9.755 × 0.6^2 = 3.512 N.mm

Load falls through a height

When the load falls through a height of 60 mm,

$$\sigma = \frac{W}{A}\left(1 + \sqrt{1 + \frac{2AEh}{WL}}\right)$$

$$= \frac{1200}{4000}\left(1 + \sqrt{1 + \frac{2 \times 4000 \times 205\,000 \times 25}{1200 \times 1000}}\right) = 0.3(1 + 184.8) = 55.7 \text{ MPa}$$

Note that with only a 60-mm drop, the maximum stress is almost 180 times more than that for static stress.

Strain energy = 9.755 × 55.7^2 = 30 265 N.mm or 30.265 N.m

Example 3.12 An unknown weight falls through a height of 20 mm on a collar rigidly fixed to the lower end of a vertical bar that is 3 m long and of 30 mm in diameter. The maximum instantaneous extension of the bar is measured to be 3.2 mm. Determine the amount of the unknown weight and the corresponding stress induced. E = 205 GPa.

Solution

Given An unknown weight falling through a height

L = 3000 m $\qquad \Delta$ = 3.2 mm

h = 20 mm $\qquad E$ = 205 000 MPa

d = 30 mm

To find Weight and stress

$$\text{Maximum stress induced} = E \cdot \varepsilon = E \cdot \frac{\Delta}{L} = 205\,000 \times \frac{3.2}{3000} = 218.7 \text{ MPa}$$

Equating the work done and strain energy

$$W(h + \Delta) = \frac{\sigma^2}{2E} \times AL$$

or

$$W(20 + 32) = \frac{218.7^2}{2 \times 205\,000} \times \frac{\pi}{4} \times 30^2 \times 3000$$

$$W = 10\,660 \text{ N} \quad \text{or} \quad 10.66 \text{ kN}$$

Example 3.13 A 1.2-kN of weight falls through a height of 20 mm on a rigid collar attached at the bottom end of a vertically suspended 2.5-m long steel tie bar. It is desired that the ratio of the instantaneous extension of the bar to its length should not exceed 1/2000. Determine the minimum cross-sectional area of the bar. Take E =200 GPa.

Solution

Given A known weight falling through a height

L = 2500 m $\qquad \Delta/L$ = 1/2000

h = 20 mm $\qquad E$ = 200 GPa

W = 1200 N

To find Minimum cross-sectional area

$$\text{Maximum stress induced} = E \times \varepsilon = E \times \frac{\Delta}{L} = 200\,000 \times \frac{1}{2500} = 80 \text{ MPa}$$

$$\text{Instantaneous extension} = \frac{1}{2000} \times 2500 = 1.25 \text{ mm}$$

Equating the work done and strain energy

$$W(h+\Delta) = \frac{\sigma^2}{2E} \times AL$$

or
$$1200(20 + 1.25) = \frac{80^2}{2 \times 200\,000} \times A \times 2500$$

or
$$A = 637.5 \text{ mm}^2$$

Example 3.14 A bar, 3.2 m long and 16 mm in diameter, hangs vertically and has a collar attached at the lower end. Determine the maximum stress induced when a weight of 80 kg falls from a height of 32 mm on the collar.

If the bar is turned down to half the diameter along half of its length, what will be the value of the maximum stress and the extension? E =205 GPa.

Solution

Given A known weight falling through a height

h =32 mm L = 3.2 m

E = 205 GPa d = 16 mm

W = 80 × 9.81 = 784.8 N

To find

— Maximum stress induced

— Maximum stress and extension when half the length is turned down to half the diameter.

$$A = (\pi/4)16^2 = 64\,\pi$$

Maximum stress

$$\sigma = \frac{W}{A}\left(1+\sqrt{1+\frac{2AEh}{WL}}\right) = \frac{784.8}{64\pi}\left(1+\sqrt{1+\frac{2\times 64\pi\times 205\,000\times 32}{784.8\times 3200}}\right)$$

$$= 3.903(1 + 32.4) = 130.4 \text{ MPa}$$

Note that with only 32-mm drop the maximum stress is 32.4 times more than that for static stress.

When bar is turned down

Diameter of the reduced section = 16/2 = 8 mm

Area of the reduced section = $(\pi/4).8^2 = 16\,\pi$

When the bar is turned down to half the diameter along half of its length, let P be the equivalent load to induce the same maximum stress.

∴
$$\Delta = \frac{P\times 1600}{16\pi\times 205\,000} + \frac{P\times 1600}{64\pi\times 205\,000}$$

$$= 0.000\,194\,P = kP \qquad \text{..... (Taking } k = 0.000\,194)$$

Using the energy equation

$$W(h+\Delta) = \frac{1}{2}P\cdot\Delta$$

$$W(h+kP) = \frac{1}{2}P\cdot kP$$

$$P^2 - 2WP - 2Wh/k = 0 \qquad \text{(multiplying throughout by } 2/k)$$

Solving,

$$P = \frac{2W + \sqrt{(-2W)^2 - 4(-2Wh/k)}}{2} = W + \sqrt{W^2 + \frac{2Wh}{k}}$$

$$= 784.8 + \sqrt{784.8^2 + \frac{2 \times 784.8 \times 32}{0.000\,194}}$$

$$= 784.8 + \sqrt{784.8^2 + 258.9 \times 10^6}$$

$$= 784.8 + 16109.2$$

$$= 16\,894 \text{ N}$$

The maximum stress (in the smaller section) = 16 894/16π = 336 MPa

The maximum extension = kp = 0.000 194 × 16 894 =3.28 mm

Example 3.15 A lift is operated by three ropes each having 28 wires of 1.4-mm diameter. The cage weighs 1.2 kN and the weight of the rope is 4.2 N/m length. Determine the maximum load carried by the lift if each wire is 36 m long and the lift operates (i) without any drop of load, and (ii) with a drop of 96 mm of the load during operations.

E_{rope} = 72 GPa and allowable stress = 115 MPa

Solution

Given A lift operated by three ropes

W = 1200 N	w = 4.2 N/m
n = 28	d = 1.4 mm
L = 36 m	σ_{max} = 115 MPa
h = 96 mm	E_{rope} = 72 GPa

To find Maximum load carried by lift if

— there is no drop

— a drop of 96 mm

Total area of cross-section, $A = \frac{\pi}{4}(1.4)^2 \times 3 \times 28 = 129.3 \text{ mm}^2$

Calculation of equivalent static load to be carried

The maximum stress occurs at the top of the wire rope where the maximum weight of the rope acts.

Maximum load = weight of cage + weight of rope

= 1200 + 3 × 36 × 4.2 = 1653.6 N

Initial stress in the rope, $\sigma = \frac{1653.6}{129.3} = 12.8$ MPa

Equivalent static stress available for carrying the load = 115 – 12.8 = 102.2 MPa

Thus, equivalent static load that can be carried,

$$P_e = 102.2 \times 129.3 = 13\,214 \text{ N}$$

The extension of the rope, $\Delta = \frac{\sigma \cdot L}{E} = \frac{102.2 \times 36\,000}{72\,000} = 51.1$ mm

With no drop

Let W be the weight which can be applied suddenly, $W \cdot \Delta = \frac{1}{2} P \cdot \Delta$

or W = 13 214/2 = 6607 N or 6.607 kN

With 96-mm drop

Let W be the weight,

$$W(h+\Delta) = \frac{1}{2} P \cdot \Delta$$

or
$$W(96+51.1) = \frac{1}{2} \times 13\,214 \times 51.1$$

or
$$W = 2295 \text{ N} \quad \text{or} \quad 2.295 \text{ kN}$$

Example 3.16 A vertical composite tie bar rigidly fixed at the upper end consists of a steel rod of 16-mm diameter enclosed in a brass tube of 16-mm internal diameter and 24-mm external diameter, each being 2-m long. Both are fixed together at the ends. The tie bar is suddenly loaded by a weight of 8 kN falling through a distance of 4 mm. Determine the maximum stresses in the steel rod and the brass tube.

E_s = 205 GPa and E_b = 100 GPa

Solution

Given A tie bar rigidly fixed at the upper end as shown in Fig. 3.13.

d_s = 16 mm $\quad$ d_{bi} = 16 mm

d_{bo} = 24 mm $\quad$ L = 2000 mm

W = 8000 N $\quad$ h = 4 mm

E_s = 205 GPa $\quad$ E_b = 100 GPa

To find Maximum stresses in the steel rod and the brass tube

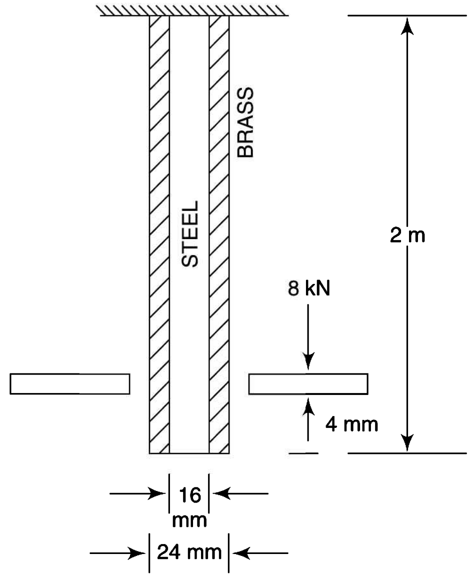


Fig. 3.13

$$A_s = (\pi/4)16^2 = 64\,\pi;\ A_b = (\pi/4)(24^2 - 16^2) = 80\,\pi$$

Let x = Extension of bar in mm

$$\sigma_s = \frac{E_s \cdot x}{L} \text{ and } \sigma_b = \sigma_b = \frac{E_b \cdot x}{L}$$

Strain energy

$$\text{Strain energy of the bar} = \frac{\sigma_s^2}{2E_s} A_s L + \frac{\sigma_b^2}{2E_b} A_b L$$

$$= \frac{E_s^2 x^2}{L^2 \cdot 2E_s} A_s L + \frac{E_b^2 x^2}{L^2 \cdot 2E_b} A_b L$$

$$= \frac{E_s x^2}{2L} A_s + \frac{E_b x^2}{2L} A_b = \frac{x^2}{2L}(E_s A_s + E_b A_b)$$

$$= \frac{x^2}{2 \times 2000}(205\,000 \times 64\pi + 100\,000 \times 80\pi) = 16\,588\, x^2 \text{ N} \cdot \text{mm}$$

Potential energy

Potential energy lost by the weight = $W(h + x)$ = 8000 (4 + x) N.mm

Equating strain and potential energies

$$16587\,x^2 = 8000\,(4 + x)$$

or
$$x^2 - 0.4823x - 1.9292 = 0$$

or
$$x = \frac{1}{2}(0.4823 + \sqrt{0.4823^2 + 4 \times 1.9292}) = 1.6508 \text{ mm}$$

$$\sigma_s = \frac{E_s \cdot x}{L} = \frac{205\,000 \times 1.6505}{2000} = 169.2 \text{ MPa}$$

$$\sigma_b = \frac{E_b \cdot x}{L} = \frac{100\,000 \times 1.6505}{2000} = 82.54 \text{ MPa}$$

Summary

1. The work done by the load in straining the material of a body is stored within it in the form of energy known as *strain energy*.
2. Resilience is defined as the strain energy per unit volume in simple tension or compression and is equal to $\sigma^2/2E$.
3. Proof resilience is the value of resilience at the elastic limit or at proof stress. It is also known as the *modulus of resilience*.
4. Modulus of toughness is the strain energy per unit volume required to cause the material to rupture. It is the area under the stress–strain diagram up to rupture point. Strain energy, $U = \frac{1}{2} \cdot P \cdot \Delta = \frac{1}{2} \times$ stress $\times$ strain $\times$ volume $= \frac{\sigma^2}{2E} \times$ volume
5. Strain energy per unit volume (3-D),
$$U = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$$
6. Shear strain energy per unit volume $= \frac{\tau^2}{2G}$
7. Shear strain energy per unit volume (3-D)
$$U = \frac{1}{12G}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$
8. Maximum stress induced in a body due to suddenly applied load is twice the stress induced by the load of the same magnitude applied gradually.
9. In case of impact or shock loading, $\sigma = \frac{W}{A}\left(1 + \sqrt{1 + \frac{2AEh}{WL}}\right)$

Objective Type Questions

1. The strain energy stored in a bar is given by

(a) $\dfrac{PL}{AE}$ (b) $\dfrac{PL^2}{2AE}$ (c) $\dfrac{P^2L}{AE}$ (d) $\dfrac{P^2L}{2AE}$

2. Strain energy of a member is given by

(a) $\dfrac{\sigma^2}{2E} \times$ volume (b) $\dfrac{P^2L}{2AE}$

(c) $\dfrac{1}{2} \times$ stress $\times$ strain $\times$ volume (d) all of these

3. Modulus of resilience is
(a) percentage of elongation of an elastic body
(b) strain energy stored in the elastic body
(c) strain energy per unit volume of the elastic body

4. Proof resilience is the maximum energy stored at
(a) plastic limit (b) limit of proportionality (c) elastic limit

5. Modulus of toughness is the area of the stress–strain diagram up to
(a) rupture point (b) yield point
(c) limit of proportionality (d) none of these

6. Shear strain energy per unit volume is given by

(a) $\dfrac{\tau^2}{4G}$ (b) $\dfrac{\tau^2}{2G}$ (c) $\dfrac{2\tau^2}{3G}$ (d) $\dfrac{\tau}{4G}$

7. Strain energy of a bar of conical section is

(a) $\dfrac{P^2L}{\pi EdD}$ (b) $\dfrac{2PL}{\pi EdD}$ (c) $\dfrac{2P^2L}{EdD}$ (d) $\dfrac{2P^2L}{\pi EdD}$

8. Strain energy stored in a body due to suddenly applied load compared to when applied slowly is

(a) twice (b) four times (c) eight times (d) half

Answers:
1. (d) 2. (d) 3. (c) 4. (c) 5. (a) 6. (b)
7. (d) 8. (a)

Review Questions

3.1 What is strain energy of a material? Derive the expressions for the same in different forms.
3.2 Define the terms: resilience, proof resilience, modulus of resilience.
3.3 Show that the proof resilience is the area under the stress-strain diagram upto elastic limit of a material.
3.4 Derive expressions for the strain energy in a three-dimensional stress system?
3.5 What is shear strain energy? Find its value per unit volume of the material.
3.6 Derive the relation for shear strain energy for a three-dimensional stress system.
3.7 What is the value of maximum stress induced in a body when the load is applied suddenly?
3.8 Deduce the relation for stress in case of impact and shock loading.

Numerical Problems

3.1 A steel bar 150-mm^2 of cross-section elongates 0.05 mm over a 50-mm gauge length under an axial load of 30 kN. Determine its strain energy. If the load at the elastic limit is 50 kN, find the proof resilience and the elongation. (750 N.mm, 2.083 N.m, 0.0833 mm)

3.2 A 1.6-m long bar is applied an axial pull such that the maximum stress induced is 140 MPa (Fig. 3.14). The larger and the smaller areas of cross-section are 240 mm^2 and 120 mm^2. Determine the strain energy stored in the bar. (5.737 N.m)

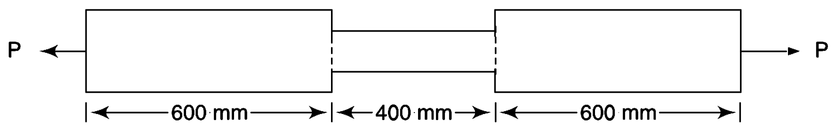


Fig. 3.20

3.3 A load of 22 kN is lowered by a steel rope at the rate of 750 mm/s. The diameter of the rope is 28 mm. When the length of the rope unwound is 12 m, the rope suddenly gets jammed. Find the instantaneous stress developed in the rope. Also, calculate the instantaneous elongation of the rope. E = 205 GPa. (187.1 MPa, 10.95 mm)

3.4 A weight of 2 kN falls 24 mm on to a collar fixed to a steel bar 14 mm in diameter and 5.5 m long. Determine the maximum stress induced in the bar. E_s = 205 GPa. (166 MPa)

3.5 A 100-N weight falls by gravity through a vertical distance of 5 m when it is suddenly stopped by a collar at the end of vertical rod of 20-mm diameter and 10-m length. The upper end of the bar is rigidly fixed. Calculate the strain induced in the bar due to impact. E = 200 GPa. (0.001 263)

3.6 A rectangular block is subjected to three mutually normal stresses of magnitude 60 MPa, 70 MPa and 80 MPa. Determine the strain energy and shear strain energy per unit volume. Take E = 200 GPa and Poisson's ratio 0.3. (0.0153 N.mm, 0.65×10^{-3} N.mm)

3.7 A weight of 800 N falls 30 mm on to a collar fixed to a steel bar of 1.2-m length. The steel bar is of 24 mm diameter for half of its length and 12 mm for the rest half. Determine the maximum stress and the extension in the bar. E_s = 205 GPa. (347.7 MPa; 1.272 mm)

3.8 A lift is operated by two ropes each 20 m long and consisting of 30 wires of 1.5 mm diameter. The weight of the cage is 1 kN and the rope weighs 3.6 N/m length. Determine the maximum load that the lift can carry if it drops through 120 mm during operations. E (rope) = 78 GPa and allowable stress = 125 MPa. (1.188 kN)

3.9 A vertical tie rod consists of a 3-m long steel rod and of 24-mm diameter encased throughout in a brass tube of 24-mm internal diameter and 36-mm external diameter. The rod is rigidly fixed at the top end. The composite tie rod is suddenly loaded by a weight of 13.5 kN falling freely through 6 mm before being stopped by the tie. Determine the maximum stresses in steel and the brass. E_s = 205 GPa and E_b = 98 GPa. (143.8 MPa; 68.76 MPa)

Chapter 4

Shear Force and Bending Moment

A structural element which is subjected to loads transverse to its axis is known as a *beam*. In general, a beam is either free from any axial force or its effect is negligible. Analysis of beams involves the determination of shear force, bending moment and the deflections at various sections. This chapter deals with the finding of shear force and bending moment at a section of different kinds of beams. Analysis of beams to find the deflections is dealt in a later chapter.

Usually, a beam is considered horizontal and the loads vertical. Other cases are considered as exceptions. A *concentrated* or *point load* is assumed to act at a point, though in practice it may be distributed over a small area. A *distributed load* is one which is spread over some length of the beam. The rate of loading may be uniform or may vary from point to point.

4.1 TYPES OF SUPPORTS AND BEAMS

A beam may have the following kinds of supports:

(i) Roller Support When a beam rests on a sliding surface such as a roller or any flat surface like a masonry wall, the support is known as a roller support. A roller support can sustain a force only normal to its surface as the possible movement on the supporting surface does not allow any resistance along that direction. Thus, it has a reaction normal to the surface only and the reaction along the rolling surface is zero. A roller support allows the rotation of the body. The conventional methods of showing a roller support are shown in Fig. 4.1a.

(ii) Hinged Support and Hinged Joint In a hinged support, no translational displacement of the beam is possible, however, it is free to rotate. A hinged support can sustain reactions in any direction, i.e., it can have components in vertical as well as horizontal directions. As a hinged support can rotate, in beams with *hinged* joints, the bending moment at the hinge is taken to be zero. Sometimes, beams are also provided with hinged joints. In such cases, there is also no reaction which is there in case of a hinged support.

The conventional methods of showing a hinged support and a hinged joint are shown in Fig. 4.1b.

(iii) Fixed or Encastre or Built-in Support A beam built into a rigid support which does not allow any type of movement or rotation is known as *fixed or encastre or built-in support* (Fig. 4.1c). A fixed support exerts a fixing moment and a reaction on the beam.

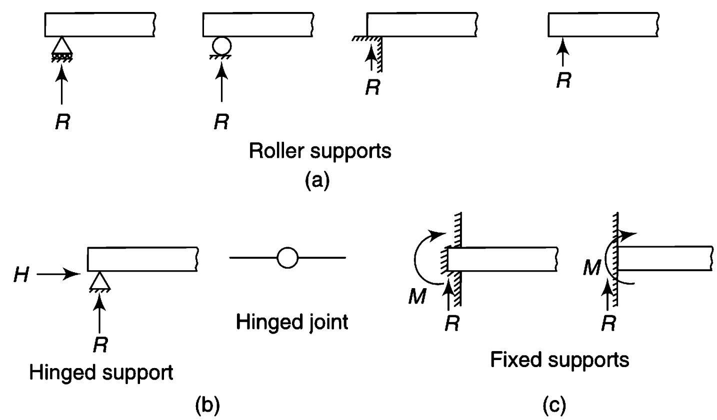


Fig. 4.1

Depending upon the type of supports, the beams are classified as follows:

(i) *Simply Supported Beam* When both the supports of beams are roller supports or one support is roller and the other hinged, the beam is known as a simply supported beam (Fig. 4.2a).

(ii) *Cantilever* A beam with one end fixed and the other end free is called a *cantilever* (Fig. 4.2b). There is a vertical reaction and moment at the fixed end (known as *fixing moment*).

(iii) A beam with both ends fixed is known as *fixed beam* (Fig. 4.2c).

(iv) Beams with one end fixed and the other simply supported are known as *propped cantilever* (Fig. 4.2d).

(v) Beams supported at more than two sections are known as *continuous beams* (Fig. 4.2e).

Generally beams with more than two reaction components cannot be analysed using the equations of static equilibrium alone and are known as *statically indeterminate beams*.

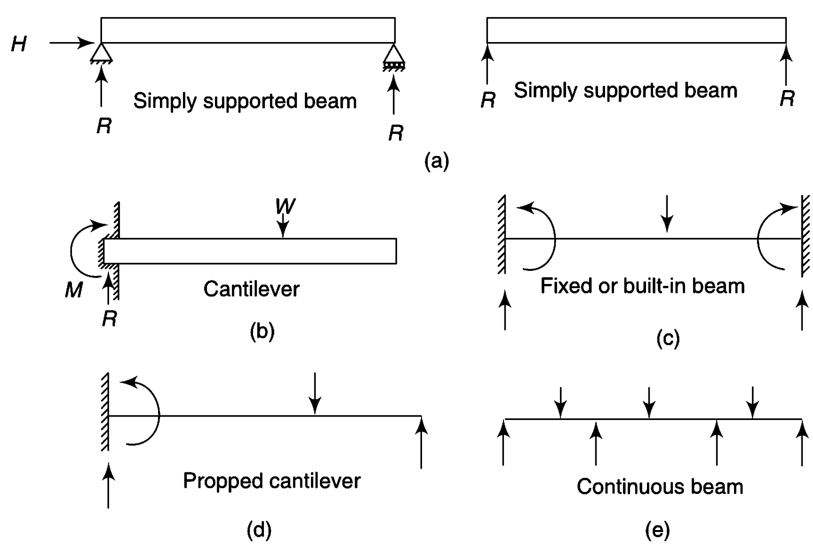


Fig. 4.2

4.2 SHEAR FORCE

Shear force is the unbalanced vertical force on one side (to the left or right) of a section of a beam and is the sum of all the normal forces on one side of the section. It also represents the tendency of either portion of the beam to slide or shear laterally relative to the other. Remember that a *force* at a section means a force of a certain magnitude acting at that point, whereas the *shear force* at a section means the sum of all the forces on one side of the section.

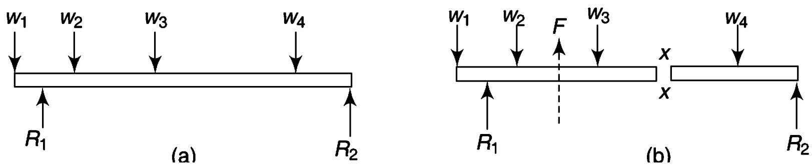


Fig. 4.3

Consider the beam as shown in Fig. 4.3a. It is simply supported at two points and carries four loads. The reactions at the supports are R_1 and R_2. Now if the beam is imagined to cut at section *x-x* into two portions (Fig. 4.3b), the resultant of all the forces (loads as well as reaction of support) to the left of the section is F (assuming upwards). Also, as the beam is in equilibrium, the resultant of the forces to the right of *x-x* must also be F downwards. The force F is known as shear or shearing force (SF).

Shear force is considered positive when the resultant of the forces to the left of a section is upwards or to the right downwards.

A *Shear Force Diagram* (*SFD*) shows the variation of shear force along the length of a beam.

4.3 BENDING MOMENT

Bending moment at a section of a beam is defined as the algebraic sum of the moments about the section of all the forces on one side of the section.

If the moment M about the section *x-x* of all the forces to the left is clockwise (Fig. 4.4), then for the equilibrium, the moment of the forces to the right of *x-x* must be M counter-clockwise.

Bending moment is considered positive if the moment on the left portion is clockwise or on the right portion counter-clockwise. This is usually referred as *sagging* bending moment as it tends to cause concavity upwards. A bending moment causing convexity upwards is taken as negative bending moment and is called *hogging* bending moment.

A *Bending Moment Diagram* (*BMD*) shows the variation of bending moment along the length of a beam.

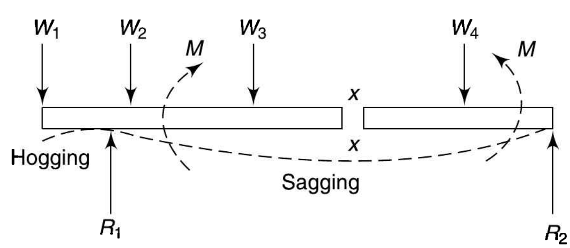


Fig. 4.4

4.4 RELATION BETWEEN LOAD SHEAR FORCE AND BENDING MOMENT

Consider a small length δx cut out from a loaded beam at a distance x from a fixed origin O (Fig. 4.5). Let

w = mean rate of loading on the length δx

F = shear force at the section x

$F + \delta F$ = shear force at the section $x + \delta x$

M = bending moment at the section x

$M + \delta M$ = bending moment at the section $x + \delta x$

Total load on the small length = $w \cdot \delta x$ acting approximately through the centre C (if the load is uniformly distributed, it will be exactly acting through C).

For equilibrium of the element of length δx, equating vertical forces,

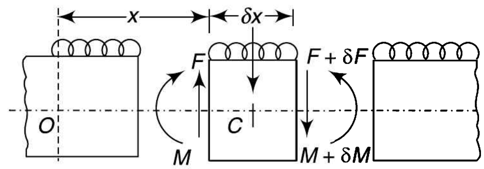


Fig. 4.5

$$F = w\delta x + (F + \delta F) \qquad \text{or} \qquad w = -\frac{dF}{dx} \tag{4.1}$$

i.e., *rate of change of shear force (or slope of the shear force curve) is equal to intensity of loading.*

Taking moments about C, $M + F \cdot \dfrac{\delta x}{2} + (F + \delta F) \cdot \dfrac{\delta x}{2} - (M + \delta M) = 0$

Neglecting the product and squares of small quantities,

$$F = \frac{dM}{dx} \tag{4.2}$$

i.e., *rate of change of bending moment curve is equal to the shear force.*

- At a point where $dM/dx = 0$, i.e., shear force is zero, the bending moment will have the maximum value.
- The point of zero bending moment, i.e., where the type of bending changes from sagging to hogging is called a point of *inflection* or *contraflexure.*

  Integrating Eq. 4.1 between two values of x at two sections A and B,

$$F_a - F_b = \int_a^b w dx$$

which is the area under the load distribution diagram.

Similarly, Integrating Eq. 4.2 between two values of x at two sections A and B,

$$M_b - M_a = \int_a^b F dx$$

This shows that the variation of bending moment between two sections is equal to the area under the shear force diagram.

Also, as $$F = \frac{dM}{dx}, w = -\frac{dF}{dx} = -\frac{d^2M}{dx^2}$$

4.5 SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR CANTILEVERS

A cantilever may carry concentrated or uniformly distributed loads.

Concentrated Loads

Assume a cantilever of length l carrying a concentrated load W at its free end as shown in Fig. 4.6a.

(i) Shear Force Diagram Consider a section at a distance x from the free end. The force to the right of the section is W, downwards and is constant along the whole length of the beam or for all values of x. Therefore, the shear force will be considered positive and the shear force diagram is a horizontal straight line as shown in Fig. 4.6b.

(ii) Bending Moment Diagram Taking moments about the section, $M = W \cdot x$

As the moment on the right portion of the section is clockwise, the bending moment diagram is negative. The bending moment can also be observed as hogging, and thus negative. The bending moment diagram is thus an inclined line increasing with the value of x (Fig. 4.6c).

Maximum bending moment = $W \cdot l$ at the fixed end.

(iii) Reaction and the Fixing Moment From equilibrium conditions, the reaction at the fixed end is W and the fixing moment applied at the fixed end = Wl

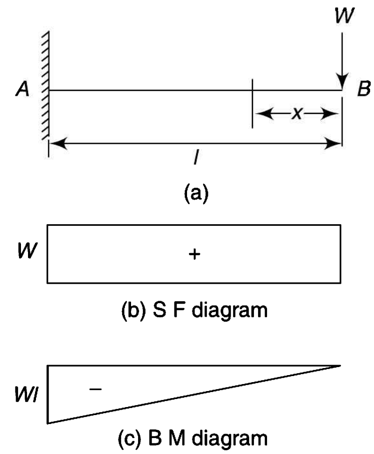


Fig. 4.6

Uniformly Distributed Load

Assume a cantilever of length l carrying a uniformly distributed load w per unit length across the whole span as shown in Fig. 4.7a.

(i) Shear Force Diagram Consider a section at a distance x from the free end. The force to the right of the section is wx, downwards and varies linearly along the whole length of the beam. Therefore, the shear force is positive and the shear force diagram is a straight line as shown in Fig. 4.7b.

(ii) Bending Moment Diagram The force wx to the right of the section can be assumed to be acting as a concentrated load at a point at a distance $x/2$ from the free end.

Taking moments about the section, $M = W \cdot x$

As the moment on the right portion is clockwise, the bending moment diagram is negative (hogging). The bending moment diagram is parabolic and increases with the value of x (Fig. 4.7c).

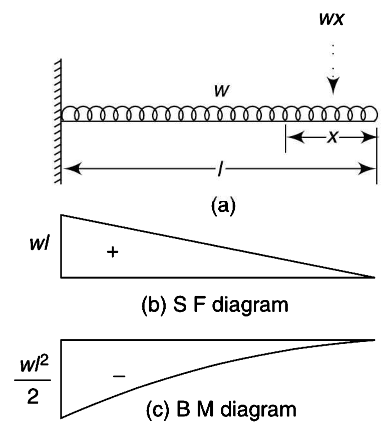


Fig. 4.7

$$\text{Maximum bending moment} = \frac{wl^2}{2} \text{ at the fixed end.}$$

Example 4.1 || A cantilever of 10-m span carries loads of 4 kN and 6 kN at 2 m and 6 m respectively from the fixed end along with another load of 6 kN at the free end. Draw the shear force and bending moment diagrams.

Solution

Given A cantilever loaded with concentrated loads as shown in Fig. 4.8a.

To find To draw shear force and bending moment diagrams

Shear force diagram

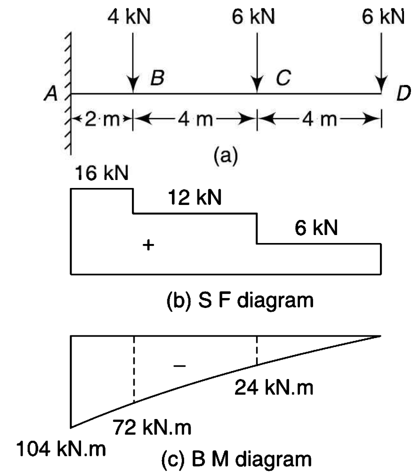
- Portion CD

  Consider a section at a distance x from the free end.

  The force to the right of the section, $F_x = 6$ kN

  It is constant between C and D.
- Similarly, for portion BC: $F_x = 6 + 6 = 12$ kN (constant)
- Portion AB: $F_x = 6 + 6 + 4 = 16$ kN (constant)

  The shear force diagram thus consists of several rectangles having different ordinates (Fig. 4.8b). It can be observed that the shear force undergoes a sudden change when passing through a load point.



(a)
(b) S F diagram
(c) B M diagram
Fig. 4.8

Bending moment diagram

- Portion CD: Taking moments about a section, $M = 6x$, i.e., it is linear.

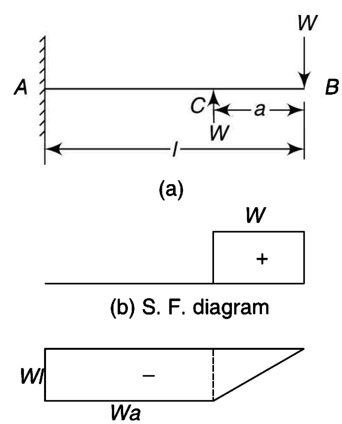
  At D, $x = 0$ and $M_d = 0$; At C, $x = 4$ m and $M_c = 24$ kN · m
- Portion BC: Taking moments about a section, $M_x = 6x + 6(x - 4)$ (linear)

  At C, $x = 4$ m and $M_c = 24$ kN · m; At B, $x = 8$ m and $M_b = 72$ kN · m;
- Portion AB: $M_x = 6x + 6(x - 4) + 4(x - 8)$ (linear)

  At B, $x = 8$ m and $M_c = 72$ kN.m; At A, $x = 10$ m and $M_d = 104$ kN.m;

The bending moment diagram is a series of straight lines between the loads (Fig. 4.8c).

Example 4.2 A cantilever of span l is to withstand a downward acting load W at the free end and an upward acting load W at a distance a from the free end. Draw the shear force and bending moment diagrams.



(a)
(b) S. F. diagram
(c) B. M. diagram
Fig. 4.9

Solution

Given A cantilever loaded with concentrated loads as shown in Fig. 4.9a.
To find To draw shear force and bending moment diagrams

Shear force diagram

- Portion CB: $F_x = W$ (constant)
- Portion AC: $F_x = W - W = 0$

Shear force diagram is shown in Fig. 4.9b.

Bending moment diagram

- Portion CB: $M_x = Wx$ (linear); $M_b = 0$; $M_c = W \cdot a$
- Portion AC: $M_x = Wx - W(x - a) = Wa$ (constant)

  Bending moment diagram is shown in Fig. 4.9c.

Example 4.3 Draw the shear force and bending moment diagrams in the following cases of cantilevers:

(i) Span of 10 m with uniformly distributed load of 3 kN/m for 6 m starting from the free end.
(ii) Span of 10 m with uniformly distributed load of 3 kN/m for 6 m starting from the fixed end.
(iii) Span of 14 m with uniformly distributed load of 3 kN/m for 6 m starting from 4 m and ending at 10 m from the fixed end.

Solution

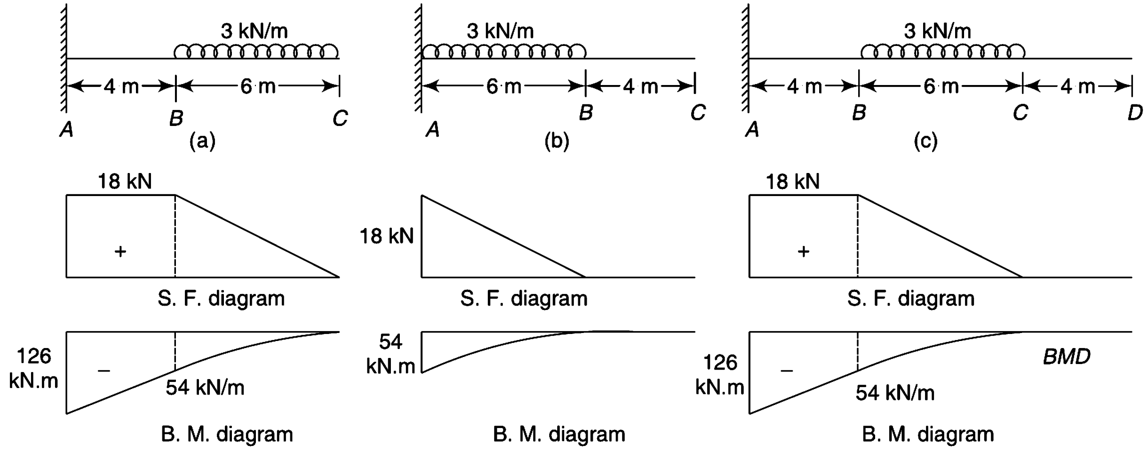


Fig. 4.10

Given Three cantilevers loaded with uniformly distributed loads as shown in Figs. 4.10 a, b and c.
To find To draw shear force and bending moment diagrams

(i) The load diagram for the first case is shown in Fig. 4.10a.
Shear force diagram
- Portion *BC:* $F_x = 3x$ (linear); $F_c = 0$; $F_b = 18$ kN;
- Portion *AB:* $F_x = 18$ kN (constant)

Bending moment diagram
- Portion *BC:* $M_x = 3x \cdot (x/2) = 1.5x^2$ (parabolic); $M_c = 0$; $M_b = 54$ kN·m
- Portion *AB:* $M_x = 18(x - 3)$ (linear); $M_b = 54$ kN·m; $M_a = 126$ kN·m

Shear force and bending moment diagrams are shown below the load diagram in Fig. 4.10a.

(ii) The load diagram for the second case is shown in Fig. 4.10b.
Shear force diagram
- Portion *BC:* $F_x = 0$; $F_c = F_b = 0$
- Portion *AB:* At distance x from *B*, $F_x = 3x$; (linear); $F_b = 0$; $F_a = 18$ kN·m

Bending moment diagram
- Portion *BC:* $M_x = 0$; $M_b = 0$
- Portion *AB:* At distance x from *B*, $M_x = 3x \cdot (x/2) = 1.5x^2$ (parabolic)

$$M_b = 0;\ M_a = 54 \text{ kN} \cdot \text{m}$$

Shear force and bending moment diagrams are shown below the load diagram in the Fig. 4.10b.

(iii) The load diagram for the last case is shown in Fig. 4.10c.
Shear force and bending moment diagrams

There will not be any shear force and bending moment in the portion *CD* and for the portion *AC*, the shear force and bending moment diagrams will as in case (i). Shear force and bending moment diagrams are shown below the load diagram in Fig. 4.10c.

Example 4.4 A cantilever of 14-m span carries loads of 6 kN, 4 kN, 6kN and 4 kN at 2 m, 4 m, 7 m, and 14 m respectively from the fixed end. It also has a uniformly distributed load of 2 kN/m run for the length between 4 m and 8 m from the fixed end. Draw the shear force and bending moment diagrams.

Solution

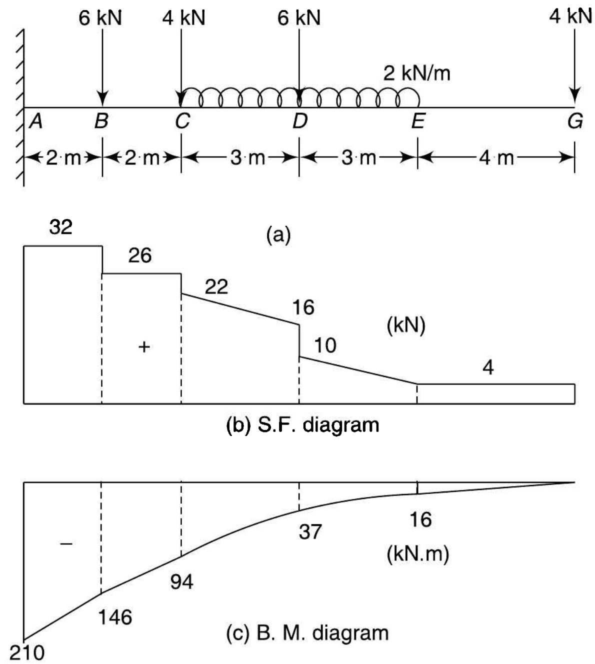


Fig. 4.11

Given A cantilever loaded with point loads and a uniformly distributed load as shown in Fig. 4.11a.
To find To draw shear force and bending moment diagrams

Shear force diagram

- Portion EG: $F_x = 4$ kN (constant);
- Portion DE: $F_x = 4 + 2(x - 4)$ (linear); $F_e = 4$ kN; $F_d = 10$ kN
- Portion CD: $F_x = 4 + 2(x - 4) + 6$; (linear); $F_d = 16$ kN; $F_c = 22$ kN
- Portion BC: $F_x = 22 + 4$; (constant); $F_c = F_b = 26$ kN;
- Portion AB: $F_x = 26 + 6$; (constant); $F_a = F_b = 32$ kN;
  Shear force diagram has been shown in Fig. 4.11b.

Bending moment diagram

- Portion EG: $M_x = 4x$ (linear); $M_g = 0$; $M_e = 16\ \text{kN}\cdot\text{m}$
- Portion DE: $M_x = 4x + \dfrac{2(x-4)^2}{2}$ …(parabolic); $M_e = 16\ \text{kN}\cdot\text{m}$; $M_d = 37\ \text{kN}\cdot\text{m}$
- Portion CD: $M_x = 4x + \dfrac{2(x-4)^2}{2} + 6(x - 7)$ (parabolic)

$$M_{d(x=7)} = 37\ \text{kN}\cdot\text{m};\ M_{c(x=10)} = 94\ \text{kN}\cdot\text{m}$$

- Portion BC: $M_x = 4x + 2 \times 6(x - 7) + 6(x - 7) + 4(x - 10)$ (linear)

$$M_{c(x=10)} = 94\ \text{kN}\cdot\text{m};\ M_{b(x=12)} = 146\ \text{kN}\cdot\text{m};$$

- Portion AB: $M_x = 4x + 2 \times 6(x-7) + 6(x-7) + 4(x-10) + 6(x-12)$ (linear)

$$M_{b(x=12)} = 146 \text{ kN}\cdot\text{m}; \; M_{a(x=14)} = 210 \text{ kN}\cdot\text{m}$$

Bending moment diagram has been shown in Fig. 4.11c.

4.6 SHEAR FORCE AND BENDING MOMENT DIAGRAMS FOR SIMPLY SUPPORTED BEAMS

A simply supported beam may carry concentrated or uniformly distributed loads.

Concentrated Loads

Assume a simply supported beam of length l carrying a concentrated load W at a distance a from end A as shown in Fig. 4.12a.
Let the distance CB be b.
First it is required to find the reactions at the supports.
Taking moments about end $A = R_b \cdot l - W \cdot a = 0$

or
$$R_b = \frac{Wa}{l}; \qquad \text{Similarly, } R_a = \frac{Wb}{l}$$

(i) Shearing force diagram

- *Portion BC:* Consider a section at a distance x from the end B. The force to the right of the section is R_b upwards and is constant along the length up to the point C on the beam. Therefore, the shear force will be negative and the shear force diagram is a horizontal straight line as shown in Fig. 4.12b.
- *Portion AC*: At a section at a distance x from the end B, the force to the right of section is

$$= R_b - W = \frac{Wa}{l} - W = \frac{Wa - Wl}{l} = -W\left(\frac{l-a}{l}\right) = R_a \text{ (downwards)}$$

Thus, the shear force will be positive and the shear force diagram is a horizontal straight line as shown in Fig. 4.12b.

Note that the portion AB can also be taken first and the forces to the left of any section may be considered. In that case, the force to the left is R_a and upwards and the shear force positive, i.e., the same as before.

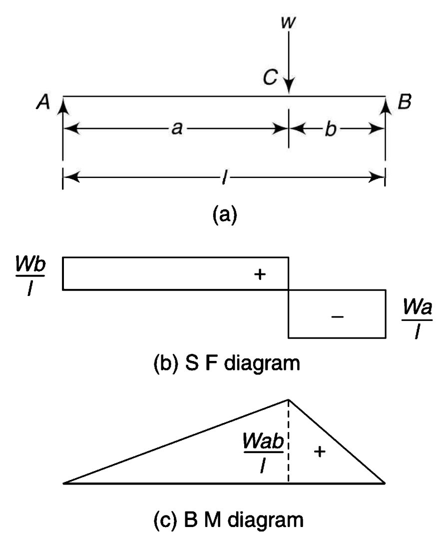


Fig. 4.12

(ii) Bending moment diagram

Portion BC: Consider a section at a distance x from the end B.

Taking moments about the section, $M = R_b \cdot x = \dfrac{Wax}{l}$

$$M_b = 0; \; M_c = Wab/l$$

As the moment on the right portion of the section is counter-clockwise, the bending moment is positive. The bending moment can also be observed as sagging, and thus positive. Therefore the bending moment is linear and increases with the value of x (Fig. 4.12c).

Portion AC: Consider a section at a distance x from the end A.

$$M = R_a \cdot x = \frac{Wbx}{l}; M_a = 0; M_c = Wab/l$$

The moment on the left portion of the section is clockwise, the bending moment is positive. The bending moment can also be observed as sagging, and thus positive.

- If the load is at the mid-span, $a = b = l/2$

  And thus the bending moment at the mid-point, $M = \dfrac{W(l/2)(l/2)}{l} = \dfrac{Wl}{4}$

  which is maximum for any position of the load on beam.

Uniformly Distributed Load

Assume a simply supported beam of length l carrying a uniformly distributed load w per unit length as shown in Fig. 4.13a.

Total load = wl; $R_a = R_b = wl/2$

(i) Shear force diagram

At a section at a distance x from A

$$F_x = R_a - wx = \frac{wl}{2} - wx = w\left(\frac{l}{2} - x\right) \text{(linear)}; F_{a(x=0)} = \frac{wl}{2}; F_{b(x=l)} = -\frac{wl}{2}$$

Shear force diagram is shown in Fig. 4.13b.

(ii) Bending moment diagram

The bending moment at a section is found by treating the distributed load as acting at its centre of gravity.

$$M_x = R_a \cdot x - wx \cdot \frac{x}{2} = \frac{wl}{2}x - \frac{wx^2}{2} = \frac{wx}{2}(l - x) \quad \text{(parabolic)}$$

$$M_{a(x=0)} = 0; \quad M_{b(x=l)} = 0;$$

For maximum value, $F = \dfrac{dM}{dx} = 0$ or $\dfrac{wl}{2} - wx = 0$ or $x = l/2$

Thus maximum bending moment, $M_{(x = l/2)} = \dfrac{wl^2}{8}$

Bending moment diagram is shown in Fig. 4.13c.

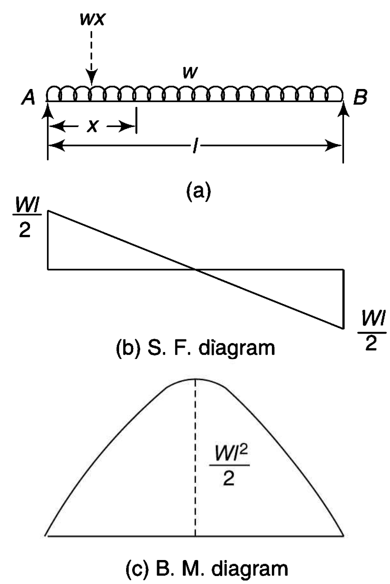


Fig. 4.13

Example 4.5 A simply supported beam of 8-m length carries three point loads of 8 kN, 4 kN and 10 kN at 2 m, 5 m and 6 m respectively from the left end. Draw the shear force and bending moment diagrams.

Solution

Given A simply supported beam loaded with point loads as shown in Fig. 4.14a.

To find To draw shear force and bending moment diagrams

Reactions at supports

Taking moments about A, $R_b \times 8 = 8 \times 2 + 4 \times 5 + 10 \times 6$

or $R_b = 12$ kN

$\therefore$ $R_a = 8 + 4 + 10 - 12 = 10$ kN

Shear force diagram

- Portion *AC:* $F_x = 10$ kN (constant);
- Portion *CD:* $F_x = 10 - 8 = 2$ kN (constant)
- Portion *DE:* $F_x = 2 - 4 = -2$ kN (constant)
- Portion *EB:* $F_x = -2 - 10 = -12$ kN (constant)

  Shear force diagram has been shown in Fig. 4.14b.

Bending moment diagram

- Portion *AC:* $M_x = 10x$ (linear); $M_{a(x=0)} = 0$; $M_{c(x=2)} = 20$ kN · m
- Portion *CD:* $M_x = 10x - 8(x - 2)$ (linear); $M_{c(x=2)} = 20$ kN · m; $M_{d(x=5)} = 26$ kN · m
- Portion *DE:* $M_x = 10x - 8(x - 2) - 4(x - 5)$ (linear);
  $M_{d(x=5)} = 26$ kN · m
  $M_{e(x=6)} = 24$ kN · m
- Portion *EB:* $M_x = 10x - 8(x - 2) - 4(x - 5) - 10(x - 6)$ (linear)
  $M_{e(x=6)} = 24$ kN · m; $M_{b(x=6)} = 0$

Bending moment diagram has been shown in Fig. 4.14b.

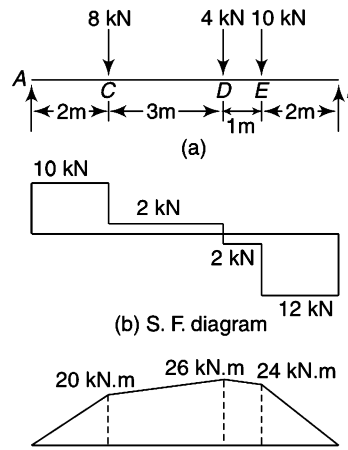


Fig. 4.14

Example 4.6 A 10-m long simply supported beam carries two point loads of 10 kN and 6 kN at 2 m and 9 m respectively from the left end. It also has a uniformly distributed load of 4 kN/m run for the length between 4 m and 7 m from the left end. Draw shear force and bending moment diagrams.

Solution

Given A simply supported beam loaded with point loads and a uniformly distributed load as shown in Fig. 4.15a.

To find To draw shear force and bending moment diagrams

Support reactions

Taking moments about *A*, $R_b \times 10 = 10 \times 2 + 4 \times 3 \times 5.5 + 6 \times 9$ or $R_b = 14$ kN

$\therefore$ $R_a = 10 + 12 + 6 - 14 = 14$ kN

Shear force diagram

- Portion *AC:* $F_x = 14$ kN (constant);
- Portion *CD:* $F_x = 14 - 10 = 4$ kN (constant)
- Portion *DE:* $F_x = 4 - 4(x - 4)$ (linear) $F_{d(x=4)} = 4$ kN; $F_{e(x=7)} = -8$ kN

  Shear force is zero at, $4 - 4(x - 4)$ or $x = 5$ m
- Portion *EF:* $F_x = 4 - 12 = -8$ kN (constant)
- Portion *FB:* $F_x = -8 - 6 = -14$ kN (constant)

Shear force diagram has been shown in Fig. 4.10b.

Bending moment diagram

- Portion *AC:* $M_x = 14x$ (linear); $M_{a(x=0)} = 0$; $M_{c(x=2)} = 28$ kN · m
- Portion *CD:* $M_x = 14x - 10(x - 2)$ (linear); $M_{c(x=2)} = 28$ kN · m; $M_{d(x=4)} = 36$ kN · m

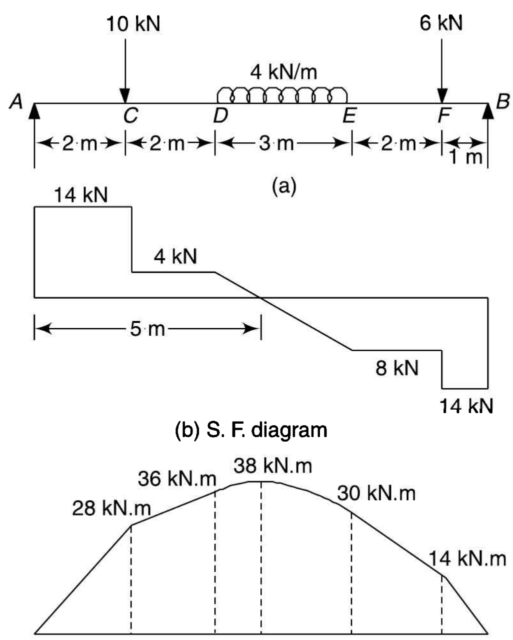


Fig. 4.15

- Portion *DE*: $M_x = 14x - 10(x-2) - \dfrac{4(x-4)^2}{2}$ ...(parabolic)

Bending moment is maximum at $x = 5$ m (at a point where shear force is zero),

$$\therefore \qquad M_{\max} = 14 \times 5 - 10(5-2) - \frac{4(5-4)^2}{2} = 38 \text{ kN} \cdot \text{m}$$

$$M_{d(x=4)} = 36 \text{ kN}\cdot\text{m}; M_{e(x=7)} = 30 \text{ kN}\cdot\text{m}$$

- Portion *EF*: $M_x = 14x - 10(x-2) - 12(x-5.5)$ ...(linear)

$$M_{e(x=7)} = 30 \text{ kN}\cdot\text{m}; M_{f(x=6)} = 14 \text{ kN}\cdot\text{m}$$

- Portion *FA*: $M_x = 14x - 10(x-2) - 12(x-5.5) - 6(x-9)$ ...(linear)

$$M_{e(x=7)} = 14 \text{ kN}\cdot\text{m}; M_{f(x=6)} = 0$$

Bending moment diagram has been shown in Fig. 4.10b.

4.7 BEAMS WITH OVERHANGS

Sometimes, there are beams with overhangs which may extend at one end or at both ends. For analysis of such beams, the drawing of shear force and bending moment diagrams may start from the left or the right-hand end depending upon the convenience. A typical case of a uniformly distributed loaded beam with equal overhangs is discussed as follows.

Figure 4.16a shows a beam with equal overhangs. Let w be the uniformly distributed load on the beam.

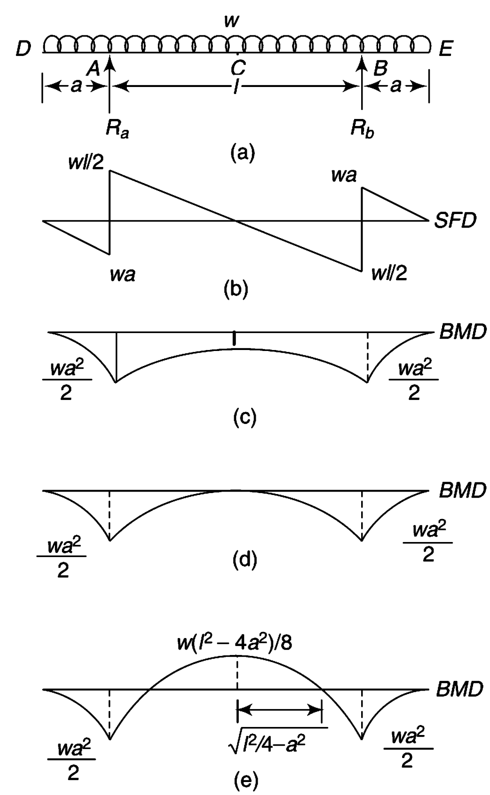


Fig. 4.16

As the overhangs are equal, $R_a = R_b = \dfrac{w(l+2a)}{2}$

(i) Shear force diagram

- Portion *DA*: $F_x = -wx$ (linear);
  $F_d = 0; F_a = -wa$
- Portion *AB*: $F_x = -wx + \dfrac{w(l+2a)}{2}$ (linear);
  $$F_{a(x=a)} = \frac{wl}{2}; F_{b(x=l+a)} = -\frac{wl}{2}$$
- Portion *BE*: $F_x = -wx + \dfrac{w(l+2a)}{2} + \dfrac{w(l+2a)}{2}$
  $$= -wx + w(l+2a) \quad \text{(linear)}$$
  $$F_{b(x=l+a)} = wa; F_{e(x=l+2a)} = 0;$$

Shear force diagram is shown in Fig. 4.16b.

(ii) Bending moment diagram

- Portion *DA*: $M_x = -\dfrac{wx^2}{2}$ (parabolic);

$$M_d = 0;\ M_a = -\frac{wa^2}{2}$$

- Portion AB: $M_x = -\dfrac{wx^2}{2} + \dfrac{w(l+2a)}{2}(x-a)$ (parabolic)

$$M_a = -\frac{wa^2}{2};\ M_{b(x=l+a)} = -\frac{wa^2}{2}$$

- Portion BE: Bending moment will be reducing to zero in a parabolic manner at E. It is convenient to consider it from end E. Then $M_x = -wx^2/2$.

At midpoint C,

$$M_{c(x\,=\,a\,+\,l/2)} = -\frac{w(a+l/2)^2}{2} + \frac{w(l+2a)}{2}\cdot\frac{l}{2} = -\frac{w}{2}\left(a^2 + \frac{l^2}{4} + al - \frac{l^2}{2} - al\right) = \frac{w}{8}(l^2 - 4a^2)$$

The shape of bending moment curve between A and B will depend upon the value of $(l^2 - 4a^2)$.

(i) If $(l^2 - 4a^2) < 0$, i.e., when $l < 2a$, M_c is negative which means bending moment will be negative throughout (Fig. 4.16c).

(ii) If $(l^2 - 4a^2) < 0$, i.e., when $l = 2a$, M_c is zero which means bending moment will just touch the span at the mid point (Fig. 4.16d).

(iii) If $(l^2 - 4a^2) < 0$, i.e., when $l > 2a$, M_c is positive which means there are to be two points of contraflexture between A and B which can be found by putting the expression for bending moment between A and B equal to zero (Fig. 4.16e), i.e.,

$$-\frac{wx^2}{2} + \frac{w(l+2a)}{2}(x-a) = 0$$

$$x^2 - (2a+l)x = -a(2a+l)$$

Adding $(a + l/2)^2$ on both sides,

$$x^2 - 2\left(a+\frac{l}{2}\right)x + \left(a+\frac{l}{2}\right)^2 = -a(2a+l) + \left(a+\frac{l}{2}\right)^2$$

or
$$\left[x - \left(a+\frac{l}{2}\right)\right]^2 = \frac{l^2}{4} - a^2$$

or
$$x - \left(a+\frac{l}{2}\right) = \pm\sqrt{\frac{l^2}{4} - a^2}$$

As $a + l/2$ is the distance DC, the points of contraflexture are at distance $\pm\sqrt{\dfrac{l^2}{4} - a^2}$ from the mid point of the beam.

Example 4.7 || A 14-m long simply supported beam has overhangs of 3 m on the left end and 2 m on the right end. It carries loads of 15 kN at the left end, 30 kN on the right end and two other loads of 15 kN and 30 kN at 6 m and 10 m respectively from the left end. Draw the shear force and the bending moment diagrams.

Solution

Given A simply supported beam with overhangs and loaded with point loads as shown in Fig. 4.17a.

To find To draw shear force and bending moment diagrams

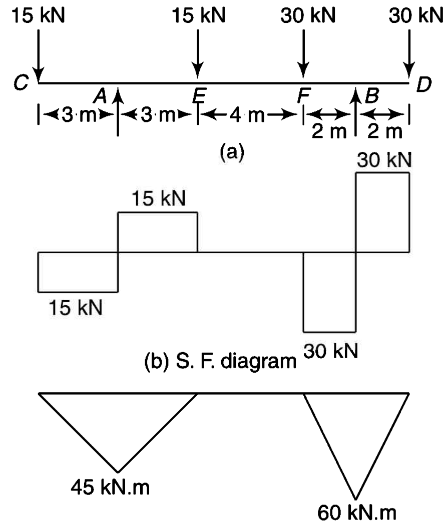


Fig. 4.17

Support reactions

Taking moments about B,

$$R_a \times 9 - 15 \times 12 - 15 \times 6 - 30 \times 2 + 30 \times 2 = 0$$

or $\quad R_a = 30$ kN

and $\quad R_b = 15 \times 2 + 30 \times 2 - 30 = 60$ kN

Shear force diagram

- Portion CA: $F_x =$ kN
- Portion AE: $F_x = -15 + 30 = 15$ kN
- Portion EF: $F_x = 15 - 15 = 0$ kN
- Portion FB: $F_x = -30$ kN
- Portion BD: $F_x = -30 + 60 = 30$ kN
  Shear force diagram is shown in Fig. 4.17b.

Bending moment diagram

- Portion CA: $M_x = -15x$ (linear), $M_{c(x=0)} = 0$, $M_{a(x=3)} = -45$ kN · m
- Portion AE: $M_x = -15x + 30(x-3)$ …(linear), $M_{a(x=3)} = -45$ kN · m, $M_{e(x=6)} = 0$
- Portion EF: $M_x = -15x + 30(x-3) - 15(x-6)$ …(linear), $M_{e(x=6)} = 0$, $M_{f(x=10)} = 0$
  Thus there is not bending moment in the portion EF
- Portion FB: $M_x = -15x + 30(x-3) - 15(x-6) - 30(x-10)$ … (linear)

$$M_{f(x=10)} = 0,\ M_{b(x=12)} = -60 \text{ kN} \cdot \text{m}$$

- Portion BD: $M_x = -15x + 30(x-3) - 15(x-6) - 30(x-10) + 60(x-12)$ … (linear)

$$M_{b(x=12)} = -60 \text{ kN} \cdot \text{m},\ M_{b(x=14)} = 0 \text{ kN} \cdot \text{m}$$

In the portion BFD, it is convenient to consider the bending moments from the right end D. Thus

- Portion BD: $M_x = -30x$ (linear), $M_{d(x=0)} = 0$, $M_{b(x=2)} = -60$ kN · m
- Portion FB: $M_x = -30x + 60(x-2)$ (linear), $M_{b(x=2)} = -60$ kN · m, $M_{f(x=4)} = 0$
- Bending moment diagram is shown in Fig. 4.17c.

Example 4.8 A simply supported beam of 7-m span with overhangs rests on supports which are 4 m apart. The left end overhanging is 2 m. The beam carries loads of 30 kN and 20 kN on the left and the right ends respectively apart from a uniformly distributed load of 25 kN/m between the supporting points. Draw the shear force and bending moment diagrams.

Solution

Given A simply supported beam with overhangs and loaded with point loads and a uniformly distributed load as shown in Fig. 4.18a.

To find To draw shear force and bending moment diagrams

Support reactions

Taking moments about B,

$$R_a \times 4 - 30 \times 6 - 25 \times 4 \times 2 + 20 \times 1 = 0$$

or $\quad R_a = 90$ kN

and $\quad R_b = 30 + 25 \times 4 + 20 - 90 = 60$ kN

Shear force diagram

- Portion CA: $F_x = -30$ kN (constant)
- Portion AB: $F_x = -30 + 90 - 25x = 60 - 25(x - 2)$ (linear)

$$F_{a(x=2)} = 60 \text{ kN} \; ; \; F_{b(x=6)} = -40 \text{ kN}$$

Shear force is zero at, $60 - 25(x - 2) = 0$ or $x = 4.4$ m

- Portion BD: $F_x = -40 + 60 = 20$ kN (constant)

Shear force diagram is shown in Fig. 4.18b.

Bending moment diagram

- Portion CA: $M_x = -30x$ (linear), $M_{c(x=0)} = 0$, $M_{a(x=2)} = -60$ kN·m
- Portion AB: $M_x = -30x + 90(x - 2) - \dfrac{25(x-2)^2}{2}$ (parabolic)

$M_{a(x=2)} = -60$ kN·m, $M_{b(x=6)} = -20$ kN·m

Bending moment is maximum at 4.4 m and is equal to

$$M_{4.4} = -30 \times 4.4 + 90(4.4 - 2) - \frac{25(4.4-2)^2}{2}$$

$= 5.5$ kN·m

Bending moment is zero at

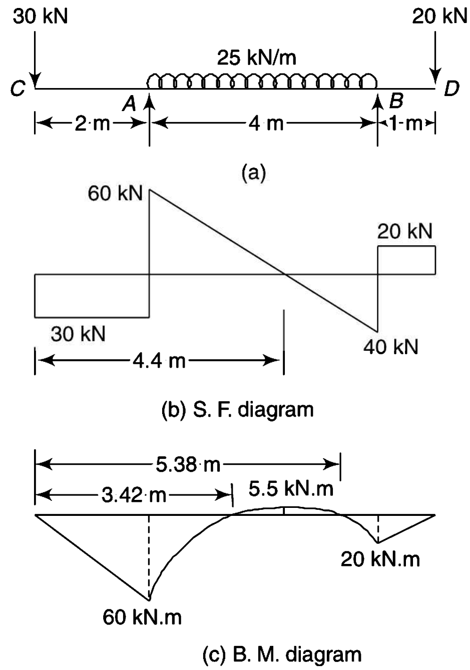


Fig. 4.18

$$-30x + 90(x - 2) - \frac{25(x-2)^2}{2} = 0$$

or $$-30x + 90(x - 2) - 12.5(x^2 - 4x + 4) = 0$$

or $$-30x + 90x - 180 - 12.5x^2 + 50x - 50 = 0$$

or $$12.5x^2 - 110x + 230 = 0$$

$$x = 5.38 \text{ m and } 3.42 \text{ m}$$

- Portion BD: $M_x = -30x + 90(x - 2) - 25 \times 4(x - 4) + 60(x - 6)$ ...(linear)

$$M_{b(x=6)} = -20 \text{ kN·m}, \; M_{d(x=7)} = 0$$

- For portion BD it is convenient to consider x from the right end D, i.e.,

$$M_x = -20x \qquad \text{...(linear)}$$

$$M_{d(x=0)} = 0, \; M_{b(x=1)} = -20 \text{ kN·m}$$

Bending moment diagram is shown in Fig. 4.18c.

Example 4.9 || A simply supported beam of length L carries a uniformly distributed load of w per unit length. It has movable supports and equal overhangs on either side. If the supports are to be so adjusted that the maximum bending moment is the minimum possible, determine the position of the supports. Also, draw shear force and bending moment diagrams.

Solution

Given A simply supported beam with equal overhangs, movable supports and loaded with a uniformly distributed load as shown in Fig. 4.19a.

To find

— Position of supports for maximum bending moment to be least
— To draw shear force and bending moment diagrams.

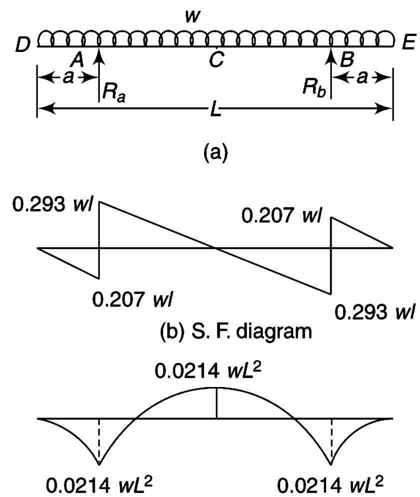


Fig. 4.19

Reactions at supports

If overhangs are small ($L > 2a$), there is sagging bending moment at the midspan and hogging bending moment over the supports (refer Section 4.7). The maximum bending moment will be the minimum if the magnitudes of the sagging and the hogging bending moments are equal.

Let the overhanging on each side be of length a.

$$R_a = R_b = \frac{wL}{2}$$

Determination of position of supports

Position of supports can be obtained by equating sagging and hogging bending moments, i.e.,

Sagging bending moment at the centre = Hogging bending moment over the supports

$$-w \cdot \frac{L}{2} \cdot \frac{L}{4} + \frac{wL}{2}\left(\frac{L}{2} - a\right) = \frac{wa^2}{2}$$

$$L^2 - 4La = 4a^2$$

or $$L^2 - 4La + 4a^2 = 4a^2 + 4a^2 \quad \text{(adding } 4a^2 \text{ on both sides)}$$

or $$(\mathrm{L} - 2a)^2 = 8a^2$$

or $$L = 2.828a + 2a \quad \text{or} \quad a = 0.207\,L$$

$$\text{Maximum bending moment} = \frac{wa^2}{2} = \frac{w(0.207L)^2}{2} = 0.0214wL^2$$

Shear force diagram

- Portion DA: $F_x = -wx$ (linear)

$$F_d = 0;\ F_a = -wa = -w \times 0.207\,L = -0.207L$$

- Portion AB: $F_x = -wx + \dfrac{wL}{2}$ (linear)

$$F_a = -w \times 0.207L + 0.5wL = 0.293wL;\ F_b = -0.293w\mathrm{L};$$

Shear force and bending moment diagrams are shown in Figs. 4.19b and c respectively.

Example 4.10 A 20-m long girder carrying a uniformly distributed load of w kN/m is to be supported on two piers that are 12 m apart in such a way that the maximum bending moment is as small as possible. Determine the distance of piers from the ends of the girder and the maximum bending moment. Draw the shear force and bending moment diagrams.

Given A simply supported overhanging beam (girder) with movable piers, 12 m apart, and loaded with a uniformly distributed load as shown in Fig. 4.20a.

To find

— Distance of piers from the ends for maximum bending moment to be minimum
— To draw shear force and bending moment diagrams.

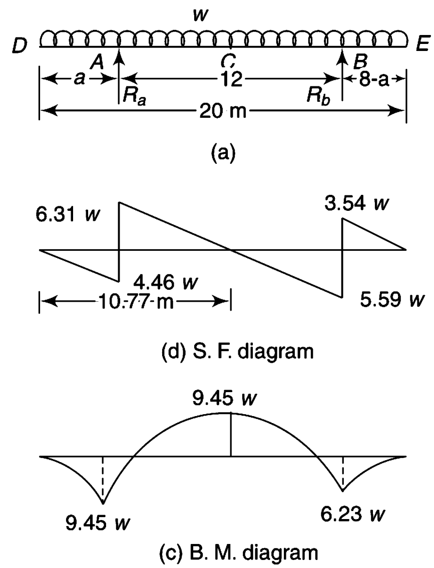


Fig. 4.20

Let distance of pier at A be a m from the end D, the other will be at B, $(8 - a)$ m from end E. Let C be the midpoint of DE.

Distance $CB = CE - BE = 10 - (8 - a) = 2 + a$

Support reactions

Taking moments about B,

$$12R_a = 20w(2 + a) \quad \text{or} \quad R_a = \frac{5w(2+a)}{3}$$

$$12R_b = 20w - \frac{5w(2+a)}{3}$$

Determination of distance of piers from the ends

To find the distance of piers from the ends for maximum bending moment to be least, the magnitudes of the sagging and the hogging bending moment are made equal. Thus finding bending moment in various portions:

- *Portion DA:* $M_x = -\dfrac{wx^2}{2}$ (parabolic); $M_d = 0$; $M_a = -\dfrac{wa^2}{2}$ (negative value)

- *Portion AB:* $M_x = -\dfrac{wx^2}{2} + \dfrac{5w(2+a)}{3}(x - a)$

It is maximum when $dM/dx = 0$ or $-wx + \dfrac{5w(2+a)}{3} = 0$

or $\quad -3x + 5(2 + a) = 0$ or $x = 5(2 + a)/3$

$$\therefore \quad \text{Maximum bending moment} = -\frac{w[(5/3)(2+a)]^2}{2} + \frac{5w(2+a)}{3}\left(\frac{5}{3}(2+a) - a\right)$$

$$= -\frac{25w}{18}(2+a)^2 + \frac{5w(2+a)}{9}(10+2a)$$

$$= \frac{5w}{18}[(2+a)(-10-5a+20+4a)]$$

$$= \frac{5w}{18}[(2+a)(10-a)]$$

$$= \frac{5w}{18}(20+8a-a^2)$$

Equating the positive and negative bending moments,

$$\frac{5w}{18}(20 + 8a - a^2) = \frac{wa^2}{2} \quad \text{or } 5(20 + 8a - a^2) = 9a^2 \text{ or } 14a^2 - 40a - 100 = 0$$

or
$$a = \frac{40 \pm \sqrt{1600 + 4 \times 14 \times 100}}{28} = \frac{40 \pm 84.85}{28}$$

As negative value of a is not practical, taking positive sign.

$$a = 4.46 \text{ m}$$

Thus distance of piers from the ends = 4.46 m and (8 – 4.46) =3.54 m

$$R_a = \frac{5w(2+a)}{3} = \frac{5w(2+4.46)}{3} = 10.77w$$

and $$R_b = 20w - 10.77w = 9.27w$$

Bending moment diagram

$$M_{\max} = \frac{wa^2}{2} = \frac{w \times 4.46^2}{2} = 9.45w \text{ at } A$$

and at $$x = \frac{5}{3}(2 = a) = \frac{5}{3}(2 + 4.46) = 10.77 \text{ m}$$

- Portion *AB*: $M_x = -\dfrac{wx^2}{2} + 10.77w(x - 4.46)$ (parabolic); $M_{b(x=4.46+12)} = -6.23w$
- Portion *BE*: $M_x = -\dfrac{wx^2}{2} + 10.77\text{w}(x - 4.46) + 9.23w\,(x - 16.46)$ (parabolic)

$$M_{e(x=20)} = 0;\ M_{b(x=4.46+12)} = -6.23w$$

M_b can also be considered from end *E*, $M_b = -\dfrac{(8-4.46)^2 w}{2} = -6.26w$

Shear force diagram

- Portion *DA*: $F_x = -wx$ (linear)

  $F_d = 0;\ F_a = -wa = -w \times 4.46 = -4.46w$
- Portion *AB*: $F_x = -wx + 10.77w$ (linear)

  $F_{a(x=4.46)} = 6.31w\,;\ F_{b(x=4.46+12)} = 5.69w$
- Portion *BE*: $F_x = -wx + 10.77w + 9.23w$ (linear)

  $F_{b(x=4.46+12)} = 3.54\,w;\ F_{e(x=20)} = 0$

Shear force and bending moment diagrams are shown in Figs. 4.20b and c respectively.

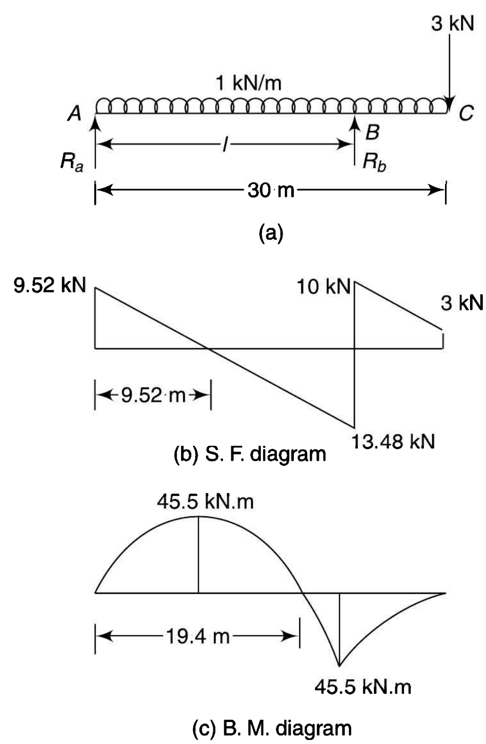


Fig. 4.21

Example 4.11 A 30-m long horizontal beam carries a uniformly distributed load of 1 kN/m on the whole length alongwith a point load of 3 kN at the right end. The beam is freely supported at the left end. Determine the position of the second support so that the maximum bending moment on the beam is as small as possible. Also, draw the shear force and bending moment diagrams indicating main values.

Solution

Given A simply supported beam with a right end overhang as shown in Fig. 4.21a. The right support movable.

To find

— Position of right support for maximum bending moment to be least

— To draw shear force and bending moment diagrams.

The maximum bending moment is as small as possible if the positive bending moment between A and B is equal to negative bending moment at the support B. Let the distance between the two supports be l.

Reaction at support A

Taking moments about B,

$$R_a l - 30(l - 15) + 3(30 - l) = 0 \text{ or } R_a = \frac{33l - 540}{l} \tag{i}$$

Maximum bending moment between A and B

Maximum bending moment between A and B will be where shear force is zero i.e.

$$R_a - wx = 0 \quad \text{or} \quad \frac{33l - 540}{l} - 1 \times x = 0 \quad \text{or} \quad x = \frac{33l - 540}{l} \tag{ii}$$

$$\text{Maximum bending moment} = R_a x - \frac{wx^2}{2}$$

$$= \frac{(33l - 540)}{l} \cdot \frac{(33l - 540)}{l} - \frac{1 \times (33l - 540)^2}{2}$$

$$= \frac{(33l - 540)^2}{l^2} - \frac{1 \times (33l - 540)^2}{2l^2} = \frac{(33l - 540)^2}{2l^2} \tag{iii}$$

$$\text{Bending moment at support } B \text{ (negative)} = 3\,(30 - l) + \frac{1(30 - l)^2}{2}$$

$$= (30 - l)\left(3 + 15 - \frac{l}{2}\right) = (30 - l)\left(18 - \frac{l}{2}\right) = 540 - 33l + \frac{l^2}{2} \tag{iv}$$

Equating positive and negative bending moments

Equating the maximum positive and negative bending moment,

$$\frac{(33l - 540)^2}{2l^2} = \frac{l^2}{2} - 33l + 540 \text{ or } (33l - 540)^2 = l^4 - 66l^3 + 1080l^2$$

or $$l^4 - 66\,l^3 - 9\,l^2 + 35\,640\,l - 291\,600 = 0$$

Solving by trial and error, $l \approx 23$ m

Thus maximum bending moment is at $x = \dfrac{33 \times 23 - 540}{23} = 9.52$ m ...[from (ii)]

$$M_{max} = \frac{(33l - 540)^2}{2l^2} = \frac{(33 \times 23 - 540)^2}{2 \times 23^2} = 45.3 \text{ kN·m} \quad \text{...[from (iii)]}$$

$$R_a = \frac{33l - 540}{l} = \frac{33 \times 23 - 540}{23} = 9.52 \text{ kN} \quad \text{...[from (i)]}$$

and $$R_b = 30 \times 1 + 3 - 9.52 = 23.48 \text{ kN}$$

Shear force diagram

- Portion AB: $F_x = 9.52 - 1 \cdot x$ (linear)

$$F_a = 9.52 \text{ kN}; \; F_{b(x=23)} = -13.48 \text{ kN}$$

- Portion BC: $F_x = 9.52 - 1 \cdot x + 23.48$ (linear)

$$F_{b(x=23)} = 10 \text{ kN} \; ; F_{c(x=30)} = 3 \text{ kN}$$

Bending moment diagram

Bending moment at support $B = 3\,(30 - 23) + \dfrac{1(30-23)^2}{2} = 45.5 \text{ kN}\cdot\text{m}$ ...[from (iv)]

(Equal to maximum bending moment)

Bending moment between A and B is zero at $R_a x - \dfrac{wx^2}{2} = 0$

or $$9.52x - \frac{1 \times x^2}{2} = 0 \; x = 19.4 \text{ m}$$

Shear force and bending moment diagrams are shown in Figs. 4.21b and c respectively.

4.8 BEAMS WITH VARYING DISTRIBUTED LOAD

When beams are loaded with varying distributed loads, the variation may be linear or parabolic and the intensity of loading at a section has to be found. The procedure is explained in the following examples.

Example 4.12 A cantilever is loaded with a distributed load of linearly varying intensity that varies linearly from zero at the free end to w per unit length at the fixed end. Draw the shear force and bending moment diagrams.

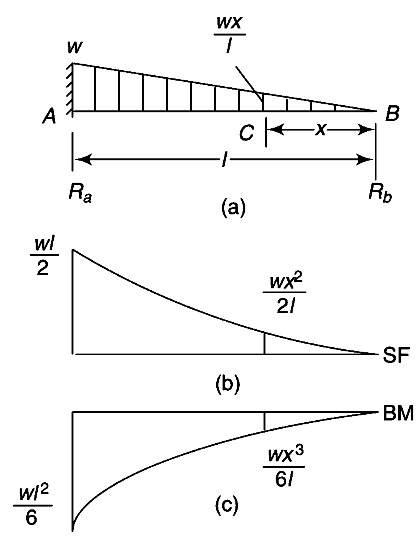


Fig. 4.22

Solution

Given A cantilever with a distributed load of linearly varying intensity as shown in Fig. 4.22a.

To find To draw shear force and bending moment diagrams.

Intensity of loading at any cross-section C at a distance x from free end $= \dfrac{w}{l} \cdot x$

Shear force diagram

At a distance x from B, $F_x = \dfrac{1}{2}\dfrac{wx}{l} \cdot x = \dfrac{wx^2}{2l}$ (parabolic); $F_b = 0$; $F_a = \dfrac{wl}{2}$

Shear force diagram is shown in Fig. 4.22b.

Bending moment diagram

Bending moment at C = load on $CB \times$ distance of centre of load
= (average intensity $\times$ distance CB) $\times$ $CB/3$

$$= \frac{wx}{2l} \cdot x \cdot \frac{x}{3} = \frac{wx^3}{6l} \quad \text{(cubic)}$$

$$M_b = 0; M_a = \frac{wl^2}{6}$$

(The expression for bending moment can also be found by integrating the expression for shear force, i.e., $M_x = \dfrac{wx^2}{6l} + C$ and at B; $x = 0$; $C = M_b = 0$)

Bending moment diagram is shown in Fig. 4.22c.

Note that the total distributed load acting on any portion is equal to the area of the load diagram on that portion.

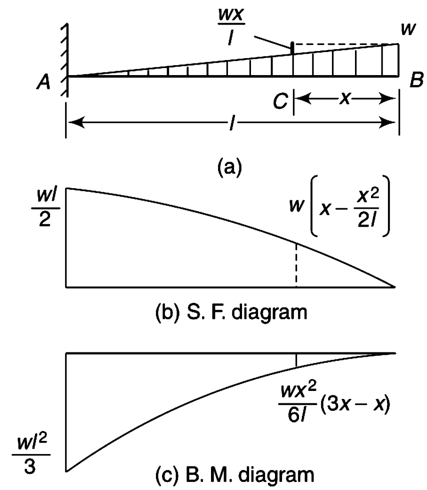


Fig. 4.23

Example 4.13 || A cantilever has a distributed load of linearly varying intensity with zero at the fixed end and w per metre at the free end. Draw the shear force and bending moment diagrams.

Solution

Given A cantilever with a distributed load of linearly varying intensity as shown in Fig. 4.23a.

To find To draw shear force and bending moment diagrams

Intensity of loading at any cross-section C at a distance x from free end $= \frac{w}{l}(l - x) = w - \frac{wx}{l}$

Shear force diagram

At a distance x from B,

$$F_x = \text{Area of rectangle on } CB - \text{area of upper small triangle on } CB$$

$$= wx - \frac{wx}{l} \cdot \frac{x}{2} = w\left(x - \frac{x^2}{2l}\right) \text{ (parabolic)}$$

(or by integrating the expression for load)

$$F_b = 0; F_a = wl/2$$

Shear force diagram is shown in Fig. 4.23b.

Bending moment diagram

$$\text{Bending moment at } C = wx \cdot \frac{x}{2} - \frac{wx}{l} \cdot \frac{x}{2} \cdot \frac{x}{3} = \frac{wx^2}{6l}(3l - x) \qquad \text{(Cubic)}$$

(or by integrating the expression for shear force)

$$M_b = 0; M_a = \frac{wl^2}{3}$$

Bending moment diagram is shown in Fig. 4.23c.

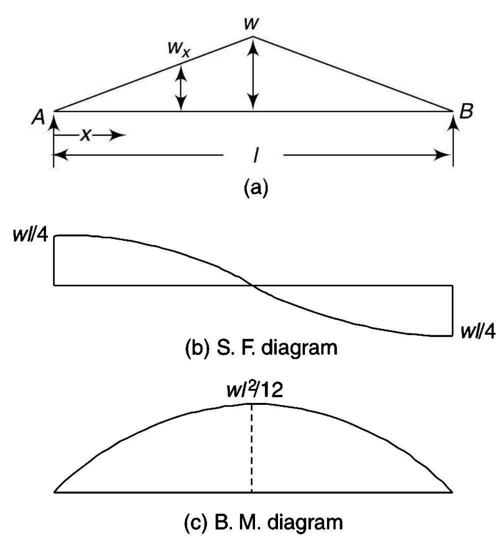


Fig. 4.24

Example 4.14 || A simply supported beam has a distributed load of linearly varying intensity with zero at each end to w per unit run at the midspan. Draw the shear force and bending moment diagrams.

Solution

Given A simply supported beam with a distributed load of linearly varying intensity as shown in Fig. 4.24a.

To find To draw shear force and bending moment diagrams.

$$R_a = R_b = \frac{1}{2}\frac{wl}{2} = \frac{wl}{4}$$

Intensity of loading at a distance x from $A = \dfrac{w}{l/2}x = \dfrac{2wx}{l}$

Shear force diagram

$$F_x = R_a - \frac{2wx}{l}\cdot\frac{x}{2} = \frac{wl}{4} - \frac{wx^2}{l} \quad \text{(parabolic)}$$

At midspan, $F_{l/2} = \dfrac{wl}{4} - \dfrac{wl}{4} = 0$

Bending moment diagram

$$M_x = \frac{wl}{4}x - \frac{wx^2}{l}\frac{x}{3} = \frac{wlx}{4} - \frac{wx^3}{3l} \quad \text{(cubic)}$$

Bending moment is maximum at midspan, $M_{\max} = \dfrac{wl^2}{8} - \dfrac{wl^2}{24} = \dfrac{wl^2}{12}$

Shear force and bending moment diagrams are shown in Fig. 4.24b and c.

Example 4.15 A beam *ABC* is 36 m in length and is supported at points *A* and *B*, 24 m apart. The beam carries a load of 48 kN at 8 m from *A* along with a distributed load, the intensity of which varies linearly from zero at *A* and *C* to 12 kN/m at *B*. Draw the shear force and bending moment diagrams. Also, determine the maximum bending moment on the beam.

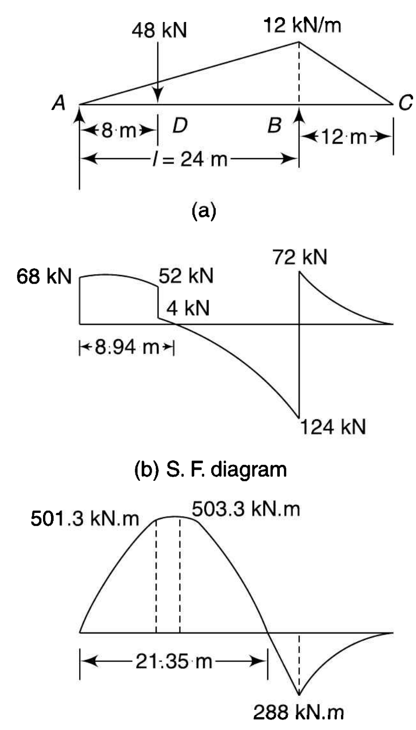


Fig. 4.25

Solution

Given A simply supported beam with a right overhang and a distributed load of linearly varying intensity as shown in Fig. 4.25a.

To find

— To draw shear force and bending moment diagrams

— Maximum bending moment

Support reactions

Taking moments about B,

$$R_a \times 24 = 48 \times 16 + \frac{24\times 12}{2}\times\frac{24}{3} - \frac{12\times 12}{2}\times\frac{12}{3}$$

or $\quad R_a = 68$ kN and $R_b = \left(48 + \dfrac{36\times 12}{2}\right) - 68 = 196$ kN

Intensity of distributed loading at any cross-section at a distance x from $A = (w/l)\cdot x = wx/l$

Shear force diagram

- Portion *AD*: $F_x = R_a - \dfrac{1}{2}\dfrac{wx}{l}x = 68 - \dfrac{12x^2}{2\times 24} = 68 - \dfrac{x^2}{4}$ (parabolic)

$$F_a = 68 \text{ kN};\ F_d = 52 \text{ kN}$$

- Portion DB: $F_x = 68 - \dfrac{x^2}{4} - 48$ ...(parabolic); $F_d = 4$ kN; $F_b = -124$ kN

  Between DB, shear force is zero when $F_x = 68 - x^2/4 - 48 = 0$ or $x = 8.94$ m
- Portion BC: It is convenient to deal this portion by using variable x from the end C.

$$\frac{1}{2}\frac{wx}{l}x = \frac{12x^2}{2 \times 12} = \frac{x^2}{2} \quad \text{... (parabolic); } F_c = 0;\ F_b = 72 \text{ kN}$$

Shear force diagram is shown in Fig. 4.25b.

Bending moment diagram

- Portion AD: $M_x = 68x - \dfrac{x^2}{4}\cdot\dfrac{x}{3} = 68x - \dfrac{x^3}{12}$ ...(cubic); $F_a = 0$; $F_d = 501.3$ kN·m
- Portion DB: $M_x = 68x - 48(x-8) - \dfrac{x^3}{12}$ (cubic)

$$M_{d(x=8)} = 501.3 \text{ kN}\cdot\text{m};\ M_{b(x=24)} = -288 \text{ kN}\cdot\text{m}$$

$$M_{\max(x\,=\,8.94)} = 68x - 48(x-8) - \frac{x^3}{12} = 68 \times 8.94 - 48(8.94 - 8) - \frac{8.94^3}{12} = 503.3 \text{ kN}\cdot\text{m}$$

Bending moment is zero at $68x - 48(x-8) - \dfrac{x^3}{12} = 0$ or $x^3 - 240x - 4608 = 0$

Solving by trial and error or with a calculator by setting it in the equation mode,

$$x = 21.35 \text{ m}$$

- Portion BC: x from the end C.

$$M_x = -\frac{x^2}{2}\frac{x}{3} = -\frac{x^3}{6} \text{ ...(cubic); } F_c = 0;\ F_b = -288 \text{ kN}\cdot\text{m}$$

Bending moment diagram is shown in Fig. 4.25c.

Example 4.16 A simply supported beam has distributed a load of linearly varying intensity with zero at one end to w per unit run at the other. Draw the shear force and bending moment diagrams.

Solution

Given A simply supported beam with a distributed load of linearly varying intensity as shown in Fig. 4.26a.

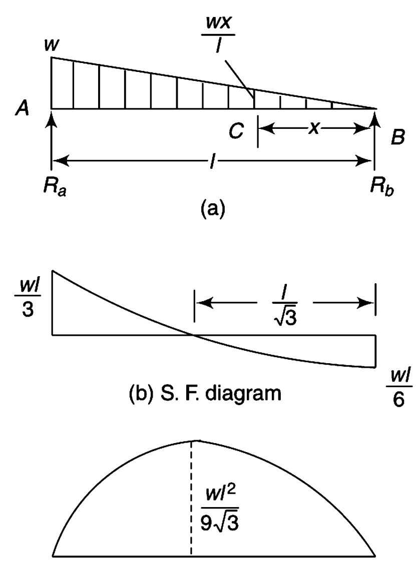
To find To draw shear force and bending moment diagrams

Support reactions

Taking moments about B,

$$R_a \times l = \frac{wl}{2}\cdot\frac{2l}{3} \text{ or } R_a = \frac{wl}{3} \text{ and } R_b = \frac{wl}{2} - \frac{wl}{3} = \frac{wl}{6}$$

Intensity of loading at a distance x from end $B = \dfrac{w}{l}x$



(a)

(b) S. F. diagram

(c) B. M. diagram

Fig. 4.26

Shear force diagram

$$F_x = -R_b + \frac{1}{2}\frac{wx}{l}x = -\frac{wl}{6} + \frac{wx^2}{2l} \text{ (parabolic);}$$

$F_b = -wl/6$; $F_a = wl/3$

Shear force is zero at $-\frac{wl}{6} + \frac{wx^2}{2l} = 0$ or $x = \frac{l}{\sqrt{3}}$

Bending moment diagram

$$M_x = \frac{wl}{6}x - \frac{wx^2}{2l}\frac{x}{3} = \frac{wlx}{6} - \frac{wx^3}{6l} \text{ (cubic); } M_b = 0;\ M_a = 0$$

Maximum bending moment at $x = \frac{l}{\sqrt{3}}$, $M_{max} = \frac{wl^2}{6\sqrt{3}} - \frac{wl^2}{6 \times 3\sqrt{3}} = \frac{wl^2}{9\sqrt{3}}$

Shear force and bending moment diagrams have been shown in Fig. 4.26b and c respectively.

Example 4.17 A beam of 9-m span supports a concrete wall of 160 mm thickness. The height of the wall is 1 m at the left end and increases to 2 m at the right end. The beam has two supports, one at 2 m from the left end and the other at 1 m from the right end. Find the maximum bending moment on the beam if the concrete weighs 25 kN/m³. Draw the shear force and bending moment diagrams.

Solution

Given A simply supported beam with overhangs and a distributed load of linearly varying intensity as shown in Fig. 4.27a.

To find To draw shear force and bending moment diagrams

Support reactions

Intensity of load at C = Volume per m length × 25 = (1 × 1 × 0.16) × 25 = 4 kN/m

Intensity of load at D = (1 × 2 × 0.16) × 25 = 8 kN/m

The loading can be divided into

(i) uniformly distributed load of 4 kN/m and

(ii) a triangular load varying from zero at A to 4 kN/m at B

Taking moments about A,

$$R_b \times 6 = (4 \times 9)\left(\frac{9}{2} - 2\right) + \frac{1}{2}(4 \times 9)\left(\frac{2}{3} \times 9 - 2\right)$$

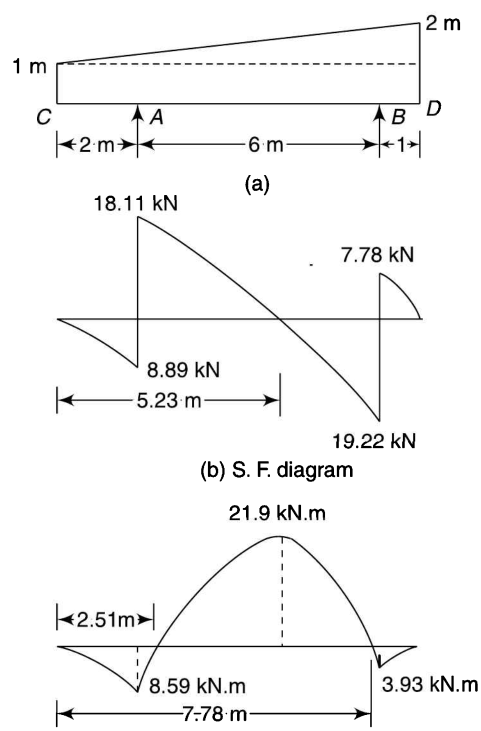
$R_b = 27$ kN

and $R_a = 4 \times 9 + (4 \times 9/2) - 27 = 27$ kN

Shear force diagram

- Portion CA: $F_x = -4x - \frac{4}{9}x \cdot \frac{x}{2} = -4x - \frac{2x^2}{9}$

  (parabolic); $F_c = 0$; $F_{a(x=2)} = -8.89$ kN



(c) B. M. diagram

Fig. 4.27

- Portion *AB:* $F_x = -4x - \dfrac{2x^2}{9} + 27$ (parabolic);

  $F_{a(x=2)} = 18.11$ kN; $F_{b(x=8)} = -19.22$ kN

- Portion *BD:* $F_x = -4x - \dfrac{2x^2}{9} + 27 + 27$ (parabolic); $F_{b(x=8)} = 7.78$ kN; $F_{d(x=9)} = 0$

  It is convenient to deal this portion by using variable x from end D.

  Between A and B, shear force is zero when $-4x - \dfrac{2x^2}{9} + 27 = 0$

  or $\quad 2x^2 + 36x - 243 = 0$ or $x = \dfrac{-36 \pm \sqrt{36^2 + 4 \times 2 \times 243}}{4} = 5.23$ m

  Figure 4.27b shows the shear force diagram.

Bending moment diagram

- Portion *CA:* $M_x = -4 \cdot \dfrac{x^2}{2} - \dfrac{2}{9} \cdot \dfrac{x^3}{3} = -2x^2 - \dfrac{2x^3}{27}$; $M_c = 0$; $M_{a(x=2)} = -8.59$ kN · m

- Portion *AB:* $M_x = -2x^2 - \dfrac{2x^3}{27} + 27(x-2)$; $M_{a(x=2)} = -8.59$ kN · m; $M_{b(x=8)} = -3.93$ kN · m

- Portion *BD:* $M_x = -2x^2 - \dfrac{2x^3}{27} + 27(x-2) + 27(x-8)$

  $$M_{b(x=8)} = -3.93 \text{ kN} \cdot \text{m};\ M_{d(x=9)} = 0$$

  It is convenient to deal this portion by using variable x from end D, i.e.,

  $$M_x = -8 \cdot \frac{x^2}{2} + \frac{2x^3}{27} = -4x^2 + \frac{2x^3}{27}$$

  Points of contraflexure between A and B are given by,

  $$-2x^2 - \frac{2x^3}{27} + 27(x-2)$$

  or
  $$x^3 + 27x^2 - 364.5x + 729 = 0$$

  Solving by trial and error or by using the equation mode of a calculator,

  $$x = 2.51 \text{ m and } 7.78 \text{ m}$$

  Maximum bending moment is where shear force is zero, i.e., at $x = 5.23$ m

  Thus $M_{max} = -2(5.23)^2 - \dfrac{2(5.23)^3}{27} + 27(5.23 - 2) = 21.9$ kN · m

  The bending moment diagram has been shown in Fig. 4.27c.

Example 4.18 || A 4-m long simply supported beam is loaded with a distributed load that varies parabolically from zero at each end to a maximum at the midspan. Draw the shear force and bending moment diagrams if the total load on the beam is 80 kN.

Solution

Given A simply supported beam with a distributed load varying parabolically as shown in Fig. 4.28a.

To find To draw shear force and bending moment diagrams

Let the intensity of loading at the midspan be w

The total load $= \frac{2}{3}wL$ or $80 = \frac{2}{3}w \times 4$ or $w = 30$ kN/m

Taking C as origin, the equation of parabola, $w' = kx'^2 = k(2 - x)^2$

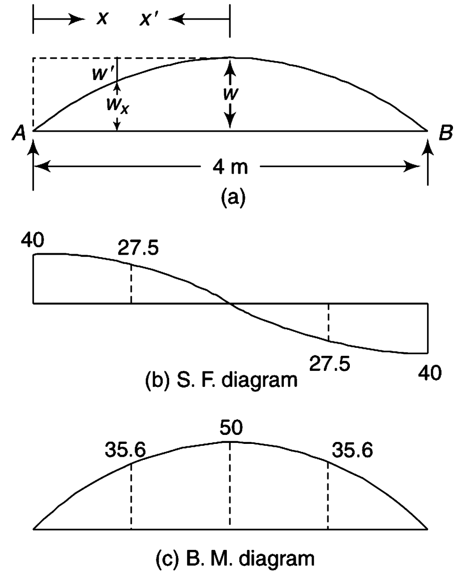


Fig. 4.28

Intensity of loading at a section

Intensity of loading at distance x from A,

$w_x = w - w' = 30 - k(2 - x)^2$

At A, the intensity of loading is zero, $0 = 30 - k(2)^2$ or $k = 7.5$

Thus $w_x = 30 - 7.5(2 - x)^2$

Shear force diagram

$$F_x = 40 - \int[30 - 7.5(2 - x)^2]dx$$
$$= 40 - \int[30 - 7.5(4 - 4x + x^2)]dx$$
$$= \int[30 - 30 + 30x - 7.5x^2]dx^3$$
$$= \int[30x - 7.5x^2]dx$$
$$= 40 - 15x^2 + 2.5x^3 + C_1$$

At $x = 0$, $F_x = 40$ kN, $\therefore$ $C_1 = 0$

Thus $F_x = 2.5x^3 - 15x^2 + 40$

At midspan, $F_x = 0$

Bending moment diagram

$$M_x = \int F_x dx = \int(40 - 15x^2 + 2.5x^3)dx$$
$$= 40x - 5x^3 + 0.625x^4 + C_2$$

At $x = 0$, $M_x = 0$, $\therefore C_2 = 0$

Thus $M_x = 0.625x^4 - 5x^3 + 40x$

Shear forces and bending moments at various sections are calculated as given in the following table:

x (m)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4
F (kN)	40	36.6	27.5	14.7	0	–14.7	–27.5	–36. 6	–40
M(kN · m)	0	19.4	35.6	46.3	50	46.3	35.6	19.4	0

Shear force and bending moment diagrams are shown in Fig. 4. 28a and b respectively.

4.9 BEAMS SUBJECTED TO COUPLES

Let a beam be subjected to a couple M as shown in Fig. 4.29a.

To find reactions at the supports, take moments about B,

$$R_a \cdot l = M \text{ or } R_a = M/l \quad \text{and} \quad R_b = -R_a = -M/l$$

The shear force diagram is a rectangle as shown in Fig. 4.29b.

Bending moment diagram

- Portion AC: $M_x = R_a \cdot x = \dfrac{Mx}{l}$ (linear);

$$M_a = 0;\ M_c = \frac{Ma}{l}$$

- Portion CB: $M_x = R_a x - M = \dfrac{Mx}{l} - M = M\left(\dfrac{x}{l} - 1\right)$ (linear)

$$M_{c(x=a)} = M\left(\frac{a}{l} - 1\right) = -M\left(\frac{l-a}{l}\right) \text{ and } M_{b(x=l)} = 0$$

Thus, bending moment diagram will be as shown Fig. 4.29c.

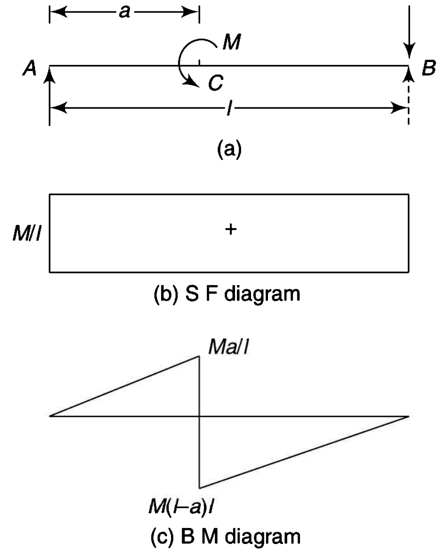


Fig. 4.29

Example 4.19 || Draw shear force and bending moment diagrams for a 6-m long simply supported beam that carries a point load of 12 kN and a clockwise couple at 2 m from the left end.

Solution

Given A simply supported beam carrying a point load and a couple as shown in Fig. 4.30a.

To find To draw shear force and bending moment diagrams

Taking moments about B,

$$R_a \times 6 + 24 - 12 \times 4 = 0 \quad \text{or} \quad R_a = 4 \text{ kN}$$
$$R_b = 12 - 4 = 8 \text{ kN}$$

Shear force diagram

- Portion AC: $F_x = 4$ kN (constant)
- Portion CB: $F_x = 4 - 12 = -8$ kN (constant)

Shear force diagram is shown in Fig. 4.30b.

Bending moment diagram

- Portion AC: $M_x = 4x$ (linear); $M_a = 0$; $M_{c(x=2)} = 8$ kN · m
- Portion CB; $M_x = 4x + 24 - 12(x - 2)$ ...(linear); $M_c = 32$ kN · m; $M_{b(x=6)} = 0$

Bending moment diagram is shown in Fig. 4.30c.

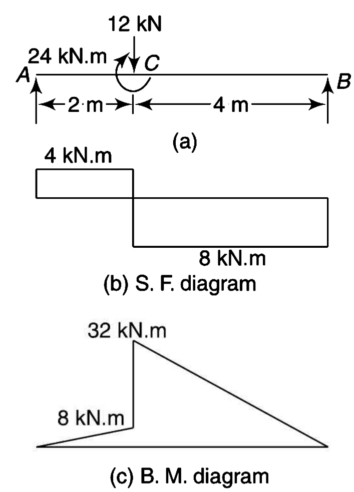


Fig. 4.30

Example 4.20 || A 10-m long simply supported beam carries a point load of 4 kN at 8 m from the left end along with a uniformly distributed load of 4 kN/m intensity for 3 m length starting from the left end. The beams also acted upon by a clockwise couple of 10 kN · m at midpoint of the span. Draw the shear force and bending moment diagrams.

Solution

Given A simply supported beam carrying a point load, a uniformly distributed load and a couple as shown in Fig. 4.31a.

To find To draw shear force and bending moment diagrams

Taking moments about B,

$$R_a \times 10 = 4 \times 3 \times 8.5 - 10 + 4 \times 2 \text{ or } R_a = 10 \text{ kN}$$

$$R_b = 4 \times 3 + 4 - 10 = 6 \text{ kN}$$

Shear force diagram

- Portion AC: $F_x = 10 - 4x$; $F_a = 10$ kN; $F_{a(x=3)} = -2$ kN
  Shear force is zero at $10 - 4x = 0$ or $x = 2.5$ m
- Portion CE: $F_x = 10 - 4 \times 3 = -2$ kN (constant)
- Portion EB: $F_x = -2 - 4\text{kN} = -6$ kN (constant)
  Shear force diagram is shown in Fig. 4.24b.

Bending moment diagram

- Portion AC: $M_x = 10x - 4\dfrac{x^2}{2} = 10x - 2x^2$ (parabolic);
  $M_a = 0$; $M_{c(x=3)} = 12$ kN·m

  Maximum bending moment between AC, $M_{(x=2.5\text{m})} = 10 \times 2.5 - 2 \times 2.5^2 = 12.5$ kN·m
- Portion CD: $M_x = 10x - 12(x - 1.5)$ (linear); $M_c = 12$ kN·m; $M_{d(x=5)} = 8$ kN·m
- Portion DE: $M_x = 10x - 12(x - 1.5) + 10$ (linear)
  $M_{d(x=5)} = 18$ kN·m; $M_{e(x=8)} = 12$ kN·m
- Portion EB: $M_x = 10x - 12(x - 1.5) + 10 - 4(x - 8)$
  $M_{e(x=8)} = 12$ kN·m; $M_{b(x=10)} = 0$
  Bending moment diagram has been shown in Fig. 4.24c.

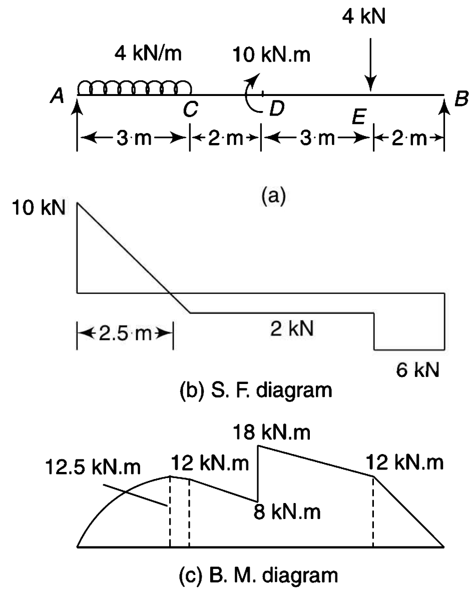


Fig. 4.31

Example 4.21 A simply supported beam of 8-m span carries a uniformly distributed load of 2 kN per m on the left half of the beam. It also has a point load of 25 kN at 6 m from the left end. In addition, the beam is also subjected to a counter-clockwise couple of 20 kN·m at the left end and a clockwise couple of 30 kN·m at the right end. Draw the shear force and the bending moment diagrams indicating the principal values.

Solution

Given A simply supported beam carrying a point load, a uniformly distributed load and end couples as shown in Fig. 4.32a.

To find To draw shear force and bending moment diagrams

The loading diagram is shown in Fig. 4.32a.

Taking moments about B,

$$R_a \times 8 = 20 + 4 \times 2 \times 6 + 25 \times 2 - 30$$

or $\quad R_a = 11$ kN

$$R_b = 4 \times 2 + 25 - 11 = 22 \text{ kN}$$

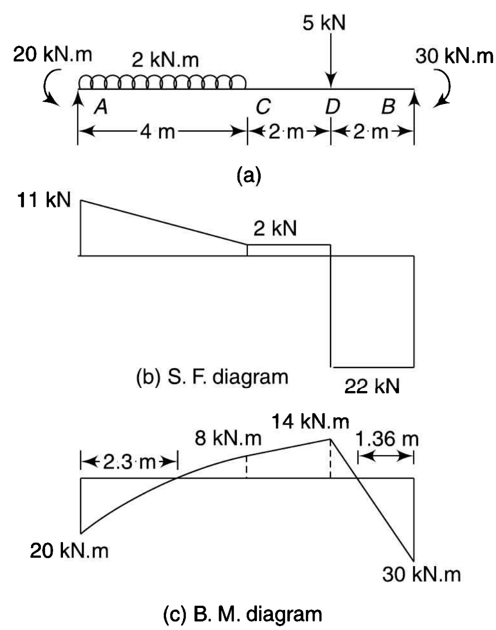


Fig. 4.32

Shear force diagram

- Portion AC: $F_x = 11 - 2x$ (linear); $F_a = 11$ kN; $F_{c(x=4)} = 3$ kN
- Portion CD: $F_x = 11 - 8 = 3$ kN (constant)
- Portion DB: $F_x = 3 - 25 = -22$ kN (constant)

Figure 4.32 b shows the shear force diagram.

Bending moment diagram

- Portion AC: $M_x = 11x - 20 - 2\dfrac{x^2}{2} = 11x - 20 - x^2$ (parabolic)

$$M_a = -20 \text{ kN}\cdot\text{m};\ M_{c(x=4)} = 8 \text{ kN}\cdot\text{m}$$

  Bending moment is zero at $11x - 20 - x^2 = 0$ or $x = 2.3$ m
- Portion CD: $M_x = 11x - 20 - 8(x - 2)$ (linear); $M_c = 8$ kN · m; $M_{d(x=6)} = 14$ kN · m
- Portion DB: $M_x = 11x - 20 - 8(x - 2) - 25(x - 6)$ (linear)

$$M_{d(x=6)} = -16 \text{ kN}\cdot\text{m};\ M_{e(x=8)} = -30 \text{ kN}\cdot\text{m}$$

  This is zero at $11x - 20 - 8(x - 2) - 25(x - 6) = 0$
  or $x = 6.64$ m or $8 - 6.64 = 1.36$ m from B.
  Bending moment diagram is shown in Fig. 4.32c.

Example 4.22 Draw the shear force and bending moment digrams for a beam 5 m long and loaded as shown in Fig. 4.33.

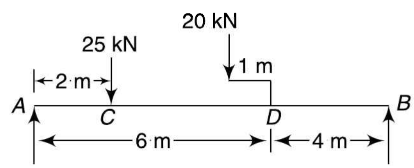


Fig. 4.33

Solution

Given A simply supported beam carrying loads as shown in Fig. 4.33.

To find To draw shear force and bending moment diagrams

The load diagram is reproduced in Fig. 4.34a. The effect of the bracket is to apply a load of 10 kN and a bending moment of (20 × 1) kN · m at the point D (Fig. 4.34b).

Taking moments about B,

$$R_a \times 10 - 25 \times 8 - 20 \times 4 - 20 = 0$$

or $$R_a = 30 \text{ kN and } R_b = 25 + 20 - 30 = 15 \text{ kN}$$

Shear force diagram

- Portion AC: $F_x = 30$ kN (constant)
- Portion CD: $F_x = 30 - 25$kN (constant)
- Portion DB: $F_x = 5 - 20 = -15$ kN (constant)

  Shear force diagram is shown in Fig. 4.34c.

Bending moment diagram

- Portion AC: $M_x = 30x$ (linear)

$$M_{a(x=0)} = 0;\ M_{c(x=2)} = 60 \text{ kN}\cdot\text{m}$$

- Portion CD: $M_x = 30x - 25(x - 2)$ (linear)

$$M_{c(x=2)} = 60 \text{ kN};\ M_{d(x=6)} = 80 \text{ kN}\cdot\text{m}$$

- Portion DB: $M_x = 30x - 25(x - 2) - 20 - 20(x - 6)$ (linear)

$$M_{d(x=6)} = 60 \text{ kN}\cdot\text{m};\ M_{b(x=10)} = 0$$

Bending moment diagram is shown in Fig. 4.34d.

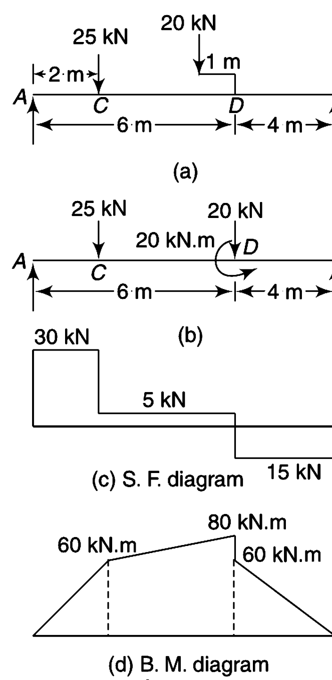


Fig. 4.34

Example 4.23 A 14-m long simply supported beam with an overhang at the right end is loaded as shown in Fig. 4.35. It carries a load of 12 kN applied through a bracket and also a uniformly distributed load for 8 m of length from the right end. Draw the shear force and bending moment diagrams.

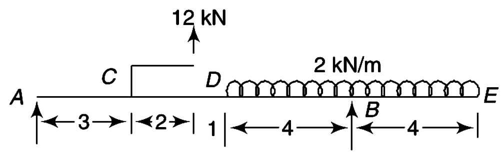


Fig. 4.35

Solution

Given A simply supported beam with a right overhang and carrying loads as shown in Fig. 4.35.
To find To draw shear force and bending moment diagrams

The load diagram is reproduced in Fig. 4.36a. The effect of the bracket is to apply a load of 12 kN and a bending moment of (12 × 2) kN · m at the point C (Fig. 4.36b).
Taking moments about B,

$$R_a \times 10 = 12 \times 7 - 24 \times 0$$

or $\quad R_a = 6$ kN and $R_b = 12 + 16 - 6 = 22$ kN

Shear force diagram

- Portion AC: $F_x = 6$ kN (constant)
- Portion CD: $F_x = 6 - 12 = -6$ kN (constant)
- Portion DB: $F_x = -6 - 2(x - 6)$ (linear)

$$F_{d(x=6)} = -6 \text{ kN};\ F_{b(x=10)} = -14 \text{ kN}$$

- Portion BE: $F_x = -6 - 2(x - 6) + 22$ (linear)

$$F_{b(x=10)} = 8 \text{ kN};\ F_{b(x=14)} = 0$$

Shear force diagram is shown in Fig. 4.36c.

Bending moment diagram

- Portion AC: $M_x = 6x$ (linear)

$$M_{a(x=o)} = 0;\ M_{c(x=3)} = 18 \text{ kN} \cdot \text{m}$$

- Portion CD: $M_x = 6x - 12(x - 3) + 24$ (linear)

$$M_{c(x=3)} = 42 \text{ kN};\ M_{d(x=6)} = 24 \text{ kN} \cdot \text{m}$$

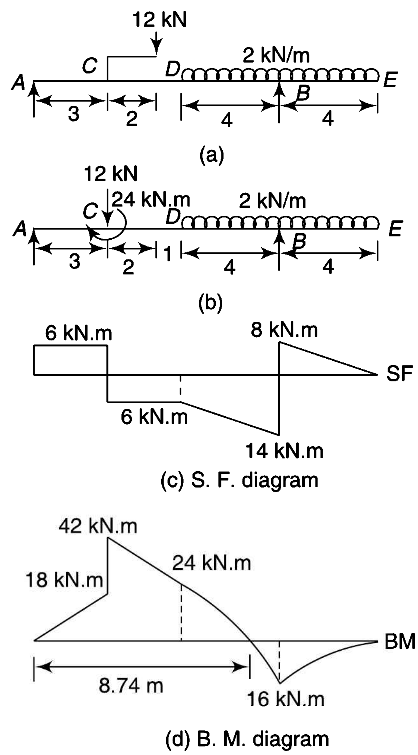


Fig. 4.36

- Portion DB: $M_x = 6x - 12(x - 3) + 24 - \dfrac{2(x-6)^2}{2}$ (parabolic)

$$M_{d(x=6)} = 24 \text{ kN};\ M_{b(x=10)} = -16 \text{ kN}$$

It is zero at $6x - 12(x - 3) + 24 - \dfrac{2(x-6)^2}{2} = 0$ or $-6x + 60 - x^2 - 36 + 12x = 0$

or $\quad x^2 - 6x - 24 = 0$ or $x = \dfrac{6 \pm \sqrt{36 + 4 \times 24}}{2} = 3 \pm 5.74 = 8.74$ m (taking positive value)

- Portion BE: *from end B,* $M_x = -\dfrac{2x^2}{2} = -x^2$ (parabolic)

$$F_{b(x=4)} = -16 \text{ kN};\ F_{e(x=0)} = 0$$

Shear force diagram is shown in Fig. 4.35d.

4.10 BEAMS WITH HINGED JOINTS

When beams are provided with hinged joints, only shear force with no bending moment can exist at the joints. The procedure is explained in the subsequent examples.

Example 4.24 Draw shear force and bending moment diagrams for the beam shown in Fig. 4.37.

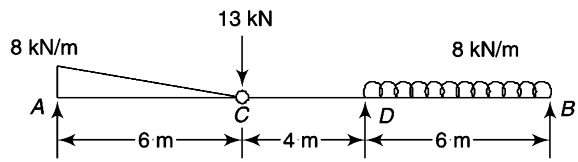


Fig. 4.37

Solution

Given A beam with three supports and a hinged joint carrying loads as shown in Fig. 4.37.
To find Shear force and bending moment diagrams

The load diagram is reproduced in Fig. 4.38a. There is no bending moment at hinged point C.

Support reactions

Taking moments about C for the left portion,

$$R_a \times 6 = \left(\frac{1}{2} \times 8 \times 6\right)\left(\frac{2}{3} \times 6\right) \text{ or } R_a = 16 \text{ kN}$$

Taking moments about C for the right portion,

$$R_b \times 10 + R_d \times 4 = 6 \times 8 \times (4 + 3) = 336$$

or $\quad R_d + 2.5R_b = 84 \qquad$ (i)

Total load, $R_a + R_d + R_b = 0.5 \times 8 \times 6 + 13 + 48 = 85$

or $\quad R_a + R_b = 85 - 16 = 69 \qquad$ (ii)

Subtracting (ii) and (i), $1.5R_b = 15$ or $R_b = 10$ kN

$$R_d = 69 - 10 = 59 \text{ kN}$$

Thus $R_a = 16$ kN ; $R_b = 10$ kN and $R_d = 59$ kN

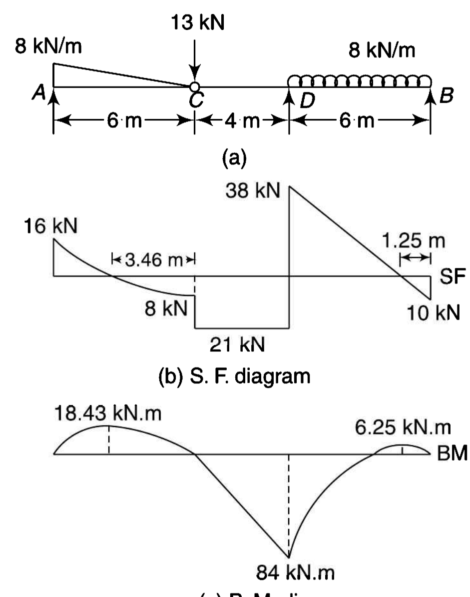


Fig. 4.38

Shear force diagram

- Portion DB: $F_x = -10 + 8x$ (linear); $F_b = -10$ kN; $F_{d(x=6)} = 38$ kN
  Shear force is zero at $-10 + 8x = 0$ or $x = 1.25$ m
- Portion CD: $F_x = 38 - 59 = -21$ kN (constant)
- Portion AC: $F_x = -21 + 13 + \frac{8}{6}x \cdot \frac{x}{2} = -8 + \frac{2x^2}{3}$ (x from C) (linear)

$$F_c = -8 \text{ kN}; F_{a(x=6)} = 16 \text{ kN}$$

Shear force is zero at $-8 + \dfrac{2x^2}{3} = 0$ or $x = 3.46$ m

Figure 4.37b shows the shear force diagram.

Bending moment diagram

- Portion DB: $M_x = 10x - 8\frac{x^2}{2}$ (parabolic); $M_b = 0$; $M_{d(x=6)} = -84$ kN · m

  Maximum bending moment $M_{(x=1.25)} = 10 \times 1.25 - 8 \times \frac{1.25^2}{2} = 6.25$ kN·m

- Portion CD: $M_x = -10x + 48(x-3) - 59(x-6)$ (linear)

$$M_{d(x=6)} = 84 \text{ kN}; M_{c(x=10)} = 0$$

It is convenient to consider x from C, then

$$M_x = 10(x+10) - 48(x+7) + 59(x+4) - 13x - \left(\frac{8}{6} \cdot x \cdot \frac{x}{2}\right)\frac{x}{3} \qquad \text{(cubic)}$$

$$M_c = 0; M_{a(x=6)} = 0$$

Maximum bending moment is at $x = 3.46$ m where shear force is zero,

$$M_{(x\,=\,3.463)} = 10(3.46+10) - 48(3.46+7) + 59(3.46+4) - 13 \times 3.46 - (2/9)3.46^3$$
$$= 18.48 \text{ kN} \cdot \text{m}$$

As bending moment at A as well as at C is zero, the part AC may be assumed as simply supported beam. Using the results of Example 5.11,

Maximum bending moment occurs at $\frac{l}{\sqrt{3}} = \frac{6}{\sqrt{3}} = 3.46$ m from C

Maximum bending moment $= \frac{wl^2}{9\sqrt{3}} = \frac{8X6^2}{9\sqrt{3}} = 18.48$ kN·m

Bending moment diagram is shown in Fig. 4.38c.

Example 4.25 Two beams AB and BC are joined together by a frictionless hinged joint to form beam ABC as shown in Fig. 4.39.

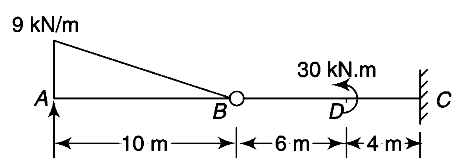


Fig. 4.39

Solution

Given Two beams joined together with a hinged joint carrying a varying load and a couple as shown in Fig. 4.39.
To find Shear force and bending moment diagrams

The load diagram is reproduced in Fig. 4.40a. There is no bending moment at the hinged point B.

Taking moments about B for the left portion,

$$R_a \times 10 = \frac{10 \times 9}{2} \times \frac{10 \times 2}{3} = 30 \text{ or } \quad R_a = 30 \text{ kN}$$

$$R_c = \frac{10 \times 9}{2} - 30 = 15 \text{ kN}$$

Shear force diagram

Intensity of loading at distance x from $B = \frac{9}{10} \cdot x$ (Fig. 4.39b)

Total triangular load = 45 kN

- Portion *AB* (*x* from *B*):

$$F_x = 30 - (45 - \text{Triangular load } BEF)$$

$$= -15 + \frac{9x}{10}\cdot\frac{x}{2} = -15 + 0.45x^2 \quad \text{(linear)}$$

$$F_{b(x=0)} = -15 \text{ kN}; \; F_{a(x=10)} = 30 \text{ kN}$$

Shear force is zero at $-15 + 0.45x^2 = 0$ or $x = 5.77$ m

- Portion *BDC* (*x* from *B*): $F_x = -15$ kN (constant)

Figure 4.40c shows the shear force diagram.

Bending moment diagram

- Portion *AB* (*x* from *B*): $M_x = 15x - 0.15x^3$ (cubic)

$$M_b = 0; \; M_{a(x=10)} = 0$$

Maximum bending moment

$$M_{(x=5.77)} = 15 \times 5.77 - 0.15 \times \frac{5.77^3}{2} = 72.1 \text{ kN}\cdot\text{m}$$

As bending moment at *B* is zero, the part *BC* may be assumed as a cantilever.

- Portion *BD* (*x* from *B*): $M_x = -15x$ (linear)

$$M_{b(x=0)} = 0; \qquad M_{d(x=6)} = -90 \text{ kN}\cdot\text{m}$$

- Portion *DC* (*x* from *B*): $M_x = -15x - 30$ (linear)

$$M_{d(x=6)} = -120; \qquad M_{c(x=10)} = -180 \text{ kN}\cdot\text{m}$$

As a check, taking moments of all the forces about C,

$$-30 \times 20 + 45 \times \left(10 + \frac{2}{3} \times 10\right) + 30 = -180 \text{ kN}\cdot\text{m}$$

Figure 4.40d shows the bending moment diagram.

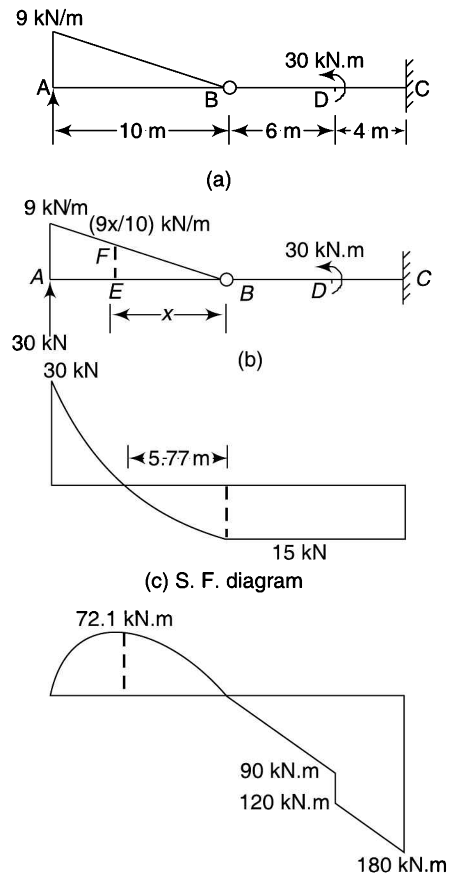


Fig. 4.40

4.11 INCLINED LOADING

In case the loads on the beam are not perpendicular to the axis of the beam, the loading can be resolved axially and transversely to the beam. The transverse component of the load produces shear force and bending moment whereas the axial component produces pull or push. This horizontal reaction is taken by the hinged support only as a roller support is unable to do so.

Example 4.26 A 10-m long horizontal beam is loaded as shown in Fig. 4.41. Draw the axial force, shear force and the bending moment diagrams.

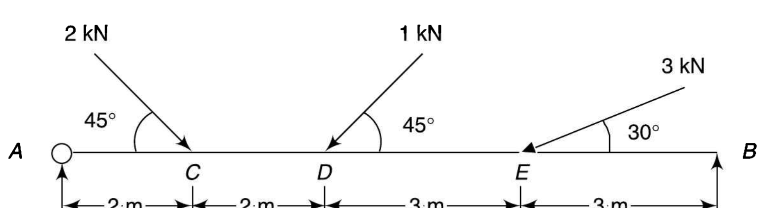


Fig. 4.41

Solution

Given A simply supported beam loaded with inclined loads as shown in Fig. 4.41.
To find Axial force, shear force and bending moment diagrams

The load diagram is reproduced in Fig. 4.42a. The inclined forces can be replaced by their horizontal and vertical components as shown in Fig. 4.42b.

Taking moments about the hinged end A,

$$R_b \times 10 = 1414 \times 2 + 707 \times 4 + 1500 \times 7 \text{ or } R_b = 1615.6 \text{ N}$$

and $$R_a = 1414 + 707 + 1500 - 1615.6 = 2005.4 \text{ N}$$

Total horizontal load on the beam = 1414 – 707 – 2598 = –1891 N

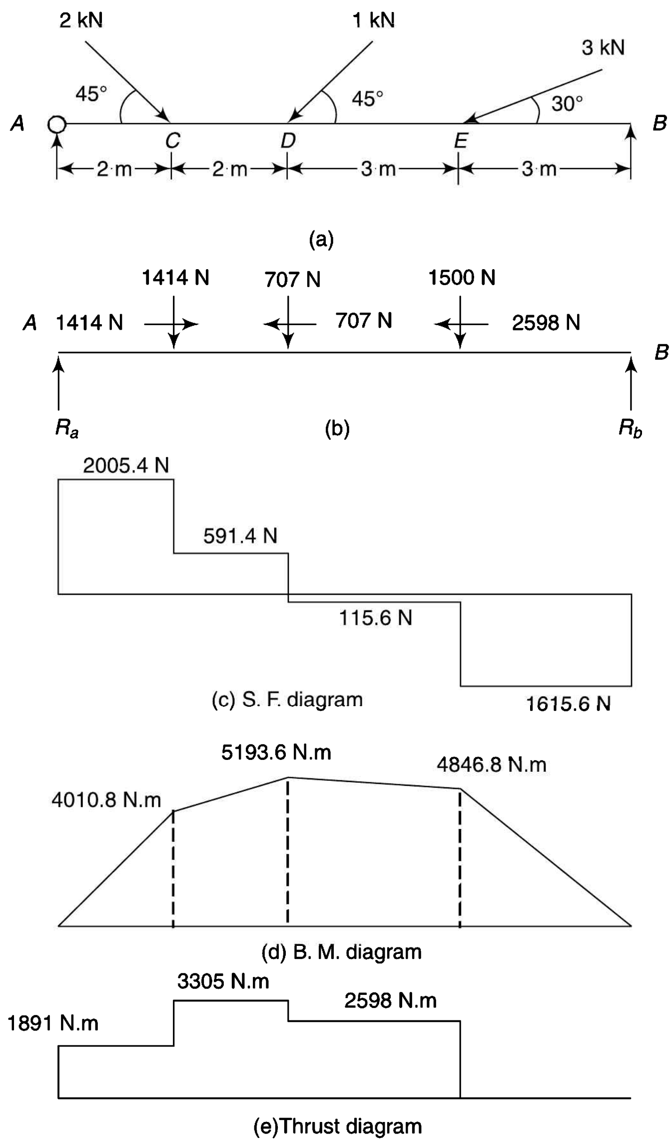


Fig. 4.42

The roller support at B does not provide any horizontal reaction. Thus the horizontal reaction at the hinged support at A = 1891 N

Shear force diagram

- Portion AC: $F_x = 2005.4$ N
- Portion CD: $F_x = 2005.4 - 1414 = 591.4$ N
- Portion DE: $F_x = 591.4 - 707 = -115.6$ N
- Portion EB: $F_x = -115.6 - 1500 = -1615.6$ N

  Shear force diagram is shown in Fig. 4.42c.

Bending moment diagram

- Portion AC: $M_x = 2005.4x$; $M_a = 0$; $M_{c(x=2)} = 4010.8$ N.m
- Portion CD: $M_x = 2005.4x - 1414\,(x-2)$; $M_{c(x=2)} = 4010.8$ N.m; $M_{d(x=4)} = 5193.6$ N.m
- Portion DE: $M_x = 2005.4x - 1414\,(x-2) - 707\,(x-4)$;

  $$M_{d(x=4)} = 5193.6 \text{ N.m};\ M_{e(x=7)} = 4846.8 \text{ N.m}$$
- Portion EB: $M_x = 2005.4x - 1414\,(x-2) - 707\,(x-4) - 1500(x-7)$;

  $$M_{e(x=7)} = 4846.8 \text{ N.m};\ M_b = 0$$

  Bending moment diagram is shown in Fig. 4.42d.

Thrust diagram (Axial force diagram)

- Portion AC: $F_x = 1891$ N
- Portion CD: $F_x = 1891 + 1414 = 3305$ N
- Portion DE: $F_x = 3305 - 707 = 2598$ N
- Portion EB: $F_x = 2598 - 2598 = 0$

  Figure 4.42e shows the thrust diagram.

4.12 LOADING AND BENDING MOMENT DIAGRAMS FROM SHEAR FORCE DIAGRAM

The loading and bending moment diagrams can easily be drawn from the shear force diagram if the following points are kept in mind:

The shear force diagram consists of

— rectangles in case of point loads
— inclined lines for the portion of uniformly distributed load
— parabolic curve for the portion of triangular or trapezium load

The bending moment diagram consists of

— inclined lines in case of point loads
— parabolic curve for the portion of uniformly distributed load
— cubic curve for the portion of triangular or trapezium load

Example 4.27 Figure 4.43 shows a shear force diagram for a beam resting on two supports, one being at the left-hand end. Determine the loading on the beam from the shear force diagram. Also, draw the bending moment diagram.

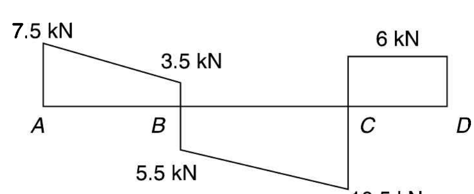


Fig. 4.43

Solution

Given Shear force diagram of a beam as shown in Fig. 4.43. The beam rests on two supports one on the left end.

To find Load and bending moment diagrams

Loading diagram

The shear force diagram is reproduced in Fig. 4.44a. It is given that one support is at the left end. On inspection of the given shear force diagram it is easy to understand that the other support is at the point at C where the diagram indicates an upward force or reaction of the support. As there are drops at points B and D in the shear force diagram, there are point loads at these points.

Thus reaction at A = 7.5 kN

- As there is a drop of 4 kN in a span length of 4 m in a linear manner, there is uniformly distributed load between AB at the rate of 1 kN/m.
- As there is total drop of (3.5 + 5.5 = 9) kN at point B, there is a point load of 9 kN at B.
- Between BC, there is drop of 5 kN in a span length of 5 m in a linear manner, there is uniformly distributed load between AB at the rate of 1 kN/m.
- There is an upward force of (10.5 + 6 =16.5 kN) at C which indicates the reaction of the support.
- There is drop of 6 kN at D indicating a point load of 6 kN. Load diagram is shown in Fig. 4.44b.

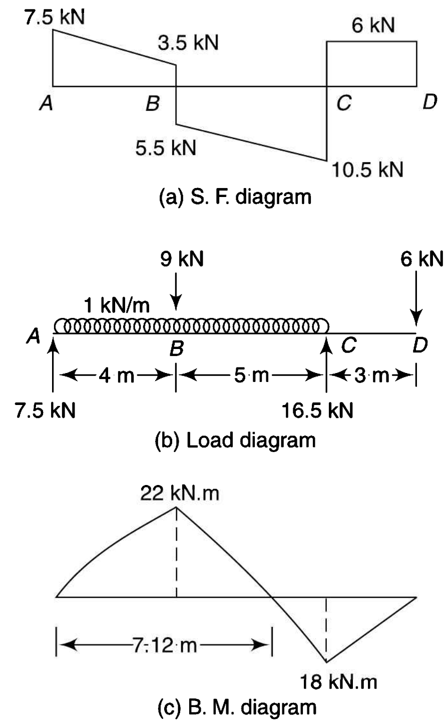


Fig. 4.44

Bending moment diagram

Portion AB: $M_x = 7.5x - \dfrac{1 \times x^2}{2}$ (parabolic);

$$M_a = 0;\ M_{b(x=4)} = 22 \text{ kN}\cdot\text{m}$$

Portion BC: $M_x = 7.5x - \dfrac{x^2}{2} - 9(x-4)$ (parabolic);

$$M_{b(x=4)} = 22 \text{ kN}\cdot\text{m};\ M_{c(x=9)} = -18 \text{ kN}\cdot\text{m}$$

It is zero when $7.5x - \dfrac{x^2}{2} - 9(x-4) = 0$ or $x = 7.12$ m

Portion CD: $M_x = 7.5x - 9(x-4.5) - 9(x-4) + 16.5(x-9)$ (linear)

$$M_{c(x=9)} = -18 \text{ kN}\cdot\text{m};\ M_{d(x=12)} = 0$$

Bending moment diagram is shown in Fig. 4.44c.

Summary

1. A structural element which is subjected to loads transverse to its axis is known as a *beam*.
2. A beam with both of its ends on simple supports is known as a *simply supported beam*. Each support exerts a reaction on the beam.
3. A beam with one end fixed and the other end free is called a *cantilever.* There is a vertical reaction and moment at the fixed end (known as *fixing moment*).
4. Generally, beams with more than two reaction components cannot be analysed using the equations of static equilibrium alone and are known as *statically indeterminate beams*.

5. *Shear force* is the unbalanced vertical force on one side (to the left or right) of a section of a beam and is the sum of all the normal forces on one side of the section.
6. Shear force is considered positive when the resultant of the forces to the left of a section is upwards or to the right downwards.
7. A *shear force diagram* shows the variation of shear force along the length of a beam.
8. *Bending moment* (B M) at a section of a beam is the algebraic sum of the moments about the section of all the forces on one side of the section.
9. The bending moment causing concavity upwards is referred as *sagging* bending moment and is taken positive. A bending moment causing convexity upwards is taken as negative and is called *hogging* bending moment.
10. A *bending moment diagram* shows the variation of bending moment along the length of a beam.
11. Rate of change of shear force (or slope of the shear force curve) is equal to intensity of loading.
12. Rate of change of bending moment curve is equal to the shear force.
13. The point of zero bending moment, i.e., where the type of bending changes from sagging to hogging is called a point of *inflection* or *contraflexture*.
14. The variation of bending moment between two sections is equal to the area under the shear force diagram.
15. The variation of shear force between two sections is equal to the area under the load distribution diagram.
16. In case the loads on the beam are not perpendicular to the axis of the beam, the loading can be resolved axially and transversely to the beam.
17. The transverse component of the load produces shear force and bending moment whereas the axial component produces pull or push. This horizontal reaction is taken by the hinged support only as a roller support is unable to do so.
18. The shear force diagram consists of rectangles in case of point loads, inclined lines for the portion of uniformly distributed load and parabolic curve for the portion of triangular or trapezium load.
19. The bending moment diagram consists of inclined lines in case of point loads, parabolic curve for the portion of uniformly distributed load and cubic curve for the portion of triangular or trapezium load.

Objective Type Questions

1. Wherever the bending moment is maximum the shear force is
(a) zero (b) also maximum (c) minimum (d) of any value

2. The point of contraflexure lies where
(a) shear force changes sign (b) bending moment is zero or changes sign
(c) shear force is zero (d) bending moment is maximum

3. For a moving load on a simply supported beam, the maximum bending moment occurs
(a) at the supports (b) under the load (c) at the midspan

4. The bending moment diagram for a cantilever subjected to bending moment at the free end is
(a) triangle (b) rectangle (c) parabola (d) elliptical

5. The variation of shear force due to a uniformly distributed load is by
(a) cubic law (b) parabolic law (c) linear law (d) uniform law

6. The variation of bending moment due a point load is by
(a) cubic law (b) parabolic law (c) linear law (d) uniform law

7. The variation of bending moment due uniformly distributed load is by
(a) cubic law (b) parabolic law (c) linear law (d) uniform law

8. Maximum bending moment in a cantilever carrying a point load at the free end occurs at the
(a) free end (b) mid-span (c) fixed end

9. At the point of application of a point load on a beam there is
(a) maximum bending moment
(b) sudden change of shape of bending moment diagram
(c) maximum deflection
(d) point of contraflexure

10. Maximum bending moment in a cantilever carrying a uniformly distributed load is
(a) $\frac{wl^2}{4}$ (b) $\frac{wl^2}{8}$ (c) $\frac{wl^3}{4}$ (d) $\frac{wl^2}{2}$

11. Maximum bending moment in a simply supported beam carrying a point load at mid-span is
(a) $Wl/2$ (b) $Wl/4$ (c) $Wl/6$ (d) $Wl/8$

12. Maximum bending moment in a simply supported beam carrying a point load W at a distance a from one end of the span l is
(a) $\frac{Wa(l-a)}{l}$ (b) $\frac{Wa^2}{l}$ (c) $\frac{Wa(l-a)^2}{l}$ (d) $\frac{Wa(l-a)}{l^2}$

13. Maximum bending moment in a simply supported beam carrying a uniformly distributed load is
(a) $\frac{wl^2}{4}$ (b) $\frac{wl^2}{8}$ (c) $\frac{wl^2}{12}$ (d) $\frac{wl}{4}$

14. Variation of shear force in a cantilever carrying a load the intensity of which varies uniformly from zero at the free end to w per unit run at the fixed end is by
(a) cubic law (b) parabolic law (c) linear law (d) none of these

15. Variation of bending moment in a cantilever carrying a load the intensity of which varies uniformly from zero at the free end to w per unit run at the fixed end is by
(a) cubic law (b) parabolic law (c) linear law (d) none of these

16. A simply supported beam carries a couple at a point on its span, the shear force
(a) varies by cubic law (b) varies by parabolic law
(c) varies linearly (d) is uniform throughout

Answers

1. (a)	2. (b)	3. (c)	4. (b)	5. (c)	6. (c)
7. (b)	8. (c)	9. (b)	10. (d)	11. (b)	12. (a)
13. (b)	14. (b)	15. (a)	16. (d)		

Review Questions

4.1 What is a beam? What do you mean by a *statically indeterminate beam*?
4.2 What are the main types of supports? Distinguish between roller and hinged supports.
4.3 How are beams classified? Give a brief account.
4.4 Define the terms 'shear force' and 'bending moment.' How are they considered positive and negative? What are sagging and hogging bending moments?
4.5 Derive the relation between bending moment and shear force in a beam. What do you mean by point of *inflection* or *contraflexure*?
4.6 What is the relation between shear force and loading function of a beam?
4.7 Indicate the shapes of shear force diagram in case of uniformly distributed load and for triangular loads.
4.8 Indicate the shapes of bending moment diagram for point loads and uniformly distributed loads.

Numerical Problems

4.1 A 5-m long cantilever beam carries a point load of 3 kN at the free end alongwith three more point loads of 2 kN, 2 kN and 1 kN at 1 m, 3 m and 4 m respectively from the fixed end. A uniformly distributed load of 2 kN/m also acts on the beam starting from 2 m and ending at 4 m from the fixed end. Draw the shear force and bending moment diagrams.

4.2 A simply supported beam of 8-m span carries point loads of 24 kN and 40 kN at distances of 2 m from each end. Draw the shear force and bending moment diagrams.

4.3 A 6-m long simply supported beam carries concentrated loads of 4 kN, 10 kN and 8 kN at 1m, 3 m and 4 m from the left-hand end. Draw shear force and bending moment diagrams.

4.4 A 12-m long beam simply supported at the ends carries a point load of 40 kN at 3 m from the left end and a uniformly distributed load of 10 kN/m on the right half of the span. Draw the shear force and bending moment diagrams indicating principal values.

4.5 A simply supported beam has a span of 9 m and carries a uniformly distributed load of 20 kN/m over the whole span alongwith two point loads of 30 kN and 40 kN at 6 m and 7.5 m respectively from the left-hand support. Draw the shear force and bending moment diagrams indicating the values at the point loads.

4.6 A 10-m long beam is simply supported at 1 m from the left end and 3 m from the right end. Concentrated loads of 6 kN and 8 kN act on the beam at the left end and right end respectively. Draw the shear force and bending moment diagrams.

4.7 A 6-m long simply supported beam carries a point load of 25 kN at the right end and a uniformly distributed load of 15 kN/m on the whole span. The two supports are 4 m apart, the left-hand support being at the left end. Draw shear force and bending moment diagrams.

4.8 The two supports of a simply supported beam are 5 m apart. The beam is 8 m long with two overhangs of 2 m on the left end and 1 m on the right end. The beam carries concentrated loads of 40 kN at the left end and 20 kN at the right end. In addition, it also carries 40 kN load at the midspan and 20 kN at 2 m from the right end of the beam. Draw shear force and bending moment diagrams for the beam.

4.9 A simply supported beam of 9-m length carries a point load of 10 kN at the right end and a uniformly distributed load of 30 kN/m for a distance of 3 m starting from left end. The supports of the beam are 6 m apart, the left end support being at the left end. Draw the shear force and bending moment diagrams indicating main values.

4.10 A 10-m long beam ABC is simply supported at A and B, B being 2 m from the right end of the beam. It carries point loads of 8 kN and 4 kN at distances 3 m and 5 m from A. The beam also has two uniformly distributed loads of intensity 4 kN/m for a distance of 4 m starting from A and of 6 kN/m on BC. Draw shear force and bending moment diagrams indicating principal values.

4.11 A simply supported beam of 10-m length carries a uniformly distributed load throughout its length. The supports of the beam are to be 6 m apart. Determine the position of the supports with respect to the ends so that the bending moment on the beam is the least possible. (2.23 m, 1.77 m)

4.12 A 15-m long girder carries a uniformly distributed load of w kN/m and is supported on two piers 9 m apart in such a way that the maximum bending moment is as small as possible. Determine the distance of piers from the ends of the girder and the maximum bending moment. (2.656 m ; 3.344 m)

4.13 A 23-m long simply supported beam has its supports 15 m apart, the left hand support being at the left end of the beam. The beam carries a load of 16 kN at 5 m from left end, and a distributed load the intensity of which varies linearly from zero at each end to 8 kN/m at the right hand support. Draw the shear force and bending moment diagrams. Find the magnitude and position of the maximum bending moment. (114.9 kN · m, 5.81 m)

4.14 A 5-m long overhanging beam of negligible weight has its supports 4 m apart, the overhang being on the left end. It carries a uniformly varying load the intensity of which varies linearly from zero at the left end to 60 kN/m at the right end. Draw the shear force and bending moment diagrams indicating salient values.

4.15 A 6-m long simply supported beam carries a load of 1.5 kN at 5 m from the left end. It also carries a distributed load of 3 kN between 1 m and 4 m from the left end, the intensity of which increases linearly from zero at start to the maximum value at the end. Draw the shear force and bending moment diagrams.

4.16 A cantilever beam carries a distributed load the intensity of which varies linearly from 10 kN/m at the fixed end to 5 kN/m at the free end. Draw the shear force and bending moment diagrams.

4.17 A 6-m long simply supported beam is loaded with distributed load that varies parabolically from zero at each end to a maximum at the midspan. If the total load is to be 7.2 kN, draw the shear force and bending moment diagrams indicating principal values.

4.18 A simply supported beam ABC of 10 m span is supported at A and B, A and B being 8 m apart. The beam carries a load of 4 kN at a distance of 6 m from A and another of the same magnitude at the right end. A counter-clockwise couple of 8 kN · m also acts at a distance of 3 m from A. Draw the shear force and bending moment diagrams indicating principal values.

4.19 A simply supported beam of 8 m span carries a uniformly distributed load of 10 kN/m over the left half and a counter-clockwise couple at 6 m from the left end. The reaction at the left support is found to be 55 kN. Draw the shear force and bending moment diagrams.

Chapter 5

Bending Stress in Beams

If a member is subjected to equal and opposite couples acting in the same longitudinal planes, the member is said to be in *pure bending*. If a constant bending moment (no shear force) acts on some length of a beam, the stresses set up on any cross-section on that part of the beam constitute a pure couple, the magnitude of which is equal to the bending moment. The internal stresses developed in the beam are known as *flexural* or *bending stresses*. In Fig. 5.1a, the portion *CD* of the simply supported beam is under pure bending as there exists no shear force. If the end sections of a straight beam are considered to remain plane, the beam under the action of bending moment bends in such a way that the inner or the concave edge of cross-section undergoes compression and the outer or the convex edge, tension. There is an intermediate surface known as *neutral surface*, at which the stress is zero. An axis obtained by intersection of the neutral surface, and a cross-section is known as *neutral axis* about which the bending of the surface takes place (Fig. 5.1b).

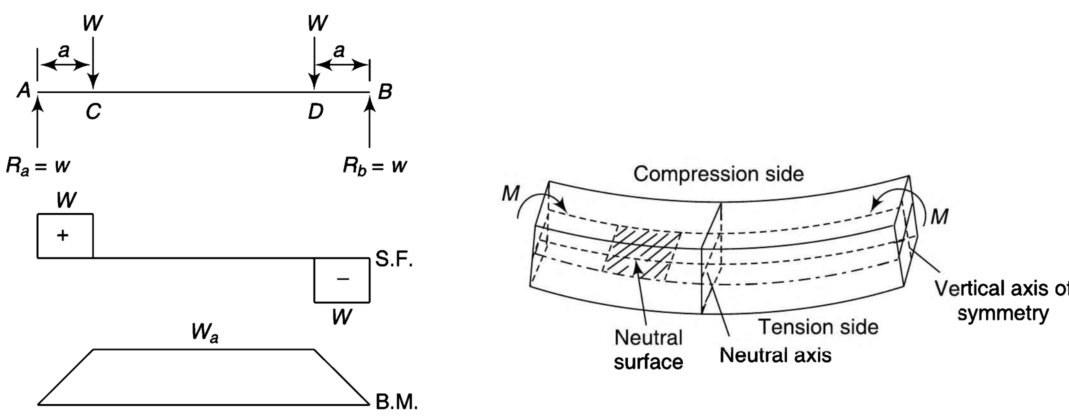


Fig. 5.1

5.1 THEORY OF SIMPLE BENDING

The following theory is applicable to the beams subjected to simple or pure bending when the cross-section is not subjected to a shear force since that will cause a distortion of the transverse planes. The assumptions being made are as under:

(i) The material is homogeneous and isotropic, i.e., it has the same values of Young's modulus in tension and compression.

(ii) Transverse planes remain plane and perpendicular to the neutral surface after bending.

(iii) Initially, the beam is straight and all longitudinal filaments are bent into circular arcs with a common centre of curvature which is large compared to the dimensions of the cross-section.

(iv) The beam is symmetrical about a vertical longitudinal plane passing through vertical axis of symmetry for horizontal beams.

(v) The stress is purely longitudinal and the stress concentration effects near the concentrated loads are neglected.

Consider a length of beam under the action of a bending moment M as shown in Fig. 5.2a. N-N is the original length considered of the beam. The neutral surface is a plane through X-X. In the side view NA indicates the neutral axis. O is the centre of curvature on bending (Fig. 5.2b).

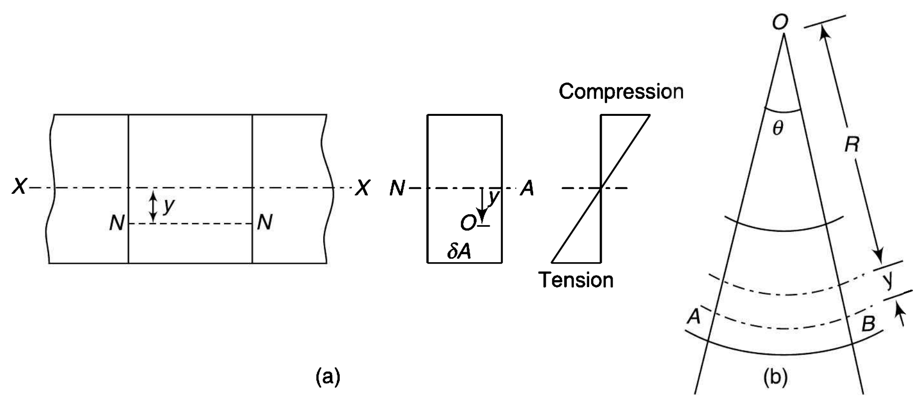


Fig. 5.2

Let R = radius of curvature of the neutral surface

θ = angle subtended by the beam length at centre O

σ = longitudinal stress

A filament of original length NN at a distance y from the neutral axis will be elongated to a length AB

The strain in $AB = \dfrac{AB - NN}{NN}$ (original length of filament AB is NN)

or
$$\frac{\sigma}{E} = \frac{(R + y)\theta - R\theta}{R\theta} = \frac{y}{R}$$

or
$$\frac{\sigma}{y} = \frac{E}{R} \tag{5.1}$$

Also
$$\sigma = y\frac{E}{R} \propto y \qquad \text{(As } E/R \text{ is constant)}$$

Thus stress is proportional to the distance from the neutral axis NA. This suggests that for the sake of weight reduction and economy, it is always advisable to make the cross-section of beams such that most of the material is concentrated at the greatest distance from the neutral axis. Thus there is universal adoption of the I-section for steel beams.

Now let δA be an element of cross-sectional area of a transverse plane at a distance y from the neutral axis NA (Fig. 5.2).

For pure bending,

Net normal force on the cross-section = 0

or $$\int \sigma \cdot dA = 0$$

or $$\int \frac{E}{R} y \cdot dA = 0 \text{ or } \frac{E}{R}\int y \cdot dA = 0$$

or $$\int y \cdot dA = 0 \tag{5.2}$$

This indicates the condition that the neutral axis passes through the centroid of the section.

Also, bending moment = moment of the normal forces about neutral axis

or $$M = \int (\sigma \cdot dA) y = \int \frac{E}{R} y \cdot dA \cdot y = \frac{E}{R}\int y^2 \cdot dA$$

$$= \frac{EI}{R} \tag{5.3}$$

where $I = \int y^2 \cdot dA$ and is known as the *moment of inertia* or *second moment of area* of the section.

From Eqs. 5.1, 5.2 and 5.3,

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R} \tag{5.4}$$

Conventionally, y is taken positive when measured outwards from the centre of curvature.

The relation derived above is based on the theory of pure bending. In practice, however, a beam is subjected to bending moment and shear force simultaneously. But it will also be observed (Section 6.1) that in most of the cases of continuous loading, the greatest bending moment occurs where the shear force is zero which corresponds to the condition of simple or pure bending. Thus, the theory and the equations obtained above can safely be used with a reasonable degree of approximation for the design of beams and structures.

Section Modulus The ratio I/y where y is the farthest or the most distant point of the section from the neutral axis is called section modulus. It is denoted by Z.

$$Z = \frac{\text{Moment of inertia about neutral axis}}{\text{Distance of farthest point from neutral axis}}$$

Thus $$M = \sigma \cdot \frac{I}{y} = \sigma \cdot Z \tag{5.5}$$

Moment of Resistance The maximum bending moment which can be carried by a given section for a given maximum value of stress is known as the *moment of resistance* (M_r).

5.2 MOMENT OF INERTIA

Moment of inertia of a rigid body is obtained by summing up the products of its various particles with the square of their distances from a given axis.

Parallel Axis Theorem

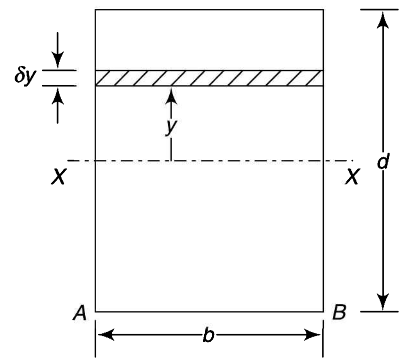


Fig. 5.3

The moment of inertia about any axis parallel to the centroidal axis is equal to the moment of inertia through the centroidal axis plus the product of the area of the figure and the square of the distance between the two axes.

Moment of inertia of different sections is given below.

(i) Rectangle Consider an elementary strip of δy thickness at distance y from XX (Fig. 5.3).

Then moment of inertia about centroidal axis x-x,

$$I_x = \int_{-d/2}^{d/2} y^2 \cdot bdy = b\int_{-d/2}^{d/2} y^2 dy = b\left(\frac{y^3}{3}\right)_{-d/2}^{d/2} = \frac{bd^3}{12}$$

By parallel axis theorem, $I_{ab} = I_{xx} + Ax^2 = \dfrac{bd^3}{12} + bd \cdot \left(\dfrac{d}{2}\right)^2 = \dfrac{bd^3}{3}$

$$\text{Section modulus, } Z_x = \frac{I_{xx}}{y_{\max}} = \frac{bd^3/12}{d/2} = \frac{bd^2}{6} \tag{5.6}$$

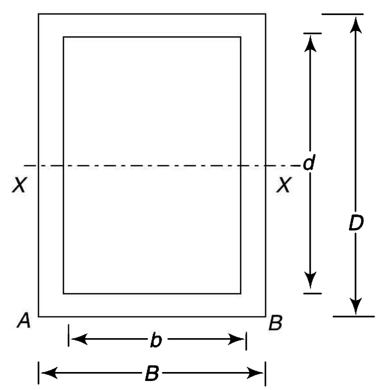


Fig. 5.4

(ii) Hollow Rectangle About centroidal axis X-X (Fig. 5.4),

$$I_{xx} = \frac{BD^3}{12} - \frac{bd^3}{12} = \frac{BD^3 - bd^3}{12}$$

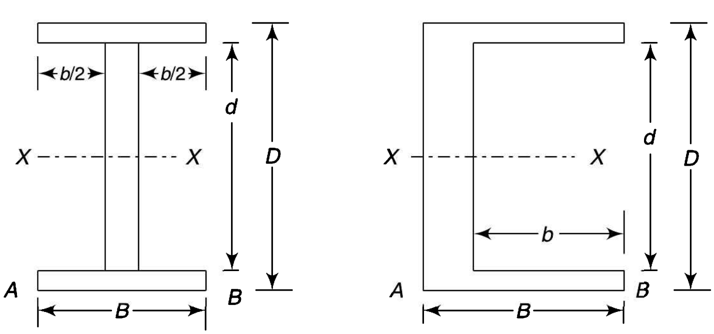
$$Z_x = \frac{(BD^3 - bd^3)/12}{D/2} = \frac{(BD^3 - bd^3)}{6D} \tag{5.7}$$

(iii) I-Section About centroidal axis XX (Fig. 5.5a),

$$I_{xx} = \frac{BD^3 - bd^3}{12} \tag{5.8}$$

where $b = B$ – thickness of the web

$d = D - 2 \times$ thickness of flange



(a) (b)

Fig. 5.5

$$Z_x = \frac{(BD^3 - bd^3)/12}{D/2} = \frac{(BD^3 - bd^3)}{6D} \tag{5.9}$$

For a channel section (Fig. 5.5b), the same expressions are valid.

(iv) Triangular Section About centroidal axis *XX* (Fig. 5.6),

$$I_{xx} = \frac{bd^3}{36} \tag{5.10}$$

About the base *AB*, $$I_{ab} = \frac{bd^3}{12} \tag{5.11}$$

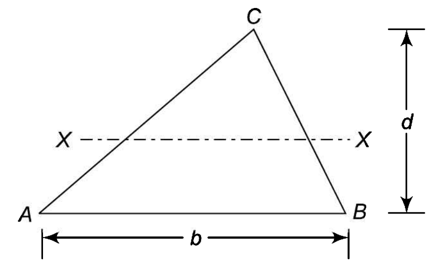


Fig. 5.6

(v) Circular Section About centroidal axis *XX* (Fig. 5.7a),

$$I_{xx} = \frac{\pi d^4}{64} \tag{5.12}$$

$$Z_x = \frac{I_{xx}}{y_{max}} = \frac{\pi d^4/64}{d/2} = \frac{\pi d^3}{32} \tag{5.13}$$

(vi) Hollow Circular Section

(Fig. 5.7b), $I_{xx} = \dfrac{\pi(D^4 - d^4)}{64}$

$$Z_x = \frac{I_{xx}}{y_{max}} = \frac{\pi(D^4 - d^4)/64}{D/2} = \frac{\pi}{32}\left(\frac{D^4 - d^4}{D}\right) \tag{5.14}$$

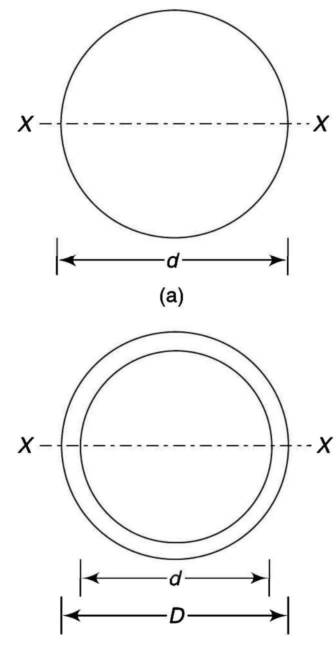


Fig. 5.7

Example 5.1 || Figure 5.8 shows a simply supported 200-mm wide, 300-mm deep and 4-m long beam. Determine the bending stress at the point *C* which is 60 mm below the top surface and 1.2 m from the left support.

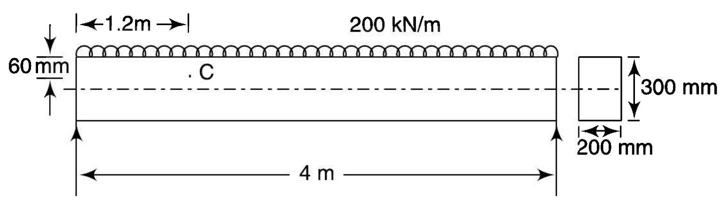


Fig. 5.8

Solution

Given A simply supported beam

$b = 200$ mm $d = 300$ mm

$w = 200$ kN/m $L = 4$ m

To find Bending stress at the point *C*, 60 mm below top surface and 1.2 m from left support

Calculation of bending moment

Reaction at each support, $R = \dfrac{200 \times 4}{2} = 400$ kN

Moment of inertia of the section = $\dfrac{200 \times 300^3}{12}$ = 450×10^6 mm^4

Bending moment at cross-section at $C = R \cdot x - \dfrac{wx^2}{2} = 400 \times 1.2 - \dfrac{200 \times 1.2^2}{2} = 336$ kN·m

Calculation of bending stress

Distance of C from the neutral axis = 150 – 60 = 90 mm

Now, $\dfrac{\sigma}{y} = \dfrac{M}{I}$ or $\sigma = \dfrac{336 \times 10^6}{450 \times 10^6} \times 90 = 67.2$ MPa

Example 5.2 A 120-mm wide and 10-mm thick steel plate is bent into a circular arc of 8 m radius. Determine the maximum value of stress produced. Also find the bending moment which will produce the maximum stress. E = 200 GPa.

Solution

Given A thick steel plate bent into a circular arc

b = 120 mm $\quad$ d = 10 mm

R = 8 m $\quad$ E = 200 GPa

To find Bending stress and bending moment

We have, $\dfrac{\sigma}{y} = \dfrac{M}{I} = \dfrac{E}{R}$

Thus, $\sigma = \dfrac{E}{R} y = \dfrac{200\,000}{8\,000} \times 5 = 125$ MPa

and $M = \dfrac{E}{R} I = \dfrac{200\,000}{8000} \times \dfrac{120 \times 10^3}{12} = 250\,000$ N·mm or 250 N·m

Example 5.3 A simply supported cast-iron square beam of 800-mm length and 15 mm × 15 mm in section fails on applying a load of 360 N at the midspan. Find the maximum uniformly distributed load that can be applied safely to a 40-mm wide, 75-mm deep and 1.6-m long cantilever made of the same material.

Solution

Given A simply supported beam

b = 15 mm $\quad$ d = 15 mm

l = 800 mm $\quad$ W = 360 N

To find Uniformly distributed load to be applied safely to a 40-mm wide, 75-mm deep and 1.6-m long cantilever made of the same material

Simply supported beam

Moment of resistance ($\sigma \cdot Z$) = Maximum bending moment

$$\sigma \cdot \frac{bd^2}{6} = \frac{Wl}{4}$$

or $\sigma \cdot \dfrac{15 \times 15^2}{6} = \dfrac{360 \times 800}{4}$ or σ = 128 MPa

Cantilever

b = 40 mm $\quad$ d = 75 mm

l = 1600 mm

Let the loading be w per m run on the cantilever to break it.

Maximum bending moment = $\dfrac{wl^2}{2} = \dfrac{w \times 1.6^2}{2}$ = $1.28w$ N·m

(In the above relation as w is in N/m, l has to be in consistent units i.e. in m)

Moment of resistance = $\sigma \cdot Z = \sigma \cdot \dfrac{bd^2}{6} = 128 \times \dfrac{40 \times 75^2}{6}$ = 4800×10^3 N·mm

or 4800 N·m

Equating moment of resistance

$1.28w = 4800$ or $w = 3750$ N/m

Example 5.4 || A floor carries a load of 8 kN/m^2 and is supported by joists that are 120 mm wide and 240 mm deep over a span of 6 m. Determine the spacing centre to centre of the joists if the maximum allowable bending stress is 10 MPa.

Solution

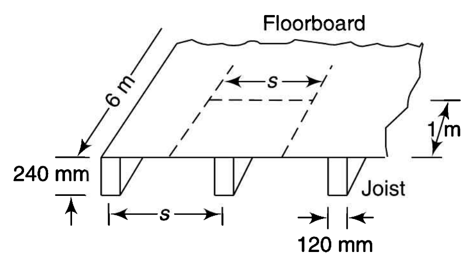


Fig. 5.9

Given A floor supported by joists as shown in Fig. 5.9,
To find Spacing s of joists centre to centre

Let the spacing of the joists be s m (Fig. 5.9),
Loading on the joist
Loading on the joist per unit length

w = Area supported by joist per unit length
× Load/unit area
= (Spacing of the joists × 1) × Load/unit area
= $s \times 8000 = 8000\,s$ N/m

Moment of resistance

Maximum bending moment $\dfrac{wl^2}{8} = \dfrac{8000s \times 6^2}{8}$ = $36\,000s$ N·m

$\therefore$ moment of resistance = $\sigma \cdot \dfrac{bd^2}{6} = 10 \times \dfrac{120 \times 240^2}{6}$

= $11\,520 \times 10^3$ Nmm or 11 520 N·m

Equating the two, $36\,000s = 11\,520$ or $s = 0.32$ m or 320 mm

Example 5.5 || A 200 mm × 80 mm *I*–beam is to be used as a simply supported beam of 6.75 m span.The web thickness is 6 mm and the flanges are of 10-mm thickness. Determine what concentrated load can be carried at a distance of 2.25 m from one support if the maximum permissible stress is 80 MPa.

Solution

Given An I-beam as shown in Fig. 5.10
To find Maximum point load at a distance of 2.25 m from one support, maximum permissible stress 80 MPa.

$$I = \frac{1}{12}(80 \times 200^3 - 74 \times 180^3) = 17.37 \times 10^6 \text{ mm}^4$$

Let W kN be the concentrated load so that the reaction at the supports are $W/3$ and $2W/3$ as shown in Fig. 5.9b.

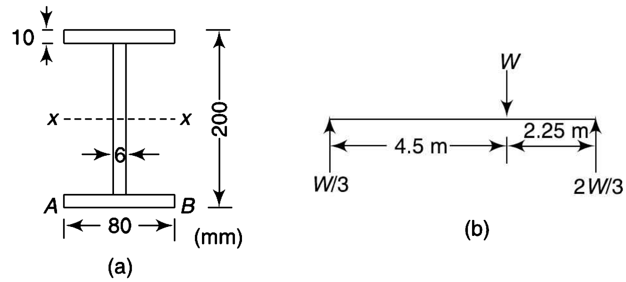


Fig. 5.10

Determination of load

Maximum bending moment = $\dfrac{W}{3} \times 4.5 = 1.5W$ kN·m or $1.5W \times 10^6$ N·mm

Now, $\dfrac{\sigma}{y} = \dfrac{M}{I}$ or $\dfrac{80}{100} = \dfrac{1.5W \times 10^6}{17.37 \times 10^6}$ or $W = 9.264$ kN

Example 5.6 A rectangular beam is to be cut out of a cylindrical log of wood with diameter *d*. Determine the ratio of depth to width of the strongest beam which can be had from log of wood.

Solution

Given A cylindrical log of wood of diameter d

To find Ratio of depth to width of strongest beam to be cut from log of wood

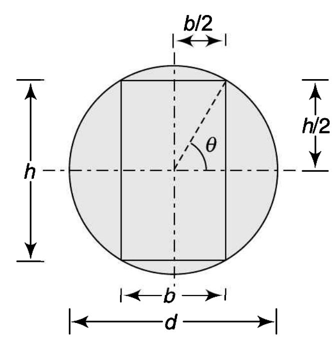


Fig. 5.11

Let d be the diameter of the cylindrical section of log and b and h, the width and the depth of the beam (Fig. 5.11).

Then, $b = 2 \times (d/2) \cos\theta = d\cos\theta$ and $h = 2 \times (d/2)\sin\theta = d\sin\theta$

Section modulus of beam

$$\text{Section modulus, } Z = \frac{bh^2}{6} = \frac{(d\cos\theta)(d\sin\theta)^2}{6} = \frac{d^3\cdot\cos\theta\cdot\sin^2\theta}{6}$$

Strongest section of beam

For the beam to be strongest, section modulus must be maximum.

i.e., $$\frac{dZ}{d\theta} = 0 \quad \text{or} \quad \frac{d^3}{6}(-\sin^3\theta + \cos\theta\cdot 2\sin\theta\cos\theta) = 0$$

or $$\sin^2\theta = 2\cos^2\theta \quad \text{or} \quad \tan^2\theta = 2 \quad \text{or} \quad \tan\theta = \sqrt{2}$$

which gives, $$\sin\theta = \sqrt{\frac{2}{3}} \quad \text{and} \quad \cos\theta = \frac{1}{\sqrt{3}}$$

$\therefore$ $$b = \frac{d}{\sqrt{3}} = 0.577\,d \quad \text{and} \quad h = \sqrt{\frac{2}{3}}d = 0.817d$$

Example 5.7 A hollow circular bar used as a beam has its outside diameter thrice the inside diameter. It is subjected to a maximum bending moment of 60 kN·m. Determine the inside diameter of the beam if the permissible bending stress is limited to 120 MPa.

Solution

Given A hollow circular beam,

$D = 3d$ $\quad M = 60$ kN·m $\quad \sigma = 120$ MPa

To find Inside diameter d

Section modulus of beam

$$Z_x = \frac{\pi}{32}\left(\frac{D^4 - d^4}{D}\right) = \frac{\pi}{32}\left(\frac{(3d)^4 - d^4}{3d}\right) = \frac{5\pi d^3}{6}$$

Applying bending equation

$$Z_x = \frac{M}{\sigma} = \frac{60 \times 10^6}{120} = 500 \times 10^3 \text{ mm}^3$$

$$\therefore \quad \frac{5\pi d^3}{6} = 500 \times 10^3 \quad \text{or} \quad d = 57.6 \text{ mm}$$

Example 5.8 || A 280 mm × 120 mm I–secion beam is to be used as a cantilever of 3.6-m length. Find the uniformly distributed load which can be carried by the beam if the permissible stress is 125 MPa. $I = 75 \times 10^6$ mm^4.

If the cantilever is strengthened by 10-mm thick steel plates welded at the top and bottom flanges to withstand a 40% increased load, find the width of the plates and the length over which the plates should extend, the maximum stress being the same.

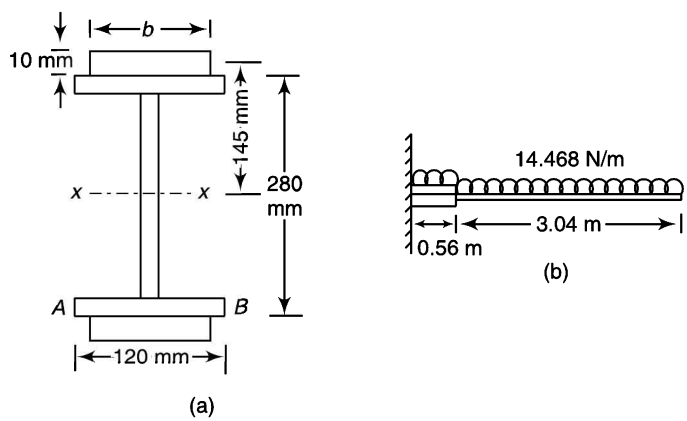


Fig. 5.12

Solution

Given A 280 mm × 120 mm cantilever of I-section

$I = 75 \times 10^6$ mm^4 $\quad \sigma = 125$ MPa

To find

— uniformly distributed load for simple beam

— width and length of 10 mm thick plates at top and bottom flanges for 40% increased load

Refer Fig. 5.12a,

Let w be the uniformly distributed load in N per m length.

Moment of resistance of I-section

Moment of resistance, $M_r = \dfrac{wl^2}{2} = \dfrac{w \times 3.6^2}{2} = 6.48w$ N·m or $6480w$ N·mm

Now, $\dfrac{\sigma}{y} = \dfrac{M}{I}$ or $\dfrac{125}{140} = \dfrac{6480w}{75 \times 10^6}$ or $w = 10\,334$ N/m

Thus $M_r = 6.48 \times 10\,334 = 66\,964$ N·m

Moment of resistance of strengthened section

When load increases by 40%, $w = 10\,334 \times 1.4 = 14\,468$ N/m

$$y_{max} = 140 + 10 = 150 \text{ mm}$$

$$M_r = \frac{wl^2}{2} = \frac{14\,468 \times 3.6^2}{2} = 93\,750 \text{ N·m} = 93.75 \times 10^6 \text{ N·mm}$$

Now, $$\frac{\sigma}{y} = \frac{M}{I} \text{ or } \frac{125}{150} = \frac{93.75 \times 10^6}{I} \text{ or } I = 112.5 \times 10^6 \text{ mm}^4$$

Required increase in moment of inertia = $112.5 \times 10^6 - 75 \times 10^6 = 37.5 \times 10^6$ mm^4

Calculation for width and length of plates

Let b be the width of the steel plates (Fig. 5.12b)

Then $2\left(\dfrac{b \times 10^3}{12} + b \times 10 \times 145^2\right) = 37.5 \times 10^6$ or $b = 89.1$ mm

With increased load but without cover plates, let the value of bending moment reaches the maximum permissible value 66 964 N·m at a distance x m from the free end, i.e.,

$$M = \frac{wx^2}{2} = \frac{14\,468x^2}{2} = 66\,964 \text{ or } x = 3.04 \text{ m}$$

The maximum value of the bending moment is at the fixed end. Thus the plates are to be fixed for a distance 3.6 – 3.04 = 0.56 m from the fixed end.

Example 5.9 || A 430 mm × 160 mm *I*–beam is to be used as a simply supported beam of span 8 m. The web thickness is 10 mm and the flanges are of 15-mm thickness. The beam carries a uniformly distributed load of 60 kN/m over the whole span. Find whether the maximum bending stress is within permissible limits of 180 MPa or not. If not, find the width of 12-mm thick cover plates to be welded to each flange for the section to be safe. Also, find the length over which the plates should extend.

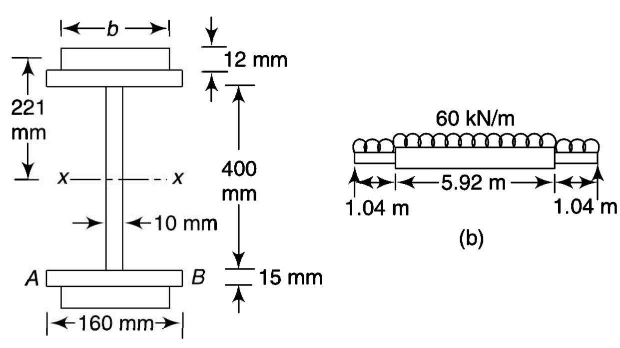


Fig. 5.13

Solution

Given An *I*-beam which can be strengthened by thick plates as shown in Fig. 5.13a

$$\sigma = 180 \text{ MPa}$$

To find

— maximum bending stress less than 180 MPa or not
— width and length of cover plates for safe stress limit

Refer Fig. 5.13a,

Check for limiting stress value for I-beam

$$I = \frac{160 \times 430^3 - 150 \times 400^3}{12} = 260.1 \times 10^6 \text{ mm}^4$$

$$\text{Maximum bending moment, } M = \frac{wl^2}{8} = \frac{60 \times 8^2}{8} = 480 \text{ kN·m}$$

$$\text{or} \quad 480 \times 10^6 \text{ N·mm}$$

Now, $\frac{\sigma}{y} = \frac{M}{I}$ or $\frac{\sigma}{215} = \frac{480 \times 10^6}{260.1 \times 10^6}$ or $\sigma = 396.8$ MPa

Thus the stress produced is greater than the permissible stress.

Required increase in moment of inertia

The maximum bending moment the section can withstand

$$M = \frac{\sigma}{y} I = \frac{180}{215} \times 260.1 \times 10^6 = 217.76 \times 10^6 \text{ N·mm} \quad \text{or} \quad 217.76 \text{ kN·m}$$

When the cover plates are fixed,

$$y_{max} = 215 + 12 = 227 \text{ mm}$$

Required moment of inertia of the beam to withstand the maximum bending moment,

$$\frac{\sigma}{y} = \frac{M}{I} \quad \text{or} \quad \frac{180}{227} = \frac{480 \times 10^6}{I} \quad \text{or} \quad I = 605.3 \times 10^6 \text{ mm}^4$$

Required increase in moment of inertia = $605.3 \times 10^6 - 260.1 \times 10^6 = 345.2 \times 10^6$ mm^4

Calculation for width and length of plates

Let b be the width of the steel plates

Thus $$2\left(\frac{b \times 12^3}{12} + b \times 12 \times 221^2\right) = 345.2 \times 10^6 \quad \text{or} \quad b = 294.4 \text{ mm}$$

Let x m be the distance from the support at which the bending moment reach the value 217.76 kN·m without cover plates,

$$M = \frac{wl}{2} x - \frac{wx^2}{2} = \frac{60 \times 8}{2} x - \frac{60x^2}{2} = 217.76$$

or $$x^2 - 8x + 7.259 = 0$$

or $$x = \frac{8 \pm \sqrt{64 - 29.036}}{2} \quad \text{or} \quad x = 6.96 \text{ m} \quad \text{or} \quad 1.04$$

Thus length of cover plates required = 6.96 – 1.04 =5.92 m and it is fixed at a distance 1.04 from each end support (Fig. 5.13b).

Example 5.10 A tube of uniform thickness has a section in the shape of regular hexagon as shown in Fig. 5.14. The tube rests in a horizontal position on two supports that are 2.4 m apart and carries two loads each of 24 kN between the supports at equal distances from the ends. Determine the least distance between the loads if the permissible bending stress is 120 MPa.

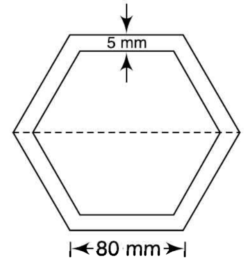


Fig. 5.14

Solution

Given A beam of hexagonal hollow section (Fig. 5.15) carrying two loads at equal distances from the ends

$$\sigma = 120 \text{ MPa}$$

To find Least distance between the loads

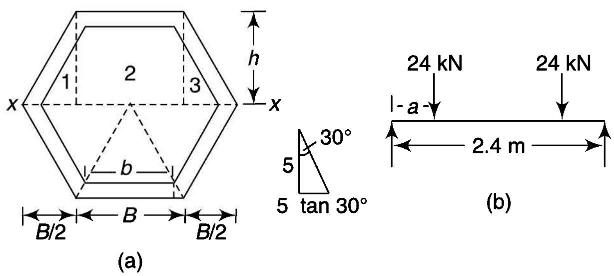


Fig. 5.15

Refer Fig. 5.15a.

Calculation of section modulus

Let B = side of the outer hexagon

and b = side of the inner hexagon

For the outer hexagon, $h = B \sin 60° = 0.866B$

$$I_{xx} = 2\left[\frac{(B/2)h^3}{12} + \frac{Bh^3}{3} + \frac{(B/2)h^3}{12}\right] = \frac{5Bh^3}{6} = \frac{5B(0.866B)^3}{6} = 0.541B^4$$

for area (1) (2) (3)

Similarly for the inner hexagon, $I_{xx} = 0.541b^4$

$$B = 80 \text{ mm};\ h = 80 \sin 60° = 80 \times 0.866 = 69.28 \text{ mm}$$

$$b = B - 2(5 \tan 30°) = 80 - 5.77 = 74.23 \text{ mm}$$

Thus for the hollow hexagonal tube

$$I_{xx} = 0.541(B^4 - b^4) = 0.541(80^4 - 74.23^4) = 5734 \times 10^6 \text{ mm}^4$$

$$Z_{xx} = \frac{I_{xx}}{h} = \frac{5734 \times 10^6}{69.28} = 82\,766 \text{ mm}^3$$

Calculations for least distance between loads

Figure 5.15b shows the loading on the beam. Let a m be the distance of each load from the respective end support.

Each reaction = 24 kN

Maximum bending moment (at the centre)

$$= 24 \times \frac{2.4}{2} - 24 \times \left(\frac{2.4}{2} - a\right) = 28.8 - 28.8 + 24a = 24a \text{ kN·m}]$$

or $\quad 24a \times 10^6$ N·mm

Now, $\qquad M = \sigma \cdot Z$

or $\qquad 24a \times 10^6 = 120 \times 82\,766 \quad$ or $\quad a = 0.414$ m

Least distance between the loads = 2.4 – 2 × 0.414 = 1.572 m

Example 5.11 || Figure 5.16 shows the section of a beam. Determine the ratio of its moment of resistance to bending in the *y*-plane to that in the *x*-plane if the maximum bending stress remains same in the two cases.

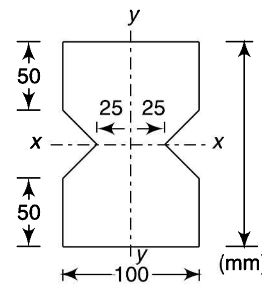


Fig. 5.16

Solution

Given A beam section as shown in Fig. 5.16

To find Ratio of moment of resistance to bending in *y*-plane to that in *x*-plane

Refer Fig. 5.17,

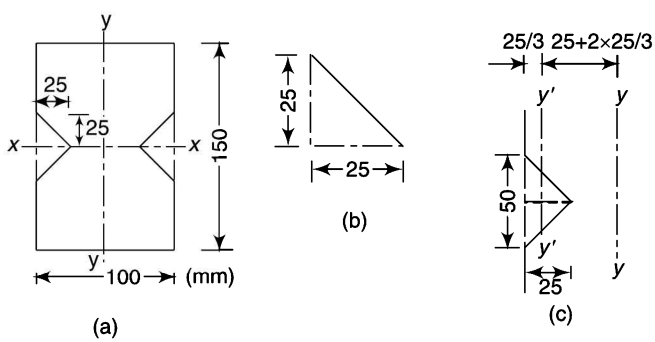


Fig. 5.17

Section modulus in x-plane

MOI of the section about *xx* = *MOI* of complete rectangle about *xx* – *MOI* of four triangles about *xx*

$$I_{xx} = \underbrace{\frac{1}{12} \times 100 \times 150^3}_{\text{(rectangle)}} - 4 \times \underbrace{\frac{25 \times 25^3}{12}}_{\text{(Fig.5.17b)}} = 27.995 \times 10^4 \text{ mm}^4$$

$$Z_{xx} = \frac{I_{xx}}{y_{\max}} = \frac{27.995 \times 10^6}{75} = 0.373 \times 10^6 \text{ mm}^3$$

Section modulus in y-plane

MOI of the section about *yy* = *MOI* of complete rectangle about *yy* – *MOI* of two isosceles triangles about *yy*

$$I_{yy} \text{ of triangle shown in Fig. (5.17c) about } y'y' = \frac{50 \times 25^3}{36}$$

$$I_{yy} \text{ of triangle about } yy \text{ (Fig. 5.17c)} = \frac{50 \times 25^3}{36} + \frac{50 \times 25}{2}\left(25 + \frac{50}{3}\right)^2$$

Thus for complete section,

$$I_{yy} = \frac{1}{12} \times 150 \times 100^3 - 2\left[\frac{50 \times 25^3}{36} + \frac{50 \times 25}{2}\left(25 + \frac{50}{3}\right)^2\right]$$

$$= 12.5 \times 10^6 - 2.214 \times 10^6 = 10.286 \times 10^6 \text{ mm}^4$$

$$Z_{yy} = \frac{10.286 \times 10^6}{50} = 0.0257 \times 10^6 \text{ mm}^3$$

Ratio of moment of resistances

$$\frac{M_{yy}}{M_{xx}} = \frac{Z_{yy}}{Z_{xx}} = \frac{0.2057 \times 10^6}{0.373 \times 10^6} = 0.552$$

Example 5.12 Figure 5.18 shows the section of a beam. Determine the ratio of its moment of resistance to bending in the *y-y* plane to that in the *x-x* plane if the maximum bending stress remains same in the two cases.

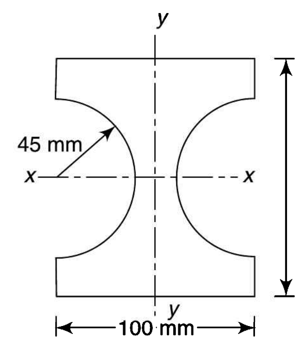


Fig. 5.18

Solution

Given A beam section as shown in Fig. 5.18

To find Ratio of moment of resistance to bending in y-plane to that in x-plane

Refer Fig. 5.19,

Section modulus in x-plane

MOI of the section about xx

= *MOI* of complete rectangle about xx – *MOI* of two semicircles about xx

$$I_{xx} = \underbrace{\frac{1}{12} \times 100 \times 150^3}_{\text{(full rectangle)}} - \underbrace{2\left[\frac{1}{2} \cdot \frac{\pi}{64}(90)^4\right]}_{\text{(two semicircles)}} = 24.9 \times 10^6 \text{ mm}^4$$

$$Z_{xx} = \frac{24.9 \times 10^6}{75} = 0.332 \times 10^6 \text{ mm}^3$$

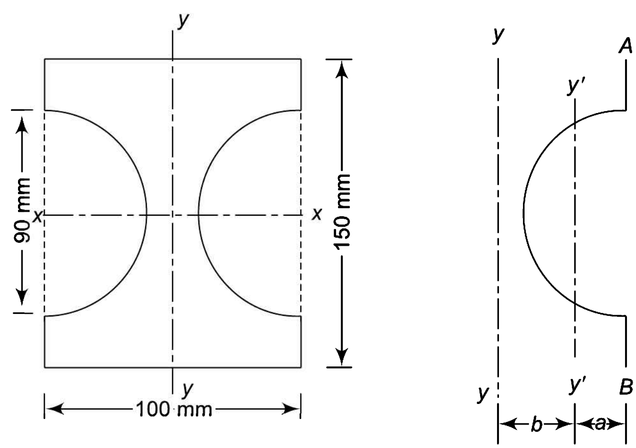


Fig. 5.19

Section modulus in y-plane

Let I = moment of inertia of each semicircle about yy

Then, $I_{yy} = \dfrac{1}{12} \times 150 \times 100^3 - 2I = 12.5 \times 10^6 - 2I$

Let $y'y'$ be the neutral axis of each semi-circle.

Area of each semi-circle = $\dfrac{1}{2} \times \dfrac{\pi}{4} \times 90^2 = 3181 \text{ mm}^2$

Distance a = distance of centroid of semi-circle from the centre

$$= \frac{4r}{3\pi} = \frac{4 \times 45}{3\pi} = 19.1 \text{ mm}$$

I = MOI of each semicircle about axis yy

= MOI of each semicircle about its own neutral axis + Area $\times$ b^2

= (MOI of each semicircle about AB – Area $\times$ a^2) + Area $\times$ b^2

$$= \frac{1}{2}\left(\frac{\pi}{64} \times 90^4\right) - 3181 \times 19.1^2 + 3181 \times (50 - 19.1)^2$$

$= 3.487 \times 10^6 \text{ mm}^4$

$$I_{yy} = 12.5 \times 10^6 - 2 \times 3.487 \times 10^6 = 5.526 \times 10^6 \text{ mm}^4$$

$$Z_{yy} = \frac{5.526 \times 10^6}{50} = 0.1105 \times 10^6 \text{ mm}^4$$

Ratio of moment of resistances

$$\frac{M_{yy}}{M_{xx}} = \frac{Z_{yy}}{Z_{xx}} = \frac{0.1105 \times 10^6}{0.332 \times 10^6} = 0.333$$

Example 5.13 || The tension flange of a cast iron *I*-section beam is 240 mm wide and 50 mm deep, the compression flange is 100 mm and 20 mm deep whereas the web is 300 mm × 30 mm. Find the load per m run which can be carried over a 4 m span by a simply supported beam if the maximum permissible stresses are 90 MPa in compression and 24 MPa in tension.

Solution

Given An *I*-section beam as shown in Fig. 5.20a.

Permissible stresses 90 MPa in compression and 24 MPa in tension.

To find Load per m run for 4 m span

Moment of inertia of the section

$$A = 100 \times 20 + 300 \times 30 + 240 \times 50 = 23\,000 \text{ mm}^2$$

$$y = \frac{2000 \times 360 + 9000 \times 200 + 12\,000 \times 25}{23\,000} = 122.6 \text{ mm}$$

$$I_{ab} = \underbrace{\frac{1}{3} \times (240 - 30) \times 50^3}_{\text{Fig.5.20(b) (for } p)} + \underbrace{\frac{1}{3} \times 30 \times (300 + 50)^3}_{(\text{for } q)} + \underbrace{\frac{1}{12} X\, 100\, X\, 20^3 + 2000\, X\, 360^2}_{(\text{for } r)}$$

$= 696.77 \times 10^6 \text{ mm}^4$

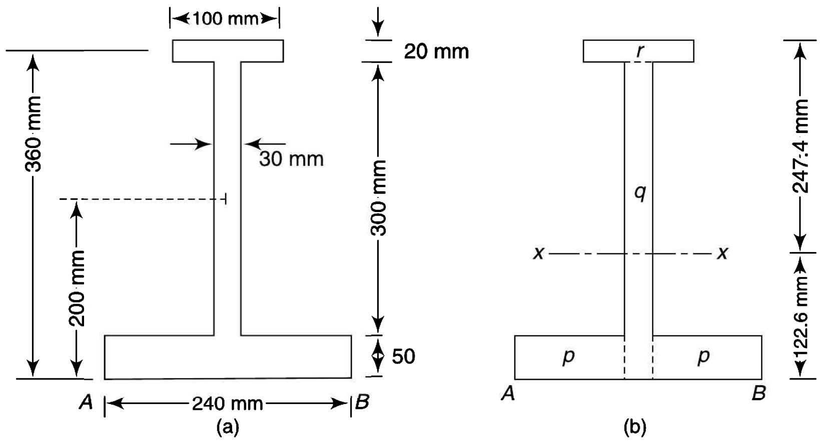


Fig. 5.20

I_{xx} = MOI of the section about AB – Area × 122.6^2
$= 696.77 \times 10^6 - 23000 \times 122.6^2 = 351.06 \times 10^6$ mm^4

Moment of resistance

Assuming that the stress in tension reaches its maximum value,

$$\frac{\sigma_c}{\sigma_t} = \frac{y_c}{y_t} \quad \text{or} \quad \frac{\sigma_c}{24} = \frac{(370 - 122.6)}{122.6}$$

or $\sigma_c = 48.43$ MPa

which is within the safe limits.

Moment of resistance $M_r = \frac{\sigma}{y} \cdot I = \frac{24}{122.6} \times 351.06 \times 10^6 = 68.72 \times 10^6$ N·mm

Calculation for uniformly distributed load

Let w be the uniformly distributed load on the beam,

$$\text{Maximum bending moment} = \frac{wl^2}{8} = \frac{w \times 4^2}{8} = 2w \text{ N·m}$$

or $2000w$ N·mm

Thus $2000w = 68.72 \times 10^6$

or $w = 34\,360$ N/m or 34.36 kN/m

Example 5.14 The tension flange of a cast iron *I*-section beam is 100 mm wide and 20 mm deep, the compression flange is 50 mm wide and 20 mm deep whereas the web is 100 mm × 20 mm. If a similar *I*-section is welded on the top of it to form a symmetrical section, find the ratio of the moment of resistance of this section to that of the previous section assuming the allowable stress in tension and compression to be the same.

Solution

Given An *I*-section beam as shown in Fig. 5.21a

To find Ratio of moment of resistance of a symmetrical section formed by welding a similar section at top of given section to that of the previous section, allowable stress in tension and compression same

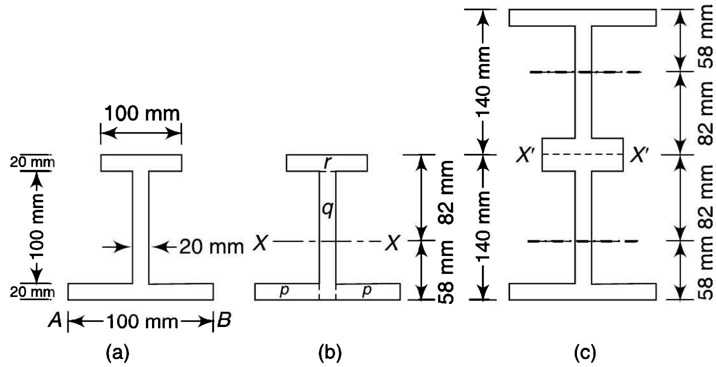


Fig. 5.21

Single section

Refer Fig. 5.21,

$$A = 50 \times 20 + 100 \times 20 + 100 \times 20 = 5000 \text{ mm}^2$$

$$y = \frac{1000 \times 130 + 2000 \times 70 + 2000 \times 10}{5000} = 58 \text{ mm}$$

$$I_{ab} = \underbrace{\frac{1}{3} \times (100 - 20) \times 20^3}_{\text{Fig. 5.21b (for } p)} + \underbrace{\frac{1}{3} \times 20 \times (100 + 20)^3}_{(\text{for } q)} + \underbrace{\frac{1}{12} \times 50 \times 20^3 + 1000 \times 130^2}_{(\text{for } r)}$$

$$= 28.67 \times 10^6 \text{ mm}^4$$

I_{xx} = MOI of the section about AB – Area $\times$ 58^2

$$= 28.67 \times 10^6 - 500 \times 58^2 = 11.58 \times 10^6 \text{ mm}^4$$

$$\text{Moment of resistance } M_r = \frac{\sigma}{y} \cdot I = \frac{\sigma}{82} \times 11.85 \times 10^6 = 144\,512\ \sigma \text{ N}\cdot\text{mm}$$

Double section

The centroid of the symmetric section is at the centre of the figure.

I_{xx} = 2 [MOI of lower half section about $X'X'$] (Fig. 5.21c)

= 2 [MOI of lower half section about XX + Area $\times$ 82^2]

$$= 2[11.85 \times 10^6 + 500 \times 82^2] = 90.94 \times 10^6 \text{ mm}^4$$

$$\text{Moment of resistance } M_r = \frac{\sigma}{y} \cdot I = \frac{\sigma}{140} \times 90.94 \times 10^6 = 649\,571\ \sigma \text{ N}\cdot\text{mm}$$

$$\text{Ratio} = \frac{649\,571}{144\,512} = 4.495$$

Example 5.15 || Compare the moment of resistances of a beam of square section placed with two sides horizontal to that with a diagonal horizontal for the same stress in each case.

Solution

Given A beam of square section

To find Ratio of moment of resistances of square section placed with two sides horizontal to that with a diagonal horizontal for the same stress

Refer Fig. 5.22,

Two sides horizontal

$$Z = \frac{1}{6}bd^2 = \frac{b^3}{6}$$

Diagonal horizontal

$$I_{xx} = 2[MOI \text{ of triangle } ABC \text{ about } AC]$$

$$= 2\left[\frac{1}{12}(\sqrt{2}b)(b/\sqrt{2})^3\right] = \frac{b^4}{12}$$

$$\therefore \quad Z = \frac{b^4/12}{b/\sqrt{2}} = \frac{b^3}{6\sqrt{2}}$$

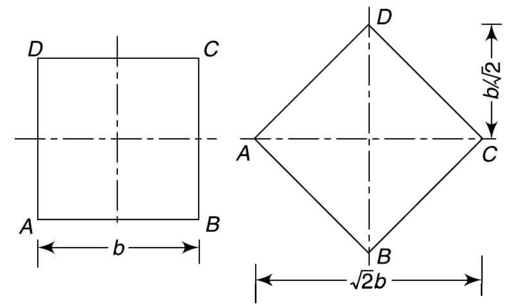


Fig. 5.22

$$\text{Ratio of moment of resistances} = \frac{b^3/6}{b^3/6\sqrt{2}} = 1.414$$

Thus arrangement of two sides horizontal is stronger by 41.4%.

Example 5.16 A beam of trapezoidal section is subjected to sagging bending moment (the larger side at the bottom). Determine the ratio of lengths of parallel sides for maximum economy. The permissible stresses in tension and compression are 60 MPa and 75 MPa respectively.

Solution

Given A beam of trapezoidal section

To find Ratio of lengths of parallel sides for maximum economy

Maximum permissible stresses: in tension 60 MPa, in compression 75 MPa

As the beam is subjected to sagging bending moment, the upper fibres are in compression and the lower in tension. For maximum economy, the maximum values of stresses in tension and compression must reach simultaneously.

Let the height of neutral axis from the bottom fibre be a (Fig. 5.23).

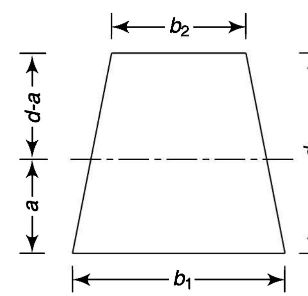


Fig. 5.23

Ratio of tensile and compressive stresses

Then $\dfrac{\sigma_c}{\sigma_t} = \dfrac{y_c}{y_t}$ or $\dfrac{75}{60} = \dfrac{d-a}{a}$ or $\dfrac{d-a}{a} = \dfrac{5}{4}$ or $\dfrac{d}{a} - 1 = \dfrac{5}{4}$ or $a = \dfrac{4}{9}d$

From the geometry

$$a = \frac{d}{3}\left(\frac{b_1 + 2b_2}{b_1 + b_2}\right) \quad \text{or} \quad \frac{d}{3}\left(\frac{b_1 + 2b_2}{b_1 + b_2}\right) = \frac{4}{9}d \quad \text{or} \quad \frac{b_1 + 2b_2}{b_1 + b_2} = \frac{4}{3}$$

Finding the ratio

Subtracting 1 from both sides, $\dfrac{b_1 + 2b_2}{b_1 + b_2} - 1 = \dfrac{4}{3} - 1$

or $\dfrac{b_2}{b_1 + b_2} = \dfrac{1}{3}$ or $b_1 + b_2 = 3b_2$ or $b_1 = 2b_2$ or $\dfrac{b_1}{b_2} = 2$

Example 5.17 Compare the flexural strength of three beams of equal weight with the following specifications:

(i) *I*-section with length of web, thickness of each flange and the thickness of web in terms of width *b* of the flanges being 1.7*b*, 0.15*b* and 0.1*b* respectively.
(ii) Rectangular section of depth equal to twice the width
(iii) Solid circular section

Solution

Given Three beams of *I*-section, rectangular section and circular section, equal weight
To find To compare the flexural strength of three beams, i.e., to find M_1/M_2 and M_1/M_3

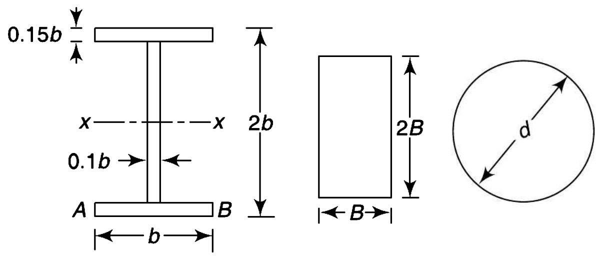


Fig. 5.24

Refer Fig. 5.24,

I-section

$$\text{Area} = 2(b \times 0.15b) + 0.1b \times 1.7b = 0.3b^2 + 0.17b^2 = 0.47b^2$$

$$I_{xx} = \frac{1}{12}[b \times (2b)^3 - (b - 0.1b)(2b - 0.3b)^3] = 0.2982b^4$$

$$Z_1 = \frac{0.2982b^4}{b} = 0.2982b^3$$

Rectangular section

Weight is the same which means area is the same as before. Let *B* be the base width of the rectangular section,

i.e., $$B(2B) = 0.47^2 \text{ or } B = 0.4848b \qquad (\text{as } h = 2b)$$

$$Z_2 = \frac{1}{6} \times 0.4848b \times (2 \times 0.4848b)^2 = 0.07596b^3$$

$$\frac{M_1}{M_2} = \frac{Z_1}{Z_2} = \frac{0.2982b^3}{0.07596b^3} = 3.926$$

Solid circular section

Weight and thus area is the same. Let *d* be the diameter of the circle,

i.e., $$\frac{\pi}{4}d^2 = 0.47b^2 \quad \text{or} \quad d = 0.7736b$$

$$Z_3 = \frac{\pi}{32}(0.7736b)^3 = 0.04545b^3$$

$$\frac{M_1}{M_3} = \frac{Z_1}{Z_3} = \frac{0.2982b^3}{0.04545b^3} = 6.561$$

Example 5.18 A long rod of uniform rectangular section is bent to form a circular arc. The thickness of the section is t. Show that the longitudinal surface strain in the rod is given by $\varepsilon = 4td/l^2$, if the displacement d of the midpoint of the length of the rod is small as compared to length l.

Solution

Given A long rod of uniform rectangular section bent to form a circular arc.

To find To show that longitudinal surface strain in the rod, $\varepsilon = 4td/l^2$

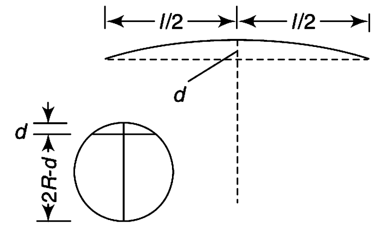


Fig. 5.25

Refer Fig. 5.25,

Let R be the radius of curvature of the rod on bending.

From property of chords of a circle,

$$d(2R-d) = \frac{l}{2}\cdot\frac{l}{2} \quad \text{or} \quad 2Rd - d^2 = \frac{l^2}{4} \quad \text{or} \quad 2Rd \approx \frac{l^2}{4} \quad \text{or} \quad R \approx \frac{l^2}{8d}$$

..... (d being small, its square is neglected)

From bending formula

$$\frac{\sigma}{y} = \frac{E}{R}; \quad \therefore \quad \varepsilon = \frac{\sigma}{E} = \frac{y}{R} = \frac{t/2}{l^2/8d} = \frac{4td}{l^2}$$

Example 5.19 Show that

(i) the moment of resistance of a beam of square cross-section with its diagonal in the plane of bending is increased by flattening the top and bottom corners (Fig. 5.26)

(ii) the moment of resistance is maximum when $y = 8/9b$

Also, find the percentage increase in the moment of resistance.

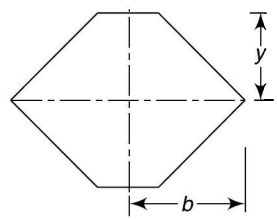


Fig. 5.26

Solution

Given A beam of square section with its diagonal in the plane of bending

To find

— To show that moment of resistance increases by flattening the top and bottom corners

— Moment of resistance is maximum when $y = 8/9b$

— Percentage increase in the moment of resistanc e

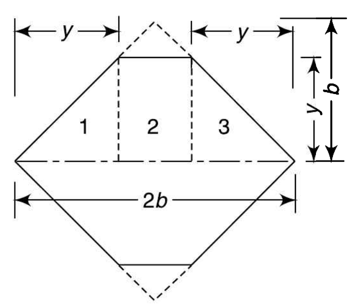


Fig. 5.27

Refer Fig. 5.27,

b is the length of the semidiagonal of the square.

Section modulus

Moment of inertia of the flattened square

$$= 2\left[\frac{y\cdot y^3}{12} + \frac{(2b-2y)}{3} + \frac{y\cdot y^3}{12}\right] = \frac{y^4}{6} + \frac{4by^3}{3} - \frac{4y^4}{3} + \frac{y^4}{6} = \frac{4by^3}{3} - y^4$$

For area (1) (2) (3)

$$Z = \frac{1}{y}\left(\frac{4by^3}{3} - y^4\right) = \frac{4by^2}{3} - y^3 \tag{i}$$

Maximum value of section modulus

It is maximum when $\dfrac{dZ}{dy} = \dfrac{8by}{3} - 3y^2 = 0 \quad \text{or} \quad 8b = 9y \quad \text{or} \quad y = \dfrac{8}{9}b$

Thus maximum value of section modulus,

$$Z = \frac{4b}{3}\left(\frac{8b}{9}\right)^2 - \left(\frac{8b}{9}\right)^3 = \frac{256b^3}{243} - \frac{512b^3}{729} = \frac{256b^3}{729}$$

Section modulus of uncut square

Section modulus of uncut square ($y = b$),

$$Z = \frac{4b^3}{3} - b^3 = \frac{b^3}{3} \qquad \ldots\text{[by inserting } y = b \text{ in (i)]}$$

As $\frac{256b^3}{729}$ is greater than $\frac{b^3}{3}$, the moment of resistance of flattened square is more than the uncut square.

Percent increase of moment of resistance

Increase in section modulus of flattened square $= \frac{256b^3}{729} - \frac{b^3}{3} = \frac{13b^3}{729}$

Percentage increase $= \frac{13b^3/729}{b^3/3} \times 100 = 5.35\%$

The moment of resistance of the flattened square increases since the removal of the small corner areas decrease the extreme distance in higher proportion than it reduces the moment of inertia of the section.

Example 5.20 A cast-iron channel supported at two points, 11 m apart, carries water as shown in Fig. 5.28. Determine the maximum depth of water in the channel if the tensile and compressive bending stresses are not to exceed 18 MPa and 48 MPa respectively. Water weighs 9.81 kN/m^3 and the cast iron 68 kN/m^3.

Solution

Given A cast iron channel as shown in Fig. 5.28.

Specific weight of water = 9.81 kN/m^3

Specific weight of cast iron = 68 kN/m^3

To find Maximum depth of water, maximum tensile and compressive bending stresses 18 MPa and 48 MPa respectively

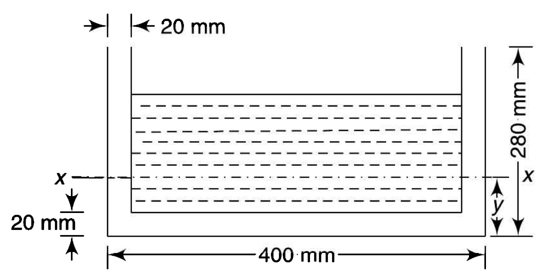


Fig. 5.28

Moment of resistance of channel section

Taking moments about the lower edge of the cast iron channel,

$$y = \frac{360 \times 20 \times 10 + 2 \times 280 \times 20 \times 140}{360 \times 20 + 280 \times 20} = 89.13 \text{ mm}$$

$$I = \left[\frac{360 \times 20^3}{12} + 360 \times 20(89.13 - 10)^2\right] + 2\left[\frac{20 \times 280^3}{12} + 20 \times 280(140 - 89.13)^2\right]$$

$$= 45.32 \times 10^6 + 102.6 \times 10^6 = 147.48 \times 10^6 \text{ mm}^4$$

If tensile stress reaches to maximum value of 18 MPa,

$$\text{Compressive stress} = 18 \times \frac{280 - 89.13}{89.13} = 38.6 \text{ MPa}$$

which is well within permissible limits.

$$\text{Thus moment of resistance} = \frac{18 \times 147.48 \times 10^6}{89.13} = 29.784 \times 10^6 \text{ N·mm}$$

or $\quad 29\,784$ N·m

Calculation for maximum depth of water

Let w be the total weight (channel + water) N per m length of the channel.

Then $\dfrac{w \times 11^2}{8} = 29\,784 \quad$ or $\quad w = 1969.2$ N/m

Let x m be the level of water in the channel.

$$\text{Weight of channel + weight of water = Total weight/m}$$

$$(0.36 \times 0.02 + 2 \times 0.28 \times 0.02) \times 68\,000 + 0.36x \times 9810 = 1969.2$$

$$1251.2 + 3531.6x = 1969.2$$

or $\quad x = 0.203$ m $\quad$ or $\quad 203$ mm

Example 5.21 The cross-section of a beam is shown in Fig. 5.29. Determine the moment of resistance of the section about the horizontal neutral axis for both positive and negative bending moment. The permissible stresses in tension and compression are 24 MPa and 85 MPa respectively.

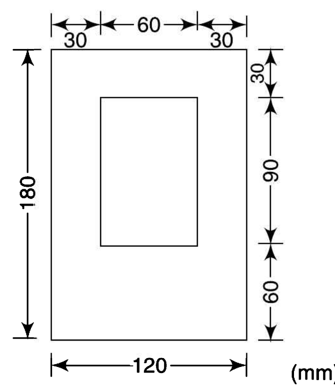


Fig. 5.29

Solution

Given Cross-section of a beam as shown in Fig. 5.29

Permissible stress in tension = 24 MPa

Permissible stress in compression = 85 MPa

To find Moment of resistance of section about horizontal neutral axis for both positive and negative bending moment

Refer Fig. 5.30,

Moment of inertia of the section

Distance of the centroid of the net section from the bottom edge,

$$y = \frac{120 \times 180 \times 90 - 60 \times 90 \times (60 + 45)}{120 \times 180 - 60 \times 90} = 85 \text{ mm}$$

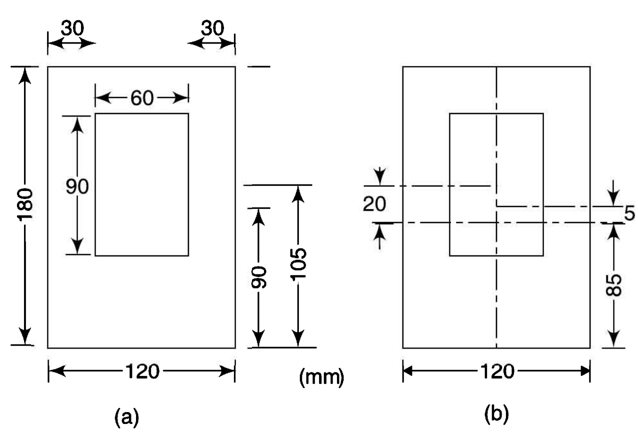


Fig.5.30

$$I = \left[\frac{120 \times 180^3}{12} + 120 \times 180 \times 5^2\right] - \left[\frac{60 \times 90^3}{12} + 60 \times 90 \times 20^2\right]$$

$$= 58.86 \times 10^6 - 5.805 \times 10^6 = 53.055 \times 10^6 \text{ mm}^4$$

Moment of resistance for sagging bending moment

For *sagging* bending moment, tensile stress is at the bottom edge. When it reaches to maximum value of 24 MPa, corresponding compressive stress at the upper edge will be (24X95/85= 26.8 MPa which is below the maximum value.

$$M_r = \sigma \cdot Z = 24 \times \frac{53.055 \times 10^6}{85} = 14.98 \times 10^6 \text{ N}\cdot\text{mm} \quad \text{or} \quad 14.98 \text{ kN}\cdot\text{m}$$

Moment of resistance for hogging bending moment

For *hogging* bending moment, tensile stress is at the upper edge. When it reaches to maximum value of 24 MPa, corresponding compressive stress at the lower edge will be (24 × 85/95= 21.47 MPa which is below the maximum value.

$$M_r = \sigma \cdot Z = 24 \times \frac{53.055 \times 10^6}{95} = 13.403 \times 10^6 \text{ N}\cdot\text{mm} \quad \text{or} \quad 13.403 \text{ kN}\cdot\text{m}$$

5.3 BEAMS WITH UNIFORM BENDING STRENGTH

Usually, the beams are designed on the basis of maximum bending stress occurring at any cross-section of the beam and a constant cross-section is provided throughout the length of the beam. However, as the actual moment and thus the stress varies and is less at all other cross-sections along the length of the beam, a beam with constant cross-section or with uniform moment of resistance is uneconomical. In beams with heavy loads, beams may be designed on the basis of variation in the bending moment. Such beams will have the same maximum bending stress all along the length and are known as *beams with uniform bending strength.* This can be achieved by having either a uniform width of the section or a uniform depth.

Beam with Constant Width and of Varying Depth

Let l be the length and b the constant width of a beam with uniform strength. Also let the depth be d_x at a distance x from the support.

Then, moment of resistance of the section, $M_r = \sigma Z_x = \sigma \dfrac{b{d_x}^2}{6}$

Moment of resistance of the section will depend upon the loading on the beam.

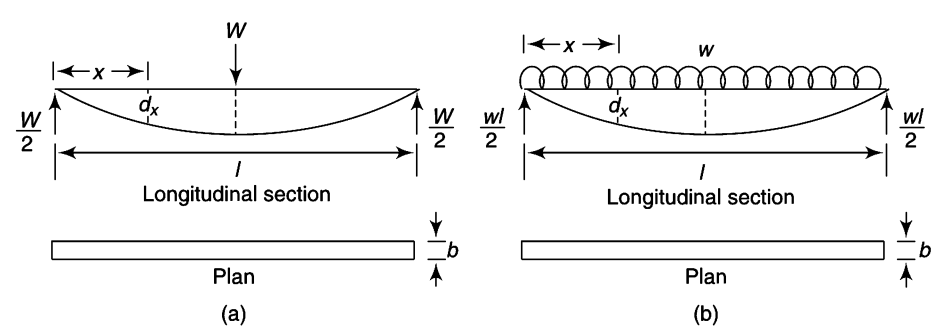


Fig. 5.31

Concentrated load W at the midspan (Fig. 5.31a)

Bending moment at the section $= \frac{W}{2}x$

Equating the moment of resistance and the bending moment,

$$\sigma\frac{bd_x^2}{6} = \frac{W}{2}x \quad \text{or} \quad d_x^2 = \frac{3W}{\sigma \cdot b}x \quad \text{or} \quad d_x = \sqrt{\frac{3W}{\sigma \cdot b} \cdot x} = k\sqrt{x}$$

where k is a constant. The expression indicates that the variation is parabolic.

At the centre $d_x = k\sqrt{l/2}$

Uniformly distributed load throughout (Fig. 5.31b)

Bending moment at the section $= \frac{wl}{2}x - \frac{wx^2}{2} = \frac{wx}{2}(l - x)$

Equating the moment of resistance and the bending moment,

$$\sigma\frac{bd_x^2}{6} = \frac{wx}{2}(l - x) \quad \text{or} \quad d_x^2 = \frac{3wx(l - x)}{\sigma \cdot b} \quad \text{or} \quad d_x = \sqrt{\frac{3wx(l - x)}{\sigma \cdot b}}$$

The expression indicates that the variation is parabolic.

At the centre, $x = l/2$, $d_x = \frac{l}{2}\sqrt{\frac{3w}{\sigma \cdot b}}$

Beam with Constant Depth and of Varying Width

Let l be the length and d the constant depth of a beam with uniform strength. Also, let the width be b_x at a distance x from the support.

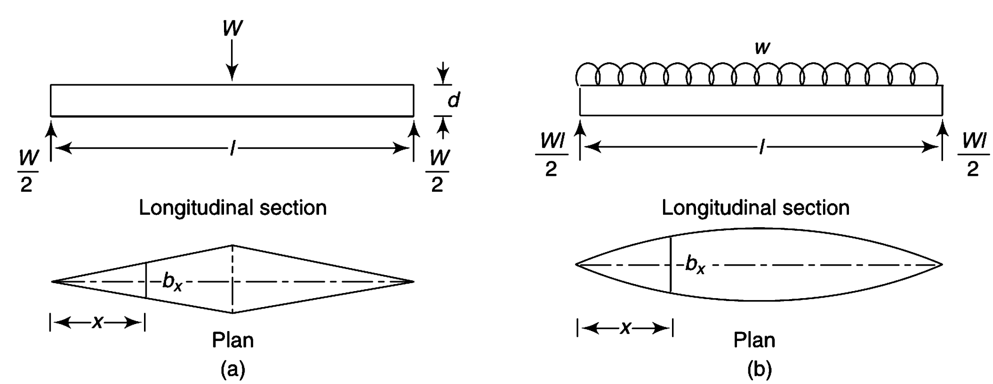


Fig. 5.32

For concentrated load W at midspan (Fig. 5.32a)

$$\sigma\frac{b_x d^2}{6} = \frac{W}{2}x \quad \text{or} \quad b_x = \frac{3W}{\sigma \cdot d^2}x \quad \text{or} \quad b_x = k'x$$

where k' is a constant. The expression indicates that the variation is linear. It is zero at the supports and $k'l/2$ at the centre.

For uniformly distributed load throughout (Fig. 5.32b),

$$\sigma \frac{b_x d^2}{6} = \frac{wx}{2}(l - x) \quad \text{or} \quad b_x = \frac{3wx(l - x)}{\sigma \cdot d^2}$$

The expression indicates that the variation is parabolic.

At the centre, $x = l/2$, $b_x = \dfrac{3wl^2}{4\sigma \cdot d^2}$

Example 5.22 A simply supported beam of span l carries a concentrated load which can be placed anywhere on the span. If the beam is designed for uniform strength keeping the width constant throughout, determine the variation of depth along the length of the beam. Also deduce an expression for variation of depth in terms of depth at midspan.

Solution

Given A simply supported beam of span l carrying a point load which can be placed anywhere on the span.

To find

— Expression for variation of depth along the length for uniform strength

— Expression for variation of depth in terms of depth at midspan

Expression for variation of depth

Let the load W be placed at a distance x from a support. The maximum bending moment occurs under the load which is given by

$$M_x = \frac{Wx(l - x)}{l}$$

and moment of resistance of the section, $M_r = \sigma Z_x = \sigma \dfrac{bd_x^2}{6}$

Equating the moment of resistance and the bending moment,

$$\sigma \frac{bd_x^2}{6} = \frac{Wx(l - x)}{l} \quad \text{or} \quad d_x = \sqrt{\frac{6Wx(l - x)}{lb\sigma}} \tag{i}$$

which is the required expression. However, it can also be expressed in terms of depth of beam at the midspan as follows:

Expression for variation of depth in terms of depth at midspan

Let d be the depth of the beam at the midspan.

From above, $$\sigma = \frac{6Wx(l - x)}{lbd_x^2}$$

When the load is at the midspan, $\sigma = \dfrac{3Wl}{2bd^2}$

Inserting this value of σ in (i),

$$d_x = \sqrt{\frac{6Wx(l - x)}{lb} \cdot \frac{2bd^2}{3Wl}} = \frac{2d}{l}\sqrt{x(l - x)}$$

5.4 FLITCHED OR COMPOSITE BEAMS

Beams made up of two different materials such as wooden beams reinforced by steel plates are known as *flitched* or *composite beams*. The two materials are rigidly connected together at the interface so that the strains are equal at the interface. If transverse sections remain plane after the bending, then strain is proportional to the distance from the common neutral axis.

Let the two materials be denoted by subscripts t and s. Then at any common surface or fibre,

$$\text{Strain} = \frac{\sigma_t}{E_t} = \frac{\sigma_s}{E_s} \quad \text{or} \quad \sigma_s = \frac{E_s}{E_t}\sigma_t = m \cdot \sigma_t \tag{5.15}$$

where m = modular ratio E_s/E_t.
Let y_t and y_s be the distances of the extreme fibres of timbre and steel respectively from the neutral axis and σ_{tm} and σ_{sm} the maximum stresses in the same. As strain is proportional to the distance from the common neutral axis,

$$\therefore \qquad \frac{\varepsilon_s}{\varepsilon_t} = \frac{y_s}{y_t} \quad \text{or} \quad \frac{\sigma_{sm}/E_s}{\sigma_{tm}/E_t} = \frac{y_s}{y_t} \quad \text{or} \quad \frac{\sigma_{sm}}{\sigma_{tm}} = \frac{E_s}{E_t} \cdot \frac{y_s}{y_t}$$

$$M_r = \frac{\sigma_{tm} I_t}{y_t} + \frac{\sigma_{sm} I_s}{y_s} = \frac{\sigma_{tm} I_t}{y_t} + \left(\frac{E_s}{E_t} \cdot \frac{\sigma_{tm}}{y_t}\right) I_s = \frac{\sigma_{tm}}{y_t}\left(I_t + \frac{E_s}{E_t} \cdot I_s\right) = \frac{\sigma_{tm}}{y_t}(I_t + m \cdot I_s) \tag{5.16}$$

$(I_t + mI_s)$ is known as *equivalent moment of inertia* of the cross-section in terms of timbre, i.e. if the beam is assumed to be made up of timbre, the moment of resistance is the same as of the flitched beam. Equivalent section can be achieved by multiplying by m the dimensions of steel in the direction parallel to the neutral axis.

It can also be shown that $M_r = \dfrac{\sigma_{sm}}{y_s}\left(I_s + \dfrac{E_t}{E_s} \cdot I_t\right) = \dfrac{\sigma_{sm}}{y_s}\left(I_s + \dfrac{I_t}{m}\right)$

In a flitched beam, the stronger material (with higher Young's modulus) bears larger stresses than the weaker material (with smaller Young's modulus) at the same distance from the neutral axis.

Example 5.23 A timbre beam, 100 mm wide by 200 mm, deep is reinforced by bolting on two steel plates each 10 mm thick one on either side of the beam. Find the maximum stress attained in the steel and the moment of resistance of the section if

(i) the plates are 200 mm deep
(ii) the plates are 160 mm deep and are symmetrically placed

The maximum stress in the timbre is to be 8 MPa and the Young's modulus of steel is 20 times that of timber.

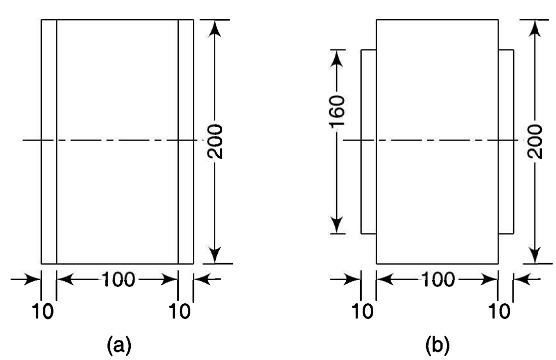


Fig. 5.33

Solution

Given A timbre beam 100 mm wide by 200 mm deep reinforced with steel plates as shown in Fig. 5.33a and b.

$\sigma_{t\max} = 8$ MPa $\qquad E_s = 20\,E_t$

To find

— Maximum stress attained in the steel

— Moment of resistance of the section

When steel plates are 200 mm deep

Refer Fig. 5.33a,

$$\sigma_s = \text{m} \cdot \sigma_t = 20 \times 8 = 160 \text{ MPa}$$

$$M_r = \frac{\sigma_t}{y}(I_t + mI_s) = \frac{8}{100}\left(\frac{100 \times 200^3}{12} + 20 \times 2 \times \frac{10 \times 200^3}{12}\right) = 26.667 \times 10^6 \text{ N}\cdot\text{mm}$$

or 26.667 kN·m

When steel plates are 160 mm deep

Refer Fig. 5.33b,

At the extreme fibre of steel, the stress in timbre $= 8 \times \frac{80}{100} = 6.4$ MPa

$$\sigma_s = m \cdot \sigma_t = 20 \times 6.4 = 128 \text{ MPa}$$

$$M_r = \frac{\sigma_t}{y}(I_t + mI_s) = \frac{8}{100}\left(\frac{100 \times 200^3}{12} + 20 \times 2 \times \frac{10 \times 160^3}{12}\right) = 16.256 \times 10^6 \text{ N}\cdot\text{mm}$$

or 16.256 kN·m

Example 5.24 || A timbre beam of 140 mm wide and 180 mm deep section is reinforced by 10 mm × 3140 mm steel plates at the top and bottom. The beam is subjected to a bending moment of 24 kN·m.

Determine

(i) the stresses in the beam

(ii) moment of resistance of the beam

The stresses in the timbre should not exceed 8 MPa. Take E_s = 210 GPa and E_t = 15 GPa.

Solution

Given A timbre beam of section 140 mm wide and 180 mm deep is reinforced by 10 mm × 140 mm steel plates at the top and bottom as shown in Fig. 5.34a.

To find

— Stresses in beam

— Moment of resistance of beam

Stresses in beam

The given beam section (Fig. 5.34a) can be transformed to an equivalent steel section (Fig. 5.34b).

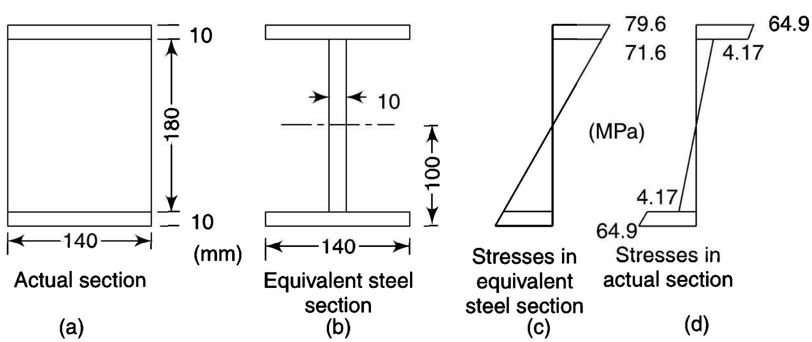


Fig. 5.34

Modular ratio, $m = 210/15 = 14$
Equivalent width of timber section = 140/14 = 10 mm

$$\text{Equivalent moment of inertia} = \frac{10 \times 180^3}{12} + 2\left(\frac{140 \times 10^3}{12} + 140 \times 10 \times 95^2\right)$$

$$= 30.15 \times 10^6 \text{ mm}^4$$

Now, $\frac{\sigma}{y} = \frac{M}{I}$ or $\frac{\sigma}{y} = \frac{24X10^6}{30.15X10^6} = 0.796$ or $\sigma = 0.796\,y$

At extreme fibre of steel, $\sigma_s = 0.796 \times 100 = 79.6$ MPa
At extreme fibre of timbre, $\sigma_s = 0.796 \times 90 = 71.6$ MPa
$\sigma_t = 71.6/m = 71.6/14 = 5.11$ MPa

The variation in stresses in the beam of equivalent steel section is indicated in Fig. 5.34c. Stresses in the actual section are indicated in Fig. 5.34d in which the stresses in the timber are reduced in modular ratio.

Moment of resistance

For the given value of the bending moment, the maximum bending stress induced in the timber is less than the permissible value. When it reaches to maximum value we have

$$M_r = 24 \times 8/5.11 = 37.57 \text{ kN·m}$$

Example 5.25 A composite beam is made up of two timber joists that are 50 mm wide and 100 mm deep with a 10-mm thick and 80-mm deep steel plate placed symmetrically between them. Assuming a maximum stress in the timber joist as 7 MPa, determine the maximum stress reached in the steel. Also, determine the moment of resistance of the section. $E_s = 20\,E_t$.

Solution

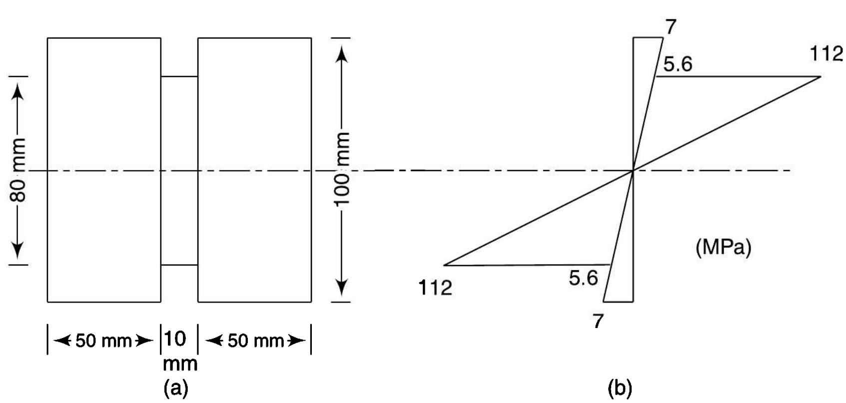


Fig. 5.35

Given A composite beam made up of timber joists and steel plate as shown in Fig. 5.35a.
$E_s = 20\,E_t$; Maximum stress in timber joist 7 MPa

To find

— Maximum stress in steel
— Moment of resistance of section

Stresses in beam

Maximum stress in steel = $20 \times 7 \times \dfrac{40}{50} = 112$ MPa

Stresses in the actual section are indicated in Fig. 5.35b.

$$M_r = \frac{\sigma_t}{y}(I_t + mI_s) = \frac{7}{50}\left(2 \times \frac{50 \times 100^3}{12} + 20 \times \frac{10 \times 80^3}{12}\right) = 2.36 \times 10^6 \text{ N·mm}$$

Example 5.26 A composite beam is formed by joining firmly a steel rod of 150-mm diameter inside a brass tube of 200-mm diameter. If the permissible bending stresses in steel and brass are 140 MPa and 70 MPa respectively. Determine the resisting moment of the composite beam. Take $E_s = 2E_b$.

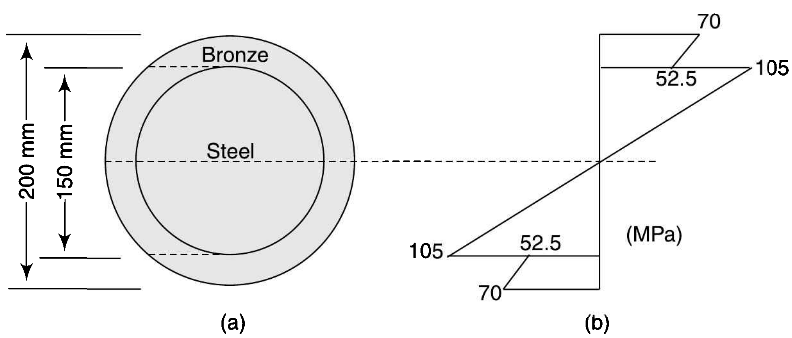


Fig. 5.36

Solution

Refer Fig. 5.36.

Given A composite beam of steel rod and bronze tube as shown in Fig. 5.36a. $E_s = 2\,E_b$. Permissible bending stresses in steel and brass are 140 MPa and 70 MPa respectively.

To find Moment of resistance of section

$$m = E_s/E_b = 2$$

Maximum stresses in steel and bronze

As the permissible values of stresses in the two materials are prescribed, it has to be seen which material attains the maximum permissible value first. On that basis the resisting moment has to be found.

If the stress in steel is to reach to the maximum value, the maximum value of stress in the bronze =

$$\frac{\sigma_{sm}}{y_s} = m \cdot \frac{\sigma_{bm}}{y_b}$$

or
$$\sigma_{bm} = \frac{\sigma_{sm}}{m} \cdot \frac{y_b}{y_s} = \frac{140}{2} \cdot \frac{100}{75} = 93.3 \text{ MPa}$$

which is more than the permissible limit. Thus the maximum limiting value of stress in the bronze is reached first. At that point, the maximum value of stress in the steel will be

$$\frac{\sigma_{sm}}{y_s} = m \cdot \frac{\sigma_{bm}}{y_b} \quad \text{or} \quad \sigma_{sm} = m \cdot \sigma_{bm} \cdot \frac{y_s}{y_b} = 2 \times 70 \times \frac{75}{100} = 105 \text{ MPa}$$

Stresses in the actual section are indicated in Fig. 5.36b.

Calculations for resisting moment

$$I_b = \frac{\pi \times (200^4 - 150^4)}{64} = 53.69 \times 10^6 \text{ mm}^4$$

$$I_s = \frac{\pi \times 150^4}{64} = 24.85 \times 10^6 \text{ mm}^4$$

$$M_{rb} = \sigma_b \cdot \frac{I_b}{y_b} = 70 \times \frac{53.69 \times 10^6}{100} = 37.58 \times 10^6 \text{ N·mm}$$

$$M_{rs} = \sigma_s \cdot \frac{I_s}{y_s} = 105 \times \frac{24.85 \times 10^6}{75} = 34.79 \times 10^6 \text{ N·mm}$$

Total resisting moment $= 37.58 \times 10^6 + 34.79 \times 10^6 = 72.37 \times 10^6$ N·mm

or $$M_r = \frac{\sigma_b}{y}(I_b + mI_s) = \frac{70}{100}(53.69 \times 10^6 + 2 \times 24.85 \times 10^6) = 72.37 \times 10^6 \text{N·mm}$$

or $$M_r = \frac{\sigma_s}{y_s}(I_s + I_b/m) = \frac{105}{75}(24.85 \times 10^6 + 53.69 \times 10^6/2) = 72.37 \times 10^6 \text{N·mm}$$

Stresses in the actual section are indicated in Fig. 5.36b.

Example 5.27 Two rectangular bars, one of brass and the other of steel, each 36 mm by 9 mm are placed together to form a beam of 36 mm by 18 mm depth on two supports 800 mm apart, the steel bar being at the bottom. Find the maximum central load if the bars are

(i) separate and can bend independently

(ii) firmly secured to each other throughout the length

Maximum permissible stress in steel is 102 MPa and in brass 72 MPa. Take E_s = 204 MPa and E_b = 85 MPa.

Solution

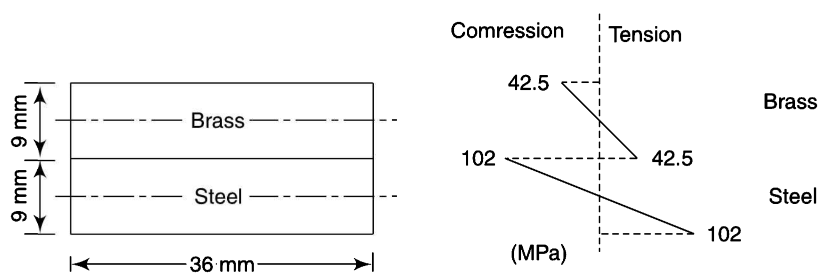


Fig. 5.37

Given Two rectangular bars, one of brass and the other of steel to form a beam as shown in Fig. 5.37. Maximum permissible stress in steel 102 MPa and in brass 72 MPa. E_s = 204 MPa, E_b = 85 MPa

To find

Maximum central load if the bars are

— separate and can bend independently

— firmly secured throughout the length

Bars are separate

When the bars are separate and can bend independently, each will have it own neutral axis.

We have $$\frac{\sigma}{y} = \frac{E}{R}$$

Assuming the same radius of curvature for the two bars,

$$R = \frac{y_s}{\sigma_s}E_s = \frac{y_b}{\sigma_b}E_b \quad \text{or} \quad \frac{\sigma_s}{\sigma_b} = \frac{y_s E_s}{y_b E_b} = \frac{E_s}{E_b} = \frac{204}{85} = 2.4 \qquad \ldots..(y_s = y_b)$$

If the stress in steel reaches to maximum value, the stress induced in brass

$$= 102/2.4 = 42.5 \text{ MPa}$$

$$M_r = M_{\text{steel}} + M_{\text{brass}}$$

$$= 102 \times \frac{36 \times 9^2}{6} + 42.5 \times \frac{36 \times 9^2}{6} = 49\,572 + 20\,655 = 70\,227 \text{ N·mm}$$

For a central load, $\frac{Wl}{4} = \frac{W \times 0.8}{4} = 70\,227$ or $W = 351.1$ N

Bars are firmly secured

When the bars are firmly secured to each other throughout the length, they will bend about a common neutral axis x-x.

Equivalent brass section

Figure 5.38a shows the equivalent section in terms of brass. The dimensions of the steel parallel to the neutral axis are increased in the modular ratio 2.4.

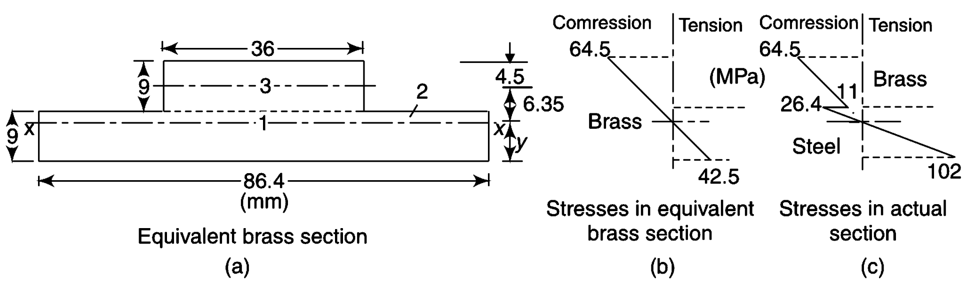


Fig. 5.38

Distance of the centroid of the net section from the bottom edge,

$$x = \frac{(36 \times 9) \times 13.5 + (86.4 \times 9) \times 4.5}{36X9 + 86.4 \times 9} = 7.15 \text{ mm}$$

$$I = \left[\frac{86.4 \times 7.15^3}{3} + \frac{86.4 \times 1.85^3}{3}\right] + \left[\frac{36 \times 9^3}{12} + (36 \times 9) \times (18 - 7.15 - 4.5)^2\right]$$

For area (1) (2) (3)

$$= 25\,961 \text{ mm}^4$$

If the stress in steel is to reach to the maximum value, corresponding stress in brass = 102/2.4 = 42.5 MPa

This is at the bottom edge. At the upper edge, the stress = 42.5 × (18 – 7.15)/7.15 = 64.5 MPa. The stress variation in the equivalent section (brass) is indicated in Fig. 5.38b. Actual stresses are shown in Fig. 5.38c.

$$\text{At the interface, Stress in brass} = 64.5 \times \frac{10.85 - 9}{10.85} = 11 \text{ MPa}$$

$$\text{Stress in steel} = 102 \times \frac{9 - 7.15}{7.26} = 26.4 \text{ MPa (or } 11 \times 2.4 = 26.4)$$

$$M_r = \frac{\sigma_1}{y} I = \frac{64.5}{10.85} \times 25\,961 = 154\,330 \text{ N·mm or } 154.33 \text{ N·m}$$

For a central load, $\dfrac{Wl}{4} = \dfrac{W \times 0.8}{4} = 154.33$ or $W = 771.65$ N

Equivalent steel section

Instead of considering an equivalent brass section (Fig. 5.39a), an equivalent steel section may also be considered as follows:

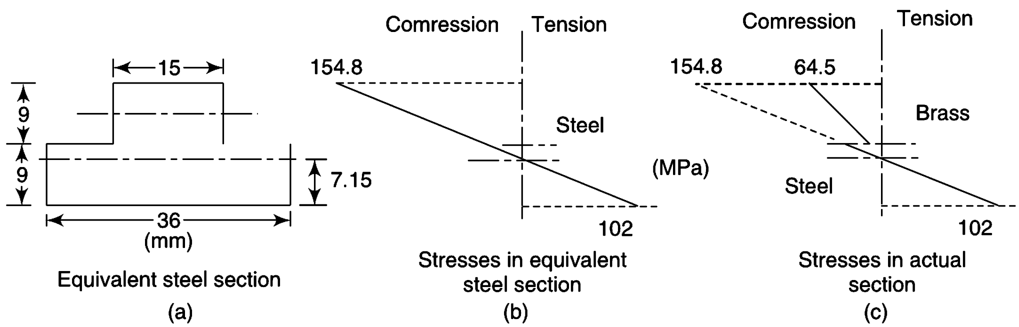


Fig.5.39

Distance of the centroid of the net section from the bottom edge,

$$y = \frac{(36 \times 9) \times 4.5 + (15 \times 9) \times 13.5}{36 \times 9 + 15 \times 9} = 7.15 \text{ mm}$$

$$I = \left[\frac{36 \times 7.15^3}{3} + \frac{36 \times 1.85^3}{3}\right] + \left[\frac{15 \times 9^3}{12} + (15 \times 9) \times (18 - 7.15 - 4.5)^2\right]$$

$$= 10\,817 \text{ mm}^4$$

If the stress in steel is to reach to the maximum value at the bottom edge, at the upper edge, the stress = 102 × (18 – 7.15)/7.15 = 154.8 MPa. The stress variation in the equivalent section (steel) are indicated in Fig. 5.39b. Actual stresses are shown in Fig. 5.39c.

$$M_r = \frac{\sigma_1}{y} I = \frac{102}{7.15} \times 108\,17 = 154\,330 \text{ N·mm} \quad \text{or} \quad 154.33 \text{ N·m}$$

Example 5.28 A composite beam is made of two timber joists of width *B* and depth *D* with a steel plate of thickness *t* and depth *d* placed symmetrically between them. Assuming a maximum stress in the timber joist as 8 MPa and in the steel as 132 MPa, determine

(i) the ratio of *D*/*d* if the maximum allowable stress in the steel and timber reach simultaneously

(ii) the ratio of *B*/*t* so that the moment of resistance of timbre alone is equal to that of steel alone.

Take E_s = 210 GPa and E_t = 10.5 GPa.

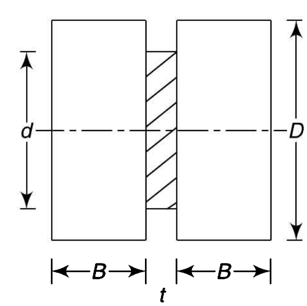


Fig. 5.40

Solution

Given A composite beam made of two timber joists and a steel plate as shown in Fig. 5.40. Maximum stress in the timber joist 8 MPa and in steel 132 MPa. E_s = 210 GPa and E_t = 10.5 GPa.

To find

- — Ratio of D/d if maximum allowable stress in steel and timber reach simultaneously
- — Ratio of B/t so that moment of resistance of timbre alone is equal to that of steel alone.

Ratio of D/d

Let the stress in extreme timbre fibre be σ_t.

Then the stress in timber at extreme surface of steel, $\sigma_t' = \sigma_t \cdot \dfrac{d/2}{D/2} = \sigma_t \cdot \dfrac{d}{D}$

If the stress in the extreme steel fibre is to reach the maximum value,

$$\sigma_s = m\sigma_t' \quad \text{or} \quad 132 = \frac{210}{10.5} \times 8 \times \frac{d}{D} \quad \text{or} \quad \frac{D}{d} = 1.212$$

Ratio of B/t

$$M_r\,(\text{timber}) = M_r\,(\text{steel})$$

$$2\sigma_t Z_t = \sigma_s Z_s \qquad \text{or} \qquad 2\sigma_t \cdot \frac{BD^2}{6} = \sigma_s \cdot \frac{td^2}{6}$$

or

$$\frac{B}{t} = \frac{1}{2}\left(\frac{d}{D}\right)^2 \frac{\sigma_s}{\sigma_t} = \frac{1}{2}\left(\frac{33}{40}\right)^2 \frac{132}{8} = 5.615$$

Example 5.29 || A composite beam is made up of two timber joists that are 80 mm wide and 180 mm deep with a 20-mm thick and 80-mm deep steel plate placed symmetrically between them and firmly attached to both in such a way that the plate is recessed into grooves cut in the inner faces of the joists so that the overall dimensions of the built-up section become 160 mm by 180 mm. Assuming a maximum stress in the timber joist as 7.2 MPa, determine the maximum stress reached in the steel and the moment of resistance of the section. Take $E_s = 20\ E_t$.

Solution

Given A composite beam of two timber joists and a steel plate as shown in Fig. 5.41.

$\sigma_{t\ \max} = 7.2$ MPa $\qquad E_s = 20\ E_t$.

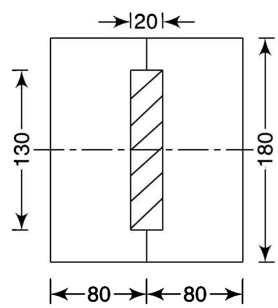


Fig. 5.41

To find

- — Maximum stress in the steel
- — Moment of resistance

Maximum stress in steel

When the stress in the timber is maximum, i.e., 7.2 MPa at the outermost fiber,

stress at 65 mm from the neutral axis = $7.2 \times \dfrac{65}{90} = 5.2$ MPa

Moment of resistance

Maximum stress in the steel = 5.2 × 20 = 104 MPa

$$M_{r(\text{timber})} = \frac{1}{6} \times 7.2 \times 160 \times 180^2 - \frac{1}{6} \times 5.2 \times 20 \times 130^2 = 5.928X10^6 \text{ N·mm}$$

$$M_{r(\text{steel})} = \frac{1}{6} \times 104 \times 20 \times 130^2 = 5.859 \times 10^6 \text{ N·mm}$$

Total moment of resistance

$$= 5.928 \times 10^6 + 5.859 \times 10^6 = 11.787 \times 10^6 \text{ N·mm} \quad \text{or} \quad 11.787 \text{ kN·m}$$

Example 5.30 A composite beam carries a uniformly distributed load of 800 N/m over the whole span of 10-m. The beam is made up of two timber joists, each 80 mm × 180 mm, with a 10-mm thick and 130-mm deep steel plate bolted between them as shown in Fig. 5.42. Find the maximum bending stresses in both the materials. Given $E_s = 20\ E_t$.

Solution

Given A composite beam of two timber joists and a steel plate as shown in Fig. 5.42.

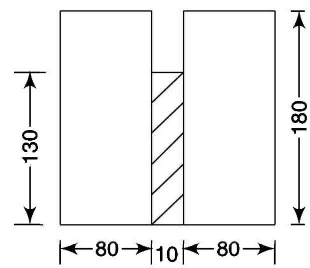


Fig. 5.42

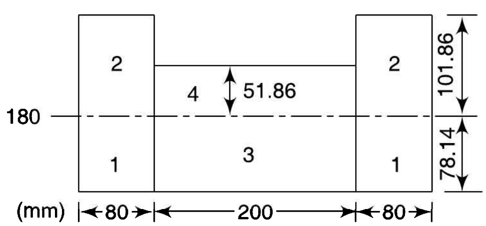
$$w = 800\ \text{N/m} \qquad l = 10\ \text{m} \qquad E_s = 20\ E_t.$$

To find Maximum bending stresses

$$\text{Maximum bending moment} = \frac{wl^2}{8} = \frac{800 \times 10^2}{8} = 10\ 000\ \text{N·m}$$

Equivalent timber section

Figure 5.43 shows the equivalent timber section. The steel plate is equivalent to a timber section of width 20 × 10 = 200 mm and depth 130 mm.



Equivalent timber section

Fig. 5.43

Height of neutral axis from the bottom suface,

$$y = \frac{2 \times 80 \times 180 \times 90 + 200 \times 130 \times 65}{2 \times 80 \times 180 + 200 \times 130} = 78.14\ \text{mm}$$

$$I = \underset{\text{For area (1)}}{\frac{2 \times 80 \times 78.14^3}{3}} + \underset{(2)}{\frac{2 \times 80 \times 101.86^3}{3}} + \underset{(3)}{\frac{200 \times 78.14^3}{3}} + \underset{(4)}{\frac{200 \times 51.86^3}{3}}$$

$$= 122.93 \times 10^6\ \text{mm}^4$$

Calculation for stresses

Maximum tensile stress in timber (at bottom edge)

$$= \frac{10\ 000 \times 10^3}{122.93 \times 10^6} \times 78.14 = 6.356\ \text{MPa}$$

Maximum tensile stress in steel (at bottom edge) = 6.356 × 20 = 127.13 MPa

Maximum compressive stress in timber (at top edge)

$$= \frac{10\ 000 \times 10^3}{122.93 \times 10^6} \times 101.86 = 8.286\ \text{MPa} \quad \text{or} \quad 6.356 \times \frac{101.86}{78.14} = 8.286\ \text{MPa}$$

Maximum compressive stress in timber at the level of top fiber of steel

$$= 8.286 \times \frac{51.86}{101.86} = 4.219 \text{ MPa}$$

Maximum compressive stress in steel = 4.219 × 20 = 84.37 MPa

Example 5.31 A straight bimetallic rectangular composite bar of width b and thickness $2t$ is made up of a strip of steel of rectangular section of width b and thickness t joined along its length by a strip of brass having the same dimensions. The bar is uniformly heated and is allowed to bend freely. Show that it bends to a radius

$$R = \frac{E_b^2 + E_s^2 + 14E_bE_s}{12E_bE_s(\alpha_b - \alpha_s)} \cdot \frac{t}{T}$$

where α_b and α_s are the coefficients of linear expansions of brass and steel respectively and T is the rise in temperature.

Solution

Given A straight bimetallic rectangular composite bar made of steel and brass strips as shown in Fig. 5.44. Bar is uniformly heated through $T°$, free to bend.

To find To show that the bar bends to a radius, $R = \dfrac{E_b^2 + E_s^2 + 14E_bE_s}{12E_bE_s(\alpha_b - \alpha_s)} \cdot \dfrac{t}{T}$

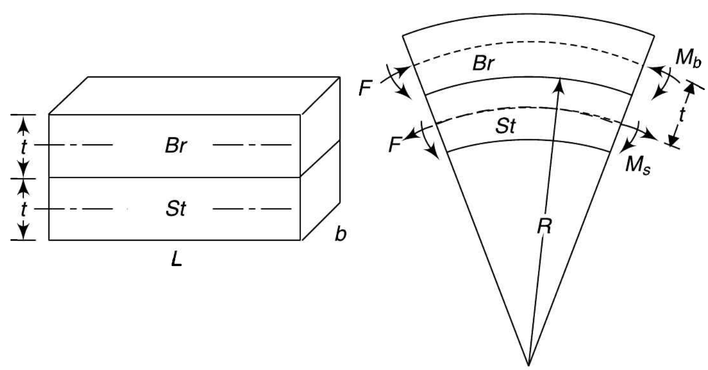


Fig. 5.44

Equilibrium equation

Let α_b be greater than α_s.

A force at the interaction of the two strips will tend to compress the brass and elongate the steel. Let this internal force be F. This force induces a direct load F at the centre of each section alongwith a bending moment in each strip (Fig. 5.44).

Bending moment due to this force = $F \cdot t$

Assuming R to be the same for both strips, i.e., much larger as compared to t,

$$M_b = \frac{I_bE_b}{R} = \frac{bt^3E_b}{12R}; \text{ Similarly, } M_s = \frac{bt^3E_s}{12R}$$

For equilibrium, $F \cdot t = M_b + M_s$

or $$F \cdot t = \frac{bt^3}{12R}(E_b + E_s) \quad \text{or} \quad F = \frac{bt^2}{12R}(E_b + E_s) \tag{i}$$

Linear strains

Linear strain at the central axis of brass $= \dfrac{(R + t/2)\theta - R\theta}{R\theta} = \dfrac{t}{2R}$

Linear strain at the central axis of steel $= \dfrac{(R - t/2)\theta - R\theta}{R\theta} = \dfrac{t}{2R}$

Difference in linear strains at the central axis $= \dfrac{t}{2R} - \left(-\dfrac{t}{2R}\right) = \dfrac{t}{R}$

Compatibility equation

The compatibility equation is,

$$\frac{t}{R} = \left(\alpha_b T - \frac{\sigma_b}{E_b}\right) - \left(\alpha_s T - \frac{\sigma_s}{E_s}\right) = (\alpha_b - \alpha_b)T - \frac{F}{bt}\left(\frac{1}{E_b} + \frac{1}{E_s}\right)$$

or

$$(\alpha_b - \alpha_b)T = F \cdot \frac{1}{bt}\left(\frac{1}{E_b} + \frac{1}{E_s}\right) + \frac{t}{R}$$

$$= \frac{bt^2}{12R}(E_b + E_s) \cdot \frac{1}{bt}\left(\frac{E_s + E_b}{E_b \cdot E_s}\right) + \frac{t}{R} \qquad \text{...[from(i)]}$$

$$= \frac{t}{12R}\left[\frac{(E_b + E_s)^2}{E_b.E_s} + 12\right] = \frac{t}{12R}\left[\frac{E_b^2 + E_s^2 + 2E_bE_s + 12E_b \cdot E_s}{E_b \cdot E_s}\right]$$

$$= \frac{t}{12R}\left[\frac{E_b^2 + E_s^2 + 14E_bE_s}{E_b \cdot E_s}\right]$$

or

$$R = \frac{E_b^2 + E_s^2 + 14E_bE_s}{12E_bE_s(\alpha_b - \alpha_s)} \cdot \frac{t}{T}$$

Example 5.32 || A 4-mm thick and 240-mm long straight bimetallic rectangular composite bar is made up by joining a steel strip of 240-mm width and 2-mm thick rectangular section to a brass strip having the same dimensions. The bar is uniformly heated and is freely allowed to bend. Determine the maximum clearance between the bar and the surface due to rise in temperature of 120°.

$\alpha_b = 18 \times 10^{-6}$/°C; $\alpha_s = 10.5 \times 10^{-6}$/°C; $E_b = 95$ Gpa; $E_s = 210$ GPa

Solution

Given A straight bimetallic rectangular composite bar made of steel and brass strips, each 240 mm wide and 2 mm thick; bar is uniformly heated to 120°, free to bend

$\alpha_b = 18 \times 10^{-6}$/°C $\qquad \alpha_s = 10.5 \times 10^{-6}$/°C

$E_b = 95$ GPa $\qquad E_s = 210$ GPa

To find Maximum clearance between the bar and the surface

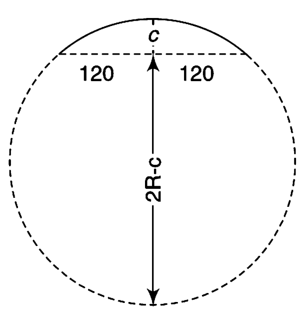


Fig. 5.45

Radius of curvature

Using the relation obtained in the previous example,

$$R = \frac{E_b^2 + E_s^2 + 14E_bE_s}{12E_bE_s(\alpha_b - \alpha_s)} \cdot \frac{t}{T}$$

$$= \frac{95\,000^2 + 210\,000^2 + 14 \times 95\,000 \times 210\,000}{12 \times 95\,000 \times 210\,000 \times (18 - 10.5) \times 10^{-6}} \cdot \frac{2}{120} = 3086 \text{ mm}$$

Maximum clearance

Let c be the clearance. From Fig. 5.45,

$$(2R - c)\, c = 120 \times 120 \qquad \text{or} \qquad 2 \times 3086c = 14400$$

or $c = 2.33$ mm (approx.) ...(c is very small compared to R, thus c^2 is neglected)

5.5 REINFORCED CONCRETE BEAMS

Concrete is a material which has compressive strength but is very weak in tension. At times it develops cracks, thus reducing its tensile strength to zero. To compensate for this weakness of concrete, steel reinforcement is done on the tension side of concrete beams and to have the maximum advantage, it is put at the maximum distance from the neutral axis of the beam (Fig. 5.46),

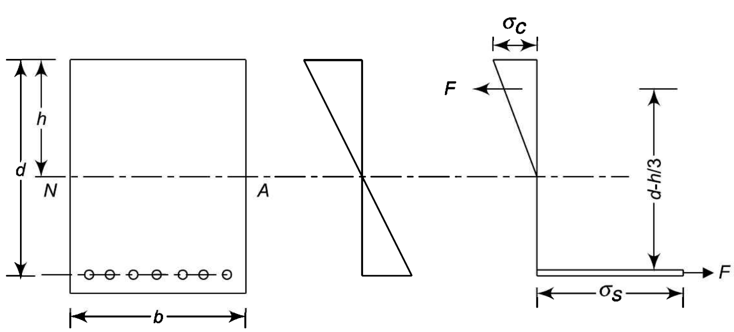


Fig. 5.46

The following assumptions are made in the reinforced concrete beams:

1. Zero stress in the concrete on tension side
2. uniform stress in the steel
3. stress proportional to strain in the concrete
4. strain proportional to distance from neutral axis

Assumption 3 is not true as concrete does not obey the Hooke's law. However, a mean value may be taken of the modulus over the range of stress used. The last assumption is true for pure bending and it also implies that there is not relative slip between concrete and steel.

Consider the case of rectangular section as shown in Fig. 5.46.

Let d = depth of reinforcement measured from compression edge

h = distance of neutral axis from the compression edge

σ_c = maximum stress in the concrete

σ_s = maximum stress in the steel

A_s = area of steel reinforcement

m = modular ratio E_s/E_c

As strains are considered proportional to distance from neutral axis,

$$\frac{\sigma_s}{\sigma_c} = \frac{E_s \varepsilon_s}{E_c \varepsilon_c} = m\frac{d-h}{h}$$

If the beam is acted upon by a pure bending moment M, then
Resultant force in concrete = resultant force in steel

$$F = \sigma_s A_s = \frac{\sigma_c}{2} \cdot bh \qquad \text{.........}(\sigma_c/2 \text{ is the mean value of stress in the concrete})$$

Moment of resistance, $M = F \cdot \left(d - \frac{h}{3}\right) = \sigma_s A_s \left(d - \frac{h}{3}\right) = \frac{\sigma_c}{2} \cdot bh \cdot \left(d - \frac{h}{3}\right)$ (5.17)

Example 5.33 A reinforced concrete beam of 100-mm wide and 150-mm deep rectangular section is of 3 m length. It has steel placed at 15 mm above the tension face. Determine the maximum stresses produced in the concrete and the steel if the beam is simply supported at the ends and has central load of 4 kN. Area of the steel is 250 mm² and the modular ratio 18.

Solution

Given A reinforced concrete beam

$W = 4$ kN $\quad A_s = 250$ mm² $\quad m = 18$

$b = 100$ mm $\quad$ depth = 150 mm

To find

Maximum stresses in concrete and steel

Calculation for distance of neutral axis

(Refer Fig. 5.46) $d = 150 - 15 = 135$ mm

We have,
$$\frac{\sigma_s}{\sigma_c} = m\frac{d-h}{h} = 18 \times \frac{(135-h)}{h} \tag{i}$$

Also,
$$\sigma_s A_s = \frac{\sigma_c}{2} \cdot bh \quad \text{or} \quad \frac{\sigma_s}{\sigma_c} = \frac{bh}{2A_s} = \frac{100h}{2 \times 250} \tag{ii}$$

From (i) and (ii),

$$\frac{100h}{2 \times 250} = \frac{18(135-h)}{h}$$

or
$$0.2h^2 = 2430 - 18h$$

or
$$h^2 + 90h - 12\,150 = 0$$

or
$$h = \frac{-90 \pm \sqrt{8100 + 4 \times 12\,150}}{2} = 74.1 \text{ mm}$$

Calculations for stresses in steel and concrete

$$M = \frac{Wl}{4} = \frac{4 \times 3}{4}\ 3 \text{ kN} \cdot \text{m}$$

Also,
$$M = \sigma_s A_s \left(d - \frac{h}{3}\right)$$

or
$$3 \times 10^6 = \sigma_s \times 250 \times \left(135 - \frac{74.1}{3}\right) \quad \text{or} \quad \sigma_s = 108.8 \text{ MPa}$$

and $$M = \frac{\sigma_c}{2} \cdot bh \cdot \left(d - \frac{h}{3}\right)$$

or $$3 \times 10^6 = 3 \times 10^6 = \frac{\sigma_c}{2} \times 100 \times 74.1 \times \left(135 - \frac{74.1}{3}\right) \quad \text{or} \quad \sigma_c = 7.34 \text{ MPa}$$

5.6 UNSYMMETRICAL BENDING

Unsymmetrical bending involves the cases in which either the bending moments do not act in a plane of symmetry of the member or the member does not possess a symmetric cross-sectional area.

In deriving the relation for pure bending, one of the assumptions is that the section is symmetrical about a vertical plane passing through the vertical axes of symmetry and the bending couple acts in that plane. A vertical axis of symmetry is perpendicular to the neutral axis passing through the centroid. Owing to symmetry of such a member and of the loading, the member remains symmetric with respect to the vertical plane and is bent in that plane. If the vertical plane is not a plane of symmetry, the member cannot be expected to bend in that plane.

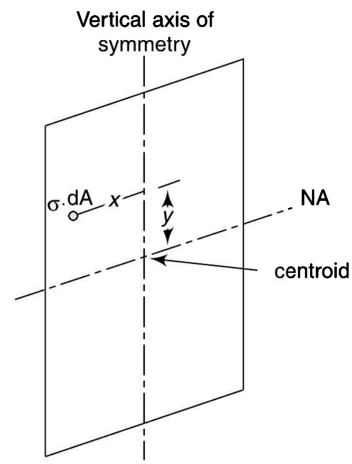


Fig. 5.47

For symmetric bending, the summation of moments of all the elementary forces about the vertical axis of symmetry must be zero, i.e.,

$$\int \sigma dA \cdot x = 0 \quad \text{or} \quad \int \sigma x \, dA \cdot = 0$$

or $$\frac{E}{R}\int y \, x \, dA = 0 \qquad \left(\because \frac{\sigma}{y} = \frac{E}{R}\right)$$

or $$\int xydA = 0 \tag{5.18}$$

which is the necessary condition to use the bending equation derived earlier. However, this condition is found to be satisfied for a set of two perpendicular axes for all types of symmetric and unsymmetric sections. The integral $\int xydA$ (also denoted by I_{xy}) is known as *product of inertia.* The two axes for which it is zero for a section are known as *principal axes* or *principlal centroidal axes* of the cross-section. The moments of inertia of an area about its principal axes are known as *principal moments of inertia*. If a cross-section has an axis of symmetry, then it can easily be shown that this satisfies the condition for a principal axis. The other principal axis will be at right angle through the centroid.

- Note that the product of inertia $I_{xy} = \int xydA$ is not the product of $I_x = \int y^2dA$ and $I_y = \int x^2dA$. Thus whereas I_{xy} is zero for principal axes, I_x and I_y will have certain values known as *principal moments of inertia*.
- In case of unsymmetrical bending it is assumed that there is no twisting of the members due to unsymmetric shear stresses. It is observed that if the load is applied through a particular point known as *shear centre*, there will not be any torsion or twisting of a member due to shear stresses. The shear centre may lie in or outside the section. If the load is not applied through the shear centre, there is twisting of the beam due to unbalanced moment caused by the shear force acting on the section. For sections symmetrical about an axis, the shear centre lies on the axis of symmetry. For sections having two axes of symmetry, the shear centre lies at the intersection of these axes and thus coincides with the centroid. (Refer Section 6.5 for details).

Example 5.34 A 4-m long simply supported beam of 80 mm width and 100 mm depth carries a load of 10 kN at the midspan. The load is inclined at 30° to the vertical longitudinal plane and the line of action of the load passes through the centroid of the rectangular section of the beam. Determine the stresses at all the corners of the section.

Solution

Given A simply supported beam carrying an inclined load as shown in Fig. 5.48a

$L = 4$ m $\qquad W = 10$ kN

To find Stresses at all corners of the section

The section being symmetrical, the centroid is at the centre of the rectangle and the coordinate axes *x-x* and *y-y* are also the principal axes (Fig. 5.48).

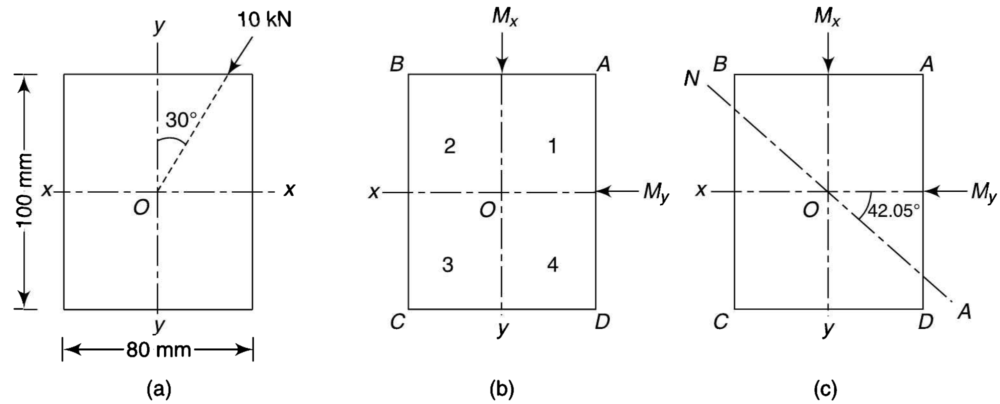


Fig. 5.48

Moment of inertia

$$I_x = \frac{80 \times 100^3}{12} = 6.667 \times 10^6 \text{ mm}^4$$

and
$$I_y = \frac{100 \times 80^3}{12} = 4.267 \times 10^6 \text{ mm}^4$$

Bending moments

Maximum bending moment $= \dfrac{10 \times 4}{4} = 10$ kN·m or 10×10^6 N·mm

Resolving into components,

$M_x = 10 \times 10^6 \cos 30° = 8.66 \times 10^6$ N·mm (due to vertical component of load)

$M_y = 10 \times 10^6 \sin 30° = 5 \times 10^6$ N·mm (due to horizontal component of load)

As it is a simply supported beam,

- the vertical load component induces compressive stress in the upper half and tensile stress in the lower half, and
- the horizontal load component induces compressive stress in the right half and tensile stress in the left half.

Bending stresses

The bending stress at any point (x, y) in the section consists of two parts, one due to bending about the axis *x-x* and the other due to the bending about *y-y*, i.e.,

$$\sigma = \frac{M_x}{I_x} \cdot y + \frac{M_y}{I_y} \cdot x$$

As both the components are to give tensile stress in the 3rd quadrant, x and y both can be assumed positive in this quadrant. Thus assume the following for positive and negative directions for x and y:

- x positive towards left of O and negative towards right of O
- y positive downward from O and negative upwards from O

Thus

— in 1st , x and y both negative

— in 2nd, x positive and y negative

— in 3rd , x and y both positive and

— in 4th , x negative and y positive

Therefore,

$$\sigma_a = \frac{8.66 \times 10^6}{6.667 \times 10^6} \cdot (-40) + \frac{5 \times 10^6}{4.267 \times 10^6} \cdot (-50)$$

$$= -1.299 \times 40 - 1.172 \times 50 = -111.83 \text{ MPa}$$

$$\sigma_b = 1.299 \times 40 + 1.172 \times (-50) = 64.95 - 46.88 = -18.07 \text{ MPa}$$

$$\sigma_c = 1.299 \times 40 + 1.172 \times 50 = 64.95 + 46.88 = 111.83 \text{ MPa}$$

$$\sigma_d = 1.299 \times (-40) + 1.172 \times 50 = -64.95 + 46.88 = 18.07 \text{ MPa}$$

Neutral axis

As the stress at the neutral axis is zero, the equation for the neutral axis is $\sigma = 0$

or $\quad 1.299\, y + 1.172 \times = 0$

or $\quad \tan\alpha = \dfrac{y}{x} = -\dfrac{1.172}{1.299} = -0.902 \quad$ or $\quad \alpha = -42.05°$

As it is negative, it is taken at an angle of 42.05° with the x-axis in such a way that it passes through a quadrant in which either x or y is negative. The neutral axis has been shown in the figure. Finally, on that side of the neutral axis which has 3rd quadrant (both x and y positive), there will be tensile stress and on the other compressive stress.

Example 5.35 || A simply supported I-beam of 2-m span carries a central load of 4 kN. The load acts through the centroid, the line of action is inclined at 30° to the vertical direction. Determine the maximum stress.

Solution

Given A simply supported I-beam carrying an inclined load as shown in Fig. 5.49

$$L = 2 \text{ m} \qquad W = 4 \text{ kN}$$

To find Maximum stress

As the section is symmetrical, the centroid is at the centre of the web and x-x and the coordinate axes y-y are also the principal axes.

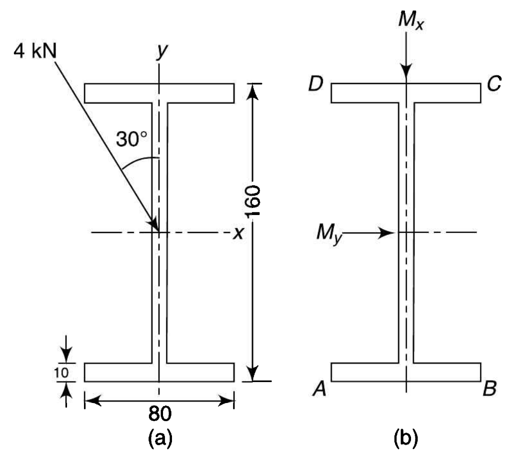


Fig. 5.49

Moment of inertia

$$I_x = 2\left(\frac{80 \times 10^3}{12} + 80 \times 10 \times 75^2\right) + \frac{8 \times 140^3}{12} = 10.843 \times 10^6\,\text{mm}^4$$

$$I_y = 2 \times \frac{10 \times 80^3}{12} + \frac{140 \times 8^3}{12} = 0.859 \times 10^6\,\text{mm}^4$$

Bending moments

Maximum bending moment $= \dfrac{4 \times 2}{4} = 2\ \text{kN·m}$

$M_x = 2 \times \cos 30° = 1.732\ \text{kN·m}$ (to induce tensile stress in lower half)

$M_y = 2 \times \sin 30° = 1\ \text{kN·m}$ (to induce tensile stress in right half)

Bending stresses

x and *y* both are positive in the lower right half (4th) quadrant.

The maximum tensile stress occurs at *B*,

$$\sigma = \frac{M_x}{I_x} \cdot y + \frac{M_y}{I_y} \cdot x = \frac{1.732 \times 10^6}{10.843 \times 10^6} \times 80 + \frac{1 \times 10^6}{0.859 \times 10^6} \times 40 = 59.34\ \text{MPa}$$

The maximum compressive stress occurs at *D* and is 59.34 MPa

Example 5.36 Figure 5.50 shows a cantilever beam of *I*-section loaded by two inclined loads . Determine the stresses at all the four corners.

Solution

Given A cantilever beam of *I*-section carrying two inclined loads as shown in Fig. 5.50

To find Stresses at all the four corners

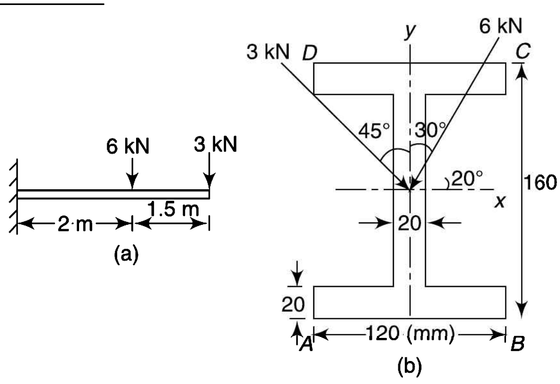


Fig. 5.50

As the section is symmetrical, the centroid is at the centre of the web and the coordinate axes *x-x* and *y-y* are also the principal axes (Fig. 5.51a).

Moment of inertia

$$I_x = 2\left(\frac{120 \times 20^3}{12} + 120 \times 20 \times 70^2\right) + \frac{20 \times 120^3}{12} = 26.56 \times 10^6\,\text{mm}^4$$

$$I_y = 2 \times \frac{20 \times 120^3}{12} + \frac{120 \times 20^3}{12} = 5.84 \times 10^6 \text{ mm}^4$$

Bending moments

Remembering that it is a cantilever,

$$M_x = 3 \times 3.5 \cos 45° + 6 \times 2 \cos 30° = 1.732 \text{ kN·m}$$

(to induce tensile stress in upper half)

$$M_y = 3 \times 3.5 \sin 45° - 6 \times 2 \sin 30° = 1.425 \text{ kN·m}$$

(to induce tensile stress in left half)

Thus x and y both are positive in the upper left (2nd) quadrant.

Bending stresses

The maximum tensile stress occurs at D,

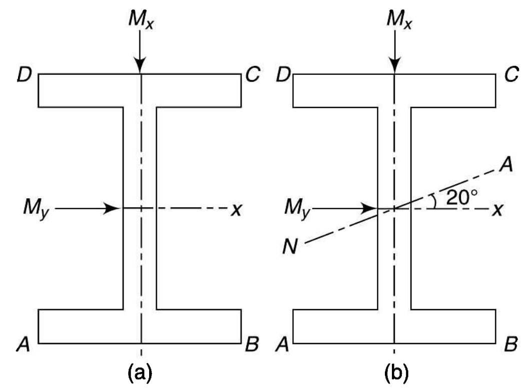


Fig. 5.51

Now, $$\sigma = \frac{M_x}{I_x} \cdot y + \frac{M_y}{I_y} \cdot x$$

$$\sigma_a = -\frac{17.827 \times 10^6}{26.56 \times 10^6} \times 80 + \frac{1.425 \times 10^6}{5.84 \times 10^6} \times 60$$

$$= -0.671 \times 80 + 0.244 \times 60$$

$$= -53.66 + 14.64 = -39.02 \text{ MPa (compressive)}$$

$$\sigma_b = -53.66 - 14.64 = -68.3 \text{ MPa (compressive)}$$

$$\sigma_c = 53.66 - 14.64 = 39.02 \text{ MPa (tensile)}$$

$$\sigma_d = 53.66 + 14.64 = 68.3 \text{ MPa (tensile)}$$

Neutral axis

Inclination of the neutral axis with x-axis is given by,

$$\sigma = \frac{M_x}{I_x} \cdot y + \frac{M_y}{I_y} \cdot x \quad \text{or} \quad -0.671y + 0.244 \cdot x = 0$$

or $$\tan \alpha = \frac{y}{x} = -\frac{0.244}{0.671} = -0.364 \quad \text{or} \quad \alpha = -20°$$

As it is negative, it is taken at an angle of 20° with the x-axis in such a way that it passes through a quadrant in which either x or y is negative. The neutral axis has been shown in the figure. On upper side of the neutral axis, there will be tensile stress and on the lower compressive stress.

Example 5.37 Figure 5.52 shows the cross-section of a 3 m long simply supported beam of T-section carrying a central load inclined at 30° to the y-axis. Determine the maximum load the beam can sustain if the maximum tensile and compressive stresses are not allowed to exceed 40 MPa and 80 MPa respectively. Locate the neutral axis also. The load passes through the centroid of the section.

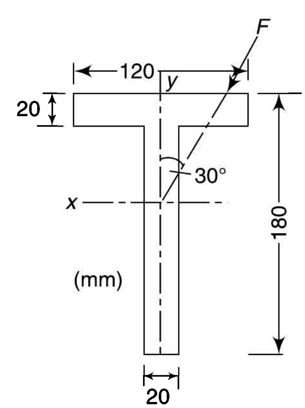


Fig. 5.52

Solution

Given A simply supported beam of T-section as shown in Fig. 5.52. Maximum tensile stress is 40 MPa and maximum compressive stress is 80 MPa.

$$L = 3 \text{ m}$$

To find

— Maximum load
— Location of neutral axis

Moment of inertia

As the section has y-axis as axis of symmetry, it is a principal axis. The other principal axis will be at a perpendicular through the centroid (Fig. 5.53a).

$$y = \frac{120 \times 20 \times 10 + 20 \times 160 \times 100}{120 \times 20 + 20 \times 160} = 61.43 \text{ mm}$$

$$I_x = \frac{120 \times 20^3}{12} + 120 \times 20 \times 51.43^2 + \frac{20 \times 160^3}{12} + 20 \times 160 \times (100 - 61.43)^2$$

$$= (6.428 + 11.587) \times 10^6 \text{ mm}^4 = 18.015 \times 10^6 \text{ mm}^4$$

$$I_y = \frac{20 \times 120^3}{12} + \frac{160 \times 20^3}{12} = 2.987 \times 10^6 \text{ mm}^4$$

Let F be the maximum load at the centre in kN.

Bending moments

$$\text{Maximum bending moment} = \frac{F \times 3}{4} = 0.75F \text{ kN}\cdot\text{m}$$

$M_x = 0.75F \cos 30° = 0.65F$ kN·m

(to induce tensile stress in lower half)

$M_y = 0.75F \sin 30° = 0.375F$ kN·m

(to induce tensile stress in left half)

Thus x and y both are positive in the lower left (3rd) quadrant.

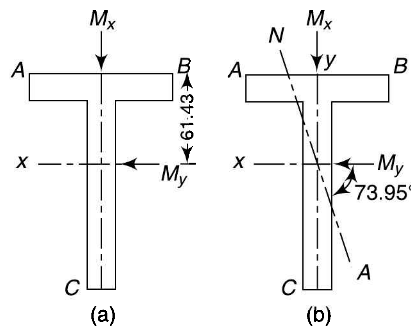


Fig. 5.53

Neutral axis

The equation for the neutral axis is $\sigma = 0$ or $\dfrac{M_x}{I_x} \cdot y + \dfrac{M_y}{I_y} \cdot x = 0$

or $$\frac{0.65F \times 10^6}{18.015 \times 10^6} \cdot y + \frac{0.375F \times 10^6}{2.987 \times 10^6} \cdot x = 0$$

or $$0.0361\, F \cdot y + 0.1255\, F \cdot x = 0$$

or $$\tan\alpha = \frac{y}{x} = -\frac{0.1255}{0.0361} = -3.476 \quad \text{or} \quad \theta = -73.95°$$

Neutral axis will pass through 2nd and 4th quadrant as shown in Fig. 5.48b.

Bending stresses

The maximum tensile stress occurs at A or C, i.e.,

$$\sigma_a = 0.0361F \times (-61.43) + 0.1255F \times 60 = -2.216F + 7.533F$$

or $$40 = 5.317\, F \quad \text{or} \quad F = 7.523 \text{ kN}$$

and $$\sigma_c = 0.0361F \times (180 - 61.43) + 0.1255F \times 10 = 4.278F + 1.255F$$

or $$40 = 5.533\, F \quad \text{or} \quad F = 7.229 \text{ kN}$$

Maximum compressive stress will be at B,

$$\sigma_b = -.0361F \times (-61.43) + 0.1255F \times (-60) = -2.216F - 7.533F$$

or $$-80 = -9.749\, F \quad \text{or} \quad F = 8.206 \text{ kN}$$

Thus the maximum load can be 7.229 kN

Example 5.38 || Figure 5.54 shows the cross-section of a cantilever channel beam that is 3 m in length. The beam supports a load of 2 kN inclined at 30° to the longitudinal plane at the free end. Locate the position of the neutral axis and find the maximum tensile stress acting on the built-in end. Assume the line of action of load to pass through the shear centre.

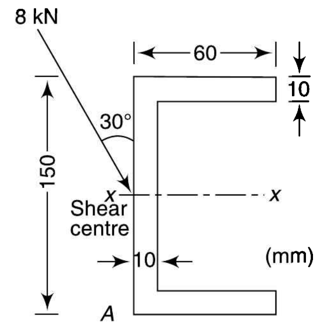


Fig. 5.54

Solution

Given A cantilever channel as shown in Fig. 5.54. As line of action of load passes through shear centre, no twisting moment.

$L = 3$ m $\qquad W = 2$ kN

To find

— Position of neutral axis
— Maximum tensile stress

As the section has x-axis as axis of symmetry, it is a principal axis. The other principal axis will be at right angle through the centroid (Fig. 5.55).

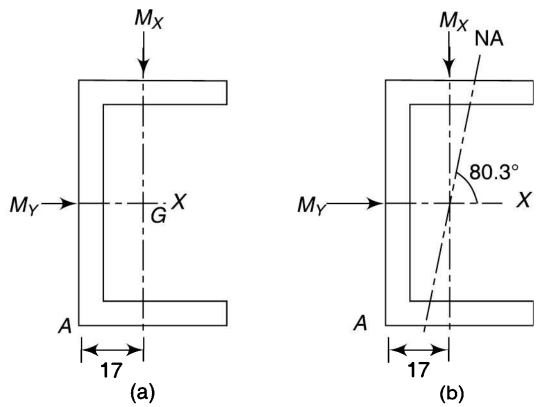


Fig. 5.55

Moment of inertia

$$y = \frac{150 \times 10 \times 5 + 2 \times 50 \times 10 \times 35}{150 \times 10 + 2 \times 50 \times 10} = 17 \text{ mm}$$

$$I_x = \frac{10 \times 150^3}{12} + 2\left(\frac{50 \times 10^3}{12} + 50 \times 10 \times 70^2\right) = 7.721 \times 10^6 \text{ mm}^4$$

I_x can also be found as under:

$$I_x = \frac{60 \times 150^3}{12} - \frac{50 \times 130^3}{12} = 7.721 \times 10^6 \text{ mm}^4$$

$$I_y = \frac{150 \times 10^3}{12} + 150 \times 10 \times (17 - 5)^2 + 2 \times \frac{10 \times 50^3}{12} + 2 \times 10 \times 50 \times (43 - 25)^2$$

$$= 0.761 \times 10^6 \text{ mm}^4$$

or $\qquad$ $$I_y = 2 \times \frac{10 \times 43^3}{3} + \frac{150 \times 17^3}{3} - \frac{130 \times 7^3}{3} = 0.761 \times 10^6 \text{ mm}^4$$

Bending moments

Maximum bending moment = 2 × 3 = 6 kN·m

Resolving into components,

$M_x = 6 \cos 30° = 5.2$ kN·m (due to vertical component of load)

$M_y = 6 \sin 30° = 3$ kN·m (due to horizontal component of load)

As it is a cantilever, both of these components induce tensile stress in the upper left quadrant i.e. above x-axis and left of y-axis in the quadrant containing point A.

Position of neutral axis

The equation for the neutral axis is $\sigma = 0$ or $\frac{M_x}{I_x} \cdot y + \frac{M_y}{I_y} \cdot x = 0$

or $$\frac{5.2 \times 10^6}{7.721 \times 10^6} \cdot y + \frac{3 \times 10^6}{0.761 \times 10^6} \cdot x = 0$$

or $$1.101y + 3.942x = 0$$

or $$\tan \alpha = \frac{y}{x} = -\frac{3.942}{1.101} = -3.58 \quad \text{or} \quad \theta = -74.4°$$

Neutral axis passes through 2nd and 4th quadrant.

Maximum tensile stress

Maximum tensile stress at A = 1.101 × 75 + 3.942 × 17 = 149.6 MPa

5.7 DETERMINATION OF PRINCIPAL AXES

Sometimes, in case of unsymmetrical sections, the directions of the principal axes are not known. In such cases, the direction of these can be found as follows:

Let OX and OY be any two perpendicular axes through the centroid and OU and OV the principal axes (Fig. 5.56). Also let the inclination of OU with OX be θ.

Let δA be an elemental area and

x and y = coordinate of the area relative to OX, OY

u and v = coordinate of the area relative to OU, OV

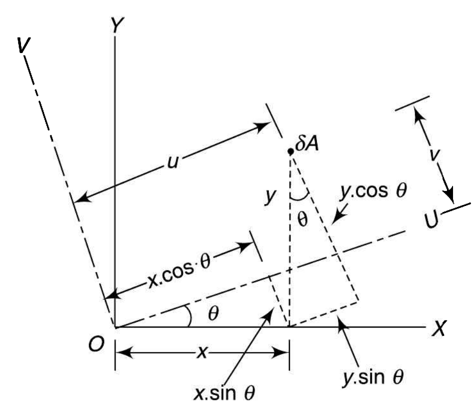


Fig. 5.56

Then $u = x \cos \theta + y \sin \theta$

And $v = x \cos \theta + y \sin \theta$

Product of inertia =

$$I_{uv} = \int uv dA = \int (x \cos \theta + y \sin \theta)(y \cos \theta - x \sin \theta)\, dA$$

$$= \int (xy \cos^2 \theta - x^2 \sin \theta \cos \theta + y^2 \sin \theta \cos \theta - xy \sin^2 \theta)\, dA$$

$$= \sin \theta \cos \theta \left(\int y^2 dA - \int x^2 dA\right) + (\cos^2 \theta - \sin^2 \theta) \int xy dA$$

$$= \sin \theta \cos \theta \left(\int y^2 dA - \int x^2 dA\right) + \left(\frac{1 + \cos \theta}{2} - \frac{1 - \cos \theta}{2}\right) \int xy dA$$

$$\left(\frac{1}{2} \sin 2\theta\right)(I_x - I_y) + \cos 2\theta \cdot I_{xy} \tag{5.19}$$

Applying the condition for principal axes, i.e., $I_{uv} = 0$

or $$\left(\frac{1}{2}\sin 2\theta\right)(I_x - I_y) + \cos 2\theta \cdot I_{xy} = 0$$

or $$\sin 2\theta(I_x - I_y) = -2\cos 2\theta \cdot I_{xy} \tag{i}$$

or $$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} \tag{5.20}$$

If I_x, I_y and I_{xy} are calculated, θ can easily be calculated.

The principal moments of inertia I_u and I_v can then be calculated as follows:

Now, $$I_u = \int v^2 \cdot dA$$

$$= \int(y\cos\theta - x\sin\theta)^2 dA = \int(y^2\cos^2\theta + x^2\sin^2\theta - 2xy\sin\theta\cos\theta)dA$$

$$= \cos^2\theta \cdot I_x + \sin^2\theta \cdot I_y - \sin 2\theta \cdot I_{xy} \tag{5.21}$$

$$= \frac{1+\cos 2\theta}{2}\cdot I_x + \frac{1-\cos 2\theta}{2}\cdot I_y - \sin 2\theta \cdot \frac{\sin 2\theta(I_x - I_y)}{-2\cos 2\theta} \quad \text{[From (i)]}$$

$$= \frac{1}{2}(I_x + I_y) + \frac{1}{2}\cos 2\theta\,(I_x - I_y) + \frac{\sin^2 2\theta\,(I_x - I_y)}{2\cos 2\theta}$$

$$= \frac{1}{2}(I_x + I_y) + (I_x - I_y)\frac{\cos^2 2\theta + \sin^2 2\theta}{2\cos 2\theta}$$

$$= \frac{1}{2}[(I_x + I_y) + \sec 2\theta\,(I_x - I_y)] \tag{5.21a}$$

In case I_x and I_y are equal, angle 2θ is 90° and $\cos 2\theta = 0$. In such cases Eq. 5.21 should be used to find I_u as Eq. 5.21a involves division by zero and thus accurate results are not obtained.

$$I_v = \int u^2 \cdot dA$$

$$= \int(x\cos\theta + y\sin\theta)^2 dA = \int(x^2\cos^2\theta + y^2\sin^2\theta + 2xy\sin\theta\cos\theta)\,dA$$

$$= \cos^2\theta \cdot I_y + \sin^2\theta \cdot I_x + \sin 2\theta \cdot I_{xy} \tag{5.22}$$

Simplifying as above,

$$I_v = \frac{1}{2}[(I_x + I_y) - \sec 2\theta(I_x - I_y)] \tag{5.22a}$$

Adding 5.21a and 5.22a,

$$I_u + I_v = I_x + I_y \tag{5.23}$$

I_{xy} for a rectangle with sides parallel to the principal axes can be found as below (Fig. 5.57):

Let h and k be the x- and y-coordinates of the centroid of the rectangle, then

$$I_{xy} = \iint xy \cdot dy \cdot dx = \left(\frac{x^2}{2}\right)_{k-b/2}^{k+b/2} \times \left(\frac{y^2}{2}\right)_{h-d/2}^{h+d/2} = kb \times hd = bd \cdot hk = A \cdot hk \tag{5.24}$$

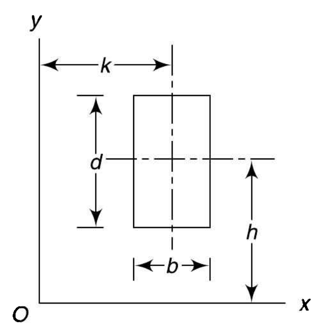


Fig. 5.57

If the origin is at the centroid of the rectangle, $I_{xy} = 0$ as h and k are both zero.

Procedure for the Analysis of Bending Stress in Unsymmetrical Sections

- Locate the x- and y-axis by taking moments of the areas about the vertical and horizontal edges.
- Calculate I_x, I_y and I_{xy}.
- Locate the u-axis by using Eq. 5.20. As this equation is derived by taking u-axis in the counter-clockwise direction with x-axis, a positive value of θ is to be taken counter-clockwise and negative in the clockwise direction to locate u-axis relative to x-axis.
- Find I_u from Eq. 5.21 or 5.21a. If I_x and I_y are equal Eq. 5.21 will give the accurate result. Find I_v using Eqs. 5.22, 5.22a or 5.23.
- Find M and its components M_u and M_v.
- In the four quadrants made by u and v-axes, mark the one in which both components give tensile stresses. This can be judged by considering the load component relative to u and v axes as well as whether the beam is simply supported or is a cantilever. u and v both are taken positive in that quadrant. Similarly, mark the quadrant in which both components give compressive stresses and thus both u and v are negative. In the rest of the two quadrants, mark the sign of u and v according to that on one side of u-axis v is to be positive and on the other negative. Similarly, on one side of v-axis u is to be positive and on the other negative.
- Use the relation $\sigma = \dfrac{M_u}{I_u} \cdot v + \dfrac{M_v}{I_v} \cdot u$ to find the stress at any point in any quadrant by inserting various values.
- Inclination of the neutral axis can be found from the relation, $\tan^{-1}\alpha = v/u$. It gives inclination with the u-axis. A negative value of the angle would mean the axes passes through a quadrant in which either u or v is negative. On one side of the neutral axis, the stresses are to be tensile and on the other compressive.

Example 5.39 An angle beam of 300 mm × 300 mm size is loaded as shown in Fig. 5.58. Determine the direction of the neutral axis and the bending stresses at A, B and C. The line of action of the load passes through the shear centre of the cross-section.

Solution

Given An angle beam as shown in Fig. 5.58.

To find

— Position of neutral axis

— Bending stresses at A, B and C

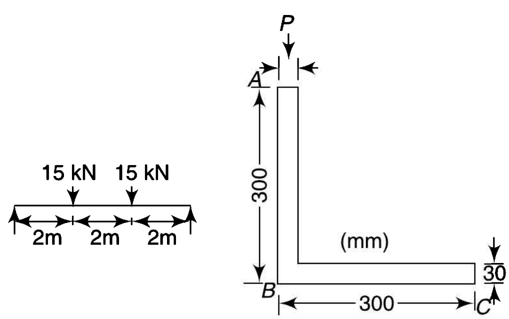


Fig. 5.58

As line of action of load passes through the shear centre, no twisting is there.

Moment of inertia about x- and y-axis

$$y = \frac{300 \times 30 \times 150 + 270 \times 30 \times 15}{300 \times 30 + 270 \times 30} = 86.05 \text{ mm}$$

As it is symmetric angle section, $x = 86.05$ mm (Fig. 5.59)

$$I_x = \frac{300 \times 30^3}{12} + 300 \times 30 \times 71.05^2 + \frac{30 \times 270^3}{12} + 30 \times 270 \times 78.95^2$$
$$= 145.804 \times 10^6 \text{ mm}^4$$

Due to symmetry, $I_y = 145.804 \times 10^6$ mm^4

$$I_{xy} = A \cdot h \cdot k = 300 \times 30(-71.05)(63.95) + 30 \times 270(-71.05)(78.95)$$
$$= -86.329 \times 10^6 \text{ mm}^4$$

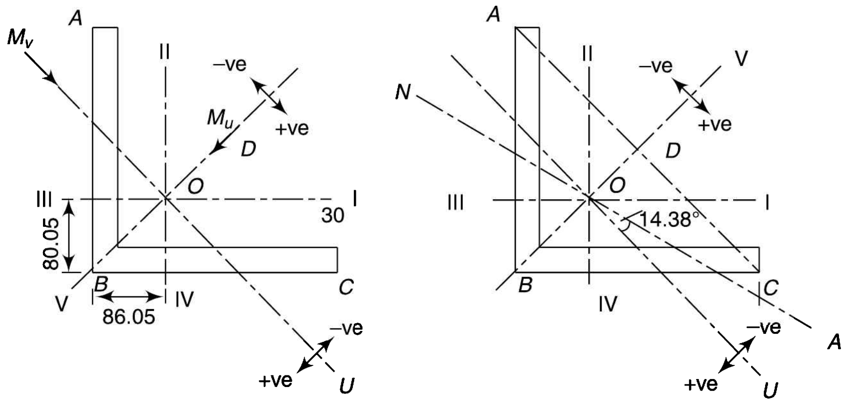


Fig. 5.59

Principal axes

Inclination of principal u-axis,

$$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} = \frac{2 \times (-86.329) \times 10^6}{0} = -\infty \quad 2\theta = -90° \quad \text{or} \quad \theta = -45°$$

u- and v-axis alongwith four quadrants with respect to these axes are shown in Fig. 5.59.

Moment of inertia about principal axes

I_u can be found from Eq. 5.21,

$$I_u = \cos^2(-45°) \times 145.804 \times 10^6 + \sin^2(-45°) \times 145.804 \times 10^6 - \sin 90° \times (-86.329 \times 10^6)$$

$$= 145.804 \times 10^6 - 86.329 \times 10^6$$

$$= 59.475 \times 10^6 \text{ mm}^4$$

$$I_v = I_x + I_y - I_u = 145.804 \times 10^6 + 145.804 \times 10^6 - 59.475 \times 10^6$$

Bending moments

Maximum bending moment = 15 × 2 = 30 kN·m

M_u = 30 cos 45° = 21.21 kN·m (to induce tensile stress in 3rd and 4th quadrants)

$M_v = M_u$ = 21.21 kN·m (to induce tensile stress in 1st and 4th quadrants)

Thus u and v both are positive in the 4th quadrant.

Bending stresses

For B, $v = OB$ = 2 × 86.05 cos 45° = 121.69 mm; u= 0

$$\sigma_b = \frac{M_u}{I_u} \cdot v + \frac{M_v}{I_v} \cdot u = \frac{21.21 \times 10^6}{59.475 \times 10^6} \times 121.69 + 0 = 43.397 \text{ MPa (tensile)}$$

For A, $v = AD = BD$ = 212.69 mm

$$u = OD = BD - OB = 300 \cos 45° - 121.69 = 212.13 - 121.69 = 90.44 \text{ mm}$$

$$\sigma_a = -\frac{21.21 \times 10^6}{59.475 \times 10^6} \times 90.44 - \frac{21.21 \times 10^6}{232.133 \times 10^6} \times 212.13$$

$$= -0.3566 \times 90.44 - 0.0914 \times 212.13 = -32.253 - 19.382$$

$$= -51.635 \text{ MPa (compressive)}$$

For C, $u = OD = 90.44$ mm; $v = CD = AD = 212.69$ mm; v negative, u positive.

$$\sigma_c = -32.253 + 19.382 = -12.871 \text{ MPa (compressive)}$$

Position of neutral axis

Inclination of the neutral axis, $\tan^{-1}\alpha = \dfrac{v}{u} = -\dfrac{0.0914}{0.3566} = -0.2563$

or $\alpha = -14.38°$

Neutral axis is shown in the Fig. 5.59b. As angle α is negative, the neutral axis has to pass through 1st and 3rd quadrants.

Example 5.40 A 60 mm × 40 mm × 6 mm angle is used as a cantilever with the 40 mm leg horizontal and on the top. The length of the cantilever is 600 mm. Determine the position of the neutral axis and the maximum stress developed if a load of 1 kN is applied at the free end. The line of action of the load passes through the shear centre of the cross-section.

Solution

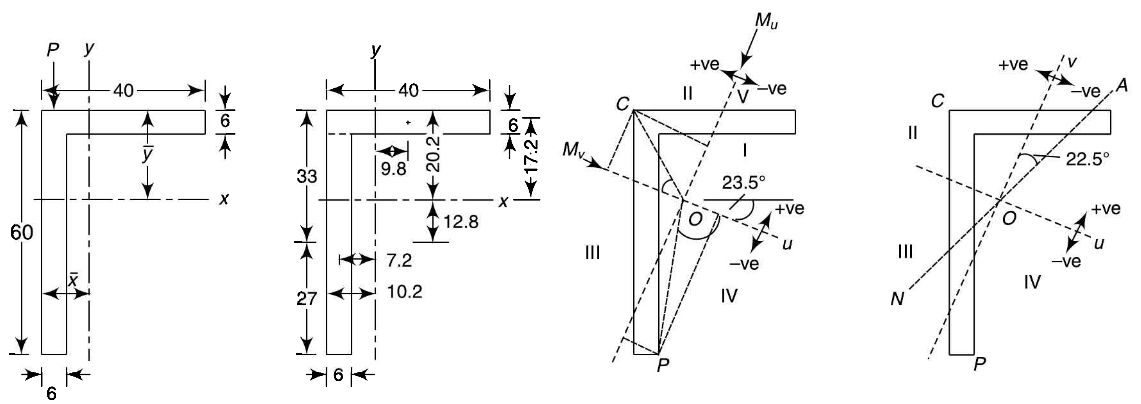


Fig. 5.60

Given An angle beam as shown in Fig. 5.60.

$L = 600$ mm $\quad\quad W = 1$ kN

To find

— Position of neutral axis
— Maximum stress

Figure 5.60 shows the section and the load.

To locate the centroid, take moments about the left and upper edge,

$$\bar{x} = \frac{40 \times 6 \times 20 + 54 \times 6 \times 3}{40 \times 6 + 54 \times 6} = 10.2 \text{ mm}$$

$$\bar{y} = \frac{40 \times 6 \times 3 + 54 \times 6 \times (27 + 6)}{40 \times 6 + 54\ X\ 6} = 20.2 \text{ mm}$$

Moment of inertia about x-and y-axis

$$I_x = \frac{6 \times 54^3}{12} + 6 \times 54 \times 12.8^2 + \frac{40 \times 6^3}{12} + 40 \times 6 \times 17.2^2 = 203\,537 \text{ mm}^4$$

$$I_y = \frac{54 \times 6^3}{12} + 54 \times 6 \times 7.2^2 + \frac{6 \times 40^3}{12} + 6 \times 40 \times 9.8^2 = 72\,818 \text{ mm}^4$$

$$I_{xy} = 54 \times 6 \times (-7.2)(-12.8) + 40 \times 6 \times 9.8 \times 17.2 = 70\,314 \text{ mm}^4 \qquad \text{...(Eq. 5.24)}$$

Position of principal axes

$$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} = \frac{2 \times 70\,314}{72\,818 - 203\,537} = -1.0758;\ 2\theta = -47° \quad \text{or} \quad \theta = -23.5°$$

Take u-axis at 23.5° in the clockwise direction with the x-axis. Four quadrants with respect to u and v axes are also shown in Fig. 5.55.

Moment of inertia about principal axes

$$I_u = \frac{1}{2}[(I_x + I_y) + \sec 2\theta(I_x - I_y)]$$

$$= \frac{1}{2}[(203\,537 + 72\,818) + \sec(2 \times 23.5°)(203\,537 - 72\,818)]$$

$$= \frac{1}{2}(27\,6355 + 191\,671) = 234\,013 \text{ mm}^4$$

$$I_v = \frac{1}{2}(27\,6355 - 191\,671) = 42\,342 \text{ mm}^4$$

Bending moments

Maximum bending moment = $1000 \times 600 = 600 \times 10^3$ N·m

Resolving about uu and vv,

$$M_u = 600 \times 10^3 \cos 23.5° = 550.2 \times 10^3 \text{ N·m}$$

(being cantilever, it induces tensile stresses in 1st and 2nd quadrants)

$$M_v = 600 \times 10^3 \cos 23.5° = 239.2 \times 10^3 \text{ N·m}$$

(being cantilever, it induces tensile stresses in 2nd and 3rd quadrants)

Thus u and v both are positive in the 2nd quadrant.

Position of neutral axis

$$\sigma = \frac{M_u}{I_u} \cdot v + \frac{M_v}{I_v} \cdot u = \frac{550.2 \times 10^3}{234\,013} \cdot v + \frac{239.2 \times 10^3}{42\,342} \cdot u = 2.351\, v + 5.649\, u$$

The equation for the neutral axis is $\sigma = 0$

or $\quad 2.351\, v + 5.649\, u = 0 \quad$ or $\quad \tan \alpha = \frac{u}{v} = -0.416 \quad$ or $\quad \alpha = -22.5°$

Neutral axis is shown in the Fig. 5.55. As angle α is negative, the neutral axis has to pass through 1st and 3rd quadrants.

Maximum tensile stress

Maximum tensile stress will be at C the coordinates of which can be found as below:

$$OC = \sqrt{10.2^2 + 20.2^2} = 22.63 \text{ mm}$$

$$\angle COD = \sin^{-1}\frac{CD}{OD} = \sin^{-1}\frac{20.2}{22.63} = \sin^{-1} 0.893 = 63.2^o$$

$$\angle COV = 63.2° - 23.5 = 39.7°$$

Thus $v = 22.63 \times \sin 39.7° = 14.46$ mm and $u = 22.63 \times \cos 39.7° = 17.41$ mm
[*Alternatively*, Refer Fig. 5.61,

$$v = CK = EO = DG - GH = 20.2 \cos 23.5° - 10.2 \sin 23.5° = 14.46 \text{ mm}$$

$$u = CE = CD + DE = CD + HO = 20.2 \sin 23.5° + 10.2 \cos 23.5° = 17.41 \text{ mm}]$$

$$\sigma = 2.351\, v + 5.649\, u = 2.351 \times 14.46 + 5.649 \times 17.41 = 132.3 \text{ MPa}$$

Maximum compressive stress

Maximum compressive stress will be at P the coordinates of which are,

$$OC = \sqrt{4.2^2 + 39.8^2} = 40.021 \text{ mm}$$

$$\angle POD = \sin^{-1}\frac{PE}{OP} = \sin^{-1}\frac{39.8}{40.021} = \sin^{-1} 0.9945 = 84°$$

$$\angle POV' = 180° - 84° - 23.5° = 72.5°$$

Thus $v = 40.021 \times \sin 72.5° = 38.17$ mm and $u = 40.021 \times \cos 72.5° = 12.02$ mm
[*Alternatively*, Refer Fig. 5.61,

$$v = PS = PT + TS = PT + QR = 4.2 \sin 23.5° + (60 - 20.2) \cos 23.5° = 38.17 \text{ mm}$$

$$u = PM = SO = RO - RS = 39.8 \sin 23.5° - 4.2 \cos 23.5° = 12.02 \text{ mm}]$$

$$\sigma = 2.351 \times 38.17 + 5.649 \times 12.02 = 147.64 \text{ MPa}$$

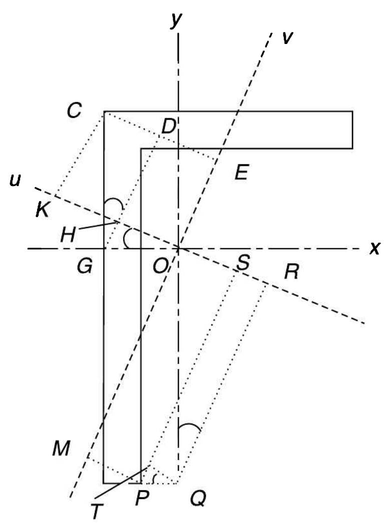


Fig. 5.61

Example 5.41 An angle beam of 140 mm × 100 mm × 10 mm is subjected to a point load of 7 kN as shown in Fig. 5.62. Determine the values of the maximum tensile and compressive bending stresses developed if the line of action of the load passes through the shear centre of the cross-section.

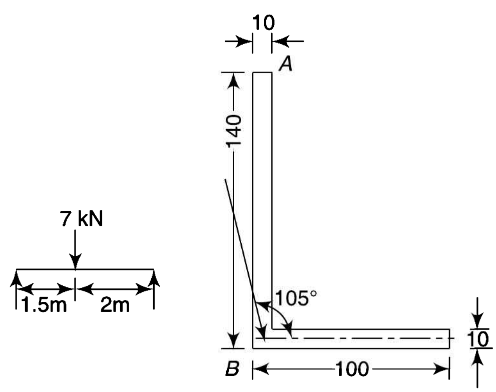
Solution

Given An angle beam as shown in Fig. 5.62.
To find Maximum tensile and compressive stresses

To locate the centroid, take moments about the left and lower edges,

$$\bar{x} = \frac{90 \times 10 \times 55 + 140 \times 10\ X\ 5}{90 \times 10 + 140 \times 10} = 24.56 \text{ mm}$$

$$\bar{y} = \frac{90 \times 10 \times 5 + 140 \times 10 \times 70}{90 \times 10 + 140 \times 10} = 44.56 \text{ mm}$$



(a) (b)

Fig. 5.62

Centroidal axes x and y are shown in Fig. 5.63a.

Moment of inertia about x- and y-axis

$$I_x = \frac{90 \times 10^3}{12} + 90 \times 10 \times 39.56^2 + \frac{10 \times 140^3}{12} + 10 \times 140 \times (70 - 44.56)^2 = 4.609 \times 10^6 \text{ mm}^4$$

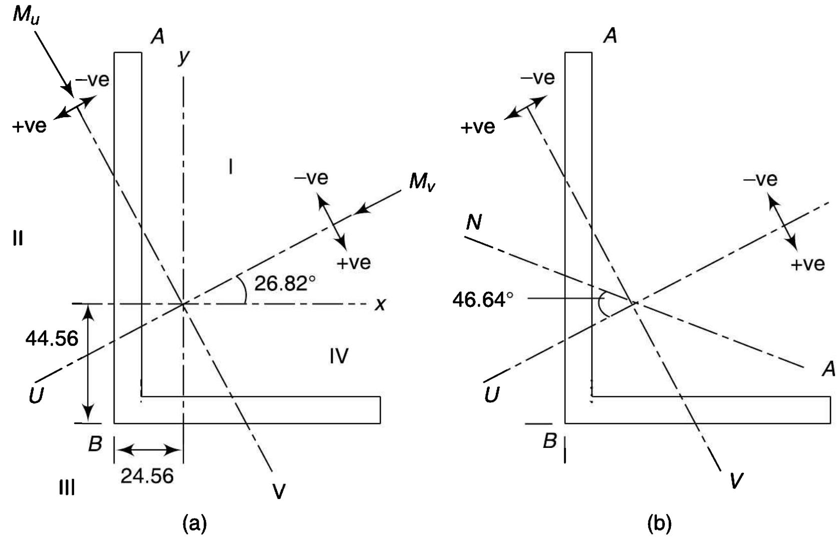


Fig. 5.63

$$I_y = \frac{10 \times 90^3}{12} + 10 \times 90 \times (55 - 24.56)^2 + \frac{140 \times 10^3}{12} + 140 \times 10 \times 19.56^2 = 1.989 \times 10^6 \text{ mm}^4$$

$$I_{xy} = 90 \times 10 \times (55 - 24.56)(-39.56) + 140 \times 10 \times (70 - 44.56)(-19.56) = -1.78 \times 10^6 \text{ mm}^4$$

Position of principal axes

$$\tan 2\theta = \frac{2I_{xy}}{I_y - I_x} = \frac{-2 \times 1.78 \times 10^6}{1.989 \times 10^6 - 4.609 \times 10^6} = 1.359 \quad \text{or} \quad 2\theta = 53.64° \quad \text{or} \quad \theta = 26.82°$$

Take u-axis at 26.82° in the counter-clockwise direction with the x-axis. Four quadrants with respect to u and v axes are also shown in Fig. 5.63a.

Moment of inertia about principal axes

$$I_u = [\cos^2(26.82°) \times 4.609 + \sin^2(26.82°) \times 1.989 - 2 \times (-1.78) \sin 26.82° \times \cos 26.82°]10^6$$

$$= 5.509 \times 10^6 \text{ mm}^4$$

$$I_v = I_x + I_y - I_u = 4.609 \times 10^6 + 1.989 \times 10^6 - 5.509 \times 10^6 = 1.089 \times 10^6 \text{ mm}^4$$

Bending moments

Maximum bending moment on the beam $= R_a \times 1.5 = \dfrac{7 \times 2}{3.5} \times 1.5 = 6$ kN·m

Angle between the load and u-axis = 105° – 26.56° = 78.18°

Resolving about uu and vv respectively,

$M_u = 6 \sin 78.18° = 5.873$ kN·m (to induce tensile stress in 3rd and 4th quadrants)

$M_v = 6 \cos 78.18° = 1.229$ kN·m (to induce tensile stress in 2nd and 3rd quadrants)

Thus u and v both are positive in the 3rd quadrant.

Position of neutral axis

$$\sigma = \frac{M_u}{I_u} \cdot v + \frac{M_v}{I_v} \cdot u = \frac{5.873 \times 10^6}{5.509 \times 10^6} \cdot v + \frac{1.229 \times 10^3}{1.089} \cdot u = 1.066\, v + 1.129\, u$$

The equation for the neutral axis is $\sigma = 0$

or $\quad 1.066\, v + 1.129\, u = 0 \quad$ or $\quad \tan^{-1} \alpha = \frac{v}{u} = -\frac{1.129}{1.066} = -1.059$ or $= -46.64°$

Neutral axis is shown in the Fig. 5.63. As angle α is negative, the neutral axis has to pass through 2nd and 4th quadrants.

Maximum tensile stress

Maximum tensile stress will be at B the coordinates of which can be found as below:

For B (Fig. 5.64),

$$OB = \sqrt{24.56^2 + 44.56^2} = 50.88 \text{ mm}$$

$$\angle BOD = \sin^{-1} \frac{BD}{OB} = \sin^{-1} \frac{44.56}{50.88} = \sin^{-1} 0.8758 = 61.14°$$

$$\angle UOB = 61.14° - 26.82° = 34.32°$$

Thus $v = 50.88 \sin 34.32° = 28.69$ mm and $u = 50.88 \cos 34.32° = 42.02$ mm

$$\sigma = 1.066 \times 28.69 + 1.129 \times 42.02 = 78.02 \text{ MPa}$$

Maximum compressive stress

Maximum compressive stress will be at A the coordinates of which are,

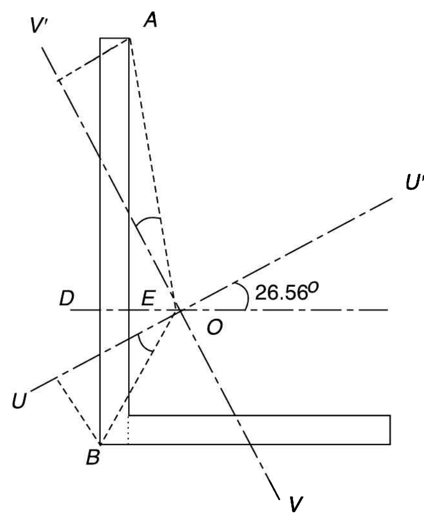


Fig. 5.64

$$OA = \sqrt{14.56^2 + (140 - 44.56)^2} = 96.544 \text{ mm}$$

$$\angle AOE = \cos^{-1} \frac{OE}{OA} = \cos^{-1} \frac{14.56}{96.544} = \cos^{-1} 0.1508 = 81.33°$$

$$\angle AOV' = 81.33° - (90° - 26.82°) = 18.15°$$

Thus $v = 96.544 \times \cos 18.15° = 91.74$ mm and $u = 96.544 \times \sin 18.15° = 30.07$ mm

$$\sigma = 1.066 \times 91.74 + 1.129 \times 30.07 = 131.74 \text{ MPa}$$

5.8 ELLIPSE OF INERTIA OR MOMENTAL ELLIPSE

This is a graphical method which can be used to find moments of inertia about two mutually perpendicular axes through the centroid when moments of inertia about the principal axes are known. The method is as follows:

From Eqs. 5.19, 5.21 and 5.22,

$$I_{uv} = \left(\frac{1}{2} \sin 2\theta\right)(I_x - I_y) + \cos 2\theta \cdot I_{xy}$$

$$I_u = \cos^2 \theta \cdot I_x + \sin^2 \theta \cdot I_y - \sin 2\theta \cdot I_{xy}$$

$$I_v = \cos^2 \theta \cdot I_y + \sin^2 \theta \cdot I_x$$

The above equations relate the moments of inertia I_u and I_v about any two mutually perpendicular axes u and v if the moment of inertia I_x and I_y about any other two mutually perpendicular axes x and y are known.

Assume the axes x and y to be principal axes. Then the product of inertia I_{xy} is zero, and thus the above equations are changed to

$$I_{uv} = \left(\frac{1}{2}\sin 2\theta\right)(I_x - I_y)$$

$$I_u = \cos^2\theta \cdot I_x + \sin^2\theta \cdot I_y$$

$$I_v = \cos^2\theta \cdot I_y + \sin^2\theta \cdot I_x$$

The first two equations are similar to equations for shear stress and direct stress at an angle when a material is subjected to two perpendicular stresses (refer Section 2.8).

Draw two circles with O as centre and radii equal to I_x and I_y taken to a suitable scale (Fig. 5.65) and complete the diagram.

Now,

$$\begin{aligned} OP &= OD + DP \\ &= OD + DG\cos\theta \\ &= I_y + (DE\cos\theta)\cos\theta \\ &= I_y + (I_x - I_y)\cos^2\theta \\ &= I_x\cos^2\theta + I_y(1-\cos^2\theta) \\ &= I_x\cos^2\theta + I_y\sin^2\theta \\ &= I_u \qquad \text{(Refer Eq.2.5)} \end{aligned}$$

and

$$\begin{aligned} PG &= DG\sin\theta \\ &= (DE\cos\theta)\sin\theta \\ &= (I_x - I_y)\cos\theta\sin\theta \\ &= \frac{1}{2}(\sigma_x - \sigma_y)\sin 2\theta \\ &= I_{uv} \end{aligned}$$

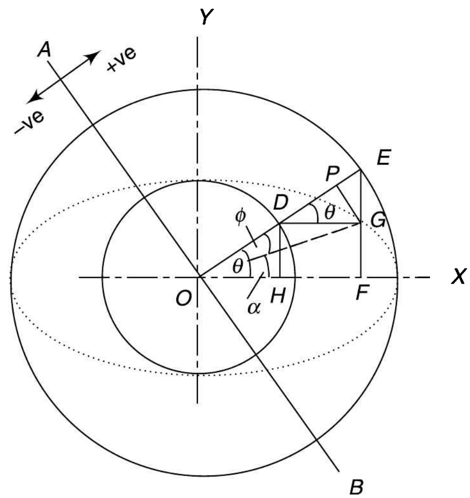


Fig. 5.65

The diagram is known as *ellipse of inertia*.

I_v can be found from the relation, $I_v = I_x + I_y - I_u$

Example 5.42 || For a standard I section, $I_x = 55.63 \times 10^6$ mm^4 and $I_y = 11.36 \times 10^6$ mm^4. Determine the moments of inertia about two perpendicular axes one of which is inclined at 30° to the x-axis axis. Also, find the product of inertia.

Solution

Moments of inertia can be found analytically or by drawing ellipse of inertia (Fig. 5.66).

Given An I-section

$$I_x = 55.63 \times 10^6\,\text{mm}^4 \qquad I_y = 11.36 \times 10^6\,\text{mm}^4$$

To find

— Moment of inertia about two perpendicular axes, one at 30° to the x-axis

— Product of inertia

Analytical solution

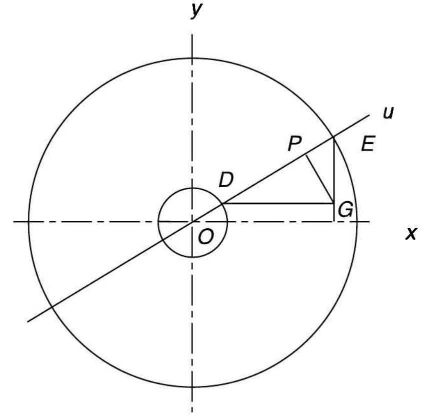


Fig. 5.66

$$I_u = I_x \cos^2\theta + I_y \sin^2\theta$$
$$= 55.63 \times 10^6 \cos^2 30^\circ + 11.36 \times 10^6 \sin^2 30^\circ$$
$$= 44.56 \times 10^6 \text{ mm}^4$$
$$I_{uv} = \left(\frac{1}{2}\sin 2\theta\right)(I_x - I_y)$$
$$= \left(\frac{1}{2}\sin 60^\circ\right)(55.63 - 11.36) \times 10^6$$
$$= 19.17 \times 10^6 \text{ mm}^4$$
$$I_v = I_x + I_y - I_u = (55.63 + 11.36 - 44.56)10^6$$
$$= 22.43 \times 10^6 \text{ mm}^4$$

From ellipse of inertia

$$I_u = OP = 44.4 \text{ mm}^4$$
$$I_v = PG = 19.3 \text{ mm}^4$$
$$I_v = I_x + I_y - I_u = (55.63 + 11.36 - 44.4)10^6 = 22.6 \times 10^6 \text{ mm}^4$$

5.9 COMBINED DIRECT AND BENDING STRESS

Let a column be acted upon by a thrust F, the line of action of which cuts the cross-section at a point on the x-axis at a distance h from the centroid O (Fig. 5.67a). If two equal and opposite forces are introduced along the axis of the column, then the system becomes equivalent to a load at O that produces a uniform direct stress alongwith a bending moment $F.h$ about y-axis, producing a varying bending stress (Fig. 5.67b).

Then the combined compressive stress at any point at a distance x from YY,

$$\sigma = \frac{F}{A} \pm \frac{Fh}{I_y} \cdot x \quad \text{where } A \text{ is the cross-sectional area}$$

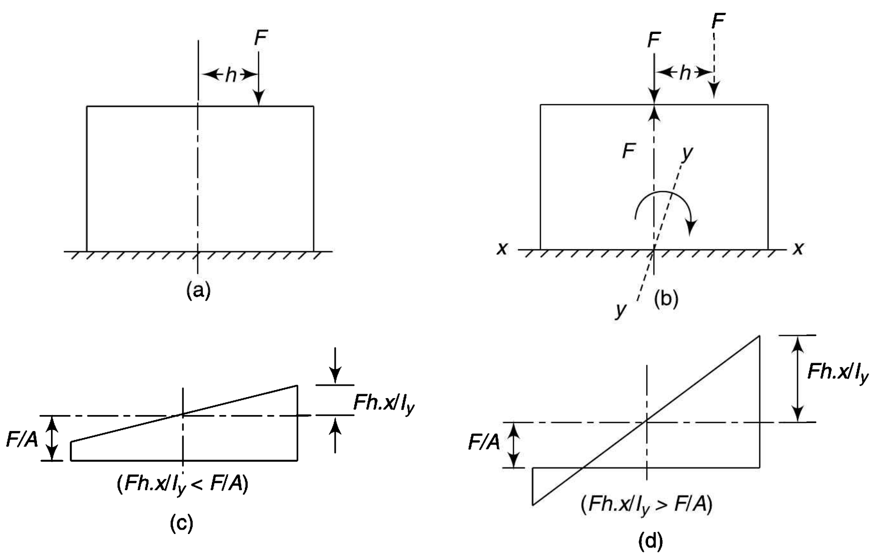


Fig. 5.67

If x is taken positive on the same side of y-axis as the load, then the bending stress will be of the same type as the direct stress and sign taken as positive (compressive). Thus maximum compressive stress occurs at the right-hand edge of the section. On the left side, x is negative and thus the sign taken is negative. Thus, at the left-hand edge of the section, the type of stress will depend upon whether the latter term on the right-hand side of the above equation is more or less as compared to the first term. If it is less, the type remains the same (Fig. 5.67c) as for direct stress otherwise it is of the opposite type (Fig. 5.67d).

In case a load is eccentric to both the axes (Fig. 5.68), then it will be equivalent to a central load as well as with a bending moment about the two axes, e.g., if the line of action of a load F is at distances h and k from the principal axes OY and OX respectively, the load is equivalent to a central load F alongwith a bending moment Fh about Oy and Fk about OX. Thus

$$\sigma = \frac{F}{A} + \frac{F \cdot h}{I_y} \cdot x + \frac{F \cdot k}{I_x} \cdot y \tag{5.25}$$

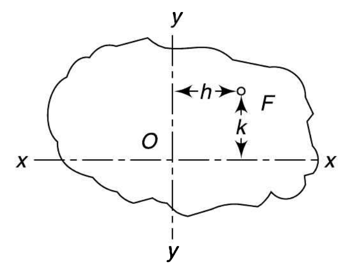


Fig. 5.68

x and y are taken positive when on the same side of their respective axes Oy and Ox as the load. It also implies that the maximum stress will occur at a point in the same quadrant in which the load is and the minimum in the opposite quadrant.

In masonry columns, it is desirable that no tensile stresses are set up. It can be shown that this can be ensured if the line of action of the load lies within a central area of the section. This central area or region is known as the *kern*, *kernel* or the *core* of the section. The extent of the *kern* depends upon the profile of the section.

Middle Third Rule for Rectangular Sections

In case of rectangular sections of width b and depth d, we have (Fig. 5.69),

$$\sigma = \frac{F}{A} + \frac{F \cdot h}{I_y} \cdot x + \frac{F \cdot k}{I_x} \cdot y$$

$$= \frac{F}{bd} + \frac{F \cdot h}{b^3 d/12} \cdot x + \frac{F \cdot k}{bd^3/12} \cdot y$$

$$= \frac{F}{bd} + \frac{12F \cdot h}{b^3 d} \cdot x + \frac{12F \cdot k}{bd^3} \cdot y$$

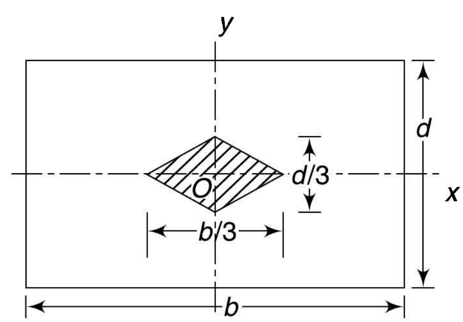


Fig. 5.69

Tensile stresses may be developed if x and y are negative, i.e.,

$$\sigma = \frac{F}{bd} - \frac{12F \cdot h}{b^3 d} \cdot x - \frac{12F \cdot k}{bd^3} \cdot y$$

In the limiting case, $x = b/2$ and $y = d/2$; and $\sigma = \frac{F}{bd} - \frac{12F \cdot h}{b^3 d} \cdot \frac{b}{2} - \frac{12F \cdot k}{bd^3} \cdot \frac{d}{2} = 0$

or

$$dh + bk = \frac{bd}{6} \tag{5.26}$$

The equation gives the limiting values of h and k. In each quadrant, the load should lie within the line produced by this equation.

When $k = 0$, $h = b/6$ and when $h = 0$, $k = d/6$. which shows that when the load lies on either axis, it should lie within the middle third to avoid the tensile stress in the cross-section. For intermediate positions, the load must lie within the diamond area.

Middle Quarter Rule for Circular Sections

For circular sections of diameter d and eccentricity e of the load (Fig. 5.70), the minimum stress is

$$\sigma = \frac{F}{\frac{\pi}{4}d^2} - \frac{F.e}{\frac{\pi}{64}d^4}\cdot\frac{d}{2} \text{ for axis along any diameter,}$$

In the limiting case, $$\sigma = \frac{F}{\frac{\pi}{4}d^2} - \frac{F.e}{\frac{\pi}{64}d^4}\cdot\frac{d}{2} = 0$$

or $$e = \frac{d}{8} \qquad (5.27)$$

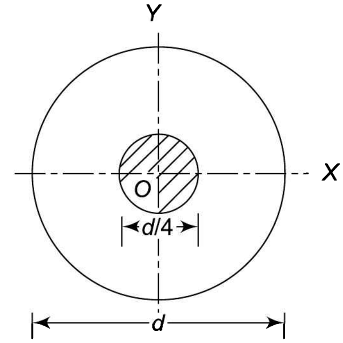


Fig.5.70

The equation gives the limiting value of eccentricity and for all possible positions of the load, it is a circle of diameter $d/4$ with centre O and the load must lie within this circle.

Example 5.43 A short column of rectangular section 160 mm × 120 mm carries a load of 200 kN. The load point is at a point 40 mm from the longer side and 70 mm from the shorter side. Determine the maximum tensile and compressive stresses in the section

Solution

Given A short column of rectangular section carrying a load as shown in Fig. 5.71
To find Maximum tensile and compressive stresses

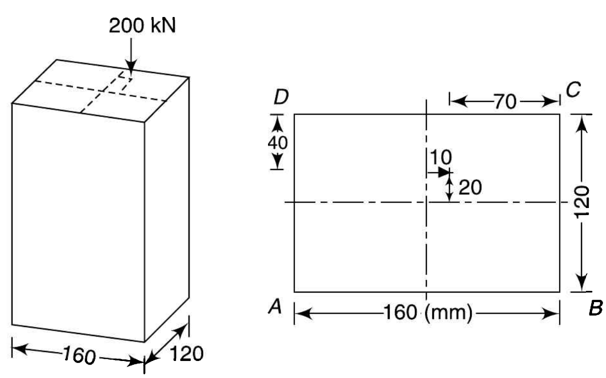


Fig. 5.71

$$I_x = \frac{160 \times 120^3}{12} = 23.04 \times 10^6 \text{ mm}^4;\ I_y = \frac{120 \times 160^3}{12} = 40.96 \times 10^6 \text{ mm}^4$$

Direct stress

$$\sigma_c = \frac{200 \times 10^3}{160 \times 120} = 10.417 \text{ MPa} \ldots \text{(compressive)}$$

Bending stress

Bending stress at A or C,

$$\sigma_b = \frac{200 \times 10^3 \times 10}{40.96 \times 10^6} \times 80 + \frac{200 \times 10^3 \times 20}{23.04 \times 10^6} \times 60$$

$$= 3.906 + 10.417 = 14.323 \text{ MPa}$$

It is compressive at C and tensile at A.

Maximum stresses

Maximum compressive stress is at C =10.417 + 14.323 = 24.74 MPa

Maximum tensile stress is at A = –10.417 (compressive) + 14.323 (tensile) = 3.906 MPa

Example 5.44 A 5 kN vertical load is applied on a wooden column of rectangular cross-section as shown in Fig. 5.72. Determine the stresses at the points *A, B, C* and *D*. Also locate the neutral axis of section.

Solution

Given A short column of rectangular section carrying a load as shown in Fig. 5.72

To find

— stresses at points A, B, C and D

— Location of neutral axis

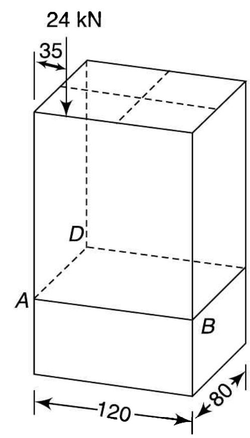


Fig. 5.72

$$I_x = \frac{120 \times 80^3}{12} = 5.12 \times 10^6 \text{ mm}^4; \; I_y = \frac{80 \times 120^3}{12} = 11.52 \times 10^6 \text{ mm}^4$$

Refer Fig. 5.73,

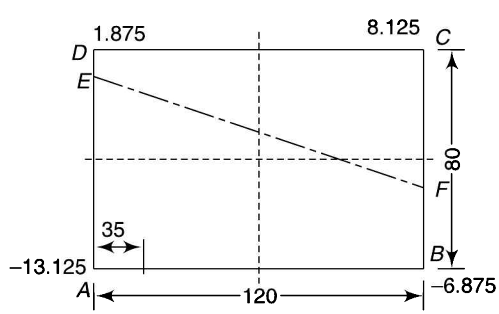


Fig. 5.73

Direct stress

$$\sigma_c = \frac{24 \times 10^3}{120 \times 80} = 2.5 \text{ MPa} \ldots \text{(compressive)}$$

Bending stresses

$$\text{Bending stress at } A = -\frac{24\times10^3\times25}{11.52\times10^6}\times60-\frac{24\times10^3\times40}{5.12\times10^6}\times40$$

$$= -3.125 - 7.5 = -10.625 \text{ MPa (compressive)}$$

Bending stress at B = +3.125 – 7.5 = –4.375 MPa (compressive)

Bending stress at C = 3.125 + 7.5 = 10.625 MPa (tensile)

Bending stress at D = -3.125 + 7.5 = 4.375 MPa (tensile)

Resultant stresses

At A = –2.5 – 10.625 = –13.125 (compressive)

At B = –2.5 – 4.375 = –6.875 (compressive)

At C = –2.5 + 10.625 = 8.125 (tensile)

At D = –2.5 + 4.375 = 1.875 (tensile)

Neutral axis

It can be observed that the stress will be zero at some points between AD and BC. Let E and F be the points of zero stress between AD and BC respectively and as the stress distribution is linear,

$$AE = \frac{13.125}{13.125+1.875}\times 80 = 70 \text{ mm}$$

$$BF = \frac{6.875}{6.875+8.125}\times 80 = 36.7 \text{ mm}$$

The neutral axis has been shown in Fig. 5.73.

Example 5.45 A short column of external and internal diameters as D and d respectively carries an eccentric load W. Determine the maximum eccentricity which the load can have without producing tension on the cross-section of the column.

Solution Refer Fig. 5.74,

Solution

Given A short hollow column of circular section carrying a load as shown in Fig. 5.74

To find Maximum eccentricity for no tensile stress

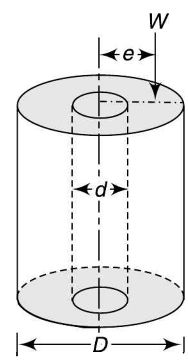


Fig. 5.74

$$I = \frac{\pi(D^4 - d^4)}{64}$$

Direct stress

$$\sigma = \frac{W}{\frac{\pi}{4}(D^2-d^2)} = \frac{4W}{\pi(D^2-d^2)}$$

Bending stress

Let the maximum permissible eccentricity be e,

$$\text{Bending stress} = \sigma_b = \frac{W\cdot e}{\frac{\pi}{64}(D^4-d^4)}\cdot\frac{D}{2} = \frac{32W\cdot e\cdot D}{\pi(D^4-d^4)}$$

Determination of maximum eccentricity

To avoid tension in the cross-section,

$$\frac{4W}{\pi(D^2-d^2)} = \frac{32W\cdot e\cdot D}{\pi(D^4-d^4)} \quad\text{or}\quad 1 = \frac{8\cdot e\cdot D}{D^2+d^2} \quad\text{or}\quad e = \frac{D^2+d^2}{8D}$$

Example 5.46 A 2.5-m long and 30-mm thick tie bar is of tapered section. The longitudinal section tapers from a depth of 60 mm to 180 mm at the ends as shown in Fig. 5.75. A load of 54 kN acts through the centroid of the smaller end parallel to the top edge. Determine the position and the magnitude of the maximum tensile stress.

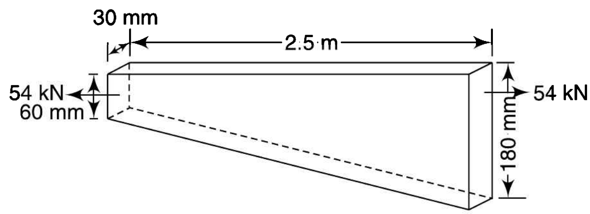


Fig. 5.75

Solution

Given A tapered tie bar carrying a tensile load as shown in Fig. 5.75

To find Position and magnitude of maximum tensile stress

Refer Fig. 5.76,

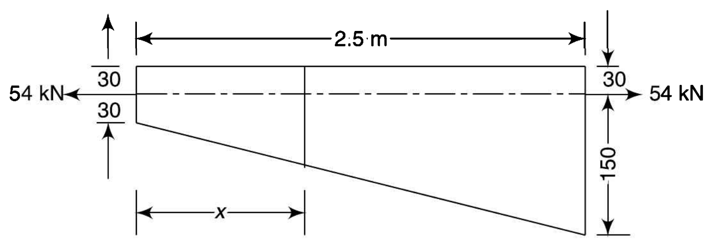


Fig. 5.76

Consider a section at distance x m from the smaller end.

Depth of the section $= 60 + (180 - 60) \cdot \dfrac{x}{2.5} = 60 + 48x = 12(5 + 4x)$ mm

Depth of centroid from top edge $= 6(5 + 4x) = (30 + 24x)$ mm

Eccentricity of the load $= 30 + 24x - 30 = 24\,x$

Area of the section $= 30 \times 12(5 + 4x) = 360\,(5 + 4x)$

Moment of inertia of section

$$I = \frac{30 \times \{12^3 (5 + 4x)^3\}}{12} = 4320(5 + 4x)$$

Tensile stress at the section

$$\text{Tensile stress, } \sigma = \frac{54\,000}{360(5 + 4x)} + \frac{54\,000 \times 24x}{4320(5 + 4x)^3} \times (30 + 24x)$$

$$= \frac{150}{(5 + 4x)} + \frac{1800x}{(5 + 4x)^2}$$

For maximum value of stress

For maximum value, $\dfrac{d\sigma}{dx} = 0$

or $$-\frac{150\times 4}{(5+4x)^2}+\frac{1800[(5+4x)^2-x\cdot 2(5+4x)\cdot 4}{(5+4x)^4}=0$$

Multiplying throughout by $(5 + 4x)^3$,

$$-600(5 + 4x) + 1800[(5 + 4x) - 8x] =0$$
$$-3000 - 2400x + 1800(5 - 4x) = 0$$
$$-3000 - 2400x + 9000 -- 7200x = 0$$
$$9600x = 6000$$
$$x = 0.625 \text{ m}$$
$$\sigma=\frac{150}{(5+4\times 0.625)}+\frac{1800x}{(5+4\times 0.625)^2}=40 \text{ MPa}$$

Example 5.47 A steel rod is placed inside a brass tube. Both are of the same length and have their axes parallel to each other but are 5 mm apart. The inside and outside diameters of the tube are 48 mm and 56 mm respectively whereas diameter of the rod is 28 mm. The ends of both are covered by rigid plates through which a compressive force of 50 kN is applied acting along the axis of the tube. Find the maximum and minimum longitudinal stresses in the tube and the rod. E_s = 210 MPa and E_b = 95 MPa.

Solution

Given A brass tube with an eccentric steel rod inside and a centric load as shown in Fig. 5.77.

E_s = 210 MPa $\qquad E_b$ = 95 MPa

To find Maximum and minimum longitudinal stresses

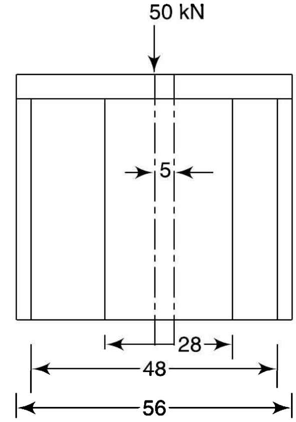


Fig. 5.77

Let F_s = direct load at the axis of the steel rod

F_b = direct load at the axis of the brass tube

M_s = bending moment on the steel rod

M_b = bending moment on the brass tube

Equilibrium equations

$$F_s + F_b = 50000 \qquad \text{(i)}$$

and $$M_s + M_b = F_s \times 5 \qquad \text{(ii)}$$

$$A_s=\frac{\pi}{4}\times 28^2=615.8 \text{ mm}^2 \text{ and } A_b=\frac{\pi}{4}\times(56^2-48^2)=652.5 \text{ mm}^2$$

$$I_s=\frac{\pi}{64}\times 28^4=30\,172 \text{ mm}^4 \text{ and } I_b=\frac{\pi}{64}\times(56^4-48^4)=222\,173 \text{ mm}^4$$

As the end plates are rigid, the tube and the rod can be assumed to bend together with same radius of curvature, thus

$$R=\frac{E_sI_s}{M_s}=\frac{E_bI_b}{M_b} \text{ or } \frac{210\,000\times 30\,172}{M_s}=\frac{95\,000\times 222\,173}{M_b} \text{ or } M_b=3.331M_s$$

From (ii), $M_s + 3.331M_s = F_s \times 5$ or $M_s = 1.154\,F_s$

Compatibility equation

Reduction in length of rod = Reduction in length of tube

Strain due to compressive stress in the rod + Strain due to bending moment in the rod

= Strain due to compressive stress in the tube

$$\frac{F_s}{E_s A_s} + \frac{M_s}{E_s I_s} y = \frac{F_b}{E_b A_b} \qquad (l \text{ is same})$$

or $$\frac{F_s}{210\,000 \times 615.8} + \frac{1.154 F_s}{210\,000 \times 30\,172} \times 5 = \frac{50\,000 - F_s}{95\,000 \times 653.5}$$

$$F_s + 0.1178 F_s = 104\,150 - 2.083 F_s$$

or $$F_s + 32\,539 F_s \text{ and } F_b = 17\,461 \text{ N}$$

or $$M_s = 1.154 \times 32\,359 \text{ N·mm} \quad \text{and} \quad M_s = 3.331 \times 37\,550 = 125\,079 \text{ N·mm}$$

Calculations for stresses

Maximum stress in the steel rod $= \dfrac{F_s}{A_s} + \dfrac{M_s}{I_s} y = \dfrac{32\,539}{615.8} + \dfrac{37\,550}{30\,172} \times \left(\dfrac{28}{2}\right)$

$= 52.84 + 17.42 = 70.26$ MPa

Minimum stress in the steel rod $= 52.84 - 17.42 = 35.42$ MPs

Maximum stress in the brass tube $= \dfrac{F_b}{A_b} + \dfrac{M_b}{I_b} y = \dfrac{17\,461}{653.8} + \dfrac{125\,079}{222\,173} \times \left(\dfrac{56}{2}\right)$

$= 26.71 + 15.76 = 42.47$ MPa

Minimum stress in the steel rod $= 26.71 - 15.76 = 10.95$ MPa

Example 5.48 A cast-iron column of external and internal diameters as 100 mm and 80 mm respectively carries a central load of 150 kN and an eccentric load W at 160 mm from the axis. Determine the maximum value of W if the permissible stresses are 140 MPa in compression and 35 MPa in tension for cast iron. Also, find the tensile stress value for this load.

Solution

Given A cast-iron hollow column with an eccentric load as shown in Fig. 5.78. Permissible stresses 140 MPa in compression and 35 MPa in tension

To find Maximum load W and tensile stress for this load

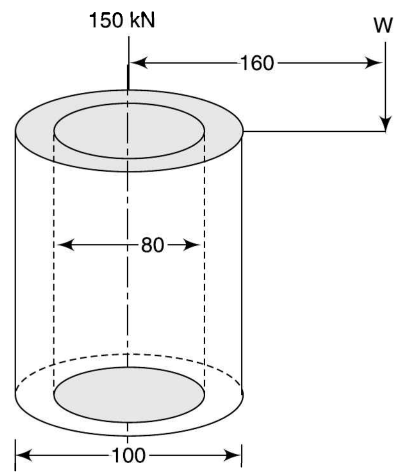


Fig. 5.78

All diameters being principal axes, let W lie as shown in Fig. 5.78.

$$A = \frac{\pi}{4}(100^2 - 80^2) = 2827 \text{ mm}^2$$

$$I = \frac{\pi}{64}(100^4 - 80^4) = 2.898 \times 10^6 \text{ mm}^4$$

Bending stress

Maximum bending stress $= \dfrac{160W}{2.898 \times 10^6} \times 50 = 0.002\,76\,W$

Maximum compressive stress

Permissible stress in compression, $140 = \dfrac{150\,000 + W}{2827} + 0.002\,76W$

$$395\,780 = 150\,000 + W + 7.8\,W$$

or $$8.8W = 245\,780 \quad \text{or} \quad W = 27\,930 \text{ N}$$

Maximum tensile stress

Permissible tensile stress, $35 = -\dfrac{150\,000 + 27\,930}{2827} + 0.002\,76W$

or $\quad 98\,945 = -150\,000 - W + 7.8\,W$

or $\quad 6.8W = 248\,945 \quad$ or $\quad W = 36\,610$ N

Thus maximum value of load = 27 930 N

Tensile stress for maximum load

$$\text{Tensile stress for this load} = -\frac{150\,000 + 27\,930}{2827} + 0.002\,76 \times 27\,930$$

$$= -62.94 + 77.09 = 14.15 \text{ MPa}$$

Example 5.49 A bar of *T*-section symmetrical about the vertical centre line has the flange 160-mm wide and 20-mm thick and the web 120-mm deep and 20-mm thick. The member is acted upon by a longitudinal pull *P* which acts on the section at a point on the vertical central line and is 50 mm from the bottom edge of the web. Determine the magnitude of the maximum pull which can be applied if the maximum allowable tensile stress on the section is 80 MPa. Also, find the minimum stress on the section when the pull *P* is transmitted.

Solution

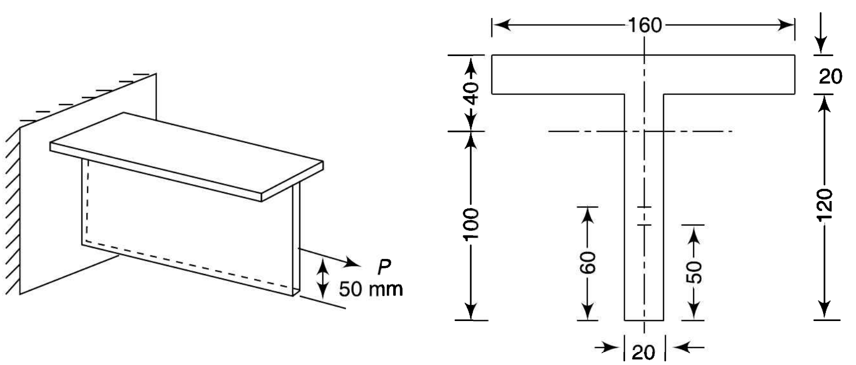


Fig. 5.79

Given A bar of *T*-section symmetrical about the vertical centre line with a pull as shown in Fig. 5.79. Maximum allowable tensile stress 80 MPa

To find

— Maximum pull

— Minimum stress on the section for this pull

$$A = 160 \times 20 + 20 \times 120 = 5600 \text{ mm}^2$$

$$\bar{y} = \frac{160 \times 20 \times 10 + 20 \times 120 \times 80}{160 \times 20 + 20 \times 120} = 40 \text{ mm}$$

Moment of inertia

$$I_x = \frac{160 \times 20^3}{12} + 160 \times 20 \times 30^2 + \frac{20 \times 120^3}{12} + 20 \times 120 \times (100 - 60)^2$$

$$= 9.707 \times 10^6 \text{ mm}^4$$

Bending moment

$e = 100 - 50 = 50$ mm

$\therefore \quad$ Bending moment, $M = 50P$

Maximum stress

$$\sigma_{max} = \frac{P}{A} + \frac{M}{I}y = \frac{P}{5600} + \frac{50P}{9.707 \times 10^6} \times (140 - 40) = 80 \text{ MPa}$$

$$P \times 0.000\,694 = 80 \quad \text{or} \quad P = 115\,330 \text{ N} \quad \text{or} \quad 115.33 \text{ kN}$$

Minimum stress

$$\text{Minimum stress on the section} = \frac{P}{A} - \frac{M}{I}y = \frac{115\,330}{5600} - \frac{50 \times 115\,330}{9.707 \times 10^6} \times 40 = -3.17\text{MPa}$$

i.e., 3.17 MPa compressive

Example 5.50 A rectangular footing 3 m × 2 m carries four vertical point loads of 200 kN, 150 kN, 300 kN and 130 kN as shown in Fig. 5.80. Neglecting the weight of the footing, determine the magnitudes of the stresses at four corners. Also find the location of an additional load of magnitude 300 kN which will make the stresses uniform at all corners.

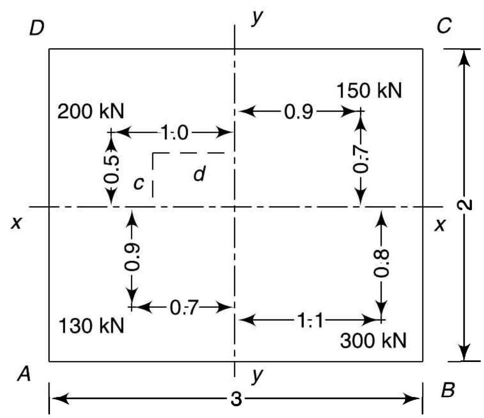


Fig. 5.80

Solution

Given A rectangular footing carrying four vertical point loads as shown in Fig. 5.80

To find

— Magnitude of the stresses at four corners

— Location of additional load of 300 kN to make the stresses uniform at all corners.

Section modulus, $I_x = \dfrac{3 \times 2^3}{12} = 2 \text{ m}^4$; $I_y = \dfrac{2 \times 3^3}{12} = 4.5 \text{ m}^4$

Direct stress

$$\text{Direct stress} = \frac{200 + 150 + 300 + 130}{3 \times 2} = 130 = 130 \text{ kN/m}^2$$

Bending stresses

Maximum bending moment due to loads eccentric about the x-axis,

$$= (130 \times 0.9 + 300 \times 0.8) - (150 \times 0.7 + 200 \times 0.5) = 152 \text{ kN·m}$$

Maximum bending stress, $\sigma_x = \dfrac{152}{2} \times 1 = 76 \text{ kN/m}^2$

This is compressive in the lower half.

Maximum bending moment due to loads eccentric about the y-axis

$$= (150 \times 0.9 + 300 \times 1.1) - (130 \times 0.7 + 200 \times 1) = 174 \text{ kN·m}$$

Maximum bending stress, $\sigma_x = \dfrac{174}{4.5} \times 1.5 = 58 \text{ kN/m}^2$

This is compressive in the right half.

Resultant stresses

At A = 130 + 76 – 58 = 148 kN/m^2 (compressive)

At B = 130 + 76 + 58 = 264 kN/m^2 (compressive)

At $C = 130 - 76 + 58 = 112$ kN/m^2 (compressive)

At $D = 130 - 76 - 58 = -4$ kN/m^2 (tensile)

Location of additional load

To have uniform stresses at the four corners, the bending moment about the two axes must be made zero. Let the additional load of 300 kN be placed at distances c and d from the two axes as shown in the figure. The quadrant chosen for the additional load is such that the bending moments due to the additional load about the two axes are of the opposite direction of the bending moments obtained above.

Thus about x-axis, $300 \times c = 152$ or $c = 0.507$ m

About y-axis, $300 \times d = 174$ or $d = 0.58$ m

5.10 MASONARY DAMS

A dam is a structure to store water and to provide head to a power house. Consider a dam as shown in Fig. 5.81. Let H and h be the height of the dam and of the water on the upstream side respectively. There are two main forces acting on the dam: force of static water pressure and the weight of the dam.

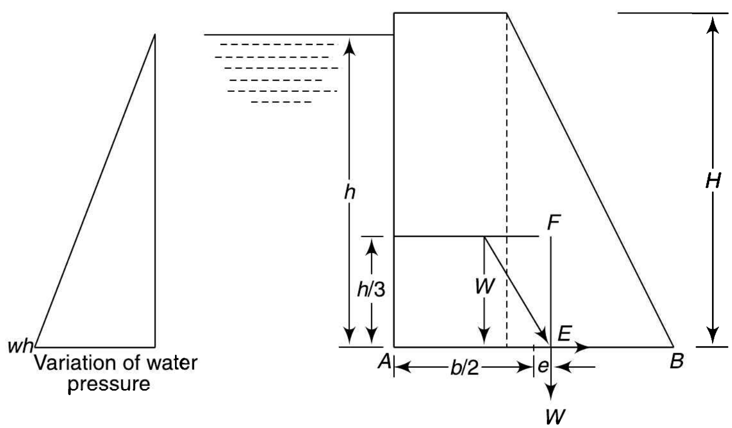


Fig. 5.81

Consider a unit length of the dam. Let W be the weight of dam acting downwards through centre of gravity of weight of the dam.

If w is the specific weight of water, then wh is the water pressure at the bottom of the dam and its variation is linear, zero at the surface of water and maximum (wh) at the bottom of dam.

Total water force acting on the dam,

$$F = \text{Area} \times \text{mean pressure} = h \cdot 1 \times \frac{wh}{2} = \frac{wh^2}{2} \tag{5.28}$$

As the water pressure varies linearly from top to bottom, the centre of force is at $H/3$ from the bottom of the dam.

If b is the width of the dam at the bottom,

Resultant thrust, $R = \sqrt{W^2 + F^2}$. Let it intersect the bottom of the dam at E.

- Horizontal component of R at E is equal to F. This is resisted by the frictional force at the base against sliding.

- Vertical component of R at E is equal to W and is balanced by the resultant of stress forces at the bottom.

The vertical component W at E induces

- normal compressive stress at the base due to load W and
- normal compressive and tensile stresses due to bending moment induced due to eccentricity of load.

$$\text{Eccentricity of loading, } e = AE - b/2$$

$$\text{The bending moment } = W \cdot e$$

Then the combined stress at any point at a distance x from midpoint of the base,

$$\sigma = \frac{W}{A} \pm \frac{W \cdot e}{I_y} \cdot x = \frac{W}{1.b} \pm \frac{W \cdot e}{1 \cdot b^3/12} \cdot x = \frac{W}{b} \pm \frac{12W \cdot e}{b^3} \cdot x \tag{5.29}$$

Tensile stresses may be developed if x is negative. At point A,

$$\sigma = \frac{W}{b} - \frac{12W \cdot e}{b^3} \cdot \frac{b}{2} = \frac{W}{b}\left(1 - \frac{6e}{b}\right) \tag{5.30}$$

This indicates that point E must lie between $b/6$ from the midpoint of the base, otherwise the term in the bracket will become negative and tensile stress will occur at the base.

Example 5.51 || A masonry dam is 10 m high. Its width at the top and bottom is 2 m and 6 m respectively. Its water face is vertical and retains water to a depth of 9 m. Determine the maximum and minimum stress values at the base. The weight of the masonry is 22 kN/m³ and specific weight of water 9.81 kN/m³.

Solution

Given A masonry dam as shown in Fig. 5.82

To find Maximum and minimum stresses at the base

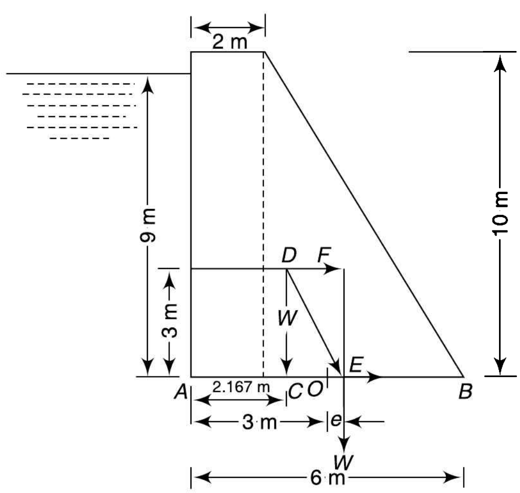


Fig. 5.82

Considering 1 m length of the dam,

Vertical force (weight) of Dam

$$W = \left(\frac{6+2}{2} \times 10 \times 1\right) \times 22 = 880 \text{ kN}$$

Let the centre of gravity of weight is at a distance $\bar{x}$ from vertical face at A.
Taking moments about A,

$$2\times 10\times\frac{2}{2}+\frac{(6-2)\times 10}{2}\left[2+\frac{(6-2)}{3}\right]=\left[2\times 10+\frac{(6-2)\cdot 10}{2}\right]\bar{x}$$

or $\quad 20 + 20 \times 3.33 = 40\,\bar{x} \quad$ or $\quad \bar{x} = 2.167$ m

Horizontal force of water

Force due to water pressure, $F=\dfrac{wh^2}{2}=\dfrac{9.81\times 9^2}{2}=397.3$ kN

It acts at a height of $h/3$ or 3 m from the base.

Eccentricity

Let the resultant of W and F strikes the base at E.

Now, $\quad \dfrac{CE}{CD}=\dfrac{F}{W} \quad$ or $\quad CE=\dfrac{397.3}{880}\times 3=1.354$ m

Eccentricity, $e = AC + CE - AO = 2.167 + 1.354 - 3 = 0.521$

Maximum and minimum stresses

Then maximum compressive stress at B,

$$\sigma=\frac{W}{b}\left(1+\frac{6e}{b}\right)=\frac{880}{6}\left(1+\frac{6\times 0.521}{6}\right)=223\text{ kN/m}^2$$

and minimum compressive stress at A,

$$\sigma=\frac{W}{b}\left(1-\frac{6e}{b}\right)=\frac{880}{6}\left(1-\frac{6\times 0.521}{6}\right)=70.3\text{ kN/m}^2$$

Example 5.52 A masonry dam trapezoidal in section is 12 m high. It is 2 m wide at the top and 8 m wide at the bottom. The face exposed to water has a slope of 1 horizontal to 12 vertical. The water level in the dam is up to the top of dam. The weight of the masonry is 25 kN/m^3 and specific weight of water is 9.81 kN/m^3. Determine the maximum and minimum stress values at the base. Also, check the stability of dam if the coefficient of friction of the dam and the soil is 0.6.

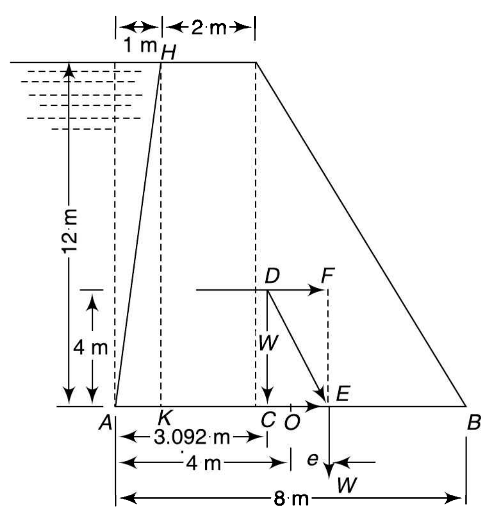


Fig. 5.83

Solution

Given A masonry dam as shown in Fig. 5.83

To find

— Maximum and minimum stresses at the base

— To check the stability

Considering 1 m length of the dam,

Total vertical weight

$$W=\left(\frac{8+2}{2}\times 12\times 1\right)\times 25=1500\text{ kN}$$

Weight of water acting as weight on the dam, i.e., weight of water contained in the area AHK,

$$F_w = \frac{12 \times 1}{2} \times 9.81 = 58.86 \text{ kN}$$

Total vertical weight = 1500 + 58.86 = 1558.86 kN

Taking moments of all the forces about A,

$$1558.86 \times \bar{x} = \left(58.86 \times \frac{1}{3}\right) + \left[\left(\frac{12 \times 1}{2} \times 25\right) \times \frac{2}{3}\right]$$

$$+ \left[(12 \times 2 \times 25) \times \left(1 + \frac{2}{2}\right)\right] + \left[\left(\frac{5 \times 12}{2} \times 25\right) \times \left(3 + \frac{5}{3}\right)\right]$$

$$= 19.6 + 100 + 1200 + 1633.3$$

$$= 4819.6$$

or $\qquad \bar{x} = 3.092$ m

Horizontal force

Horizontal water force, $F = \dfrac{wh^2}{2} = \dfrac{9.81 \times 12^2}{2} = 706.32$ kN

It acts at a height of $h/3$ or 4 m from the base.

Eccentricity

Now, $\dfrac{CE}{CD} = \dfrac{F}{W}$ or $CE = \dfrac{706.32}{1558.86} \times 4 = 1.812$ m

Eccentricity, $e = AC + CE - AO = 3.092 + 1.812 - 4 = 0.904$ m

Maximum and minimum stresses

Maximum compressive stress at B,

$$\sigma = \frac{W}{b}\left(1 + \frac{6e}{b}\right) = \frac{1558.86}{8}\left(1 + \frac{6 \times 0.904}{8}\right) = 337 \text{ kN/m}^2$$

Maximum compressive stress at A,

$$\sigma = \frac{W}{b}\left(1 - \frac{6e}{b}\right) = \frac{1558.86}{6}\left(1 - \frac{6 \times 0.904}{6}\right) = 62.7 \text{ kN/m}^2$$

Checking the stability

- As $e = 0.904$ m lies between $b/6$, i.e., between 8/6 or 1.333 m, no tension is present at the base as also observed above.
- Frictional force = $\mu W = 0.6 \times 1558.86 = 935.3$ kN is more than horizontal force 706.32 kN. Thus dam is safe against sliding.

$$\text{Factor of safety against sliding} = \frac{935.3}{706.3} = 1.32$$

- Clockwise moments about $B = 706.32 \times 4 = 2825.3$ kN·m
  Counter-clockwise moments about $B = 1558.86 \times (8 - 3.092) = 7650.9$ kN·m
  As the moments due to the weight of the dam are more than overturning moments, the dam is safe against overturning.

$$\text{Factor of safety} = \frac{7650.9}{2825.3} = 2.71$$

5.11 RETAINING WALLS

Materials having loose particles such as sand, clay, earth, gravel, ash and mud take the shape of a cone when placed freely on a surface. The angle of the cone varies from material to material. This means that if such a material is placed near a wall, the volume of material which is prevented from moving in the conical shape towards the wall will exert pressure on the wall. The intensity of pressure on the wall varies zero at the top of earth to a maximum at the bottom (Fig. 5.84).

For construction of roads in hilly areas, usually, retaining walls are must in order to sustain lateral pressure of soil or other materials. The material retained by the wall is known as backfill which may have its top surface horizontal or inclined. The backfill above the horizontal plane at the top of a wall is known as surcharge. In the design of retaining walls, it is necessary to determine the lateral pressure exerted by the retaining mass of soil.

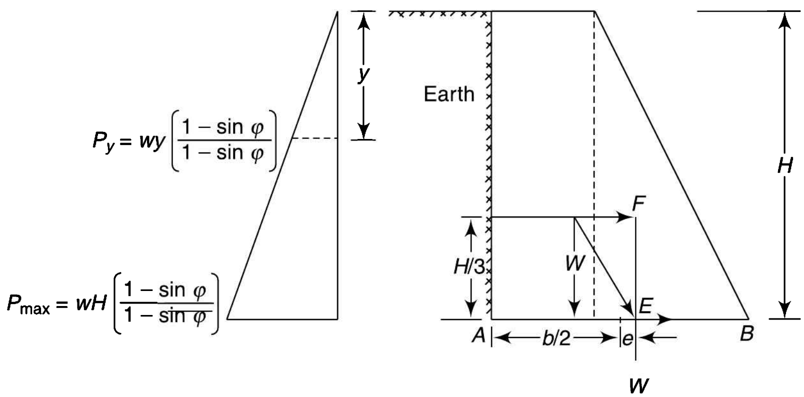


Fig. 5.84

Angle of Repose It is the natural slope of the materials when placed on a horizontal surface in the absence of any external acting force. For most of the materials, its value varies between 20° and 40°.

Rankine's Formula This formula provides the earth pressure at any depth y from the top of the retaining wall. It is given by

$$p_y = wy\left(\frac{1-\sin\varphi}{1-\sin\varphi}\right) \tag{5.31}$$

where w is the specific weight of earth and φ is the angle of repose. If H is the height of the wall, then maximum pressure at the bottom is

$$p_{\max} = wH\left(\frac{1-\sin\varphi}{1-\sin\varphi}\right) \tag{5.32}$$

Total horizontal force= Average pressure × height

$$= \frac{1}{2}\cdot wH\left(\frac{1-\sin\varphi}{1-\sin\varphi}\right)\cdot H$$

$$= \frac{wH^2}{2}\left(\frac{1-\sin\varphi}{1-\sin\varphi}\right) \tag{5.33}$$

This force acts at $H/3$ from the bottom.

Example 5.53 || A 12-m high masonry retaining wall of trapezoidal section has a width of 2 m at the top and 8 m at the bottom. The retaining surface is vertical and the wall retains earth which is level up to the top. Determine the maximum and minimum stresses at the base. The densities of the earth and the masonry are 14 kN/m^3 and 25 kN/m^3 respectively and the angle of repose of the earth is 30°.

Solution

Given A masonry retaining wall as shown in Fig. 5.85
To find Maximum and minimum stresses at the base

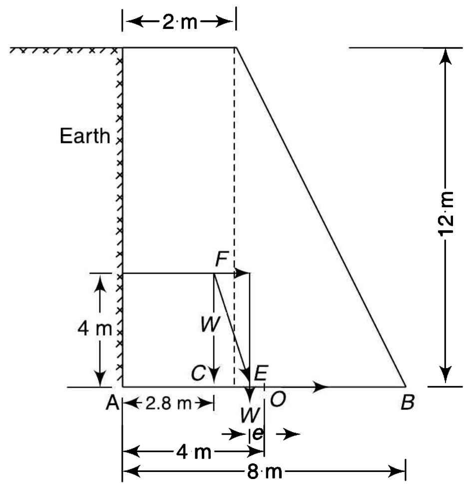


Fig. 5.85

Considering 1 m length of the dam,
Weight of wall

$$W = \left(\frac{8+2}{2} \times 12 \times 1\right) \times 25 = 1500 \text{ kN}$$

Taking moments about AD,

$$2 \times 12 \times \frac{2}{2} + \frac{6 \times 12}{2}\left[2 + \frac{6}{3}\right] = \left[2 \times 12 + \frac{6 \times 12}{2}\right]\bar{x}$$

or $\quad 24 + 36 \times 4 = 60\,\bar{x} \quad$ or $\quad \bar{x} = 2.8$ m

Horizontal force

Force due to earth pressure, $F = \dfrac{wH^2}{2}\left(\dfrac{1-\sin\varphi}{1-\sin\varphi}\right)$

$$= \frac{14 \times 12^2}{2}\left(\frac{1-\sin 30°}{1-\sin 30°}\right)$$

$$= 336 \text{ kN}$$

It acts at a height of $h/3$ or 4 m from the base.
Eccentricity
Let the resultant of W and F strikes the base at E.

Now, $\quad \dfrac{CE}{CD} = \dfrac{F}{W} \quad$ or $\quad CE = \dfrac{336}{1500} \times 4 = 0.896$ m

Eccentricity, $e = AC + CE - AO = 2.8 + 0.896 - 4 = -0.304$ m
Maximum and minimum stresses
Maximum compressive stress at A,

$$\sigma = \frac{W}{b}\left(1 + \frac{6e}{b}\right) = \frac{1500}{8}\left(1 + \frac{6 \times 0.304}{8}\right) = 230.3 \text{ kN/m}^2$$

and minimum compressive stress at B,

$$\sigma = \frac{W}{b}\left(1 - \frac{6e}{b}\right) = \frac{1500}{8}\left(1 - \frac{6 \times 0.304}{8}\right) = 144.8 \text{ kN/m}^2$$

Example 5.54 || A masonry chimney that is 20 m high tapers from 2.4 m external diameter at the base to 1.6-m diameter at the top. The weight of the chimney is 1200 kN. Determine the uniform horizontal wind pressure so that no tension occurs at the base.

Solution

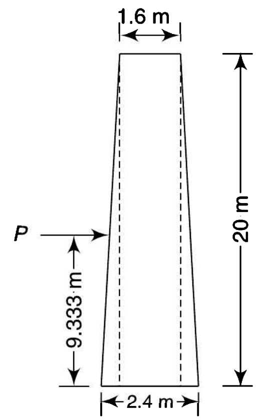


Fig. 5.86

Given A masonry chimney as shown in Fig. 5.86

To find Horizontal wind pressure for no tension at base

$$\text{Area of base} = \frac{\pi}{4}(2.4^2 - 1.6^2) = 2.513 \text{ m}^2$$

$$I = \frac{\pi}{64}(2.4^4 - 1.6^4) = 1.307 \text{ m}^4$$

Direct (vertical) stress

$$\text{Direct stress} = \frac{1200}{2.513} = 477.517 \text{ kN/m}^2$$

Maximum bending moment

$$\text{Projected area} = \frac{2.4 + 1.6}{2} \times 20 = 40 \text{ m}^2$$

Let p be the uniform horizontal wind pressure. It acts on the projected area chimney.

Total wind pressure = $40p$

Height of the centroid of the projected area or of the trapezium,

$$\bar{y} = \frac{20 \times 1.6 \times 10 + \left(\frac{1}{2} \times 0.8 \times 20\right) \times \frac{20}{3}}{\frac{2.4 + 1.6}{2} \times 20} = 9.333 \text{ m}$$

Maximum bending moment = $40p \times 9.333 = 373.33p$ kN·m

Bending stress

$$\text{Maximum bending stress} = \frac{373.33p}{1.307} \times 1.2 = 342.77p \text{ kN·m}$$

Calculation for wind pressure

For no tension at the base, $342.77p = 477.512$

$$p = 1.115 \text{ kN/m}^2$$

Summary

1. If a constant bending moment (no shear force) acts on some length of a beam, the stresses set up on any cross-section on that part of the beam constitute a pure couple, the magnitude of which is equal to the bending moment.
2. The beam under the action of bending moment bends in such a way that the inner or the concave edge of cross-section undergoes compression and the outer or the convex edge, tension.
3. An intermediate surface at which the stress is zero is known as *neutral surface*. An axis obtained by intersection of the neutral surface and a cross-section is known as *neutral axis* about which the bending of the surface takes place.
4. The assumptions made in the theory of bending are that the material is homogeneous and isotropic, transverse planes remain plane and perpendicular to the neutral surface after bending, initially the beam is straight and all longitudinal filaments are bent into circular arcs with a common centre of curvature which is large compared to the dimensions of the cross-section and the stress is purely longitudinal and the stress concentration effects near the concentrated loads are neglected.

5. The governing relation in the theory of bending is $\sigma/y = M/I = E/R$, Conventionally, y is taken positive when measured outwards from the centre of curvature.
6. The ratio I/y where y is the farthest or the most distant point of the section from the neutral axis and is denoted by Z is called section modulus. Thus $M = \sigma Z$.
7. The maximum bending moment which can be carried by a given section for a given maximum value of stress is known as the *moment of resistance*.
8. Moment of inertia of a rigid body is obtained by summing the products of its various particles with the square of their distances from a given axis.
9. The *parallel axis theorem* states that the moment of inertia about any axis parallel to the centroidal axis is equal to the moment of inertia through the centroidal axis plus the product of the area of the figure and the square of the distance between the two axes.
10. The moment of inertia and section modulus of different sections are

    Rectangle: $I_{xx} = bd^3/12$, $I_{ab} = bd^3/3$ and $Z_x = bd^2/6$

    Hollow rectangle: $I_{xx} = (BD^3 - bd^3)/12$, $Z_x = (BD^3 - bd^3)/6D$

    I-section: $I_{xx} = (BD^3 - bd^3)/12$, $Z_x = (BD^3 - bd^3)/6D$

    Triangular section: $I_{xx} = bd^3/36$; $I_{ab} = bd^3/12$

    Circular section: $I_{xx} = \pi d^4/64$; $Z_x = \pi d^3/32$

    Hollow circular section: $I_{xx} = \pi(D^4 - d^4)/64$; $Z_x = \pi(D^4 - d^4)/32D$
11. Beams made up of two different materials such as wooden beams reinforced by steel plates are known as *flitched* or *composite beams*.
12. At any common surface in a flitched beam, Strain = $\sigma_1/E_1 = \sigma_2/E_2$
13. Moment of resistance of a flitched beam, $M_r = \sigma_{1m}(I_1 + mI_2)/y_1$ where m = modular ratio E_2/E_1 and $(I_1 + mI_2)$ is known as *equivalent moment of inertia* of the cross-section
14. To compensate for the weakness of concrete, steel reinforcement is done on the tension side of concrete beams and to have the maximum advantage it is put at the greatest distance from the neutral axis of the beam.
15. The integral $\int xydA = 0$ is known as *product of inertia* and the axes for which it is zero for a section are known as *principal axes* of the cross-section.
16. The limitation of the theory of bending moment is that it can be applied only to the case of bending about a principal axis.
17. The directions of the principal axes can be found from the relation $\tan 2\theta = 2I_{xy}(I_y - I_x)$; where I_{xy} for a rectangle with sides parallel to the principal axes and is given by, $I_{xy} = A \cdot hk$

    Then $I_u = \frac{1}{2}[(I_x + I_y) + \sec 2\theta\,(I_x - I_y)]$ and $I_v = \frac{1}{2}[(I_x + I_y) - \sec 2\theta(I_x - I_y)]$
18. *Ellipse of inertia* or *momental ellipse* is graphical method which can be used to find moments of inertia about two mutually perpendicular axes through the centroid when moments of inertia about the principal axes are known.
19. In masonry columns, it is desirable that no tensile stresses are set up. It is ensured if the line of action of the load lies within a central area of the section. In rectangular sections, it should lie within the middle third and in circular sections within middle quarter or within a circle of diameter $d/4$ with centre O.

Objective Type Questions

1. A beam is said to be loaded in pure bending if
   (a) shear force and bending moment are uniform throughout
   (b) shear force is zero and bending moment is uniform throughout
   (c) shear force can vary but bending moment is uniform throughout
2. The moment of inertia of rectangular lamina of side d and b about centroidal axis parallel to side d is
   (a) $\frac{bd^3}{12}$ (b) $\frac{bd^3}{36}$ (c) $\frac{db^3}{12}$ (d) $\frac{db^3}{36}$
3. *Neutral axis* in a beam carries ____bending stress.
   (a) maximum (b) minimum (c) zero
4. In simple bending of beams, the stress in the beam
   (a) is constant (b) varies linearly (c) varies parabolically
5. In a transversally loaded beam, the maximum tensile stress occurs at the
   (a) top edge (b) bottom edge (c) neutral axis (d) none of these
6. In a transversally loaded beam, the maximum compressive stress occurs at the
   (a) top edge (b) bottom edge (c) neutral axis (d) none of these
7. The relation governing the simple bending of beam is
   (a) $\frac{\sigma}{y} = \frac{M}{E} = \frac{I}{R}$ (b) $\frac{\sigma}{y} = \frac{M}{R} = \frac{E}{I}$ (c) $\frac{\sigma}{E} = \frac{M}{I} = \frac{y}{R}$ (d) $\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$
8. The diameter of kern of a circular cross-section of diameter d is
   (a) $\frac{d}{2}$ (b) $\frac{d}{3}$ (c) $\frac{d}{4}$ (d) $\frac{2d}{3}$
9. The stress in a beam is less if its section modulus is
   (a) high (b) low (c) zero
10. E*quivalent moment of inertia* of the cross-section in terms of timber of a flitched beam made up of steel and timbre is ($m = E_s/E_t$)
    (a) $(I_t + m/I_s)$ (b) $(I_t + I_s/m)$ (c) $(I_t + mI_s)$ (d) $(I_t + 2mI_t)$
11. E*quivalent moment of inertia* of the cross-section in terms of steel of a flitched beam made up of steel and timbre is ($m = E_s/E_t$)
    (a) $(I_s + mI_t)$ (b) $(I_s + I_t/m)$ (c) $(I_s + m/I_t)$ (d) $(I_s + 2mI_t)$

Answers

1. (b)	2. (c)	3. (c)	4. (b)	5. (b)	6. (a)
7. (d)	8. (c)	9. (a)	10. (c)	11. (b)	

Review Questions

5.1 What do you mean by the terms 'neutral axis' and 'neutral surface'?
5.2 Develop the theory of simple bending clearly stating the assumptions made.
5.3 Prove the relation $\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R}$ for simple bending.
5.4 Define the term 'moment of resistance'.
5.5 What do you mean by the term 'flitched beams'? Develop a relation for the moment of resistance for such a beam.

5.6 What are reinforced concrete beams? Where are they used? How do you find the moment of resistance of such beams?

5.7 What are meant by the terms '*principal axis*', '*principal moments of inertia*' and '*product of inertia*'?

5.8 How do you find the principal moments of inertia of an area?

5.9 Describe the method to find the moments of inertia about two mutually perpendicular axes through the centroid when the moments of inertia about the principal axes are known.

5.10 What is the middle third rule for rectangular sections?

Numerical Problems

5.1 A 250-mm deep and 150-mm wide rectangular beam is subjected to a maximum bending moment of 250 kN·m. Determine the maximum stress produced in the beam and the radius of curvature for the portion of the beam where bending is maximum. (160 MPa, 156.24 m)

5.2 A rectangular beam is to be cut out of a cylindrical log of wood with a diameter of 300 mm. Determine the depth and width of the strongest beam which can be had from the log of wood. (173.1 mm, 245.1 mm)

5.3 A hollow circular bar used as a beam has outside diameter twice of the inside diameter. If it is subjected to a maximum bending moment of 40 kN·m and the allowable bending stress is 100 MPa, determine the inside diameter of the bar. (81.6 mm)

5.4 A cast-iron pipe of 200-mm internal diameter and 220-mm external diameter is supported at two points, 8 m apart. Determine the maximum stress in the pipe material when it runs full. The density of cast iron is 70 kN/m^3 and of water 9.81 kN/m^3. (18.59 MPa)

5.5 The tension flange of a girder of I-section is 240 mm × 40 mm, whereas the compression flange 120 mm × 20 mm. The web is 300 mm deep and 20 mm thick. If the girder is used as a simply supported beam of 8 m span, determine the load per m run if the allowable stress is 90 MPa in compression and 30 MPa in tension. (9.277 KN/m)

5.6 A wooden beam is 120 mm deep and 80 mm wide with a semicircular groove of 20-mm radius at the centre of each side. The beam is simply supported on a span of 3 m and is acted upon by a point load of 450 N at a distance of 1 m from one end and a uniformly distributed load of 500 N/m over the whole span. Determine the maximum stress developed in the section. (4.27 MPa)

5.7 A metallic tube of 30-mm outside diameter and 20-mm inside diameter is simply supported on a span of 1 m. A load of 1 kN at the midspan is just sufficient to induce the stress to the elastic limit. Four such tubes are joined together to form a single beam, the centres of the tubes forming a square of 30 mm side and two sides of this square are horizontal. Determine the maximum central load which can be put on the beam if the stress is not allowed to exceed the elastic limit. (7.54 kN)

5.8 A simply supported beam weighs W and it carries a concentrated load W' at the midspan. If the error in calculating the bending stress is e% when the weight of the beam is neglected, show that the ratio of W'/W is $(100 - e)/2e$.

5.9 A 60-mm wide and 80-mm deep timber beam is reinforced by riveting on two flitches, each 60 mm by 5 mm in section. Determine the moment of resistance when the flitches are attached symmetrically on the sides. Allowable stress for timbre is 8 MPa. Also find the maximum stress in the steel. $E_s = 210$ GPa and $E_t = 14$ GPa. (10.52 kN·m; 90 MPa)

5.10 A timber of section 500 mm × 500 mm is reinforced by securing two steel plates of 500 mm × 8 mm size at the top and bottom surfaces. The composite section is used as a beam of 6-m span and is loaded with a uniformly distributed load of 80 kN/m run. Determine the maximum bending stresses in steel and timber at the midspan. $E_s = 200$ GPa and $E_t = 10$ GPa. (119.7 MPa, 5.8 MPa)

5.11 A compound beam is made up of two bars joined together, one of steel and the other of brass. The width of each bar is b whereas the thicknesses are t and t' to make the overall thickness $t + t'$. Determine the ratio of t to t' so that the neutral axis of the section is at the dividing line of the two bars. Take $E_s = 2\,E_b$. (0.707)

5.12 A composite section is made of a solid bronze rod of 20-mm diameter surrounded by a cylinder of 28-mm external diameter. The allowable stresses in bronze and steel are 100 MPa and 150 MPa respectively. Calculate the moment of resistance of the section. $E_s = 1.75\,E_b$. (287.2 kN·mm)

5.13 A flitched beam is made up of two timber joists, each 80 mm wide and 240 mm deep, with a steel plate placed symmetrically in between and clamped to the joists. Find the total moment of resistance of the section if the permissible stress in the joist is 9 MPa. $E_s = 20\,E_t$. (19.97 kN·m)

5.14 A simply supported 100-mm wide and 120-mm deep horizontal beam is 5 m long. It carries a load of 12 kN at the midspan. The load lies in a plane inclined at 30° to the vertical longitudinal plane. The load line passes through the centroid of the rectangular section of the beam. Calculate the stresses at all the corners of the section. Show the variation of stresses in the section on a diagram.
(± 91.6 MPa, ± 16.6 MPa)

5.15 In a compression testing specimen of 15-mm diameter, the line of thrust is parallel to the axis of the specimen but is displaced from it. Determine the displacement of the line of action of thrust from the axis if the allowable stress is 20 per cent more than the mean stress on a normal section.
(0.375 mm)

5.16 A short column of 30-mm internal diameter and 60-mm external diameter carries an eccentric load of 120 kN. Determine the maximum eccentricity which the load can have without producing tension on the cross-section of the column. (9.375 mm)

5.17 A vertical flag staff is 10 m high and is of square section throughout. The section at the bottom is 160 mm × 160 mm and tapers uniformly to 80 mm × 80 mm at the top. Determine the maximum stress due to bending if a horizontal pull of 1 kN is applied at the top, the direction of loading being along a diagonal of the section. (24.55 MPa)

5.18 A brass tube of 60-mm outer diameter and 50-mm inner diameter encloses a steel rod of 30-mm diameter. The tube and the rod are of the same length with their axes parallel to each other and 5 mm apart. The ends of both are covered by rigid plates through which a compressive force of 60 kN is applied along the axis of the tube. Find the maximum and the minimum longitudinal stresses in the tube and the rod. $E_b = 95$ GPa and $E_s = 205$ GPa.
(In tube: 39.93 MPa and 13.15 MPa; In rod: 66.88 MPa and 38 MPa)

5.19 A 40-kN load acts on a short column of rectangular cross-section 80 mm × 60 mm at a point which is 30 mm from the shorter side and 15 mm from the longer side. Find the maximum tensile and compressive stresses in the section. (10.4 MPa; 27.1 MPa)

Chapter **6**

Shear Stress in Beams

While discussing the theory of simple bending in the previous chapter, it was assumed that no shear force acts on the section. However, when a beam is loaded, the shear force at a section is always present along with the bending moment. It is, therefore, important to study the variation of shear stress in a beam and to know its maximum value within safe limits. It is observed that in most cases, the effect of shear stress is quite small as compared to the effect of bending stress and may be ignored. In some cases, however, it may be desirable to consider its effect also. In most cases, beams are designed for bending stresses and checked for shear stresses. This chapter discusses the shear stress and its variation across the section.

A shear force in a beam at any cross-section sets up shear stress on transverse sections, the magnitude of which varies across the section. In the analysis, it is assumed that the shear stress is uniform across the width and does not affect the distribution of bending stress. The latter assumption is not strictly true as the shear stress causes a distortion of transverse planes and they do not remain plane.

As every shear stress is accompanied by an equal complimentary shear stress, shear stress on transverse planes has complimentary shear stress on longitudinal or horizontal planes parallel to the neutral axis.

6.1 VARIATION OF SHEAR STRESS

Let a beam be under transverse loading (Fig. 6.1). To find the value of shear stress on an element, consider a small block $QRST$ of length δx of the beam cut horizontally. Figures 6.2a and b show the front and the end views of the beam.

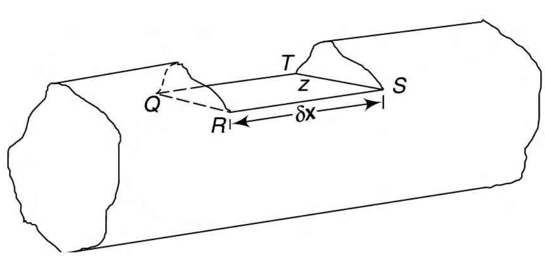


Fig. 6.1

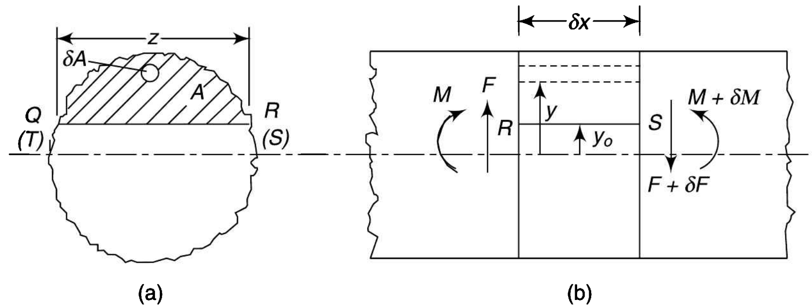


Fig. 6.2

Figure 6.2b also shows two transverse sections of the beam. As shear force and bending moment varies along the length of a beam, let

F, $F + \delta F$ = shear forces at the two sections

M, $M + \delta M$ = bending moments at the two sections

Consider the equilibrium of a block above the surface $QRST$.

Let y_o be the distance of the surface from the neutral axis and z be the width of the cross-section at this position.

Consider an elemental area δA at a distance y from the neutral axis.

Let σ and $\sigma + \delta\sigma$ be the bending stress on the transverse plane as a result of bending moment (Fig. 6.3).

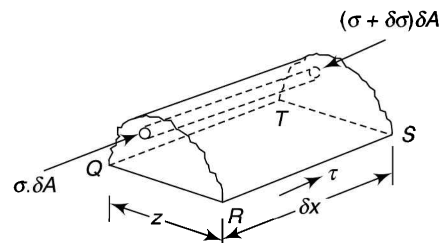


Fig. 6.3

Then force applied on the left end of the elemental area = $\sigma \cdot \delta A = \dfrac{My}{I} \cdot \delta A$

Force applied on the right end of the elemental area = $(\sigma + \delta\sigma) \cdot \delta A = \dfrac{(M + \delta M)y}{I} \cdot \delta A$

$\therefore$ net force applied on the elemental area δA towards left

$$\frac{(M + \delta M)y}{I} \cdot \delta A - \frac{My}{I} \cdot \delta A = \frac{\delta M \cdot y}{I} \cdot \delta A$$

Total force applied on the gross area of the block = $\displaystyle\int \frac{\delta M \cdot y}{I} \cdot \delta A$

This force on the block tends to slide the block towards left. This is resisted by the horizontal shear force on the surface area of block $QRST$. If τ is the average shear stress value at the surface,

Net shear force on the surface = shear stress × area = $\tau \cdot z \cdot \delta x$

Equating the two forces, $\displaystyle \tau \cdot z \cdot \delta x = \int \left(\frac{\delta M \cdot y}{I} \right) \cdot dA$

or $$\tau = \frac{\delta M}{\delta x \cdot z \cdot I} \int y \cdot dA = \left(\frac{\delta M}{\delta x} \right) \frac{1}{zI} \cdot A\bar{y}$$

But $$\frac{\delta M}{\delta x} = F; \quad \therefore \quad \tau = F \cdot \frac{A\bar{y}}{zI} \qquad (6.1)$$

In the above relation,

— z is the actual width of the section at the position where τ is to be calculated

— I is the total moment of inertia about the neutral axis

— $A\bar{y}$ is the moment of the shaded area about the neutral axis

As the horizontal shear stress must be complimentary to transverse shear stress and the two must be equal in magnitude, Eq. 6.1 also represents the average value of shear stress along the line *RS*.

- As the upper and lower faces of a beam are free surfaces, i.e., no forces are exerted on these faces, horizontal and thus transverse shear stresses are zero along the upper and lower edges of the transverse section (Fig. 6.4).
- It may appear from Eq. 6.1 that shear stress will be maximum along the neutral axis. This cannot be concluded as such since the shear stress depends upon the width *z* as well which may vary in irregular sections.
- In case of a beam in pure bending, i.e., when the beam is subjected to equal and opposite couples and no shear force ($F = 0$), the transverse shear stress is zero.

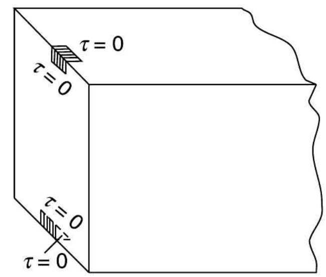


Fig. 6.4

6.2 SHEAR STRESS VARIATION IN DIFFERENT SECTIONS

Variation of shear stress in different type of sections is discussed below:

Rectangular Section

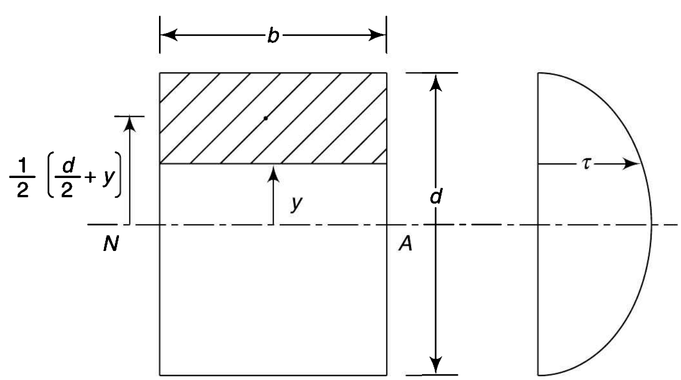


Fig. 6.5

At a distance *y* from neutral axis (Fig. 6.5),

$$\tau = F \cdot \frac{A\bar{y}}{zI} = F \cdot \frac{b\left(\frac{d}{2} - y\right) \cdot \left(\frac{d/2 + y}{2}\right)}{b \cdot \left(\frac{bd^3}{12}\right)} = \frac{6F}{bd^3} \cdot \left(\frac{d^2}{4} - y^2\right) \qquad (6.2)$$

This indicates that there is parabolic variation of shear stress with *y*.

At neutral axis ($y = 0$), Shear stress = $\frac{3}{2} \cdot \frac{F}{bd}$; it is the maximum shear stress.

Usually, *F/bd* is known as the *mean stress* and thus $\tau_{max} = 1.5\ \tau_{mean}$

I-section

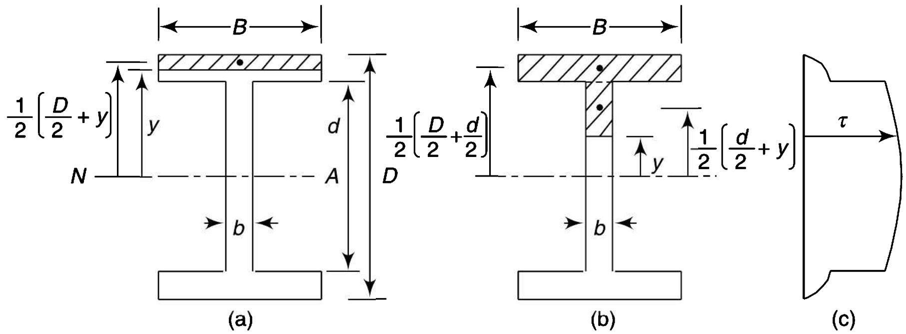


Fig. 6.6

In the flange, at a distance y from neutral axis (Fig. 6.6a),

$$\tau_f = F \cdot \frac{A\bar{y}}{zI} = F \cdot \frac{B\left(\frac{D}{2} - y\right)\left[\frac{1}{2}\left(\frac{D}{2} + y\right)\right]}{B \cdot I} = \frac{F}{2I} \cdot \left(\frac{D^2}{4} - y^2\right)$$

At $y = D/2$, $\sigma = 0$

At $y = d/2$, $\tau = \dfrac{F}{8I}(D^2 - d^2)$ (6.3)

In the web, at a distance y from neutral axis (Fig. 6.6b),

$$\tau_w = F \cdot \frac{A\bar{y}}{zI} = F \cdot \frac{A\bar{y}\text{ (for flange)} + A\bar{y}\text{(for web)}}{zI}$$

$$= F \cdot \frac{B\left(\frac{D}{2} - \frac{d}{2}\right)\left[\frac{1}{2}\left(\frac{D}{2} + \frac{d}{2}\right)\right] + b\left(\frac{d}{2} - y\right)\left[\frac{1}{2}\left(\frac{d}{2} + y\right)\right]}{b \cdot I}$$

$$= \frac{F}{bI} \cdot \left[B\left(\frac{D-d}{2}\right)\left(\frac{D+d}{4}\right) + \frac{b}{2}\left(\frac{d-2y}{2}\right)\left(\frac{d+2y}{2}\right)\right]$$

$$= \frac{F}{8I}\left[\frac{B}{b}(D^2 - d^2) + (d^2 - 4y^2)\right] \tag{6.4}$$

Maximum shear stress = $\dfrac{F}{8I} \cdot \left[\dfrac{B}{b}(D^2 - d^2) + d^2\right]$ at the neutral axis ($y = 0$) (6.5)

At the top of web, $y = d/2$, $\dfrac{F}{8I} \cdot \left[\dfrac{B}{b}(D^2 - d^2)\right] = \dfrac{B}{b} \cdot \tau_f$ (6.6)

Shear stress distribution is shown in Fig. 6.6c.

It is to be noted that the shear force carried by the flanges is small as compared to that carried by the web. This is true especially in case of widely used thin sections. In such cases the shear force carried by the web may be 95% of the total. Also the variation of stress over the web width is comparatively small (about 25% to 30%). Thus, usually for design purposes, assumption is taken that all the shear force is carried by the web uniformly over it.

Square with a Diagonal Horizontal

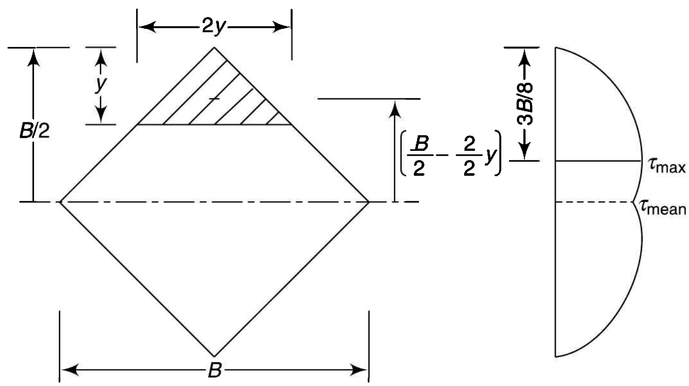


Fig. 6.7

Refer Fig. 6.7,

Moment of inertia about the neutral axis, $I = 2\left[\frac{B\cdot(B/2)^3}{12}\right] = \frac{B^4}{48}$

$$F\cdot\frac{A\bar{y}}{zI} = F\cdot\frac{\frac{2y\cdot y}{2}\left(\frac{B}{2}-\frac{2}{3}y\right)}{(2y)\cdot\left(\frac{B^4}{48}\right)} = \frac{24Fy}{B^4}\cdot\left(\frac{B}{2}-\frac{2}{3}y\right)$$

$$= \frac{4Fy}{B^4}(3B - 4y) \text{ i.e. it is parabolic.} \tag{6.7}$$

- At $y = 0$, $\tau = 0$
- At neutral axis, $y = B/2$, $\tau = \dfrac{2F}{B^2}$ (6.8)

  If b is the side of the square, $B = \sqrt{2}b$,

- At neutral axis, $\tau = \dfrac{2F}{(\sqrt{2}b)^2} = \dfrac{F}{b^2} = \dfrac{F}{\text{area}} = \tau_{mean}$ (6.9)

- For maximum value, $\dfrac{d\tau}{dy} = \dfrac{d}{dy}(3By - 4y^2) = 0$ or $3B - 8y = 0$

  or $y = \dfrac{3}{8}B = \dfrac{3\sqrt{2}b}{8}$

$$\tau_{max} = \frac{4Fy}{B^4}(3B-4y) = \frac{4F(3/8)B}{B^4}\left(3B - \frac{4\times 3}{8}B\right) = \frac{9F}{4B^2} = \frac{9F}{4(\sqrt{2}b)^2} = \frac{9F}{8b^2} = \frac{9}{8}\tau_{mean} \tag{6.10}$$

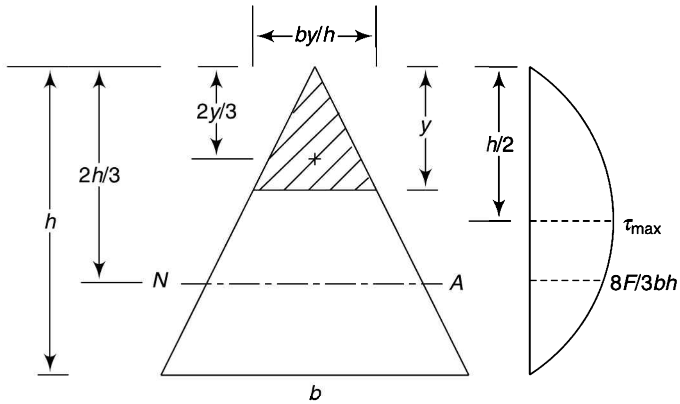


Fig. 6.8

Triangular Section

Refer Fig. 6.8,

$$\tau = F \cdot \frac{A\bar{y}}{zI} = \frac{F\left(\frac{1}{2}\cdot y \cdot \frac{by}{h}\right)\left(\frac{2}{3}h - \frac{2}{3}y\right)}{\frac{by}{h}\cdot\frac{bh^3}{36}} = \frac{12Fy}{bh^3}(h-y) \tag{6.11}$$

For maximum value, $\frac{d\tau}{dy} = \frac{d}{dy}(hy - y^2) = 0$ or $h - 2y = 0$ or $y = h/2$

$$\tau_{\max} = \frac{12Fh/2}{bh^3}\left(h - \frac{h}{2}\right) = \frac{3F}{bh} = \frac{3}{2}\cdot\frac{F}{bh/2} = 1.5\tau_{\text{mean}} \tag{6.12}$$

At neutral axis, $\tau = \frac{12F(2h/3)}{bh^3}\left(h - \frac{2h}{3}\right) = \frac{8F}{3bh}$

Hexagonal Section

Refer Fig. 6.9,

$$I = 2\left[\frac{b}{2}\cdot\frac{d^3}{12} + \frac{bd^3}{3} + \frac{b}{2}\cdot\frac{d^3}{12}\right] = \frac{5}{6}bd^3 = \frac{5}{6}b\left(\frac{\sqrt{3}}{2}b\right)^3 = \frac{5\sqrt{3}}{16}b^4$$

Now, $A\bar{y}$ = Moment of shaded area about neutral axis

Consider a strip of thickness δy at a height y from neutral axis and parallel to it,

$$z = 2b - 2\frac{y}{\sqrt{3}} = \frac{2}{\sqrt{3}}(\sqrt{3}b - y)$$

Area of the strip = $z \cdot \delta y$

Moment of elementary area about neutral axis = $z \cdot \delta y \cdot y$

Moment of whole of the shaded area about neutral axis,

$$A\bar{y} = \int_y^{\sqrt{3}b/2} \frac{2}{\sqrt{3}}(\sqrt{3}b - y)\, dy \cdot y = \frac{2}{\sqrt{3}}\int_y^{\sqrt{3}b/2}(\sqrt{3}b - y)\cdot y \cdot dy$$

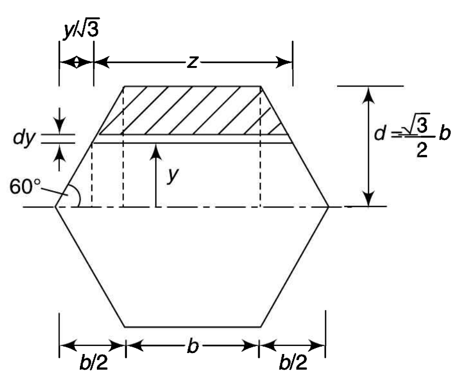


Fig. 6.9

$$\frac{2}{\sqrt{3}}\left[\sqrt{3}b\frac{y^2}{2}-\frac{y^3}{3}\right]_y^{\sqrt{3}b/2}=\frac{2}{\sqrt{3}}\left[\sqrt{3}\cdot\frac{3}{8}b^3-\frac{3\sqrt{3}}{24}b^3-\sqrt{3}b\frac{y^2}{2}+\frac{y^3}{3}\right]$$

$$\frac{2}{\sqrt{3}}\left[\frac{\sqrt{3}}{4}b^3-\frac{\sqrt{3}}{2}by^2+\frac{y^3}{3}\right]$$

$$\tau=F\cdot\frac{A\bar{y}}{zI}=\frac{F\frac{2}{\sqrt{3}}\left[\frac{\sqrt{3}}{4}b^3-\frac{\sqrt{3}}{2}by^2+\frac{y^3}{3}\right]}{\frac{5\sqrt{3}}{16}b^4\times\frac{2}{\sqrt{3}}(\sqrt{3}b-y)}=\frac{F\frac{2}{\sqrt{3}}\left[\frac{\sqrt{3}}{4}b^3-\frac{\sqrt{3}}{2}by^2+\frac{y^3}{3}\right]}{\frac{5}{8}b^4(\sqrt{3}b-y)} \tag{6.13}$$

At neutral axis, $y = 0$, $\tau=\dfrac{4F}{5\sqrt{3}b^2}=0.4618\dfrac{F}{b^2}$ (6.14)

Average shear stress $=\dfrac{F}{4\left(\frac{1}{2}\frac{b}{2}d\right)+2bd}=\dfrac{F}{3bd}=\dfrac{F}{3b(\sqrt{3}b/2)}=\dfrac{2F}{3\sqrt{3}b^2}=0.385\dfrac{F}{b^2}$ (6.15)

$$\frac{\sigma_{\text{n.a.}}}{\sigma_{\text{ave.}}}=\frac{0.4618}{0.385}=1.2$$

Shear stress at $y = b/2$, $\tau=\dfrac{F\frac{2}{\sqrt{3}}\left[\frac{\sqrt{3}}{4}b^3-\frac{\sqrt{3}}{2}b\left(\frac{b}{2}\right)^2+\frac{1}{3}\left(\frac{b}{2}\right)^3\right]}{\frac{5}{8}b^4\left(\sqrt{3}b-\frac{b}{2}\right)}$

$$=\frac{F(0.5-0.25+0.048)}{0.77b^2}=0.387\frac{F}{b^2}$$

$$\frac{\sigma_{y=0.5b}}{\sigma_{\text{av}}}=\frac{0.387}{0.385}\approx 1$$

Circular Section

Refer Fig. 6.10,

$$z=2\sqrt{r^2-y^2} \quad \text{or} \quad z^2=4(r^2-y^2)$$

or $\quad 2z\cdot dz=-8ydy$

or $\quad ydy=-z\cdot dz/4$ (i)

Now, $A\bar{y}$ = Moment of shaded area about neutral axis

Consider a strip of thickness δy at a height y from neutral axis and parallel to it,

Area of the strip $= z\cdot\delta y$

Moment of elementary area about neutral axis $= z\cdot\delta y\cdot y$

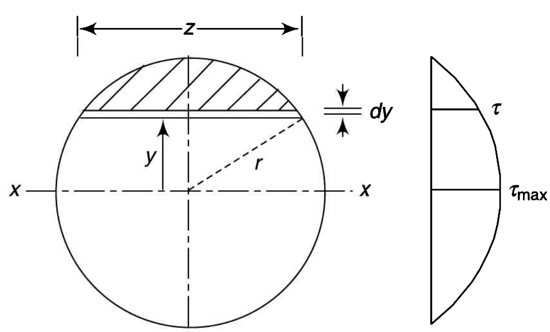


Fig. 6.10

Moment of whole of the shaded area about neutral axis,

$$A\bar{y} = \int_y^r z \cdot dy \cdot y = -\frac{1}{4}\int_z^0 z \cdot z \cdot dz = \frac{1}{4}\int_0^z z^2 \cdot dz = \frac{z^3}{12}$$

$$\tau = F \cdot \frac{1}{zI}\frac{z^3}{12} = F \cdot \frac{1}{I}\frac{z^2}{12} = \frac{F}{3I}(r^2 - y^2) \tag{6.16}$$

Thus, shear stress variation is parabolic in nature.

$$\tau_{\max(y=0)} = F \cdot \frac{d^2/4}{3\,(\pi d^4/64)} = \frac{4}{3}\frac{F}{(\pi d^2/4)} = \frac{4}{3}\tau_{av} \tag{6.17}$$

Example 6.1 A 100 mm × 40 mm *I*-beam is subjected to a shear force of 15 kN. Find the transverse shear stress at the neutral axis and at the top of the web. Compare it with the mean stress on the assumption of uniform distribution over the web. What is the percentage of shear force carried by the web? Moment of inertia of the section is 1.1 × 10^6 mm^4, web thickness is 3 mm and flange thickness is 4 mm.

Solution

Given An *I*-beam as shown in Fig. 6.11a.

$F = 15$ kN $\qquad I = 1.1 \times 10^6$ mm^4

To find

— Transverse shear stress at neutral axis
— Mean stress on assumption of uniform distribution over web
— Percentage of shear force carried by web

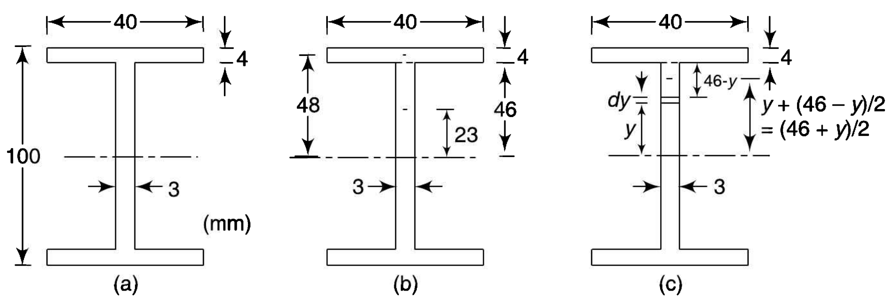


Fig. 6.11

Shear stress

At the neutral axis, $\tau = F\dfrac{A\bar{y}}{zI} = 15\,000 \times \dfrac{40 \times 4 \times 48 + 3 \times 46 \times 23}{3 \times 1.1 \times 10^6}$ (Refer Fig. 6.11b)

$= 49.34$ MPa

At the top of web, $\tau = 15\,000 \times \dfrac{40 \times 4 \times 48}{3 \times 1.1 \times 10^6} = 34.9$ MPa

Shear stress for uniform distribution over web

Assuming all the shear force carried uniformly by the web, $\tau = \dfrac{150\,000}{3 \times 92} = 54.35$ MPa

Shear force carried by the web

To find the total shear force carried by the web, assume an elementary length of the web *dy* at a distance *y* from the neutral axis (Fig. 11.c).

Shear stress in the elementary length,

$$\tau = F\frac{A\bar{y}}{zI} = 15\,000 \times \frac{40 \times 4 \times 48 + (46 - y) \times 3 \times [(46 + y)/2]}{3 \times 1.1 \times 10^6} \quad \text{(Refer Fig. 6.11c)}$$

$$= \frac{(10\,854 - 1.5y^2)}{220}$$

Shear force in the elementary length = Shear force × Area = $\tau \times b \cdot dy$

Total shear force carried by the web $= \int_{-d/2}^{d/2} \tau \cdot b \cdot dy$

$$= \int_{-46}^{46} \frac{(10\,854 - 1.5y^2)}{220} \times 3 \times dy$$

$$= \frac{1}{73.3}\int_{-46}^{46} (10\,854 - 1.5y^2)dy = \frac{1}{73.3}\left(10\,854y - \frac{1.5y^3}{3}\right)_{-46}^{46} = 12\,290 \text{ N}$$

Percentage of shear force carried by the web

Percentage of total load $= \dfrac{15\,000 - 12\,290}{15\,000} = 82\%$

Example 6.2 A simply supported beam of 2-m span carries a uniformly distributed load of 140 kN per m over the whole span. The cross-section of the beam is a *T*-section with a flange width of 120 mm, web and flange thickness of 20 mm and overall depth of 160 mm. Determine the maximum shear stress in the beam and draw the shear stress distribution for the section.

Solution

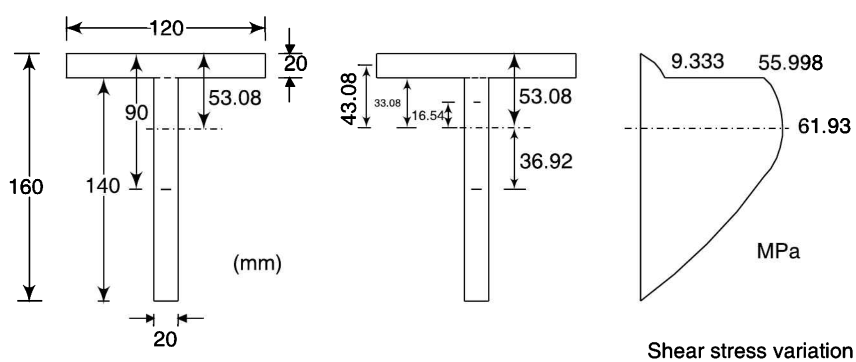


Fig. 6.12

Given A simply supported beam carrying uniformly distributed load as shown in Fig. 6.12.

$w = 140$ kN/m $\qquad L = 2$ m

To find

— Maximum shear stress

— To draw shear stress distribution

Reaction at each end of the beam $= \dfrac{140 \times 2}{2} = 140$ kN

Thus maximum shear force = 140 × 1 =140 kN

Moment of inertia

Refer Fig. 6.12,

Taking moments about the top edge,

$$\bar{y} = \frac{120 \times 20 \times 10 + 140 \times 20 \times 90}{120 \times 20 + 140 \times 20} = 53.08 \text{ mm}$$

$$I = \underbrace{\frac{120(20)^3}{12} + 120 \times 20 \times 43.08^2}_{\text{(flange)}} + \underbrace{\frac{20(140)^3}{12} + 20\ X\ 140\ X\ 36.92^2}_{\text{(web)}}$$

$$= 12.924 \times 10^6 \text{ mm}^4$$

Shear stress

Shear stress in the flange at the junction of flange and web

$$= F \cdot \frac{A\bar{y}}{zI} = \frac{140\,000 \times (120 \times 20) \times 43.08}{12.924 \times 10^6 \times 120} = 9.333 \text{ MPa}$$

Shear stress in the web at the junction $= 9.333 \times \dfrac{120}{20} = 56$ MPa

Maximum shear stress (at neutral axis) $= \dfrac{140\,000 \times [120 \times 20 \times 43.08 + (20 \times 33.08) \times 16.54]}{12.924 \times 10^6 \times 20} = 61.93$ MPa

Example 6.3 A cast-iron bracket of I-section has its top flange as 200 mm × 40 mm, bottom flange as 120 mm × 40 mm and the web as 300 mm × 40 mm. The overall depth of the section is 380 mm. The bracket is subjected to bending. If the maximum tensile stress in the top flange is not to exceed 15 MPa, determine the bending moment the section can take. If the beam is subjected to a shear force of 150 kN, sketch the stress distribution over the depth of the section.

Solution

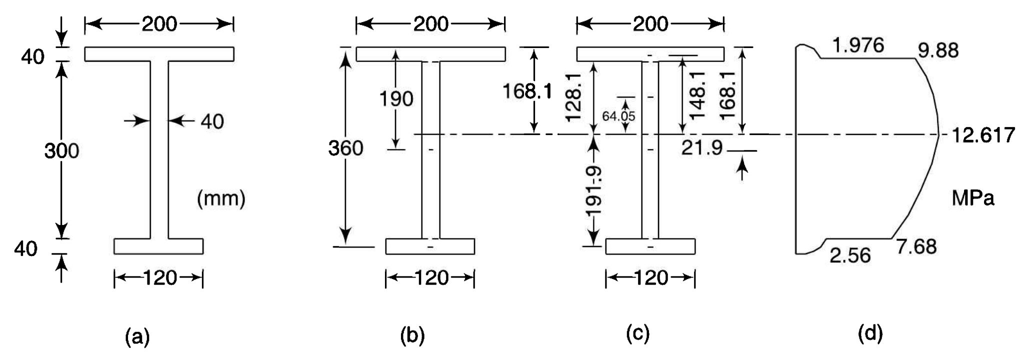


Fig. 6.13

Given A cast iron bracket of *I*-section as shown in Fig. 6.13a

$F = 150$ kN $\qquad \sigma = 15$ MPa

To find

— Maximum bending moment for the section

— To sketch the stress distribution

Refer Fig. 6.13b,

Taking moments about the top edge,

$$\bar{y} = \frac{200 \times 40 \times 20 + 300 \times 40 \times 190 + 120 \times 40 \times 360}{200 \times 40 + 300 \times 40 + 120 \times 40} = 168.1 \text{ mm}$$

Moment of inertia

Refer Fig. 6.13c,

$$I = \underbrace{\frac{200 \times 40^3}{12} + 200 \times 40 \times 148.1^2}_{\text{(upper flange)}} + \underbrace{\frac{40 \times 300^3}{12} + 40 \times 300 \times 21.9^2}_{\text{(web)}}$$

$$+ \underbrace{\frac{120 \times 40^3}{12} + 120 \times 40 \times 191.9^2}_{\text{(lower flange)}}$$

$$= 176.54 \times 10^6 + 95.76 \times 10^6 + 177.4 \times 10^6 = 449.7 \times 10^6 \text{ mm}^4$$

Maximum bending moment for the section

$$M = \frac{15 \times 449.7 \times 10^6}{168.1} = 40.128 \times 10^6 \text{ N.mm} \quad \text{or} \quad 40.128 \text{ kN.m}$$

Shear stresses

In upper flange at junction with web, $\tau = \dfrac{150 \times 10^3 (200 \times 40) \times 148.1}{449.7 \times 10^6 \times 200} = 1.976$ MPa

In web at junction with upper flange, $\tau = 1.975 \times \dfrac{200}{40} = 9.88$ MPa

At neutral axis (maximum),

$$\tau = \frac{150 \times 10^3 (200 \times 40 \times 148.1) + 40 \times 128.1 \times 64.05)}{449.7 \times 10^6 \times 40} = 12.617 \text{ MPa}$$

In lower flange at junction with web,

$$\tau = \frac{150 \times 10^3 (40 \times 120 \times 191.9)}{449.7 \times 10^6 \times 120} = 2.56 \text{ MPa}$$

In web at junction with lower flange, $\tau = 2.56 \times \dfrac{120}{40} = 7.68$ MPa

The stress distribution over the depth of the section has been shown in Fig. 6.13d.

Example 6.4 || A square of 20-mm side is used as a beam with its diagonal in the horizontal position. If the vertical shear force at a section is 2 kN, determine the value and the location of the maximum shear stress occurring in the cross-section. Also, find the shear stress at the neutral axis.

Solution

Given A beam of square section of 20-mm side with its diagonal in horizontal position.

$F = 2$ kN

To find

— Maximum shear stress

— Shear stress at neutral axis

Refer Fig. 6.14, using the results obtained in Section 6.2, (taking y from upper corner).

At $y = 0$, $\tau = 0$

Shear stress at neutral axis

$$\text{At neutral axis, } \tau = \tau_{mean} = \frac{2000}{20^2} = 5 \text{ MPa}$$

Maximum shear stress

For maximum value,

$$y = \frac{3b}{8\sqrt{2}} = \frac{3\sqrt{2} \times 20}{8} = 10.6 \text{ mm}$$

$$\tau_{max} = \frac{9}{8}\tau_{mean} = \frac{9}{8} \times 5 = 5.625 \text{ MPa}$$

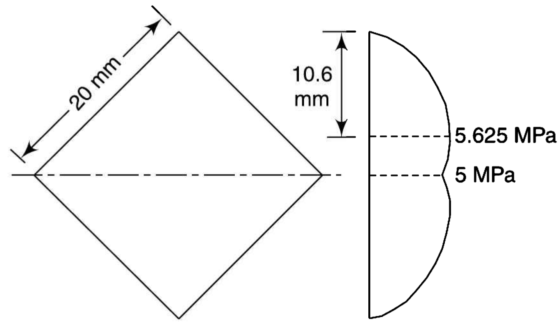


Fig. 6.14

Example 6.5 A steel bar as shown in Fig. 6.15 is subjected to a shear force of 10 kN. Plot the shear stress distribution of the section indicating principal values.

Solution

Given A steel bar as shown in Fig. 6.15

$F = 10$ kN

To find To plot shear stress distribution

Moment of inertia

$$I = \frac{100 \times 140^3}{12} - \frac{\pi(80)^4}{64} = 20.856 \times 10^6 \text{ mm}^4 \qquad \text{(Fig. 6.16a)}$$

(full rectangle) (two semi-circles)

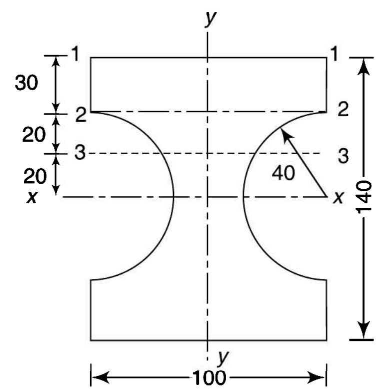


Fig. 6.15

Calculations for shear stresses

- At 1-1, $\tau = 0$
- At 2-2, $\tau = \dfrac{10\,000 \times 100 \times 30 \times 55}{20.856 \times 10^6 \times 100} = 0.79$ MPa
- At 3-3, in order to find shear stress at levels between x-x and 3-3, moment of area above that level about the neutral axis is needed.

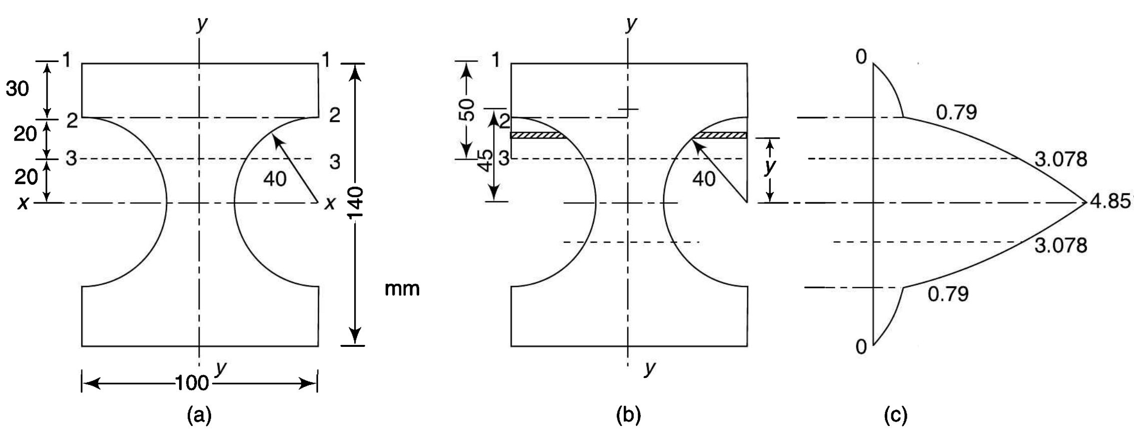


Fig. 6.16

Consider a strip of elementary height δy between 2-2 and 3-3 in the semicircular portions as shown in Fig. 6.16b.

$$\text{Width of the strip} = 2\sqrt{40^2 - y^2}$$

$$\text{Area of the strip} = 2\sqrt{40^2 - y^2}\ \delta y$$

$$\text{Moment of area about neutral axis} = 2\sqrt{40^2 - y^2} \cdot \delta y \cdot y$$

$$\text{Moment of total curved area between 2 and 3} = \int_{20}^{40} 2\sqrt{40^2 - y^2}\, y \cdot dy$$

Thus moment of area of the section above 3-3
= Moment of area of rectangle – Moment of total curved area between 2-2 and 3-3

$$A\bar{y} = 100 \times 50 \times 45 - \int_{20}^{40} 2\sqrt{40^2 - y^2}\, y \cdot dy$$

$$= 225\,000 + \int_{20}^{40} [(1600 - y^2)^{1/2} \cdot (-2y)]dy = 225\,000 + \frac{2}{3}\left[(1600 - y^2)^{3/2}\right]_{20}^{40}$$

$$= 225\,000 + \frac{2}{3} \times [0 - (1200)^{1.5}] = 197\,287 \text{ mm}^3$$

Width at 3-3, $b = 100 - 2\sqrt{40^2 - 20^2} = 30.72$ mm

$$\tau = \frac{10\,000 \times 197\,287}{20.856 \times 10^6 \times 30.72} = 3.078 \text{ MPa}$$

- At neutral axis,

$$= 245\,000 - \int_{0}^{40} [(1600 - y^2)^{1/2} . (-2y)]dy = 245\,000 + \frac{2}{3}\left[(1600 - y^2)^{3/2}\right]_{0}^{40}$$

$$= 225\,000 + \frac{2}{3} \times [0 - (1600)^{1.5}] = 202\,333 \text{ mm}^3$$

Width at neutral axis = 100 – 80 = 20 mm

$$\tau = \frac{10 \times 10^3 \times 202\,333}{20.856 \times 10^6 \times 20} = 4.851 \text{ MPa}$$

Shear stress distribution is shown in Fig. 6.16c.

Example 6.6 A beam of rectangular cross-section 160 mm wide and 300 mm deep is of 4-m span and is loaded with a central point load of 50 kN. Determine the bending and shear stresses at the top, 100 mm and 40 mm from the neutral axis of the section and at the neutral axis. Consider the bending moment at the mid cross-section of the beam. Also find the principal planes and the principal stresses at these points and plot the variations along the section.

Solution

Given A beam of rectangular cross-section as shown in Fig. 6.17a

$F = 50$ kN $\qquad L = 4$ m

To find

At the top, 100 mm and 40 mm from the neutral axis and at the neutral axis of beam:

— Bending and shear stresses

— principal planes and principal stresses

— to plot the variations along the section

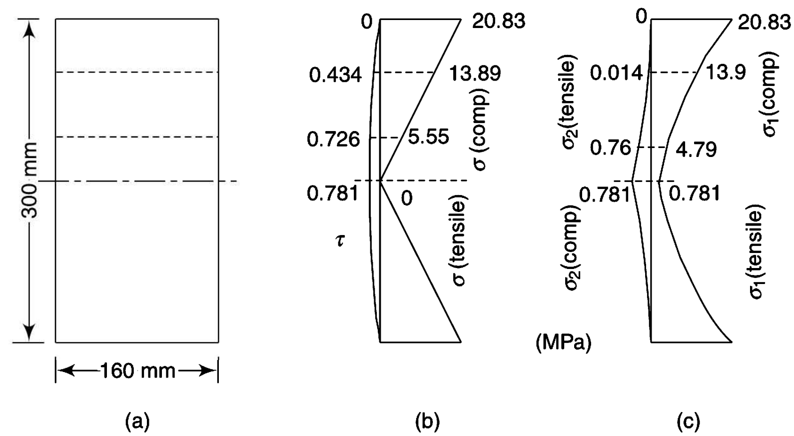


Fig. 6.17

Refer Fig. 6.17a,

$$I = \frac{160 \times 300^3}{12} = 360 \times 10^6 \text{ mm}^4$$

Bending moment at the mid cross-section of beam

Shear force = 50/2 = 25 kN

Bending moment = 25 × 2 = 50 kN.m = 50 × 10^6 N.mm

At the top

$$\text{Bending stress, } \sigma = \frac{50 \times 10^6 \times 150}{360 \times 10^6} = 20.83 \text{ MPa (Compressive)}$$

Shear stress, $\tau = 0$,

Principal stresses, $\sigma_1 = 20.83$ MPa; $\sigma_2 = 0$

At* 100 *mm from neutral axis

$$\text{Bending stress, } \sigma = 20.83 \times \frac{100}{150} = 13.89 \text{ MPa}$$

$$\text{Shear stress, } \tau = \frac{F \cdot A\bar{y}}{Ib} = \frac{25 \times 10^3 \times (160 \times 50 \times 125)}{360 \times 10^6 \times 160} = 0.434 \text{ MPa}$$

$$\text{Principal stresses, } \sigma_{1,2} = \frac{13.89}{2} \pm \sqrt{\left(\frac{13.89}{2}\right)^2 + 0.434^2} = 6.945 \pm 6.959$$

$$\sigma_1 = 13.9 \text{ MPa} \quad \text{and} \quad \sigma_2 = -0.014 \text{ MPa}$$

$$\tan 2\theta = \frac{2 \times 0.434}{13.89} = 0.0625$$

or $$\theta = 1.79° \quad \text{and} \quad 91.79°$$

At 40 mm from neutral axis

Bending stress, $\sigma = 20.83 \times \dfrac{40}{150} = 5.55$ MPa

Shear stress, $\tau = \dfrac{25 \times 10^3 \times (160 \times 110 \times 95)}{360 \times 10^6 \times 160} = 0.726$ MPa

Principal stresses, $\sigma_{1,2} = \dfrac{5.55}{2} \pm \sqrt{\left(\dfrac{5.55}{2}\right)^2 + 0.726^2} = 2.775 \pm 2.015$

$$\sigma_1 = 4.79 \text{ MPa and } \sigma_2 = -0.76 \text{ MPa}$$

$$\tan 2\theta = \frac{2 \times 0.726}{5.55} = 0.262$$

or $$\theta = 7.33° \quad \text{and} \quad 97.33°$$

At the neutral axis

Bending stress, $\sigma = 0$

Shear stress, $\tau = \dfrac{25 \times 10^3 \times (160 \times 150 \times 75)}{360 \times 10^6 \times 160} = 0.781$ MPa

Principal stresses, $\sigma_1 = \sigma_2 = 0.781$ MPa

$$\theta_1 = \theta_2 = 45°$$

At the neutral axis, there is a state of simple shear. Principal stresses are 0.781 MPa compressive along one diagonal plane and 0.781 MPa tensile along another diagonal plane.

Figure 6.17b shows the plot of bending and shear stresses along the section of the beam, whereas Fig. 6.17c is the variation of principal stresses along the section.

Example 6.7 || A 320 mm × 160 mm *I*-section joist has 20-mm thick flanges and a 15-mm thick web. At a certain cross-section it is acted upon by a bending moment of 100 kN.m and a shear force of 200 kN. Determine the principal stresses

(i) At the top
(ii) In the flanges at 140 mm from neutral axis
(iii) In the web at 140 mm from neutral axis
(iv) At the neutral axis

Plot the variations along the section.

Solution

Given A joist of I-section as shown in Fig. 6.18a

M = 100 kN.m $\qquad F$ = 200 kN

To find

— Principal stresses at top, in flanges and web at 140 mm from neutral axis and at neutral axis
— To plot the variations along the section

Refer Fig. 6.18, $I = \dfrac{160 \times 320^3}{12} - \dfrac{145 \times 280^3}{12} = 171.65 \times 10^6 \text{ mm}^4$

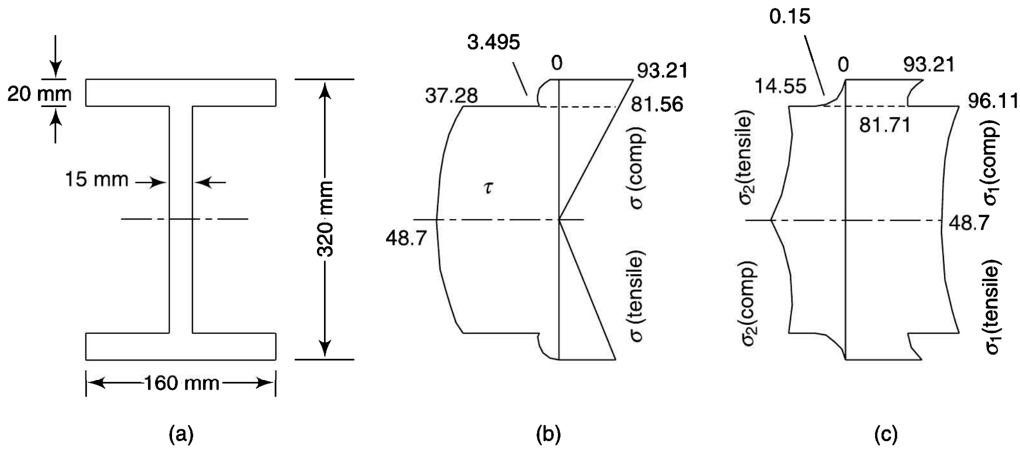


Fig. 6.18

At the top

Bending stress, $\sigma = \dfrac{100 \times 10^6 \times 160}{171.65 \times 10^6} = 93.21$ MPa

Shear stress, $\tau = 0$,

Principal stresses, $\sigma_1 = 93.21$ MPa; $\sigma_2 = 0$

In flanges at 140 mm from neutral axis

Bending stress, $\sigma = 93.21 \times \dfrac{140}{160} = 81.56$ MPa

Shear stress, $\tau = \dfrac{F}{8I}(D^2 - d^2)$ ...(Eq. 6.3)

$$= \frac{200 \times 10^3}{8 \times 171.65 \times 10^6}(320^2 - 280^2) = 3.495 \text{ MPa}$$

Principal stresses, $\sigma_{1,2} = \dfrac{81.56}{2} \pm \sqrt{\left(\dfrac{81.56}{2}\right)^2 + 3.495^2} = 40.78 \pm 40.93$

$= 81.71$ MPa and -0.15 MPa

In web at 140 mm from neutral axis

Bending stress, $\sigma = 81.56$ MPa

Shear stress, $\tau = 3.495 \times \dfrac{160}{15} = 37.28$ MPa

Principal stresses, $\sigma = \dfrac{81.56}{2} \pm \sqrt{\left(\dfrac{81.56}{2}\right)^2 + 37.28^2} = 40.78 \pm 55.33$

$= 96.11$ MPa and -14.55 MPa

At neutral axis

Bending stress, $\sigma = 0$

Shear stress, $\tau = \dfrac{200 \times 10^3 (160 \times 20 \times 150 + 140 \times 15 \times 70)}{171.65 \times 10^6 \times 15} = 48.7$ MPa

Principal stresses, σ_1 = 48.7 MPa; σ_2 = –48.7 MPa

Figure 6.18b shows the plot of bending and shear stresses along the section of the beam, whereas Fig. 6.18c is the variation of principal stresses along the section.

6.3 BUILT-UP BEAMS

It can be noted that the shear stress is caused by the difference in the bending stresses between two sections of a beam subjected to bending (Section 6.2). Also, the shear stress never occurs alone, but in pairs, i.e., complimentary longitudinal shear stress accompanies the transverse shear stress. The assumptions of homogeneity and continuity in deriving the bending equation imply that there is no slip at the interfaces of different layers of the beam. Thus, a beam should have sufficient strength to resist the longitudinal shear stresses otherwise there can be slipping of different layers of the beam at the interfaces (Fig. 6.19a). However, if slipping is prevented by some means, the bending behaviour of the beam would be as shown in Fig. 6.19b.

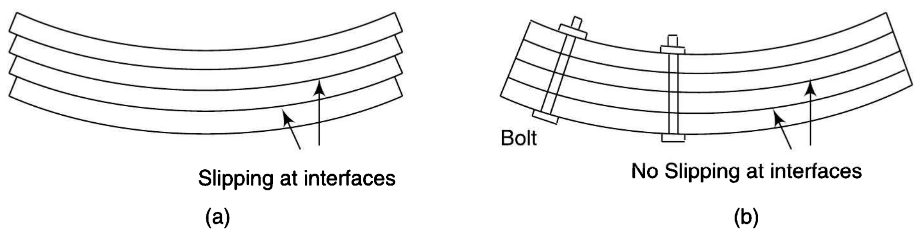


Fig. 6.19

In case of built-up steel sections, adequate connectors such as riveting, bolting or welding must be provided to resist the slip due to interfacial shear. Similarly, in case of timber sections, nailing or bolting may be provided. In steel-concrete composite beams, shear connectors such as studs or steel sections are welded to the steel beam that project into the concrete slab.

When girders are built up from plates and angles (Fig.6.20a), the load is calculated by considering the distance equal to the pitch of the rivets. The area used in a part of the cross-section is that which comes away

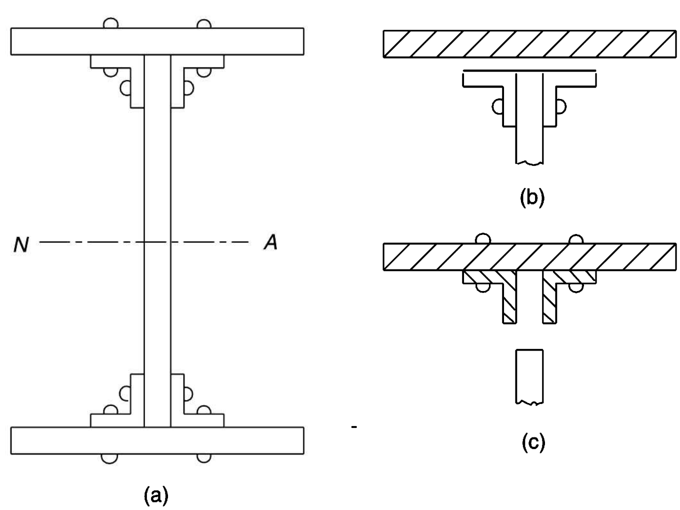


Fig. 6.20

when that particular set of rivets is removed, e.g., for rivets holding the flange to the angle sections, the area is of the flange sections as flange comes away on removing the rivets (Fig.6.20b) and in case of rivets holding the angles to the web, the area is of flange and of the angles as they are separated from the web (Fig. 6.20c).

Usually, the flange rivets are put in a zig-zag way so that they do not occur together in one cross-section.

Example 6.8 An *I*-section beam is built up of a 200 mm × 10 mm web plate with 120 mm × 20 mm flange plates secured by rivets through 40 mm × 40 mm × 6 mm angle sections as shown in Fig. 6.21. Determine the maximum uniformly distributed load which can be applied over a span of 10 m if the permissible bending stress is 90 MPa. Find also the pitch of the rivets. The rivets used are of 10-mm diameter in a zig-zag manner with permissible shear stress of 70 MPa and bearing pressure of 140 MPa.

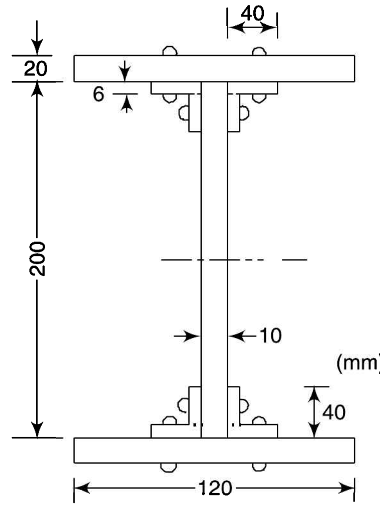


Fig. 6.21

Solution

Given A built up beam of I-section as shown in Fig. 6.21.

σ = 90 MPa $\quad\quad$ τ = 70 MPa

L = 10 m $\quad\quad$ p = 140 MPa

To find

— Maximum uniformly distributed load

— Pitch of rivets

Let the maximum uniformly distributed load be w N/m,

$$\text{Maximum bending moment, } M = \frac{wl^2}{8} = \frac{w \times 10^2}{8} = 12.5w \text{ N}\cdot\text{m}$$

Moment of inertia

Moment of inertia will consist of the following (Fig. 6.22):

- for the web, $I_w = \dfrac{10 \times 180^3}{12}$ = 4.86 × 10⁶ mm⁴ (allowing for two rivets, b = 200-20)

- for the angles,

$$I_a = 4\left[\underbrace{\frac{(40-10)\times 6^3}{12} + (40-10)\times 6 \times 97^2}_{\text{(Horizontal leg)}} + \underbrace{\frac{6\times(34-10)^3}{12} + 6\times(34-10)\times 77^2}_{\text{(Vertical leg)}}\right]$$

$$= 10.22 \times 10^6 \text{ mm}^4$$

- for the flanges, $I_f = 2\left[\dfrac{(120-10)\times 20^3}{12} + 110 \times 20 \times 110^2\right] = 26.69 \times 10^6 \text{ mm}^4$

(zig-zag riveting and thus allowing for one rivet)

Total $I = 4.86 \times 10^6 + 10.22 \times 10^6 + 26.69 \times 10^6 = 41.77 \times 10^6 \text{ mm}^4$

Calculation of uniformly distributed load

$$\text{Bending stresses, } \sigma = \frac{My}{I} \quad \text{or} \quad 90 = \frac{12.5w \times 120}{41.77 X 10^6} \quad \text{or} \quad w = 2506 \text{ N/m}$$

Calculations for pitch of flange rivets

Maximum shear force = $\dfrac{wl}{2} = \dfrac{2506 \times 10}{2}$ = 12 530 N

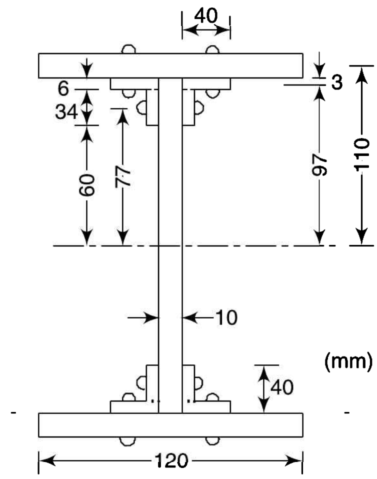


Fig. 6.22

Two rivets are in single shear,

Permissible load per pitch length = $2 \times \dfrac{\pi}{4} \times 10^2 \times 70$ = 11 000 N

Two rivets are being crushed,

Permissible load per pitch length = 2 × 10 × 6 × 140 = 16 800 N

As shear load is smaller, so permissible load = 11 000 N

$A\bar{y}$ = (120 – 10) × 20 × 110 = 242 000 mm^3

Let p be the pitch of rivets, Interfacial shear stress = $\dfrac{FA\bar{y}}{Ib}$

Interfacial shear load per pitch length = $\dfrac{FA\bar{y}}{Ib} Xbp = \dfrac{FA\bar{y}}{I} \cdot p$

$$\therefore \quad \frac{12\,530 \times 242\,000}{41.77 \times 10^6} \times p = 11\,000 \quad \text{or} \quad p = 151.5 \text{ mm}$$

Calculations for pitch of web rivets

One rivet is in double shear,

Permissible load per pitch length = $2 \times \dfrac{\pi}{4} \times 10^2 \times 70$ = 11 000 N

One rivet is being crushed,

Permissible load per pitch length = 10 × 10 × 140 = 14 000 N

[Area being crushed is the area of the web plate (10 mm) which is less than the combined area of two angles (6 + 6) mm]

As shear load is smaller, so permissible load = 11 000 N

$$A\bar{y} = 242\,000 + 2 \times 30 \times 6 \times 97 + (34 - 10) \times 6 \times 77 = 299\,096 \text{ mm}^3$$

$$\text{Thus} \quad \frac{12\,530 \times 299\,096}{41.77 \times 10^6} \times p = 11\,000 \quad \text{or} \quad p = 122.6 \text{ mm}$$

Example 6.9 || A 4-m long cantilever is fabricated from five 40-mm high and 120-mm wide timber planks fastened together by vertical bolts of 20-mm diameter. The overall cross-section of the cantilever is 120 mm × 200 mm. The bolts are provided at a spacing of 110 mm. The cantilever is loaded with a uniformly distributed load of 2.5 kN per m including its own weight. Determine the shear stresses at all the four planes of contact of the planks in a bolt located at a distance 2 m from the support.

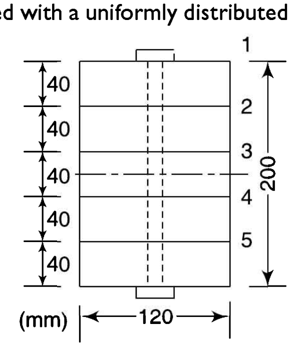


Fig. 6.23

Solution

Given A cantilever fabricated from five timber planks fastened together by vertical bolts of 20 mm diameter as shown in Fig. 6.23.

w = 2.5 kN/m $\qquad$ L = 4 m

To find Shear stresses at all four planes of contact at a section at 2 m from support

Shear force at 2 m from the support = 2500 × 2 =5000 N

Moment of inertia of the section = $\dfrac{120 \times 200^3}{12}$ = 80 × 10^6 mm^4

Area of the bolt = $\dfrac{\pi}{4}$ × 20^2 = 314.2 mm^2

***Shear stress at level* 2**

Horizontal shear stress at level 2, $\tau = \dfrac{F \cdot A\bar{y}}{Ib} = \dfrac{5000 \times 120 \times 40 \times 80}{80 \times 10^6 \times 120}$ = 0.2 MPa

Load carried by each bolt = τ × area = τ × b × spacing

Shear stress in the bolt = $\dfrac{\tau \times b \times \text{spacing}}{\text{Area of bolt}} = \dfrac{0.2 \times 120 \times 110}{314.2}$ =8.402 MPa

Shear stress at level 5 is equal to that at level 2.

***Shear stress at level* 3**

Horizontal shear stress at level 3, $\tau = \dfrac{5000 \times 120 \times 80 \times 60}{80 \times 10^6 \times 120}$ = 0.3 MPa

Shear stress in the bolt = $\dfrac{0.3 \times 120 \times 110}{314.2}$ =12.603 MPa

Shear stress at level 4 is equal to that at level 3.

Example 6.10 A composite beam consists of a timber section of 180 mm × 140 mm bonded with 10 mm × 140 mm steel plates at top and bottom. Determine the stresses in the beam when it is subjected to a shear force of 100 kN. Also find the spacing of bolts of 12-mm diameter for the shear connection between the flitches and the timber beam. Allowable shear stress in steel is 100 MPa. The Young's modulus of steel is 210 GPa and of timber, 15 GPa.

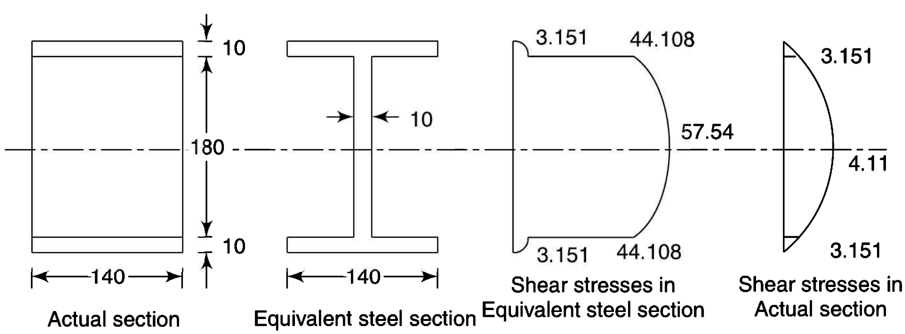


Fig. 6.24

Solution

Given A composite beam made of a timber section and steel plates at top and bottom as shown in Fig. 6.24.

σ = 100 MPa F = 100 kN

E_s = 210 GPa E_t = 15 GPa

To find

— Stresses in beam

— spacing of bolts

$$\text{Modular ratio} = \frac{E_s}{E_t} = \frac{210}{15} = 14$$

Equivalent steel section

Transforming the composite beam into an equivalent steel section,

$$I_s = \frac{10 \times 180^3}{12} + 2\left[\frac{140 \times 10^3}{12} + 140 \times 10 \times 95^2\right] = 30.153 \times 10^6 \text{ mm}^4$$

Shear stress in the equivalent steel section,

- At steel plate timber junction (in the flanges)

$$\tau = F \cdot \frac{A\bar{y}}{Iz} = 100 \times 10^3 \times \frac{140 \times 10 \times 95}{30.153 \times 10^6 \times 140} = 3.151 \text{ MPa}$$

- At steel plate timber junction (in the web) = 3.151 × 14 = 44.108 MPa
- At neutral axis, $\tau = \dfrac{100 \times 10^3}{30.153 \times 10^6 \times 10}(140 \times 10 \times 95 + 10 \times 90 \times 45) = 57.45$ MPa

Shear stress in the actual section at neutral axis = 57.54/14 = 4.11 MPa

Calculations for spacing of bolts

Interfacial stress = 3.151 MPa

Interfacial force, $F = 3.151\ Xbp$ ...(p is the pitch)

Strength of bolt in single shear = $\dfrac{\pi}{4} \times 12^2 \times 100 = 11\ 310$ N

Thus 3.151 × 140 × p or $p = 25.64$ mm

Example 6.11 A timber beam simply supported at the ends has a span of 3 m and is subjected to a load of 800 N at the centre. The 40-mm wide and 120-mm deep beam is of rectangular section. It is reinforced by screwing a 6-mm thick and 40-mm wide steel plate at the bottom. The screws are of 5-mm diameter and pitched 60 mm apart. Determine the maximum stresses in the steel plate and the timber. Also find the maximum shear stress in the screws. Take $E_s/E_t = 16$.

Solution

Given A simply supported timber beam reinforced by a steel plate with screws as shown in Fig. 6.25. Screws diameter 5 mm and pitched at 60 mm apart.

$L = 3$ m $\qquad F = 800$ N

$E_s/E_t = 16$

To find

— Maximum stresses in steel and timber

— Maximum shear stress in screws

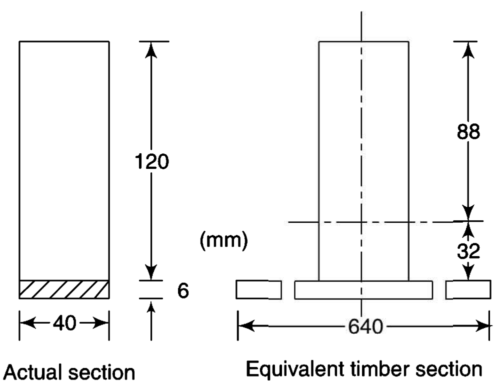


Fig. 6.25

Equivalent timber section

The equivalent timber section has been shown in the figure.

Width at the bottom is 40 × 16 = 640 mm.

$$\text{Taking moments about top of section, } \bar{y} = \frac{40 \times 120 \times 60 + 640 \times 6 \times 123}{40 \times 120 + 640 \times 6} = 88 \text{ mm}$$

$$I_{xx} = \frac{40 \times 88^3 + (640 \times 38^3 - 600 \times 32^3)}{3} = 14.239 \times 10^6 \text{ mm}^4$$

Maximum bending moment, $M_{max} = \frac{Wl}{4} = \frac{800 \times 3}{4} = 600 \text{ N·m} = 600 \times 10^3 \text{ N·mm}$

Bending stresses

Maximum compressive stress in the timber,

$$\sigma = \frac{600 \times 10^3 \times 88}{14.239 \times 10^6} = 3.708 \text{ MPa}$$

Maximum tensile stress in the timber $\sigma = \frac{600 \times 10^3 \times 32}{14.239 \times 10^6} = 1.348 \text{ MPa}$

Maximum tensile stress in the steel $\sigma = 1.348 \times \frac{38}{32} \times 16 = 25.617 \text{ MPa}$

Shear stress in screws

Area of cross-section of the screw $= \frac{\pi}{4} \times 5^2 = 19.6 \text{ mm}^2$

Pitch, $p = 60$ mm

Maximum horizontal force $= \tau \times b \times p = 60\, b \cdot \tau$

$$= 60b \cdot \frac{FA\bar{y}}{Ib} = \frac{60F}{I} A\bar{y} = \frac{60 \times (800/2)}{14.239 \times 10^6} \times (640 \times 40) \times (38 - 3) = 226.5 \text{ N}$$

Maximum shear stress in the screws $= \frac{226.5}{19.6} = 11.56 \text{ MPa}$

6.4 SHEAR STRESS IN THIN SECTIONS

Consider a thin channel section under transverse loading (Fig. 6.26a). Shear stress distribution along thickness t of the channel section may be assumed constant due to assumption of thin section. To find the value of shear stress on an element, consider a small block $QRST$ of length δx of the beam cut vertically and consider its equilibrium as was done in Section 6.1 (Fig. 6.26b).

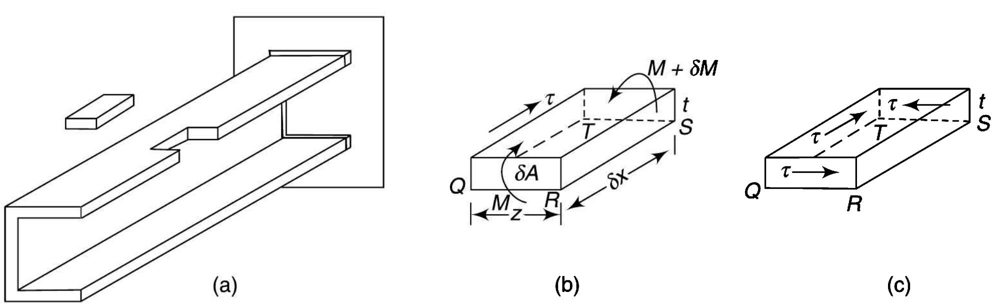


Fig. 6.26

∴ net force applied on the elemental area δA towards left

$$= \frac{(M + \delta M)y}{I} \cdot \delta A - \frac{My}{I} \cdot \delta A = \frac{\delta M \cdot y}{I} \cdot \delta A$$

Total force applied on the gross area of the block $= \int \frac{\delta M \cdot y}{I} \cdot \delta A$

This force on the block tends to slide the block towards left which is resisted by the shear force due to complimentary shear stress τ in the vertical plane at QT.
Net shear force at the surface = shear stress × area = $\tau \cdot t \cdot \delta x$

Equating the two forces, $\tau \cdot t \cdot \delta x = \int \left(\frac{\delta M \cdot y}{I}\right) \cdot dA$

or $$\tau = \frac{\delta M}{\delta x \cdot t \cdot I} \int y \cdot dA = \left(\frac{\delta M}{\delta x}\right) \frac{1}{tI} \cdot A\bar{y}$$

or $$\tau = F \cdot \frac{A\bar{y}}{tI} \quad \text{or} \quad \tau = F \cdot \frac{tz\bar{y}}{tI} = F \cdot \frac{z\bar{y}}{I} \tag{6.18}$$

This indicates that the magnitude of the complimentary shear stress varies linearly as the value of z i.e. zero at the tip of the flange and maximum at the point of joining with the web. As the horizontal outer surfaces of the flange are free surfaces, no shear stress acts on these surfaces. Thus, shear stress in the transverse planes, the magnitude of which is equal to the complimentary shear stress, will also vary as the distance from the tip of the flange. Figure 6.26c shows the shear and complementary shear stress on the vertical planes on elements at R or S.

- In the lower leg of the channel, the direction of shear stress is opposite to that in the upper leg. This is essential for force balance in the horizontal direction (Fig. 6.27a).
- In case of I-sections, the direction of shear stress in the two legs of the upper flange will be opposite to the each other. The distribution in the lower legs will be as shown in Fig. 6.27b.
- In case of channels and I-sections with thin flanges, the vertical stress values are quite small and usually are ignored. However, the horizontal components are significant and are taken into account.

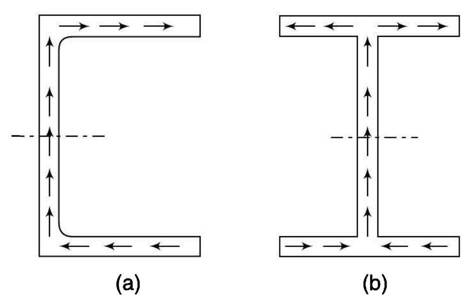


Fig. 6.27

Thin Circular Tube

If the thickness of a circular tube is small then the fact that the shear stress follows the direction of boundary can be used to find the same.

Figure 6.28 shows two transverse sections of a thin circular beam at a distance δx apart. Let A and B be two symmetrically placed positions at angle θ from the vertical. Let the shear stress at these positions be τ. Let σ and $\sigma + \delta\sigma$ be the bending stress on two transverse sections on an elemental area δA at an angle φ from the vertical.

Then force applied on the left end of the elemental area $= \sigma \cdot \delta A = \frac{My}{I} \cdot \delta A$

Force applied on the right end of the elemental area $= (\sigma + \delta\sigma) \cdot \delta A = \frac{(M + \delta M)y}{I} \cdot \delta A$

$\therefore$ net force applied on the elemental area δA towards left

$$= \frac{(M + \delta M)y}{I} \cdot \delta A - \frac{My}{I} \cdot \delta A = \frac{\delta M \cdot y}{I} \cdot \delta A$$

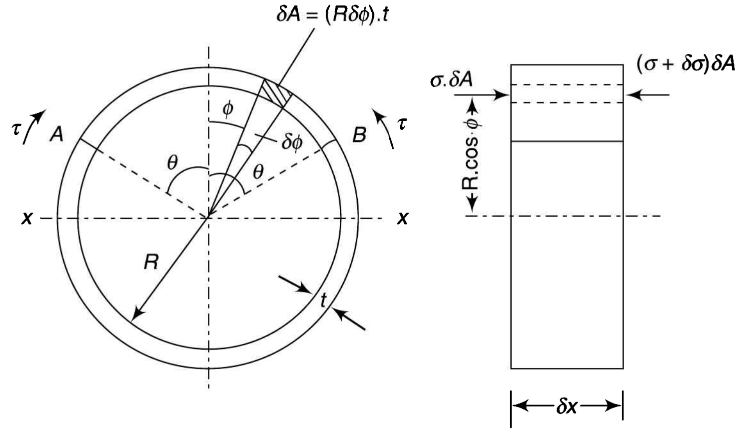


Fig. 6.28

Total force applied on the total area of the block = $\int \frac{\delta M \cdot y}{I} \cdot \delta A$

This force on the block tends to slide the block above the area towards the left which is resisted by the shear force due to complimentary shear stress τ in the horizontal plane.

Net shear force at the surface = shear stress × area = $2\tau \cdot t \cdot \delta x$ (t is the thickness)

Equating the two forces = $2\tau \cdot t \cdot \delta x = \int \left(\frac{\delta M \cdot y}{I}\right) \cdot dA$

or $$\tau = \frac{\delta M}{2\delta x \cdot t \cdot I} \int y \cdot dA = \frac{\delta M}{2\delta x \cdot t \cdot I} \int y \cdot (R \cdot d\varphi \cdot t) = \frac{F}{2I} \int (R \cdot d\varphi) y$$

In this expression I is the moment of inertia of the total area about the neutral axis x-x which can be found as follows:

$$I = \frac{1}{2} \times \text{Polar moment of inertia}$$

$$= \frac{1}{2} \times \text{Area} \times (\text{mean radius})^2 = \frac{1}{2} \cdot 2\pi Rt \cdot R^2 = \pi t R^3$$

Thus, $$\tau = \frac{F}{2\pi t R^3} \int_{-\theta}^{\theta} (R \cdot d\varphi)(R \cos \varphi) = \frac{F}{2\pi Rt} \int_{-\theta}^{\theta} \cos \varphi \cdot d\varphi$$

$$= \frac{F}{2\pi Rt} (\sin \varphi)_{-\theta}^{\theta} = \frac{F \sin \theta}{\pi Rt}$$

At neutral axis, $\tau = \dfrac{F}{\pi Rt}$ ... ($\because \theta = 90°$)

Mean value of shear stress = $\dfrac{F}{\text{Total area}} = \dfrac{F}{2\pi Rt}$ (6.19)

i.e., shear stress at neutral axis is twice the mean value.

Example 6.12 A beam section as shown in Fig. 6.29 has a constant thickness of 2 mm. The shear force at the section is 8 kN. Sketch the shear stress distribution in the section of the beam.

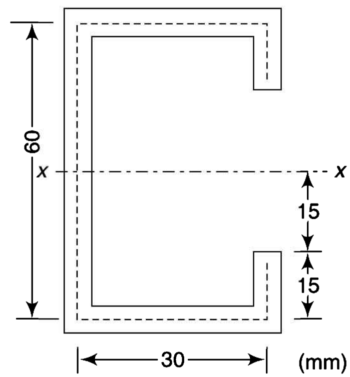


Fig. 6.29

Solution

Given A beam section as shown in Fig. 6.29

$F = 8$ kN $\qquad t = 2$ mm

To find Shear stress distribution

$$I_x = \frac{2\times 60^3}{12} + 2\left(\frac{30\times 2^2}{12} + 30\times 2\times 30^2\right) + 2\left(\frac{2\times 15^3}{12} + 2\times 15\times 22.5^2\right) = 175\,540 \text{ mm}^4$$

Shear stresses

$$\tau_1 = F\cdot\frac{A\bar{y}}{zI} = 0$$

$$\tau_2 = \frac{8\times 10^3(2\times 15\times 22.5)}{175\,540\times 2} = 15.38 \text{ MPa}$$

$$\tau_3 = 15.38 + \frac{8\times 10^3(30\times 2\times 30)}{175\,540\times 2} = 56.4 \text{ MPa}$$

$$\tau_4 = 56.4 + \frac{8\times 10^3(30\times 2\times 15)}{175\,540\times 2} = 76.9 \text{ MPa}$$

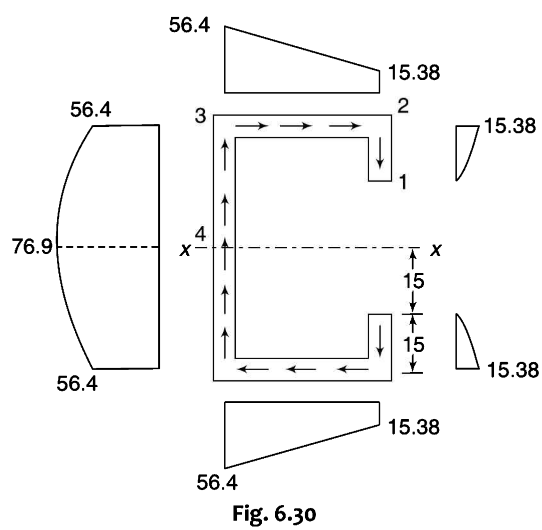


Fig. 6.30

Shear stress distribution has been shown in Fig. 6.30.

6.5 SHEAR CENTRE

So far, the analysis for the shear stresses due to transverse loading was limited to members having a vertical plane of symmetry. The load is applied in that plane and the member is observed to bend in the plane of loading. Consider the case of a channel with its web in the horizontal position (Fig. 6.31). When it is loaded through its vertical line of symmetry, the vertical shear force is taken by two flanges only and web is considered to take only a marginal part of vertical shear force which is ignored. Thus, vertical force in each flange is $F/2$. The transverse shear force in flanges will be negligible and in the web it is symmetrically distributed about the vertical line of symmetry in opposite directions.

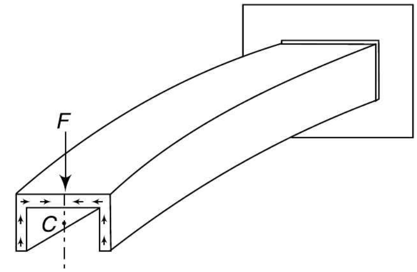


Fig 6.31

However, there can be transverse loading on thin-walled members which do not have vertical plane of symmetry. In Fig. 6.32a the channel is turned through 90°. Now, though the line of action of the loading may still be passing through the centroid of the end section, the member is observed to bend and twist under the loading because now the vertical shear force is taken by the web only and the vertical shear force taken by flanges is negligible (Fig. 6.32b). The transverse shear force in the web is negligible but in the flanges they form a couple formed by horizontal forces in the upper and lower legs of the channel. In such cases, the channel section will twist unless the line of action of load is displaced as shown in Fig. 6.32c.

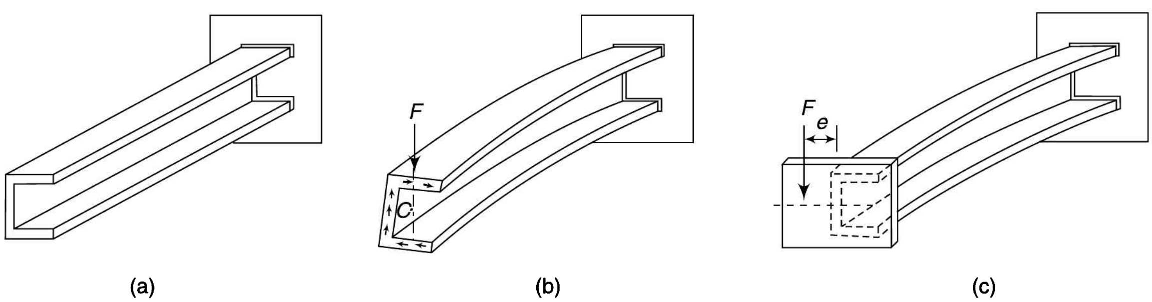


Fig. 6.32

Shear centre is the point in or outside a section through which the shear force applied produces no torsion or twist of the member.

- For a beam with two axes of symmetry, the shear centre coincides with the centroid.
- For sections having one axis of symmetry, shear centre does not coincide with the centroid, though it lies on the axis of symmetry.

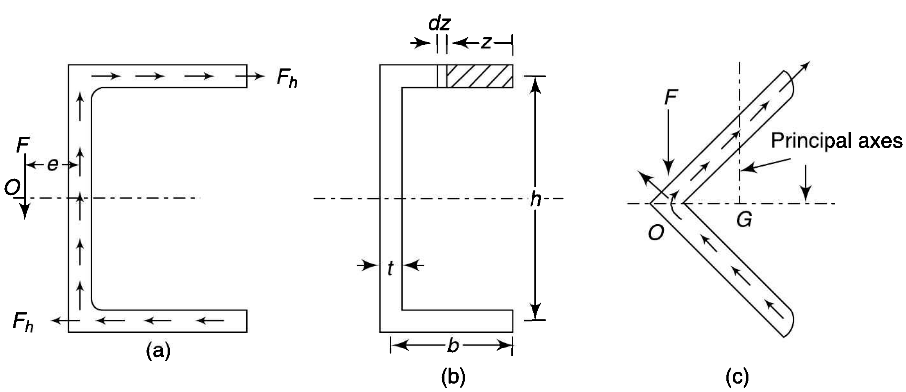


Fig. 6.33

Consider the channel section shown in Fig. 6.33a. Assuming uniform thickness t of the web and flanges,

Moment of inertia, $$I_x = \frac{t \times h^3}{12} + 2\left[\frac{b \times t^3}{12} + bt\left(\frac{h}{2}\right)^2\right] \approx \frac{th^2}{12}(h + 6b)$$

(Neglecting moment of inertia of flanges about their axes, i.e., neglecting the first term in the bracket)

Consider an elementary length dz of the flange at a distance z from the tip (Fig. 6.33b),

Shear stress in the elementary length, $\tau = F \cdot \frac{(tz)\bar{y}}{tI} \cdot = \cdot F \cdot \frac{(tz)}{tI} \cdot \frac{h}{2} = \frac{Fzh}{2I}$

Shear force in the elementary length $= \tau \times$ area $= \frac{Fzh}{2I} \cdot (t \cdot dz) = \frac{Fht}{2I} \cdot (z \cdot dz)$

Total force in each flange, $$F_f = \frac{Fht}{2I}\int_0^b z \cdot dz = \frac{Fht}{2I}\left[\frac{z^2}{2}\right]_0^b = \frac{Fhtb^2}{4I} \qquad (6.20)$$

Now, if the force F acts through the vertical axis of the web, no moments result from the vertical forces whereas clockwise moments give rise to a clockwise couple which can twist the cross-section of the channel. However, if the line of application of the vertical force F is displaced to the left at a distance e from the vertical axis of the web, the clockwise couple due to force in the flanges can be made to balance with the counter-clockwise couple to external force F and vertical force in the web, i.e.,

$$\frac{Fhtb^2}{4I} \times h \quad \text{or} \quad e = \frac{h^2tb^2}{4I} = \frac{h^2tb^2/4}{th^2(h+6b)/12} = \frac{3b^2}{h+6b}$$

In case of an equal angle section, the principal axes are as shown in Fig. 6.33c. As the shear forces due to shear stresses in the two legs as well as the external forces intersect at a common point O, no resultant moments are obtained of any force and thus the point O is the shear centre.

Example 6.13 Determine the position of the shear centre for an 80 mm by 40 mm outside by 5-mm thick channel section.

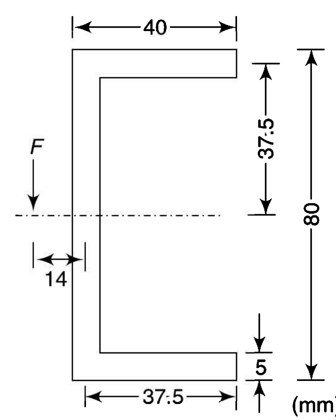


Fig. 6.34

Solution

Given A channel section as shown in Fig. 6.34

To find To locate shear centre

Refer Fig. 6.34,

Moment of inertia about x-axis

$$I_x = \frac{5 \times 80^3}{12} + 2\left[\frac{35 \times 5^3}{12} + 35 \times 5 \times 37.5^2\right] = 706\ 250 \text{ mm}^4$$

Total shear force in the flanges

$$F_f = \frac{Fhtb^2}{4I} = \frac{F \times 75 \times 5 \times 37.5^2}{4 \times 706\ 250} = 0.186\ 68\ F \qquad \text{...(Eq. 6.20)}$$

Calculation for location of shear centre

If e is the distance of the shear centre from the centre line of the web, then for equilibrium,

$$F \cdot e = F_f \times 2\bar{y} \quad \text{or} \quad F \cdot e = 0.18668\text{F} \times 2 \times 37.5 \quad \text{or} \quad e = 14 \text{ mm}$$

or by the relation, $e = \dfrac{3b^2}{h+6b} = \dfrac{3(37.5)^2}{75+6(37.5)} = 14.06$ mm

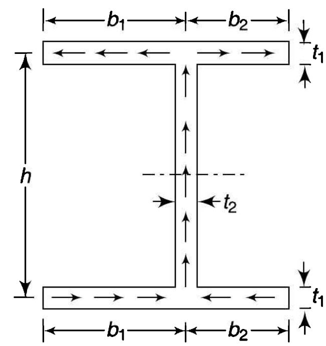


Fig. 6.35

Example 6.14 Locate the shear centre for the section shown in Fig. 6.35.

Solution

Given An *I*- section as shown in Fig. 6.35
To find To locate shear centre

Refer Fig. 6.36,
Moment of inertia about x-axis

$$I_x = 2\left[\frac{t_2h^3}{12} + (b_1+b_2)\frac{t_1^3}{12} + (b_1+b_2)t_1\left(\frac{h}{2}\right)^2\right]$$

Consider an elementary length dz of the left part of flange at a distance z from the tip.
Shear force in elementary length

Shear stress in the elementary length, $\tau = F \times \dfrac{(tz)\bar{y}}{tI} = F \times \dfrac{(t_1z)}{t_1I} \times \dfrac{h}{2} = \dfrac{Fzh}{2I}$

Shear force in the elementary length = $\tau \times$ area = $\dfrac{Fzh}{2I} \times (t_1 \times dz) = \dfrac{Fht_1}{2I} \cdot (z \cdot dz)$

Total force in left part of flange, $F_f = \dfrac{Fht_1}{2I}\displaystyle\int_0^{b_1} z \cdot dz = \dfrac{Fht_1}{2I}\left[\dfrac{z^2}{2}\right]_0^{b_1} = \dfrac{Fht_1b_1^2}{4I}$

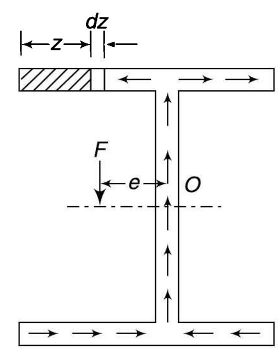


Fig. 6.36

Similarly, Total force in right part of flange, $F_f = \dfrac{Fht_1b_2^2}{4I}$

Calculation for location of shear centre
Taking moments of the shear forces about the web centre O,

$$F \cdot e = \frac{Fht_1b_1^2}{4I1} \times h - \frac{Fht_1b_2^2}{4I} \times h = \frac{Fht_1h}{4I}(b_1^2 - b_2^2)$$

$$e = \frac{h^2t_1}{4I}(b_1^2 - b_2^2)$$

Example 6.15 Locate the shear centre for the section shown in Fig. 6.37.

Solution

Given An *I*- section as shown in Fig. 6.37
To find To locate shear centre

Moment of inertia about x-axis

$$I_x = 2\left[\frac{10 \times 380^3}{12} + (100+60)\frac{20^3}{12} + (100+60) \times 20 \times 200^2\right]$$

$$= 2(45.73 + 0.107 + 128) \times 10^6 = 173.84 \times 10^6 \text{ mm}^4$$

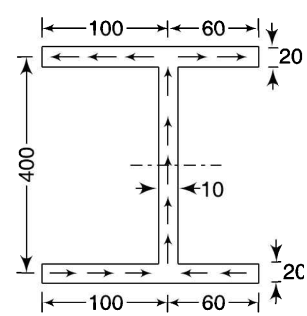


Fig. 6.37

Calculation for location of shear centre

Using the result obtained in the previous example,

$$e = \frac{h^2 t_1}{4I}(b_1^2 - b_2^2) = \frac{400^2 \times 20}{4 \times 173.84 \times 10^6}(100^2 - 60^2) = 29.45 \text{ mm}$$

Example 6.16 || Locate the shear centre for the section shown in Fig. 6.38.

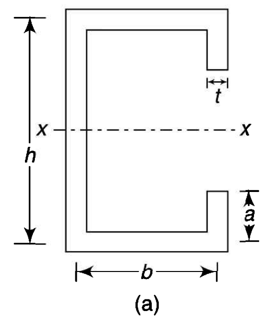


Fig. 6.38

Solution

Given A channel section as shown in Fig. 6.38

To find To locate shear centre

Consider an elementary length dy of the vertical leg of the flange at a distance y from the tip (Fig. 6.39a).

Force in the vertical leg of flanges

Shear stress in the elementary length,

$$\tau = F \cdot \frac{(ty)\bar{y}}{tI} = F \cdot \frac{(ty)}{tI} \cdot \left(\frac{h}{2} - a + \frac{y}{2}\right) = \frac{Fy}{2I} \cdot (h - 2a + y)$$

Shear force in the elementary length = $\tau \times$ area = $\dfrac{Fy}{2I} \cdot (h - 2a + y) \cdot t\,dy$

Total force in the vertical leg of each flange,

$$F_1 = \frac{Ft}{2I}\int_0^a (hy - 2ay + y^2)\,dy = \frac{Ft}{2I}\left[h\frac{y^2}{2} - 2a\frac{y^2}{2} + \frac{y^3}{3}\right]_0^a = \frac{Fta^2}{12I}(3h - 4a)$$

Force in the horizontal leg of flanges

Consider an elementary length dz of the horizontal leg of the flange at a distance z from the right end (Fig. 6.39b).

Shear stress in the elementary length, $\tau = \dfrac{F}{tI}\left[at\left(\dfrac{h}{2} - \dfrac{a}{2}\right) + tz \cdot \dfrac{h}{2}\right]$

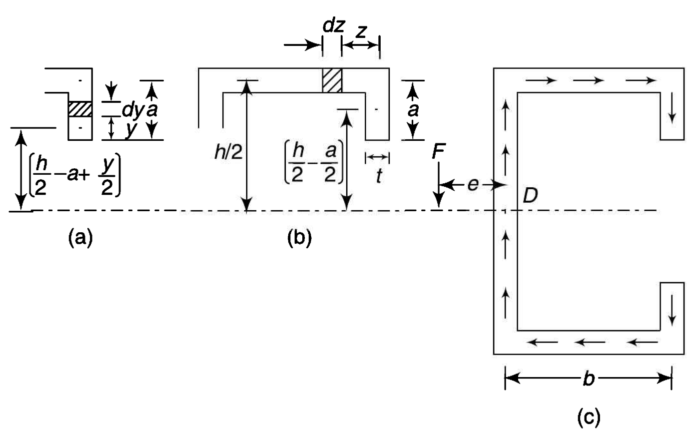


Fig. 6.39

Shear force in the elementary length = $\tau \times$ area = $\dfrac{F}{tI}\left[at\left(\dfrac{h}{2}-\dfrac{a}{2}\right)+tz\cdot\dfrac{h}{2}\right](t\cdot dz)$

Total force in horizontal leg of each flange,

$$F_2 = \frac{F}{tI}\int\left[at\left(\frac{h}{2}-\frac{a}{2}\right)+tz\cdot\frac{h}{2}\right]\cdot tdz = \frac{Ft}{2I}\int_0^b a((h-a)+hz)dz$$

$$= \frac{Ft}{2I}\left[a(h-a)\cdot z + h\frac{z^2}{2}\right]_0^b = \frac{Ftb}{4I}[2a(h-a)+hb]$$

Calculation for location of shear centre

Taking moments about the point D,

$$F\cdot e = 2F_1\cdot b + 2F_2\cdot\frac{h}{2} = 2F_1\cdot b + F_2\cdot h = 2\frac{Fta^2}{12I}(3h-4a)\cdot b + \frac{Ftb}{4I}[2a(h-a)+hb]h$$

$$e = \frac{bt}{12I}(6ha^2 - 8a^3 + 6ah^2 - 6ha^2 + 3h^2b)$$

$$= \frac{bt}{12I}(6ah^2 - 8a^3 + 3h^2b)$$

Moment of inertia

$$I_x = 2\left[\frac{ta^3}{12}+ta\left(\frac{h}{2}-\frac{a}{2}\right)^2\right]+2\left[\frac{bt^3}{12}+bt\left(\frac{h}{2}\right)^2\right]+\frac{th^3}{12}$$

$$= t\left[\frac{a^3}{6}+\frac{a}{2}(h^2-2ah+a^2)+\frac{bt^2}{6}+\frac{bh^2}{2}+\frac{h^2}{12}\right]$$

$$= \frac{t}{12}[8a^3+6ah^2-12a^2h+2bt^2+6bh^2+h^3]$$

Shear centre

$$e = \frac{b(6ah^2-8a^3+3h^2b)}{(8a^3+6ah^2-12a^2h+2bt^2+6bh^2+h^3)} \qquad \text{(Fig. 6.39c)}$$

If $a = 0$, $\quad e = \dfrac{b(3h^2b)}{(2bt^2+6bh^2+h^3)}$

and neglecting moment of inertia of flanges about their axes,

$$e = \frac{3b^2}{6b+h} \text{ as obtained earlier.}$$

Example 6.17 Locate the shear centre for the section shown in Fig. 6.40

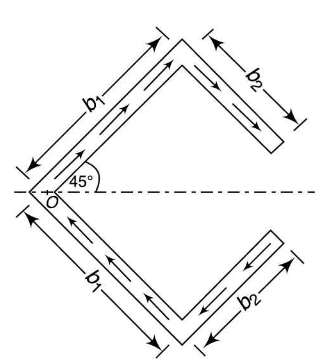


Fig. 6.40

Solution

Given A section as shown in Fig. 6.40

To find To locate shear centre

Refer Figs. 6.41a and b.

Let u and v be axes along the length of the flanges as shown in Fig. 6.41b.

I_{u1} and I_{v1} are the moment of inertia of rectangular legs about the axes u_1 and v_1 respectively.

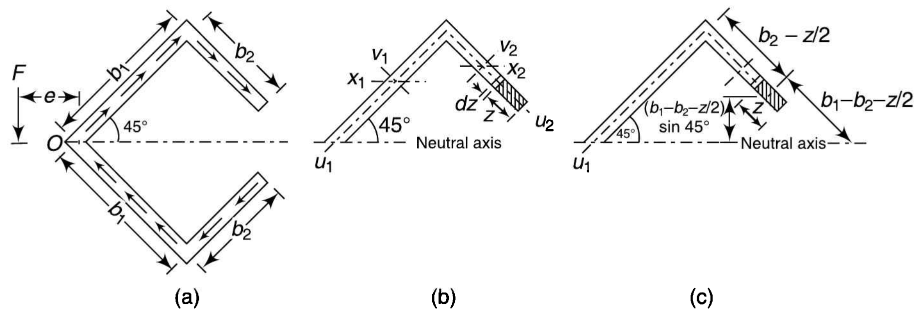


Fig. 6.41

Moment of inertia of length b_1 about axis x_1

$$I_{x1} = I_{u1}\cos^2 45° + I_{v1}\sin^2 45° = \frac{I_{u1}+I_{v1}}{2} = \frac{1}{2}\left(\frac{b_1 t^3}{12} + \frac{t b_1^3}{12}\right) = \frac{b_1 t}{24}(t^2 + b_1^2)$$

Moment of inertia of length b_1 about neutral axis

$$I_{na1} = I_{x1} + b_1 t\left(\frac{b_1}{2}\sin 45°\right)^2 = \frac{b_1 t}{24}(t^2 + b_1^2) + b_1 t\left(\frac{b_1}{2}\sin 45°\right)^2 = \frac{b_1 t}{24}(t^2 + 4b_1^2)$$

Moment of inertia of length b_2 about axis x_2

$$I_{x2} = I_{u2}\cos^2 45° + I_{v2}\sin^2 45° = \frac{I_{u2}+I_{v2}}{2} = \frac{1}{2}\left(\frac{b_2 t^3}{12} + \frac{t b_2^3}{12}\right) = \frac{b_2 t}{24}(t^2 + b_2^2)$$

Moment of inertia of length b_2 about neutral axis

$$I_{na2} = I_{x2} + b_2 t\left(b_1 \sin 45° - \frac{b_2}{2}\sin 45°\right)^2 = \frac{b_2 t}{24}(t^2 + b_2^2) + \frac{b_2 t}{8}(2b_1 - b_2)^2$$

$$= \frac{b_2 t}{24}(t^2 + b_2^2) + \frac{b_2 t}{8}(4b_1^2 - 4b_1 b_2 + b_2^2) = \frac{b_2 t}{24}(t^2 + 12b_1^2 + 4b_2^2 - 12 b_1 b_2)$$

Total moment of inertia about neutral axis

Total moment of inertia of the section = $2I_{na1} + 2I_{na2}$

$$I_{na} = \frac{b_1 t}{12}(t^2 + 4b_1^2) + \frac{b_2 t}{12}(t^2 + 12b_1^2 + 4b_2^2 - 12 b_1 b_2) \qquad \text{(i)}$$

Shear force in the short leg of flange

Consider an elementary length dz of the short leg of the section at a distance z from the tip (Fig. 6.41c),

Shear stress in the elementary length,

$$\tau = F\cdot\frac{(tz)\bar{y}}{tI} = F\cdot\frac{z}{I}\left[b_1 - \left(b_2 - \frac{z}{2}\right)\right]\sin 45° = \frac{Fz}{2\sqrt{2}\cdot I}\cdot(2b_1 - 2b_2 + z)$$

Shear force in the elementary length $= \tau \times \text{area} = \dfrac{Fz}{2\sqrt{2} \times I} \cdot (2b_1 - 2b_2 + z) \cdot tdz$

Total shear force in the short leg,

$$F_1 = \frac{Ft}{2\sqrt{2} \cdot I} \int_0^{b_2} (2b_1 z - 2b_2 z + z^2) \cdot dz = \frac{Ft}{2\sqrt{2} \cdot I} \left[b_1 b_2^2 - b_2^3 + \frac{b_2^3}{3} \right]_0^{b_2} = \frac{Ftb_2^2}{6\sqrt{2}.I}(3b_1 - 2b_2)$$

There is no need to find the shear force in the long leg of the flange as the line of force passes through the point O and moments are zero.

Calculation for location of shear centre

Taking moments about the point O,

$$F \cdot e = 2\left[\frac{Ftb_2^2}{6\sqrt{2} \cdot I}(3b_1 - 2b_2) \cdot b_1 \right]$$

$$e = 2\left[\frac{tb_1 b_2^2}{3\sqrt{2} \cdot I}(3b_1 - 2b_2) \right]$$

Example 6.18 Locate the shear centre for the arc of a circle.

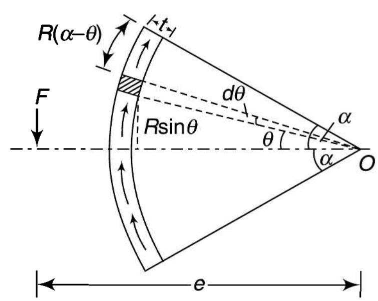


Fig 6.42

Solution

Given Arc of a circle

To find To locate shear centre

Consider an elementary length $R \cdot d\theta$ of the arc of the circle at an angular distance θ from the centre line of the sector (Fig. 6.42),

Shear force in elementary length

Shear stress in the elementary length,

$$\tau = F \cdot \frac{[R(\alpha - \theta)t]\bar{y}}{tI} = \frac{F}{I}[R(\alpha - \theta)](R\sin\theta)$$

Shear force in the elementary length $= \tau \times \text{area} = \dfrac{F}{I}[R(\alpha - \theta)](R\sin\theta)(t \cdot R \cdot d\theta)$

$$= \frac{FtR^3}{I} \cdot (\alpha - \theta)\sin\theta \cdot d\theta$$

Total force

Total force in the arc of the circle,

$$F_1 = \frac{FtR^3}{I} \int_{-\alpha}^{+\alpha} (\alpha - \theta)\sin\theta \cdot d\theta = \frac{FtR^3}{I}\left[\{-(\alpha - \theta)\cos\theta\}_{-\alpha}^{+\alpha} - \int_{-\alpha}^{+\alpha} \cos\theta \cdot d\theta \right]$$

$$= \frac{FtR^3}{I}[2\alpha\cos\alpha - 2\sin\alpha] = \frac{2FtR^3}{I}(\alpha\cos\alpha - \sin\alpha)$$

The expression inside the bracket is negative, which will give a value of e on the other side of O. Taking a positive value, $F_1 = \dfrac{2FtR^3}{I}(\sin\theta - \alpha\cos\alpha)$

Calculation for location of shear centre

Taking moments about the point O,

$$FXe = \left[\frac{2FtR^3}{I}(\sin\alpha - \alpha\cos\alpha)\right]R$$

Thus
$$e = \frac{2tR^4}{I}(\sin\alpha - \alpha\cos\alpha)$$

$$I_x = \int \text{shaded area} \times y^2 = \int_{-\alpha}^{+\alpha}(t\cdot R\cdot d\theta)(R\sin\theta)^2 = R^3t\int_{-\alpha}^{+\alpha}\sin^2\theta\cdot d\theta = R^3t\int_{-\alpha}^{+\alpha}\frac{1-\cos 2\theta}{2}$$

$$= R^3t\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right)_{-\alpha}^{+\alpha} = R^3t\left(\alpha - \frac{\sin 2\alpha}{2}\right) = R^3t(\alpha - \sin\alpha\cos\alpha)$$

$$\therefore \quad e = \frac{2tR^4}{R^3t(\alpha - \sin\alpha\cos\alpha)}(\sin\alpha - \alpha\cos\alpha) = \frac{2R(\sin\alpha - \alpha\cos\alpha)}{(\alpha - \sin\alpha\cos\alpha)}$$

- If $2\alpha = \pi$, i.e., for a semi-circular arc, $e = \dfrac{4R}{\pi}$ (Fig. 6.43a)
- If there is a slit in a cylindrical tube, $2\alpha = 2\pi$ and $e = 2R$ (Fig. 6.43b)

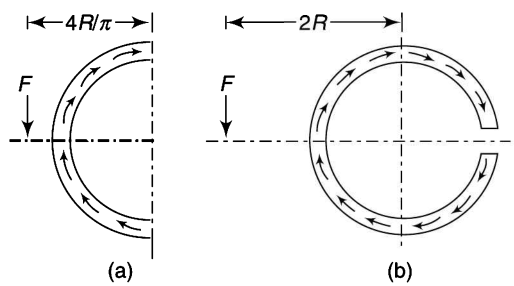


Fig. 6.43

Summary

1. Shear force in a beam at any cross-section sets up shear stress on transverse sections, the magnitude of which varies across the section which is given by $\tau = F\cdot A\bar{y}/zI$
2. For a rectangular section, $\tau_{max} = 1.5\ \tau_{mean}$
3. For a circular section, $\tau_{max} = (4/3)\ \tau_{mean}$
4. For a thin circular tube, $\tau_{max} = 2\ \tau_{mean}$
5. For a square with a diagonal horizontal, $\tau_{max} = (9/8)\ \tau_{mean}$
6. For a triangular section, $\tau_{max} = 1.5\ \tau_{mean}$
7. In built-up steel sections adequate connectors such as riveting, bolting or welding are provided to resist the slip due to interfacial shear.
8. *Shear centre* is the point in or outside a section through which the shear force applied produces no torsion or twist of the member.
9. For a beam with two axes of symmetry, the shear centre coincides with the centroid.

Objective Type Questions

1. Shear stress at a distance y from neutral axis of a cross-section is
   (a) $F \cdot \frac{z\bar{y}}{AI}$ (b) $F \cdot \frac{A\bar{y}}{zI}$ (c) $z \cdot \frac{FA}{I\bar{y}}$ (d) $F \cdot \frac{zI}{A\bar{y}}$
2. In a beam of I-section, the maximum shear stress is carried by the
   (a) web (b) upper flange (c) lower flange
3. Ratio of maximum to average shear stress in a rectangular section is
   (a) 1.2 (b) 1.5 (c) 2 (d) 2.5
4. Ratio of maximum to average shear stress in a triangular cross-section is
   (a) 1.2 (b) 1.5 (c) 2 (d) 2.5
5. Ratio of maximum to average shear stress in a rectangular section with a diagonal horizontal is
   (a) 5/4 (b) 4/5 (c) 8/9 (d) 9/8
6. Ratio of shear stress at neutral axis to average shear stress in a hexagonal section is
   (a) 1.2 (b) 1.5 (c) 2 (d) 2.5
7. Ratio of maximum shear stress to average shear stress is 4/3 in a _____section.
   (a) rectangular (b) triangular (c) circular (d) hexagonal
8. The nature of shear stress distribution in a rectangular beam is
   (a) uniform (b) linear (c) parabolic (d) elliptic
9. Maximum shear stress in a rectangular beam occurs at
   (a) neutral axis (b) bottom edge (c) top edge
10. Shear centre is the point in or outside a section through which the shear force applied produces _____ in the beam
    (a) only twisting (b) only bending
    (c) twisting and bending (d) no twisting and bending
11. Shear centre of a semicircular arc is at
    (a) $4r/\pi$ (b) $3r/\pi$ (c) $2r/\pi$ (d) r/π

Answers

1. (b) 2. (a) 3. (b) 4. (b) 5. (d) 6. (a)
7. (c) 8. (c) 9. (a) 10. (b) 11. (a)

Review Questions

6.1 What assumptions are taken in the analysis of shear stress in beams?
6.2 Find the governing relation to find the shear stress in beams. Deduce a relation for the shear stress across a rectangular section. What is the maximum value of shear stress?
6.3 Establish relations to find the shear stress across sections of (i) an I-section, and (ii) a circular section. What is the maximum value in each case?
6.4 Find a relation for the shear stress in a thin circular tube. Also, express it in terms of mean shear stress.
6.5 Develop relations for the shear stress across sections of (i) a square with a diagonal horizontal (ii) a triangle, side as base, and (iii) a hexagon, side as base.
6.6 What do you mean by shear centre? Explain with the help of examples.
6.7 A single channel section when used as a beam with its web vertical and acted upon by vertical load will be in torsion unless the load is applied through a particular point outside the section. Why?

Numerical Problems

6.1 A 1.3-m long simply supported timber beam carries a concentrated load W at the midspan. The cross-section of the beam is 150 mm wide and 250 mm deep. Find the value of W if the permissible working stresses are 7 MPa in bending and 1 MPa in shear. (33.65 kN)

6.2 A 50-mm wide and 120-mm deep I-beam is acted upon by a shear force of 10 kN. The web thickness is 3.5 mm and the flange thickness is 5.5 mm. Determine the transverse shear stress at the neutral axis and at the top of the web. Compare the same with the mean stress on the assumption of uniform distribution over the web. Also find the percentage of shear force carried by the web. Moment of inertia of the section is 2.2×10^6 mm^4 and the area is 940 mm^2.

(27.2 MPa, 20.45 MPa; Mean stress, 26.2 MPa; 82%)

6.3 A simply supported timber beam is 2 m long and is of rectangular section with depth twice the width. It carries a point load of 10 kN at the centre and a uniformly distributed load of 10 kN/m over the whole length. Find the suitable cross-section of the beam if the permissible shear stress for the timber is 0.8 MPa and bending stress in tension or compression is 10 MPa. (118.6 mm $\times$ 237.2 mm)

6.4 A 2-m long timber beam of 100-mm width and 150-mm depth supports a uniformly distributed load. Determine the maximum load that can be carried by the beam if the allowable stresses are 28 MPa longitudinally and 2 MPa in transverse shear. (20 kN/m)

6.5 The width of flanges of an I-section is b and the overall depth is $2b$. The thickness t of the web and the flanges is uniform. Neglecting the higher powers of t, find the ratio of the intensities of the maximum shear stress and the mean shear stress. (2.25)

6.6 A 300 mm $\times$ 150 mm I-girder having 12-mm thick flanges and 8-mm thick web is subjected to a shear force of 150 kN at a particular section. Determine the value of the maximum shear stress in the flange. Also find the ratio of the maximum shear stress to minimum shear stress in the web.

(2.92 MPa,1.295)

6.7 A 200-mm wide and 250-mm deep T-sectional beam has a web and flanges thickness of 50-mm. It is subjected to a vertical shear force of 100 kN. Determine the shear stress at the junction of the web and the flange. Also find the shear stress at the neutral axis. Moment of inertia about the horizontal neutral axis is 113.4×10^6. (2.76 MPa, 11.02 MPa, 11.36 MPa)

6.8 A beam is of triangular cross-section with a base of 100 mm and a height of 120 mm, the lower surface being horizontal. If the shear force on a section is 20 kN, draw the distribution of shear stress in the beam. (Maximum value 5 MPa, at neutral axis 4.44 MPa)

6.9 A channel section of a beam is 120 mm $\times$ 60 mm with a uniform thickness of 15 mm. Draw the distribution of the shear stress at a section where a shear force of 50 kN is acting. Also determine the ratio of the maximum and the mean shear stresses.

(At neutral axis 28.19 MPa, at flange junction 5.33 MPa and 21.33 MPa)

6.10 A beam of 4-m span is in the form of a cross bar as shown in Fig. 6.44. It carries a uniformly distributed load of 40 kN/m. Draw the shear stress distribution in the beam.

(At neutral axis 4.03 MPa, at 10 mm from neutral axis 32.6 MPa, 3.62 MPa)

6.11 A beam of square cross-section of 100-mm side is used with a diagonal in a horizontal position. The vertical shear force at a section is 150 kN. Determine the shear stress at the neutral axis and the value of the maximum shear stress and its location.

(15 MPa, 16.88 MPa , 53.03 mm)

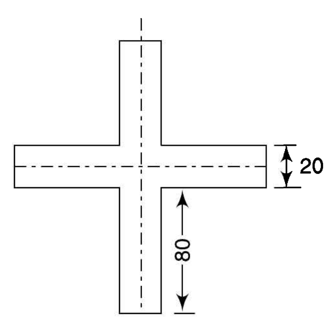


Fig. 6.44

6.12 Determine the approximate position of shear centre for a channel section of 60 mm $\times$ 60 mm outside and of 5 mm thickness. (24.8 mm)

6.13 Show that the shear centre of an arc of a circle of radius *r* placed symmetrically about a horizontal line and making an angle of α on each side is given by $\left[e = \dfrac{2r(\sin\alpha - \alpha\cos\alpha)}{\alpha - \sin\alpha\cos\alpha}\right]$ where e is the distance of the shear centre from the centre of the arc.

6.14 Determine the position of the shear centre of a thin arc of a circle which subtends an angle of (i) 60° (ii) 90° at the centre and has a mean radius of 100 mm. (102.8 mm, 106.3 mm)

Chapter 7

Slope and Deflection

As a load is applied on a beam, it deflects. The deflection can be observed and measured directly whereas other parameters such as shear force, bending moment and stresses can only be calculated. Though it is important that the cross-section of a beam is strong enough to withstand the bending stresses and shear stresses, i.e., it is based on *strength criterion*, the deflections must also be restricted. Excessive deflections can cause visible or invisible cracks in beams. Also, excessive deflections perceptible by the naked eye give a feeling of unsafe structure to the occupants of the building causing adverse effects on their health. Thus, it is extremely important to have the knowledge of maximum deflection in a beam under the given loading. The maximum deflection of a beam must not exceed a given limit. The designing of a beam from this aspect is known as *stiffness criterion*. In this chapter, the governing differential equation of beams is formulated and various methods of solution are discussed. The basic method involves integrating the differential equation whereas in other methods the integral is obtained indirectly. The relation obtained provides the *elastic curve*, i.e., the curve into which the axis of the beam is transformed under the loading.

7.1 BEAM DIFFERENTIAL EQUATION

As mentioned above, the deflection profile of a beam is known as its *elastic curve*. If a beam is subjected to pure bending, it is bent into a circular arc and the radius of bending or the radius of curvature is given by $\frac{M}{I} = \frac{E}{R}$ or $R = \frac{EI}{M}$. However, the radius of curvature may not be constant at all the points as the beam may not be subjected to pure bending, which is generally the case.

Consider a segment PQ of infinitesimal length ds of the elastic curve of a beam as shown in Fig. 7.1. Let R be the radius of curvature and $d\theta$ the included angle of the segment distance.

Then, the length $ds = R \cdot d\theta$

As ds is an infinitesimal length, it can be assumed to be the hypotenuse of a right-angled triangle DEF as shown in the figure.

The slope of the curve at the point P with coordinates (x, y) is given by

$$\tan \theta = \frac{dy}{dx} \qquad \text{(i)}$$

where dx and dy represent the projected lengths of the segment ds along X- and Y-axes respectively.

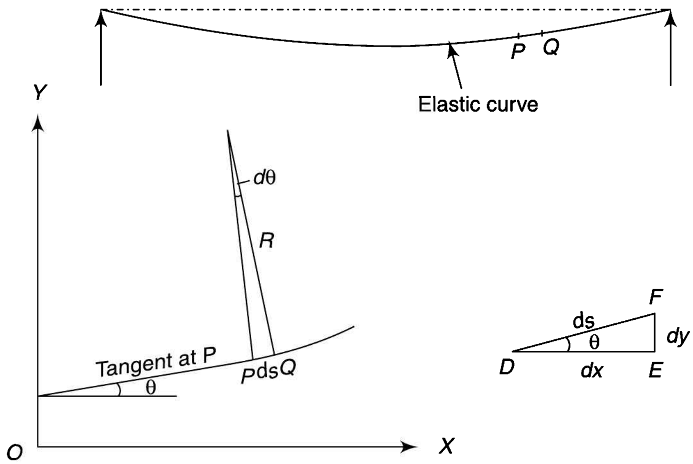


Fig. 7.1

Differentiating (i) with respect to x,

$$\sec^2\theta \cdot \frac{d\theta}{dx} = \frac{d^2y}{dx^2} \text{ or } \sec^2\theta . \frac{ds}{R} \cdot \frac{1}{dx} = \frac{d^2y}{dx^2} \text{ or } \frac{\sec^3\theta}{R} = \frac{d^2y}{dx^2} \ldots \left(\because \frac{ds}{dx} = \sec\theta\right)$$

or $$\frac{d^2y}{dx^2} = \frac{(1+\tan^2\theta)^{3/2}}{R} \qquad [\sec\theta = (1 + (\tan^2\theta)^{1/2}]$$

Usually, R is very large as compared to beam span and $\tan\theta$ or dy/dx, the slope of the tangent to the curve at any point is extremely small (of the order 0.001), the square will be still smaller as compared to 1 and thus can be neglected.

or $$\frac{d^2y}{dx^2} = \frac{1}{R} = \frac{M}{EI}$$

or $$EI\frac{d^2y}{dx^2} = M \qquad (7.1)$$

The above equation is the *governing differential equation* of the beam and takes into account the effect of bending moment only. The effect of shear on the deflection is extremely small and usually neglected.

From Eq. 7.1, the moment sustained by an element of the beam is proportional to the product EI; the larger the EI, the larger is the moment. Thus, the product EI is an index of the bending (flexural) strength of an element is called the *flexural rigidity* of the element.

7.2 SLOPE AND DEFLECTION AT A POINT

There are several methods to find the slope and deflection profile of a beam from the governing differential equation. However, the following are the main methods:

(i) Double integration method
(ii) Macaulay's method
(iii) Moment-area method
(iv) Strain energy method
(v) Conjugate beam method

Sign Convention

Consider a beam (Fig. 7.2a) and a cantilever (Fig. 7.2b). It is usual to take the origin either at A or B.

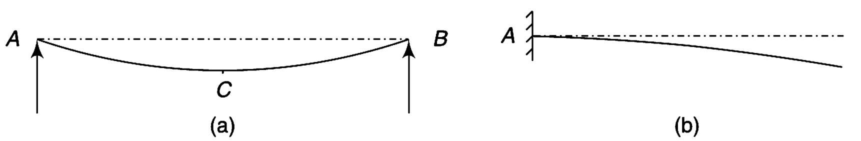


Fig. 7.2

(i) Origin at A If the origin is taken at A, x is taken positive towards right and y positive upwards. Then

- Deflection is negative or it is downwards in both cases
- The slope is negative between AC and positive between CB for beam and negative for cantilever throughout

(ii) Origin at B If the origin is taken at B, x is taken positive towards left and y positive upwards. Then

- Deflection is negative or it is downwards in both cases
- The slope is positive between AC and negative between CB for beam and positive for cantilever throughout

7.3 DOUBLE INTEGRATION METHOD

In this method, the equation of the elastic curve is integrated twice to obtain the deflection of the beam at any cross-section. The constants of integration are found by applying the end conditions.

$$EI\frac{dy}{dx} = \int M \cdot dx + C_1 \quad \text{from which slope can be calculated at any point}$$

and $$EI \cdot y = \iint (M \cdot dx) + C_1 x + C_2 \text{ from which deflection is known at any point.}$$

Constants of integration involved are obtained from the end conditions.

Deflections for different types of loading on the cantilevers and simply supported beams can be found as given below:

(A) Cantilevers

(i) Concentrated Load at Free End Let A be the fixed end and B the free end of a cantilever of uniform cross-section (Fig. 7.3a).

Also let

l = length of the cantilever

I = moment of inertia of the section about neutral axis

Assuming the origin to be at B, consider a section of the cantilever from the free end B,

Bending moment at the section = $-Wx$ (being hogging)

or $$EI\frac{d^2y}{dx^2} = -Wx$$

- Integrating, $$EI\frac{dy}{dx} = -\frac{Wx^2}{2} + C_1$$

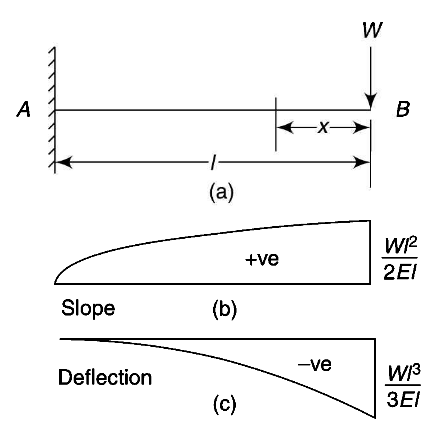


Fig. 7.3

At $x = l$, $\dfrac{dy}{dx} = 0$, $\therefore\ C_1 = \dfrac{Wl^2}{2}$

$$\therefore \quad EI\frac{dy}{dx} = -\frac{Wx^2}{2} + \frac{Wl^2}{2}$$

Thus slope is given by, $\dfrac{dy}{dx} = \dfrac{W}{2EI}(l^2 - x^2)$ (7.2)

- Integrating Eq. 7.2 again, $EIy = -\dfrac{Wx^3}{6} + \dfrac{Wl^2}{2}x + C_2$

At $x = l, y = 0$, $\therefore\ C_2 = -\dfrac{Wl^3}{3}$; $\therefore\ EIy = -\dfrac{Wx^3}{6} + \dfrac{Wl^2}{2}x - \dfrac{Wl^3}{3}$

Thus deflection is given by, $y = -\dfrac{W}{6EI}(2l^3 - 3l^2x + x^3)$ (7.3)

- At the free end ($x = 0$), the slope and the deflection are maximum.

Slope $= \dfrac{Wl^2}{2EI}$ and Deflection $= -\dfrac{Wl^3}{3EI}$ (downwards) (7.4)

The slope and deflection are shown in Fig. 7.3b and c respectively.

Origin at the fixed end If the origin is taken at the fixed end A (Fig.7.4a),
Bending moment at the section $= -W(l - x)$ (being hogging)
or $EIy'' = -W(l - x)$

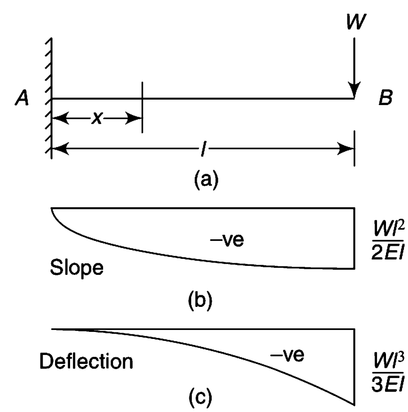


Fig. 7.4

- Integrating, $EIy' = -W\left(lx - \dfrac{x^2}{2}\right) + C_1$

At $x = 0$, $\dfrac{dy}{dx} = 0$, $\therefore\ C_1 = 0$; Thus $EIy' = -W\left(lx - \dfrac{x^2}{2}\right)$

Integrating again, $EIy = -W\left(\dfrac{lx^2}{2} - \dfrac{x^3}{6}\right) + C_2$

At $x = 0, y = 0$, $\therefore\ C_2 = 0$; Thus $EIy = -W\left(\dfrac{lx^2}{2} - \dfrac{x^3}{6}\right)$

Therefore, slope and deflection are given by,

$$y' = -\frac{W}{2EI}(2lx - x^2) \qquad (7.5)$$

and

$$y = -\frac{W}{6EI}(3lx^2 - x^3) \qquad (7.6)$$

- At the free end, $x = l$, the slope and the deflection are maximum and are given by

$$\text{Slope} = -\frac{Wl^2}{2EI} \text{ and Deflection} = -\frac{Wl^3}{3EI} \qquad (7.6a)$$

The slope and deflection are shown in Fig. 7.4b and c respectively.

(ii) Concentrated Load not at Free End Between AC, at any section at a distance x from A (Fig.7.5a), $M = -W(a - x)$

The equation of slope and elastic curve can be obtained as in previous case in the form $y' = -\frac{W}{2EI}(2ax - x^2)$ and $y = -\frac{W}{6EI}(3ax^2 - x^3)$, i.e., by replacing l by a.

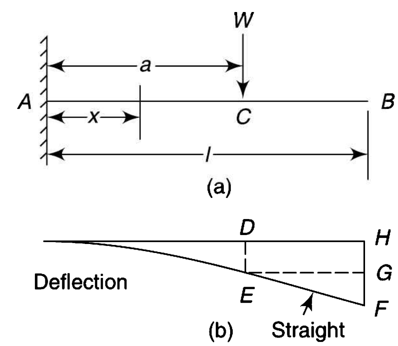


Fig. 7.5

At C, $x = a$, $y' = -\frac{Wa^2}{2EI}$ and $y_c = -\frac{Wa^3}{3EI}$

Between CB, at any section at a distance x from A, $M = 0$,

$$\therefore \quad EI\frac{d^2y}{dx^2} = 0 \quad \text{or} \quad \frac{dy}{dx} = C_1 \quad \text{or} \quad \frac{d^2y}{dx^2} = 0$$, i.e., the slope is constant between CB and is equal to slope at C or the portion of cantilever from C to B remains straight with slope $\frac{dy}{dx} = y' = \frac{GF}{GE} = -\frac{Wa^2}{2EI}$ or $GF = y' \cdot GE$ (Fig. 7.5b).

Deflection at B = Deflection at $C + GF$ = Deflection at $C + y' \cdot GE$

$$= -\frac{Wa^3}{3EI} - \frac{wa^2}{2EI} \cdot (l - a) \tag{7.7}$$

If W is at the midpoint, deflection $= \left[\frac{W(l/2)^3}{3EI} + \frac{W(l/2)^2}{2EI} \cdot \frac{l}{2}\right] = \frac{5Wl^3}{48EI}$ (7.7a)

(iii) Uniformly Distributed Load on Whole Span Let the origin be at the free end (Fig. 7.6a). At a section at a distance x from the free end,

$$EI\frac{d^2y}{dx^2} = M = -\frac{wx^2}{2}$$

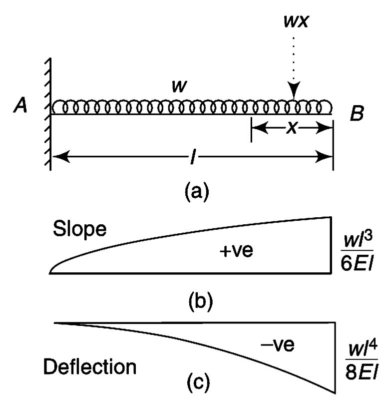


Fig. 7.6

- Integrating, $EI\frac{dy}{dx} = -\frac{wx^3}{6} + C_1$

At $x = l$, $\frac{dy}{dx} = 0$, $\therefore$ $C_1 = \frac{wl^3}{6}$

Thus, $EI\frac{dy}{dx} = -\frac{wx^3}{6} + \frac{wl^3}{6} = \frac{w}{6}(l^3 - x^3)$

- Integrating again, $EI \cdot y = -\frac{wx^4}{24} + \frac{wl^3}{6}x + C_2$

At A, $x = l$, $y = 0$, $\therefore$ $0 = -\frac{wl^4}{24} + \frac{wl^3}{6} \cdot l + C_2$ or $C_2 = -\frac{wl^4}{8}$

Thus, $EI \cdot y = -\frac{wx^4}{24} + \frac{wl^3}{6}x - \frac{wl^4}{8}$

Therefore, slope and deflection are given by,

$$\frac{dy}{dx} = \frac{w}{6EI}(l^3 - x^3) \quad \text{and} \quad y = -\frac{w}{24EI}(x^4 - 4l^3x + 3l^4) \tag{7.8}$$

- Maximum slope $= \dfrac{wl^3}{6EI}$ at $x = 0$ (7.9)

  Maximum deflection $= -\dfrac{wl^4}{8EI}$ at $x = 0$ (7.9a)

  Slope and deflection are shown in Fig. 7.6b and c respectively.

- If origin is taken at the fixed end, slope and deflection can be worked out to be

$$y' = -\frac{w}{6EI}(3l^2x - 3lx^2 + x^3); \qquad y = -\frac{w}{24EI}(6l^2x^2 - 4lx^3 + x^4) \tag{7.10}$$

(iv) Uniformly Distributed Load on a part of Span from Fixed End Refer Fig. 7.7. Origin at A,

At C, $\dfrac{dy}{dx} = -\dfrac{wa^3}{6EI}$ and $y_c = -\dfrac{wa^4}{8EI} \cdots (l = x = a)$ (Refer Eqs.7.10)

Between CB, at any section at a distance x from A, $M = 0$,

$$\therefore \quad EI\frac{d^2y}{dx^2} = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} = 0 \quad \text{or} \quad \frac{dy}{dx} = C_1$$

i.e. the slope is constant between CB and is equal to slope at C.

$$\frac{dy}{dx} = y' = \frac{GF}{GE} = -\frac{wa^3}{6EI} \quad \text{or} \quad GF = y' \cdot GE$$

Deflection at B = Deflection at $C + GF$

= Deflection at $C + y' \cdot GE$

$$= -\frac{wa^4}{8EI} - \frac{wa^3}{6EI}\cdot(l - a) \tag{7.11}$$

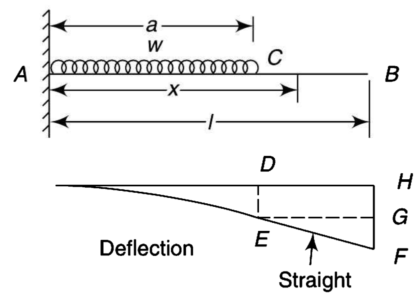


Fig. 7.7

(v) Uniformly Distributed Load on a Part of Span from Free End The slope and the deflection at B can be found by first considering the cantilever loaded for the whole span (Fig. 7.8a) and then deducting the effect for the span loaded from A to C upwards (Fig. 7.8b).

Thus slope, $\dfrac{dy}{dx} = \dfrac{wl^3}{6EI} - \dfrac{w(l - a)^3}{6EI}$ (7.12)

Deflection can be found as follows,

- For whole span having uniformly distributed load, $y_b = \dfrac{wl^4}{8EI}$ (downwards)

- For span loaded between AC,

  $\dfrac{w(l - a)^4}{8EI} + \dfrac{w(l - a)^3}{6EI}\cdot a$ (upwards) (Refer Eq.7.11)

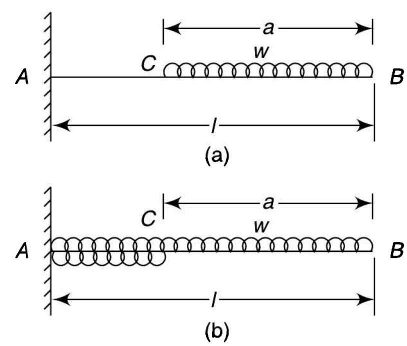


Fig. 7.8

Thus deflection of B (downwards) $= \dfrac{wl^4}{8EI} - \left[\dfrac{w(l - a)^4}{8EI} + \dfrac{w(l - a)^3}{6EI}\cdot a\right]$ (7.12a)

(vi) Couple at the Free End Refer Fig. 7.9, $EI\dfrac{d^2y}{dx^2} = -M$ ...(being sogging)

- Integrating, $EI\dfrac{dy}{dx} = -Mx + C_1$

  At $x = 0$, $\dfrac{dy}{dx} = 0$, $\therefore$ $C_1 = 0$; Thus, $EI\dfrac{dy}{dx} = -Mx$

- Integrating again, $EI \cdot y = -M\dfrac{x^2}{2} + C_2$

  At $x = 0, y = 0$, $\therefore$ $C_2 = 0$; Thus, $EIy = -\dfrac{M}{2}x^2$

  Therefore, slope and deflection curves are given by

$$\frac{dy}{dx} = -\frac{M}{EI}x \text{ (linear) and } y = -\frac{M}{2EI}x^2 \text{ (parabola)} \tag{7.13}$$

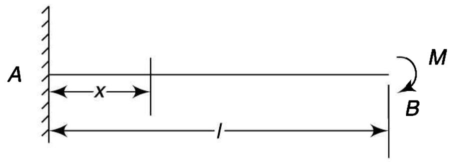


Fig. 7.9

(vii) Distributed Load of Varying Intensity, Zero at Free End Figure 7.10 shows the beam with distributed load of varying intensity.

Intensity of loading at any cross-section C at a distance x from free end $= \dfrac{wx}{l}$

Bending moment at C = load on CB X distance of centre of load

$$= \left(\frac{1}{2}\frac{wx}{l}\cdot x\right)\cdot\frac{x}{3} = \frac{wx^3}{6l}$$

Bending moment is negative, being hogging.

$$\therefore \qquad EI\frac{d^2y}{dx^2} = -\frac{wx^3}{6l}$$

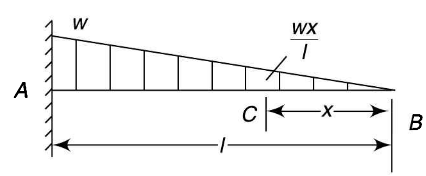


Fig. 7.10

- Integrating, $EI\dfrac{dy}{dx} = -\dfrac{wx^4}{24l} + C_1$

  At $x = l$, $\dfrac{dy}{dx} = 0$, $\therefore$ $C_1 = \dfrac{wl^3}{24}$

  Thus, $EI\dfrac{dy}{dx} = -\dfrac{wx^4}{24l} + \dfrac{wl^3}{24}$

- Integrating again, $EI \cdot y = -\dfrac{wx^5}{120l} + \dfrac{wl^3x}{24} + C_2$

  At $x = l, y = 0$, $\therefore$ $0 = -\dfrac{wl^4}{120} + \dfrac{wl^4}{24} + C_2$ or $C_2 = -\dfrac{wl^4}{30}$

  Thus, $EI \cdot y = -\dfrac{wx^5}{120l} + \dfrac{wl^3x}{24} - \dfrac{wl^4}{30}$

  Therefore, slope and deflection at free end i.e. at $x = 0$,

$$\frac{dy}{dx} = \frac{wl^3}{24EI} \text{ and } y = -\frac{wl^4}{30EI} \tag{7.14}$$

(vii) Distributed Load of Varying Intensity, Zero at Fixed End The deflection at B can be found by first considering the cantilever loaded for the whole span with uniformly distributed load w and then deducting the effect for the span loaded from A to B upwards with a distributed load of varying intensity from zero at free end and w at the fixed end (Fig. 7.11).

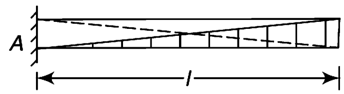


Fig. 7.11

(a) For a whole span having a uniformly distributed load, $y_b = -\dfrac{wl^4}{8EI}$ (Refer Eq. 7.10)

(b) For span loaded with varying intensity, $y_b = -\dfrac{wl^4}{30EI}$ (Refer Eq. 7.14)

Thus deflection of $B = -\dfrac{wl^4}{8EI} - \left(-\dfrac{wl^4}{30EI}\right) = -\dfrac{11wl^4}{120EI}$ (7.15)

(B) Simply Supported Beam

(i) Concentrated Load at Midspan Figure 7.12a shows a simply supported beam AB of span l and carrying a load W at the mid point C.

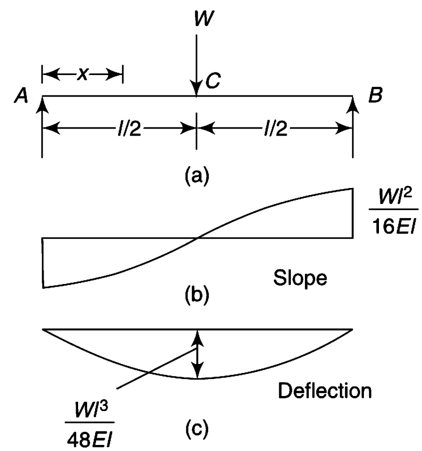


Fig. 7.12

$\therefore \quad R_a = R_b = W/2$

Consider a section from A (origin at A),

$$M = \frac{W}{2}x \qquad \text{(Positive, being sagging)}$$

$$\therefore \qquad EI\frac{d^2y}{dx^2} = \frac{W}{2}x$$

- Integrating, $EI\dfrac{dy}{dx} = \dfrac{Wx^2}{4} + C_1$

At $x = \dfrac{l}{2}$, $\dfrac{dy}{dx} = 0$, $\therefore\ C_1 = -\dfrac{Wl^2}{16}$ $\therefore\ EI\dfrac{dy}{dx} = \dfrac{Wx^2}{4} - \dfrac{Wl^2}{16}$

- Integrating again, $EIy = \dfrac{Wx^3}{12} - \dfrac{Wl^2x}{16} + C_2$

At $x = 0$, $y = 0$, $\therefore\ EIy = \dfrac{Wx^3}{12} - \dfrac{Wl^2x}{16}$

Therefore, slope and deflection are given by,

$$\frac{dy}{dx} = -\frac{W}{16EI}(l^2 - 4x^2) \text{ and } y = -\frac{W}{48EI}(3l^2x - 4x^3)$$

- At A, $x = 0$, $\therefore$ slope $= -\dfrac{Wl^2}{16EI}$ (7.16)

Deflection at $C = -\dfrac{W}{48EI}\left(3l^2 \cdot \dfrac{l}{2} - 4 \cdot \dfrac{l^3}{8}\right) = -\dfrac{Wl^3}{48EI}$ (7.17)

Slope and deflection are shown in Fig. 7.12b and c respectively.

- Slope and deflection for the portion CB is symmetric as for AC. However, equations for the portion CB with A as origin can also be formed in the following form:

$$\frac{dy}{dx} = -\frac{W}{16EI}(4x^2 - 8lx + 3l^2) \text{ and } y = -\frac{W}{48EI}(4x^3 + 9l^2x - l^3 - 12lx^2) \quad (7.18)$$

(ii) Eccentric Concentrated Load The section of zero slope is not defined in this case as the load is not symmetric (Fig. 7.13). Therefore, the bending moment should be developed for both the segments AC and CB separately.

Taking moments about B,

$$R_a \times l = Wb \text{ or } R_a = \frac{Wb}{l}; \quad R_b = W - \frac{Wb}{l} = \frac{W(l-b)}{l} = \frac{Wa}{l}$$

- Bending moment for segment $AC = \dfrac{Wbx}{l}$

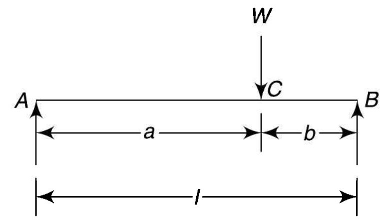


Fig. 7.13

or $$EIy'' = \frac{Wbx}{l}$$

Integrating, $$EIy' = \frac{Wbx^2}{2l} + C_1 \quad \text{(i)}$$

Integrating again, $$EIy = \frac{Wbx^3}{6l} + C_1x + C_2$$

At A, $x = 0, y = 0$; $\therefore C_2 = 0$

Thus $$EIy = \frac{Wbx^3}{6l} + C_1x \quad \text{(ii)}$$

- Bending moment for segment $CB = \dfrac{Wbx}{l} - W(x-a)$

or $$EIy'' = \frac{Wbx}{l} - W(x-a)$$

Integrating, $$EIy' = \frac{Wbx^2}{2l} - W\frac{(x-a)^2}{2} + C_3 \quad \text{(iii)}$$

Integrating again, $$EIy = \frac{Wbx^3}{6l} - W\frac{(x-a)^3}{6} + C_3x + C_4 \quad \text{(iv)}$$

- Slope at C from (i) = Slope at C from (iii)

$$\frac{Wbx^2}{2l} + C_1 = \frac{Wbx^2}{2l} - W\frac{(x-a)^2}{2} + C_3$$

or $$\frac{Wbx^2}{2l} + C_1 = \frac{Wbx^2}{2l} + C_3 \quad (x = a)$$

or $C_1 = C_3$

- Deflection at C from (ii) = Deflection at C from (iv)

$$\frac{Wbx^3}{6l} + C_1x = \frac{Wbx^3}{6l} - W\frac{(x-a)^3}{6} + C_3x + C_4$$

Now at C, $x = a$, $C_1 = C_3$; Therefore, $C_4 = 0$

- At B, $x = l$, $y = 0$; $\therefore$ from (iv),

$$0 = \frac{Wbl^3}{6l} - W\frac{(l-a)^3}{6} + C_3 l \qquad (C_4 = 0)$$

or
$$\frac{Wbl^3}{6l} - W\frac{b^3}{6} + C_3 l = 0$$

or
$$C_3 = -\frac{Wbl^2}{6l} + W\frac{b^3}{6l} = -\frac{Wb}{6l}(l^2 - b^2)$$

Also
$$C_1 = C_3 = -\frac{Wb}{6l}(l^2 - b^2)$$

Thus elastic curve of the beam is

$$EIy = \frac{Wbx^3}{6l} - \frac{Wb}{6l}(l^2 - b^2)x \qquad \text{for } x \le a \text{ [From (ii)]} \tag{7.19}$$

and
$$EIy = \frac{Wbx^3}{6l} - W\frac{(x-a)^3}{6} - \frac{Wb}{6l}(l^2 - b^2)x \qquad \text{for } x \ge a \text{ [From (iv)]} \tag{7.20}$$

- The value of x for maximum deflection can be obtained as under: Let $a > b$,

From (i) inserting the value of constant C_1, $EIy' = \dfrac{Wbx^2}{2l} - \dfrac{Wb}{6l}(l^2 - b^2)$

At maximum deflection, slope = 0

or $\quad 0 = \dfrac{Wbx^2}{2l} - \dfrac{Wb}{6l}(l^2 - b^2) \quad$ or $\quad x = \sqrt{\dfrac{l^2 - b^2}{3}}$

Thus maximum deflection, $EIy = \dfrac{Wb}{6l}\left(\sqrt{\dfrac{l^2 - b^2}{3}}\right)^3 - \dfrac{Wb}{6l}(l^2 - b^2)\sqrt{\dfrac{l^2 - b^2}{3}}$

$$= -\frac{Wb(l^2 - b^2)^{3/2}}{9\sqrt{3} \cdot EIl} \tag{7.21}$$

Distance of point of maximum deflection from the centre $= \sqrt{\dfrac{l^2 - b^2}{3}} - \dfrac{l}{2}$

The maximum value of which ca be $= \dfrac{l}{\sqrt{3}} - \dfrac{l}{2} \approx \dfrac{l}{13}$ (7.22)

Thus the maximum deflection always lies within $l/13$ of the centre.

(iii) Uniformly Distributed Load on Whole Span Figure 7.14 shows a simply supported beam AB of span l and carrying a uniformly distributed load w over the whole span.

$\therefore \qquad R_a = R_b = wl/2$

Consider a section of the beam from A (origin at A),

$$EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2}$$

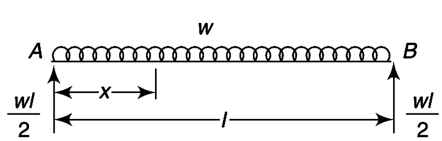


Fig. 7.14

- Integrating, $EI\dfrac{dy}{dx} = \dfrac{wlx^2}{4} - \dfrac{wx^3}{6} + C_1$

At $x = \dfrac{l}{2}, \dfrac{dy}{dx} = 0$, $\therefore\ 0 = \dfrac{wl}{4}\cdot\dfrac{l^2}{4} - \dfrac{w}{6}\cdot\dfrac{l^3}{8} + C_1$ or $C_1 = -\dfrac{wl^3}{24}$

$\therefore\quad EI\dfrac{dy}{dx} = \dfrac{wlx^2}{4} - \dfrac{wx^3}{6} - \dfrac{wl^3}{24}$

- Integrating again, $EIy = \dfrac{wlx^3}{12} - \dfrac{wx^4}{24} - \dfrac{wl^3}{24}x + C_2$

At $x = 0, y = 0$, $\therefore\ C_2 = 0$ $\therefore\ EIy = \dfrac{wlx^3}{12} - \dfrac{wx^4}{24} - \dfrac{wl^3}{24}x$

- The maximum deflection is at the midspan, i.e., at $x = l/2$,

$$y_{max} = \frac{1}{EI}\left[\frac{wl}{12}\left(\frac{l}{2}\right)^3 - \frac{w}{24}\left(\frac{l}{2}\right)^4 - \frac{wl^3}{24EI}\left(\frac{l}{2}\right)\right] = -\frac{5}{384EI}wl^4 \qquad (7.23)$$

- Slope at A, $(x = 0)$, $EI\dfrac{dy}{dx} = -\dfrac{wl^3}{24}$ or $\dfrac{dy}{dx} = -\dfrac{wl^3}{24EI}$ (7.24)

(iv) Distributed Load of Varying Intensity Refer Fig. 7.15.

Intensity of load at a distance x from $A = \dfrac{wx}{l}$

Taking moments about B, $R_a \times l = \dfrac{wl}{2}\cdot\dfrac{l}{3}$ or $R_a = \dfrac{wl}{6}$; $R_b = \dfrac{wl}{2} - \dfrac{wl}{6} = \dfrac{wl}{3}$

Intensity of load at a distance x from $A = \dfrac{wx}{l}$

Bending moment at the section at a distance x from $A = \dfrac{wl}{6}x - \dfrac{wx}{l}\cdot\dfrac{x}{2}\cdot\dfrac{x}{3} = \dfrac{wl}{6}x - \dfrac{wx^3}{6l}$

$$\therefore \qquad EI\frac{d^2y}{dx^2} = \frac{wlx}{6} - \frac{wx^3}{6l}$$

- Integrating, $EI\dfrac{dy}{dx} = \dfrac{wlx^2}{12} - \dfrac{wx^4}{24l} + C_1$
- Integrating again, $EIy = \dfrac{wlx^3}{36} - \dfrac{wx^5}{120l} + C_1x + C_2$

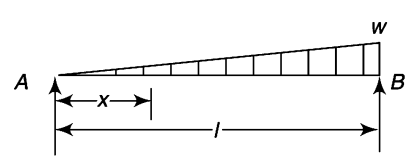


Fig. 7.15

At $x = 0, y = 0$, $\therefore\ C_2 = 0$

At $x = l, y = 0$, $\therefore\ 0 = \dfrac{wl^4}{36} - \dfrac{wl^4}{120} + C_1 l$ or $C_1 = -\dfrac{7wl^3}{360}$

Thus $\quad EIy = \dfrac{wlx^3}{36} - \dfrac{wx^5}{120l} - \dfrac{7wl^3}{360}x$ (7.25)

To find maximum deflection, equate the slope to zero.

i.e. $$EI\frac{dy}{dx}=\frac{wlx^2}{12}-\frac{wx^4}{24l}-\frac{7wl^3}{360}=0$$

or $$30l^2x^2-15x^4-7l^4=0$$

or $$15x^4-30l^2x^2+7l^4=0$$

Let $x = kl$

Then $15k^4 - 30k^2 + 7 = 0$; Solving, $k^2=\dfrac{30\pm\sqrt{900-4\times7\times15}}{30}=0.2697$

(considering the feasible value of k, it cannot be more than 1)

or $x^2/l^2 = 0.2697$

or $x = 0.5193l$

Thus for maximum deflection, $EIy=\dfrac{wl(0.5193l)^3}{36}-\dfrac{w(0.5193l)^5}{120l}-\dfrac{7wl^3}{360}(0.5193l)$

or $EIy = -0.00652wl^4$ or $y_{max}=-0.00652\dfrac{wl^4}{EI}$ (7.26)

(v) Couple at One End Assume a simply supported beam of length l having a clockwise couple M at the left end (Fig. 7.15a).

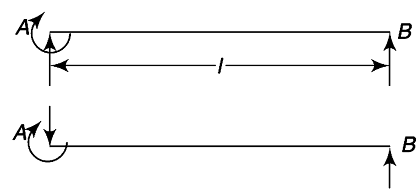


Fig. 7.15a

Taking moments about A, $R_b \times l = M$ or $R_b=\dfrac{M}{l}$ $(\uparrow)$

Similarly, $R_a=\dfrac{M}{l}$ $(\downarrow)$

At any section x from A, $EI\dfrac{d^2y}{dx^2}=-\dfrac{M}{l}x+M$

Integrating, $$EI\frac{dy}{dx}=-\frac{Mx^2}{2l}+Mx+C_1 \quad \text{(i)}$$

Integrating again, $$EIy=-\frac{Mx^3}{6l}+\frac{M}{2}x^2+C_1x+C_2 \quad \text{(ii)}$$

- At A, $x = 0$, $y = 0$

$$EIy=-\frac{Mx^3}{6l}+\frac{M}{2}x^2+C_1x+C_2 \text{ or } C_2=0$$

At B, $x = l$, $y = 0$

or $$0=-\frac{Mx^3}{6l}+\frac{M}{2}x^2+C_1x$$

or $$C_1=\frac{Ml^2}{6l}-\frac{M}{2l}l^2=-\frac{Ml}{3}$$

Thus slope and deflection equations are

$$EI\frac{dy}{dx} = -\frac{Mx^2}{2l} + Mx - \frac{Ml}{3} = -\frac{M}{6l}(3x^2 - 6lx + 2l^2)$$

and $$EIy = -\frac{Mx^3}{6l} + \frac{M}{2}x^2 + C_1 x = -\frac{M}{6l}(x^3 - 3lx^2 + 2l^2 x)$$

Slope at $A = -\frac{M}{6lEI}(2l^2) = -\frac{Ml}{3EI}$

Slope at $B = -\frac{M}{6lEI}(3l^2 - 6l^2 + 2l^2) = \frac{Ml}{6EI}$

Maximum deflection will be where slope is zero, i.e.,

$$3x^2 - 6lx + 2l^2 = 0 \text{ or } x = 0.423\, l$$

Thus maximum deflection,

$$y_{\max} = -\frac{M}{6EIl}(x^3 - 3lx^2 + 2l^2 x)$$

$$= -\frac{M}{6EIl}\left[(0.423l)^3 - 3l \times (0.423l)^2 + 2l^2 \times 0.423\right]$$

$$= -\frac{0.64Ml^2}{6EI}$$

Example 7.1 || Two parallel steel cantilevers each of length l, one above the other project horizontally from a vertical wall. Their free ends are connected together by a vertical steel tie rod of length a. A load W is applied at the mid point of the lower beam. Show that the pull in the rod is given by, $P = \frac{5}{32} \cdot \frac{Wl^3}{l^3 + (6aI)/(\pi d^2)}$ where d is the diameter of the rod and I the moment of inertia of the section of each beam about its neutral axis.

If the length of each cantilever is 3 m and that of the tie rod is 2.4 m, find the proportion of the load W carried by the tie bar. The diameter of the rod is 20 mm and the moment of inertia of each cantilever is 28×10^6 mm^4.

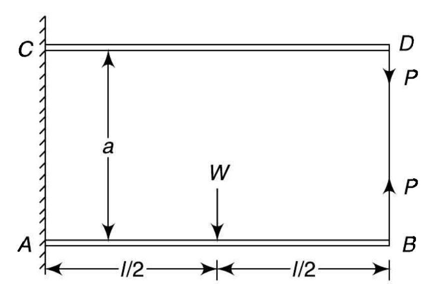


Fig. 7.16

Solution

Given Two parallel steel cantilevers one above the other, connected together by a vertical steel tie rod as shown in Fig. 7.16.

$l = 3$ m $\qquad a = 2.4$ m

$I = 28 \times 10^6$ mm^4 $\qquad d = 20$ mm

To find

— To show that pull in rod is given by, $P = \frac{5}{32} \cdot \frac{Wl^3}{l^3 + (6aI)/(\pi d^2)}$

— Proportion of load carried by tie bar.

Deflection of cantilevers

For cantilever CD, $y_1 = \frac{Pl^3}{3EI}$ (downwards) (Eq. 7.6a)

For cantilever AB, $y_2 = \dfrac{5Wl^3}{48EI} - \dfrac{Pl^3}{3EI}$ (downwards) ... (Eq. 7.7a)

Extension of tie rod

$$\delta = \frac{P \cdot a}{(\pi/4)d^2 E}$$

Compatibility equation

$$y_2 - y_1 = \delta$$

or
$$\frac{5Wl^3}{48EI} - \frac{Pl^3}{3EI} - \frac{Pl^3}{3EI} = \frac{P \cdot a}{(\pi/4)d^2 E}$$

or
$$P\left(\frac{2l^3}{3I} + \frac{4a}{\pi d^2}\right) = \frac{5Wl^3}{48I}$$

or
$$P \times 48I \times \frac{2}{3I}\left(l^3 + \frac{6aI}{\pi d^2}\right) = 5Wl^3 \quad \text{or} \quad P = \frac{5}{32} \cdot \frac{Wl^3}{l^3 + (6aI)/(\pi d^2)}$$

Numerical

$$I = 28 \times 10^6 \text{ mm}^4 = 28 \times 10^{-6} \text{ m}^4$$

$$P = \frac{5}{32} \cdot \frac{Wl^3}{l^3 + \dfrac{6aI}{\pi d^2}} = \frac{5}{32} \cdot \frac{W \times 3^3}{3^3 + \dfrac{6 \times 2.4 \times 28 \times 10^{-6}}{\pi (0.02)^2}} = 0.154\ W$$

Example 7.2 A uniform circular bar of length l and diameter d is extended by an amount δ under a tensile load F. Show that if the bar is used as a beam simply supported at its ends and carries a central load W, the maximum deflection is given by, $y = \dfrac{W \cdot \delta \cdot l}{3Fd^2}$.

If $l = 60d$ and maximum bending stress due to W is equal to 0.8 times the tensile stress due to pull F, determine the ratio y/δ.

Solution

Given A uniform circular bar of length l and diameter d extends by δ under tensile load F.

To find

— To show that if bar is used as beam simply supported at its ends carrying a central load W, the maximum deflection is, $y = \dfrac{W \cdot \delta \cdot l}{3Fd^2}$.

— Ratio y/δ if $l = 60d$ and maximum bending stress is 0.8 times the tensile stress due to pull F.

Due to tensile load, extension of the bar, $\delta = \dfrac{Fl}{AE} = \dfrac{4Fl}{\pi d^2 E}$ or $E = \dfrac{4Fl}{\pi d^2 \delta}$

Maximum deflection

Due to central load, maximum deflection at the centre of the beam

$$y = \frac{Wl^3}{48EI} = \frac{Wl^3}{48 \times \dfrac{4Fl}{\pi d^2 \delta} \cdot \dfrac{\pi d^4}{64}} = \frac{W\delta l^2}{3Fd^2} \qquad \text{(i)}$$

Ratio of W/F from condition $\sigma_b = 0.8\sigma$

Maximum bending stress, $\sigma_b = \dfrac{M}{Z} = \dfrac{Wl}{4} \cdot \dfrac{32}{\pi d^3} = \dfrac{8W \times 60d}{\pi d^3} = \dfrac{480W}{\pi d^2}$

Tensile stress, $\sigma = \dfrac{F}{\pi d^2/4} = \dfrac{4F}{\pi d^2}$

According to the given condition,

$$\sigma_b = 0.8\sigma \quad \text{or} \quad \frac{480W}{\pi d^2} = 0.8 \times \frac{4F}{\pi d^2} \quad \text{or} \quad \frac{W}{F} = \frac{1}{150}$$

Ratio y/δ

From (i), $\dfrac{y}{\delta} = \dfrac{W}{F} \cdot \dfrac{l^2}{3d^2} = \dfrac{W}{F} \cdot \dfrac{(60d)^2}{3d^2} = 1200 \times \dfrac{W}{F} = 1200 \times \dfrac{1}{150} = 8$

Example 7.3 An 80-mm wide and 180-mm deep cantilever is of 3 m span. It carries a uniformly distributed load of 6 kN/m intensity on a 2-m length of the span starting from the free end. Determine the slope and the deflection at the free end. E = 205 GPa.

Solution

Given A cantilever carrying a uniformly distributed load as shown in Fig. 7.17.

$$E = 205 \text{ GPa}$$

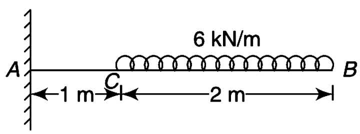


Fig. 7.17

To find Slope and deflection at free end

$$w = 6 \text{ kN/m} = 6 \text{ N/mm}$$

$$I = \frac{80 \times 180^3}{12} = 38.88 \times 10^6 \text{ mm}^4$$

Slope at free end

$$\frac{dy}{dx} = \frac{wl^3}{6EI} - \frac{w(l-a)^3}{6EI} \qquad \text{...(Eq. 7.12)}$$

or

$$\frac{dy}{dx} = \frac{6}{6 \times 205\,000 \times 38.88 \times 10^6}[3000^3 - (3000 - 2000)^3]$$

$$= 0.003\ 26 \text{ rad}$$

Deflection at free end

$$\text{Deflection of } B \text{ (downwards)} = \frac{wl^4}{8EI} - \left[\frac{w(l-a)^4}{8EI} + \frac{w(l-a)^3}{6EI} \cdot a\right] \qquad \text{...(Eq. 7.12a)}$$

$$= \frac{6}{2 \times 205\,000 \times 38.88 \times 10^6}\left[\frac{3000^4}{4} - \frac{(3000-2000)^4}{4} - \frac{(3000-2000)^3}{3} \times 2\right]$$

$$= 7.4 \text{ mm}$$

Example 7.4 A simply supported beam of length L has equal overhangs at the two ends. The distance between the two supports is l. Determine the ratio of l to L so that the downward deflection at the centre is equal to upward deflection at the ends due to a central concentrated load.

Solution

Given A simply supported beam with equal overhangs and a central point load as shown in Fig. 7.18.

To find Ratio of l to L so that downward deflection at centre is equal to upward deflection at ends

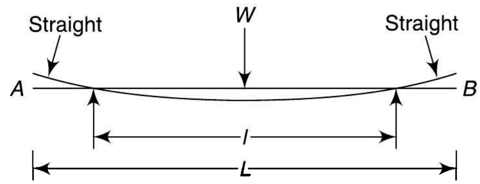


Fig. 7.18

Deflections due to central concentrated load

$$\text{Deflection at the centre} = \frac{Wl^3}{48EI}$$

$$\text{Deflection at end} = \text{Slope at the support} \times \text{Overhang} = \frac{Wl^2}{16EI} \cdot \frac{L-l}{2}$$

Equating the deflections

$$\frac{Wl^2}{16EI} \cdot \frac{L-l}{2} = \frac{Wl^3}{48EI} \quad \text{or} \quad \frac{L-l}{2} = \frac{l}{3} \text{ rad}$$

or

$$3L - 3l = 2l \quad \text{or} \quad l/L = 0.6$$

Example 7.5 A horizontal steel beam with a moment of inertia of 60×10^6 mm^4 carries a uniformly distributed load of 40 kN over its length of 3 m. The beam is supported by three 1.6-m long vertical tie rods, one at each end and one in the middle. The diameter of each of the end rods is 20 mm and that of the middle rod is 24 mm. Determine the deflection at the centre of the beam below the end points and the stress in each of the rod. Take E for the rods as 205 GPa.

Solution

Given A horizontal steel beam supported by three tie rods and carrying a uniformly distributed load as shown in Fig. 7.19.

$W = 40$ kN $\qquad E = 205$ GPa

To find

— Deflection at centre of beam below the end points

— stress in each rod

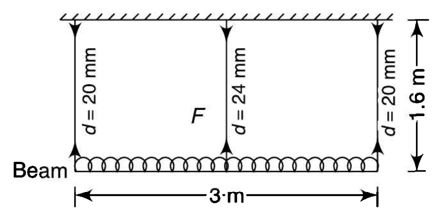


Fig. 7.19

Let F kN be the load in the middle rod

Compatibility equation

Elongation of middle rod – Elongation of end rod = Deflection at centre due to u.d.l. – Upward deflection at centre due to load F in central rod

$$\frac{F \times l}{\text{Area} \times E} - \frac{(W-F)l}{2 \times \text{Area} \times E} = \frac{5WL^3}{384EI} - \frac{FL^3}{48EI} \qquad \ldots(W = wl)$$

$$\frac{F \times 1600}{\pi \times 12^2 \times E} - \frac{(40-F) \times 1600}{2 \times (\pi \times 10^2)E} = \frac{5 \times 40 \times 3000^3}{384E \times 60 \times 10^6} - \frac{F \times 3000^3}{48E \times 60 \times 10^6}$$

$$3.54F - 101.86 + 2.55F = 234.38 - 9.38F$$

$$F = 21.73 \text{ kN}$$

Stresses in rods

$$\text{Stress in the middle rod} = \frac{F}{A} = \frac{21\,730}{\pi \times 12^2} = 48.03 \text{ MPa}$$

$$\text{Stress in the end rods} = \frac{(40\,000 - 21\,730)}{2 \times \pi \times 10^2} = 29.08 \text{ MPa}$$

Deflection at centre of beam

Deflection of centre relative to ends = difference of elongations of tie rods

$$= \frac{48.03 \times 1600 - 29.08 \times 1600}{205\,000} = 0.148 \text{ mm}$$

Example 7.6 A beam *AB* of length *l* is simply supported on two props at the ends and carries a uniformly distributed load of *w* per unit length. If a prop is provided at the centre of the span so that it supports the beam to the level of end supports, calculate

(i) the prop reaction and slope at each end
(ii) the position and magnitude of maximum deflection
(iii) the reaction of the central prop if all the props are elastic with stiffness s
(iv) the reaction of the central prop if end props are rigid and only the central prop is elastic with stiffness s

Solution

Given A propped beam carrying a uniformly distributed load as shown in Fig. 7.20.

To find

— Prop reaction and slope at each end
— position and magnitude of maximum deflection
— reaction of central prop if props are elastic
— reaction of central prop if end props rigid and central prop is elastic

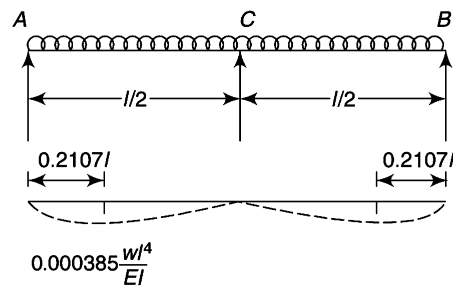


Fig.7.20

Let the reaction of the prop at the centre be R.

Prop reaction

Deflection at C due to uniformly distributed load only $= \dfrac{5}{384}\dfrac{wl^4}{EI}$

Deflection due to R only $= \dfrac{Rl^3}{48EI}$ (upwards)

Equating the two, $\dfrac{Rl^3}{48EI} = \dfrac{5}{384}\dfrac{wl^4}{EI}$ or $R = \dfrac{5wl}{8}$

Reaction of end supports $= \dfrac{wl - 5wl/8}{2} = \dfrac{3wl}{16}$

Calculations for slope at each end

At a distance x from A, $EI\dfrac{d^2y}{dx^2} = \dfrac{3wlx}{16} - \dfrac{wx^2}{2}$

Integrating, $EI\dfrac{dy}{dx} = \dfrac{3wlx^2}{32} - \dfrac{wx^3}{6} + C_1$

At $x = \dfrac{l}{2}$, slope = 0, $\therefore\ 0 = \dfrac{3wl}{32}\dfrac{l^2}{4} - \dfrac{w}{6}\dfrac{l^3}{8} + C_1$ or $C_1 = -\dfrac{wl^3}{384}$

Thus $EI\dfrac{dy}{dx} = \dfrac{3wlx^2}{32} - \dfrac{wx^3}{6} - \dfrac{wl^3}{384}$

$\therefore$ Slope at each end, i.e. at $x = 0$, $\dfrac{dy}{dx} = -\dfrac{wl}{384EI}$

Maximum deflection

Integrating the slope equation, $EIy = \dfrac{wl}{32}x^3 - \dfrac{w}{24}x^4 - \dfrac{wl^3}{384}x + C_2$

At $x = 0, y = 0$; $\therefore C_2 = 0$

Thus deflection equation is $EIy = \dfrac{wl}{32}x^3 - \dfrac{w}{24}x^4 - \dfrac{wl^3}{384}x$

Maximum deflection will be where slope is zero, i.e.

$$0 = \frac{3wlx^2}{32} - \frac{wx^3}{6} - \frac{wl^3}{384} \quad \text{or} \quad 36x^2 - 64x^3 - l^3 = 0$$

Solving by trial and error, $x = 0.2107l$

For maximum deflection, $EIy = \dfrac{wl}{32}(0.2107l)^3 - \dfrac{w}{24}(0.2107l)^4 - \dfrac{wl^3}{384}(0.2107l)$

Thus maximum deflection $= 0.000\,385\,\dfrac{wl^4}{EI}$

If props are elastic

Downward deflection at C due to uniformly distributed load – upward deflection at C due to prop reaction = yielding of central prop – yielding of end props

$$\frac{5}{384}\frac{wl^4}{EI} - \frac{Rl^3}{48EI} = \frac{R}{s} - \frac{wl - R}{2s}$$

$$\frac{Rl^3}{48EI} + \frac{3R}{2s} = \frac{5}{384}\frac{wl^4}{EI} + \frac{wl}{2s}$$

Multiplying throughout by $48EI/l^3$,

$$R\left(1 + \frac{72EI}{sl^3}\right) = wl\left(\frac{5}{8} + \frac{24EI}{sl^3}\right) \quad \text{or} \quad R = \frac{wl\left(\dfrac{5}{8} + \dfrac{24EI}{sl^3}\right)}{\left(1 + \dfrac{72EI}{sl^3}\right)}$$

If only central prop is elastic

$$\frac{5}{384}\frac{wl^4}{EI} - \frac{Rl^3}{48EI} = \frac{R}{s} \quad \text{or} \quad \frac{Rl^3}{48EI} + \frac{R}{s} = \frac{5}{384}\frac{wl^4}{EI}$$

Multiplying throughout by $48EI/l^3$,

$$R\left(1 + \frac{48EI}{sl^3}\right) = \frac{5wl}{8} \quad \text{or} \quad R = \frac{5wl}{8\left(1 + \dfrac{48EI}{sl^3}\right)}$$

Example 7.7 A horizontal beam, simply supported at its ends carries a load of varying intensity which varies uniformly from 10 kN /m at one end to 50 kN/m at the other. Find the central deflection if the span is 9 m in length and is 500 mm deep. Take maximum bending stress as 80 MPa and E = 210 GPa.

Solution

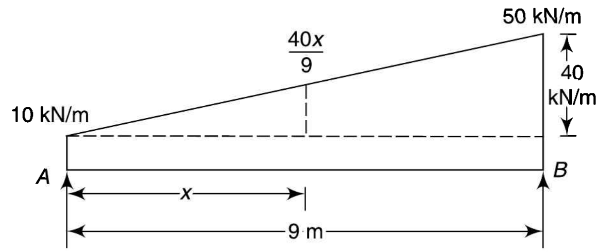


Fig. 7.21

Given A simply supported beam carrying a load of varying intensity as shown in Fig. 7.21.

$\sigma_b = 80$ MPa $\qquad d = 500$ mm $\qquad E = 210$ GPa

To find Central deflection

Divide the loading diagram into two portions: one with a uniform rate of 10 kN/m and the second with a varying load of 0 to 40 kN/m across the span.

Reactions

Taking moments about A, $R_b \times 9 = 10 \times 9 \times 4.5 + \frac{1}{2} \times 9 \times 40 \times \frac{2}{3} \times 9$

$$R_b = 165 \text{ kN and } R_a = \frac{50+10}{2} \times 9 - 165 = 105 \text{ kN}$$

Calculations for maximum bending moment

At a distance x from A,

$$\text{Shear force} = 105 - 10x - \frac{1}{2} x \cdot \frac{40x}{9} = 105 - 10x - 2.222x^2$$

$$\text{Bending moment} = 105x - 10 \times \frac{x^2}{2} - 2.222 \times \frac{x^3}{3} = 105x - 5x^2 - 0.741x^3$$

Maximum bending moment occurs at zero shear force, i.e., at

$$105 - 10x - 2.222x^2 = 0 \text{ or } 2.222x^2 + 10x - 105 = 0 \text{ or } x^2 + 4.5x - 47.25 = 0$$

or $\quad x = \dfrac{-4.5 \pm \sqrt{4.5^2 + 4 \times 47.25}}{2} = -2.25 + 7.233 = 4.983$ m (Taking the feasible value)

$\therefore$ maximum bending moment $= 105x - 5x^2 - 0.741x^3$

$$= 105 \times 4.983 - 5(4.983)^2 - 0.741(4.983)^3 = 307.6 \text{ kN·m or } 307.6 \times 10^6 \text{ N·mm}$$

Moment of inertia

$$\text{Maximum bending stress} = \frac{M_{max}}{I} \cdot y_{max}$$

or $\quad 80 = \dfrac{307.6 \times 10^6}{I} \times \left(\dfrac{500}{2}\right)$ $\quad$ or $\quad I = 961.25 \times 10^6$ mm^4

Determination of central deflection

Now, $EI\dfrac{d^2y}{dx^2} = 105x - 5x^2 - 0.741x^3$

Integrating, $EI\dfrac{dy}{dx} = 105 \times \dfrac{x^2}{2} - 5 \times \dfrac{x^3}{3} - 0.741 \times \dfrac{x^4}{4} + C_1$

Integrating again, $EIy = 105 \times \dfrac{x^3}{6} - 5 \times \dfrac{x^4}{12} - 0.741 \times \dfrac{x^5}{20} + C_1 x + C_2$

At $x = 0, y = 0$; $\therefore\ C_2 = 0$

At $x = 9$ m, $y = 0$; $0 = 105 \times \dfrac{9^3}{6} - 5 \times \dfrac{9^4}{12} - 0.741 \times \dfrac{9^5}{20} + C_1 \times 9$ or $C_1 = -870.7$

Thus deflection is given by, $EIy = 105 \times \dfrac{x^3}{6} - 5 \times \dfrac{x^4}{12} - 0.741 \times \dfrac{x^5}{20} - 870.7x$

At the centre; $EIy = 105 \times \dfrac{4.5^3}{6} - 5 \times \dfrac{4.5^4}{12} - 0.741 \times \dfrac{4.5^5}{20} - 870.7 \times 4.5 = -2562.7\ \text{kN}\cdot\text{m}^2$

$$y = \frac{2562.7 \times 10^{12}}{210\,000 \times 961.25 \times 10^6} = 12.7\ \text{mm}$$

7.4 MACAULAY'S METHOD

While applying the double integration method, a separate expression for the bending moment is needed to be written for each section of the beam, each producing a different equation with its own constants of integration. The method is convenient for simple cases but in complex cases it becomes cumbersome. In Macaulay's method, a single equation is written for the bending moment for all the portions of the beam. The equation is formed in such a way that the same constants of integration are applicable to all portions.

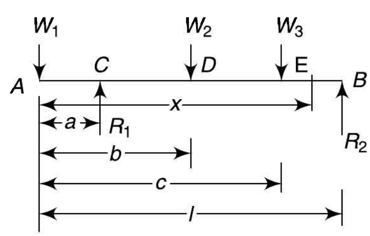


Fig. 7.22

Consider a simply supported beam AB of length l as shown in Fig. 7.22. Taking A as origin and writing the expression for the bending moment in the last portion of the beam,

$$EI\frac{d^2y}{dx^2} = M = -W_1 x \mid + R_1(x-a) \mid - W_2(x-b) \mid - W_3(x-c)$$

In the above expression, there are separation lines.

- The portion to the left of the first separation line is valid for the portion AC.
- The portion to the left of the second separation line is valid for the portion CD.
- The portion to the left of the third separation line is valid for the portion DE.
- The whole of the expression is valid for the portion EB.

It may be noted that the same expression is applicable to all the portions of the beam if all negative terms inside the brackets are omitted for a particular section. If x is less than c, then the last term is omitted. If x is less than b, then the last two terms are omitted and so on. While integrating, the brackets are integrated as a whole, i.e.,

$$EI\frac{dy}{dx} = -W_1\frac{x^2}{2} + C_1 \Big| + \frac{R_1}{2}(x-b)^2 \Big| - \frac{W_2}{2}(x-b)^2 \Big| - \frac{W_3}{2}(x-c)^2$$

and $$EIy = -W_1\frac{x^3}{6} + C_1x + C_2 \Big| + \frac{R_1}{6}(x-a)^3 \Big| - \frac{W_2}{6}(x-b)^3 \Big| - \frac{W_3}{6}(x-c)^3$$

constants C_1 and C_2 are evaluated from the end conditions.

Consider the case of a simply supported beam having an eccentric load W (Fig. 7.23).

Taking moments about B, $R_a \times l = Wb$ or $R_a = \dfrac{Wb}{l}$

Similarly, $R_b = \dfrac{Wa}{l}$

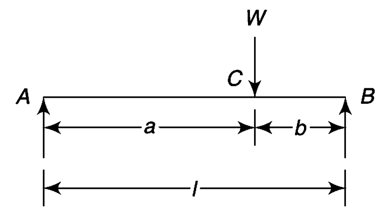


Fig. 7.23

- Bending moment for any segment $= \dfrac{Wbx}{l} \Big| - W(x-a)$

or $$EIy'' = \frac{Wbx}{l} \Big| - W(x-a)$$

The portion to the left of this line is valid for the portion AC whereas the whole expression is valid for the portion CB.

Integrating, $$EIy' = \frac{Wbx^2}{2l} + C_1 \,|\, - W\frac{(x-a)^2}{2}$$

Integrating again, $$EIy = \frac{Wbx^3}{6l} + C_1x + C_2 \,|\, - W\frac{(x-a)^3}{6}$$

- At A, $x = 0$, $y = 0$; The equation for the elastic curve for the portion AC is

$$EIy = \frac{Wbx^3}{6l} + C_1x + C_2\,; \quad \therefore \quad \frac{Wbx^3}{6l} + C_1x + C_2 = 0 \quad \text{or} \quad C_2 = 0$$

At B, $x = l$, $y = 0$; The whole equation is valid for the portion CB

$$\therefore \quad 0 = \frac{Wbl^2}{6} + C_1l - W\frac{b^3}{6} \quad \text{or} \quad C_1 = -\frac{Wb}{6l}(l^2 - b^2)$$

Thus the slope and deflection are given by

$$EIy' = \frac{Wbx^2}{2l} - \frac{Wb}{6l}(l^2 - b^2) \Big| - W\frac{(x-a)^2}{2}$$

and $$EIy = \frac{Wbx^3}{6l} - \frac{Wb}{6l}(l^2 - b^2)x \Big| - W\frac{(x-a)^3}{6}$$

- Deflection under the load, $EIy = \dfrac{Wba^3}{6l} - \dfrac{Wb}{6l}(l^2 - b^2)a = -\dfrac{Wab}{6l}(l^2 - b^2 - a^2)$

$$= -\frac{Wab}{6l}[(a+b)^2 - b^2 - a^2] = -\frac{Wab}{6l}[a^2 + b^2 + 2ab - b^2 - a^2] = -\frac{Wa^2b^2}{3l} \qquad (7.27)$$

- The value of x for maximum deflection can be obtained as is done in the previous section.

Example 7.8 A simply supported beam of 8-m length carries two point loads of 64 kN and 48 kN at 1 m and 4 m respectively from the left-hand end. Find the deflection under each load and the maximum deflection. E = 210 GPa and I = 180 × 10^6 mm^4.

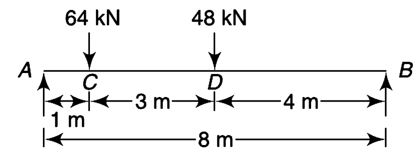


Fig. 7.24

Solution

Refer Fig. 7.24.

Given A simply supported beam carrying point loads as shown in Fig. 7.24

E = 210 GPa $\qquad I$ = 180 × 10^6 mm^4.

To find

— Deflection under each load

— Maximum deflection

Taking moments about B, $R_a \times 8 = 64 \times 7 + 48 \times 4$ or $R_a = 80$ kN

and $R_b = 64 + 48 - 80 = 32$ kN

Deflection equation

At any section x from A, $EI\dfrac{d^2y}{dx^2} = 80x \,|- 64(x-1)\,|- 48(x-4)$

Integrating, $EI\dfrac{dy}{dx} = \dfrac{80x^2}{2} + C_1 \left| - \dfrac{64(x-1)^2}{2} \right| - \dfrac{48(x-4)^2}{2}$

$$= 40x^2 + C_1 |-32(x-1)^2| - 24(x-4)^2 \qquad \text{(i)}$$

Integrating again, $EIy = \dfrac{40x^3}{3} + C_1x + C_2 \left| - \dfrac{32(x-1)^3}{3} \right| - \dfrac{24(x-4)^3}{3}$ (ii)

- At A, $x = 0$, $y = 0$; the equation for the elastic curve for the portion AD is

  $EIy = \dfrac{40x^3}{3} + C_1x + C_2$; or $C_2 = 0$

  At B, $x = 8$ m, $y = 0$; The whole equation is valid for the portion DB i.e.

  $$EIy = \frac{40x^3}{3} + C_1x - \frac{32(x-1)^3}{3} - \frac{24(x-4)^3}{3} \qquad ......(C_2 = 0)$$

  or $\quad 0 = \dfrac{40 \times 8^3}{3} + C_1 \times 8 - \dfrac{32(8-1)^3}{3} - \dfrac{24(8-4)^3}{3}$ or $C_1 = -332$

  Thus (ii) becomes $EIy = \dfrac{40x^3}{3} - 332x \left| - \dfrac{32(x-1)^3}{3} \right| - \dfrac{24(x-4)^3}{3}$

Deflection under the loads

- Deflection under 64-kN load,

  $$EIy = \frac{40 \times 1^3}{3} - 332 \times 1 = -318.7 \text{ kN.m}^3 \text{ or } -318.7 \times 10^{12} \text{ N·mm}^3$$

  and $\quad y = -\dfrac{318.7 \times 10^{12}}{210\,000 \times 180 \times 10^6} = -8.43$ mm or 8.43 mm (numerically)

- Under 48-kN load,

$$EIy = \frac{40 \times 4^3}{3} - 332 \times 4 - \frac{32 \times 3^3}{3} = -762.7 \text{ kN.m}^3 \text{ or } -762.7 \times 10^{12} \text{ N.mm}^3$$

and $$y = -\frac{762.7 \times 10^{12}}{210\,000 \times 180 \times 10^6} = -20.18 \text{ mm} \quad \text{or} \quad 20.18 \text{ mm} \qquad \text{(numerically)}$$

Maximum deflection

Assuming maximum deflection to be between DC and differentiating (i),

$$0 = 40x^2 - 332 - 32(x-1)^2 \text{ or } 0 = 40x^2 - 332 - 32x^2 - 32 + 64x$$

or $$x^2 + 8x - 45.5 = 0 \text{ or } x = \frac{-8 \pm \sqrt{8^2 + 4 \times 45.5}}{2} = 3.84 \text{ m}$$

$$EIy = \frac{40 \times 3.84^3}{3} - 332 \times 3.84 - \frac{32 \times 2.84^3}{3} = -764.2 \text{ kN}\cdot\text{m}^3 \text{ or } -764.2 \times 10^{12} \text{ N}\cdot\text{mm}^3$$

and $$y_{max} = -\frac{764.2 \times 10^{12}}{210\,000 \times 180 \times 10^6} = -20.22 \text{ mm} \quad \text{or} \quad 20.22 \text{ mm} \qquad \text{(nemerically)}$$

Example 7.9 || A beam of length l and hinged at the two ends carries a clockwise couple M at a distance a from the left end. Determine the slope at each end and the deflection at the point of application of the couple.

Also deduce the expressions for slopes and deflection if the couple acts at the midspan of the beam.

Solution

Refer Fig. 7.25

Given A hinged beam carrying a couple as shown in Fig. 7.25.

To find

— Slope at each end

— Deflection at point of couple

— Slope and deflection if couple acts at midspan

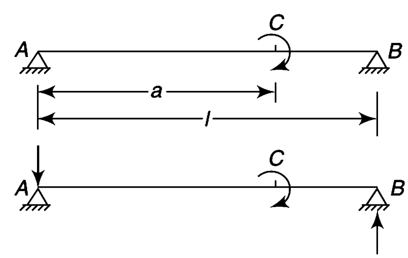


Fig. 7.25

Taking moments about A, $R_b \times l = M$ or $R_b = \frac{M}{l}$ (upwards)

Similarly, $R_a = \frac{M}{l}$ (downwards)

Slope and deflection equations

At any section x from A, $$EI\frac{d^2y}{dx^2} = -\frac{M}{l}x \,\Big|\, + M(x-a)^\circ \qquad \text{(Note the writing of last term)}$$

Integrating, $$EI\frac{dy}{dx} = -\frac{Mx^2}{2l} + C_1 \,\Big|\, + M(x-a) \qquad \text{(i)}$$

Integrating again, $$EIy = -\frac{Mx^3}{6l} + C_1x + C_2 \,\Big|\, + \frac{M}{2}(x-a)^2 \qquad \text{(ii)}$$

- At A, $x = 0, y = 0$; The equation for the elastic curve for the portion AC is

$$EIy = -\frac{Mx^3}{6l} + C_1x + C_2 \quad \text{or} \quad C_2 = 0$$

At B, $x = l$, $y = 0$; The whole equation is valid for the portion CB

or $\quad 0 = -\dfrac{Mx^3}{6l} + C_1 x + \dfrac{M}{2}(x-a)^2$

or $\quad C_1 = \dfrac{Ml^2}{6l} - \dfrac{M}{2l}(l^2 - 2al + a^2) = -\dfrac{M}{6l}(2l^2 - 6al + 3a^2)$

Thus slope and deflection equations are

$$EI\frac{dy}{dx} = -\frac{Mx^2}{2l} - \frac{M}{6l}(2l^2 - 6al + 3a^2)\Bigg| + M(x-a)$$

and
$$EIy = -\frac{Mx^3}{6l} - \frac{M}{6l}(2l^2 - 6al + 3a^2)x\Bigg| + \frac{M}{2}(x-a)^2$$

- ***Slope at the ends***

Slope at A, $\dfrac{dy}{dx} = -\dfrac{M}{6lEI}(2l^2 - 6al + 3a^2)$

Slope at B, $EI\dfrac{dy}{dx} = -\dfrac{Ml}{2} - \dfrac{M}{6l}(2l^2 - 6al + 3a^2) + M(l-a)$

or $\quad \dfrac{dy}{dx} = -\dfrac{M}{6EIl}(3l^2 + 2l^2 - 6al + 3a^2 - 6l^2 + 6al) = -\dfrac{M}{6EIl}(3a^2 - l^2)$

- ***Deflection at couple point***

Deflection at C, $EIy = -\dfrac{Ma^3}{6l} - \dfrac{M}{6l}(2l^2 - 6al + 3a^2)a$

or $\quad EIy = -\dfrac{Ma}{6l}(2l^2 - 6al + 3a^2 + a^2)$

or $\quad y = -\dfrac{Ma}{3EIl}(l^2 - 3al + 2a^2) = -\dfrac{Ma}{3EIl}(l-a)(l-2a)$

Couple at midspan

If C is the midpoint of the span,

Slope at A, $\quad \dfrac{dy}{dx} = -\dfrac{M}{6lEI}\left(2l^2 - \dfrac{6l^2}{2} + \dfrac{3l^2}{4}\right) = \dfrac{Ml}{6EI}$

Slope at B, $\quad \dfrac{dy}{dx} = -\dfrac{M}{6EIl}\left(\dfrac{3l^2}{4} - l^2\right) = \dfrac{Ml}{24EI}$

Deflection at C, $\quad y = -\dfrac{Ma}{3EIl}(l-a)(l-2a) = 0 \quad \ldots.(l = 2a)$

Example 7.10 || A beam is loaded as shown in Fig. 7.26. Determine the deflection of the points C and D. Take EI = 90 000 kN·m^2.

Solution

Given A beam loaded as shown in Fig. 7.26.

To find Deflection of points C and D

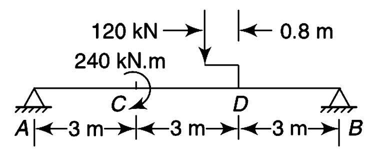


Fig. 7.26

The effect of the bracket is to apply a load of 120 kN and a bending moment of (120 × 0.8 = 96) kN.m at the point D (Fig. 7.27).

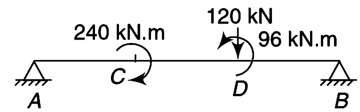


Fig. 7.27

Taking moments about B,

$$R_a \times 9 + 240 - 120 \times 3 - 96 = 0$$

or $$R_a = 24 \text{ kN and } R_b = 120 - 24 = 96 \text{ kN}$$

Deflection equation

At any section x from A,

$$EI\frac{d^2y}{dx^2} = 24x \,|+ 240(x-3)^\circ - 120(x-6)\,| - 96(x-6)^\circ$$

(Note the writing of second and last terms)

Integrating, $$EI\frac{dy}{dx} = 12x^2 + C_1 \Big| + 240(x-3) - 60(x-6)^2 \,| - 96(x-6) \quad \text{(i)}$$

Integrating again, $EIy = 4x^3 + C_1x + C_2| + 120(x-3)^2 - 20(x-6)3| -48(x-6)^2$ (ii)

- At A, $x = 0$, $y = 0$; The equation for the elastic curve for the portion AC is

$$EIy = 4x^3 + C_1x + C_2 \text{ or } C_2 = 0$$

At B, $x = 9$ m, $y = 0$; The whole equation is valid for the portion DB

$EIy = 4x^3 + C_1x + 120(9-3)^2 - 20(9-6)^3 - 48(9-6)^2$

or $2916 + C_1 \times 9 + 4320 - 540 - 432 = 0$ or $C_2 = -696$

Thus, the deflection equations is

$$EIy = 4x^3 - 696x| + 120(x-3)^2 - 20(x-6)^3| -48(x-6)^2$$

Deflections at C and D

Deflection at the point C, $EIy = 4x^3 - 696x = 4 \times 3^3 - 696 \times 3 = -1980$

or $$y = \frac{1980}{90\,000} = 0.022 \text{ m} = 22 \text{ mm}$$

Deflection at point D, $EIy = 4 \times 6^3 - 696 \times 6 + 120(6-3)^2 = -2232$

or $$y = \frac{-2232}{90\,000} = -0.0248 \text{ m} \quad \text{or} \quad 0.248 \text{ mm (numerically)}$$

Example 7.11 || A cantilever beam of 9-m length is acted upon by three couples, one counter-clockwise of 80 kN.m at the free end, second clockwise of 200 kN.m at 3 m from the free end and the third counter-clockwise of 300 kN·m at 6 m from the free end. Determine the slope and deflection of the beam at couple points. Take EI = 90 000 kN.m^2.

Solution

Given A cantilever beam acted upon by three couples as shown in Fig. 7.28a.

To find Slope and deflection of beam at couple points

Reaction couple at A = 300 +80 – 200 =180 kN·m (clockwise) (Fig. 7.28b)

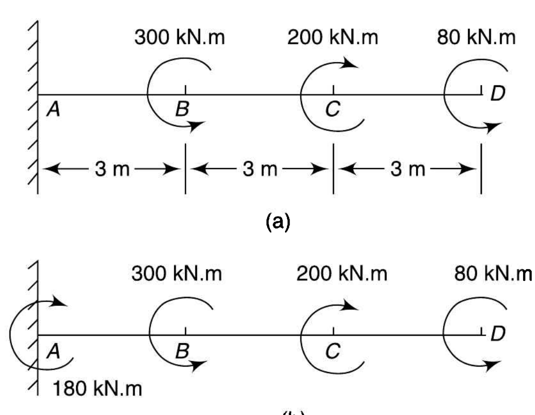


Fig. 7.28

Slope and deflection equations

At any section x from A,

$$EI\frac{d^2y}{dx^2} = 180 \mid - 300(x-3)^\circ \mid + 200(x-6)^\circ$$

(Note the writing of last two terms)

Integrating, $EI\frac{dy}{dx} = 180x + C_1 \mid - 300(x-3) + 200(x-6)$ (i)

At $x = 0$, $\frac{dy}{dx} = 0$; $\therefore C_1 = 0$

Integrating (i) again, $EIy = 90x^2 + C_2|-150(x-3)^2| + 100(x-6)^2$ (ii)

At $x = 0$, $y = 0$; $\therefore C_2 = 0$

Slopes at couple points

Slope at B, $EI\frac{dy}{dx} = 180 \times 3 = 540$ or $\frac{dy}{dx} = \frac{540}{90\,000} = 0.006$ rad

Slope at C, $EI\frac{dy}{dx} = 180 \times 6 - 300 \times 3 = 180$ or $\frac{dy}{dx} = \frac{180}{90\,000} = 0.002$ rad

Slope at D,

$$EI\frac{dy}{dx} = 180 \times 9 - 300 \times 6 + 200 \times 3 = 540 \quad \text{or} \quad \frac{dy}{dx} = \frac{540}{90\,000} = 0.006 \text{ rad}$$

Deflections at couple points

Deflection at B, $EIy = 90x^2 = 90 \times 3^2 = 810$ or $y = \frac{810}{90\,000} \times 1000$ m or 9 mm

Deflection at C, $EIy = 90 \times 6^2 - 150 \times 3^2 = 1890$ or $y = \frac{1890}{90\,000} \times 1000 = 21$ mm

Deflection at D,

$$EIy = 90 \times 9^2 - 150 \times 6^2 + 100 \times 3^2 = 2790 \quad \text{or} \quad y = \frac{2790}{90\,000} \times 1000 = 31 \text{ mm}$$

Example 7.12 The distance between the supports of a simply supported beam is l. The beam has two equal overhangs of length a over each support. The beam carries a point load $2W$ at the centre and a point load W at each end. Determine

(i) the slopes at the ends and at the supports
(ii) the deflection at the ends and at the centre

Solution

Given A simply supported beam with two overhangs and loaded as shown in Fig. 7.29.

To find

— slopes at the ends and at supports
— deflection at ends and at centre

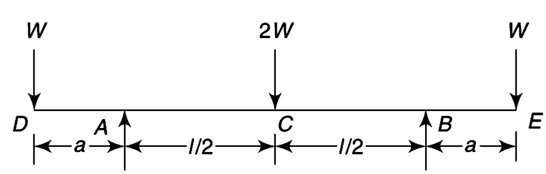


Fig. 7.29

By symmetry, $R_a = R_b = 2W$

Slope equation

Considering left half of the beam,

Bending moment at any section x from D, $EI\dfrac{d^2y}{dx^2} = -Wx \mid + 2W(x-a)$

Integrating, $$EI\frac{dy}{dx} = -\frac{Wx^2}{2} + C_1 \Bigg| + W(x-a)^2 \qquad \text{(i)}$$

At $x = \dfrac{l}{2} + a$ $\dfrac{dy}{dx} = 0$; $\therefore$ $0 = -\dfrac{W}{2}\left(\dfrac{l}{2}+a\right)^2 + C_1 \Bigg| + W\left(\dfrac{l}{2}\right)^2$

or $$C_1 = \frac{W}{2}\left(\frac{l}{2}+a\right)^2 - W\left(\frac{l}{2}\right)^2 = \frac{W}{8}(4a^2 + 4al - l^2)$$

(i) becomes, $$EI\frac{dy}{dx} = -\frac{Wx^2}{2} + \frac{W}{8}(4a^2 + 4al - l^2) \Bigg| + W(x-a)^2$$

Slopes at ends and supports

Slope at $x = 0$, i.e., at D, $\dfrac{dy}{dx} = \dfrac{W}{8EI}(4a^2 + 4al - l^2)$ (consider only first portion)

Slope at E is same as at D.

Slope at A, ($x = a$), $\dfrac{dy}{dx} = -\dfrac{Wa^2}{2EI} + \dfrac{W}{8EI}(4a^2 + 4al - l^2)) = \dfrac{Wl}{8EI}(4a - l))$

Slope at B is same as at A.

Deflection equation

Integrating again, $EIy = -\dfrac{Wx^3}{6} + \dfrac{W}{8}(4a^2 + 4al - l^2)x + C_2 \Bigg| + \dfrac{W}{3}(x-a)^3$

At $x = a$; $y = 0$,

$$0 = -\frac{Wa^3}{6} + \frac{W}{8}(4a^2 + 4al - l^2)a + C_2$$

or $$C_2 = \frac{Wa^3}{6} - \frac{W}{8}(4a^2 + 4al - l^2)a = -\frac{Wa}{24}(8a^2 + 12al - 3l^2)$$

$$EIy = -\frac{Wx^3}{6} + \frac{W}{8}(4a^2 + 4al - l^2)x - \frac{Wa}{24}(8a^2 + 12al - 3l^2) \Bigg| + \frac{W}{3}(x-a)^3$$

Deflection at the ends and at the centre

Deflection at D, ($x = 0$), $EIy = -\dfrac{Wa}{24}(8a^2 + 12al - 3l^2)$ (consider only first portion)

Deflection at E is same as at D.

Deflection at the centre, $x = \dfrac{l}{2} + a$,

$$EIy = -\frac{W}{6}\left(\frac{l}{2}+a\right)^3 + \frac{W}{8}(4a^2 + 4al - l^2)\left(\frac{l}{2}+a\right) - \frac{Wa}{24}(8a^2 + 12al - 3l^2) \Bigg| + \frac{W}{3}\left(\frac{l}{2}\right)^3$$

$$Ely = -\frac{W}{6}\left(\frac{l^3}{8} + a^3 + \frac{3al^2}{4} + \frac{3a^2l}{2}\right) + \frac{W}{16}(4a^2l + 4al^2 - l^3)$$

$$+\frac{W}{8}(4a^3 + 4a^2l - al^2) - \frac{W}{24}(8a^3 + 12a^2l - 3al^2)\Bigg| + \frac{Wl^3}{24}$$

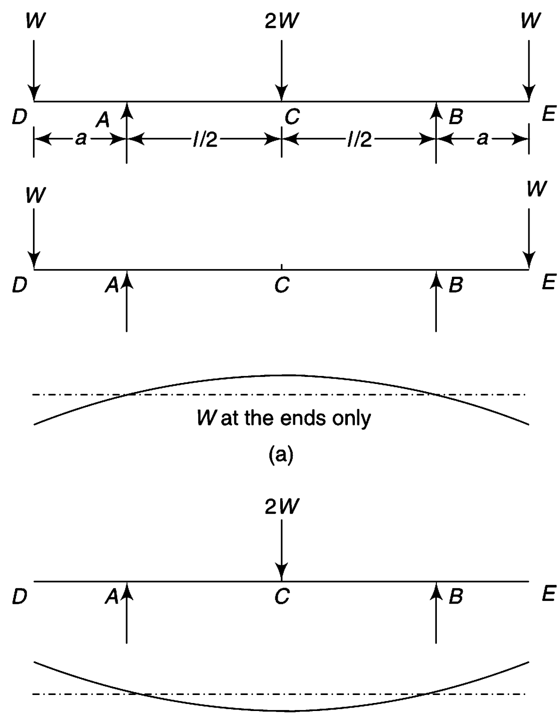
$$y = \frac{Wl^2}{24EI}(3a - l)$$

Alternative solution

Using the principle of superposition,
assume the loading to be a combination of the following loads:

(i) W at the ends only (Fig.7.30a)

(ii) $2W$ at the centre only (Fig. 7.30b)



W at the ends only

(a)

2 *W* at the centre only

(b)

Fig. 7.30

W at the ends

Reaction of support at $A = W$

Bending moment in portion $DA = Wx$

Bending moment in portion $AC = Wx - W(x - a) = Wa$ = constant

Since the loading on the beam is symmetrical, the slope at C must be zero. As in a cantilever the slope at the fixed end is zero, the portion AC may be considered as a cantilever with fixed end at C and A as free end subjected to a constant bending moment Wa.

$$\therefore \quad \text{slope at } A = -\frac{M}{EI}x = -\frac{Wa}{EI}\left(-\frac{l}{2}\right) = \frac{Wal}{2EI} \qquad \text{(Eq. 7.13)}$$

Slope at D = Slope at A + Slope at D assuming AD as a cantilever fixed at A,

$$= \frac{Wal}{2EI} + \frac{Wa^2}{2EI} = \frac{Wa}{2EI}(l + a)$$

Deflection at D = Deflection at D due to slope at A + Deflection at D assuming AD as cantilever fixed at A

$$= \frac{Wal}{2EI} \cdot a + \frac{Wa^3}{3EI} = \frac{Wa^2}{6EI}(3l + 2a)$$

Deflection at C = Deflection of A relative to C considering AC as cantilever fixed at C

$$= \frac{M}{EI}\frac{x^2}{2} = \frac{Wa}{2EI}\left(\frac{l}{2}\right)^2 = \frac{Wal}{8EI}$$

2W at the centre

Reaction of support at $A = W$

Now AB is simply supported beam with DA and BE overhangs.

$$\therefore \quad \text{slope at } A = \text{Slope at } D = \frac{2Wl^2}{16EI} = \frac{Wl^2}{8EI}$$

$$\text{Deflection at } D = \text{Slope at } A \times \text{Overhang} = \frac{Wl^2}{8EI} \cdot a = \frac{Wal^2}{8EI}$$

$$\text{Deflection at } C = \frac{2Wl^3}{48EI} = \frac{Wl^3}{24EI}$$

Net slopes and deflections

$$\text{Slope at } A = \frac{Wal}{2EI} - \frac{Wl^2}{8EI} = \frac{Wl}{8EI}(4a - l)$$

$$\text{Slope at } D = \frac{Wa}{2EI}(l + a) - \frac{Wl^2}{8EI} = \frac{Wl}{8EI}(4a^2 + 4al - l^2)$$

$$\text{Deflection at } D = \frac{Wa^2}{6EI}(3l + 2a) - \frac{Wal^2}{8EI} = \frac{Wa}{24EI}(8a^2 + 12al - 3l^2)$$

$$\text{Deflection at } C = \frac{Wal^2}{8EI} - \frac{Wl^3}{24EI} = \frac{Wl^2}{24EI}(3a - l)$$

Example 7.13 The distance between the supports of a simply supported beam is l. The beam has two equal overhangs of length $l/3$ over each support. The beam carries a point load $2W$ at the centre and a point load W at each end. Determine

(i) the slopes at the ends and at the supports

(ii) deflection at the ends and at the centre

Solution

Given A simply supported beam with two overhangs and loaded as shown in Fig. 7.31.

To find

— slopes at the ends and at supports
— deflection at ends and at centre

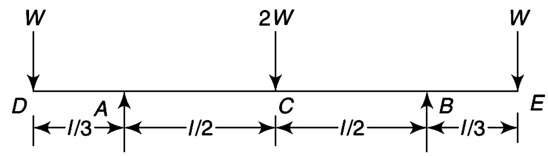


Fig. 7.31

The loaded beam is shown in Fig. 7.31. The problem can be worked as in the previous example or by inserting directly $a = l/3$ in the results obtained in that example.

Slopes

Slope at $A = \dfrac{Wl}{8EI}(4a - l) = \dfrac{Wl^2}{24EI}$; the slope is same at B

Slope at $D = \dfrac{Wl}{8EI}(4a^2 + 4al - l^2) = \dfrac{7}{72}\dfrac{Wl^2}{EI}$; the slope is same at E

Deflections

Deflection at $D = \dfrac{Wa}{24EI}(8a^2 + 12al - 3l^2) = \dfrac{17}{648}\dfrac{Wl^3}{EI}$, the deflection is same at E

Deflection at $C = \dfrac{Wal^2}{8EI} - \dfrac{Wl^3}{24EI} = \dfrac{Wl^2}{24EI}(3a - l) = 0$

Example 7.14 A simply supported beam with distance between the supports as L has an overhang on the right at a distance l from the right support. The beam carries a point load W at the right end.

Determine

(i) slopes over each support
(ii) slope at the right end
(iii) deflection at the right end
(iv) maximum deflection between the supports
(v) above values if $l = L/4$

Solution

Given A simply supported beam with a right hand overhang and a load at right end loaded as shown in Fig. 7.32.

To find

— slope over each support and at right end
— deflection at right end and maximum deflection between supports
— above values for $l = L/4$

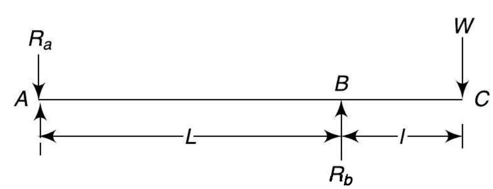


Fig. 7.32

Taking moments about B,

$$R_aL + W \times l = 0 \quad \text{or} \quad R_a = -\frac{Wl}{L} \quad \text{and} \quad R_b = W - \left(-\frac{Wl}{L}\right) = \frac{W(L+l)}{L}$$

Slope and deflection equations

This shows that reaction R_a is actually downwards.

At any distance x from A,

$$EI\frac{d^2y}{dx^2} = -\frac{Wl}{L}x \Big| + \frac{L+l}{L}W(x - L)$$

Integrating, $EI\dfrac{dy}{dx} = -\dfrac{Wl}{2L}x^2 + C_1 \Big| + \dfrac{L+l}{2L}W(x-L)^2$

Integrating again, $EIy = -\dfrac{Wl}{6L}x^3 + C_1x + C_2 \Big| + \dfrac{L+l}{6L}W(x-L)^3$

At A, $x = 0, y = 0$; $C_2 = 0$ (Consider first portion only)

At $x = L, y = 0$, $0 = -\dfrac{Wl}{6L}L^3 + C_1L$ or $C_1 = \dfrac{WlL}{6}$

Thus the slope equation is $EI\dfrac{dy}{dx} = -\dfrac{Wl}{2L}x^2 + \dfrac{WlL}{6} \Big| + \dfrac{L+l}{2L}W(x-L)^2$

Slopes over each support and at right end

Slope at A, $(x = 0)$, $\dfrac{dy}{dx} = \dfrac{WlL}{6EI}$

Slope at B, $(x = L)$, $\dfrac{dy}{dx} = \dfrac{1}{EI}\left(-\dfrac{WlL}{2} + \dfrac{WlL}{6}\right) = -\dfrac{WlL}{3EI}$

Slope at C, $(x = L+l)$, $\dfrac{dy}{dx} = \dfrac{1}{EI}\left(-\dfrac{Wl}{2L}(L+l)^2 + \dfrac{WlL}{6} + \dfrac{L+l}{2L}Wl^2\right) = -\dfrac{Wl}{6EI}(2L+3l)$

Deflection at right end

Deflection equation is $EIy = -\dfrac{Wl}{6L}x^3 + \dfrac{WlL}{6}x \Big| + \dfrac{L+l}{6L}W(x-L)^3$

Deflection at C, $(x = L + l)$,

$$y = \frac{1}{EI}\left[-\frac{Wl}{6L}(L+l)^3 + \frac{WlL}{6}(L+l) + \frac{L+l}{6L}Wl^3\right]$$

$$= \frac{1}{EI}\left[-\frac{Wl}{6L}\{(L+l)^3 - L^2(L+l) - l^2(L+l)\}\right]$$

$$= \frac{1}{EI}\left[-\frac{Wl}{6L}(L^3 + l^3 + 3lL^2 + 3l^2L - L^3 - L^2l - l^2L - l^3)\right]$$

$$= \frac{1}{EI}\left[-\frac{Wl^2}{3}(L+l)\right] = -\frac{Wl^2}{3EI}(L+l)$$

Maximum deflection between supports

Maximum deflection between A and B is where the slope is zero,

$$EI\frac{dy}{dx} = -\frac{Wl}{2L}x^2 + \frac{WlL}{6} = 0 \text{ or } x = L/\sqrt{3}$$

$$\therefore \qquad y_{max} = \frac{1}{EI}\left[-\frac{Wl}{6L}\frac{L^3}{3\sqrt{3}} + \frac{WlL}{6}\frac{L}{\sqrt{3}}\right] = \frac{WlL^2}{9\sqrt{3}EI}$$

When $l = L/4$

When $l = L/4$, the solution can be obtained either from the above results directly or by forming the slope and deflection equations for this case.

Slope at A, $\frac{dy}{dx} = \frac{WL^2}{24EI}$; Slope at B, $\frac{dy}{dx} = -\frac{WL^2}{12EI}$

Slope at C, $\frac{dy}{dx} = -\frac{Wl}{6EI}(2L + 3l) = -\frac{11WL^2}{96EI}$

Deflection at C, $y = -\frac{Wl^2}{3EI}(L + l) = -\frac{5WL^3}{192EI}$; $y_{\max} = \frac{WlL^2}{9\sqrt{3}} = \frac{WL^2}{36\sqrt{3}EI}$

Example 7.15 A beam of length l is loaded with a uniformly distributed load w per unit length. It has one support at the left end and the other at a distance $l/3$ from the right end. Determine the deflection at the right end of the beam.

Solution

Given A simply supported beam with a right-hand overhang and loaded with a uniformly distributed load as shown in Fig. 7.33.

To find Deflection at C

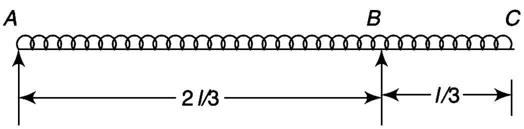


Fig. 7.33

Taking moments about A, $R_b \times \frac{2l}{3} = \frac{wl^2}{2}$ or $R_b = \frac{3wl}{4}$ and $R_a = \frac{wl}{4}$

Slope and deflection equations

At any section x from C, $EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2} \Bigg| + \frac{3wl}{4}\left(x - \frac{l}{3}\right)$

Integrating, $EI\frac{dy}{dx} = -\frac{wx^3}{6} + C_1 \Bigg| + \frac{3wl}{8}\left(x - \frac{l}{3}\right)^2$

Integrating again, $EIy = -\frac{wx^4}{24} + C_1x + C_2 \Bigg| + \frac{wl}{8}\left(x - \frac{l}{3}\right)^3$

At B, $(x = l/3)$, $y = 0$, $0 = -\frac{w}{24}\frac{l^4}{81} + C_1\frac{l}{3} + C_2$ ...(consider only first part)

or $C_1l + 3C_2 = \frac{wl^4}{648}$ (i)

At A, $(x = l)$, $y = 0$, $0 = -\frac{wl^4}{24} + C_1l + C_2 + \frac{wl}{8}\left(\frac{2l}{3}\right)^3$ or $C_1l + C_2 = \frac{wl^4}{216}$ (ii)

Subtracting (ii) from (i) and solving, $C_2 = -\frac{wl^4}{648}$ and $C_1 = \frac{wl^3}{162}$

Thus deflection $EIy = -\frac{wx^4}{24} + \frac{wl^3}{162}x - \frac{wl^4}{648} \Bigg| + \frac{wl}{8}\left(x - \frac{l}{3}\right)^3$

Deflection at the right end

To find deflection at C put $x = 0$ in the first part, $y = \frac{wl^4}{648EI}$

Example 7.16 || A beam of uniform section and length l rests on horizontal supports at its ends. It is loaded with uniformly distributed load w per unit length which extends over a length a from the right hand support. Determine the value of a so that the maximum deflection may occur at the left hand end of the load. If the maximum deflection is expressed by (wl^4/kEI), find the value of k.

Solution

Given A simply supported beam loaded with a uniformly distributed load as shown in Fig. 7.34.

To find

— Value of a so that maximum deflection occurs at left hand end of load

— If maximum deflection is expressed by (wl^4/kEI), to find k

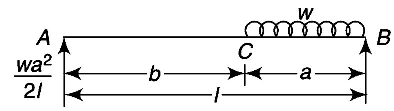


Fig 7.34

Taking moments about B, $R_a l = \dfrac{wa^2}{2}$ or $R_a = \dfrac{wa^2}{2l}$

Slope and deflection equations

At any section x from A, $EI\dfrac{d^2y}{dx^2} = \dfrac{wa^2}{2l}x \,\Big|\, - \dfrac{w(x-b)^2}{2}$

- Integrating, $EI\dfrac{dy}{dx} = \dfrac{wa^2}{4l}x^2 + C_1 \,\Big|\, - \dfrac{w(x-b)^3}{6}$
- Integrating again, $EIy = \dfrac{wa^2}{12l}x^3 + C_1x + C_2 \,\Big|\, - \dfrac{w(x-b)^4}{24}$

At $x = 0, y = 0, \therefore C_2 = 0$ (consider first part only)

At $x = l, y = 0, \therefore 0 = \dfrac{wa^2l^2}{12} + C_1 l \,\Big|\, - \dfrac{w(l-b)^4}{24}$

$$C_1 = -\frac{wa^2l}{12} + \frac{wa^4}{24l} = -\frac{wa^2}{24l}(2l^2 - a^2) \qquad \ldots\ldots (l - b = a)$$

For maximum deflection at C

If y is maximum at C, slope at $C = 0$. Inserting $x = b$ in the slope equation and equating to zero,

Thus $$0 = \frac{wa^2}{4l}b^2 - \frac{wa^2}{24l}(2l^2 - a^2) \qquad \text{(consider first part only)}$$

or $$6b^2 - (2l^2 - a^2) = 0$$

or $$6(l - a)^2 - (2l^2 - a^2) = 0$$

or $$6(l^2 + a^2 - 2la) - 2l^2 + a^2 = 0$$

or $$7a^2 - 12al + 4l^2 = 0$$

$$a = \frac{12l \pm \sqrt{144l^2 - 112l^2}}{14} = \frac{l}{14}(12 \pm 5.657)$$

For practical solution taking negative sign,

$$a = \frac{l}{14}(12 - 5.657) = 0.453l$$

Determination of value of k

$$C_1 = -\frac{w(0.453l)^2 l}{12} + \frac{w(0.453l)^4}{24l} = -0.01535wl^3$$

At C, $x = b = l - a = l - 0.453l = 0.547\ l$

$$EIy = \frac{w(0.453l)^2}{12l}(0.547l)^3 - 0.01535wl^3 \times 0.547l$$

or $$y_{\max} = \frac{wl^4}{EI}(0.0028 - 0.0084) = -\frac{wl^4}{178.6EI}$$

or $$y_{\max} = \frac{wl^4}{178.6EI} \quad \text{(numerically)}$$

$\therefore$ $$k = 178.6$$

Example 7.17 A simply supported beam of length l carries a uniformly distributed load of intensity w starting from a distance $l/4$ from the left end and ending at the midspan. Deduce the expressions for slope and deflection at any point.

Solution

Given A simply supported beam loaded with a uniformly distributed load as shown in Fig. 7.35a.
To find Expressions for slope and deflection at any point

When the load does not reach the end support, it is treated in a way that first it reaches to the end and then an upward load is superimposed for that portion where originally there is no load i.e. the loading is considered downward from C to B and upwards from D to B (Fig.7.35b).

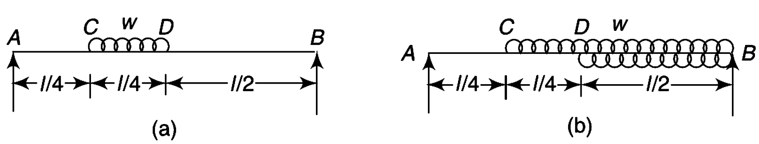


Fig. 7.35

Taking moments about B,

$$R_a \cdot l = w \cdot \frac{l}{4}\left(\frac{l}{2} + \frac{l}{8}\right) \text{ or } R_a = \frac{5wl}{32}$$

Slope and deflection equations
At any point x from A,

$$EI\frac{d^2y}{dx^2} = \frac{5wl}{32}x \Bigg| - \frac{w(x - l/4)^2}{2} \Bigg| + \frac{w(x - l/2)^2}{2}$$

Integrating, $$EI\frac{dy}{dx} = \frac{5wl}{64}x^2 + C_1 \Bigg| - \frac{w}{6}\left(x - \frac{l}{4}\right)^3 \Bigg| + \frac{w}{6}\left(x - \frac{l}{2}\right)^3$$

Integrating again, $$EIy = \frac{5wl}{192}x^3 + C_1 x + C_2 \Bigg| - \frac{w}{24}\left(x - \frac{l}{4}\right)^4 \Bigg| + \frac{w}{24}\left(x - \frac{l}{2}\right)^4$$

At $x = 0$, $y = 0$; $\therefore$ $C_2 = 0$ ...(considering first part only)

At $x = l, y = 0$; $\therefore$ $0 = \frac{5wl^4}{192} + C_1 l - \frac{w}{24}\cdot\frac{81}{256}l^4 + \frac{w}{24}\cdot\frac{1}{16}l^4$ or $C_1 = -\frac{95}{6144}wl^3$

Thus

Slope is $\frac{dy}{dx} = \frac{1}{EI}\left[\frac{5wl}{64}x^2 - \frac{95}{6144}wl^3 \middle| -\frac{w}{6}\left(x-\frac{l}{4}\right)^3 \middle| +\frac{w}{6}\left(x-\frac{l}{2}\right)^3\right]$

and deflection is

$$EIy = \frac{1}{EI}\left[\frac{5wl}{192}x^3 - \frac{95}{6144}wl^3x \middle| -\frac{w}{24}\left(x-\frac{l}{4}\right)^4 \middle| +\frac{w}{24}\left(x-\frac{l}{2}\right)^4\right]$$

Example 7.18 || A simply supported beam has its supports 8 m apart at A and B. It carries a uniformly distributed load of 6 kN/m between A and B starting from 1 m and ending at 5 m from A. The end B of the beam has an overhang of 1 m and at the free end a concentrated load of 8 kN is applied. Determine deflection of the free end and the maximum deflection between A and B. Take E = 210 GPa and $I = 20 \times 10^6$ mm^4.

Solution

Given A simply supported beam with a right hand overhang and loaded with a point load and a uniformly distributed load as shown in Fig. 7.36a.

E = 210 GPa $\quad I = 20 \times 10^6$ mm^4

To find

— Deflection of free end

— Maximum deflection between supports

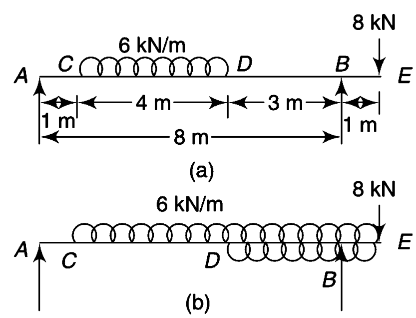


Fig. 7.36

To apply Macaulay's method, a uniformly distributed load has to be continuous up to the end E of the beam. To compensate the same, an upward uniformly distributed load has to be considered from D to E as shown in Fig.7.36b.

Taking moments about A, $R_b \times 8 = 8 \times 9 + 4 \times 6 \times 3$ or R_b = 18 kN

and $\quad R_a = 8 + 24 - 18 = 14$ kN

Slope and deflection equations

At any section x from A, $\quad EI\frac{d^2y}{dx^2} = 14x \mid -\frac{6(x-1)^2}{2} \middle| + \frac{6(x-5)^2}{2} \middle| + 18(x-8)$

Integrating, $\quad EI\frac{dy}{dx} = \frac{14x^2}{2} + C_1 \middle| -\frac{6(x-1)^3}{6} \middle| + \frac{6(x-5)^3}{6} \middle| + \frac{18(x-8)^2}{2}$

or $\quad EI\frac{dy}{dx} = 7x^2 + C_1 \mid -(x-1)^3 \mid + (x-5)^3 \mid + 9(x-8)^2$

Integrating again $\quad EIy = \frac{7x^3}{3} + C_1\cdot x + C_2 \middle| -\frac{(x-1)^4}{4} \middle| + \frac{(x-5)^4}{4} \middle| + 3(x-8)^3$

At A, $(x = 0), y = 0$, $\therefore C_2 = 0$ ...(consider only first part)

At B, $(x = 8 \text{ m}), y = 0$, $0 = \frac{7\times 8^3}{3} + 8C_1 - \frac{(8-1)^4}{4} + \frac{(8-5)^4}{4}$; $C_1 = -76.83$

$\therefore$ slope and deflection equations are as follows:

$$EI\frac{dy}{dx} = 7x^2 - 76.83 \,|- (x-1)^3 \,| + (x-5)^3 + 9(x-8)^2$$

$$EIy = \frac{7x^3}{3} - 76.83x \left| - \frac{(x-1)^4}{4} \right| + \frac{(x-5)^4}{4} \Bigg| + 3(x-8)^3$$

Deflection of free end

Deflection at E, ($x = 9$ m),

$$EIy = \frac{7\times 9^3}{3} - 76.83\times 9 - \frac{(9-1)^4}{4} + \frac{(9-5)^4}{4} + 3(9-8)^3 \Bigg| = 52.5 \text{ kN}\cdot\text{m}^3$$

$$y = \frac{52.5\times 10^{12}}{210\,000\times 20\times 10^6} = 12.5 \text{ mm}$$

Maximum deflection between supports

Let the maximum deflection be at a distance x from A. Assuming it between C and D and putting the slope equal to zero,

$$0 = 7x^2 - 76.83 - (x-1)^3 = 0 \quad \text{or} \quad (x-1)^3 - 7x^2 + 76.83 = 0$$

Solving by trial a error, $x = 3.72$

$$EIy = \frac{7\times 3.72^3}{3} - 76.83\times 3.72 - \frac{(3.72-1)^4}{4} = -179.4 \text{ kN}\cdot\text{m}^3$$

$$y = \frac{179.4\times 10^{12}}{210\,000\times 20\times 10^6} = 42.7 \text{ mm} \qquad \text{......(numerically)}$$

7.5 MOMENT–AREA METHOD (MOHR'S THEOREMS)

Figure 7.37 shows the shape of a deflected beam between two chosen points A and B. In order to find the difference in slopes of these points, consider two sections P and Q at a very small distance ds along the curve. After bending, the normal sections subtend an angle $d\theta$ at the centre of curvature.

Now, $ds = R\,d\theta$ or $d\theta = \dfrac{ds}{R} = ds\cdot\dfrac{M}{EI}$

$$\text{.....} \left(\frac{1}{R} = \frac{M}{EI}\right)$$

Since the curvature is usually very small, $ds \approx dx$.

or $d\theta = \dfrac{M}{EI}\cdot dx$

As $d\theta$ is equal to the change of slope between points P and Q, the change of slope between A and B is given by

$$\theta_A^B = \sum\nolimits_A^B d\theta = \int_A^B \frac{M\cdot dx}{EI}$$

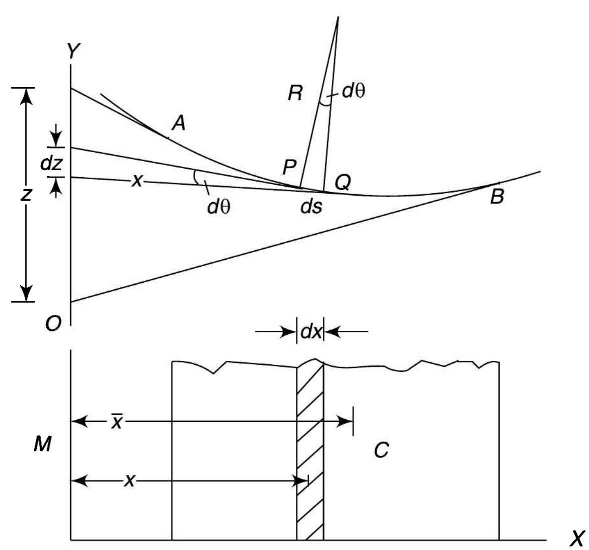


Fig. 7.37

In the above expression, $M \cdot dx$ is the area of the bending moment diagram between P and Q.

Thus $\theta_A^B = \dfrac{1}{EA} \times$ Area of bending moment diagram between A and B (7.28)

From the above, Mohr's first moment-area theorem can be stated as below:

The difference of slopes between any two points on an elastic curve of a beam is equal to the net area of the bending moment diagram between these two points divided by EI.

The above relation can also be derived from $\dfrac{d^2y}{dx^2} = \dfrac{M}{EI}$ by integrating it between A and B, i.e.,

$$\left[\frac{dy}{dx}\right]_A^B = \int_A^B \frac{Mdx}{EI} \quad \text{or} \quad \left[\frac{dy}{dx}\right]_B - \left[\frac{dy}{dx}\right]_A = \frac{A}{EI} \tag{i}$$

Now, consider any vertical line OY. Let the tangents at P and Q to the elastic line cut off an intercept dz on this line. The angle between the tangents will also be equal to $d\theta$.

Now, $$d\theta = \frac{M}{EI} \cdot dx$$

Multiplying both sides by x, $x \cdot d\theta = \dfrac{M}{EI} \cdot x \cdot dx$

As the slope everywhere is small, x represents the distance of points A or B from the vertical.

Thus the above equation may be written as

$$dz = \frac{(M \cdot dx)x}{EI}$$

The LHS of the above equation represents the vertical intercept on the line OY given by the tangents at the points P and Q. The RHS is equal to the net moment about the same line OY of the bending moment diagram between points P and Q divided by EI.

The deflection due to bending of all elemental portions between A and B will be given by integrating the above equation, i.e.,

$$z = \int \frac{(M \cdot dx)x}{EI} = \frac{A\bar{x}}{EI} \tag{7.29}$$

where

z = intercept on a vertical line OY between the tangents at any points A and B
A = area of the bending moment diagram between A and B
$\bar{x}$ = distance of the centroid of the area from the vertical line OY

The above equation leads to the statement of Mohr's second theorem:

The intercepts on a given line between the tangents to the elastic curve of a beam at any two points is equal to the net moment taken about that line of the area of the bending moment diagram between the two points divided by EI.

Usually, it is convenient to break down the bending moment diagram into a number of simple figures. The intercept z is considered positive when the tangent at B meets the line OY below the tangent at A.

This method is useful in those cases in which it can give an easier solution than mathematical treatment. In such cases, most of the times a point of zero slope is known. The deflection at any point can be found by considering the area of the bending moment diagram between that point and point of zero slope and taking moments about the point where the deflection is to be found. So, this method is useful in problems of cantilevers (zero slope at fixed ends), symmetrically loaded simply supported beams (zero slope at the centre) and built-in beams (zero slope at each end).

Examples

(i) Cantilever with a Concentrated Load at the Free End Figure 7.38 shows the cantilever with the concentrated load W at the free end and its bending moment diagram.

Area of the bending moment diagram, $A = -\frac{1}{2} l \cdot Wl = -\frac{Wl^2}{2}$

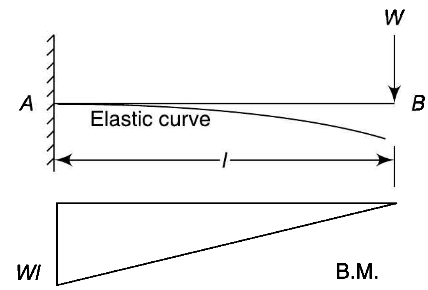


Fig. 7.38

- According to Mohr's first moment-area theorem,
  Difference of slopes between A and B
  = (Area of bending moment diagram between A and B)/EI
  As slope at A is zero,
  $\therefore$ Slope of $B = \frac{A}{EI} = -\frac{Wl^2}{2EI}$
- Deflection at B = Intercept on a vertical line at B between tangents at A and B on elastic curve
  According to Mohr's second moment-area theorem,
  Intercept on a vertical line at B made by tangents at A and B on elastic curve
  = (Net moment of area of bending moment diagram between A and B about B)/EI
  $\therefore$ Deflection at $B = \frac{A\bar{x}}{EI}$
  where $\bar{x}$ = distance of the centroid of the area from B
  Thus, deflection $= -\frac{Wl^2}{2EI} \cdot \frac{2l}{3} = -\frac{Wl^3}{3EI}$ (7.30)

(ii) Cantilever with Uniformly Distributed Load Figure 7.39 shows the cantilever with uniformly distributed load and its bending moment diagram.

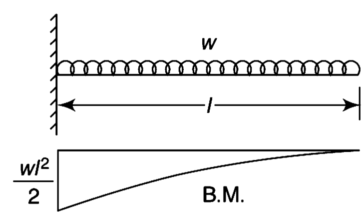


Fig. 7.39

Area of the bending moment diagram, $A = \frac{1}{3} l \cdot \frac{wl^2}{2} = \frac{wl^3}{6}$

As explained above for concentrated load,

Slope at free end $= \frac{A}{EI} = -\frac{wl^3}{6EI}$

Deflection $= \frac{A\bar{x}}{EI} = -\frac{wl^3}{6EI} \cdot \frac{3l}{4} = -\frac{wl^4}{8EI}$ ($\bar{x}$ from B) (7.31)

(iii) Simply Supported Beam with Concentrated Load at the Midspan The bending moment diagram has been shown in Fig. 7.40

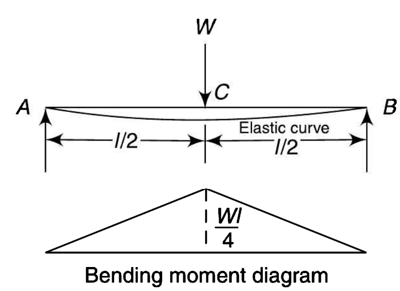


Fig. 7.40

Area of the bending moment diagram between A and C, $A = \frac{1}{2} \cdot \frac{l}{2} \cdot \frac{Wl}{4} = \frac{Wl^2}{16}$

- According to Mohr's first moment-area theorem,
  Difference of slopes between A and C
  = (Area of bending moment diagram between A and C)/EI

As slope at C is zero, (due to symmetry)

Slope at $A = \dfrac{A}{EI} = -\dfrac{Wl^2}{16EI}$

- Deflection at C with respect to A = Deflection of A with respect to C
  = Intercept on a vertical line at A between tangents at A and C on elastic curve

According to Mohr's second moment-area theorem,
Intercept on a vertical line at A made by tangents at A and C on elastic curve
= (Net moment of area of bending moment diagram between A and C about A)/EI

$\therefore$ deflection at $A = \dfrac{A\bar{x}}{EI}$

where $\bar{x}$ = distance of centroid of the bending moment diagram from $A = \dfrac{2}{3}\cdot\dfrac{l}{2} = \dfrac{l}{3}$

$$\therefore \quad \text{deflection of } A = \frac{Wl^2}{16EI}\cdot\frac{l}{3} = \frac{Wl^3}{48EI} \qquad \text{(Numerically)} \tag{7.32}$$

(iv) Simply Supported Beam with Uniformly Distributed Load As the loading is symmetrical, area of half the bending moment diagram can be considered (Fig. 7.41).

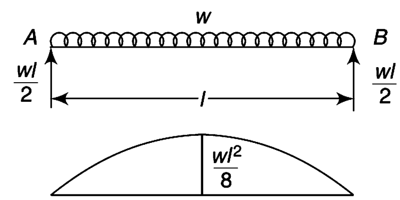


Fig. 7.41

Area of the bending moment diagram, $A = \dfrac{2}{3}\cdot\dfrac{l}{2}\cdot\dfrac{wl^2}{8} = \dfrac{wl^3}{24}$

As explained above for concentrated load,

Slope at $A = -\dfrac{A}{EI} = -\dfrac{wl^3}{24EI}$

Deflection of midpoint relative to A = Deflection of A relative to midpoint $= \dfrac{A\bar{x}}{EI}$

where $\bar{x}$ = distance of centroid of the bending moment diagram from $A = \dfrac{5}{8}\cdot\dfrac{l}{2} = \dfrac{5l}{16}$

$$\text{Deflection} = \frac{A\bar{x}}{EI} = \frac{wl^3}{24EI}\cdot\frac{5l}{16} = \frac{5wl^4}{384EI} \tag{7.33}$$

Example 7.19 || A simply supported beam of length l carries a concentrated load W at a distance a from end A. Determine

(a) a relation for the elastic curve of the beam
(b) deflection at the midspan
(c) deflection under the load
(d) maximum deflection
(e) slope at the two ends

Solution

Given A simply supported beam carrying a point load as shown in Fig. 7.42a.

To find

— A relation for elastic curve of beam
— deflection at midspan, under the load and maximum deflection
— slope at the two ends

Fig. 7.42b shows the beam alongwith the elastic curve. Let $b = l - a$.

Now, $R_a = \dfrac{Wb}{l}$ and $R_b = \dfrac{Wa}{l}$

Applying Mohr's second moment-area theorem

Consider a section D at a distance x from A between A and C.

Deflection at D, $D'D_1 = D'D'' - D''D_1$

- $D'D'' = B'B'' \cdot \dfrac{x}{l}$

  But $B'B''$ is the intercept made by tangents at A and B on the elastic curve on a vertical line at B.

  According to Mohr's second moment-area theorem, it must be equal to the net moment of area of bending moment diagram between A and B about B divided by EI.

  Thus $$B'B'' = \frac{1}{EI}\left(\frac{1}{2}\cdot Wb \cdot l \cdot \frac{l}{3} - \frac{1}{2}\cdot Wb \cdot b \cdot \frac{b}{3}\right)$$
  $$= \frac{Wb}{6EI}(l^2 - b^2)$$

  $$\therefore \quad D'D'' = \frac{Wbx}{6EIl}(l^2 - b^2)$$

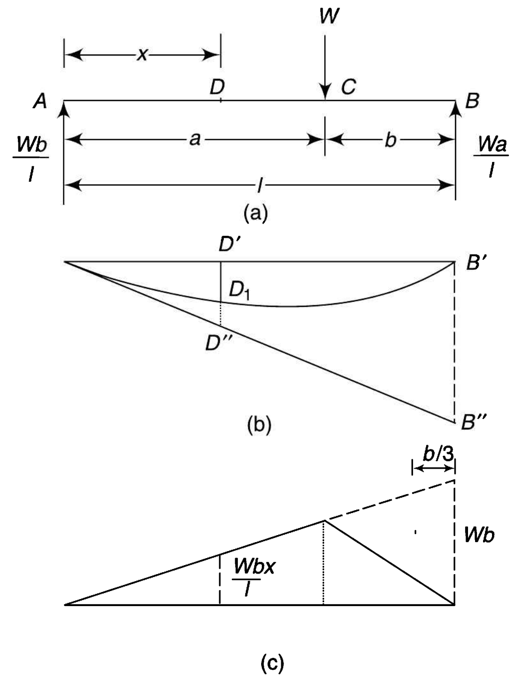


Fig. 7.42

- $D''D_1$ is the intercept made by tangents at D and A on the elastic curve on a vertical line at D.

  According to Mohr's second moment-area theorem, it must be equal to the net moment of area of bending moment diagram between A and D about D divided by EI.

  Thus $$D''D_1 = \frac{1}{EI}\left(\frac{1}{2}\cdot\frac{Wbx}{l}\cdot x \cdot \frac{x}{3}\right) = \frac{Wbx^3}{6EIl}$$

  $$\therefore \quad \text{Deflection at } D = \frac{Wbx}{6EIl}(l^2 - b^2) - \frac{Wbx^3}{6EIl} = \frac{Wbx}{6EIl}(l^2 - b^2 - x^2) \qquad \text{(i)}$$

Deflection at midspan and under the load

Deflection at the midspan ($x = l/2$)

$$= \frac{Wb}{6EIl}\left(\frac{l}{2}\right)\left(l^2 - b^2 - \frac{l^2}{4}\right) = \frac{Wb}{48EI}(3l^2 - 4b^2)$$

Deflection under the load ($x = a$)

$$= \frac{Wba}{6EIl}(l^2 - b^2 - a^2) = \frac{Wab}{6EIl}[(a+b)^2 - b^2 - a^2] = \frac{Wab}{6EIl}\cdot 2ab = \frac{Wa^2b^2}{3EIl}$$

Maximum deflection

For maximum deflection, differentiate (i) with respect to x and equate to zero,

$$\frac{d}{dx}(l^2x - b^2x - x^3) = 0 \quad \text{or} \quad l^2 - b^2 - 3x^2 = 0 \quad \text{or} \quad x = \frac{\sqrt{l^2 - b^2}}{3}$$

$$\therefore \quad \text{Maximum deflection} = \frac{Wb}{6EIl}\sqrt{\frac{l^2 - b^2}{3}}\left[l^2 - b^2 - \frac{l^2 - b^2}{3}\right]$$

$$= \frac{Wb}{6EIl}\sqrt{l^2 - b^2} \cdot \frac{2}{3\sqrt{3}}(l^2 - b^2) = \frac{Wb(l^2 - b^2)^{3/2}}{9\sqrt{3}EIl}$$

Slope at the ends

Slope at A, $\theta_a = \dfrac{Wb}{6EIl}(l^2 - b^2) = \dfrac{Wb}{6EIl}(l - b)(l + b) = \dfrac{Wab}{6EIl}(l + b)$ ... $(l - b = a)$

Slope at B with respect to A i.e. $\theta_b^a = \theta_a - (-)\theta_b = \theta_a + \theta_b$

(Slope at A and B are of opposite sign)

= Area of bending moment diagram between A and B

$$= \frac{1}{EI}\left(\frac{1}{2} \cdot Wb \cdot l - \frac{1}{2} \cdot Wb \cdot b\right) = \frac{Wb}{2EI}(l - b) = \frac{Wab}{2EI}$$

$\therefore$ Slope at B, $\theta_b = \theta_b^a - \theta_a = \dfrac{Wab}{2EI} - \dfrac{Wab}{6EIl}(l + b) = \dfrac{Wab}{6EIl}(2l - b) = \dfrac{Wab}{6EIl}(l + a)$

Example 7.20 A simply supported beam of 12-m span carries a concentrated load of 30 kN at a distance of 9 m from the end A. Determine the deflection at the load point and the slopes at the load point and at the two ends. Take $I = 2 \times 10^9$ mm^4 and $E = 205$ GPa.

Solution

Given A simply supported beam carrying a point load as shown in Fig. 7.43a.

To find

— Deflection at load point

— slopes at load point and at two ends

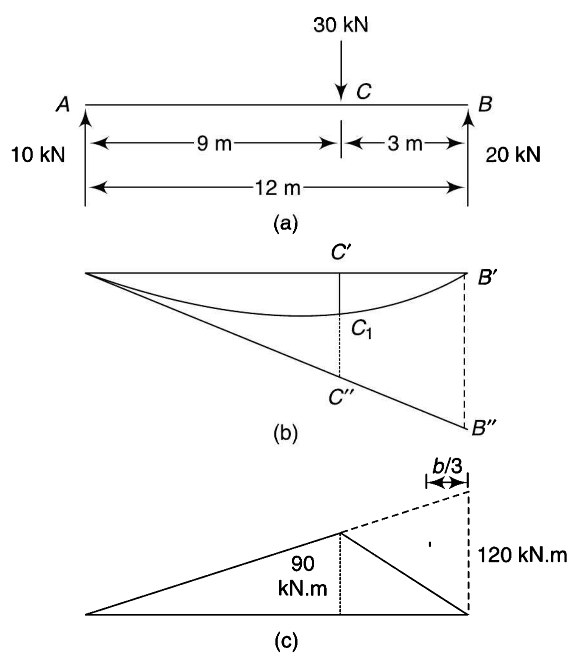


Fig. 7.43

Figure 7.43b shows the elastic curve.

Now, $R_a = \dfrac{30 \times 3}{12} = 7.5$ kN and $R_b = 30 - 7.5 = 22.5$ kN

$EI = 205\ 000 \times 2 \times 10^9 = 410 \times 10^{12}$ m²

Applying Mohr's second moment-area theorem

$$C'C'' = B'B'' \times \frac{9}{12}$$

But $B'B''$ is the intercept made by tangents at A and B on the elastic curve on a vertical line at B.

According to Mohr's second moment-area theorem, it must be equal to the net moment of area of bending moment diagram between A and B about B divided by EI.

Thus $$B'B'' = \frac{10^{12}}{EI}\left(\frac{1}{2} \times 120 \times 12 \times \frac{12}{3} - \frac{1}{2} \times 120 \times 3 \times \frac{3}{3}\right) \quad \text{(Fig. 7.43c)}$$

$$= \frac{10^{12}}{410 \times 10^{12}}(2880 - 180) = 6.585 \text{ mm}$$

or $$B'B'' = \frac{10^{12}}{EI}\left[\frac{1}{2} \times 90 \times 9 \times \left(3 + \frac{9}{3}\right) + \frac{1}{2} \times 90 \times 3 \times \frac{2 \times 3}{3}\right]$$

$$= \frac{10^{12}}{410 \times 10^{12}}(2430 + 270) = 6.585 \text{ mm}$$

$$C'C'' = B'B'' \times \frac{9}{12} = 6.585 \times \frac{9}{12} = 4.939 \text{ mm}$$

- $C''C_1$ is the intercept made by tangents at C and A on the elastic curve on a vertical line at C.

  According to Mohr's second moment-area theorem, it must be equal to the net moment of area of bending moment diagram between A and C about C divided by EI.

  Thus $$C''C_1 = \frac{10^{12}}{410 \times 10^{12}}\left(\frac{1}{2} \times 90 \times 9 \times \frac{9}{3}\right) = 2.963 \text{ mm}$$

Deflection at the load point

Deflection at $C = 4.939 - 2.963 = 1.976$ mm

Slopes at the load point and at the ends

Slope at $A = \dfrac{B'B''}{l} = \dfrac{6.585}{12\ 000} = 0.549 \times 10^{-3}$ rad

Slope at B with respect to A = Area of bending moment diagram between A and B

$$= \frac{10^9}{410 \times 10^{12}}\left(\frac{1}{2} \times 90 \times 9 + \frac{1}{2} \times 90 \times 3\right) = 1.317 \times 10^{-3} \text{ rad}$$

∴ Slope at $B = 10^3 \times (1.317 - 0.549) = 0.768 \times 10^{-3}$ rad

Slope at C with respect to A = Area of bending moment diagram between A and C

∴ Slope at $C = 10^3 \times (0.768 - 0.549) = 0.219 \times 10^{-3}$ rad

Example 7.21 12-m long horizontal cantilever *ABC* is built in at *A* and supported at *B*, 2 m from *C* by a rigid prop so that *AB* is horizontal. If *AB* and *BC* carry uniformly distributed loads of 1.2 kN/m and 2 kN/m respectively, determine the load taken by the prop.

Solution

Given A cantilever ABC with a support at B and carrying a uniformly distributed load as shown in Fig. 7.44a.

To find Load on the prop

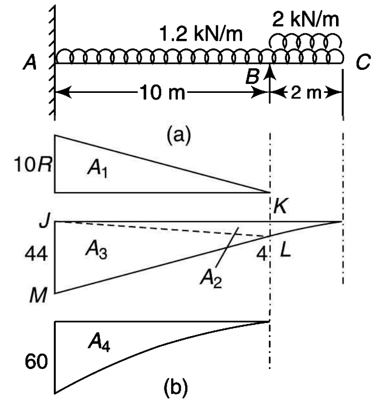


Fig. 7.44

Figure 7.44a shows the loaded beam.

Let the prop reaction be R.

The loading on the beam may be broken down into the following loads:

(i) Reaction of the prop R

(ii) Distributed load on BC

(iii) Distributed load on AB

The bending moment diagram for each of the above loads is shown in Fig. 7.44b.

As A and B remains horizontal, $\Sigma A\bar{x}$ for the bending moment diagram between AB must be zero.

Determination of areas

$$A_1 = \frac{1}{2} \times 10R \times 10 = 50\,R \text{ kN}\cdot\text{m}^2$$

Due to load on BC, bending moment at $B = -(2 \times 2) \times 1 = -4$ kN·m^2

Due to load on BC, bending moment at $A = -4 \times (10 + 1) = -44$ kN·m^2

The trapezium $JKLM$ between AB can be split into two triangles,

$$A_2 = -\frac{1}{2} \times (4 \times 10) = -20 \text{ kN}\cdot\text{m}^2$$

$$A_3 = -\frac{1}{2} \times (44 \times 10) = -220 \text{ kN}\cdot\text{m}^2$$

Due to uniformly distributed load on AB, area A_4 is a parabola.

Bending moment at $$A = \frac{1.2 \times 10^2}{2} = 60 \text{ kN}\cdot\text{m}^2$$

$$A_4 = -\frac{1}{3} \times 60 \times 10 = -300 \text{ kN}\cdot\text{m}^2$$

Applying Mohr's second moment-area theorem

According to Mohr's second moment-area theorem,

Intercept on a vertical line at B made by tangents at A and B on elastic curve

= (Net moment of area of bending moment diagram between A and B about B)/EI

But points A and B are to remain at the same level. Therefore, Net moment of area of bending moment diagram between A and B about B is zero,

or $$A_1\bar{x}_1 - A_2\bar{x}_2 - A_3\bar{x}_3 - A_4\bar{x}_4 = 0$$

or $$50R \cdot \left(\frac{2}{3} \times 10\right) = 20 \times \frac{10}{3} + 220 \times \left(\frac{2}{3} \times 10\right) + 300 \times \frac{3}{4} \times 10; \quad R = 11.35 \text{ kN}$$

Example 7.22 || A simply supported beam carrying a uniformly distributed load *w* rests on two supports symmetrically placed at the same level and has overhangs on both ends. The distance between the supports is 2*l* and the length of each overhang is *a*. Determine the ratio of *l* to *a* when

(i) the greatest downward deflection has the least value
(ii) the two ends of the beam remain horizontal

Solution

Given A simply supported beam with overhangs and a uniformly distributed load as shown in Fig. 7.45a.
To find Ratio of l to a when

— Greatest downward deflection has least value
— two ends of beam remain horizontal.

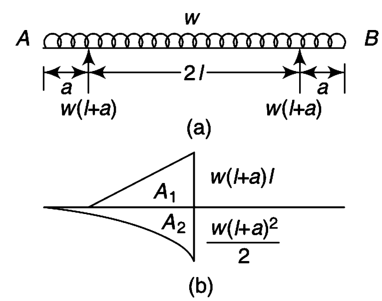


Fig. 7.45

Let w be the uniformly distributed load on the beam and the spacing between the supports be $2l$ with overhanging distance a at each end.

Then reaction at each support = $w(l + a)$

Now the maximum deflection of the beam can be at the midspan or at the free ends. However, its least value will occur when the deflections at the ends and at the centre are the same as any movement of the supports from that position will increase any of these values.

Since the slope at the centre is zero, bending moment diagram for half the beam can be considered.

Applying Mohr's second moment-area theorem

Splitting the bending moment diagram into two parts, A_1 due to the support and A_2 due to the load, then as the deflection at A and that at the mid span are the same, net value of $A\bar{x}$ is to be zero.

$$A_1\bar{x}_1 - A_2\bar{x}_2 = 0$$

where $\bar{x}_1$ and $\bar{x}_2$ is the distances of centroids of bending moment diagrams from A

or $$\left[\frac{1}{2}\cdot l\cdot w(l+a)l\right]\left(a+\frac{2}{3}l\right)=\left[\frac{1}{3}\cdot\frac{w}{2}(l+a)^2(l+a)\right]\frac{3}{4}(l+a)$$

or $$l^2\left(a+\frac{2}{3}l\right)=\left(\frac{1}{4}(l+a)^3\right)$$

or $$12l^2a + 8l^3 = 3(l^3 + a^3 + 3la^2 + 3l^2a)$$

or $$5l^3 + 3l^2a - 9la^2 - 3a^3 = 0$$

Solving by trial and error, $l = 1.24a$

Applying Mohr's first moment-area theorem

If the two ends of the beam are to remain horizontal, the slope at the two ends as well as at the mid span must be zero.

As slope is given by $-\dfrac{A}{EI}$, the net area of the bending moment diagram must be zero between either end and the mid section of the span i.e.

$$\frac{1}{2}\cdot l\cdot w(l+a)l-\frac{1}{3}\cdot\frac{w}{2}(l+a)^2(l+a)=0 \quad \text{or} \quad 3l^2 = (1+a)^2$$

or $$l + a = \sqrt{3}l \quad \text{or} \quad 0.732l = a \quad \text{or} \quad l = 1.366a$$

Example 7.23 A mild steel bar of 30-mm diameter is used to make a cantilever of uniform strength. Determine the maximum length of the cantilever to support a load of 2 kN at the free end if the allowable stress is to be 80 MPa. Also find its end deflection. E = 210 GPa.

Solution

Given A mild steel bar of 30-mm diameter as shown in Fig. 7.46

$E = 210$ GPa $\qquad \sigma = 80$ MPa

To find

— Maximum length of cantilever to be made from bar having uniform strength

— Deflection at free end

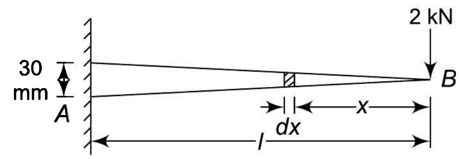


Fig. 7.46

Let l be the length. Then maximum bending moment is $2000l$ at the fixed end. Thus the the strongest section is to be at this end of 30 mm diameter (Fig. 7.46)

$$\therefore \qquad 2000l = \frac{(\pi/64)\times 30^4}{15}\times 80 \quad \text{or } l = 106 \text{ mm}$$

For uniform strength throughout

Diameter d at a distance x from the free end,

$$2000x = \frac{(\pi/64)\times d^4}{d/2}\times 80 \quad \text{or } d^3 = 254.6x$$

Moment of inertia of the bar varies throughout the length of the cantilever.

Deflection at the free end

Deflection at the free end can be found by considering a small length dx at a distance x from the free end by moment area method from the relation $z = \int \frac{(M \cdot dx)x}{EI}$ if moments are taken from the free end.

Bending moment at distance x from the free end = $2000x$

Area of the bending moment diagram = $2000x \cdot dx$

Moment of the area about the free end = $2000 \cdot x \cdot dx \cdot x = 2000x^2 \cdot dx$

$$\text{Deflection} = \frac{1}{EI}\int_0^l 2000x^2 \cdot dx = \frac{1}{210000\times(\pi/64)d^4}\cdot\int_0^{106} 2000x^2\cdot dx = \frac{0.194}{254.6^{4/3}}\int_0^{157}\frac{x^2}{x^{4/3}}\cdot dx$$

$$= 120.2\times 10^{-6}\int_0^{106} x^{2/3}\cdot dx = 120.2\times 10^{-6}\left(\frac{x^{5/3}}{5/3}\right)_0^{106} = 72.1\times 10^{-6}(2374) = 0.171$$

Example 7.24 A simply supported beam of $2l$ span carries a uniformly distributed load of intensity $2w$ over the left half and w over the right half of the span. Deduce an expression for the deflection of the beam at the centre of the span if the moment of inertia is $2I$ for the left half and I for the right half of the span.

If a prop is placed at the centre of the span to raise its level to the original position, find the force in the prop.

Solution

Given A simply supported beam with different moment of inertia of left and right halves and carrying different loads on left and right halves as shown in Fig. 7.47.

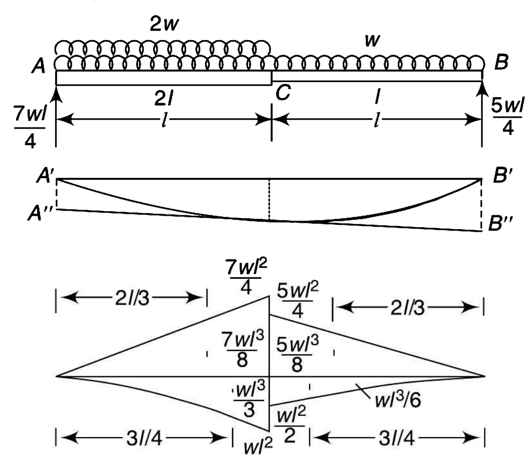


Fig. 7.47

To find

— An expression for deflection of beam at centre of span
— Force in a prop placed at centre to raise the level to original position

Figure 7.47 shows the loaded beam alongwith the bending moment diagrams for the left and the right halves of the span, the positive values being due to reactions of the supports and the negative due to loadings.

The deflection curve is also shown in the figure. It can be seen that the slope at the midspan is not zero and the deflections at the end points A and B relative to C are not equal and the deflection at C is the mean of the two values at the ends.

Taking moments about B, $R_a \cdot 2l = 2wl \cdot (3l/2) + wl \cdot (l/2)$ or $R_a = \dfrac{7wl}{4}$

$$\therefore \quad R_b = (2w \cdot l + w \cdot l) - \frac{7wl}{4} = \frac{5wl}{4}$$

Applying Mohr's second moment-area theorem to left half

According to Mohr's second moment-area theorem,

Intercept on a vertical line at A made by tangents at A and C on elastic curve (AA')

= (Net moment of area of bending moment diagram between A and C about A)/EI (i)

Bending moment at midspan due to reaction at $A = \dfrac{7wl}{4} \cdot l = \dfrac{7wl^2}{4}$

Area of bending moment diagram $= \dfrac{1}{2} \cdot \dfrac{7wl^2}{4} \cdot l = \dfrac{7wl^3}{8}$

Bending moment at midspan due to uniformly distributed load $= \dfrac{2wl^2}{2} = wl^2$

Area of bending moment diagram $= \dfrac{1}{3} wl^2 \cdot l = \dfrac{wl^3}{3}$

$$\therefore \quad A\bar{x} \text{ for left half} = \left(\frac{7wl^3}{8}\right)\left(\frac{2l}{3}\right) - \left(\frac{wl^3}{3}\right)\left(\frac{3l}{4}\right) = \frac{7wl^4}{12} - \frac{wl^4}{4} = \frac{wl^4}{3}$$

Thus from (i), $A'A'' = \dfrac{wl^4}{3E(2I)} = \dfrac{wl^4}{6EI}$

Applying Mohr's second moment-area theorem to right half

Intercept on a vertical line at B made by tangents at B and C on elastic curve (BB') = (Net moment of area of bending moment diagram between B and C about B)/EI

Net value of $A\bar{x}$ for the right half,

$$\therefore \quad A\bar{x} = \left(\frac{5wl^3}{8}\right)\left(\frac{2l}{3}\right) - \left(\frac{wl^3}{6}\right)\left(\frac{3l}{4}\right) = \frac{5wl^4}{12} - \frac{wl^4}{8} = \frac{7wl^4}{24}$$

$$\therefore \quad BB' = \frac{7wl^4}{24EI}$$

As can be observed from the figure, net deflection at C is the mean value AA' and BB'.

$$\text{Net deflection at } C = \frac{1}{2}\left(\frac{wl^4}{6EI} + \frac{7wl^4}{24EI}\right) = \frac{11wl^4}{48EI}$$

When prop is placed

Let the upward reaction of the prop at the centre be P to raise its level to the original position (Fig. 7.48). This is equivalent to a problem of simply supported beam with central load P and deflection equal to $\dfrac{11wl^4}{48EI}$.

Then downward force applied at A and $B = P/2$. The corresponding bending moment diagram is shown in Figure 7.48.

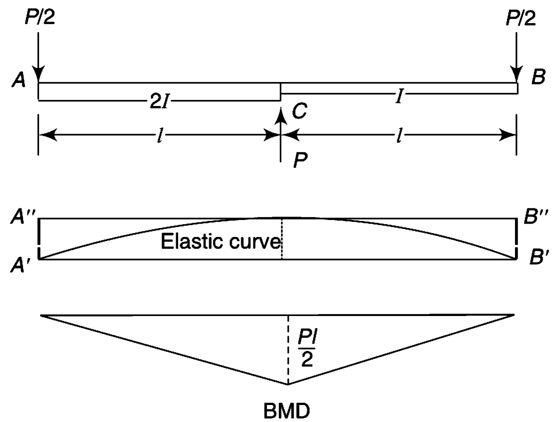


Fig. 7.48

$$\text{Deflection of } A \text{ with respect to } C = \frac{1}{E(2I)}\left(\frac{1}{2}\cdot\frac{Pl}{2}\cdot l\cdot\frac{2l}{3}\right) = \frac{Pl^3}{12EI}$$

$$\text{Deflection of } B \text{ with respect to } C = \frac{1}{EI}\left(\frac{1}{2}\cdot\frac{Pl}{2}\cdot l\cdot\frac{2l}{3}\right) = \frac{Pl^3}{6EI}$$

$$\text{Thus deflection at } C,\quad \frac{11wl^4}{48EI} = \frac{1}{2}\left(\frac{Pl^3}{12EI} + \frac{Pl^3}{6EI}\right) = \frac{Pl^3}{8EI} \quad \text{or} \quad P = \frac{11wl}{6}$$

7.6 STRAIN ENERGY DUE TO BENDING

Consider two sections of a beam a small distance dx apart (Fig. 7.49). As the distance is small, the bending moment acting may be taken to be same throughout the length dx. Let it be M. Let σ be the bending stress on an elemental cylinder of area dA at a distance y from the neutral axis.

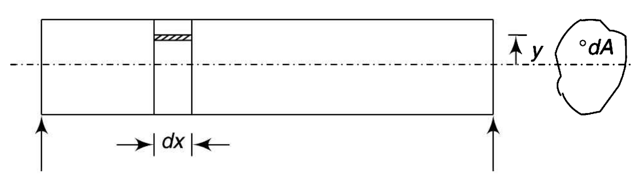


Fig. 7.49

$$\text{Strain energy in the elemental cylinder} = \frac{\sigma^2}{2E} \times \text{volume} = \frac{\sigma^2}{2E} \times dA \cdot dx$$

Strain energy of the length dx, $\delta U = \int \frac{\sigma^2}{2E} \cdot dA \cdot dx = \int \frac{1}{2E}\left(\frac{My}{I}\right)^2 \cdot dA \cdot dx = \frac{M^2 \cdot dx}{2EI^2}\int y^2 \cdot dA$

But $$\int y^2 \cdot dA = I$$

$\therefore$ $$\delta U = \frac{M^2 \cdot dx}{2EI}$$

Strain energy stored in the whole beam, $$U = \int \frac{M^2 \cdot dx}{2EI} \tag{7.34}$$

Example 7.25 Find the deflection at the free end of a cantilever which carries a point load at the free end.

Solution

Given A cantilever carrying a point load at free end as shown in Fig. 7.50.
To find Deflection at free end

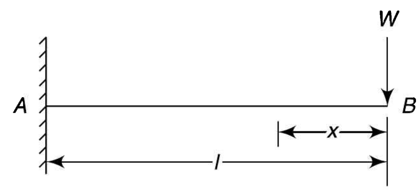


Fig. 7.50

Bending moment at a distance x from B, $M = -Wx$

Strain energy

$$U = \int_0^l \frac{M^2 \cdot dx}{2EI} = \int_0^l \frac{W^2x^2}{2EI}dx = \frac{W^2}{2EI}\left(\frac{x^3}{3}\right) = \frac{W^2l^3}{6EI}$$

Equating work done to strain energy

Work done by the load = strain energy of the beam

$$\frac{1}{2}W\delta = \frac{W^2l^3}{6EI} \quad \text{or} \quad \delta = \frac{Wl^3}{3EI}$$

Example 7.26 A simply supported beam AB of length l carries a concentrated load W at a distance a from end A and b from end B. Deduce expressions for the strain energy of the beam and the deflection under the load. Also deduce the expressions when the load is at the mid span.

Solution

Given A simply supported beam with a point load as shown in Fig. 7.51.
To find

— Strain energy of beam
— Strain energy when load is at mid span

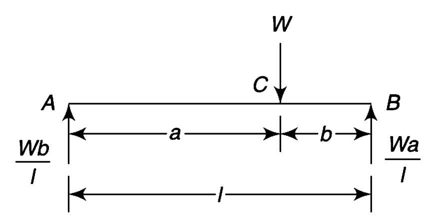


Fig. 7.51

The loaded beam is shown in Fig. 7.51. $R_a = \frac{Wb}{l}$ and $R_b = \frac{Wa}{l}$

Bending moment at a distance x from A, $M = \frac{Wb}{l}x$

Strain energy

Strain energy of section AC,

$$U = \int_0^l \frac{M^2 \cdot dx}{2EI} = \frac{1}{2EI}\left[\int_0^a \left(\frac{Wbx}{l}\right)^2 dx\right] = \frac{W^2b^2}{2EIl^2}\int_0^a x^2 dx = \frac{W^2b^2}{2EIl^2}\left(\frac{x^3}{3}\right)_0^a = \frac{W^2b^2a^3}{6EIl^2}$$

Similarly, strain energy of section BC, $U = \frac{W^2b^3a^2}{6EIl^2}$

Total strain energy $U = \dfrac{W^2b^2a^3}{6EIl^2} + \dfrac{W^2b^3a^2}{6EIl^2} = \dfrac{W^2a^2b^2(a+b)}{6EIl^2} = \dfrac{W^2a^2b^2}{6EIl}$

Equating work done to strain energy

Now, work done by the load = strain energy of the beam

$$\frac{1}{2}W\delta = \frac{W^2a^2b^2}{6EIl} \quad \text{or} \quad \delta = \frac{Wa^2b^2}{6EIl}$$

Load at midspan

$$a = b = l/2, \qquad \delta = \frac{Wl^4}{48EIl} = \frac{Wl^3}{48EI}$$

Example 7.27 || A rigid cantilever frame shown in Fig. 7.52 carries a load W at the free end. Assuming a constant value of EI, determine the vertical displacement of the free end C.

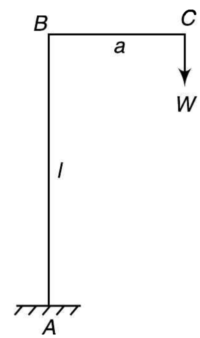


Fig. 7.52

Solution

Given A rigid cantilever frame carrying a load at free end C as shown in Fig. 7.52.
To find Vertical displacement of free end

Strain energy

- Strain energy of the section BC,
  Bending moment at a distance x from C, $M = -Wx$

$$U = \int_0^l \frac{M^2 \cdot dx}{2EI} = \frac{1}{2EI}\left[\int_0^a W^2x^2dx\right] = \frac{W^2a^3}{6EI}$$

- Strain energy of the section AB,
  Bending moment at a distance x from B, $M = -Wa$ (Fig. 7.53)

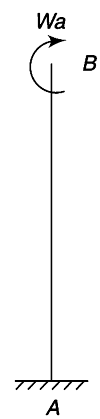


Fig. 7.53

$$U = \int_0^l \frac{M^2 \cdot dx}{2EI} = \frac{1}{2EI}\left[\int_0^l W^2a^2dx\right] = \frac{W^2a^2l}{2EI}$$

Total strain energy $U = \dfrac{W^2a^3}{6EI} + \dfrac{W^2a^2l}{2EI} = \dfrac{W^2a^2(a+3l)}{6EI}$

Equating work done to strain energy

Now, work done by the load = strain energy of the beam

$$\frac{1}{2}W\delta = \frac{W^2a^2(a+3l)}{6EI} \quad \text{or} \quad \delta = \frac{Wa^2(a+3l)}{3EI}$$

Example 7.28 || Compare the strain energy of a centrally loaded simply supported beam with that of the same beam with a uniformly distributed load. Assume the value of the maximum bending stress to be the same in the two cases.

Solution

Given A simply supported beam loaded in two ways: centrally loaded and with a uniformly distributed load.
To find To compare strain energies in the two cases if maximum bending stress is same

As maximum bending stress = M/Z, for the same beam in the two cases, maximum M has to be the same, i.e.,

$$\frac{Wl}{4} = \frac{wl^2}{8} \quad \text{or } W = wl/2$$

For central load W

$$U_1 = \frac{W^2 l^3}{96EI} \qquad \text{(Refer Example 7.16)}$$

For a simply supported beam

Figure 7.54 shows a simply supported beam of length l and carrying a uniformly distributed load w per unit length.

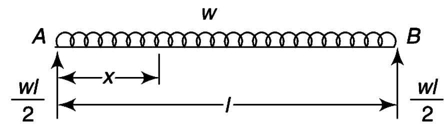


Fig. 7.54

$$M = \frac{wl}{2} \cdot x - \frac{wx^2}{2} = \frac{wx}{2}(l - x)$$

$$U_2 = \int_0^l \frac{M^2 \cdot dx}{2EI} = \int_0^l \frac{w^2}{4} x^2 (l - x)^2 \frac{dx}{2EI} = \frac{w^2}{8EI} \int_0^l (l^2 x^2 + x^4 - 2lx^3) dx$$

$$= \frac{w^2}{8EI}\left(\frac{l^2 x^3}{3} + \frac{x^5}{5} - \frac{2lx^4}{4}\right) = \frac{w^2 l^5}{240EI}$$

Comparison of strain energies

$$\therefore \qquad \frac{U_1}{U_2} = \frac{W^2 l^3}{96EI} \Big/ \frac{w^2 l^5}{240EI} = \frac{w^2 \cdot l^2 l^3}{4 \times 96EI} \Big/ \frac{w^2 l^5}{240EI} = \frac{5}{8}$$

7.7 CASTIGLIANO'S FIRST THEOREM (DEFLECTION FROM STRAIN ENERGY)

Castigliano's first theorem is stated as below:

If a structure is subjected to a number of external loads (or couples), the partial derivative of the total strain energy with respect to any load (or couple) provides the deflection in the direction of that load (or couple).

Mathematically,

Let U = total strain energy of the structure

W_1, W_2, W_3 .....External loads at points O_1, O_2, O_3.....

M_1, M_2, M_3 .....External couples at the same points

Then according to this theorem,

Deflections at respective points and directions are

$$\frac{\partial U}{\partial W_1}, \frac{\partial U}{\partial W_2}, \frac{\partial U}{\partial W_3} \ldots\ldots\ldots$$

and angular rotation of the couples at the applied points

are $$\frac{\partial U}{\partial M_1}, \frac{\partial U}{\partial M_2}, \frac{\partial U}{\partial M_3} \ldots\ldots\ldots$$

Proof Let x_1, x_2, x_3 ..... be the displacements in the direction of gradually applied loads (Fig. 7.55a).

Then,

$$U = \frac{1}{2} W_1 x_1 + \frac{1}{2} W_2 x_2 + \frac{1}{2} W_3 x_3 \ldots\ldots \qquad \text{(i)}$$

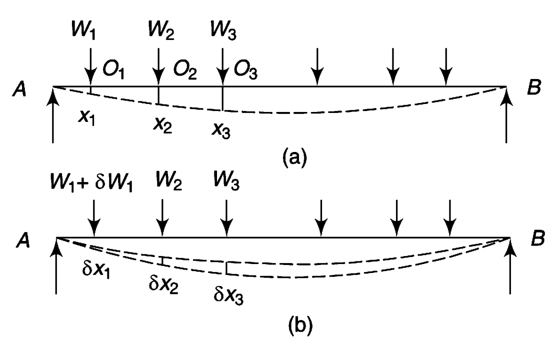


Fig. 7.55

Let W_1 be increased to $W_1 + \delta W_1$ and due to the increased load δW_1, let $\delta x_1, \delta x_2, \delta x_3 \ldots$ be the increments in $x_1, x_2, x_3 \ldots\ldots$ respectively (Fig. 7.55b).

Then Increase in external work done, $\delta U = \frac{1}{2}\delta W_1 \delta x_1 + W_1 \delta x_1 + W_2 \delta x_2 + W_3 \delta x_3 \ldots\ldots$

(All loads except δW_1 exist already and thus are treated as suddenly applied loads whereas δW_1 is gradually applied load)

In the above, $\frac{1}{2}\delta W_1 \delta x_1$ being product of small quantities may be neglected.

Thus $\quad \delta U = W_1 \delta x_1 + W_2 \delta x_2 + W_3 \delta x_3 \ldots\ldots$ (ii)

However, if the loads $W_1 + \delta W_1, W_2, W_3 \ldots\ldots$ had been applied gradually from zero,

Then, strain energy

$$= U + \delta U = \frac{1}{2}(W_1 + \delta W_1)(x_1 + \delta x_1) + \frac{1}{2}W_2(x_2 + \delta x_2) + \frac{1}{2}W_3(x_3 + \delta x_3) \ldots\ldots$$

Subtract (i) from (iii), neglecting the product of small quantities,

$$\delta U = \frac{1}{2}W_1 \delta x_1 + \frac{1}{2}\delta W_1 x_1 + \frac{1}{2}W_2 \delta x_2 + \frac{1}{2}W_3 \delta x_3 \ldots\ldots$$

or $\quad 2\delta U = W_1 \delta x_1 + \delta W_1 x_1 + W_2 \delta x_2 + W_3 \delta x_3 \ldots\ldots$ (iii)

Subtracting (ii) from (iii),

$$\delta U = \delta W_1 x_1$$

In the limit, $\frac{\partial U}{\partial W_1} = x_1$ (7.35)

In a similar way, it can be proved that $\frac{\partial U}{\partial W_2} = x_2, \frac{\partial U}{\partial W_3} = x_3$ etc.

Similarly for couples also, the theorem can be proved.

7.8 DEFLECTIONS BY CASTIGLIANO'S THEOREM

In the previous section it was observed that the deflection of a structure at a point of load W can be obtained by finding the partial derivative of the strain energy of the structure. It may be observed that though the differentiation with respect to load may be carried out before or after the integration, the calculations are simplified if the differentiation is carried out before the integration.

In case of a beam, $\quad U = \int_0^l \frac{M^2 \cdot dx}{2EI}$

and deflection at the point of application of load W_i,

$$\delta_i = \frac{\partial U}{\partial W_i}$$

$$= \int_0^l \frac{M}{EI}\frac{\partial M}{\partial W_i} dx$$

$$= \frac{1}{EI}\int_0^l M \cdot \frac{\partial M}{\partial W_i} dx \qquad (7.36)$$

In case deflection is to be found at a point where a load does not exist, a virtual load may be assumed at the point which may be put to zero before integration.

In a similar way, the slope θ_i of a beam at a point can also be obtained by applying a virtual couple M_i at the point and putting the same to zero before integration.

$$\theta_i = \frac{\partial U}{\partial M_i}$$

$$= \int_0^l \frac{M}{EI}\frac{\partial M}{\partial M_i}dx$$

$$= \frac{1}{EI}\int_0^l M \cdot \frac{\partial M}{\partial M_i}dx \qquad \text{where } M \text{ is a function of } M_i. \qquad (7.37)$$

Example 7.29 Find the deflection at the free end of a cantilever which carries a point load at the free end.

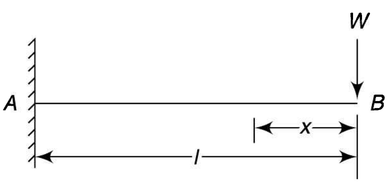


Fig. 7.56

Solution

Given A cantilever carrying a point load at free end as shown in Fig. 7.56.

To find Deflection at free end

Bending moment at a distance x from B, $M = -Wx$

Its derivative with respect to W, $\dfrac{\partial M}{\partial W} = -x$

By Castigliano's theorem

$$\delta_b = \int_0^l \frac{M}{EI}\frac{\partial M}{\partial W}dx = \int_0^l \frac{(-Wx)}{EI}(-x)dx = \frac{W}{EI}\int_0^l x^2 dx = \frac{W}{EI}\cdot\left(\frac{x^3}{3}\right)_0^l = \frac{Wl^3}{3EI}$$

As the load is directed downwards, the positive sign indicates the deflection downwards.

Example 7.30 Find the deflection at the free end of a cantilever which carries a uniformly distributed load and a point load at the free end.

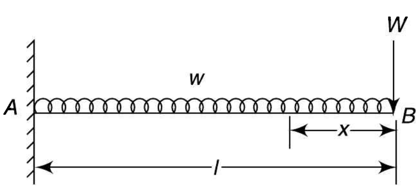


Fig. 7.57

Solution

Given A cantilever having a point load at free end and a uniformly distributed load throughout as shown in Fig. 7.57.

To find Deflection at free end

Bending moment at a distance x from B, $M = -Wx - \dfrac{wx^2}{2}$

Its derivative with respect to $\dfrac{\partial M}{\partial W} = -x$

By Castigliano's theorem

$$\delta_b = \int_0^l \frac{M}{EI}\frac{\partial M}{\partial W}dx = \int_0^l \frac{(-Wx - wx^2/2)}{EI}(-x)dx$$

$$= \frac{1}{EI}\int_0^l (Wx^2 + \frac{wx^3}{2})dx = \frac{1}{EI}\cdot\left(\frac{Wl^3}{3} + \frac{wl^4}{8}\right)$$

As the load W is directed downwards, the positive sign indicates the deflection downwards.

Example 7.31 || Determine deflection and the slope at the free end of a cantilever with uniformly distributed load on the whole span.

Solution

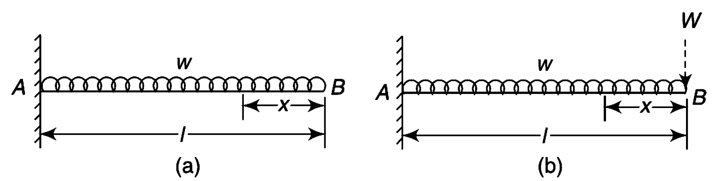


Fig. 7.58

Given A cantilever carrying a uniformly distributed load as shown in Fig. 7.58a.
To find Deflection and slope at free end

Deflection at free end

To find the deflection at the free end, it is necessary to first assume a concentrated load W at the free end and later on equating that to zero (Fig. 7.58b).

- Bending moment at a distance x from B, $M = -Wx - \dfrac{wx^2}{2}$

  Its derivative with respect to W, $\dfrac{\partial M}{\partial W} = -x$

By Castigliano's theorem

Inserting the above values in the equation, $\delta_b = \int_0^l \dfrac{M}{EI}\dfrac{\partial M}{\partial W}dx$ and making the assumed load to zero before integrating,

$$\delta_b = \int_0^l \frac{M}{EI}\frac{\partial M}{\partial W}dx = \int_0^l \frac{(-wx^2/2)}{EI}(-x)dx = \frac{w}{2EI}\int_0^l x^3 dx = \frac{w}{2EI}\cdot\left(\frac{x^4}{4}\right)_0^l = \frac{wl^4}{8EI}$$

As the load W is directed downwards, the positive sign indicates the deflection downwards.

Slope at free end

Apply a dummy clockwise couple M_b at B.

Bending moment at a distance x from B, $M = -M_b - \dfrac{wx^2}{2}$

Its derivative with respect to M_b, $\dfrac{\partial M}{\partial M_b} = -1$

By Castigliano's theorem

Inserting the above values in $\theta_b = \int_0^l \dfrac{M}{EI}\dfrac{\partial M}{\partial M_b}dx$ where M is a function of M_b and making the assumed couple to be zero before integrating,

$$\theta_b = \int_0^l \frac{M}{EI}\frac{\partial M}{\partial M_b}dx = \int_0^l \frac{(-wx^2/2)}{EI}(-1)dx = \frac{w}{2EI}\int_0^l x^2 dx = \frac{w}{2EI}\cdot\left(\frac{x^3}{3}\right)_0^l = \frac{wl^3}{6EI}$$

Example 7.32 || A simply supported beam AB of length l carries a concentrated load W at a distance a from end A and b from end B. Deduce expression for deflection under the load. Also find the deflection if the load is at the midspan.

Solution

Given A simply supported beam carrying a point load as shown in Fig. 7.59.

To find

— Deflection at load point

— deflection if load is at midspan

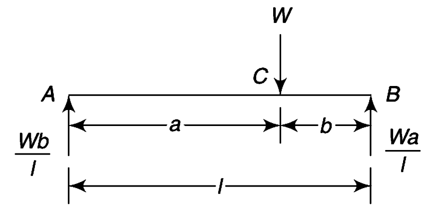


Fig.7.59

$$R_a = \frac{Wb}{l} \quad \text{and} \quad R_b = \frac{Wa}{l}$$

Bending moment at a distance x from A, $M = \dfrac{Wb}{l}x$

Its derivative with respect to W, $\dfrac{\partial M}{\partial W} = \dfrac{bx}{l}$

Similarly, for the portion BC, $M = \dfrac{Wa}{l}x$ and $\dfrac{\partial M}{\partial W} = \dfrac{ax}{l}$ (x from B)

By Castigliano's theorem

Inserting the above values in the equation, $\delta_c = \displaystyle\int_0^l \frac{M}{EI}\frac{\partial M}{\partial W}dx$

$$\delta_c = \int_0^a \frac{Wbx}{EIl}\cdot\frac{bx}{l}\cdot dx + \int_0^b \frac{Wax}{EIl}\cdot\frac{ax}{l}\cdot dx = \frac{W}{EIl^2}\left[\int_0^a b^2x^2dx + \int_0^b a^2x^2dx\right]$$

$$= \frac{W}{EIl^2}\left[b^2\left(\frac{x^3}{3}\right)_0^a + a^2\left(\frac{x^3}{3}\right)_0^b\right] = \frac{W}{3EIl^2}(b^2a^3 + a^2b^3) = \frac{Wa^2b^2}{3EIl^2}(a+b) = \frac{Wa^2b^2}{3EIl}$$

As the load is W directed downwards, the positive sign indicates the deflection downwards.

If load is at midspan

$$a = b = l/2$$

$$\therefore \qquad \delta = \frac{Wa^2b^2}{3EIl} = \frac{Wl^2l^2}{3\times4\times4EIl} = \frac{Wl^3}{48EI}$$

Example 7.33 A semicircular arc has one end hinged and the other end on rollers. The two ends being horizontal, the roller end is pulled with a horizontal force F. Find the horizontal displacement of the roller end.

Solution

Given A semi-circular arc hinged at one end and other end on rollers as shown in Fig. 7.60.

To find Horizontal displacement of roller end when pulled with a horizontal force F

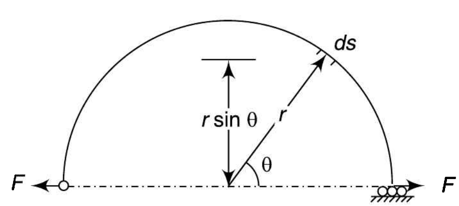


Fig. 7.60

Consider a small length ds of the arc at section C as shown in Fig. 7.59

Bending moment at the section = $F \cdot r\sin\theta$

Its derivative with respect to F, $\dfrac{\partial M}{\partial F} = r\sin\theta$

By Castigliano's theorem

$$\delta = \int_0^l \frac{M}{EI}\frac{\partial M}{\partial F} ds = 2\int_0^{\pi/2} \frac{F \cdot r\sin\theta}{EI} r\sin\theta \cdot rd\theta = \frac{2Fr^3}{EI}\int_0^{\pi/2} \sin^2\theta \cdot dx$$

$$= \frac{2Fr^3}{EI}\int_0^{\pi/2} \sin^2\theta \cdot dx = \frac{Fr^3}{EI}\int_0^{\pi/2} (1-\cos 2\theta)d\theta$$

$$= \frac{Fr^3}{EI}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi/2} = \frac{\pi Fr^3}{2EI}$$

Example 7.34 || Determine the maximum deflection of a simply supported beam of span l carrying a load of w per unit length using strain energy method. Also find the slope at the ends.

Solution

Given A simply supported beam carrying a uniformly distributed load.

To find

(i) Maximum deflection

(ii) Slope at ends

Maximum deflection is at the midspan. Thus assume a concentrated load W at this point (Fig. 7.61).

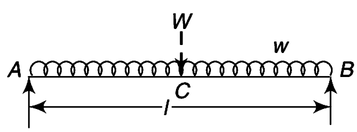


Fig.7.61

$$R_a = R_b = \frac{W + wl}{2}$$

Bending moment at a distance x from A, $M = \frac{W + wl}{2}x - \frac{wx^2}{2}$

Its derivative with respect to W, $\frac{\partial M}{\partial W} = \frac{x}{2}$

By Castigliano's theorem

Inserting the above values in the equation, $\delta_c = \int_0^{l/2} \frac{M}{EI}\frac{\partial M}{\partial W}dx$ and making the assumed load to be zero before integrating,

$$\delta_c = 2 \times \frac{1}{4}\int_0^{l/2} (wlx - wx^2)x \cdot dx = \frac{w}{2EI}\left(\frac{lx^3}{3} - \frac{x^4}{4}\right)_0^{l/2} = \frac{5wl^4}{384EI}$$

Determination of slope at A

Apply a dummy clockwise couple M_a at A.

Taking moments about B, $R_a \times l + M_a = wl \cdot \frac{l}{2}$ or $R_a = \frac{wl}{2} - \frac{M_a}{L}$

and $$R_b = wl - \left(\frac{wl}{2} - \frac{M_a}{L}\right) = \frac{wl}{2} + \frac{M_a}{L}$$

Bending moment at a distance x from A, $M = \frac{wlx}{2} - \frac{M_a x}{l} - \frac{wx^2}{2}$

Its derivative with respect to M_a, $\frac{\partial M}{\partial M_a} = -\frac{x}{l}$

By Castigliano's theorem

Inserting these values in $\theta_a = \int_0^l \frac{M}{EI}\frac{\partial M}{\partial M_a}dx$ and putting the assumed couple M_a to zero before integrating,

$$\theta_a = \frac{1}{EI}\int_0^l \left[\frac{wlx}{2} - \frac{wx^2}{2}\right]\left(-\frac{x}{l}\right) = -\frac{w}{2EIl}\left[\frac{lx^3}{3} - \frac{x^4}{4}\right]_0^l = -\frac{wl^3}{2EI}$$

Slope at B is same as at A.

Example 7.35 A cantilever of length l has a circular cross-section of a diameter d for half the distance from the fixed end and a diameter $d/2$ for the remaining length. Determine the deflection at the free end if it carries a load W at the free end.

Solution

Given A cantilever of diameter d for half span from fixed end and $d/2$ for the rest length and carrying a load W at free end as shown in Fig. 7.62.
To find Deflection at free end

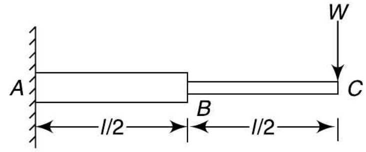


Fig. 7.62

Moment of inertia of thin section, $I = \frac{\pi(d/2)^4}{64} = \frac{\pi d^4}{1024}$

Moment of inertia of thick section $= \frac{\pi d^4}{64} = 16I$

Bending moment at a distance x from C, $M = Wx$

Its derivative with respect to W, $\frac{\partial M}{\partial W} = x$

By Castigliano's theorem

Inserting the above values in the equation, $\delta_c = \int_0^l \frac{M}{EI}\frac{\partial M}{\partial W}dx$

$$\delta_c = \int_0^{l/2}\frac{Wx}{EI}x \cdot dx + \int_{l/2}^{l}\frac{Wx}{E(16)I}x \cdot dx = \frac{W}{EI}\left[\left(\frac{x^3}{3}\right)_0^{l/2} + \left(\frac{x^3}{48}\right)_{l/2}^{l}\right]$$

$$= \frac{W}{EI}\left[\frac{l^3}{24} + \frac{7l^3}{384}\right] = \frac{23Wl^3}{384EI} = \frac{23Wl^3}{384E(\pi d^4/1024)} = \frac{184Wl^3}{3E}$$

Example 7.36 A rigid cantilever frame shown in Fig. 7.63 carries a load W at the free end. Assuming a constant value of EI, determine the vertical and horizontal displacement of the free end C.

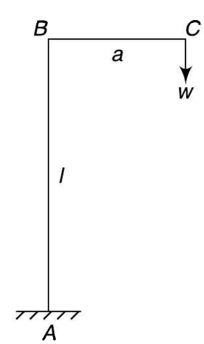


Fig. 7.63

Solution

Given A rigid cantilever frame carrying a load W at free end as shown in Fig. 7.63.
To find Vertical and horizontal displacement of free end C

For vertical displacement

Bending moment at a distance x from C, $M = -Wx$

Its derivative with respect to W, $\frac{\partial M}{\partial W} = -x$

Bending moment at a distance x from B, $M = -Wa$ (Fig. 7.64)

Its derivative with respect to W, $\dfrac{\partial M}{\partial W} = -a$

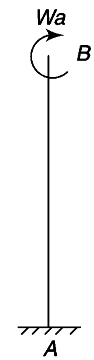


Fig. 7.64

By Castigliano's theorem

Inserting these values in the equation, $\delta_C = \int_0^l \dfrac{M}{EI}\dfrac{\partial M}{\partial W}dx$

$$\delta_C = \int_0^l \frac{M}{EI}\frac{\partial M}{\partial W}dx = \frac{1}{EI}\left[\int_0^a (-Wx)(-x)dx + \int_0^l (-Wa)(-a)dx\right]$$

$$= \frac{W}{EI}\left(\frac{a^3}{3} + a^2 l\right) = \frac{Wa^2}{EI}\left(\frac{a}{3} + l\right)$$

For horizontal displacement

Introduce a dummy horizontal force F at C,

Bending moment at a distance x from C, $M = -Wx$

Its derivative with respect to W, $\dfrac{\partial M}{\partial F} = 0$

Bending moment at a distance x from B, $M = -Wa - Fx$

Its derivative with respect to W, $\dfrac{\partial M}{\partial W} = -x$

By Castigliano's theorem

Inserting these values in the equation, $\delta_C = \int_0^l \dfrac{M}{EI}\dfrac{\partial M}{\partial W}dx$ and putting $F = 0$ before integration,

$$\delta_C = \int_0^l \frac{M}{EI}\frac{\partial M}{\partial W}dx = \frac{1}{EI}\left[\int_0^a (-Wx)(0)dx + \int_0^l (-Wa)(-x)dx\right]$$

$$= \frac{Wa}{EI}\left(\frac{l^2}{2}\right) = \frac{Wal^2}{2EI}$$

Example 7.37 A bent cantilever frame shown in Fig. 7.65 carries a load W at the free end D. Determine the vertical displacement of the free end. In the figure, a and I denote the length and the moment of inertia respectively.

Solution

Given A bent cantilever carrying a load W at free end as shown in Fig. 7.65

To find Vertical displacement of free end D

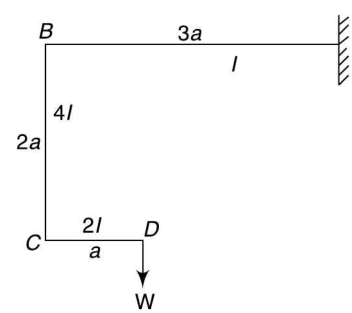


Fig. 7.65

Bending moments

Bending moment at a distance x from D, $M = -Wx$

Its derivative with respect to W, $\dfrac{\partial M}{\partial W} = -x$

Bending moment at a distance x from C, $M = -Wa$

Its derivative with respect to W, $\dfrac{\partial M}{\partial W} = -a$

Bending moment at a distance x from B, $M = -Wa + Wx = -W(a - x)$

Its derivative with respect to W, $\dfrac{\partial M}{\partial W} = -(a - x)$

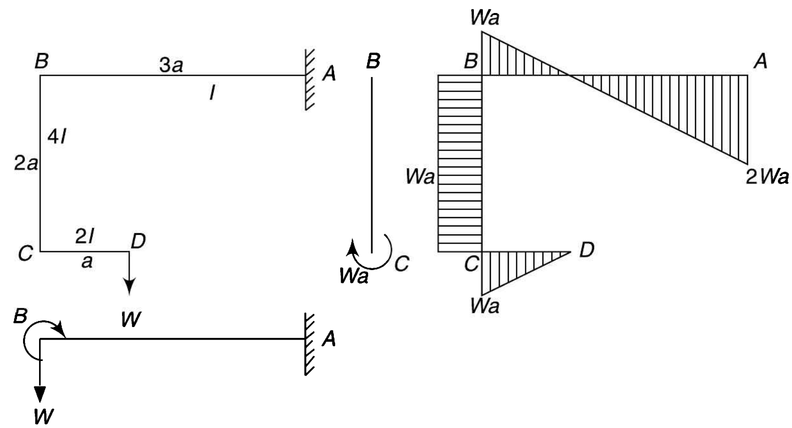


Fig. 7.66

By Castigliano's theorem

Inserting the above values in the equation, $\delta_c = \int_0^l \frac{M}{EI}\frac{\partial M}{\partial W}dx$

$$\delta_c = \frac{1}{E}\left[\int_0^a \frac{(-Wx)(-x)dx}{2I} + \int_0^{2a}\frac{(-Wa)(-a)dx}{4I} + \int_0^{3a}\frac{\{-W(a-x)\}\{-(a-x)dx\}}{I}\right]$$

$$= \frac{W}{4EI}\left[2\int_0^a x^2dx + \int_0^{2a} a^2dx + 4\int_0^{3a}(a^2 - 2ax + x^2)dx\right]$$

$$= \frac{W}{4EI}\left[2\left(\frac{x^3}{3}\right)_0^a + (a^2x)_0^{2a} + 4\left(a^2x - ax^2 + \frac{x^3}{3}\right)_0^{3a}\right]$$

$$= \frac{W}{4EI}\left[\frac{2a^3}{3} + 2a^3 + 4(3a^3)\right]$$

Example 7.38 Determine the ratio of the maximum deflection to the maximum bending stress of a simply supported beam of span *l* carrying

(i) a central load
(ii) a uniformly distributed load

Solution

Given A simply supported beam of span *l* carrying (i) a central load (ii) a uniformly distributed load.
To find Ratio of maximum deflection to maximum bending stresses

Beam with a central load

Let W be the load at the mid point of a simply supported beam of length l.

Maximum bending moment = $Wl/4$

Maximum bending stress, $\sigma = \frac{My}{I} = \frac{Wl}{4}\cdot\frac{d/2}{I} = \frac{Wdl}{8I}$

Maximum deflection, $\delta = \frac{Wl^3}{48EI}$

$$\therefore \qquad \frac{\delta}{\sigma} = \frac{Wl^3/48EI}{Wdl/8I} = \frac{l^2}{6Ed}$$

Beam with a uniformly distributed load

Let w be the uniformly distributed load on a simply supported beam of length l.

Maximum bending moment = $\dfrac{wl^2}{8}$

Maximum bending stress, $\sigma = \dfrac{My}{I} = \dfrac{wl^2}{8} \cdot \dfrac{d/2}{I} = \dfrac{wdl^2}{16I}$

Maximum deflection, $\delta = \dfrac{5}{384} \cdot \dfrac{wl^4}{EI}$

$$\therefore \qquad \frac{\delta}{\sigma} = \frac{5}{384} \cdot \frac{wl^4}{EI} \Big/ \frac{wdl^2}{16I} = \frac{5}{24}\,\frac{l^2}{Ed}$$

7.9 IMPACT LOADING ON BEAMS

In case, the load on the beam is not gradual but drops through some height, then the load causes an impact on the beam and the beam undergoes a maximum instantaneous deflection. The beam vibrates for some time and then comes to rest. Let y be the deflection at the load point when the point load W is applied gradually and δ the maximum deflection produced when the same load drops on to the beam through a distance h. Also let F be the equivalent gradually applied load which can produce the deflection δ.

Work done (potential energy loss) by the load = $W(h + \delta)$

Loss of Potential energy of load = Gain of strain energy of beam

Equating the two,
$$W(h+\delta) = \frac{1}{2} F \cdot \delta \qquad \text{(i)}$$

Now, as gradually applied loads W and F produce deflections y and δ respectively and for same cross-sectional area deflections are to be proportional to loads,

$$\therefore \qquad \frac{F}{\delta} = \frac{W}{y} \text{ or } F = \frac{W\delta}{y} \qquad (7.38)$$

Thus (i) becomes, $W(h+\delta) = \dfrac{W\delta^2}{2y}$ or $h + \delta = \dfrac{\delta^2}{2y}$ or $\delta^2 - 2y\delta - 2yh = 0$

Solving,
$$\delta = \frac{2y + \sqrt{4y^2 + 8yh}}{2} = y + \sqrt{y^2 + 2yh} \qquad (7.39)$$

Example 7.39 || In a beam, a gradually applied load W produces a deflection of 6 mm at the load point and the maximum bending stress 72 MPa. Determine the greatest height from which a load of 0.1 W can be dropped without exceeding the elastic limit of 270 MPa.

Solution

Given A gradually applied load W produces a deflection of 6 mm at load point in a beam. σ_{max} = 72 MPa

To find Height from which 0.1 W load can be dropped. σ_{max} = 270 MPa

As stress and deflection produced are proportional to loads for a gradually applied load,

Deflection due to load $0.1W = 6 \times 0.1 = 0.6$ mm

Bending stress due to load $0.1W = 72 \times 0.1 = 7.2$ MPa

As deflections are to be proportional to loads, $\dfrac{F}{\delta} = \dfrac{W}{y}$

or $$\delta = \frac{Fy}{W} = \frac{270 \times 0.6}{7.2} = 22.5 \text{ mm}$$

Determination of h

Thus $$\delta = y + \sqrt{y^2 + 2yh} \text{ or } 22.5 = 0.6 + \sqrt{0.6^2 + 2 \times 0.6 \times h}$$

or $$0.6^2 + 2 \times 0.6 \times h = (22.5 - 0.6)^2$$

or $$h = \frac{21.9^2 - 0.6^2}{1.2} = 399.4 \text{ mm}$$

Example 7.40 A simply supported 500 mm × 180 mm rolled steel beam has a span of 5 m. A load of 25 kN is dropped on to the middle of the beam from a height of 15 mm. Determine the maximum instantaneous deflection and the maximum stress produced. E = 200 GPa and $I_x = 450 \times 10^6$ mm^4.

Solution

Given

$W = 25$ kN	$h = 15$ mm
$l = 5$ m	$I_x = 450 \times 10^6$ mm^4
$H = 500$ mm	$w = 180$ mm
$E = 200$ GPa $= 200$ kN/mm^2	

To find δ **and** σ_{max}

Let δ be the maximum deflection produced when the load W drops on to the beam. Also let F be the equivalent gradually applied load which can produce the same deflection δ.

Equating potential energy to strain energy

$$W(h + \delta) = \frac{1}{2} F \cdot \delta$$

or $$2Wh + 2W\delta = F \cdot \delta$$

or $$\delta = \frac{2Wh}{F - 2W} \quad \text{(i)}$$

Also, We have $$\delta = \frac{Fl^3}{48EI} \quad \text{(Example 7.31)} \quad \text{(ii)}$$

Determination of maximum deflection

From (i) and (ii), $$\frac{Fl^3}{48EI} = \frac{2Wh}{F - 2W}$$

or $$F(F - 2W) = \frac{96WhEI}{l^3}$$

or $$F(F - 2 \times 25) = \frac{96 \times 25 \times 15 \times 200 \times 450 \times 10^6}{5000^3}$$

or $$F^2 - 50F = 25\,920$$

or $$F^2 - 50F + 625 = 25\,920 + 625 = 26\,545$$

or $$(F - 25)^2 = 162.92$$

or $$F = 162.9 + 25 = 187.9 \text{ kN}$$

$$\delta = \frac{Fl^3}{48EI} = \frac{187.9 \times 5000^3}{48 \times 200 \times 450 \times 10^6} = 5.44 \text{ mm}$$

Determination of maximum bending stress

$$Z = \frac{I}{y} = \frac{450 \times 10^6}{500/2} = 1.8 \times 10^6 \text{ mm}^3$$

Maximum bending stress $= \dfrac{M}{Z} = \dfrac{Fl}{4} \cdot \dfrac{1}{Z} = \dfrac{187.9 \times 5000}{4} \cdot \dfrac{1}{1.8 \times 10^6}$

$= 0.1304 \text{ kN/mm}^2$ or 130.4 MPa

7.10 CONJUGATE BEAM METHOD

We have, $EI \cdot \dfrac{d^2y}{dx^2} = M$ or $\dfrac{d^2y}{dx^2} = \dfrac{M}{EI}$ (i)

Differentiating it, $EI \cdot \dfrac{d^3y}{dx^3} = \dfrac{dM}{dx} = F$

Differentiating it again, $EI \cdot \dfrac{d^4y}{dx^4} = \dfrac{dF}{dx} = -w$

or $$\frac{d^4y}{dx^4} = -\frac{w}{EI} \quad \text{or} \quad \frac{d^2}{dx^2}\left(\frac{d^2y}{dx^2}\right) = -\frac{w}{EI}$$

or $$\frac{d^2}{dx^2}\left(\frac{M}{EI}\right) = -\frac{w}{EI} \quad \text{or} \quad \frac{d^2M}{dx^2} = -w \qquad \text{(ii)}$$

Comparing equations (i) and (ii), it can be observed that *M/EI* bears the same relation to *y* as *w* bears to *M*.

- Thus as indicated by (ii), if *w* indicates the actual loading, and a bending moment diagram is drawn, it provides the bending moment at any cross-section of the beam.
- In a similar way it may be said from (i) that if the bending moment diagram (*M/EI*) is assumed as the loading diagram on the beam (the beam is known as *conjugate beam*) and a new bending moment diagram is constructed from this, the diagram will be a *deflection curve.*

A similar analogy for the slope can also be deduced as follows.

$$\frac{d^2y}{dx^2} = \frac{M}{EI} \quad \text{or} \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{M}{EI} \quad \text{or} \quad \frac{d}{dx}(slope) = \frac{M}{EI} \qquad \text{(iii)}$$

Also, $$\frac{dF}{dx} = -w \qquad \text{(iv)}$$

Thus shear force diagram drawn with *M/EI* as loading will provide the slope at any section.

Because of the analogies stated above, some points to be taken care regarding the support conditions are the following:

No.	Simple (real) beam	Conjugate beam
1.	Actual loading diagram is the loading diagram.	Bending moment diagram is the loading diagram.
2.	Bending moment diagram from loading diagram provides the bending moment at any section.	Bending moment diagram from loading diagram provides the deflection at any section.
3.	Shear force diagram provides the shear stress at a section.	Shear force diagram provides the slope at a section.
4.	Shear force and bending moment at the fixed end of a cantilever exist.	SF and BM at fixed end will provide some values of slope and deflection which are not feasible. Thus a fixed end is transformed into a free end to obtain SF as well as BM as zero.
5.	Shear force and bending moment at the free end of a cantilever are zero.	SF and BM at the free end will provide zero slope and deflection which are not feasible. Thus a free end is transformed into a fixed end.
6.	Bending moment at the supports of simply supported beam is zero.	Deflection at the supports is zero. So, end conditions remain same.
7.	Shear force moment at the supports of simply supported beam exists.	Slope at the supports exists. So, end conditions remain same.
8.	An intermediate support has same slope on both sides. Also, An intermediate support has no deflection.	To have same shear force (slope) on both sides and zero bending moment (deflection), It is transformed into an intermediate hinge.
9.	A hinge support has same shear force and zero bending moment.	It is transformed into an intermediate support.

Example 7.41 Find expressions for the central deflection and the slope at the ends of a simply supported beam carrying a central load by conjugate beam method.

Solution

Given A simply supported beam carrying a central load as shown in Fig. 7.67a

To find

— Central deflection

— Slope at ends

Conjugate beam

As maximum bending moment at the centre is $Wl/4$, the same is shown as $Wl/4EI$ in the M/EI diagram (Fig. 7.67b).

Now, in the conjugate beam method, this diagram is to be considered as loading diagram and a new bending moment diagram is to be drawn which will give the deflection of the beam. For the same, first we need to find the reaction on the supports.

$$R_a = R_b = \frac{Wl}{4EI} \times \frac{l}{2} \times \frac{1}{2} = \frac{Wl^2}{16EI}$$

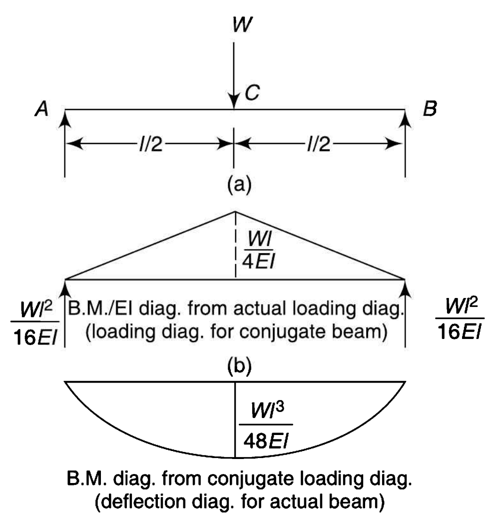


Fig. 7.67

The bending moment diagram (deflection diagram) drawn from the conjugate beam is shown in Fig. 7.67c.

Deflections

Deflection y at any point at a distance x from A

= bending moment due to load on the conjugate beam

$$= \frac{Wl^2}{16EI}x - \frac{Wl/4EI}{l/2} \cdot x \cdot \frac{x}{2} \cdot \frac{x}{3} = \frac{Wl^2}{16EI}x - \frac{W}{12EI}x^3 = \frac{W}{48EI}(3l^2 x - 4x^3)$$

Maximum deflection at the centre $= \dfrac{W}{48EI}\left[3l^2 \cdot \dfrac{l}{2} - 4\left(\dfrac{l}{2}\right)^3\right] = \dfrac{Wl^3}{48EI}$

Slopes

Slope at any point at a distance x from A

= Shearing force at the point due to load on the conjugate beam

$$= \frac{Wl^2}{16EI} - \frac{Wl/4EI}{l/2} \cdot x \cdot \frac{x}{2}$$

Slope at the ends $= \dfrac{Wl^2}{16EI}$ .....($x = 0$)

Example 7.42 || A simply supported beam has moment of inertia I for the left half and $2I$ for the right half. It carries a concentrated load W at the centre. Determine the slope at each end and the deflection at the centre.

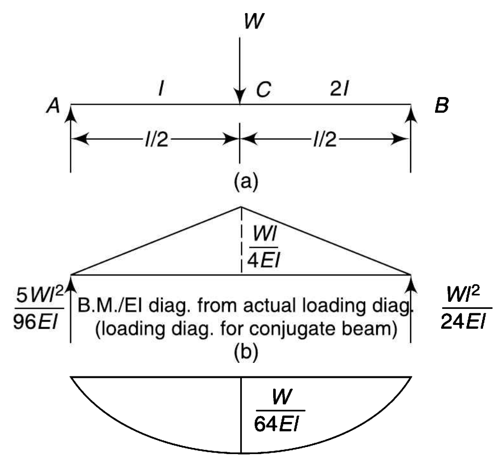
Solution

Given A simply supported beam having moment of inertia I for the left half and $2I$ for the right half and carrying a central load as shown in Fig. 7.68a.

To find

— Slope at ends and at center

— Deflection at centre



B.M. diag. from conjugate loading diag. (deflection diag. for actual beam)
(c)
Fig. 7.68

Conjugate beam

The central load on the conjugate diagram will be $Wl/4EI$ as in the previous case (Fig. 7.68b). However, thickness of this diagram on right half may be considered twice as compared to left half as it has double moment of inertia of the beam.

Taking moments about A,

$$R_b \times l = \left[\left(\frac{1}{2}\right)\left(\frac{Wl}{4EI} \cdot \frac{l}{2}\right)\right]\left(\frac{2}{3} \cdot \frac{l}{2}\right) + \left[\left(\frac{1}{2}\right)\left(\frac{Wl}{4 \times 2EI} \cdot \frac{l}{2}\right)\right]\left(\frac{l}{2} + \frac{1}{3} \cdot \frac{l}{2}\right)$$

$$= \frac{Wl^3}{48EI} + \frac{Wl^3}{48EI} \cdot = \frac{Wl^3}{24EI}$$

or $$R_b = \frac{Wl^2}{24EI}$$

$\therefore$ $$R_a = \left(\frac{1}{2}\right)\left(\frac{Wl}{4EI} \cdot \frac{l}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{Wl}{4 \times 2EI} \cdot \frac{l}{2}\right) - \frac{Wl^2}{24EI} = \frac{3Wl^2}{32EI} - \frac{Wl^2}{24EI} = \frac{5Wl^2}{96EI}$$

The bending moment diagram (deflection diagram) drawn from the conjugate beam is shown in Fig. 7.68c.

Slopes

Slope at A = shearing force at A for conjugate beam = $\dfrac{5Wl^2}{96EI}$

Slope at B = shearing force at B for conjugate beam = $\dfrac{Wl^2}{24EI}$

Slope at C = shearing force at C for conjugate beam

$$= \frac{5Wl^2}{96EI} - \left(\frac{1}{2}\right)\left(\frac{Wl}{4EI}\cdot\frac{l}{2}\right) = -\frac{Wl^2}{96EI} \text{ or } \frac{5Wl^2}{96EI} \text{ numerically}$$

Deflection

Deflection at C = bending moment at C for conjugate beam

$$= \frac{5Wl^2}{96EI}\cdot\frac{l}{2} - \left[\left(\frac{1}{2}\right)\left(\frac{Wl}{4EI}\cdot\frac{l}{2}\right)\right]\left(\frac{1}{3}\cdot\frac{l}{2}\right) = \frac{Wl^3}{64EI}$$

Example 7.43 A 10 m long simply supported beam AB carries loads of 80 kN and 60 kN at 2 m and 7 m respectively from A. E = 200 GPa and I = 150 × 10^6 mm^4. Determine the deflection and slope under the loads using conjugate beam method.

Solution

Given A simply supported beam loaded as shown in Fig. 7.69a.

E = 200 GPa I = 150 × 10^6 mm^4

To find Slope and deflection under the loads

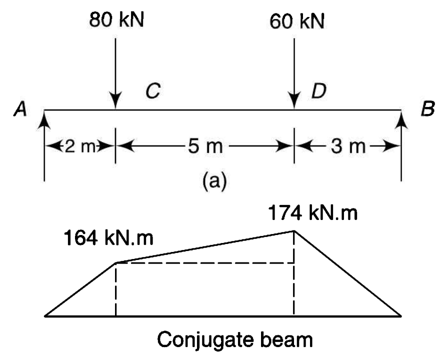


Fig. 7.69

E = 200 GPa = 200 kN/mm^2 = 200 × 10^6 kN/m^2

I = 150 × 10^6 mm^4 = 150 × 10^{-6} m^4

Taking moments about A,

$10\,R_b = 80 \times 2 + 60 \times 7$ or $R_b = 58$ kN

$R_a = 80 + 60 - 58 = 82$ kN

Bending moment at C = 82 × 2 = 164 kN·m

Bending moment at D = 58 × 3 = 174 kN·m

Conjugate beam

Bending moment (conjugate beam) diagram is shown in Fig. 7.69b.

Taking moments about B to find the reaction at A from conjugate loads,

$$10\,R_a = \left(164\times2\times\frac{1}{2}\right)\left(\frac{2}{3}+8\right) + 164\times5\left(3+\frac{5}{2}\right) + (174-164)\times5\times\frac{1}{2}\left(3+\frac{5}{3}\right) + 174\times3\times\frac{1}{2}\times2$$

$10\,R_a = 1421.3 + 4510 + 116.7 + 522$ or $R_a = 657$

$R_b = 164 \times (2/2) + 164 \times 5 + (174 - 164) \times (5/2) + 174 \times (3/2) - 657 = 613$

For conjugate beam

Shearing force at C = 657 – 164 × (2/2) = 493

Shearing force at D = –613 + 174 × (3/2) = –352

Bending moment at C = 657 × 2 – 164 × (2/3) = 1204.7

Bending moment at D = 613 × 3 – 174 × (3/2) × 1 = 1578

Slope and deflection

EI = 200 × 10^6 × (150 × 10^{-6}) = 30 000 kN·m^2

Slope at $C = 493/30\ 000 = 0.0164$ rad
Slope at $D = 352/30\ 000 = 0.0117$ rad
Deflection at $C = 1204.7/30\ 000 = 0.04016$ m $= 40.16$ mm
Deflection at $D = 1578/30\ 000 = 0.0526$ m $= 52.26$ mm

Example 7.44 || A cantilever of length l has a point load W at the free end. Determine the slope and the deflection at the free end.

Solution

Given A cantilever with a point load at free end as shown in Fig. 7.70a.
To find Slope and deflection at free end

Conjugate beam
The conjugate beam is shown in Fig. 7.70b. The fixed end is transformed into a free end and the free end into a fixed end.
Slope and deflection
Slope at the free end = shear force at B for conjugate beam = $\frac{Wl}{EI} \cdot \frac{l}{2} = \frac{Wl^2}{2EI}$

Deflection at the free end = bending moment at B for conjugate beam = $\frac{Wl^2}{2EI} \cdot \frac{2}{3} l = \frac{Wl^3}{3EI}$

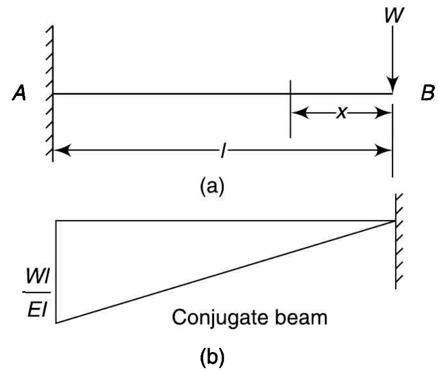


Fig. 7.70

Example 7.45 || A cantilever of length l has a point load W at a distance a from the fixed end. Determine the slope and the deflection at the free end.

Solution

Given A cantilever with a load W at a distance a from fixed end as shown in Fig. 7.71a.
To find Slope and deflection at free end

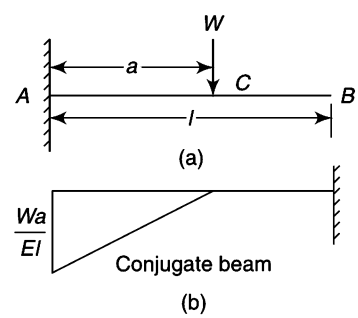


Fig. 7.71

Conjugate beam
The conjugate beam is shown in Fig. 7.71b. The fixed end is transformed into a free end and the free end into a fixed end in the conjugate beam.
Slope and deflection

Slope at the free end = shear force at B for conjugate beam = $\frac{Wa}{2} \cdot a \cdot \frac{1}{EI} = \frac{Wa^2}{2EI}$

Deflection at the free end = bending moment at B for conjugate beam = $\frac{Wa^2}{2EI}\left(l - \frac{a}{3}\right)$

Example 7.46 || A cantilever of length l is loaded with a uniformly distributed load of w per unit length. Determine the slope and the deflection at the free end.

Solution

Given A cantilever with a uniformly distributed load as shown in Fig. 7.72a.
To find Slope and deflection at free end

Conjugate beam
The bending moment diagram or the conjugate beam is shown in Fig. 7.72b. The fixed end is transformed into a free end and the free end into a fixed end.

Slope and deflection
Slope at the free end = shearing force at B for conjugate beam =

$$\frac{1}{3}\cdot l\cdot\frac{wl^2}{2}\cdot\frac{1}{EI}=\frac{Wl^3}{6EI}$$

Deflection at the free end = bending moment at B for conjugate beam $=\dfrac{wl^3}{6EI}\dfrac{3l}{4}=\dfrac{wl^4}{8EI}$

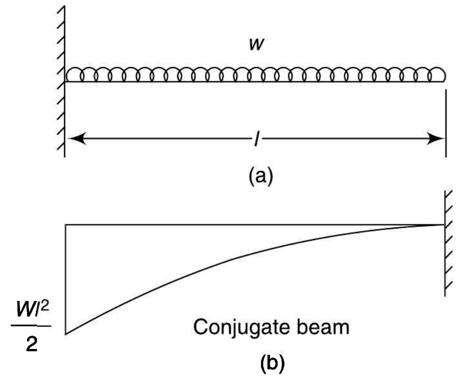


Fig. 7.72

Example 7.47 A simply supported beam *ACBD* with an overhang is supported at *A* and *B*. It carries loads of 2*W* at *C* and *W* at *D*. The distances between *AC*, *CB* and *BD* are *l*/2, *l*/2 and *l*/4 respectively. *BD* is the overhang. Determine the slopes at the load and the support points and the deflections at the load points using conjugate beam method.

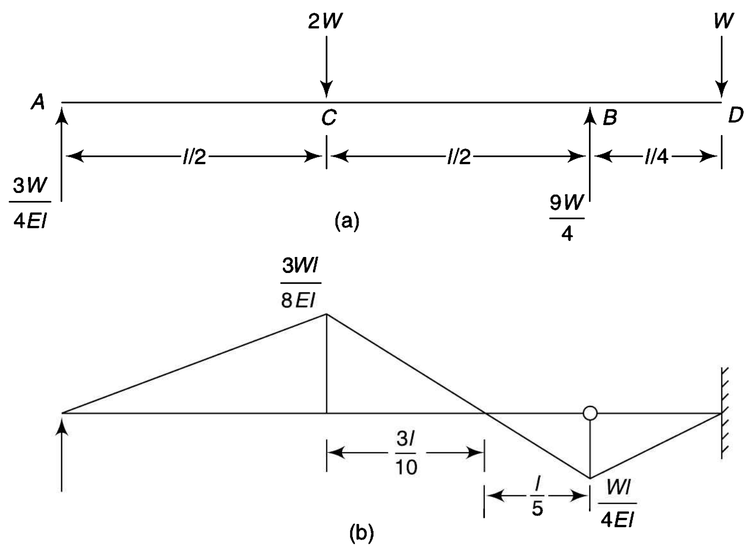


Fig. 7.73

Solution

Given A simply supported beam with right overhang and point loads as shown in Fig. 7.73a.

To find

— Slopes at load and support points
— Deflections at load points

Simply supported beam
To find reaction at the supports, take moments about B,

$$R_a\cdot l=2W\cdot\frac{l}{2}-W\cdot\frac{l}{4}\text{ or }R_a=\frac{3W}{4}\text{ and }R_b=3W-\frac{3W}{4}=\frac{9W}{4}$$

Bending moment at $C = \frac{3W}{4}\cdot\frac{l}{2} = \frac{3Wl}{8}$; Bending moment at $B = -W\cdot\frac{l}{4} = \frac{Wl}{4}$

Point of contraflexure in CB: Let this point be at a distance x from C,

$$\frac{3W}{4}\left(\frac{l}{2}+x\right) - 2Wx = 0 \text{ or } \frac{3l}{8}+\frac{3x}{4} - 2x = 0 \text{ or } x = \frac{3l}{10}$$

Conjugate beam

Figure 7.73b shows the conjugate beam (BM/EI). Note the following:

- Simple support at A remains a simple support.
- Intermediate support at B is transformed into intermediate hinge.
- Free end at D is transformed into a fixed end.

At the hinge point, bending moment is to be zero.

To find the reactions, take moments about B,

$$R_a\cdot l = \left(\frac{1}{2}\cdot\frac{l}{2}\cdot\frac{3Wl}{8EI}\right)\left(\frac{l}{2}+\frac{1}{3}\cdot\frac{l}{2}\right) + \left(\frac{1}{2}\cdot\frac{3l}{10}\cdot\frac{3Wl}{8EI}\right)\left(\frac{l}{5}+\frac{2}{3}\cdot\frac{3l}{10}\right) - \left(\frac{1}{2}\cdot\frac{l}{5}\cdot\frac{Wl}{4EI}\right)\left(\frac{1}{3}\cdot\frac{l}{5}\right)$$

$$= \left(\frac{3Wl^2}{32EI}\right)\left(\frac{2l}{3}\right) + \left(\frac{9Wl^2}{160EI}\right)\left(\frac{2l}{5}\right) - \left(\frac{Wl^2}{40EI}\right)\left(\frac{l}{15}\right)$$

$$R_a = \left(\frac{Wl^2}{16EI}\right) + \left(\frac{9Wl^2}{400EI}\right) - \left(\frac{Wl^2}{600EI}\right) \text{ or } R_a = \frac{Wl^2}{12EI}$$

$$R_b = \frac{3Wl^2}{32EI} + \frac{9Wl^2}{160EI} - \frac{Wl^2}{40EI} - \frac{Wl^2}{12EI} = \frac{Wl^2}{24EI}$$

Slopes

Slope at A = shearing force at $A = \frac{Wl^2}{12EI}$

Slope at C = shearing force at $C = \frac{Wl^2}{12EI} - \frac{3Wl^2}{32EI} = -\frac{Wl^2}{96EI}$

Slope at B = shearing force at $B = -\frac{Wl^2}{96EI} - \frac{9Wl^2}{160EI} + \frac{Wl^2}{40EI} = -\frac{Wl^2}{24EI}$

(Numerically equal to reaction of the hinge)

Slope at D = shearing force at $D = -\frac{Wl^2}{24EI} + \frac{1}{2}\cdot\frac{l}{4}\cdot\frac{Wl}{4EI} = -\frac{Wl^2}{96EI}$

(considering portion BD)

Deflections

Deflection at C = bending moment at $C = \frac{Wl^2}{12EI}\cdot\frac{l}{2} \quad \frac{3Wl^2}{32EI}\frac{l}{6} = \frac{5Wl^3}{192EI}$

Deflection at D = bending moment at $D = -\frac{Wl^2}{24EI}\cdot\frac{l}{4} + \frac{1}{2}\cdot\frac{l}{4}\cdot\frac{Wl}{4EI}\cdot\frac{2}{3}\cdot\frac{l}{4} = -\frac{Wl^3}{192EI}$

7.11 DEFLECTION DUE TO SHEAR

Shear strain energy $= \dfrac{\tau^2}{2G} \times$ volume (Eq. 3.9)

where τ is the shear stress and the G is the shear modulus of elasticity.

In a beam as the shear stress values varies along the length as well as across a cross-section, the strain energy for the whole beam,

$$U = \frac{1}{2G}\iint \tau^2 dA \cdot dx \tag{7.40}$$

where dx = an element of length of the beam

and dA = Area of an element of cross-section in the elemental length

Consider the case of a rectangular cross-section of a beam as shown in Fig. 6.5

Shear stress is given by $\tau = \dfrac{6F}{bd^3} \cdot \left(\dfrac{d^2}{4} - y^2\right)$ (Eq. 6.2)

$$\therefore \quad U = \frac{1}{2G}\int\left[\int_{-d/2}^{d/2}\left(\frac{6F}{bd^3}\right)^2\left(\frac{d^2}{4} - y^2\right)^2 b \cdot dy\right] \cdot dx$$

$$= \frac{36}{2bd^6G}\int\left[F^2\int_{-d/2}^{d/2}\left(\frac{d^4}{16} + y^4 - \frac{d^2y^2}{2}\right) \cdot dy\right] \cdot dx$$

$$= \frac{18}{bd^6G}\int\left[F^2\left(\frac{d^4y}{16} + \frac{y^5}{5} - \frac{d^2y^3}{6}\right)_{-d/2}^{d/2}\right] \cdot dx$$

$$= \frac{18}{bd^6G}\int F^2 \cdot 2\left(\frac{d^5}{32} + \frac{d^5}{160} - \frac{d^5}{48}\right) \cdot dx$$

$$= \frac{3}{5bdG}\int F^2 \cdot dx \tag{7.41}$$

The following cases may be considered:

- *Cantilever*

  (i) Load W at free end: Shear force is equal to W throughout the length.

  $$\therefore \quad U = \frac{3}{5bdG}\int F^2 \cdot dx = \frac{3W^2}{5bdG}(x)_0^l = \frac{3W^2 l}{5bdG}$$

  Deflection at the free end or load point,

  $$\delta = \frac{dU}{dW} = \frac{6Wl}{5bdG} \tag{7.42}$$

  (ii) Uniformly distributed load: Shear force varies linearly.

  Consider a load $w \cdot \delta x$, on a length δx at a distance x from the fixed end. This load may be considered as a point load and thus the deflection at the load point $= \dfrac{6w\delta x \cdot x}{5bdG}$

$\therefore$ Total deflection due to shear at the free end

$$= \int_0^l \frac{6w}{5bdG} \cdot x \cdot dx = \frac{6w}{5bdG}\left(\frac{x^2}{2}\right)_0^l = \frac{3wl^2}{5bdG} \quad (7.43)$$

- *Simply supported beam*
  (i) Load W at mid-point: Shear force is equal to $W/2$ (positive for first half) and $W/2$ (positive for second half).

$$\therefore \quad U = \frac{3}{5bdG}\int F^2 \cdot dx = 2 \times \frac{3(W/2)^2}{5bdG} \cdot (x)_0^{l/2} = \frac{3W^2 l}{20bdG}$$

Deflection at the midpoint or load point, $\delta = \dfrac{dU}{dW} = \dfrac{3Wl}{10bdG}$

Deflection at the midpoint can also be obtained assuming half the beam to be a cantilever with load $W/2$ at the free end.

and $\therefore$ deflection $= \dfrac{6(W/2)(l/2)}{5bdG} = \dfrac{3Wl}{10bdG}$

Load not centrally applied: Deflection at the load point can be obtained assuming the beam to be a cantilever fixed at one end and length upto the load point.

$$\therefore \quad \text{deflection} = \frac{6(Wl_2/l)l_1}{5bdG} = \frac{6Wl_1l_2}{5bdlG} \quad (7.44)$$

  (ii) Uniformly distributed load: Assume half the beam to be a cantilever.

$$\therefore \quad \text{deflection} = \frac{3w(l/2)^2}{5bdG} = \frac{3wl^2}{20bdG} \quad (7.45)$$

- Deflection due to shear is usually very small as compared to deflection due to bending. Therefore, usually, it is neglected.

Example 7.48 A simply supported beam of 4-m span is 100-mm wide and 200-mm deep. It carries a central load of 50 kN. Determine the deflection due to shear and the total deflection. E = 200 GPa and G = 82 GPa.

Given A simply supported beam

$W = 50$ kN	$l = 4$ m
$b = 100$ mm	$d = 200$ m
$E = 200$ GPa	$G = 82$ GPa

To find Deflection due to shear and total deflection

Solution

Deflection due to shear and bending

Deflection due to shear $= \dfrac{3Wl}{10bdG} = \dfrac{3 \times 50\,000 \times 4000}{10\ X\ 100 \times 200 \times 82\,000} = 0.0366$ mm

Deflection due to bending $= \dfrac{Wl^3}{48EI} = \dfrac{50\,000 \times 4000^3}{48 \times 200\,000 \times (100 \times 200^3/12)} = 5$ mm

Total deflection

Total deflection = 0.0366 + 5 = 5.0366 mm

Example 7.49 A 3-m long simply supported beam of 80 mm width and 160 mm depth carries a uniformly distributed load of 20 kN/m over the whole length. Determine the deflection due to shear, due to bending and the total deflection. E = 200 GPa and G = 80 GPa.

Solution

Given A simply supported beam

$w = 20$ kN/m	$l = 3$ m
$b = 80$ mm	$d = 160$ m
$E = 200$ GPa	$G = 80$ GPa

To find Deflection due to shear, due to bending and total deflection

Solution

Deflection due to shear and bending

Loading = 20 000N/m = 20 N/mm

$$\text{Deflection due to shear} = \frac{3wl^2}{20bdG} = \frac{3 \times 20 \times 3000^2}{20 \times 80 \times 160 \times 80\ 000} = 0.026 \text{ mm}$$

$$\text{Deflection due to bending} = \frac{5wl^3}{384EI} = \frac{5 \times 20 \times 3000^4}{384 \times 200\ 000 \times (80 \times 160^3/12)} = 3.862 \text{ mm}$$

Total deflection

Total deflection = 0.026 + 3.862 = 3.888 mm

7.12 MAXWELL'S RECIPROCAL DEFLECTION THEOREM

Maxwell's theorem of reciprocal displacements is stated as follows:

The deflection of any point P resulting from application of a load at any other point Q is the same as the deflection of Q resulting from the application of the same load at P.

In the mathematical form,

$$\delta_{pq} = \delta_{qp}$$

Figure 7.74a shows a structure AB. When a load W is applied at any point Q, the deflections of point Q and any other point P are δ_q and δ_p respectively. Also when the same load W is applied at point P, the deflections of the points P and Q are Δ_p and Δ_q respectively.

Now let the structure be loaded afresh as under:

- Apply a load W at point Q so that δ_q and δ_p are the deflections of points Q and P respectively.

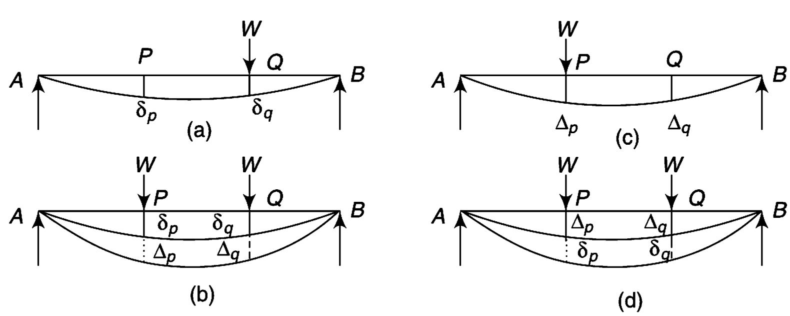


Fig. 7.74

Work done on the structure = $\frac{1}{2}W\delta_q$ (the load applied is gradual)

Now, apply another load W at point P. There will be further deflections δ_p and δ_q of the points P and Q respectively (Fig. 7.74b).

Additional work done on the structure

= Work done by load W at P + Work done by load W at Q

$= \frac{1}{2}W\Delta_p + W\cdot\Delta_q$ (load at Q is not gradual but sudden as it already exists)

Thus total work done in this type of loading = $\frac{1}{2}W\delta_q + \frac{1}{2}W\Delta_p + W\cdot\Delta_q$

- Now, load the structure in a different way. First, apply a load W at point P (Fig. 7.74c) so that Δ_p and Δ_q are the deflections of points P and Q respectively.

Work done on the structure = $\frac{1}{2}W\Delta_p$ (the load applied is gradual)

Now, apply another load W at point Q (Fig. 7.74d). There will be further deflections δ_p and δ_q of the points P and Q respectively.

Additional work done on the structure

= Work done by load W at P + Work done by load W at Q = $W\cdot\delta_p + \frac{1}{2}W\delta_q$

Thus total work done in this type of loading = $\frac{1}{2}W\Delta_p + W\cdot\delta_p + \frac{1}{2}W\delta_q$

Finally as the loading on the structure is the same in the two cases, the work done must be same. Therefore,

$$\frac{1}{2}W\delta_q + \frac{1}{2}W\Delta_q + W\cdot\Delta_q = \frac{1}{2}W\Delta_p + W\cdot\delta_p + \frac{1}{2}W\delta_q$$

or
$$\Delta_q = \delta_p$$

i.e. deflection at Q due to load W at P = deflection at P due to load W at Q

7.13 BETTI'S THEOREM OF RECIPROCAL DEFLECTIONS

It may be stated as follows:

In an elastic system, the external work done by a force acting at P during the deflections caused by another force at Q is equal to the external work done by the force at Q during the deflections caused by the force at P.

In the mathematical form, $F_p\delta_{pq} = Q_p\delta_{qp}$

Summary

1. Excessive deflections can cause visible or invisible cracks in beams. Also, excessive deflections perceptible by naked eye give a feeling of unsafe structure to the occupants of a building causing adverse effect on their health.
2. The designing of a beam from deflection aspect is known as *stiffness criterion.*
3. Deflection profile of a beam is known as its *elastic curve*.

4. Governing *differential equation* of a beam under the action of bending moment is $EI(d^2y/dx^2) = M$
5. Main methods to find the slope and deflection of a beam are double integration method, Macaulay's method, area-moment method, strain energy method and conjugate beam method.
6. In double integration method, the equation of the elastic curve is integrated twice to obtain the deflection of the beam at any cross-section. The constants of integration are found by applying the end conditions.
7. In Macaulay's method a single equation is written for the bending moment for all the portions of the beam. The equation is formed in such a way that the same constants of integration are applicable to all portions.
8. Mohr's first moment-area theorem states that *the difference of slopes between any two points on an elastic curve of a beam is equal to the net area of the bending moment diagram between these two points divided by EI.*
9. Mohr's second theorem states that *the intercepts on a given line between the tangents to the elastic curve of a beam at any two points is equal to the net moment taken about the same line of the area of the bending moment diagram between the two points divided by EI.*
10. Strain energy of a beam, $U = \int(M^2/EI) \cdot dx$
11. Castigliano's first theorem states that if a structure is subjected to a number of external loads (or couples), the partial derivative of the total strain energy with respect to any load (or couple) provides the deflection in the direction of that load (or couple).
12. In conjugate beam method, a conjugate beam is drawn by assuming the bending moment of the actual loading as the loading on this beam. The new bending moment of this beam gives the deflection curve of the actual loading and the new shear force diagram provides the slope.
13. Maxwell's theorem of reciprocal displacements states that the deflection of any point P resulting from application of a load at any other point Q is the same as the deflection of Q resulting from the application of the same load at P.
14. Betti's theorem of reciprocal deflections states that in an elastic system, the external work done by a force acting at P during the deflections caused by another force at Q is equal to the external work done by the force at Q during the deflections caused by the force at P.

Objective Type Questions

1. The designing of a beam based on bending and shear stress considerations is known as ________ criterion.
(a) stiffness (b) toughness (c) strength

2. The designing of a beam based on deflection considerations is known as ______ criterion.
(a) stiffness (b) toughness (c) strength

3. The flexural rigidity of a beam is
(a) E/I (b) EI (c) I/E (d) E^2I

4. Deflection of the free end of a cantilever having a point load at the free-end
(a) $\dfrac{Wl^3}{48EI}$ (b) $\dfrac{5Wl^3}{48EI}$ (c) $\dfrac{5Wl^3}{24EI}$ (d) $\dfrac{Wl^3}{3EI}$

5. Slope at the free end of a cantilever having a point load at the free end
(a) $\dfrac{Wl^2}{2EI}$ (b) $\dfrac{Wl^2}{3EI}$ (c) $\dfrac{Wl^2}{4EI}$ (d) $\dfrac{Wl^2}{8EI}$

6. Deflection of the free end of a cantilever having a point load at the mid-span

(a) $\frac{Wl^3}{3EI}$ (b) $\frac{5Wl^3}{24EI}$ (c) $\frac{5Wl^3}{48EI}$ (d) $\frac{Wl^3}{48EI}$

7. Maximum deflection of a cantilever having a uniformly distributed load

(a) $\frac{wl^4}{2EI}$ (b) $\frac{wl^4}{4EI}$ (c) $\frac{wl^4}{8EI}$ (d) $\frac{wl^4}{24EI}$

8. Maximum slope of a cantilever having a uniformly distributed load

(a) $\frac{wl^3}{4EI}$ (b) $\frac{wl^3}{6EI}$ (c) $\frac{wl^3}{12EI}$ (d) $\frac{wl^3}{24EI}$

9. Maximum deflection of a simply supported beam with a central point load

(a) $\frac{Wl^3}{4EI}$ (b) $\frac{Wl^3}{8EI}$ (c) $\frac{Wl^3}{24EI}$ (d) $\frac{Wl^3}{48EI}$

10. Slope at the supports of a simply supported beam with a central point load

(a) $\frac{Wl^2}{8EI}$ (b) $\frac{Wl^2}{12EI}$ (c) $\frac{Wl^2}{16EI}$ (d) $\frac{Wl^2}{24EI}$

11. Maximum deflection of a simply supported beam with a total uniformly distributed load W is

(a) $\frac{Wl^3}{384EI}$ (b) $\frac{5Wl^3}{384EI}$ (c) $\frac{Wl^3}{48EI}$ (d) $\frac{5Wl^3}{48EI}$

12. Slope at the supports of a simply supported beam with a uniformly distributed load is

(a) $\frac{wl^3}{12EI}$ (b) $\frac{wl^3}{24EI}$ (c) $\frac{wl^3}{48EI}$ (d) $\frac{wl^3}{96EI}$

Answers

1. (c)	2. (a)	3. (b)	4. (d)	5. (a)	6. (c)
7. (c)	8. (b)	9. (d)	10. (c)	11. (b)	12. (b)

Review Questions

7.1 Establish the governing differential equation of beams. What are its limitations?

7.2 What is Macaulay's method of beam deflection analysis? What are its advantages over the direct integration method?

7.3 State and prove the moment-area theorem.

7.4 State and develop the analogies between the real beam and the conjugate beam.

7.5 State and prove Castigliano's first theorem.

7.6 Deduce the expressions for deflection by the energy method.

7.7 State and prove the Maxwell's reciprocal deflection theorem.

7.8 What is Betti's theorem of reciprocal deflection?

Numerical Problems

7.1 A cantilever of 3-m span carries a uniformly distributed load of 10 kN per metre length over the entire span. Determine the deflection of the free end. $E = 200$ GPa and $I = 80 \times 10^6$ mm^4. (6.33 mm)

7.2 A cantilever of 4-m span carries a uniformly distributed load over the entire span. The deflection at the free end is 30 mm. Determine the slope at the free end. (0.01 rad)

7.3 A cantilever of 3-m span carries a uniformly distributed load of 10 kN per metre length for 2 m starting from the fixed end. Determine the slope and the deflection of the free end. $E = 200$ GPa and $I = 80 \times 10^6$ mm^4. (0.000 833 rad, 2.083 mm)

7.4 A cantilever of 3-m span carries a uniformly distributed load of 10 kN per metre length for 2 m starting from the free end. Determine the slope and the deflection of the free end. $E = 200$ GPa and $I = 80 \times 10^6$ mm^4. (0.0027 rad, 6.04 mm)

7.5 A 6-m long simply supported beam carries a point load W at the midspan. If the slope at the ends of the beam is not to exceed one degree, determine the deflection at the load point. (34.9 mm)

7.6 A 5-m long simply supported beam carries a point load of 5 kN at 3 m from the left end. Determine the slope at the left end, deflection at the load point and the maximum deflection. $E = 200$ GPa and $I = 100 \times 10^6$ mm^4. (0.000 35 rad, 0.6 mm, 0.617 mm)

7.7 The rate of loading on a simply supported beam of length l is $p \sin \pi x/l$ where x is the distance from one end. Show that the reactions at the supports are pl/π and the maximum bending moment is pl^2/π^2.

7.8 A simply supported beam with an overhang is loaded as shown in Fig. 7.75. Find the ratio of W/P to make the deflection at the free end equal to zero. (6)

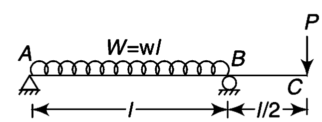


Fig. 7.75

7.9 Determine the maximum deflection of a simply supported beam of 5-m length and carrying a uniformly distributed load from zero at the ends to 8 kN/m at the centre. $EI = 2$ MN·m^2. (20.8 mm)

7.10 A simply supported horizontal beam carries a load which varies from 20 kN at one end to 50 kN at the other. Determine the central deflection if the span is 10 m and the width 420 mm. The bending stress is limited to 84 MPa. E =210 GPa. (25 mm)

7.11 A simply supported beam has a span of 15 m and carries two point loads of 4 kN and 9 kN at 6 m and 10 m respectively from one end. Find the deflection under each load and the maximum deflection. $E = 200$ GPa and $I = 400 \times 10^6$ mm^4. (9.39 mm, 8.99 mm; 10.48 mm at 7.79 m)

7.12 A beam AB of 6 m span is simply supported at the ends. It carries a concentrated load of 6 kN at a distance of 6 m from the left-hand end support and a uniformly distributed load of 2 kN/m at the right half of the beam. Find the deflection at the midspan, slope at the left-hand end and the maximum deflection. (9.97 mm, 0.0053 rad, 9.97 mm)

7.13 A simply supported beam has its supports 8 m apart at A and B. It carries a uniformly distributed load of 4 kN/m between A and B starting from 2 m and ending at 5 m from A. The end B of the beam has an overhang of 3 m and at the free end a concentrated load of 6 kN is applied. Determine deflection of the free end and the maximum deflection between A and B. Tale $E = 200$ GPa and $I = 15 \times 10^6$ mm^4. (22.7 mm, 16.1 mm)

7.14 A simply supported beam is applied a positive bending couple of 8 kN·m at the left end and another positive bending couple of 10 kN·m at the right end. If the beam is 8 m long, find the maximum deflection. $EI = 12$ MN.m^2. Use moment area method (5.76 mm)

7.15 A simply supported 8 m long beam carries a uniformly distributed load of 3 kN/m over the whole span and a concentrated load of 6 kN at the centre. Determine the central deflection and the slope at the ends. $E = 200$ GPa and $I = 100 \times 10^6$ mm^4. (11.2 mm, 0.0044 rad)

7.16 A simply supported beam of 5 m length carries a uniformly distributed load of 6 kN/m over the whole span in addition to negative bending couples of 4 kN·m at the each end. Determine the deflection at the midspan. $EI = 100$ MN·m^2. Use moment area method. (0.363 mm)

7.17 An 18-m long horizontal cantilever ABC is built-in at A and supported at B, 14 m from A by a rigid prop. AB and BC carry uniformly distributed loads of 600 N/m and 1kN/m respectively. Determine the load taken by the prop. (12.58 kN)

7.18 A cantilever of uniform strength is to be made from a mild steel bar of 40-mm diameter. A load of 3 kN is to be supported from the free end and the allowable stress is to be 75 MPa. Find the maximum length of the cantilever. Also find its end deflection. $E = 210$ GPa. (157 mm, 0.265 mm)

7.19 A simply supported beam has a span of 9 m. It carries a load of 72 kN at a distance of 2 m and another 45 kN at a distance of 5 m from the left hand support. Find the deflection and the slope under the loads by conjugate beam method. $E = 200$ GPa and $I = 150 \times 10^6$ mm^4. (33.1 mm, 48.6 mm; 0.76°, 0.21°)

7.20 A beam *AB* of 5-m span and simply supported at ends *A* and *B*. *C* is a point on the beam at a distance 4 m from *A*. The beam has moment of inertia of 80×10^6 mm^4 for the length *AC* and 20×10^6 mm^4 for the length *CB*. The beam is loaded with a point load of 6 kN at *C*. Determine the slope at *A*, deflection at the midspan and the maximum deflection. $E = 205$ GPa. Use the conjugate beam method.

(0.0014 rad, 5.5 mm, 5.8 mm)

7.21 A cantilever of length l carries a concentrated load W at the free end. The section of the cantilever is circular of diameter d for half the length from the fixed end and of diameter $d/2$ for the remaining length. Determine the slope and the deflection at the midspan and at the free end by conjugate beam method. ($3Wl^2/128EI$, $19Wl^2/128EI$; $5Wl^3/768EI$, $23Wl^2/384EI$)

Chapter **8**

Fixed and Continuous Beams

A beam is called a *fixed beam* if both of its ends are rigidly fixed so that the end slopes remain horizontal. Such beams are also known as *built-in* or *encastre beams*. Owing to fixidity, the slope of the beam is zero at each end. In Chapter 4, bending moment and shear force diagrams were drawn for the statically determinate beams in which it was possible to find the reactions by applying the equations of static equilibrium. However, beams of the fixed type are *statically indeterminate beams* in which the equations of equilibrium alone are not sufficient to determine the internal forces. In order to find the reactions at the supports, equations of deflections in addition to equations of statics are required to be used.

When a beam is carried by more than two supports, it is known as a *continuous* or *propped beam*. The analysis of propped beams and cantilevers is also similar to built–in beams. In this chapter, various methods used to determine the bending moment and shear force of fixed beams, continuous beams and propped cantilevers are discussed.

8.1 EFFECT OF FIXIDITY

A fixed beam may be assumed as equivalent to a freely supported loaded beam subjected to end couples which make the slopes at the ends zero.

Figure 8.1a shows a fixed beam with external loading. In Fig. 8.1b the same beam is shown as a simply supported beam acted upon by the external loading. Below the load diagram, the bending moment diagram due to that is also shown. This is usually referred as *free moment diagram* and the reactions as *free reactions* (R_1 and R_2). Similarly, Fig. 8.1c shows the simply supported beam acted upon by end couples introduced to bring the slopes back to zero along with the bending moment diagram below the same. This is referred as *fixing moment diagram.* In case the fixing moments at the ends are unequal, the reactions are also there ($R = (M_a - M_b)/l$, upward on the side of higher fixing moment.

The combined bending moment diagram can be obtained by adding either free moment diagram to the fixing moment diagram algebraically (Fig. 8.1d) or the fixing moment diagram to the free moment diagram algebraically (Fig. 8.1e). It can be observed that the maximum bending moment is much lesser in the fixed beam as compared to that in a simply supported beam.

There are several methods to find the values of fixing bending moments. The following are the usual methods:

1. Moment-area method

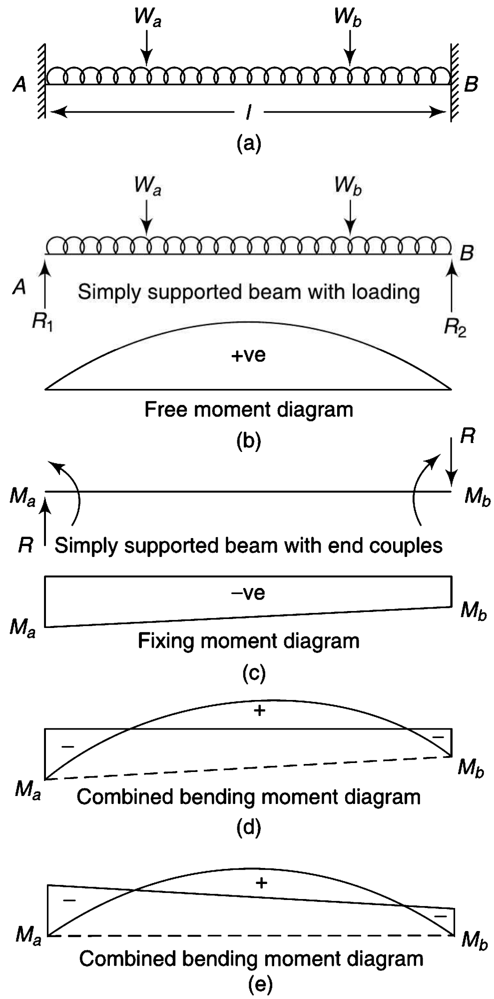


Fig. 8.1

2. Macaulay's method
3. Three-moment method
4. Moment distribution method
5. Method of flexibility coefficients

8.2 MOMENT–AREA METHOD

Referring Fig. 8.1c,

Bending moment at a distance x from the left-hand end $= Rx - M_a = \dfrac{M_a - M_b}{l} \cdot x - M_a$

This indicates a straight line variation of the bending moment from a value $-M_a$ at $x = 0$ to $-M_b$ at $x = l$.

For a beam of uniform section, as the change of slope from one end to the other is zero, net area of the bending moment diagram is zero. If the bending moment due to external loading is sagging and due to fixing moment hogging, then

Area of free moment diagram = Area of fixing moment diagram

$$A_1 = A_2 \tag{8.1}$$

Similarly, if the deflection of one end relative to other is zero, the moments of areas of the bending moment diagram about an end is zero, i.e.,

Moment of area of free moment diagram = Moment of area of fixed moment diagram

or
$$A_1\bar{x}_1 = A_2\bar{x}_2 \tag{8.2}$$

The area to be considered may be broken into parts to obtain convenient rectangles, triangles and parabolas.

Total reactions at the ends,

$$R_a = R_1 + R = R_1 + \frac{M_a - M_b}{l} \text{ and } R_b = R_2 - R = R_2 - \frac{M_a - M_b}{l}$$

Example 8.1 Determine the maximum bending moment and the deflection of a beam of length l and flexural rigidity EI. The beam is fixed at both ends and carries a concentrated load W at the midspan.

Solution

Given A beam fixed at both ends and loaded as shown in Fig. 8.2a.

To find

— Maximum bending moment
— Maximum deflection

Due to symmetry, fixing moment $M_a = M_b = M$ (say)

The free moment diagram is a triangle with maximum ordinate $Wl/4$ as shown in Fig. 8.2b.

Maximum bending moment

As the slope at A is equal to slope at $B = 0$, net area of the moment diagram must be zero, i.e.,

$$\frac{1}{2}\cdot\frac{Wl}{4}\cdot l = Ml \text{ or } M = \frac{Wl}{8}$$

The combined bending moment diagram is shown in Fig. 8.2c. The maximum bending moment is $Wl/8$ hogging (at the ends) and sagging ($wl/4$–$wl/8$, at the centre).

Deflection

Maximum deflection will be at the midspan. To find the deflection of C relative to A, Take moments of the areas of the bending moment diagram between A and C about A,

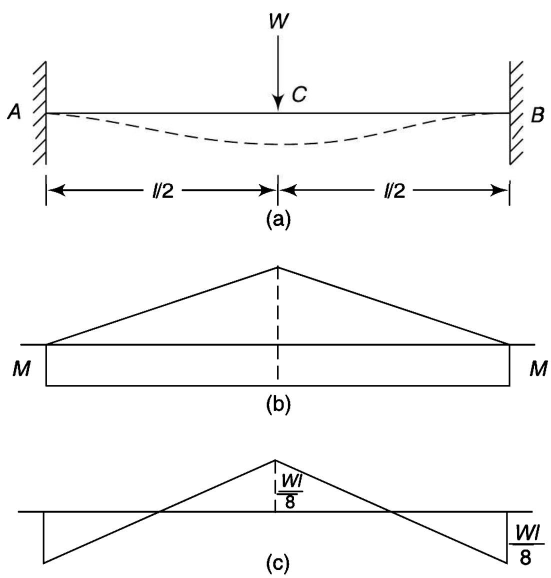


Fig. 8.2

$$y = \frac{1}{EI}\left[\left(\frac{1}{2}\cdot\frac{Wl}{4}\cdot\frac{l}{2}\right)\left(\frac{2}{3}\cdot\frac{l}{2}\right) - \left(\frac{Wl}{8}\cdot\frac{l}{2}\right)\cdot\frac{l}{4}\right] = \frac{Wl^3}{192EI} \text{ from Fig. 8.2b}$$

Example 8.2 Determine the maximum bending moment and the deflection of a beam of length l and flexural rigidity EI. The beam is fixed horizontally at both ends and carries a uniformly distributed load w over the whole span.

Solution

Given A beam fixed horizontally at both ends and loaded as shown in Fig. 8.3a.

To find

— Maximum bending moment

— Maximum deflection

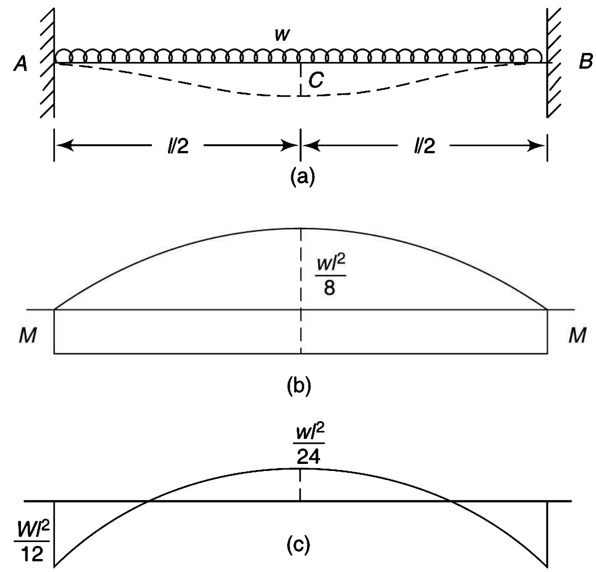


Fig. 8.3

Due to symmetry, fixing moment $M_a = M_b = M$ (say)

The free moment diagram is a parabola with maximum ordinate $Wl^2/8$ as shown in Fig. 8.3b.

Maximum bending moment

As the slope at A is equal to slope at $B = 0$, net area of the moment diagram must be zero, i.e.,

$$\frac{2}{3}\cdot\frac{wl^2}{8}l = Ml \text{ or } M = \frac{wl^2}{12}$$

The combined bending moment is shown in Fig. 8.3c. The maximum bending moment is $wl^2/12$ hogging (at the ends) and $wl^2/24$ (= $wl^2/8 - wl^2/12$) sagging (at the centre).

Deflection

To find the deflection of C relative to A, take moments of the areas between A and C about A,

$$y = \frac{1}{EI}\left[\left(\frac{2}{3}\cdot\frac{wl^2}{8}\cdot\frac{l}{2}\right)\left(\frac{5}{8}\cdot\frac{l}{2}\right) - \left(\frac{wl^2}{12}\cdot\frac{l}{2}\right)\cdot\frac{l}{4}\right] = \frac{wl^4}{384EI} \text{ from Fig. 8.3b}$$

Points of contraflexure

$$M_x = \frac{wl}{2}\cdot x - \frac{wx^2}{2} - \frac{wl^2}{12} = 0$$

or $\quad 6x_2 - 6lx + l_2 = 0$

or $\quad x = \dfrac{6l \pm \sqrt{(6l)^2 - 4\times 6\times l^2}}{2\times 6} = 0.211l \text{ or } 0.789l$

Example 8.3 A beam has its ends fixed horizontally at the same level. The beam is of length l and carries a load W at a distance a from one end and b from the other end. Determine the fixing moments at the ends. Hence show that for a distributed load on the same beam, the fixing moment at one end is given by

$$\int_0^l \frac{rx(l-x)^2}{l^2}\cdot dx$$

where r is the rate of loading at a distance x from the end under consideration.

Solution

Given A beam fixed horizontally at both ends and loaded as shown in Fig. 8.4a.

To find

— Fixing moments at ends

— To show that fixing moment for a distributed load on same beam is $\int_0^l \frac{rx(l-x)^2}{l^2} \cdot dx$ where r is rate of loading at a distance x from the end

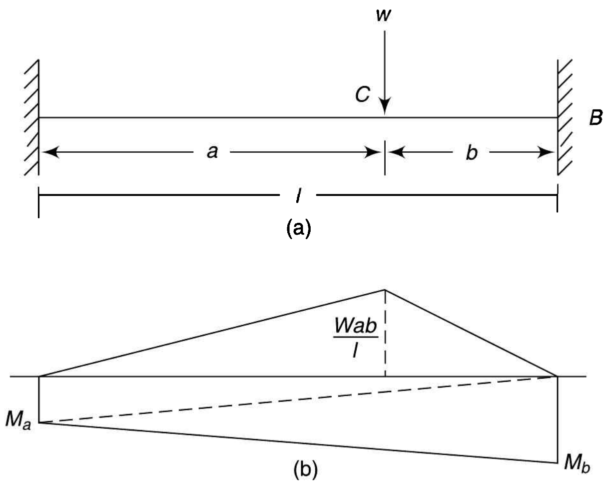


Fig. 8.4

Let the fixing moments be M_a and M_b at the two ends.

The free moment diagram is a triangle with maximum ordinate Wab/l as shown in Fig. 8.4b.

Fixing moments

As the slope at A is equal to slope at $B = 0$, net area of the moment diagram must be zero, i.e.,

$$\frac{M_a + M_b}{2} l = \frac{1}{2} \cdot \frac{Wab}{l} \cdot l \quad \text{or} \quad M_a + M_b = \frac{Wab}{l} \tag{i}$$

Deflection of B relative to A is zero, so net moments of areas about A must be zero, i.e.,

$$\frac{1}{2} \cdot M_a \cdot l \cdot \frac{l}{3} + \frac{1}{2} \cdot M_b \cdot l \cdot \frac{2l}{3} = \left(\frac{1}{2} \cdot \frac{Wab}{l} \cdot a\right)\frac{2a}{3} + \left(\frac{1}{2} \cdot \frac{Wab}{l} \cdot b\right)\left(a + \frac{b}{3}\right) \text{ from Fig. 8.4b}$$

or
$$\frac{l^2}{6} \cdot M_a + \frac{l^2}{3} \cdot M_b = \frac{Wab}{l}\left(\frac{a^2}{3} + \frac{ab}{2} + \frac{b^2}{6}\right)$$

or
$$M_a + 2M_b = \frac{Wab}{l^3}(2a^2 + 3ab + b^2) \tag{ii}$$

Subtracting (i) from (ii),

$$M_b = \frac{Wab}{l^3}(2a^2 + 3ab + b^2) - \frac{Wab}{l} = \frac{Wab}{l^3}(a^2 + 2ab + b^2 + a^2 + ab - l^2)$$

$$M_b = \frac{Wab}{l^3}[(a+b)^2 + a(a+b) - l^2] = \frac{Wab}{l^3}[l^2 + al - l^2] = \frac{Wa^2b}{l^2}$$

and from (i),
$$M_a = \frac{Wab}{l} - \frac{Wa^2b}{l^2} = \frac{Wab}{l^2}(l-a) = \frac{Wab^2}{l^2}$$

Distributed load

For a distributed load, consider a short length δx of the beam at a distance x from one end (Fig. 8.5).
Load on the length $\delta x = r \cdot \delta x$

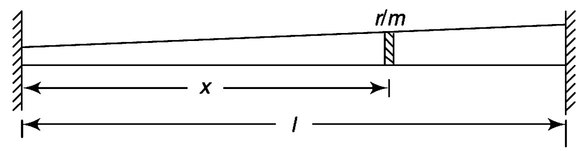


Fig. 8.5

The load on the short length may be considered as a point load.

Fixing moment due to load on this length, $\delta M_a = \dfrac{Wab^2}{l^2} = \dfrac{r \cdot dx \cdot x(l-x)^2}{l^2}$

Fixing moment due to the whole load, $M_a = \displaystyle\int_0^l \frac{r \cdot x(l-x)^2}{l^2} \cdot dx$

and $\quad \delta M_b = \dfrac{r \cdot dx \cdot x^2 \cdot (l-x)}{l^2} \quad$ or $\quad M_b = \displaystyle\int_0^l \frac{r \cdot x^2 \cdot (l-x)}{l^2} \cdot dx$

Example 8.4 A 10-m long beam has its ends fixed horizontally at the same level. It carries a load of varying intensity from zero at one end to 15 kN/m at the other. Determine the fixing moments at the ends.

Solution

Given A beam fixed horizontally loaded as shown in Fig. 8.6.

To find Fixing moments at ends

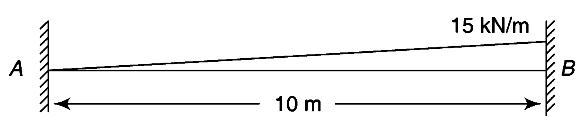


Fig. 8.6

Fixing moments

Intensity of loading at a distance x from end A (Fig. 8.6), $r = \dfrac{wx}{l}$ kN/m

$$M_a = \int_0^l \frac{wx \cdot x(l-x)^2}{l^3} \cdot dx = \int_0^l \frac{wx^2(l-x)^2}{l^3} \cdot dx$$

$$= \frac{w}{l^3}\int_0^l (l^2x^2 - 2lx^3 + x^4)dx = \frac{w}{l^3}\left(\frac{l^2x^3}{3} - \frac{2lx^4}{4} + \frac{x^5}{5}\right)_0^l = \frac{wl^2}{30} = \frac{15 \times 10^2}{30} = 50 \text{ kN}\cdot\text{m}$$

and

$$M_b = \int_0^l \frac{wx \cdot x^2 \cdot (l-x)}{l^3} \cdot dx = \int_0^l \frac{wx^3 \cdot (l-x)}{l^3} \cdot dx$$

$$= \frac{w}{l^3}\int_0^l (lx^3 - x^4)dx = \frac{w}{l^3}\left(\frac{lx^4}{4} - \frac{x^5}{5}\right)_0^l = \frac{wl^2}{20} = \frac{15 \times 10^2}{20} = 75 \text{ kN}\cdot\text{m}$$

The results of the above examples show that for standard cases, the maximum bending moment exists at one of the fixed ends. In general for a combination of downward loads, the maximum bending moment exists at the end of greater fixing moment.

Example 8.5 A beam of length l has its ends fixed. It carries a uniformly distributed load of w per unit length from one end up to the midspan. Determine the fixing moments and reactions at the supports. Also, draw bending moment and shear force diagrams.

Solution

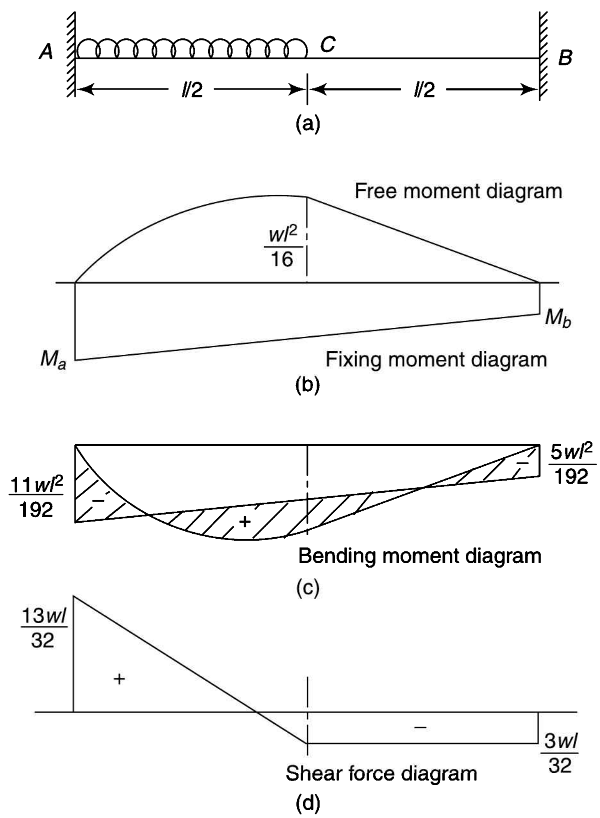


Fig. 8.7

Given A beam fixed at both ends and loaded as shown in Fig. 8.7.

To find

— Fixing moments at ends

— Reaction at supports

Fixing moments

Let the fixing moments be M_a and M_b at the two ends.

- $A_1 = A_2$

  Take moments about end A, $R_2 \cdot l - w \cdot \frac{l}{2} \cdot \frac{l}{4} = 0$ or $R_2 = \frac{wl}{8}$ and $R_1 = \frac{3wl}{8}$

  For portion AC of free moment diagram:

$$M_x = \frac{3wl}{8}x - \frac{wx^2}{2}; \quad \therefore \quad M_{c(x=l/2)} = \frac{wl^2}{16}$$

Bending moment diagrams for the free and fixing moments are shown in Fig. 8.7b.

Area of bending moment diagram,

$$A = \int_0^{l/2}\left(\frac{3wl}{8}x - \frac{wx^2}{2}\right)dx + \frac{1}{2}\cdot\frac{wl^2}{16}\cdot\frac{l}{2} = \left(\frac{3wl}{8}\cdot\frac{x^2}{2} - \frac{wx^3}{6}\right)_0^{l/2} + \frac{wl^3}{64} = \frac{wl^3}{24}$$

Thus $\dfrac{M_a + M_b}{2}l = \dfrac{wl^3}{24}$ or $M_a + M_b = \dfrac{wl^2}{12}$ (i)

- $A_1\bar{x}_1 = A_2\bar{x}_2$

$$A_1\bar{x}_1 = \int_0^{l/2} M_x \cdot x + \left(\frac{1}{2}\cdot\frac{wl^2}{16}\cdot\frac{l}{2}\right)\cdot\left(\frac{l}{2}+\frac{1}{3}\cdot\frac{l}{2}\right) = \int_0^{l/2}\left(\frac{3wl}{8}x^2 - \frac{wx^3}{2}\right)dx + \frac{wl^4}{96}$$

$$\left(\frac{3wl}{8}\cdot\frac{x^3}{3} - \frac{wx^4}{8}\right)_0^{l/2} + \frac{wl^4}{96} = \frac{7wl^4}{384}$$

Thus $\dfrac{1}{2}\cdot M_a\cdot l\cdot\dfrac{l}{3} + \dfrac{1}{2}\cdot M_b\cdot l\cdot\dfrac{2l}{3} = \dfrac{7wl^4}{384}$

or $(M_a + 2M_b)\dfrac{l^2}{6} = \dfrac{7wl^4}{384}$ or $M_a + 2M_b = \dfrac{7wl^2}{64}$ (ii)

Subtract (i) from (ii),

$$M_b = \frac{7wl^2}{64} - \frac{wl^2}{12} = \frac{5wl^2}{192} \text{ and } M_a = \frac{wl^2}{12} - \frac{5wl^2}{192} = \frac{11wl^2}{192}$$

Bending moment diagram has been shown in Fig. 8.7c.

Reactions at the ends

$$R_a = R_1 + \frac{M_a - M_b}{l} = \frac{3wl}{8} + \frac{6wl}{192} = \frac{13wl}{32}$$

and

$$R_b = R_2 - \frac{M_a - M_b}{l} = \frac{wl}{8} - \frac{6wl}{192} = \frac{3wl}{32}$$

Shear force diagram has been shown in Fig. 8.7d.

Example 8.6 A beam of 18-m span is fixed horizontally at the ends. It is acted upon by a downward point load of 18 kN at 6 m from the left end and an upward force of 12 kN at the midspan. Find the end reactions and the fixing moments. Also, draw the bending moment and shear force diagrams.

Solution

Given A beam fixed at both ends and loaded as shown in Fig. 8.8a.

To find

— End reactions

— Fixing moments

Fixing moments

Let the fixing moments be M_a and M_b at the two ends.

- $A_1 = A_2$

Take moments about end B,

$$R_1 \times 18 - 18 \times 12 + 12 \times 9 = 0 \quad\text{or}\quad R_1 = 6\text{ kN and } R_2 = 18 - 12 - 6 = 0$$

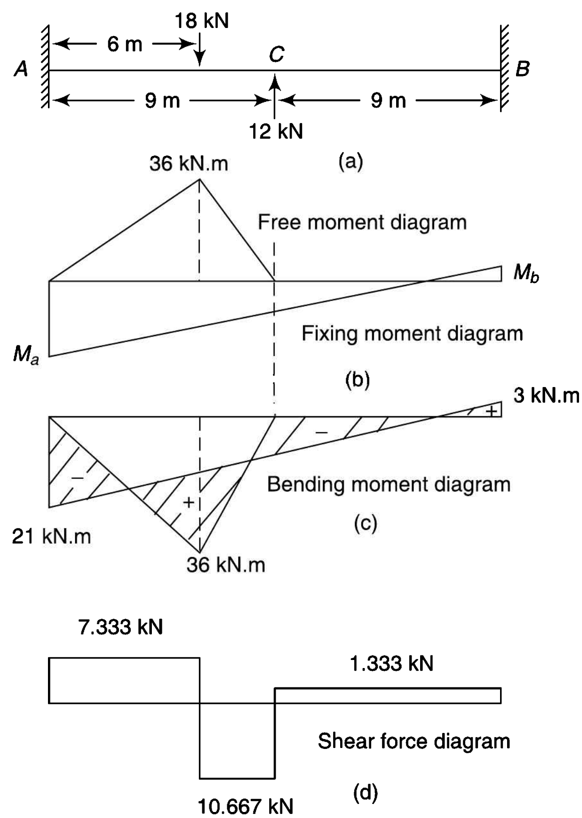


Fig. 8.8

Bending moment diagrams for the free and fixing moments are shown in Fig. 8.2b.

Thus $\dfrac{M_a + M_b}{2} \times 18 = \dfrac{36 \times 9}{2}$ or $M_a + M_b = 18$ (i)

- $A_1 \bar{x}_1 = A_2 \bar{x}_2$

$$(M_a + 2M_b)\frac{18^2}{6} = \frac{6 \times 36}{2} \times 4 + \frac{3 \times 36}{2} \times 7 \quad \text{or} \quad (M_a + 2M_b) = 15 \qquad \text{(ii)}$$

Subtract (i) from (ii),

$$M_b = -3 \text{ kN·m and from (i), } M_a + (-3) = 18 \text{ or } M_a = 21 \text{ kN·m}$$

The fixing moment at B is, therefore, in the counter-clockwise direction. The resultant bending moment diagram is shown in Fig. 8.8c.

Reactions

Reactions at the ends,

$$R_a = R_1 + \frac{M_a - M_b}{l} = 6 + \frac{21 - (-3)}{18} = 7.333 \text{ kN} \qquad \text{(upwards)}$$

and $$R_b = R_2 - \frac{M_a - M_b}{l} = 0 - \frac{21 - (-3)}{18} = -1.333 \text{ kN} \qquad \text{(downwards)}$$

Shear force diagram is shown in Fig. 8.8d.

Example 8.7 || Determine the change in the fixing moments and end reactions if one end of a fixed beam of span l sinks an amount δ below the other, the ends remaining horizontal.

Solution

Given A beam fixed at both ends, the right end sinks by an amount δ below the left end as shown in Fig. 8.9a.

To find

— Change in the fixing moments
— end reactions

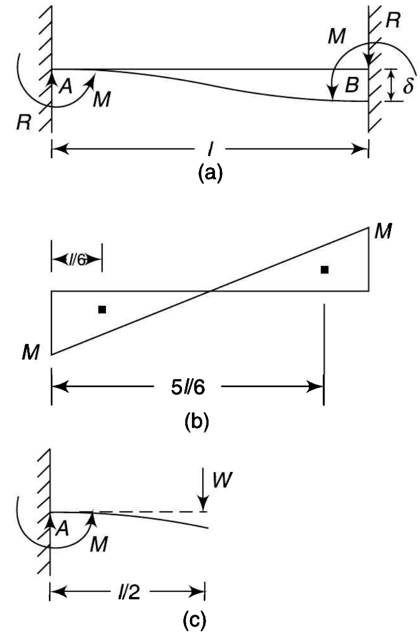


Fig. 8.9

Change in fixing moments

The shape of the deflected beam will be as shown in Fig. 8.9a. Let M be the change in the fixing moment. From the figure, it can be observed that the fixing moment must be sagging at right end and hogging at left end. For equilibrium, the reactions at the ends must act as shown in the figure. Taking moments about the end A,

$$M + M = R \times l \quad \text{or} \quad R = 2M/l$$

The bending moment diagram is shown in Fig. 8.9b.

Now, $$z = \frac{A\bar{x}}{EI} \qquad \text{(Eq. 7.29)}$$

$$\delta = \frac{\frac{1}{2}\cdot\frac{Ml}{2}\cdot\frac{5l}{6} - \frac{1}{2}\cdot\frac{Ml}{2}\cdot\frac{l}{6}}{EI} = \frac{Ml^2}{6EI}$$

(About left-hand end, $z = \delta$)

or $$M = \frac{6EI\delta}{l^2}$$

End reactions

$$R = \frac{2M}{l} = \frac{12EI\delta}{l^3}$$

Alternate solution

Considering the beam as two cantilevers of length $l/2$ carrying the end load W and the deflection at the free end $\delta / 2$ (Fig. 8.9c).

Now, deflection at the end $= \dfrac{\delta}{2} = \dfrac{W(l/2)^3}{3EI}$ ...(Eq. 7.6a)

or $W = \dfrac{12EI\delta}{l^3}$ and thus $M = W \cdot \dfrac{l}{2} = \dfrac{12EI\delta}{l^3} \cdot \dfrac{l}{2} = \dfrac{6EI\delta}{l^2}$ and $R = W = \dfrac{12EI\delta}{l^3}$

8.3 MACAULAY'S METHOD

In Chapter 7, Macaulay's method was described to find the deflection of simply supported and cantilever loaded beams. The same method may be applied to find the fixing moments, end reactions and the deflection of built-in beams. The following examples illustrate the procedure.

Example 8.8 Determine the maximum bending moment and the deflection of a beam of length l and flexural rigidity EI. The beam is fixed horizontally at both ends and carries a concentrated load W at the midspan.

Solution

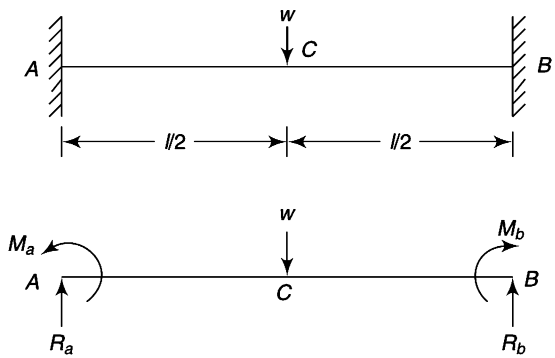


Fig. 8.10

Given A beam fixed at both ends and carrying a central point load as shown in Fig. 8.10.

To find

— Maximum bending moment

— Deflection

Due to symmetry, $R_a = R_b = W/2$ and $M_a = M_b$

Bending moments

Consider the beam to be simply supported as shown in Fig. 8.10. Taking A as origin and writing the expression for the bending moment in the last portion of the beam,

- $EIy'' = -M_a + \frac{W}{2}x \Bigg| - W\left(x - \frac{l}{2}\right)$

- Integrating, $EIy' = -M_a \cdot x + \frac{W}{2}\frac{x^2}{2} + A \Bigg| - \frac{W}{2}\left(x - \frac{l}{2}\right)^2$

  At $x = 0, y' = 0, \therefore A = 0$ ... (considering the first portion only)

  Thus $EIy' = -M_a \cdot x + \frac{Wx^2}{4} \Bigg| - \frac{W}{2}\left(x - \frac{l}{2}\right)^2$

- Integrating again, $EIy = -M_a \cdot \frac{x^2}{2} + \frac{Wx^3}{12} + B \Bigg| - \frac{W}{6}\left(x - \frac{l}{2}\right)^3$

  At $x = 0, y = 0, \therefore B = 0$ ... (considering the first portion only)

  Thus, $EIy = -M_a \cdot \frac{x^2}{2} + \frac{Wx^3}{12} \Bigg| - \frac{W}{6}\left(x - \frac{l}{2}\right)^3$

  Also, at $x = l, y' = 0$ and $y = 0$

  $\therefore \quad 0 = -M_a \cdot l + \frac{Wl^2}{4} - \frac{Wl^2}{8}$

$$\text{or} \quad M_a = \frac{Wl}{8} \text{ and } M_b = M_a = \frac{Wl}{8}$$

Thus maximum bending moment $= \dfrac{Wl}{8}$

Maximum deflection

Maximum deflection at midspan, $EIy = -\dfrac{Wl}{8} \cdot \dfrac{l^2}{8} + \dfrac{Wl^3}{96}$ or $y = -\dfrac{Wl^3}{192EI}$

Example 8.9 Determine the maximum bending moment and the deflection of a beam of length *l* and flexural rigidity *EI*. The beam is fixed horizontally at both ends and carries a uniformly distributed load *w* over the whole span.

Solution

Given A beam fixed at both ends and carrying a uniformly distributed load as shown in Fig. 8.11.

To find

— Maximum bending moment
— Deflection

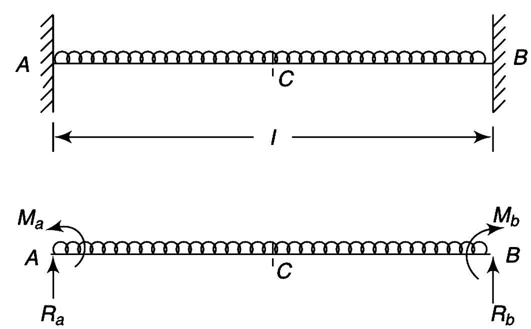


Fig. 8.11

Due to symmetry, $R_a = R_b = wl/2$ and $M_a = M_b$

Bending moments

Consider the beam to be simply supported as shown in Fig. 8.11. Taking *A* as origin and writing the expression for the bending moment in the last portion of the beam,

- $EIy'' = -M_a + \dfrac{wl}{2}x - \dfrac{wx^2}{2}$
- Integrating, $EIy' = -M_a \cdot x + \dfrac{wlx^2}{4} - \dfrac{wx^3}{6} + 0$ ...(Constant *A* will be zero)
- Integrating again, $EIy = -M_a \cdot \dfrac{x^2}{2} + \dfrac{wlx^3}{12} - \dfrac{wx^4}{24} + 0$ ...(Constant *B* will be zero)

Also, At $x = l$, $y' = 0$ and $y = 0$

$$\therefore \quad 0 = -M_a \cdot l + \frac{wl^3}{4} - \frac{wl^3}{6} \quad \text{or} \quad M_a = \frac{wl^2}{12} \text{ and } M_b = M_a = \frac{wl^2}{12}$$

Maximum deflection

Maximum deflection at midspan, $EIy = -\dfrac{wl^2}{12} \cdot \dfrac{l^2}{8} + \dfrac{wl^4}{96} - \dfrac{wl^4}{384}$ or $y = -\dfrac{Wl^4}{384EI}$

Example 8.10 A beam has its ends fixed horizontally at the same level. The beam is of length *l* and carries a load *W* at a distance *a* from one end and *b* from the other end. Determine the fixing moments at the ends. Also, find the maximum deflection and the deflection under the load.

Solution

Given A beam fixed at both ends and carrying a point load as shown in Fig. 8.12.

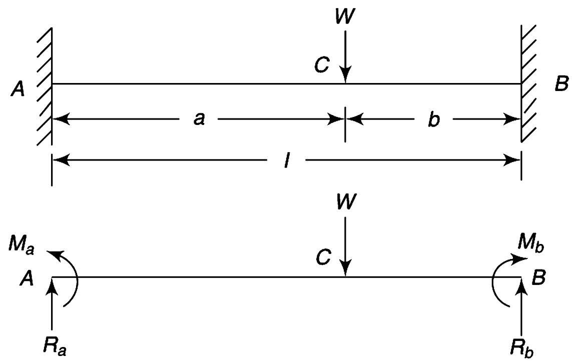


Fig. 8.12

To find

— Fixing moments
— Maximum deflection
— Deflection under load

Fixing moments

- $EIy'' = -M_a + R_a x| - W(x-a)$
- Integrating, $EIy' = -M_a \cdot x + R_a \dfrac{x^2}{2} + 0 \Big| - \dfrac{W}{2}(x-a)^2$ ...(At $x = 0$, $y' = 0$, $\therefore A = 0$)
- Integrating again, $EIy = -M_a \cdot \dfrac{x^2}{2} + R_a \dfrac{x^3}{6} + 0 \Big| - \dfrac{W}{6}(x-a)^3$ ...($B = 0$)

  Also, At $x = l$, $y' = 0$ and $y = 0$

  $$\therefore \quad 0 = -M_a \cdot l + R_a \frac{l^2}{2} - \frac{W}{2}(l-a)^2 \quad \text{or} \quad 2M_a \cdot l = R_a l^2 - Wb^2 \qquad \text{(i)}$$

  $$\text{and} \quad 0 = -M_a \cdot \frac{l^2}{2} + R_a \frac{l^3}{6} - \frac{W}{6}(l-a)^3 \quad \text{or} \quad 3M_a \cdot l^2 = R_a l^3 - Wb^3 \qquad \text{(ii)}$$

  Multiplying (i) by l and subtracting from (ii),

  $$M_a \cdot l^2 = Wb^2 (l-b) = Wab^2 \qquad \text{or} \qquad M_a = \frac{Wab^2}{l^2}$$

  From (i), $2\dfrac{Wab^2}{l^2} l = R_a l^2 - Wb^2$

  $$\text{or} \quad R_a = \frac{Wb^2}{l^3}(2a + l) = \frac{Wb^2}{l^3}(2a + a + b) = \frac{Wb^2}{l^3}(3a + b)$$

- Bending moment at right fixed end,

  $$-M_b = -\frac{Wab^2}{l^2} + \frac{Wb^2}{l^3}(3a+b)l - W(l-a) = -\frac{Wab^2}{l^2} + \frac{Wb^2}{l^2}(3a+b) - Wb$$

  $$= -\frac{Wb}{l^2}(ab - 3ab - b^2 + l^2) = -\frac{Wb}{l^2}(-2ab - b^2 + a^2 + 2ab + b^2) = -\frac{Wba^2}{l^2}$$

Maximum deflection

For maximum deflection, $y' = 0$,

$$0 = -\frac{Wab^2}{l^2}\cdot x + \frac{Wb^2}{l^3}(3a+b)\frac{x^2}{2} \quad \text{or} \quad -2al + (3a+b)x = 0 \quad \text{or} \quad x = \frac{2al}{3a+b}$$

Thus maximum deflection,

$$EIy = -\frac{Wab^2}{2l^2}\cdot\left(\frac{2al}{3a+b}\right)^2 + \frac{Wb^2}{6l^3}(3a+b)\left(\frac{2al}{3a+b}\right)^3 = -\frac{Wa^3b^2}{(3a+b)^2}\left(2-\frac{4}{3}\right)$$

or
$$y = -\frac{2}{3}\cdot\frac{Wa^3b^2}{EI(3a+b)^2}$$

Deflection under the load

For deflection under the load, $x = a$,

$$EIy = -\frac{Wab^2}{2l^2}\cdot a^2 + \frac{Wb^2}{6l^3}(3a+b)a^3 = -\frac{Wa^3b^2}{6l^3}(3l-3a-b)$$

$$= -\frac{Wa^3b^2}{6l^3}(3a+3b-3a-b)$$

or
$$y = -\frac{Wa^3b^3}{3EIl^3}$$

Example 8.11 || A built-in beam of uniform section is of 16-m span. It carries a uniformly distributed load of 12 kN/m on the left half of the section along with a 150 kN load at 12 m from the left end. Determine the end reactions and fixing moments. Also find the maximum deflection and the position where it occurs. Draw the bending moment, shear force and deflection diagrams.

$E = 205$ GPa and $I = 400 \times 10^6$ mm^4

Solution

Given A built-in beam carrying a point load and a uniformly distributed load as shown in Fig. 8.13a.

$E = 205$ GPa $\qquad I = 400 \times 10^6$ mm^4

To find

— End reactions
— Fixing moments
— Maximum deflection
— To draw bending moment, shear force and deflection diagrams

Let the fixing moments be M_a and M_b and reactions R_a and R_b. Taking the origin at the left-hand end (Fig. 8.13a),

The uniformly distributed load on the beam is equivalent to as shown in Fig. 8.13b, i.e., a downward uniformly distributed load of 12 kN up to a length x beyond 8 m and an upward uniformly distributed load for distance $(x - 8)$ m.

End reactions and fixing moments

$$EIy'' = -M_a + R_a x - \frac{12x^2}{2}\Bigg| + \frac{12(x-8)^2}{2}\Bigg| - 150(x-12)$$

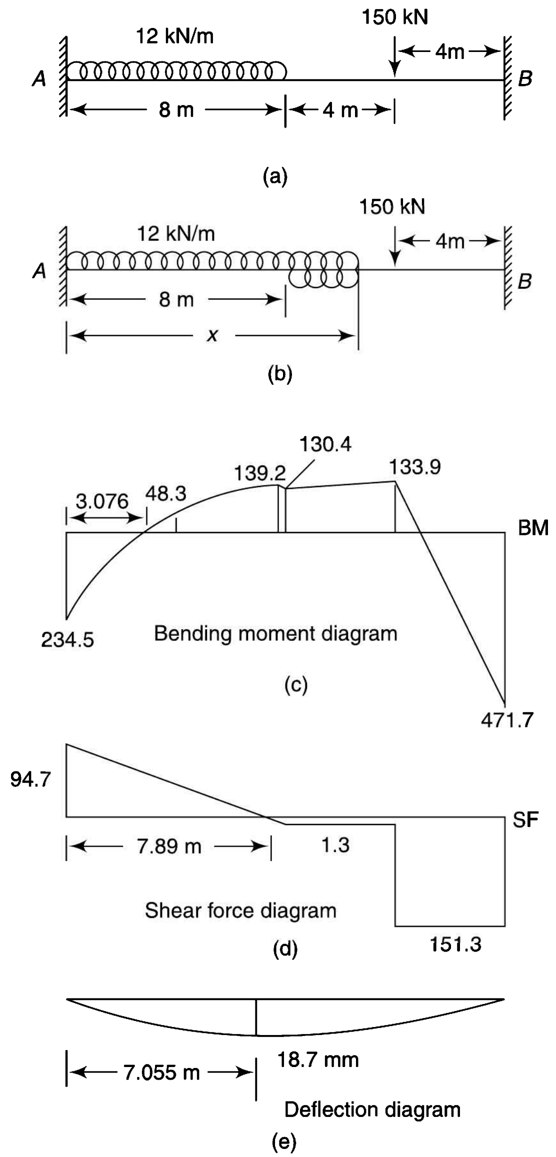


Fig. 8.13

- Integrating, $EIy' = -M_a \cdot x + R_a \dfrac{x^2}{2} - \dfrac{6x^3}{3} + 0 \Bigg| + \dfrac{6(x-8)^3}{3} \Bigg| - \dfrac{150(x-12)^2}{2}$ $\quad ...(A = 0)$

$$= -M_a \cdot x + R_a \frac{x^2}{2} - 2x^3 \Bigg| + 2(x-8)^3 \Bigg| - 75(x-12)^2$$

- Integrating again, $EIy = -M_a \cdot \dfrac{x^2}{2} + R_a \dfrac{x^3}{6} - \dfrac{x^4}{2} + 0 \Bigg| + \dfrac{(x-8)^4}{2} \Bigg| - 25(x-12)^3$ $\quad \ldots(B = 0)$

- When $x = 16$, $y' = 0$ and $y = 0$,

$$0 = -16M_a + 128R_a - 2 \times 16^3 + 2 \times 8^3 - 75 \times 4^2 \quad \text{or} \quad 8R_a - M_a = 523 \qquad \text{(i)}$$

and
$$0 = -128M_a + R_a \cdot \frac{16^3}{6} - \frac{16^4}{2} + \frac{8^4}{2} - 25 \times 4^3$$

or
$$-128M_a + 682.67R_a - 32768 + 2048 - 1600 = 0 \quad \text{or} \quad 5.333R_a - M_a = 252.5 \qquad \text{(ii)}$$

Subtracting ((ii) from (i), $2.667R_a = 252.5$

or
$$R_a = 94.7 \text{ kN}$$

From (i),
$$M_a = 8 \times 94.7 - 523 = 234.5 \text{ kN·m}$$
$$R_b = 8 \times 12 + 150 - 94.7 = 151.3 \text{ kN}$$

Bending moment at B,

$$-M_b = -234.5 + 94.7 \times 16 - \frac{12 \times 16^2}{2} + \frac{12(16-8)^2}{2} - 150(16-12)$$
$$= -471.3 \text{ kN·m}$$

Bending moment

For bending moment to be zero between 0 and 8 m,

$$EIy = -234.5 + 94.7x - 6x^2 = 0 \quad \text{or} \quad 6x^2 - 94.7x + 234.5 = 0$$

$$x = \frac{94.7 \pm \sqrt{94.7^2 - 4 \times 6 \times 234.5}}{2 \times 6} = 3.076 \text{ m}$$

Bending moment is maximum where shear force is zero,

$$R_a - wx = 0 \quad \text{or} \quad 94.7 - 12x = 0 \quad \text{or} \quad x = 7.89 \text{ m}$$

Maximum bending moment $= -234.5 + 94.7 \times 7.89 - 6 \times 7.89^2 = 139.2$ kN·m

Maximum deflection

Maximum deflection will be where slope is zero. Assuming it is within 8 m of span,

$$0 = -234.5 \cdot x + 94.5\frac{x^2}{2} - 2x^3$$

or
$$x^2 - 23.675x + 117.25 = 0$$

$$x = \frac{23.675 \pm \sqrt{23.675^2 - 4 \times 117.25}}{2} = 7.055 \text{ m}$$

$$EIy = -234.5 \times \frac{7.055^2}{2} + 94.7 \times \frac{7.055^3}{6} - \frac{7.055^4}{2}$$
$$= -1532.3 \text{ kN·m}^3 \text{ or } 1532.3 \times 10^9 \text{ kN·mm}^3$$

$$y = \frac{1532.3 \times 10^9}{205 \times 400 \times 10^6} = 18.7 \text{ mm}$$

Figures 8.13 *c*, *d* and *e* show the bending moment, shear force and deflection diagrams respectively.

8.4 CLAPEYRON'S THREE-MOMENT EQUATION

When a beam is carried by more than two supports, it is known as a *continuous beam.* The moment–area method already discussed may be extended to find a relation between bending moments at three points.

Consider a portion ABC of a continuous beam supported on three supports as shown in Fig. 8.14a. Let A_1 and A_2 be the free bending moment areas obtained by treating the beam as simply supported over two independent spans l_1 and l_2 (Fig. 8.14b). In a continuous beam, the bending moment at the three supports will not be zero, but will have some values. Let M_a, M_b and M_c be the actual bending moments at these points and thus a fixing moment diagram may be introduced as shown in Fig. 8.14c. The actual bending moment diagram will be the algebraic sum of the two diagrams.

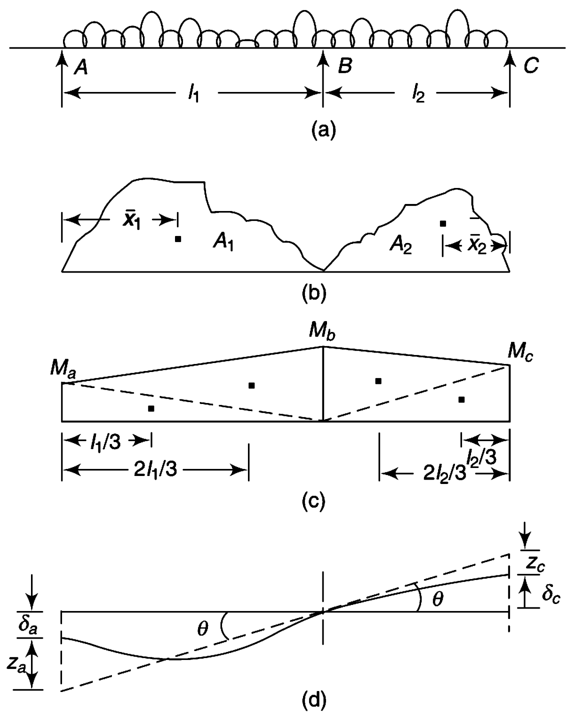


Fig. 8.14

Figure 8.14d shows the elastic line of the deflected beam, the deflections δ_a and δ_c of the supports A and C are relative to the central support, positive upwards. Let θ be the slope of the beam over the centre support and z_a and z_c the intercepts made by the tangent at the central support at the end of two spans.

As slopes are small,

$$\theta = \frac{z_a + \delta_a}{l_1} = \frac{z_c + \delta_c}{l_2} \text{ or } \frac{z_a}{l_1} + \frac{\delta_a}{l_1} = \frac{z_c}{l_2} + \frac{\delta_c}{l_2}$$

Now $\quad z = \dfrac{A\bar{x}}{EI}$ (Eq.7.29)

The intercept z on the left end is positive and on right end negative as discussed in Section 7.6.

$$\therefore \quad \frac{1}{EI_1 l_1}\left(A_1\bar{x}_1 + \frac{M_a l_1}{2}\cdot\frac{l_1}{3} + \frac{M_b l_1}{2}\cdot\frac{2l_1}{3}\right) + \frac{\delta_a}{l_1}$$

$$= -\frac{1}{EI_2 l_2}\left(A_2\bar{x}_2 + \frac{M_c l_2}{2}\cdot\frac{l_2}{3} + \frac{M_b l_2}{2}\cdot\frac{2l_2}{3}\right) + \frac{\delta_c}{l_2}$$

or

$$M_a\frac{l_1}{I_1} + 2M_b\left(\frac{l_1}{I_1} + \frac{l_2}{I_2}\right) + M_c\frac{l_2}{I_2} = -6\left(\frac{A_1\bar{x}_1}{I_1 l_1} + \frac{A_2\bar{x}_2}{I_2 l_2}\right) + 6E\left(\frac{\delta_c}{l_2} - \frac{\delta_a}{l_1}\right) \tag{8.3}$$

If the beam is uniform, $I_1 = I_2 = I$

$$M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2 = -6\left(\frac{A_1\bar{x}_1}{l_1} + \frac{A_2\bar{x}_2}{l_2}\right) + 6EI\left(\frac{\delta_c}{l_2} - \frac{\delta_a}{l_1}\right) \tag{8.4}$$

If supports are at the same level, i.e. $\delta_a = \delta_c = 0$

$$M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2 = -6\left(\frac{A_1\bar{x}_1}{l_1} + \frac{A_2\bar{x}_2}{l_2}\right) \tag{8.5}$$

Equation (8.3) is the most general form of the equation of three moments. This is also known as Clapeyron's three–moment equation.

- In case of a uniformly distributed load, the bending moment diagram is of parabolic shape with maximum value $wl^2/8$ at the mid span. Thus the term $6\left(\dfrac{A_1\bar{x}_1}{l_1}+\dfrac{A_2\bar{x}_2}{l_2}\right)$ is simplified to

$$6\left[\frac{1}{l_1}\left(\frac{2}{3}l_1\cdot\frac{w_1l_1^2}{8}\right)\cdot\frac{l_1}{2}+\frac{1}{l_2}\left(\frac{2}{3}l_2\cdot\frac{w_2l_2^2}{8}\right)\cdot\frac{l_2}{2}\right]=-\frac{w_1l_1^3}{4}-\frac{w_2l_2^2}{4}$$

Thus Eq. 8.5 may be written for a uniformly distributed load as

$$M_al_1+2M_b(l_1+l_2)+M_cl_2=-\frac{w_1l_1^3}{4}-\frac{w_2l_2^2}{4} \qquad (8.6)$$

- Clapeyron's three–moment equation can also be applied to beams fixed at one end or propped cantilevers by assuming an imaginary zero span beyond the fixed end. The fixing moment at the imaginary support of the zero span is taken to be zero.

Example 8.12 A beam of length $2l$ with uniformly distributed load of w per unit run rests on three equally spaced supports. Draw the bending moment and shear force diagrams.

Solution

Given A beam resting on three supports and carrying a uniformly distributed load as shown in Fig. 8.15a.
To find To draw bending moment and shear force diagrams.

Bending moments

For AB and BC, $M_{\max}=\dfrac{wl^2}{8}$ kN·m

Three-moment equation for uniformly distributed load when the ends are simply supported is,

$$M_al_1+2M_b(l_1+l_2)+M_cl_2=-\frac{w_1l_1^3}{4}-\frac{w_2l_2^2}{4} \qquad \text{...(Eq. 8.6)}$$

But $w_1 = w_2 = w$; $l_1 = l_2 = l$ and also bending moment at end supports, $M_a = M_c = 0$

$$\therefore \quad 4M_bl=-\frac{wl^3}{4}-\frac{wl^3}{4}=-\frac{wl^3}{2}$$

or $$M_b=-\frac{wl^2}{8}$$

Determination of reactions

Also bending moment at $B = R_al-\dfrac{wl^2}{2}$,

$$\therefore \quad R_al-\frac{wl^2}{2}=-\frac{wl^2}{8} \quad \text{or} \quad R_a=\frac{3wl}{8}$$

Also, $$R_c=R_a=\frac{3wl}{8} \text{ by symmetry.}$$

and $$R_b=2wl-2\times\frac{3wl}{8}=\frac{5wl}{4}$$

Bending moment and shear force diagrams are shown in Figs. 15b and c respectively.

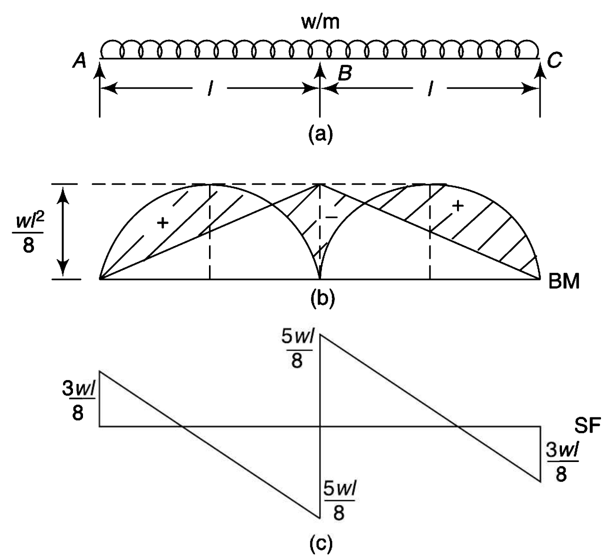


Fig. 8.15

Example 8.13 A continuous beam *ABCD* rests on four supports and the three spans are *AB* = 5 m, *BC* = 10 m and *CD* = 3 m. The uniformly distributed loads are 3 kN, 2 kN and 4 kN per metre run on *AB*, *BC* and *CD* respectively. The beam is of uniform section throughout. Determine the bending moment at the supports *B* and *C*. Also plot the bending moment and shear force diagrams.

Solution

Given A continuous beam resting on four supports and carrying uniformly distributed loads of different intensities on three spans as shown in Fig. 8.16a.

To find

— Bending moment at B and C

— To plot bending moment and shear force diagrams

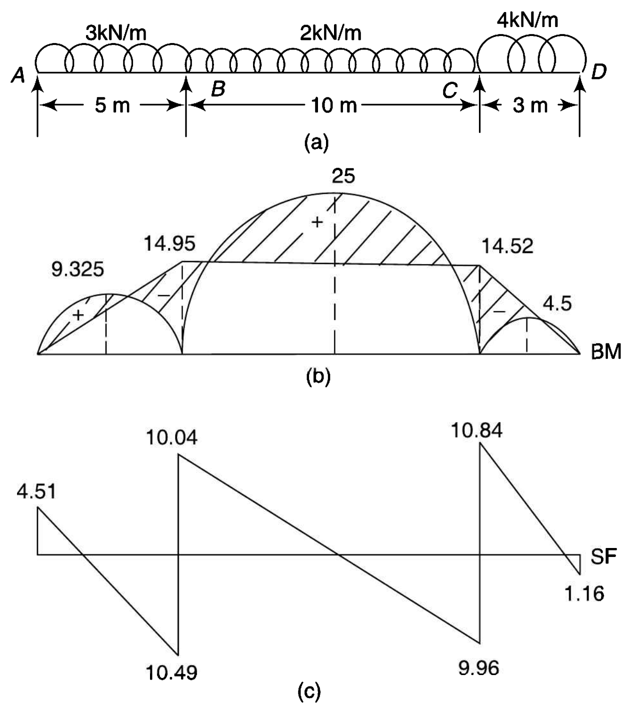


Fig. 8.16

Bending moments

For AB, $M_{max} = \dfrac{3 \times 5^2}{8} = 9.325 \text{ kN·m}$

For BC, $M_{max} = \dfrac{2 \times 10^2}{8} = 25 \text{ kN·m}$

For CD, $M_{max} = \dfrac{4 \times 3^2}{8} = 4.5 \text{ kN·m}$

Also, $M_a = M_d = 0$

- Apply the three moment equation for uniformly distributed loads to the spans AB and BC,

$$M_a \times 5 + 2M_b(5 + 10) + M_c \times 10 = -\frac{3 \times 5^3}{4} - \frac{2 \times 10^3}{4}$$

or $\quad 30M_b + 10M_c = -593.75$

or $\quad M_b + 0.333M_c = -19.79 \qquad$ (i)

- Apply the three moment equation for uniformly distributed loads to the spans BC and CD,

$$M_b \times 10 + 2M_c(10 + 3) + M_d \times 3 = -\frac{2 \times 10^3}{4} - \frac{4 \times 3^3}{4}$$

or $\quad 10M_b + 26M_c = -527$

or $\quad M_b + 2.6M_c = -52.7 \qquad$ (ii)

Subtracting (i) from (ii),

$$2.267M_c = -32.91 \quad \text{or} \quad M_c = -14.52 \text{ kN·m} \quad \text{and} \quad M_b = -14.95 \text{ kN·m}$$

Determination of reactions

- To find R_a, take moments about B,

$$R_a \times 5 - 3 \times 5 \times 2.5 = -14.95 \qquad \text{or} \qquad R_a = 4.51 \text{ kN}$$

- To find R_b, consider bending moment at C,
  $4.51 \times 15 + R_b \times 10 - 3 \times 5\,(10 + 2.5) - 2 \times 10 \times 5 = -14.52$ or $R_b = 20.53$ kN
- To find R_d, consider bending moment at C from the right end,
  $R_d \times 3 - 4 \times 3 \times 1.5 = -14.52$ or $R_d = 1.16$
  $R_c = 3 \times 5 + 2 \times 10 + 4 \times 3 - (4.51 + 20.53 + 1.16) = 20.8$ kN

Bending moment and shear force diagrams are shown in Fig. 16b and c respectively.

Example 8.14 A straight elastic beam of uniform cross-section rests on four similar elastic supports which are placed l distance apart. The supports are compressed by d units per unit of load upon them. The beam is acted upon by a uniformly distributed load of total amount W. Show that the reactions at the central supports are each $\dfrac{W(11/6 + 3EId/l^3)}{5 + 12EId/l^3}$.

Solution

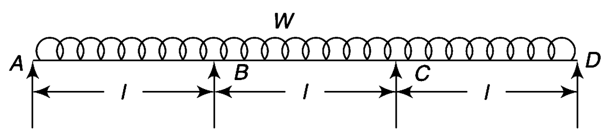


Fig. 8.17

Given A continuous beam resting on four supports and carrying uniformly distributed loads as shown in Fig. 8.17.

To find To show that each reaction at central supports is $\dfrac{W(11/6 + 3EId/l^3)}{5 + 12EId/l^3}$.

Applying the three-moment equation

Applying the three-moment equation for uniformly distributed loads to the spans AB and BC,

$$M_a l + 2M_b(l + l) + M_c l = -\frac{wl^3}{4} - \frac{wl^3}{4} + 6EI\left(\frac{\delta_c}{l_2} - \frac{\delta_a}{l_1}\right)$$

- In this equation, $M_a = 0$, $w \times 3l = W$ or $w = W/3l$
  and $M_c = M_b$; $\delta_c = 0$
  (From symmetry of the beam, B and C have same load, so level of C is the same as of B)

  $\therefore$ The above equation reduces to $5M_b = -\dfrac{Wl}{6} - 6EI\dfrac{\delta_a}{l^2}$ (i)
- Also, $R_a = R_d$; $R_b = R_c$ (From symmetry of the beam)

  Thus, $W = R_a + R_b + R_c + R_d = 2R_a + 2R_b$ or $R_a = \left(\dfrac{W}{2} - R_b\right)$

  and $\delta_a = (R_a - R_b)d = \left(\dfrac{W}{2} - R_b - R_b\right)d = \left(\dfrac{W}{2} - 2R_b\right)d$

  $\therefore$ (i) becomes, $5M_b = -\dfrac{Wl}{6} - \dfrac{6EId}{l^2}\left(\dfrac{W}{2} - 2R_b\right)$ (ii)

Bending moment at B

$$M_b = M_a + R_a l - \frac{wl^2}{2} = 0 + l\left(\frac{W}{2} - R_b - \frac{W}{6}\right) = l\left(\frac{W}{2} - R_b - \frac{W}{6}\right) \quad \text{(hogging)}$$

or $$M_b = -l\left(R_b - \frac{W}{3}\right) \tag{iii}$$

Determination of reactions

From (ii) and (iii)

$$5R_b l - \frac{5Wl}{3} = \frac{Wl}{6} + \frac{3EIdW}{l^2} - \frac{12EId}{l^2}R_b$$

or $$R_b\left(5 + \frac{12EId}{l^3}\right) = \frac{Wl}{6} + \frac{3EIdW}{l^3} + \frac{5Wl}{3} = W\left(\frac{11}{6} + \frac{3EId}{l^3}\right)$$

or $$R_b = \frac{W(11/6 + 3EId/l^3)}{5 + 12EId/l^3}$$

Example 8.15 A 26-m long continuous beam is carried on four supports, two at the ends and the two at points 7 m and 16 m from the left end. The beam carries two point loads of 10 kN at 3 m and 8 kN at 12 m from the left end, uniformly distributed load of 1 kN/m runs over the span between the two supports of the right end. Draw the bending moment and shear force diagrams.

Solution

Given A continuous beam resting on four supports and loaded as shown in Fig. 8.18a.

To find To draw bending moment and shear force diagrams

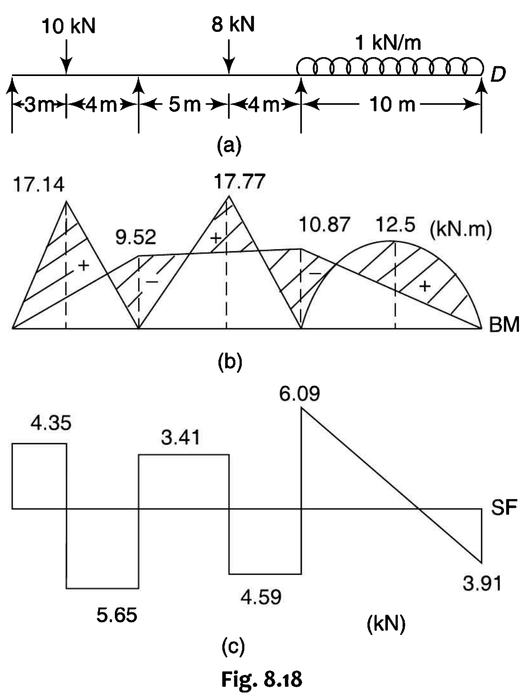


Fig. 8.18

Bending moments

- For span AB, $M_{\max} = \dfrac{Wab}{l} = \dfrac{10 \times 3 \times 4}{7} = 17.14 \text{ kN}\cdot\text{m}$

$$A\bar{x} = \frac{17.14 \times 3}{2} \times 2 + \frac{17.14 \times 4}{2} \times \left(3 + \frac{4}{3}\right) = 200 \quad (A \text{ as origin})$$

or it may be calculated as below:

$$A\bar{x} = \frac{7 \times 17.14}{2} \times \frac{7+3}{3} = 200$$

- For span BC, $M_{\max} = \dfrac{8 \times 5 \times 4}{9} = 17.77 \text{ kN}\cdot\text{m}$

$$A\bar{x} = \frac{9 \times 17.77}{2} \times \frac{9+4}{3} = 346.52 \quad (C \text{ as origin})$$

$$A\bar{x} = \frac{9 \times 17.77}{2} \times \frac{9+5}{3} = 373.17 \quad (B \text{ as origin})$$

- For span CD, $M_{\max} = \dfrac{1 \times 10^2}{8} = 12.5 \text{ kN}\cdot\text{m}$

- $M_a = M_d = 0$

Applying the three moment equation to the spans AB and BC,

$$M_a \times 7 + 2M_b(7+9) + M_c \times 9 = -\frac{6 \times 200}{7} - \frac{6 \times 346.52}{9}$$

or $\quad 0 + 32M_b + 9M_c = -402.44 \quad$ or $\quad M_b + 0.281M_c = -12.576$ (i)

Applying the three-moment diagram to the spans BC and CD,

$$M_b \times 9 + 2M_c(9+10) + M_d \times 10 = -\frac{6 \times 373.17}{9} - \frac{1 \times 10^3}{4}$$

$9M_b + 38M_c = -498.98$

$M_b + 4.222M_c = -55.42$ (ii)

Subtracting (i) from (ii), $3.941M_c = -42.844 \quad$ or $\quad M_c = -10.87$ kN·m

and $\quad M_b = -55.42 + 4.222 \times 10.87 = -9.52$ kN·m

Reactions

To find R_a, take moments about B,

$R_a \times 7 - 10 \times 4 = -9.52 \quad$ or $\quad R_a = 4.35$ kN

To find R_b, consider bending moment at C,

$4.35 \times 16 + R_b \times 9 - 10 \times 13 - 8 \times 4 = 10.87 \quad$ or $\quad R_b = 9.06$ kN

To find R_d, consider bending moment at C from the right end,

$R_d \times 10 - 10 \times 1 \times 5 = -10.87 \quad$ or $\quad R_d = 3.91$

$R_c = 10 + 8 + 10 - (4.35 + 9.06 + 3.91) = 10.68$ kN

As a check, moments at D,

$4.35 \times 26 + 9.06 \times 19 + 10.68 \times 10 - 10 \times 23 - 8 \times 14 - 10 \times 5 = 0$

Bending moment and shear force diagrams are shown in Fig. 18b and c respectively.

Example 8.16 A continuous beam *ABC* of uniform section is fixed at *A* and simply supported at *B* and *C*. The spans *AB* and *BC* are 4 m and 3 m respectively. The beam carries a uniformly distributed load of 6 kN/m throughout its span. Draw the bending moment and shear force diagrams.

Solution

Given A continuous beam fixed at *A*, simply supported at *B* and *C*, and loaded as shown in Fig. 8.19a.
To find To draw bending moment and shear force diagrams

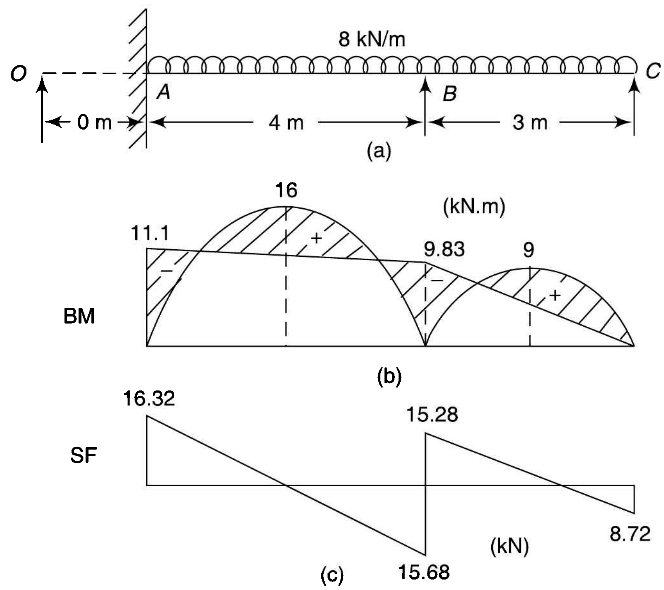


Fig. 8.19

Bending moments
Considering *AB* and *BC* as simply supported beams,

$$\text{For } AB,\ M_{\max} = \frac{8\times 4^2}{8} = 16 \text{ kN}\cdot\text{m};\quad \text{For } BC,\ M_{\max} = \frac{8\times 3^2}{8} = 9 \text{ kN}\cdot\text{m};$$

Assume an imaginary zero span *OA* beyond the fixed end. The fixing moment at the imaginary support of the zero span is taken to be zero.

Applying three-moment equations
Apply the three-moment equation for uniformly distributed loads to the spans *OA* and *AB*,

$$0 + 2M_a(0+4) + M_b \times 4 = -0 - \frac{8\times 4^3}{4}$$

or $\quad 8M_a + 4M_b = -28$

or $\quad 2M_a + M_b = -32 \quad$ (i)

Apply the three-moment diagram for uniformly distributed loads to the spans *AB* and *BC*,

$$M_a \times 4 + 2M_b(4+3) + M_c \times 3 = -\frac{8\times 4^3}{4} - \frac{8\times 3^3}{4}$$

or $\quad 4M_a + 14M_b + 0 = -182$

or $\quad 2M_a + 7M_b = -91 \quad$ (ii)

Subtracting (i) from (ii),

$$6M_b = -59 \quad \text{or} \quad M_b = -9.83 \text{ kN·m} \quad \text{and} \quad M_a = -11.1 \text{ kN·m}$$

Reactions

To find R_a, take moments about B,

$$R_a \times 4 - 11.1 - 8 \times 4 \times 2 = -9.83 \quad \text{or} \quad R_a = 16.32 \text{ kN}$$

To find R_c, consider bending moment at B,

$$R_c \times 3 - 8 \times 3 \times 1.5 = -9.83, \quad R_c = 8.72 \text{ kN}$$

$$R_b = 8 \times 7 - 16.32 - 8.72 = 30.96 \text{ kN}$$

R_b can also be found by taking moments about A,

$$R_b \times 4 + 8.72 \times 7 - 8 \times 7 \times 3.5 = -11.1 \quad \text{or} \quad R_b = 30.96 \text{ kN}$$

Bending moment and shear force diagrams are shown in Fig. 8.19b and c respectively.

Example 8.17 A continuous beam *ABCDE* is loaded as shown in Fig. 8.20. Draw the bending moment diagram for the entire beam, stating values at salient points.

Solutions

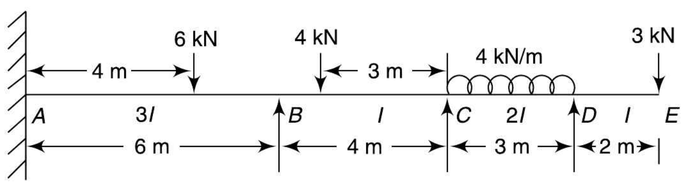


Fig. 8.20

Given A continuous beam fixed at A and loaded as shown in Fig. 8.20.
To find To draw bending moment diagram

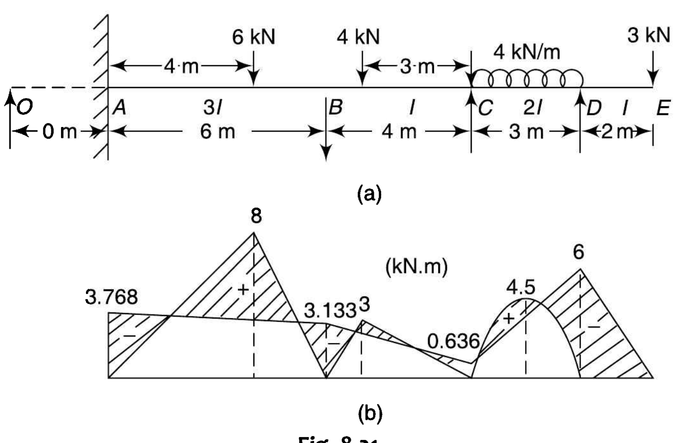


Fig. 8.21

Bending moments

- For span AB, $M_{\max} = \dfrac{Wab}{l} = \dfrac{6 \times 4 \times 2}{6} = 8 \text{ kN·m}$

$$A\bar{x} = \frac{4\times 8}{2}\times 4\times\frac{2}{3}+\frac{2\times 8}{2}\times\left(4+\frac{2}{3}\right) = 80 \qquad (A \text{ as origin})$$

or it may be calculated as $A\bar{x} = \dfrac{6\times 8}{2}\times\dfrac{6+4}{3} = 80$

$$A\bar{x} = \frac{6\times 8}{2}\times\frac{6+2}{3} = 64 \qquad (B \text{ as origin})$$

- For span BC, $M_{max} = \dfrac{4\times 1\times 3}{4} = 3\text{ kN}\cdot\text{m}$

$$A\bar{x} = \frac{4\times 3}{2}\times\frac{4+1}{3} = 10 \qquad (B \text{ as origin})$$

$$A\bar{x} = \frac{4\times 3}{2}\times\frac{4+3}{3} = 14 \qquad (C \text{ as origin})$$

- For span CD, $M_{max} = \dfrac{wl^2}{8} = \dfrac{4\times 3^2}{8} = 4.5\text{ kN}\cdot\text{m}$

$$A\bar{x} = \frac{2}{3}\times 3\times 4.5\times 1.5 = 13.5 \qquad (B \text{ or C as origin})$$

- Moment at D for the cantilever at $DE = -3\times 2 = -6$ kN·m ... (hogging)

  Assume an imaginary zero span OA beyond the fixed end (Fig. 8.21a). The fixing moment at the imaginary support of the zero span is taken to be zero.

Applying three-moment equation

Applying the three-moment equation to the spans OA and AB,

$$0 + 2M_a\left(0+\frac{6}{3I}\right) + M_b\times\frac{6}{3I} = -6\left(0+\frac{64}{3I\times 6}\right) \quad \text{or} \quad 4M_a + 2M_b = -21.33$$

or $\quad 2M_a + M_b = -10.67$ (i)

Applying the three-moment equation to the spans AB and BC,

$$M_a\times\frac{6}{3I} + 2M_b\left(\frac{6}{3I}+\frac{4}{I}\right) + M_b\frac{4}{I} = -6\left(\frac{80}{3I\times 6}+\frac{14}{I\times 4}\right)$$

or $\quad 2M_a + 12M_b + 4M_c = -47.67$ (ii)

Applying the three-moment diagram to the spans BC and CD,

$$M_b\times\frac{4}{I} + 2M_c\left(\frac{4}{I}+\frac{3}{2I}\right) - 6\times\frac{3}{2I} = -6\left(\frac{10}{I\times 4}+\frac{13.5}{2I\times 3}\right) \quad \text{or} \quad 4M_b + 11M_c = -19.5$$

or $\quad M_c = -1.773 - 0.363M_b$ (iii)

Inserting value of M_c from (iii) in (ii), $2M_a + 12M_b + 4(-1.773 - 0.363M_b) = -47.67$

$\quad 2M_a + 10.548M_b = -40.6$ (iv)

Subtracting (i) from (iv), $9.548M_b = -29.92$ or $M_b = -3.133$ kN·m

$\quad 2M_a - 3.133 = -10.67$ or $M_a = 3.768$ kN·m

and $\quad M_c = -1.773 - 0.363\times 3(-3.133) = -0.636$ kN·m

Bending moment diagram is shown in Fig. 8.21b.

8.5 MOMENT DISTRIBUTION METHOD

The moment distribution method is quite useful in the analysis of continuous beams. In this method, initially all the members of the beam are assumed to be fixed in position and the fixed end moments due to loading are found for each member. Then all the hinged joints are released by applying equal and opposite moments. The effect of the applied moment on the adjacent joints is carried over and any unbalanced moment at a joint is distributed in the spans.

Carry Over Factor

The effect of applied moment at a joint on the other joints is known as the *carry over factor.* Figure 8.22 shows a beam AB of span l, fixed at A and simply supported at B. It is applied a clockwise moment M at B. If the beam is not subjected to any external loading, the reactions at the fixed and free ends are to be equal. Let each be R. Also let the fixing moment be M_a at A.

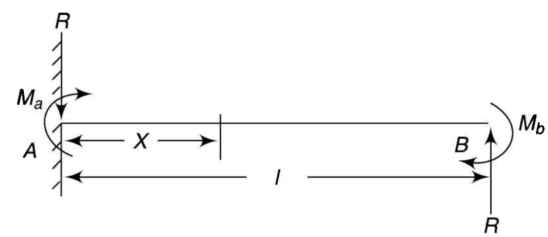


Fig.8.22

Taking moments about A, $Rl = M_b + M_a$ (i)

Consider a section at a distance x from the fixed end and take moments about the same,

$$M_x = M_a - Rx$$

Then $EI\dfrac{d^2y}{dx^2} = M_x = M_a - Rx$ ...(Eq.7.1)

Integrating, $EI\dfrac{dy}{dx} = M_a \cdot x - \dfrac{Rx^2}{2} + C_1$

At $x = 0$, $\dfrac{dy}{dx} = 0$, , $\therefore$ $C_1 = 0$; Thus, $EI\dfrac{dy}{dx} = M_a \cdot x - \dfrac{Rx^2}{2}$ (ii)

Integrating again, $EI \cdot y = M_a\dfrac{x^2}{2} - \dfrac{Rx^3}{6} + C_2$

At $x = 0, y = 0$, $\therefore$ $C_2 = 0$; Thus, $EI \cdot y = M_a\dfrac{x^2}{2} - \dfrac{Rx^3}{6}$

Also, At $x = l, y = 0$, $\therefore$ $C_2 = 0$; Thus, $0 = M_a\dfrac{l^2}{2} - \dfrac{Rl^3}{6}$ or $Rl = 3M_a$

From (i), $3M_a = M_b + M_a$ or $M_a = M_b/2$

Thus the carry over factor in this case is one–half.

- In case of a simply supported beam at both ends, there is no fixing moment at any end and thus the carry over factor is zero.

Stiffness It is the bending moment per unit slope.

From (ii), $EI\dfrac{dy}{dx} = M_a \cdot x - \dfrac{Rx^2}{2}$ or Slope $i = \dfrac{dy}{dx} = \dfrac{1}{EI}\left(M_a \cdot x - \dfrac{Rx^2}{2}\right)$

Slope at the free end, (at $x = l$),

$$i_b = \frac{1}{EI}\left(M_a \cdot l - \frac{Rl^2}{2}\right) = \frac{1}{EI}\left(M_a \cdot l - \frac{3M_a l}{2}\right) = -\frac{M_a l}{2EI} = -\frac{M_b l}{4EI}$$

or stiffness of beam, $\dfrac{M_b}{i_b} = -\dfrac{4EI}{l}$

Negative sign means that the slope is negative at B if A is the origin (refer Section 7.2).

To find the stiffness of a simply supported beam at both ends, consider the beam as shown in Fig. 8.23. Let a clockwise moment be applied at the end B. As there is no external loading, the two reactions at the ends are to be equal.

Taking moments about A, $Rl = M_b$ (i)

Consider a section at a distance x from the fixed end.

Take moments about the same, $M_x = -Rx$

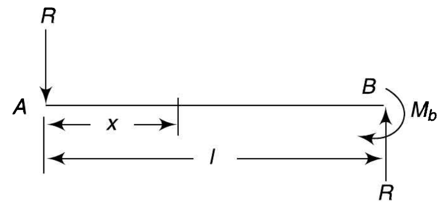


Fig. 8.23

Then $EI\dfrac{d^2y}{dx^2} = -Rx$

Integrating, $EI\dfrac{dy}{dx} = -\dfrac{Rx^2}{2} + C_1$

Integrating again, $EI \cdot y = -\dfrac{Rx^3}{6} + C_1 x + C_2$

At $x = 0$, $y = 0$, $\therefore$ $C_2 = 0$; Thus, $EI \cdot y = -\dfrac{Rx^3}{6} + C_1 x$

Also, At $x = l$, $y = 0$, Thus, $0 = -\dfrac{Rl^3}{6} + C_1 l$ or $C_1 = \dfrac{Rl^2}{6} = \dfrac{M_b l}{6}$

$\therefore$ $EI\dfrac{dy}{dx} = -\dfrac{Rx^2}{2} + \dfrac{M_b l}{6}$

Slope at B, (at $x = l$),

$$i_b = \frac{1}{EI}\left(\frac{M_b \cdot l}{6} - \frac{Rl^2}{2}\right) = \frac{1}{EI}\left(\frac{M_b \cdot l}{6} - \frac{M_b l}{2}\right) = -\frac{M_b l}{3EI}$$

or stiffness of beam, $\dfrac{M_b}{i_b} = -\dfrac{3EI}{l}$

Negative sign means that the slope is negative at B if A is the origin.

Distribution Factors

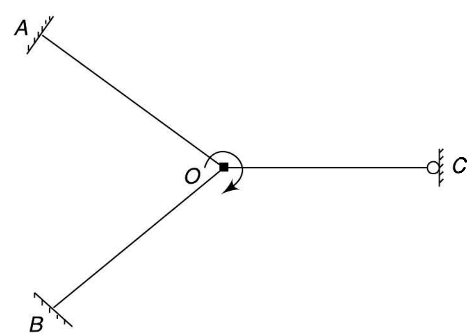


Fig.8.24

Consider three members OA, OB and OC meeting at O. Assume that members OA and OB are fixed at A and B respectively and member OC is hinged at C. Let the joint O be subjected to a moment M as shown in Fig. 8.24. Owing to this moment, each member is rotated through an equal angle θ.

Now, stiffness of member OA, $s_a = -\dfrac{4E_a I_a}{l_a}$ (as end A is fixed)

Stiffness of member OB, $s_b = -\dfrac{4E_b I_b}{l_b}$ (as end B is fixed)

Stiffness of member OC, $s_c = -\dfrac{3E_c I_c}{l_c}$ (as end C is hinged)

Combined stiffness of all the members, $s = s_a + s_b + s_c$
Applied moment, $M = s\,\theta$
Moment in OA, $M_a = s_a\theta$, Similarly, $M_b = s_b\theta$ and $M_c = s_c\theta$

Now, $\dfrac{M_a}{M} = \dfrac{s_a\theta}{s\theta} = \dfrac{s_a}{s}$ or $M_a = M\dfrac{s_a}{s}$, Similarly, $M_b = M\dfrac{s_b}{s}$ and $M_c = M\dfrac{s_c}{s}$

s_a/s, s_b/s and s_c/s are known as *distribution factors* (k) for the members OA, OB and OC respectively. Distribution factor may be defined as the ratio of stiffness of a member of a joint to the combined stiffness of all the members of the joint.

The procedure for the analysis can be summarized as follows:

1. Assume all members of the beam to be fixed in position and direction at the two ends. Write the fixed end moments due to external loads on each member.
2. As the bending moment at a hinged joint is to be zero, release all hinged joints by applying equal and opposite moment at the joint.
3. Carry over the effect (one-half) of the applied moment on the adjacent joints.
4. Distribute the unbalanced moment at a joint in the two spans in the ratio of their distribution factors.
5. Carry over the effect of the distributed moments on the adjacent joints.
6. Repeat the steps 4 and 5 till the distributed moments become small and their carry over moments can be neglected.
7. Add up all the moments at the joints. It is to be noted that the algebraic sum of the moments at a joint is to be zero, i.e., positive and negative moments must be equal.
   - Remember that as the bending moment at the end support of a simply supported beam is zero, there is to be no carry over effect of the applied moment or the distributed moments to that joint.
   - Bending moment diagram can be obtained by superimposing the diagrams for the free moments and for the fixing moments as usual.

Sign Convention

Though any sign convention may be adopted for the analysis by this method, yet according to a widely used convention, all clockwise moments at the ends are taken positive and all counter-clockwise moments negative. The same is being used in the following examples.

Example 8.18 Figure 8.24 shows three beams loaded in the same way. The first beam (Fig. 8.25a) is fixed at both ends, the second beam is fixed only at the left end (Fig. 8.25b) and the third beam is simply supported at both the ends (Fig. 8.25c). Draw the bending moment and the shear force diagrams for each type of beam by the moment distribution method.

Solution

(i) B*eam fixed at both ends*

- Maximum bending moment for the spans AB and BC treating as simply supported beams,

  For span AB, $M_{max} = \dfrac{wl^2}{8} = \dfrac{4.5 \times 4^2}{8} = 9\ \text{kN}\cdot\text{m}$

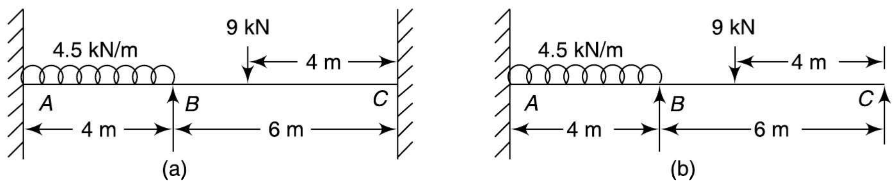


Fig. 8.25

For span BC, $\quad M_{\max} = \dfrac{Wab}{l} = \dfrac{9 \times 2 \times 4}{6} = 12 \text{ kN}\cdot\text{m}$

- For fixing moments, assume the continuous beam ABC to be made up of fixed beams AB and BC.

For span AB: Fixing moments at A, $M_a = -\dfrac{wl^2}{12} = -\dfrac{4.5 \times 4^2}{12} = -6$ kN.m

Fixing moments at B, $M_b = \dfrac{wl^2}{12} = \dfrac{4.5 \times 4^2}{12} = 6 \text{ kN}\cdot\text{m}$

For span BC: Fixing moments at B, $M_b = \dfrac{9 \times 2 \times 4^2}{6^2} = -8 \text{ kN}\cdot\text{m}$

Fixing moments at C, $M_c = \dfrac{9 \times 2^2 \times 4}{6^2} = 4 \text{ kN}\cdot\text{m}$

Distribution factors at B

Stiffness factor for AB, $s_{ba} = \dfrac{4EI}{4} = EI$ $\quad$ (as the beam is fixed at A)

Stiffness factor for AB, $s_{bc} = \dfrac{4EI}{6} = \dfrac{2EI}{3}$ $\quad$ (as the beam is fixed at C)

Distribution factor for AB, $k_{ba} = \dfrac{s_{ab}}{s_{ab} + s_{bc}} = \dfrac{EI}{EI + 2EI/3} = \dfrac{3}{5}$

Distribution factor for BC, $k_{bc} = \dfrac{s_{bc}}{s_{ab} + s_{bc}} = \dfrac{2EI/3}{EI + 2EI/3} = \dfrac{2}{5}$

	A	B	B	C
Distribution factors		3/5	2/5	
Fixed end moments	– 6	6	–8	4
Distribute		1.2*	0.8*	
Carry over	0.6**			0.4**
Net moments	–5.4	7.2	–7.2	4.4

*Distribute the negative sum in the ratio of distribution factors, i.e., –(6 – 8) × 3 / 5 = 1.2 and –(6 – 8) × 2 / 5 = 0.8 or –(6 – 8) – 1.2 = 0.8

**one–half of the values at joint *B*.

The bending moment diagram can be completed as shown in Fig. 8.26a.

Shear force diagram

To find reaction R_a, taking moments about *B*,

$$R_a \times 4 - 5.4 - 4.5 \times 4 \times 2 = -7.2 \quad \text{or} \quad R_a = 8.55 \text{ kN}$$

Similarly, $$R_c \times 6 - 4.4 - 9 \times 2 = -7.2 \quad \text{or} \quad R_c = 2.53 \text{ kN}$$

$$R_b = 4.5 \times 4 + 9 - 8.55 - 2.53 = 15.92 \text{ kN}$$

The shear force diagram can be completed as shown in Fig. 8.26b.

(ii) *Beam fixed at one end*

Fixing and free moments will be as in the first case.

Stiffness factor for *AB*, $s_{ba} = \dfrac{4EI}{4} = EI$

(as the beam is fixed at *A*)

and for *BC*, $s_{bc} = \dfrac{3EI}{6} = \dfrac{EI}{2}$

(as the beam is simply supported at *C*)

Distribution factor for *AB*, $k_{ba} = \dfrac{s_{ab}}{s_{ab} + s_{bc}} = \dfrac{EI}{EI + EI/2} = \dfrac{2}{3}$

Distribution factor for *BC*, $k_{bc} = \dfrac{s_{bc}}{s_{ab} + s_{bc}} = \dfrac{EI/2}{EI + EI/2} = \dfrac{1}{3}$

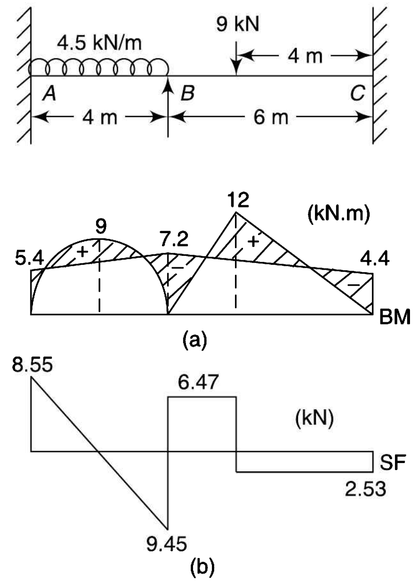


Fig.8.26

	A		B	C
Distribution factors		2/3	1/3	
Fixed end moments	– 6	6	–8	4
Release C				–4
Carry over			–2	
Net moments	– 6	6	–10	0
Distribute		2.67	1.33	
Carry over	1.33			
Final moments	–4.67	8.67	–8.67	0

The bending moment diagram can be completed as shown in Fig. 8.27a.

Shear force diagram

Taking moments about *B*,

$$R_a \times 4 - 4.67 - 4.5 \times 4 \times 2 = -8.67 \text{ or } R_a = 8 \text{ kN}$$

and $$R_c \times 6 - 9 \times 2 = -8.67 \quad \text{or} \quad R_c = 1.56 \text{ kN}$$

$\therefore$ $$R_b = 4.5 \times 4 + 9 - 8 - 1.56 = 17.44 \text{ kN}$$

R_b can also be found by taking moments about A,

$$1.56 \times 10 + R_b \times 4 - 9 \times 6 - 18 \times 2$$
$$= -4.67 \quad \text{or} \quad R_b = 17.44 \text{ kN}$$

Then R_a can also be found by taking moment about C,

$$R_a \times 10 - 4.67 - 18 \times 8 + 17.44 \times 6 - 9 \times 4 = 0$$

or $\quad R_a = 8$

The shear force diagram can be completed as shown in Fig. 8.27b.

(iii) *Simply supported beam*

Fixing and free moments will be as in the first case.

Stiffness factor for AB, $s_{ba} = \dfrac{3EI}{4}$

(as the beam is simply supported at A)

and for BC, $s_{bc} = \dfrac{3EI}{6} = \dfrac{EI}{2}$

(as the beam is simply supported at C)

Distribution factor for AB,

$$k_{ba} = \frac{s_{ab}}{s_{ab} + s_{bc}} = \frac{3EI/4}{3EI/4 + EI/2} = \frac{3}{5}$$

Distribution factor for BC, $k_{bc} = \dfrac{s_{bc}}{s_{ab} + s_{bc}} = \dfrac{EI/2}{3EI/4 + EI/2} = \dfrac{2}{5}$

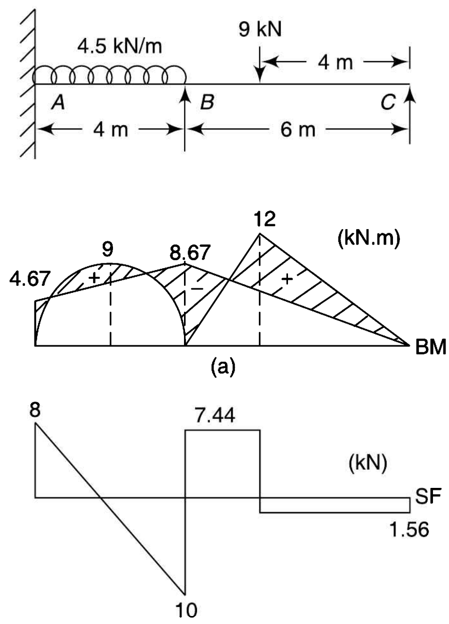


Fig. 8.27

	A	B		C
Distribution factors		3/5	2/5	
Fixed end moments	–6	6	–8	4
Release A and C	+6			–4
Carry over		3	–2	
Net moments	0	9	–10	0
Distribute		0.6	0.4	
Final moments	0	9.6	–9.6	0

The bending moment diagram can be completed as shown in Fig. 8.28a.

Shear force diagram

Taking moments at B,

$$R_a \times 4 - 4.5 \times 4 \times 2 = -9.6 \text{ or } \quad R_a = 6.6 \text{ kN}$$

and $\quad R_c \times 6 - 9 \times 2 = -9.6 \quad \text{or} \quad R_c = 1.4 \text{ kN}$

$$R_b = 4.5 \times 4 + 9 - 6.6 - 1.4 = 19 \text{ kN}$$

As a check, taking moments about A,

$$1.4 \times 10 + 19 \times 4 - 9 \times 6 - 18 \times 2 = 0$$

The shear force diagram can be completed as shown in Fig. 8.28b.

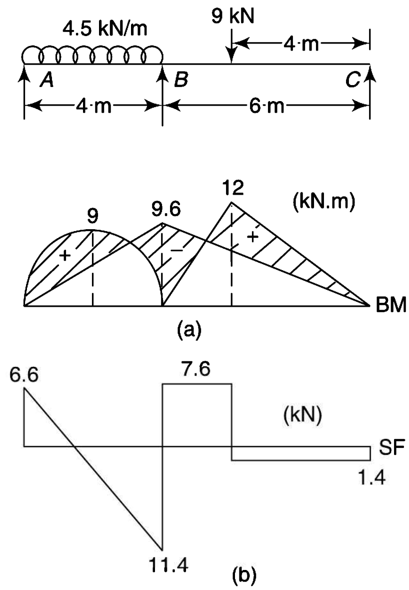


Fig. 8.28

Example 8.19 A beam *ABCD* is 15 m long and is simply supported at *A*, *B* and *C*. Span *AB* is 5 m long and carries a uniformly distributed load of 2 kN/m. Span *BC* is 8 m long and does not carry any load whereas overhanging span *CD* is 2 m long and carries a point load of 2 kN at the free end *D*. The moments of inertia of spans *AB*, *BC* and *CD* are 2*I*, 3*I* and *I* respectively. Draw the bending moment and shear force diagrams for the entire beam.

Solution

Given A continuous beam loaded as shown in Fig. 8.29a.

To find To draw bending moment and shear force diagrams

Bending moments

- Maximum bending moment for the spans *AB* and *BC* treating these as simply supported beams,

For span *AB*, $M_{\max} = \dfrac{wl^2}{8} = \dfrac{2 \times 5^2}{8} = 6.25 \text{ kN}\cdot\text{m}$

- For fixing moments, assume the continuous beam *ABC* to be made up of fixed beams *AB* and *BC*.

For span AB: Fixing moments at *A*,

$$M_a = -\frac{wl^2}{12} = -\frac{2 \times 5^2}{12} = -4.17 \text{ kN}\cdot\text{m}$$

Fixing moments at *B*, $M_b = 4.17$ kN·m

For span BC: Span *BC* does carry any load. Therefore, fixing moments at *B* and *C* are zero.

The moment at *C* for cantilever *CD* = –2 × 3 = –6 kN·m (hogging)

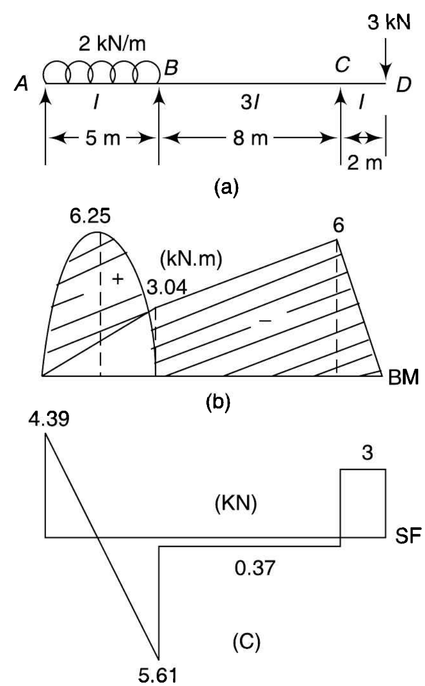


Fig. 8.29

Distribution factors

Stiffness factor for AB, $s_{ba} = \dfrac{3E \times I}{5} = \dfrac{3EI}{5}$ (as the beam is simply supported at A)

Stiffness factor for BC, $s_{bc} = \dfrac{3E \times 3I}{8} = \dfrac{9EI}{8}$ (as the beam is overhanging beyond C)

Distribution factor for AB, $k_{ba} = \dfrac{s_{ab}}{s_{ab} + s_{bc}} = \dfrac{\frac{3EI}{5}}{\frac{3EI}{5} + \frac{9EI}{8}} = \dfrac{8}{23}$

Distribution factor for BC, $k_{bc} = \dfrac{s_{bc}}{s_{ab} + s_{bc}} = \dfrac{\frac{9EI}{8}}{\frac{3EI}{5} + \frac{9EI}{8}} = \dfrac{15}{23}$ or $k_{bc} = 1 - \dfrac{8}{23} = \dfrac{15}{23}$

	A	B		C	
Distribution factors		8/23	15/23		
Fixed end moments	–4.17	4.17	0	0	–6
Release A and balance C	4.17			6	
Carry over		2.09	3		
Net moments	0	6.26	3	6	–6
Distribute		–3.22*	–6.04*		
Final moments	0	–3.04	–3.04	6	–6

*–(6.26 + 3) × 8/23 = –3.22 and –(6.26 + 3) × 15/23 = –6.04

The bending moment diagram can be completed as shown in Fig. 8.29b.

Shear force diagram

To find reaction R_a, taking moments about B,

$$R_a \times 5 - 2 \times 5 \times 2.5 = -3.04 \quad \text{or} \quad R_a = 4.39 \text{ kN}$$

Similarly,
$$R_c \times 8 - 3 \times 10 = -3.04 \quad \text{or} \quad R_c = 3.37 \text{ kN}$$

$$R_b = 2 \times 3 - 4.39 - 3.37 = 5.24 \text{ kN}$$

The shear force diagram can be completed as shown in Fig. 8.29c.

Example 8.20 A 24-m long cantilever beam *ABCDE* is fixed at *A* and simply supported at *B*, *C*, *D* and *E*. Each of the four spans is 6 m long. The beam carries point loads of 4 kN and 8 kN at 9 m and 15 m from the left end. The beam also carries uniformly distributed load of 1 kN/m between *AC* and *DE*. Draw the bending moment and shear force diagrams for the beam using moment distribution method.

Solution

Given A continuous beam loaded as shown in Fig. 8.30a.

To find To draw bending moment shear force diagrams

Bending moments

First assuming the continuous beam *ABCDE* to be made up of fixed beams *AB*, *BC*, *CD* and cantilever *DE*.

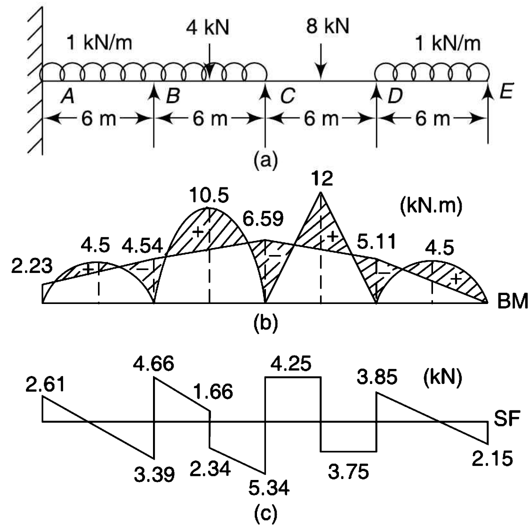


Fig. 8.30

- *For span AB*: Fixing moments at *A*,

$$M_a = -\frac{wl^2}{12} = -\frac{1 \times 6^2}{12} = -3\ \text{kN}\cdot\text{m}$$

Fixing moments at *B*, $M_b = 3$ kN·m

- *For span BC*: Fixing moments at *B*,

$$M_b = -\frac{wl^2}{12} - \frac{Wl}{8} = -\frac{1 \times 6^2}{12} - \frac{4 \times 6}{8} = -6\ \text{kN}\cdot\text{m}$$

Fixing moments at *C*, $M_c = 6$ kN·m

- *For span CD*: Fixing moments at *C*,

$$M_c = -\frac{Wl}{8} = -\frac{8 \times 6}{8} = -6\ \text{kN}\cdot\text{m}$$

Fixing moments at *D*, = $M_d = 6$ kN·m

- *For span DE*: Fixing moments at *D*,

$$M_d = -\frac{wl^2}{12} = -\frac{1 \times 6^2}{12} = -3\ \text{kN}\cdot\text{m}$$

Fixing moments at *E*, $M_e = 3$ kN·m

Distribution factors

- *At B*: Stiffness factor for *AB*, $s_{ba} = \frac{4EI}{6} = \frac{2EI}{3}$ (as the beam is fixed at *A*)

and for *BC*, $s_{bc} = \frac{4EI}{6} = \frac{2EI}{3}$ (as the beam is continuous at *C*)

Distribution factor for *AB*, $k_{ba} = \frac{s_{ab}}{s_{ab} + s_{bc}} = \frac{2EI/3}{2EI/3 + 3EI/3} = \frac{1}{2}$

Distribution factor for *BC*, $k_{bc} = \frac{s_{bc}}{s_{ab} + s_{bc}} = \frac{1}{2}$

- *At C*: Distribution factor for *BC*, $k_{cb} = \frac{1}{2}$ (as for *AB*)

Distribution factor for *CD*, $k_{cd} = \frac{1}{2}$ (as for *BC*)

- *At D*: Stiffness factor for *CD*, $s_{dc} = \frac{4EI}{6} = \frac{2EI}{3}$ (as the beam is continuous at *C*)

and for *DE*, $s_{de} = \frac{3EI}{6} = \frac{EI}{2}$ (as the beam is simply supported at *E*)

Distribution factor for *CD*, $k_{dc} = \frac{2EI/3}{2EI/3 + EI/2} = \frac{4}{7}$

Distribution factor for *DE*, $k_{de} = \frac{EI/2}{2EI/3 + EI/2} = \frac{3}{7}$

	A	B		C		D		E
Distribution factors		1/2	1/2	1/2	1/2	4/7	3/7	
Fixed end moments	–3	3	–6	6	–6	6	–3	3
Release *E*								–3
Carry over							–1.5	
Net moments	–3	3	–6	6	–6	6	–4.5	0
Distribute		1.5	1.5	0	0	–0.86	–0.64	
Carry over	0.75			0.75	–0.43			
Distribute				–0.16	–0.16			
Carry over			–0.08			–0.08		
Distribute		0.04	0.04			0.046	0.034	
Final moments	–2.25	4.54	4.54	6.59	–6.59	5.106	–5.106	0

Bending moment diagram

For span AB, $M_{\max} = \dfrac{wl^2}{8} = \dfrac{1 \times 6^2}{8} = 4.5$

For span BC, $M_{\max} = \dfrac{wl^2}{8} + \dfrac{Wl}{4} = \dfrac{1 \times 6^2}{8} + \dfrac{4 \times 6}{4} = 10.5 \text{ kN}\cdot\text{m}$

For span CD, $M_{\max} = \dfrac{Wl}{4} = \dfrac{8 \times 6}{4} = 12 \text{ kN}\cdot\text{m}$

For span DE, $M_{\max} = \dfrac{wl^2}{8} = \dfrac{1 \times 6^2}{8} = 4.5 \text{ kN}\cdot\text{m}$

Now, the bending moment diagram can be completed as shown in Fig. 8.30b.

Shear force diagram

To find reaction R_e, taking moments about D,

$$R_e \times 6 - 6 \times 3 = -5.108 \qquad \text{or} \qquad R_e = 2.15 \text{ kN}$$

Taking moments about C,

$$2.15 \times 12 - 6 \times 9 + R_d \times 6 - 8 \times 3 = -6.588 \qquad \text{or} \qquad R_d = 7.6 \text{ kN}$$

Taking moments about B,

$$2.15 \times 18 - 6 \times 15 + 7.6 \times 12 - 8 \times 9 + R_c \times 6 - 4 \times 3 - 6 \times 3 = -4.541 \quad \text{or} \quad R_c = 9.59 \text{ kN}$$

Taking moments about B,

$$R_a \times 6 - 2.229 - 6 \times 3 = -4.541 \quad \text{or} \quad R_a = 2.61 \text{ kN}$$

$$R_b = 12 + 4 + 8 + 6 - 2.15 - 7.6 - 9.59 - 2.61 = 8.05 \text{ kN}$$

As a check, taking moments about C,

$$2.61 \times 12 - 2.229 - 12 \times 6 - 4 \times 3 + 8.05 \times 6 \approx 6.588$$

The shear force diagram can be completed as shown in Fig. 8.30c.

Example 8.21 A cantilever beam *ABCDE* is loaded as shown in Fig. 8.31. The moment of inertia of various segments is also shown in the diagram. Draw the bending moment diagram for the entire beam by moment distribution method, stating values at salient points.

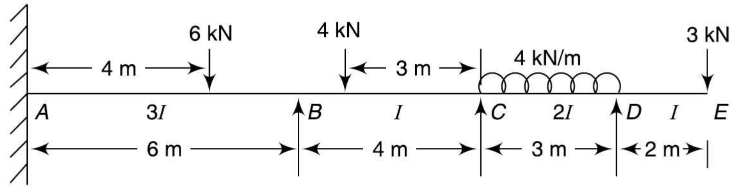


Fig. 8.31

Solution

Given A cantilever beam loaded as shown in Fig. 8.31.

To find To draw bending moment diagram

First assume the continuous beam $ABCDE$ to be made up of fixed beams AB, BC, CD and cantilever DE. Refer Fig. 8.32a,

Bending moments

- *For span AB*: Fixing moments at A,

$$M_a = \frac{Wab^2}{l^2} = \frac{6 \times 4 \times 2^2}{6^2} = -2.67 \text{ kN}\cdot\text{m}$$

Fixing moments at B,

$$M_b = \frac{Wa^2 \cdot b}{l^2} = \frac{6 \times 4^2 \times 2}{6^2} = 5.33 \text{ kN}\cdot\text{m}$$

- *For span BC*: Fixing moments at B,

$$M_b = \frac{4 \times 1 \times 3^2}{4^2} = -2.25 \text{ kN}\cdot\text{m}$$

Fixing moments at C,

$$M_c = \frac{4 \times 1^2 \times 3}{4^2} = 0.75 \text{ kN}\cdot\text{m}$$

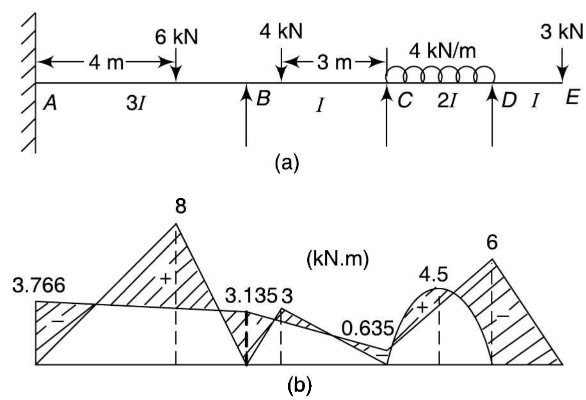


Fig. 8.32

- *For span CD*: Fixing moments at C, $M_c = -\frac{wl^2}{12} = -\frac{4 \times 3^2}{12} = -3 \text{ kN}\cdot\text{m}$

Fixing moments at D, $M_d = \frac{wl^2}{12} = \frac{4 \times 3^2}{12} = 3 \text{ kN}\cdot\text{m}$

- Moment at D for the cantilever at $DE = -3 \times 2 = -6$ kN·m

Distribution factors

- *At B*: Stiffness factor for AB, $s_{ba} = \frac{4E \times 3I}{6} = 2EI$ (as the beam is fixed at A)

and for BC, $s_{bc} = \frac{4EI}{4} = EI$ (as the beam is continuous at C)

Distribution factor for AB, $k_{ba} = \frac{s_{ab}}{s_{ab} + s_{bc}} = \frac{2EI}{2EI + EI} = \frac{2}{3}$

Distribution factor for BC, $k_{bc} = \dfrac{s_{bc}}{s_{ab} + s_{bc}} = \dfrac{EI}{2EI + EI} = \dfrac{1}{3}$

- *At C*: Stiffness factor for BC, $s_{cb} = \dfrac{4E \times I}{4} = EI$ (as the beam is continuous at B)

and for CD, $s_{cd} = \dfrac{3E \times 2I}{3} = 2EI$ (as the beam is overhanging beyond D)

Distribution factor for BC, $k_{cb} = \dfrac{EI}{EI + 2EI} = \dfrac{1}{3}$

Distribution factor for CD, $k_{cd} = \dfrac{2EI}{EI + 2EI} = \dfrac{2}{3}$

	A	*B*		*C*		*D*	
Distribution factors		2/3	1/3	1/3	2/3		
Fixed end moments	−2.67	5.33	−2.25	0.75	−3	3	−6
Balance D						+3	
Carry over					1.5		
Net moments	−2.67	5.33	−2.25	0.75	−1.5	6	−6
Distribute		−2.053	−1.027	0.25	0.5		
Carry over	−1.027		0.125	− 0.514			
Distribute		−0.083	−0.042	0.172	0.342		
Carry over	−0.041		0.086	− 0.021			
Distribute		− 0.057	−0.029	0.007	0.014		
Carry over	−0.028		0.003	−0.014			
Distribute		− 0.002	−0.001	0.005	0.009		
Final moments	−3.766	3.135	3.135	0.635	−0.635	6	−6

Bending moment diagram

For span AB, $M_{\max} = \dfrac{Wab}{l} = \dfrac{6 \times 4 \times 24}{6} = 8\ \text{kN}\cdot\text{m}$

For span BC, $M_{\max} = \dfrac{4 \times 1 \times 3}{4} = 3\ \text{kN}\cdot\text{m}$

For span CD, $M_{\max} = \dfrac{wl^2}{8} = \dfrac{4 \times 3^2}{8} = 4.5\ \text{kN}\cdot\text{m}$

Now, the bending moment diagram can be completed as shown in Fig. 8.32b.

Example 8.22 || A continuous beam *ABC* is built-in at *A* and simply supported at *B* and *C*. Span *AB* is 15 m long and carries a uniformly distributed load of 8 kN/m run and span *BC* is 12 m long and carries a point load of 80 kN at 4 m from support *B*. Draw the bending moment and shear force diagrams if the support *B* sinks 12 mm relative to *A* and *C*. *E* = 205 GPa and *I* = 560 × 10⁶ mm⁴.

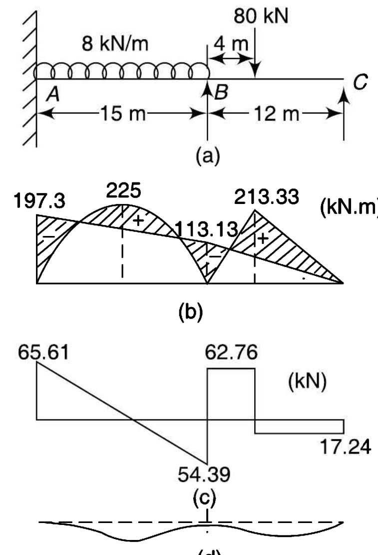


Fig. 8.33

Solution

Given A continuous beam built-in at left end and loaded as shown in Fig. 8.33a.

To find To draw bending moment and shear force diagrams

First assuming the continuous beam *ABC* to be made up of fixed beams *AB* and *BC*.

Bending moments

- *For span AB*: Fixing moments at *A*,

$$M_a = -\frac{wl^2}{12} = -\frac{8 \times 15^2}{12} = -150 \text{ kN}\cdot\text{m}$$

Fixing moments at *B*, M_b = 150 kN·m

For span BC: Fixing moments at *B*,

$$M_b = \frac{80 \times 4 \times 8^2}{12^2} = -142.22 \text{ kN}\cdot\text{m}$$

Fixing moments at *C*, $M_c = \dfrac{80 \times 4^2 \times 8}{12^2} = 71.11 \text{ kN}\cdot\text{m}$

- In span *AB*, moments at *A* and *B* due to sinking of support *B* by 12 mm,

$$M = -\frac{6EI\delta}{l^2} \qquad \text{(Example 8.7, –ve being counter-clockwise)}$$

or $$M = -\frac{6 \times 205\,000 \times 880 \times 10^6 \times 12}{15\,000^2} = -57.73 \times 10^6 \text{ N}\cdot\text{mm} \text{ or } -57.73 \text{ kN}\cdot\text{m}$$

Due to sinking of support *B* by 12 mm, moments at *B* and *C* (Refer Example 8.7),

$$M = -\frac{6EI\delta}{l^2} \qquad \text{(–ve being counter–clockwise)}$$

or $$M = \frac{6 \times 205\,000 \times 880 \times 10^6 \times 12}{12\,000^2} = 90.2 \times 10^6 \text{ N}\cdot\text{mm} \quad \text{or} \quad 90.2 \text{ kN}\cdot\text{m}$$

Distribution factors

At B: Stiffness factor for *AB*, $s_{ba} = \dfrac{4EI}{l} = \dfrac{4EI}{15}$ (as the beam is fixed at *A*)

and for *BC*, $s_{bc} = \dfrac{3EI}{l} = \dfrac{3EI}{12} = \dfrac{EI}{4}$ (as the beam is simply supported at *C*)

Distribution factor for *AB*, $k_{ba} = \dfrac{s_{ab}}{s_{ab} + s_{bc}} = \dfrac{4EI/15}{4EI/15 + EI/4} = \dfrac{16}{31}$

Distribution factor for *BC*, $k_{ba} = \dfrac{s_{ab}}{s_{ab} + s_{bc}} = \dfrac{EI/4}{4EI/15 + EI/4} = \dfrac{15}{31}$

	A		B		C
Distribution factors		16/31	15/31		
Fixed end moments	–150	150	–142.22		71.11
Moments due to sinking	–57.73	–57.73	90.2		90.2
Initial moments	–207.73	92.27	–52.02		161.31
Release C					–161.31
Carry over			–80.66		
Net moments	–207.73	92.27	–132.68		0
Distribute		20.86	19.55		
Carry over	10.43				
Final moments	–197.3	113.13	–113.13		0

Bending moment diagram

For span AB, $M_{\max} = \dfrac{wl^2}{8} = \dfrac{8 \times 15^2}{8} = 225$

For span BC, $M_{\max} = \dfrac{Wab}{l} = \dfrac{80 \times 4 \times 8}{12} = 213.33\ \text{kN}\cdot\text{m}$

Bending moment diagram can be completed as shown in Fig. 8.33b.

Shear force diagram

Taking moments about B,

$$R_c \times 12 - 80 \times 4 = -113.13 \text{ or } R_c = 17.24 \text{ kN}$$

Taking moments about A,

$$17.24 \times 27 + R_b \times 15 - 80 \times 19 - 8 \times 15 \times 7.5 = -197.3 \quad \text{or} \quad R_b = 117.15 \text{ kN}$$

$$R_a = 8 \times 15 + 80 - 17.24 - 117.15 = 65.61 \text{ kN}$$

As a check, taking moments about B,

$$65.61 \times 15 - 197.3 - 8 \times 15 \times 7.5 = -113.13 \text{ which is true.}$$

Shear force diagram can be completed as shown in Fig. 8.33c.

Figure 8.33d shows the elastic curve or the deflected shape of the beam.

8.6 METHOD OF FLEXIBILITY COEFFICIENTS

The *method of flexibility coefficients* is based on reducing an indeterminate structure to its determinate form and then applying the compatibility conditions to obtain the redundant forces. The method is also known as *flexibility method* or *compatibility method* or *method of consistent deformations*. The brief procedure to be adopted in this method is described below:

1. Note the degree of redundancy of the structure i.e. the forces and the couples, the removal of which will reduce the structure to its determinate form.
2. Reduce the structure to its determinate form by removing the redundant forces or couples.
3. Determine the deflections of the beam at the locations of removed forces (or slopes at the locations of removed couples).

4. Apply the redundant forces of unit magnitude one by one and determine the deflections at the locations of the redundant forces (or slopes at the locations of redundant couples).
5. Apply the compatibility equations and solve to determine the values of the redundant forces or couples.
6. Superimpose the effects of the applied load and the redundant forces or couples to find the final values.

The following examples will clarify the method.

Example 8.23 Draw the bending moment and shear force diagrams of a propped cantilever of length l with uniformly distributed load over the whole span. The prop is provided at the free end.

Solution

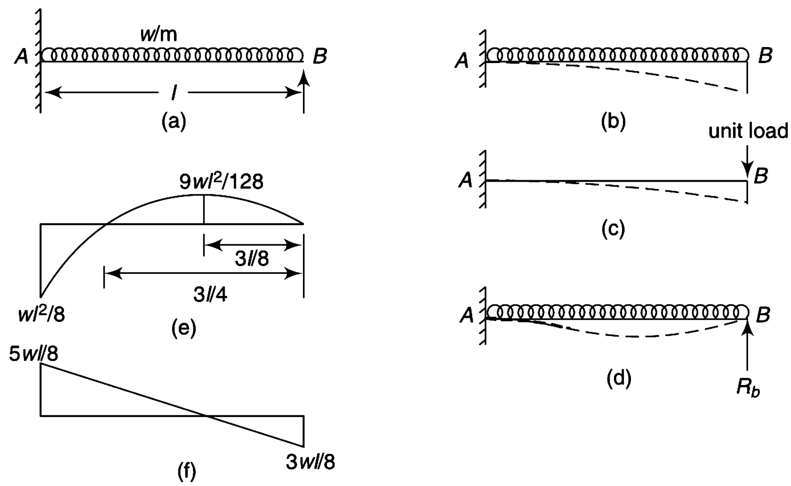


Fig. 8.34

Given A propped cantilever loaded with a uniformly distributed load as shown in Fig. 8.34a.
To find To draw bending moment and shear force diagrams

The structure involves two reactions and a couple at the fixed end. The degree of redundancy of this structure is one, i.e., by removing any one of the reactions or the couple, the structure reduces to a determinate form.

Removing the prop

Let the prop reaction be removed. The structure reduces to a simple cantilever with uniformly distributed load (Fig. 8.34b).

$$\text{The deflection at the free end} = \frac{wl^4}{8EI}$$

Assuming a unit load at free end

Assume a unit load at the free end, i.e., at the point of redundant force (Fig. 8.34c).

$$\text{The deflection at the free end} = \frac{l^3}{3EI}$$

Compatibility equation

Applying the compatibility equation, i.e., if R_b is the reaction at the prop, then deflection at the free end is to be zero.

$$\frac{wl^4}{8EI} + R_b \cdot \frac{l^3}{3EI} \text{ or } R_b = -\frac{3wl}{8}$$

Negative sign indicates that the reaction is upwards (Fig. 8.34d).

$$\text{Reaction at } A,\ R_a = wl - \frac{3wl}{8} = \frac{5wl}{8} \text{ upwards}$$

Bending moment and shear force diagrams

Bending moment at any distance x from B,

$$M_x = R_b x - \frac{wx^2}{2} = \frac{3wlx}{8} - \frac{wx^2}{2} = \frac{w}{8}(3lx - 4x^2)$$

At A, $M_a = -\dfrac{wl^2}{8}$

For point of inflection, $3lx - 4x^2 = 0$ or $x = 0.75l$

For maximum value, $\dfrac{d}{dx}(3lx - 4x^2) = 0$ or $3l - 8x = 0$ or $x = \dfrac{3l}{8}$

and maximum value $= \dfrac{w}{8}(3lx - 4x^2) = \dfrac{w}{8}\left(3l \times \dfrac{3l}{8} - 4 \times \dfrac{9l^2}{64}\right) = \dfrac{9wl^2}{64}$

The bending moment and shear force diagrams are shown in Figs. 8.34e and 8.34f respectively.

Alternate solution

- The reactions at A and B can also be found by assuming the moment at the fixed end A to be the redundant. The structure reduces to a simply supported beam with uniformly distributed load (Fig. 8.35a).

  The slope at $A = -\dfrac{wl^3}{24EI}$ (Eq. 7.24)

- Assume a unit clockwise couple at A, i.e., at the point of redundant moment (Fig. 8.35b).

  The slope at $A = -\dfrac{l}{3EI}$ (Eq. 7.7)

- Applying the compatibility equation, i.e., if M is the couple at the fixed end A, then for slope at the fixed end to be zero,

$$-\frac{wl^3}{24EI} - M \cdot \frac{1}{3EI} = 0 \quad \text{or} \quad M = -\frac{wl^2}{8}$$

  Negative sign indicates that it is counter-clockwise couple.
  Taking moments about A, (assuming R_b to be upwards)

$$\text{or} \quad R_b \cdot l - \frac{wl^2}{2} + \frac{wl^2}{8} \quad \text{or} \quad R_b = \frac{3wl}{8} \quad \text{(Fig. 8.35c)}$$

$$\text{and} \quad R_a = wl - \frac{3wl}{8} = \frac{5wl}{8} \text{ (downwards)}$$

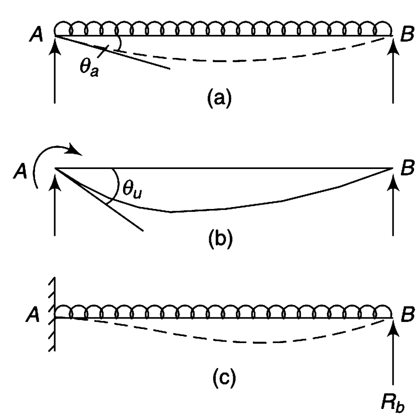


Fig. 8.35

Example 8.24 Draw the bending moment and shear force diagrams of a propped cantilever of length l with a point load at the midspan. The prop is provided at the free end.

Solution

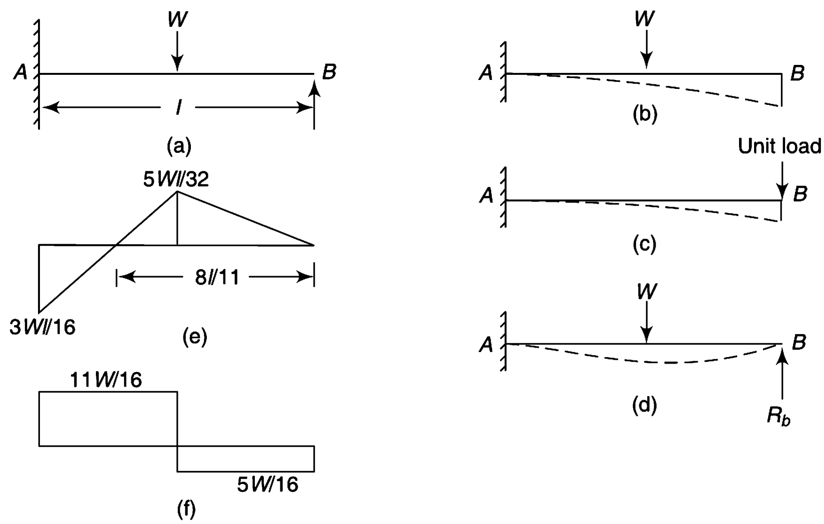


Fig. 8.36

Given A propped cantilever loaded with a point load at midspan as shown in Fig. 8.36a.
To find To draw bending moment and shear force diagrams

The structure involves two reactions and a couple at the fixed end. The degree of redundancy of this structure is one, i.e., by removing any one of the reactions or the couple, the structure reduces to a determinate form.

Removing the Prop

Let the prop reaction be removed. The structure reduces to a simple cantilever with a point load at midspan (Fig. 8.36b).

$$\text{The deflection at the free end} = \frac{5wl^3}{48EI} \qquad \text{(Eq. 7.7a)}$$

Assuming a unit load at free end

Assume a unit load at the free end, i.e., at the point of redundant force (Fig. 8.36c).

$$\text{The deflection at the free end} = \frac{l^3}{3EI} \qquad \text{(Eq. 7.6a)}$$

Compatibility equation

Applying the compatibility equation, i.e., if R_b is the reaction at the prop, then deflection at the free end is to be zero.

$$\frac{5Wl^3}{48EI} + R_b \cdot \frac{l^3}{3EI} = 0 \text{ or } R_b = -\frac{5W}{16}$$

Negative sign indicates that the reaction is upwards (Fig. 8.36d).

$$\text{Reaction at } A,\ R_a = W - \frac{5W}{16} = \frac{11W}{16} \text{ (upwards)}$$

Bending moment and shear force diagrams

Bending moment at any distance x from B (between B and C),

$$M_x = R_b x = \frac{5W}{16}x$$

$$\text{At } C,\ M_c = \frac{5W}{16}\cdot\frac{l}{2} = \frac{5Wl}{32}$$

Bending moment at any distance x from B (between C and A),

$$M_x = \frac{5W}{16}x - W\left(x - \frac{l}{2}\right) = \frac{5W}{16}x - Wx + \frac{Wl}{2} = -\frac{11W}{16}x + \frac{Wl}{2}$$

$$\text{At A},\ M_{a(x=l)} = -\frac{3W}{16}$$

Point of inflection, $-\dfrac{11W}{16}x + \dfrac{Wl}{2} = 0$ or $x = 8l/11$

The bending moment diagram is shown in Fig. 8.36e.

Figure 8.36f shows the shear force diagram.

Alternate solution

- The reactions at A and B can also be found by assuming the moment at the fixed end A to be the redundant. The structure reduces to a simply supported beam with a point load at midspan (Fig. 8.37a).

  The slope at $A = -\dfrac{Wl^2}{16EI}$ (Eq. 7.16)

- Assume a unit clockwise couple at A, i.e., at the point of redundant moment (Fig. 8.37b).

  The slope at $A = -\dfrac{l}{3EI}$ (Eq. 7.7)

- Applying the compatibility equation, i.e., if M is the couple at the fixed end A, then for slope at the fixed end to be zero,

  $$-\frac{wl^2}{16EI} - M\cdot\frac{l}{3EI} = 0 \quad\text{or}\quad M = -\frac{3wl}{16}\ \text{(clockwise)}$$

  Taking moments about A, (assuming R_b to be upwards) (Fig. 8. 37c)

  or $\quad R_b \cdot l - \dfrac{wl^2}{2} + \dfrac{wl^2}{8} \quad$ or $\quad R_b = \dfrac{3wl}{8}$

  and $\quad R_a = wl - \dfrac{3wl}{8} = \dfrac{5wl}{8}$ (upwards)

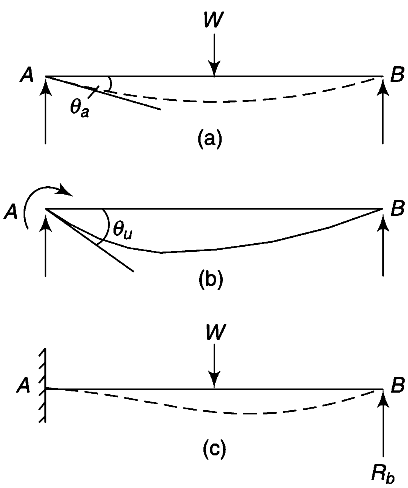


Fig. 8.37

Example 8.25 A beam of length l is fixed horizontally at both ends and carries a uniformly distributed load w over the whole span. Draw the bending moment and shear force diagrams.

Solution

Given A fixed end horizontal beam having uniformly distributed load as shown in Fig. 8.38a.

To find To draw bending moment and shear force diagrams

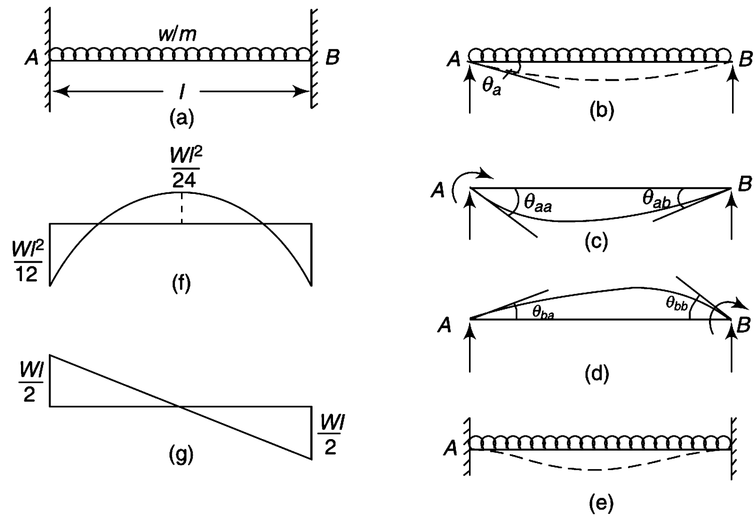


Fig. 8.38

The structure is shown in Fig. 8.38a. The structure involves two reactions and two couples at the fixed ends. The degree of redundancy of this structure is two, i.e., by removing two of the reactions or the couples, the structure reduces to a determinate form.

Removing the end couples

Let the moments at the fixed ends A and B be the redundant couples. The structure reduces to a simply supported beam with uniformly distributed load (Fig. 8.35a).

$$\text{The slope at } A = -\frac{3wl^3}{24EI} \qquad \text{(Eq. 7.24)}$$

Assuming unit couples at A and B

- Assume a unit clockwise couple at A, i.e., at the point of redundant moment (Fig. 8.38b).

$$\text{The slope at } A = -\frac{l}{3EI} \qquad \text{(Eq. 7.7)}$$

$$\text{and slope at } B = \frac{l}{6EI}$$

- Assume a unit clockwise couple at B, i.e., at the point of redundant moment (Fig. 8.38c).

$$\text{The slope at } A = \frac{l}{6EI} \qquad \text{(Eq. 7.7)}$$

$$\text{and slope at } B = -\frac{l}{3EI}$$

Compatibility equation

Applying the compatibility equation, i.e., if M_a is the clockwise couple at the fixed end A and M_b is the clockwise couple at the fixed end B, then for slope at the fixed end to be zero,

$$-\frac{wl^3}{24EI} - M_a\frac{l}{3EI} + M_b\frac{l}{6EI} = 0 \qquad \text{(i)}$$

and

$$\frac{wl^3}{24EI} + M_a\frac{l}{6EI} - M_b\frac{l}{3EI} = 0 \qquad \text{(ii)}$$

Adding (i) and (ii), $-M_a \dfrac{l}{6EI} - M_b \dfrac{l}{6EI} = 0$ or $M_a = -M_b$

From (i) $-\dfrac{wl^3}{24EI} + M_b \dfrac{l}{3EI} + M_b \dfrac{l}{6EI} = 0$

or $-\dfrac{wl^3}{24EI} + M_b \dfrac{3l}{6EI} = 0$ or $M_b = \dfrac{wl^2}{12}$ and $M_a = -\dfrac{wl^2}{12}$

Negative sign of M_a indicates that the direction of the couple at A is counter–clockwise and not clockwise as assumed. (Fig. 8.38d)

Reaction at each end = $\dfrac{wl}{2}$

Bending moment and shear force diagrams

Bending moment at any distance x from A,

$$M_x = \frac{wl}{2}\cdot x - \frac{wl^2}{12} - \frac{wx^2}{2}$$

At midspan C, $M_c = \dfrac{wl}{2}\cdot\dfrac{l}{2} - \dfrac{wl^2}{12} - \dfrac{wl^2}{8} = \dfrac{wl^2}{24}$

Point of inflection, $\dfrac{wl}{2}\cdot x - \dfrac{wl^2}{12} - \dfrac{wx^2}{2}$

$$6x^2 - 6lx + l^2 = 0 \quad \text{or} \quad x = 0.211l \quad \text{or} \quad x = 0.789l$$

The bending moment diagram is shown in Fig. 8.38e.
Figure 8.38f shows the shear force diagram.

Summary

1. A beam is called a fixed beam if both of its ends are rigidly fixed so that the end slopes remain horizontal. Such beams are also known as *built-in* or *encastre* beams. Owing to fixidity, the slope of the beam is zero at each end.
2. A fixed beam may be assumed as equivalent to a freely supported loaded beam subjected to end couples which make the slopes at the ends zero.
3. The bending moment diagram for free supports is known as *free moment diagram* and that for end couples as *fixing moment diagram.*
4. In area–moment method,

   Area of free moment diagram (A_1) = *Area of fixing moment diagram* (A_2)

   Also, if the deflection of one end relative to other is zero,

   Moment of area of free moment diagram = Moment of area of fixed moment diagram
5. Macaulay's Method can also be used to find the fixing moments, end reactions and the deflection of built-in beams.
6. When a beam is carried by more than two supports, it is known as a *continuous beam.*
7. Clapeyron's three-moment equation, general form

$$M_a \frac{l_1}{I_1} + 2M_b\left(\frac{l_1}{I_1} + \frac{l_2}{I_2}\right) + M_c \frac{l_2}{I_2} = -6\left(\frac{A_1\bar{x}_1}{I_1 l_1} + \frac{A_2\bar{x}_2}{I_2 l_2}\right) + 6E\left(\frac{\delta_c}{l_2} - \frac{\delta_a}{l_1}\right)$$

If the beam is uniform,

$$M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2 = -6\left(\frac{A_1\bar{x}_1}{l_1} + \frac{A_2\bar{x}_2}{l_2}\right) + 6EI\left(\frac{\delta_c}{l_2} - \frac{\delta_a}{l_1}\right)$$

If supports are at the same level, $M_a l_1 + 2M_b(l_1 + l_2) + M_c l_2 = -6\left(\frac{A_1\bar{x}_1}{l_1} + \frac{A_2\bar{x}_2}{l_2}\right)$

In case of uniformly distributed load, $6\left(\frac{A_1\bar{x}_1}{l_1} + \frac{A_2\bar{x}_2}{l_2}\right)$ is simplified to $\frac{w_1 l_1^3}{4} + \frac{w_2 l_2^2}{4}$

8. The moment distribution method is useful in the analysis of continuous beams.
9. The effect of applied moment at a joint on the other joints is known as the *carry over factor.*
10. *Stiffness* is the bending moment per unit slope of the span of a beam.
11. Distribution factor is the ratio of stiffness of a member of a joint to the combined stiffness of all the members of the joint.

Objective Type Questions

1. Maximum deflection of a fixed beam carrying a point load at the midspan

(a) $\frac{Wl^3}{192EI}$ (b) $\frac{Wl^3}{384EI}$ (c) $\frac{Wl^3}{48EI}$ (d) $\frac{Wl^3}{24EI}$

2. Maximum deflection of a fixed beam carrying a uniformly distributed load is

(a) $\frac{wl^4}{12EI}$ (b) $\frac{wl^4}{48EI}$ (c) $\frac{wl^4}{96EI}$ (d) $\frac{wl^4}{384EI}$

3. For a beam having a point load at the midspan, the ratio of the deflection when the two ends are simply supported to when they are fixed is

(a) 2 (b) 3 (c) 4 (d) 5

4. A continuous beam has

(a) one support (b) two supports
(c) more than two supports (d) very long span

5. Three–moment theorem for continuous beams was forwarded by

(a) Bernoulli (b) Clapeyron (c) Castigliano (d) Maxwell

Answers

1. (a) 2. (d) 3. (c) 4. (c) 5. (b)

Review Questions

8.1 What do you mean by a fixed beam? What is a continuous or propped beam? Why is it not possible to analyse these by equations of static equilibrium?

8.2 Discuss the effect of fixidity in case of a fixed beam.

8.3 In what way the moment-area is helpful in analyzing indeterminate beams. Explain.

8.4 State and deduce the Clapeyron's three-moment equation.

8.5 What are the merits and limitations of the theorem of three-moments?

8.6 Show that the maximum displacement in a fixed beam is one-fifth of that in a simply supported beam for a uniformly distributed load when both the beams have the same properties.

8.7 What is the point of contraflexture?

8.8 Find the maximum displacement of a fixed beam with a point load at the mid–span.

8.9 Draw the shear force and bending moment diagrams for a fixed beam when one of its supports sinks.
8.10 Draw the bending moment and shear force diagrams for a propped cantilever when the prop sinks.
8.11 What will be the maximum displacement of a two-span continuous beam with equal spans and subjected to a uniformly distributed load?

Numerical Problems

8.1 A-6 m long fixed beam carries a point load of 40 kN at the midspan. Determine the fixed end moments and the deflection under the load. $E = 205$ GPa and $I = 80 \times 10^6$ mm^4. (30 kN·m, 2.74 mm)

8.2 A 3-m long fixed end beam carries a point load of 40 kN at 2 m from the left end. Flexural rigidity of the beam is 8000 kN·m^2. Determine the fixed end moments, deflection under the load and the maximum deflection and its location. (8.89 kN·m, 17.78 kN·m, 0.494 mm, 0.533 mm, 1.714 m)

8.3 A 5-m long fixed end beam carries a uniformly distributed load of 6 kN per metre length over the entire span. Determine the fixing moments at the ends and deflection at the midspan. $E = 200$ GPa and $I = 40 \times 10^6$ mm^4. (12.5 kN · m, 1.22 mm)

8.4 Determine the fixing moments and the support reactions of a 6-m long beam which carries a load of 2 kN/m over the left half of span. (4.125 kN·m, 1.875 kN·m, 4.875 kN, 1.125 kN)

8.5 A beam rigidly fixed at the ends has a span of 15 m. It carries point loads of 15 kN at 3 m and 20 kN at 9 m from one end along with a uniformly distributed load of 8 kN/m over the whole span. The bending stress on the beam is limited to 105 MPa. Find the section modulus of the beam. Also sketch the bending moment diagram. (0.001 98 m^3)

8.6 A beam AB fixed at A and B is 9 m long and is propped at C distant 5 m from A. The beam carries a uniformly distributed load of 80 kN/m over the whole length. The prop at C sinks slightly such that the pressure at the prop is 200 kN. Calculate the fixing moments and the reactions at the supports A and B. (342.5 kN·m, 293.1 kN·m; 341.7 kN, 178.3 kN)

8.7 A built-in beam AB of length L carries a point load W at the midspan. The moment of inertia of the beam from the either end to a distance $l/4$ is I and for the rest of the beam it is $3I$. Find the end reactions and draw the bending moment diagram. ($3Wl/32$)

8.8 A fixed end beam has variable moment of inertia. For one half it is I_a and for the rest it is I_b. Show that the fixed end moments due to a concentrated load W applied at the midspan are $\frac{Wl}{2}\left[\frac{I_a(I_a+3I_b)}{I_a^2+I_b^2+14I_aI_b}\right]$ and $\frac{Wl}{2}\left[\frac{I_b(3I_a+I_b)}{I_a^2+I_b^2+14I_aI_b}\right]$

8.9 A built-in beam is of length l. It is loaded by two concentrated loads W each at distances $l/3$ and $2l/3$ from the left-hand end support. Find the support moments and draw the bending moment and shear force diagrams. Also find the deflection at the centre. ($2Wl/9$, $5Wl^3/648EI$)

8.10 A built-in beam of 6 m span carries a varying distributed load, the intensity of which varies from 20 kN at one end to 40 kN at the other end. Calculate the fixing moments and the reactions at the ends. (84 kN·m, 96 kN·m; 102 kN, 138 kN)

8.11 A built-in beam has a span of 22 m. It carries a load of 120 kN at 16 m from the left-hand end along with a uniformly distributed load of 10 kN/m upto 12 m from the same end of the beam. Determine the end reactions and the fixing moments. Also find the magnitude and the position of the maximum deflection. $E = 205$ Ga and $I = 800 \times 10^6$ mm^4. (115.9 kN, 104.1 kN; 446 kN·m, 196.2 kN·m; 45.1 mm, 11.5 m)

8.12 A continuous beam ABC consists of two spans, AB 5 m long and BC 7 m long. The span AB carries a point load of 100 kN at 3 m from A whereas the span BC has a point load of 140 kN at 4 m from C. Calculate the moments and the reactions at the supports. (0, 150 kN·m, 0; 10 kN, 200 kN, 30 kN)

8.13 A continuous beam ABC consists of two spans. Ends A and C are simply supported. The span AB is 4 m long and has a moment of inertia I and the span BC is 5 m long with a moment of inertia $2I$. The loading on span AB is 6 kN/m and on span BC 10 kN/m. Determine the support moments at A, B and C and draw shear force and bending moment diagrams.

(0, 25.5 kN·m, 0; 5.625 kN, 48.475 kN, 19.9 kN)

8.14 A continuous beam has three equally spaced supports with two overhangs at the two ends. The whole beam carries a uniformly distributed load. The overhangs are so selected that the reactions of the three supports are equal. Determine the length of each overhang in terms of the spacing between the supports. (0.44 of the spacing)

8.15 A straight uniform beam is of length $4l$. It is freely supported at its ends and at two intermediate supports each distant l from the either end. The end supports are unyielding while the intermediate supports are such that they deflect by an amount α for each unit of load on them. The whole beam carries a uniformly distributed load of w per unit length. Show that the reactions at the supports are

$$\frac{wl}{8}\left(\frac{7l^3 + 48EI\alpha}{4l^3 + 3EI\alpha}\right) \text{ and } \frac{wl}{8}\left(\frac{57l^3}{4l^3 + 3EI\alpha}\right).$$

8.16 A horizontal continuous beam ABC is fixed at C and simply supported at A and B. The lengths AB and BC are 5 m and 6 m respectively. A point load of 50 kN acts at a distance of 3 m from A and another of 70 kN at a distance of 3 m from C. Find the reactions on the supports and draw the shear force and bending moment diagrams. (9.98 kN, 83.37 kN, 26.65 kN)

8.17 Draw the bending moment diagram for a beam simply supported at A, B and C. The span lengths AB and BC are 12 m and 8 m with moments of inertia I and $1.5I$ respectively. The beam carries a uniformly distributed load of 4 kN/m throughout. Use moment distribution method.

(At B 48 kN·m, mid of AB 72 kN·m, mid of BC 32 kN·m)

8.18 Solve Example 8.12 by moment distribution method.

8.19 Solve Example 8.16 by moment distribution method.

8.20 A continuous beam $ABCDE$ has roller supports at B, C and D, a fixed support at A and an overhang at DE. The span lengths are $AB = BC = CD = 4$ m and $DE = 2$ m. The beam carries a uniformly distributed load of 3 kN/m over the spans $ABCD$ and of 1.5 kN/m over DE. A point load of 6 kN is also placed midway between B and C. Draw the bending moment diagram using moment distribution method. Mention the salient values.

(At A 3.43 kN·m, at supports 5.25 kN·m, 5.82 kN·m,3 kN·m;
mid of AB 6 kN·m, mid of BC 12 kN·m, mid of CD 6 kN·m)

8.21 A continuous beam ABC rigidly fixed at A have spans AB and BC of 6 m and 4 m length respectively. It carries a uniformly distributed load of 60 kN/m on the whole length. Draw the bending moment diagram for the beam if the support B sinks by 8 mm as compared to A and C. Take $E = 205$ MPa and $I = 400 \times 10^6$ mm^4. Use moment distribution method.

(At A 197.3 kN·m, At B 32.7 kN·m, mid of AB 180 kN·m, mid of BC 80 kN·m)

Chapter 9

Bending of Curved Bars

So far, analysis of stresses due to bending has been restricted to straight bars and beams. However, many times there are cases of initially curved beams or bars which are subjected to bending moments tending to increase or decrease the initial curvature. The expressions for stresses and deflections for such cases are being developed in this chapter.The discussion will be limited to curved members of uniform cross-section having a plane of symmetry in which the bending couples are applied. It is also assumed that the stresses remain within proportional limit.

If the radius of curvature of a curved bar is large as compared to the depth of cross-section or the initial curvature is small, the analysis is similar to as for pure bending. However, if the dimensions of the cross-section of the bar and the radius of curvature are of the same order of magnitude, a different method of anaysis has to be followed.

9.1 BARS OF SMALL INITIAL CURVATURE

If the curvature of a curved bar is small, i.e., radius of curvature is large compared with the dimensions of the cross-section, the analysis is similar to that for pure bending.

Let R and R' be the radii of curvature of the neutral surface before and after bending respectively under the action of pure bending moment M (Fig. 9.1). Assume that the plane sections remain plane after bending.

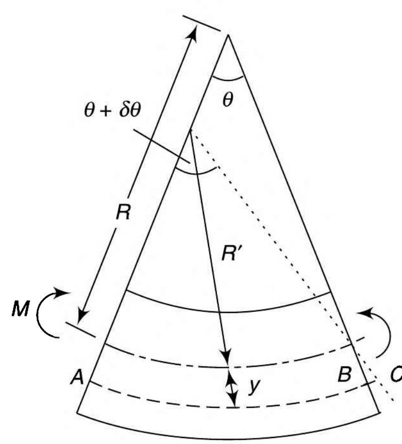


Fig. 9.1

Consider an element at a distance y from the neutral axis.

$$\text{Strain in the element} = \frac{AC - AB}{AB} = \frac{(R' + y)(\theta + \delta\theta) - (R + y)\theta}{(R + y)\theta}$$

$$= \frac{R'(\theta + \delta\theta) + y\theta + y\delta\theta - R\theta - y\theta}{(R + y)\theta}$$

$$= \frac{R'(\theta + \delta\theta) - R\theta + y\delta\theta}{(R + y)\theta}$$

As length of the neutral axis remains constant, $R'(\theta + \delta\,\theta) = R\theta$ and $\delta\theta = \dfrac{R - R'}{R'}\theta$

$$\text{Strain} = \frac{y\delta\theta}{(R+y)\theta} = \frac{y(R-R')\theta}{(R+y)\theta \cdot R'}$$

As curvature is small and thus y is very small as compared to R and thus can be neglected in the denominator,

$$\therefore \qquad \text{Strain} = \frac{y(R-R')}{RR'} = y\left(\frac{1}{R'} - \frac{1}{R}\right)$$

Normal stress (neglecting the lateral stress), $\sigma = Ey\left(\dfrac{1}{R'} - \dfrac{1}{R}\right)$ (i)

As total normal force is to be zero,

$$\int \sigma \cdot dA = E\left(\frac{1}{R'} - \frac{1}{R}\right)\int y \cdot dA = 0 \quad \text{or} \quad \int y \cdot dA = 0$$

which indicates that the neutral axis passes through the centroid of the section.

$$\text{Moment of resistance, } M = \int \sigma y dA = E\left(\frac{1}{R'} - \frac{1}{R}\right)\int y^2 \cdot dA = EI\left(\frac{1}{R'} - \frac{1}{R}\right) \qquad \text{(ii)}$$

$$\text{From (i) and (ii), } \frac{M}{I} = \frac{\sigma}{y} = E\left(\frac{1}{R'} - \frac{1}{R}\right) \qquad (9.1)$$

Strain energy of a small length δs along the neutral axis under the action of bending moment M,

$$\delta U = \frac{1}{2}M \cdot \delta\theta = \frac{1}{2}M \cdot \frac{R-R'}{R'}\theta = \frac{1}{2}(M\theta)R\left(\frac{1}{R'} - \frac{1}{R}\right) = \frac{1}{2}M\delta s\frac{M}{EI} = \frac{M^2}{2EI}\delta s \qquad (9.2)$$

Example 9.1 A piston ring is required to keep its outer surface circular in the stressed as well as in the unstressed condition when a uniform radial pressure is exerted. Express the variation of thickness of its surface along the angular direction of the piston in terms of maximum thickness. Also, deduce an expression to express initial radius in terms of final radius, maximum thickness, maximum bending stress and the modulus of elasticity.

Solution

Given A piston ring acted upon by a uniform radial pressure as shown in Fig. 9.2

To find

(i) To express variation of surface thickness in terms of maximum thickness

(ii) To express initial radius in terms of final radius, maximum thickness, maximum bending stresses and modulus of elasticity

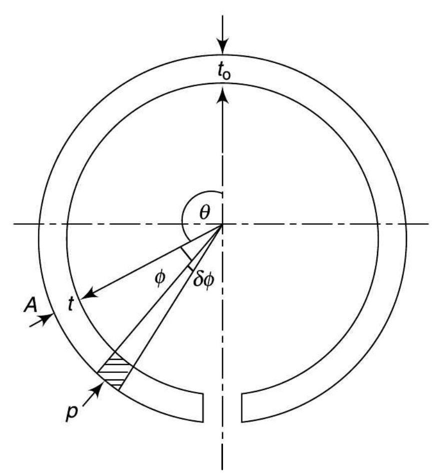


Fig. 9.2

Let R and R' be the radii of curvature of the neutral surface before and after bending respectively under the action of pure bending moment M.

Variation of thickness of the surface

If p is the uniform pressure on the outer surface,

Bending moment at a point A, $M = \int_0^{\pi-\theta} p[b(R'd\varphi)]\,R\sin\varphi$

$$= pbR'^2\int_0^{\pi-\theta}\sin\varphi \cdot d\varphi = pbR'^2(1+\cos\theta)$$

where b is the axial width of the ring.

If outer surface should remain circular in the stressed as well as unstressed condition,

Then $\dfrac{M}{I} = E\left(\dfrac{1}{R'} - \dfrac{1}{R}\right)$ = constant

or $$\frac{pbR'^2(1+\cos\theta)}{bt^3/12} = \frac{12pR'^2(1+\cos\theta)}{t^3} = C \quad \text{(i)}$$

When $\theta = 0°$, $t = t_o$ and thus $\dfrac{24pR'^2}{t_o^3} = C$

From (i), $$\frac{12pR'^2(1+\cos\theta)}{t^3} = \frac{24pR'^2}{t_o^3}$$

Thus t in terms of maximum thickness,

$$t = \sqrt[3]{\frac{1+\cos\theta}{2}} \cdot t_o$$

which gives the variation of thickness of the surface.

Expression for initial radius

Maximum bending stress at any section $= \dfrac{M}{I}\cdot\dfrac{t}{2}$

$$= \frac{12pR'^2(1+\cos\theta)}{t^3}\cdot\frac{t}{2}$$

$$= \frac{6pR'^2(1+\cos\theta)}{t^2}$$

Maximum value is at $\theta = 0°$, where $t = t_o$, and $\therefore\ \sigma_o = \dfrac{12pR'^2}{t_o^2}$

Initial radius can be found from

$$\frac{1}{R'} - \frac{1}{R} = \frac{\sigma}{Ey} \quad \text{...(Eq. 9.1)}$$

or $$\frac{1}{R'} - \frac{1}{R} = \frac{\sigma_o}{Ey_o} = \frac{12pR'^2}{t_o^2}\cdot\frac{1}{E\cdot t_o/2} = \frac{24pR'^2}{Et_o^3}$$

or $$\frac{1}{R} = \frac{1}{R'} - \frac{24pR'^2}{Et_o^3}$$

$$= \frac{1}{R'}\left(1 - \frac{24pR'^3}{Et_o^3}\right)$$

$$= \frac{1}{R'}\left(1 - \frac{24R'^3}{Et_o^3}\cdot\frac{\sigma_o t_o^2}{12R'^2}\right)$$

$$= \frac{1}{R'}\left(1 - \frac{2\sigma_o R'}{Et_o}\right)$$

The initial radius can be found from this expression if the values of final radius, maximum thickness, the maximum bending stress and the modulus of elasticity are known.

9.2 BARS OF LARGE INITIAL CURVATURE (WINKLER–BACH THEORY)

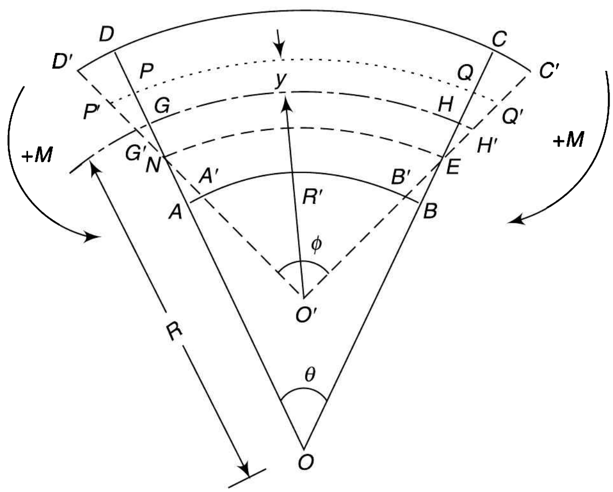


Fig. 9.3

The following assumptions are made while making the analysis:

1. Limit of proportionality does not exceed.
2. Plane transverse sections remain plane after bending.
3. The material is isotropic and obeys Hooke's law.
4. There are no radial strains.
5. No buckling failure takes place.

Let a small length *ABCD* of an initially curved beam take the shape *A′B′C′D′* after bending when a bending moment of magnitude *M* is applied (Fig. 9.3). *M* is considered positive when it further tends to increase the curvature of the beam. Let *GH* be the centroidal axis and *PQ* be any fibre at a radial distance y from it. Let *NE* be the neutral axis, i.e., no change in the length of its surface. Also, let θ and φ be the angles subtended by the small length at the centres of curvature before and after applying the bending moment.

Strain in the fibre *PQ*, $\varepsilon = \dfrac{P'Q' - PQ}{PQ} = \dfrac{(R' + y)\varphi - (R + y)\theta}{(R + y)\theta} = \dfrac{(R' + y)\varphi}{(R + y)\theta} - 1$

or
$$\frac{\varphi}{\theta} = (1 + \varepsilon)\frac{R + y}{R' + y} \qquad \text{(i)}$$

Strain in the fibre *GH* (centroidal axis), $\varepsilon' = \dfrac{G'H' - GH}{GH} = \dfrac{R'\varphi - R\theta}{R\theta} = \dfrac{R'\varphi}{R\theta} - 1$

or
$$\frac{\varphi}{\theta} = (1 + \varepsilon')\frac{R}{R'} \qquad \text{(ii)}$$

From (i) and (ii), $(1 + \varepsilon)\dfrac{R + y}{R' + y} = (1 + \varepsilon')\dfrac{R}{R'}$

or $$\varepsilon = (1+\varepsilon')\frac{(R'+y)/R'}{(R+y)/R} - 1 = (1+\varepsilon')\frac{1+y/R'}{1+y/R} - 1$$

$$= (1+\varepsilon')\frac{1+y/R'}{1+y/R} - (1+\varepsilon') + \varepsilon'$$ (Adding and subtracting ε' on RHS)

$$= (1+\varepsilon')\left(\frac{1+y/R'}{1+y/R} - 1\right) + \varepsilon'$$

$$= (1+\varepsilon')\left(\frac{1+y/R' - 1 - y/R}{1+y/R}\right) + \varepsilon'$$

$$= \varepsilon' + (1+\varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right)\frac{y}{1+y/R} \tag{9.3}$$

The resisting moment about an axis through the centroid,

$$M = \int \sigma\, dA \cdot y = E\int \varepsilon \cdot y \cdot dA$$

$$= E\int \varepsilon' y dA + E\int (1+\varepsilon')\frac{y^2}{1+y/R}\cdot\left(\frac{1}{R'} - \frac{1}{R}\right) dA$$

$$= E\varepsilon' \int y dA + E(1+\varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right)\int \frac{y^2}{1+y/R}\, dA \tag{iii}$$

Moments of total area about centroidal axis is zero, $\therefore \int y \cdot dA = 0$

Thus (iii) becomes, $$M = E(1+\varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right)\int \frac{y^2}{1+y/R} dA \tag{9.4}$$

Let $$\int \frac{y^2}{1+y/R} dA = Ap^2 \tag{9.5}$$

$\therefore$ $$M = E(1+\varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right)Ap^2 \tag{9.6}$$

or $$E(1+\varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right) = \frac{M}{Ap^2} \tag{9.6a}$$

Now, $$Ap^2 = \int \frac{y^2}{1+y/R} dA$$

$$= \int \frac{yR + y^2 - yR}{1+y/R} dA$$

$$= \int \frac{yR(1+y/R) - yR}{1+y/R} dA$$

$$= R\int \left(y - \frac{y}{1+y/R}\right) dA$$

$$= R\int ydA - R\int \frac{y}{1+y/R}dA$$

$$= 0 - R\int \frac{y}{1+y/R}dA$$

$$= -R\int \frac{y}{1+y/R}dA$$

or $$\int \frac{y}{1+y/R}dA = -\frac{Ap^2}{R} \tag{9.7}$$

Total force on cross-section $A'D'$ or $B'C'$ in the circumferential direction,

$$F = \int\sigma \cdot dA = E\int\varepsilon \cdot dA$$

$$= E\left[\int \varepsilon' dA + \int (1+\varepsilon')\frac{y}{1+y/R}\cdot\left(\frac{1}{R'}-\frac{1}{R}\right)dA\right] \quad \text{(Using Eq. 9.3)}$$

$$= E\varepsilon' A - E(1+\varepsilon')\left(\frac{1}{R'}-\frac{1}{R}\right)\frac{Ap^2}{R} \quad \text{(Using Eq. 9.7)} \tag{9.7a}$$

As transverse plane sections before bending remain plane after bending, $F = 0$

$$0 = E\varepsilon' A - E(1+\varepsilon')\left(\frac{1}{R'}-\frac{1}{R}\right)\frac{Ap^2}{R}$$

Dividing throughout by $EA(1+\varepsilon')$

or $$\frac{\varepsilon'}{1+\varepsilon'} = \left(\frac{1}{R'}-\frac{1}{R}\right)\frac{p^2}{R} \tag{9.8}$$

Now, $$\sigma = E\varepsilon = E\left[\varepsilon' + (1+\varepsilon')\frac{y}{1+y/R}\cdot\left(\frac{1}{R'}-\frac{1}{R}\right)\right] \quad \text{(Using Eq. 9.3)}$$

$$= E(1+\varepsilon')\left[\frac{\varepsilon'}{1+\varepsilon'} + \frac{y}{1+y/R}\cdot\left(\frac{1}{R'}-\frac{1}{R}\right)\right]$$

$$= E(1+\varepsilon')\left[\left(\frac{1}{R'}-\frac{1}{R}\right)\frac{p^2}{R} + \frac{y}{1+y/R}\cdot\left(\frac{1}{R'}-\frac{1}{R}\right)\right] \quad \text{(Using Eq. 9.8)}$$

$$= E(1+\varepsilon')\left(\frac{1}{R'}-\frac{1}{R}\right)\left[\frac{p^2}{R} + \frac{y}{1+y/R}\right]$$

$$= \frac{M}{Ap^2}\left[\frac{p^2}{R} + \frac{Ry}{R+y}\right] \quad \text{(Using 9.6a)}$$

$$= \frac{M}{AR}\left[1 + \frac{R^2}{p^2}\cdot\frac{y}{R+y}\right] \tag{9.9}$$

At the inside of the centroidal axis, y is negative and thus,

$$\sigma = \frac{M}{AR}\left[1 - \frac{R^2}{p^2}\frac{y}{R-y}\right]$$

The position of the neutral axis can be found from the fact that at the neutral axis, $\sigma = 0$,

$$\sigma = \frac{M}{AR}\left[1 + \frac{R^2}{p^2}\frac{y}{R+y}\right] = 0$$

or $$yR^2 = -Rp^2 - yp^2 \text{ or } y = \frac{-Rp^2}{R^2 + p^2} \tag{9.10}$$

p^2 for different sections can be evaluated easily by the following simplified expression:

$$p^2 = \frac{1}{A}\int \frac{Ry^2}{R+y}dA = \frac{1}{A}\int\left(Ry - \frac{R^2 y}{R+y}\right)dA = \frac{1}{A}\int\left[Ry - R^2\left(1 - \frac{R}{R+y}\right)\right]dA$$

$$= \frac{R}{A}\left[\int ydA - \int RdA + \int \frac{R^2}{R+y}dA\right] = \frac{R}{A}\left[0 - RA + \int \frac{R^2}{R+y}dA\right]$$

$$= \frac{R^3}{A}\int \frac{1}{R+y}dA - R^2 \tag{9.11}$$

9.3 VALUES OF p^2 FOR VARIOUS SECTIONS

Rectangular Section (Fig. 9.4)

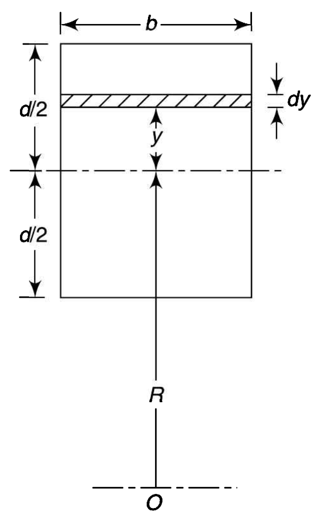


Fig. 9.4

$A = bd$ and $dA = b.dy$

$$\therefore \quad p^2 = \frac{R^3}{bd}\int_{-d/2}^{d/2}\frac{1}{R+y}b \cdot dy - R^2 = \frac{R^3}{d}\ln\left(\frac{R+d/2}{R-d/2}\right) - R^2 = \frac{R^3}{d}\ln\left(\frac{2R+d}{2R-d}\right) - R^2 \tag{9.12}$$

Trapezoidal Section (Fig. 9.5)

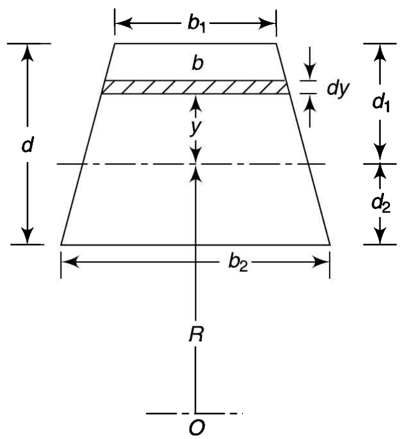


Fig. 9.5

$$dA = b \cdot dy = \left[b_1 + \frac{b_2 - b_1}{d}(d_1 - y) \right] dy$$

$$\int \frac{dA}{R+y} = \int_{-d_2}^{d_1} \left[\frac{b_1 + \dfrac{(b_2 - b_1)(d_1 - y)}{d}}{R+y} \right] dy = \int_{-d_2}^{d_1} \left[\frac{b_1 + \dfrac{(b_2 - b_1)\, d_1}{d} - \dfrac{(b_2 - b_1)\, y}{d}}{R+y} \right] dy$$

$$= \int_{-d_2}^{d_1} \left[\frac{b_1 + \dfrac{(b_2 - b_1) d_1}{d} + \dfrac{(b_2 - b_1) R}{d} - \dfrac{(b_2 - b_1) R}{d} - \dfrac{(b_2 - b_1)\, y}{d}}{R+y} \right] dy$$

[adding and subtracting $(b_2 - b_1)R/d$ in the numerator]

$$= \int_{-d_2}^{d_1} \left[\frac{\left(b_1 + \dfrac{(b_2 - b_1) d_1}{d} + \dfrac{(b_2 - b_1) R}{d} \right)}{R+y} \right] dy - \int_{-d_1}^{d_2} \frac{(b_2 - b_1)}{d} \frac{R+y}{R+y} dy$$

$$= \left(b_1 + \frac{(b_2 - b_1) d_1}{d} + \frac{(b_2 - b_1) R}{d} \right) \ln \frac{R + d_1}{R - d_1} - \frac{(b_2 - b_1)}{d} (d_1 + d_2)$$

$$= \left(b_1 + \frac{(b_2 - b_1)}{d} (R + d_1) \right) \ln \frac{R + d_1}{R - d_1} - (b_2 - b_1) \qquad (d = d_1 + d_2)$$

Thus $$p^2 = \frac{R^3}{A} \left[\left(b_1 + \frac{(b_2 - b_1)}{d} (R + d_1) \right) \ln \frac{R + d_1}{R - d_2} - (b_2 - b_1) \right] - R^2 \tag{9.13}$$

In this equation, $A = \dfrac{b_1 + b_2}{2} \cdot d, \quad d_1 = \dfrac{b_2 + 2b_1}{b_1 + b_2} \cdot \dfrac{d}{3}, \quad d_2 = \dfrac{2b_2 + b_1}{b_1 + b_2} \cdot \dfrac{d}{3} = d - d_1$

Triangular Section (Fig. 9.6)

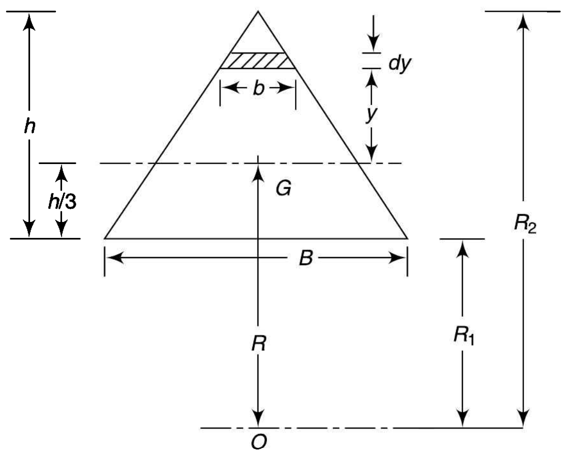


Fig. 9.6

Let $R + y = z, \quad \therefore dy = dz,$

$$dA = b \cdot dy = \frac{B}{h}(R_2 - z) \cdot dz$$

$$\int \frac{dA}{R+y} = \int_{R_1}^{R_2} \frac{B(R_2 - z)}{h} \cdot \frac{dz}{z} = \frac{B}{h}\left(\int_{R_1}^{R_2} R_2 \cdot \frac{dz}{z} - \int_{R_1}^{R_2} dz\right) = \frac{B}{h}\left[R_2 \ln \frac{R_2}{R_1} - (R_2 - R_1)\right]$$

$$p^2 = \frac{R^3}{bh/2} \cdot \frac{B}{h}\left[\left(R + \frac{2h}{3}\right)\ln\frac{(R + 2h/3)}{(R - h/3)} - h\right] - R^2$$

$$= \frac{2R^3}{h}\left[\left(\frac{3R + 2h}{3h}\right)\ln\frac{(3R + 2h)}{(3R - h)} - 1\right] - R^2 \tag{9.14}$$

Circular Cross-section (Fig. 9.7)

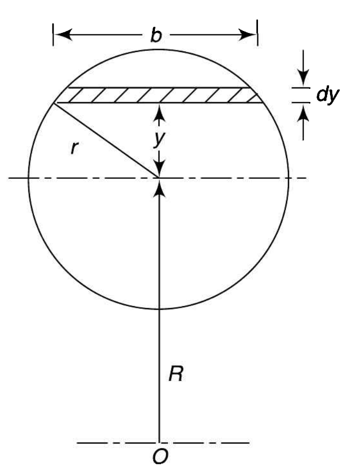


Fig. 9.7

$$dA = bdy = 2\sqrt{r^2 - y^2} \cdot dy$$

$$\int \frac{dA}{R+y} = 2\int_{-r}^{r} \frac{2\sqrt{r^2 - y^2}}{R+y}\, dy = 2\pi(R - \sqrt{R^2 - r^2}) \qquad \text{(by calculus methods)}$$

$$p^2 = \frac{R^3}{A}[2\pi(R - \sqrt{R^2 - r^2})] - R^2 = \frac{2\pi R^3}{\pi r^2}[R - (R^2 - r^2)^{1/2}] - R^2$$

$$= \frac{2R^3}{r^2}\left[R - R\left(1 - \frac{r^2}{R^2}\right)^{1/2}\right] - R^2 = \frac{2R^3}{r^2}\left[R - R\left(1 - \frac{r^2}{2R^2} - \frac{r^4}{8R^4} - \frac{r^6}{16R^6} + \ldots\right)\right] - R^2$$

$$= \frac{2R^3}{r^2}\left[\frac{r^2}{2R} + \frac{r^4}{8R^3} + \frac{r^6}{16R^5} + \ldots\right] - R^2 = R^2 + \frac{r^2}{4} + \frac{r^4}{8R^2} + \ldots - R^2 = \frac{r^2}{4} + \frac{r^4}{8R^2} + \ldots$$

Also the expression may be put in terms of d,

$$p^2 = \frac{d^2}{16} + \frac{d^4}{128R^2} + \ldots \tag{9.15}$$

I-Section (Fig. 9.8)

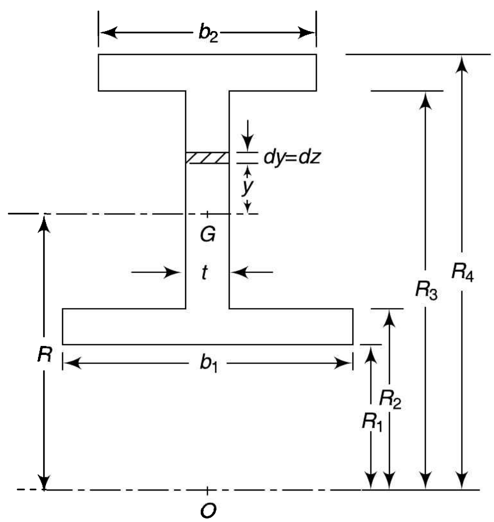


Fig. 9.8

Let $R + y = z$, $\quad \therefore\ dy = dz$

$$p^2 = \frac{R^3}{A}\left[\int_{R_1}^{R_2} \frac{dA}{R+y} + \int_{R_2}^{R_3} \frac{dA}{R+y} + \int_{R_3}^{R_4} \frac{dA}{R+y}\right] - R^2$$

$$= \frac{R^3}{A}\left[\int_{R_1}^{R_2} \frac{b_1 \cdot dz}{z} + \int_{R_2}^{R_3} \frac{t \cdot dz}{z} + \int_{R_3}^{R_4} \frac{b_2 \cdot dz}{z}\right] - R^2$$

$$= \frac{R^3}{A}\left[b_1 \ln\frac{R_2}{R_1} + t \ln\frac{R_3}{R_2} + b_2 \ln\frac{R_4}{R_3}\right] - R^2 \tag{9.16}$$

T-section (Fig. 9.9)

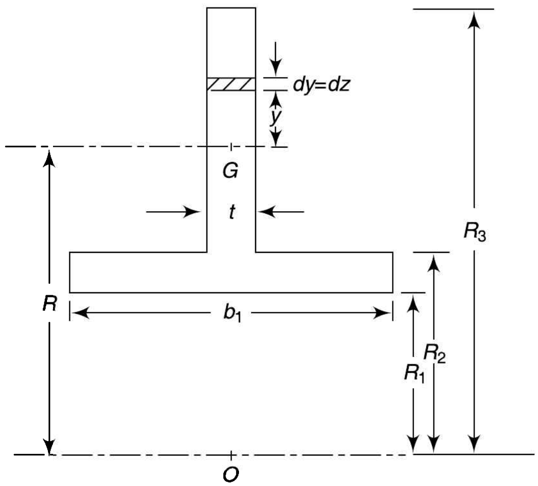


Fig. 9.9

Let $R + y = z, \quad \therefore dy = dz$

$$p^2 = \frac{R^3}{A}\left[\int_{R_1}^{R_2} \frac{dA}{R+y} + \int_{R_2}^{R_3} \frac{dA}{R+y}\right] - R^2$$

$$= \frac{R^3}{A}\left[\int_{R_1}^{R_2} \frac{b_1 \cdot dz}{z} + \int_{R_2}^{R_3} \frac{t \cdot dz}{z}\right] - R^2 = \frac{R^3}{A}\left[b_1 \ln\frac{R_2}{R_1} + t \ln\frac{R_3}{R_2}\right] - R^2 \tag{9.17}$$

Example 9.2 A curved bar of rectangular section of 30-mm width, 40-mm depth and mean radius of curvature of 60 mm is initially unstressed. If a bending moment of 400 N.m is applied to the bar which tends to straighten it, determine the stresses at the inner and outer surfaces and sketch a diagram to show the variation of stress across the section. Also find the position of the neutral axis.

Solution

Given A curved bar of rectangular section

$R = 60$ mm $\qquad M = 400$ N.m

$d = 40$ mm $\qquad w = 30$ mm

To find

(i) Stresses at inner and outer surfaces

(ii) Position of neutral axis

(iii) To sketch variation of stress across the section

$A = 40 \times 30 = 1200\ \text{mm}^2$

Calculating p^2

$$p^2 = \frac{R^3}{d}\ln\left(\frac{2R+d}{2R-d}\right) - R^2 = \frac{60^3}{40}\ln\left(\frac{2\times 60+40}{2\times 60-40}\right) - 60^2 = 143\text{ mm}^2$$

Bending moment is negative as it tends to straighten the bar. y is negative at inside and positive at outside.

Stress at outside face

$$\sigma_o = \frac{M}{AR}\left[1+\frac{R^2}{p^2}\frac{y}{R+y}\right] = \frac{-400\,000}{(40\times 30)\times 60}\left[1+\frac{60^2}{143}\cdot\frac{20}{60+20}\right]$$

$$= -5.556(1+6.294) = -40.5\text{ MPa} \qquad \text{(compressive)}$$

Stress at inside face

$$\sigma_i = \frac{-M}{AR}\left[1-\frac{R^2}{p^2}\frac{y}{R-y}\right] = \frac{-400\,000}{(40\times 30)\times 60}\left[1-\frac{60^2}{143}\cdot\frac{20}{60-20}\right]$$

$$= -5.556(1-12.587) = 64.4\text{ MPa} \qquad \text{(tensile)}$$

Position of the neutral axis

$$y = \frac{-Rp^2}{R^2+p^2} = \frac{-60\times 143}{60^2+143} = -2.29\text{ mm}$$

Sketching the variation

Figure 9.10 shows the variation of stress across the section.

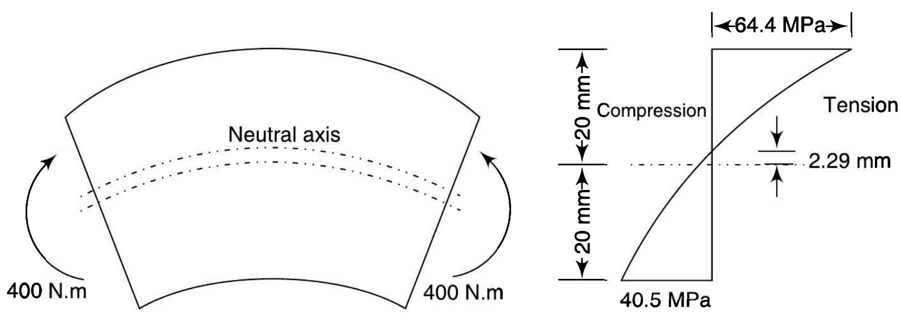


Fig. 9.10

Example 9.3 A ring having a saw cut along the horizontal diameter is made from a circular cross-section bar of 60-mm diameter. The inside diameter of the circular ring is 80 mm. It is subjected to a vertical compressive load of 15 kN. Find the stresses at the inside and the outside points along the horizontal section opposite to the saw cut. Also find the position of the neutral axis.

Solution

Given A circular ring with a saw cut along horizontal diameter subjected to a vertical compressive load as shown in Fig. 9.11

To find

(i) Stresses at P and Q

(ii) Position of neutral axis

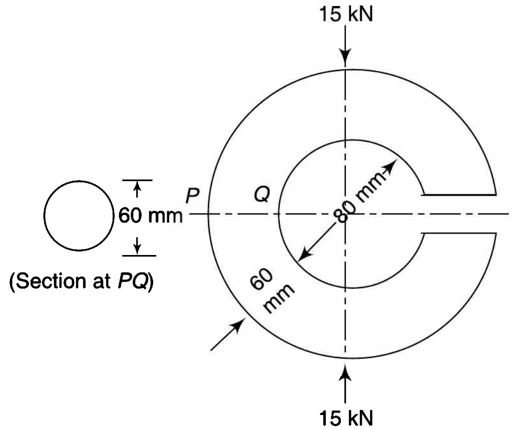


Fig. 9.11

$$R = 40+30 = 70 \text{ mm} \; ; A = \frac{\pi}{4} \times 60^2 = 2827.4 \text{ mm}^2$$

Calculating p^2

$$p^2 = \frac{d^2}{16} + \frac{d^4}{128R^2} + \ldots = \frac{60^2}{16} + \frac{1}{128} \cdot \frac{60^4}{70^2} = 245.7 \text{ mm}^2$$

As bending moment tends to increase the curvature, it is positive.

Resultant stress = Direct stress + Bending stress

Stress at outside face (P)

$$\sigma_o = -\frac{W}{A} + \frac{WR}{AR}\left[1 + \frac{R^2}{p^2}\frac{y}{R+y}\right] = \frac{W}{A} \cdot \frac{R^2}{p^2}\frac{y}{R+y} = \frac{15\,000}{2827.4} \cdot \frac{70^2}{245.7}\frac{30}{70+30}$$

$$= 105.82 \times 0.3 = 31.74 \text{ MPa} \quad \text{(tensile)}$$

Stress at inside face (Q)

$$\sigma_i = -\frac{W}{A} + \frac{WR}{AR}\left[1 - \frac{R^2}{p^2}\frac{y}{R-y}\right] = -\frac{W}{A} \cdot \frac{R^2}{p^2} \cdot \frac{y}{R-y}$$

$$= -\,105.82 \times \frac{30}{70-30} = 79.36 \text{ MPa} \quad \text{(compressive)}$$

Position of the neutral axis

Position of neutral axis = $y = \dfrac{-Rp^2}{R^2 + p^2} = \dfrac{-70 \times 245.7}{70^2 + 245.7} = -3.34$ mm

i.e., the neutral axis is at a radius of 70 – 3.34 = 66.66 mm

Example 9.4 An open ring has a *T*-section. It is subjected to a tensile load of 80 kN as shown in Fig. 9.12. Find the stresses at points *P* and *Q*.

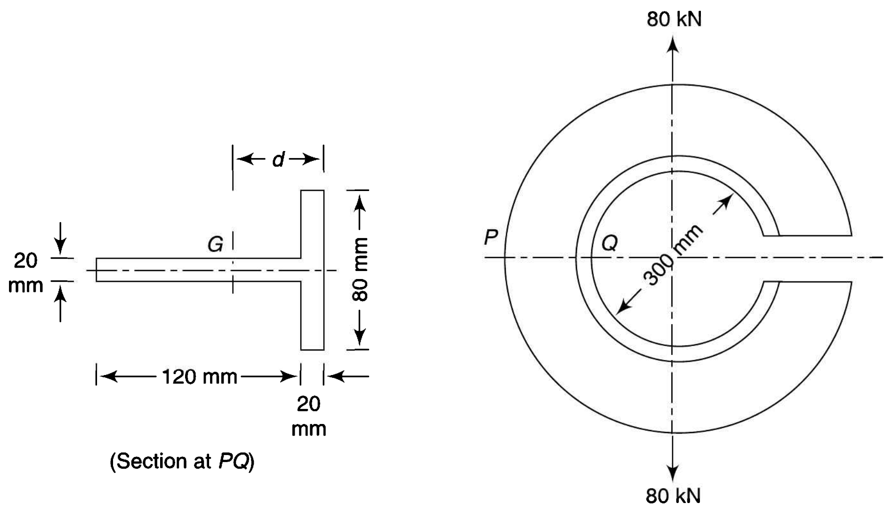


Fig. 9.12

Solution

Given An open ring of *T*-section subjected to a tensile load as shown in Fig. 9.12.

To find Stresses at *P* and *Q*

$$A = 80 \times 20 + 120 \times 20 = 4000 \text{ mm}^2.$$

$$d = \frac{80 \times 20 \times 10 + 120 \times 20 \times 80}{4000} = 52 \text{ mm}$$

Calculating p^2

$$R = 150 + 52 = 202 \text{ mm}$$

$$R_1 = 150 \text{ mm};\ R_2 = 170 \text{ mm and } R_3 = 150 + 20 + 120 = 290 \text{ mm}$$

$$p^2 = \frac{R^3}{A}\left[b_1 \ln\frac{R_2}{R_1} + t \ln\frac{R_3}{R_2}\right] - R^2 = \frac{202^3}{4000}\left[80 \ln\frac{170}{150} + 20 \ln\frac{290}{170}\right] - 202^2 = 1839.5 \text{ mm}^2$$

As bending moment tends to decrease the curvature, it is negative.

Stress at outside face (P)

$$y = 290 - 202 = 88 \text{ mm}$$

$$\sigma_o = \frac{W}{A} - \frac{WR}{AR}\left[1 + \frac{R^2}{p^2}\frac{y}{R+y}\right] = -\frac{W}{A}\cdot\frac{R^2}{p^2}\frac{y}{R+y} = -\frac{80\,000}{4000}\cdot\frac{202^2}{1839.5}\cdot\frac{88}{202+88}$$

$$= -443.64 \times 0.3034 = -134.6 \text{ MPa} \quad \text{(compressive)}$$

Stress at inside face (Q)

$$y = d = 52 \text{ mm}$$

$$\sigma_i = \frac{W}{A} - \frac{WR}{AR}\left[1 - \frac{R^2}{p^2}\frac{y}{R-y}\right] = \frac{W}{A}\cdot\frac{R^2}{p^2}\cdot\frac{y}{R-y} = 443.64 \times \frac{52}{202-52}$$

$$= 153.8 \text{ MPa} \quad \text{(tensile)}$$

Example 9.5 An open ring has an *I*-section as shown in Fig. 9.13. It is subjected to a load of 50 kN. Find the stresses at points *P* and *Q*.

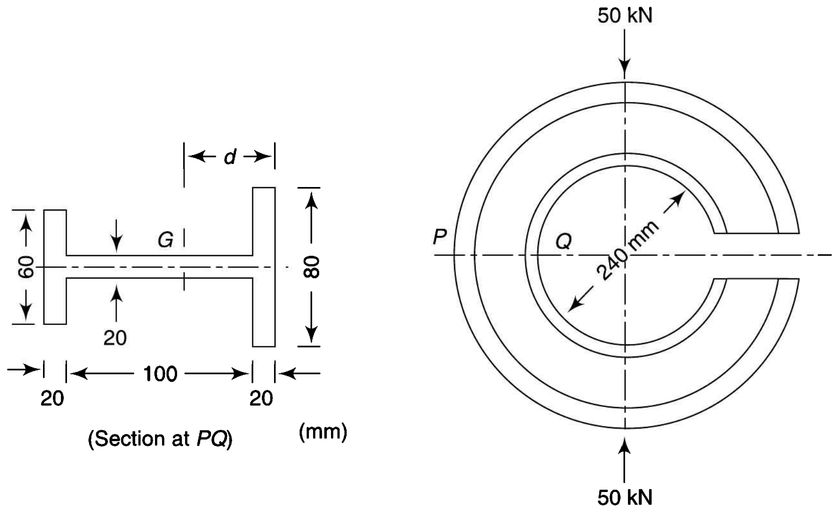


Fig. 9.13

Solution

Given An open ring of *I*-section subjected to a compressive load as shown in Fig. 9.13.
To find Stresses at *P* and *Q*

$$A = 80 \times 20 + 100 \times 20 + 60 \times 20 = 4800 \text{ mm}^2$$

$$d = \frac{80 \times 20 \times 10 + 100 \times 20 \times 70 + 60 \times 20 \times 130}{4800} = 65 \text{ mm}$$

Calculating p^2

$R = 120 + 65 = 185$ mm

$R_1 = 120$ mm; $R_2 = 140$ mm; $R_3 = 120 + 20 + 100 = 240$ mm and $R_4 = 240 + 20 = 260$ mm

$$p^2 = \frac{R^3}{A}\left[b_1 \ln\frac{R_2}{R_1} + t \ln\frac{R_3}{R_2} + b_2 \ln\frac{R_4}{R_3}\right] - R^2$$

$$= \frac{185^3}{4800}\left[80 \ln\frac{140}{120} + 20 \ln\frac{240}{140} + 60 \ln\frac{260}{240}\right] - 185^2$$

$$= 2597 \text{ mm}^2$$

Stress at outside face (P)

$$\sigma_o = -\frac{W}{A} + \frac{WR}{AR}\left[1 + \frac{R^2}{p^2}\frac{y}{R+y}\right] = \frac{W}{A}\cdot\frac{R^2}{p^2}\frac{y}{R+y} = \frac{50\,000}{4800}\cdot\frac{185^2}{2597}\frac{75}{185+75}$$

$$= 137.28 \times 0.2885 = 39.6 \text{ MPa} \quad \text{(tensile)}$$

Stress at inside face (Q)

$$\sigma_1 = -\frac{W}{A} + \frac{WR}{AR}\left[1 - \frac{R^2}{p^2}\frac{y}{R-y}\right] = -\frac{W}{A}\cdot\frac{R^2}{p^2}\cdot\frac{y}{R-y} = -137.28 \times \frac{65}{185-65}$$

$$= -74.36 \text{ MPa (compressive)}$$

Example 9.6 Determine the resultant stresses at P and Q of the frame shown in Fig. 9.14. Also find the location of the neutral axis.

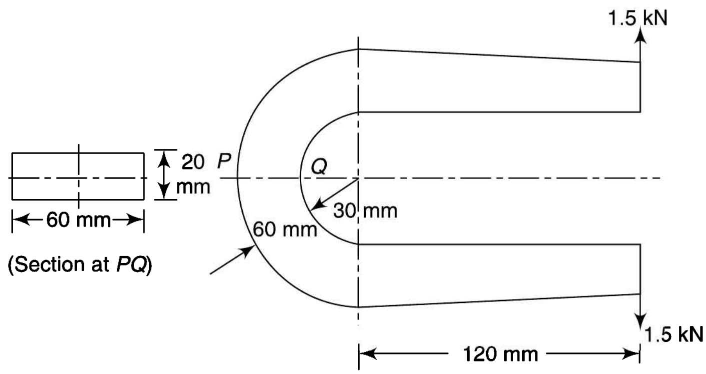


Fig. 9.14

Solution

Given A frame with loading as shown in Fig. 9.14.

To find

(i) Stresses at P and Q

(ii) Location of neutral axis

$A = 60 \times 20 = 1200\ \text{mm}^2$.

$d = 30$ mm ; $R = 30 + 30 = 60$ mm

$M = 1500 \times (120 + 60) = 270 \times 10^3\ \text{N} \cdot \text{mm}$

Calculating p^2

$$p^2 = \frac{R^3}{d}\ln\left(\frac{2R+d}{2R-d}\right) - R^2 = \frac{60^3}{60}\ln\left(\frac{2\times 60+60}{2\times 60-60}\right) - 60^2 = 355\ \text{mm}^2$$

Stress at outside face (P)

$$\sigma_o = \frac{W}{A} - \frac{M}{AR}\left[1 + \frac{R^2}{p^2}\frac{y}{R+y}\right] = \frac{1500}{1200} - \frac{270\times 10^3}{1200\times 60}\left[1 + \frac{60^2}{355}\cdot\frac{30}{60+30}\right]$$

$= 1.25 - 3.75 \times 4.38 = -15.18$ MPa (compressive)

Stress at inside face (Q)

$$\sigma_i = \frac{W}{A} - \frac{M}{AR}\left[1 - \frac{R^2}{p^2}\frac{y}{R-y}\right] = \frac{1500}{1200} - \frac{270\times 10^3}{1200\times 60}\left[1 - \frac{60^2}{355}\cdot\frac{30}{60-30}\right]$$

$= 1.25 - 3.75 \times (-9.14) = 35.52$ MPa (tensile)

Position of the neutral axis

$$y = \frac{-Rp^2}{R^2+p^2} = \frac{-60\times 355}{60^2+355} = -5.39\ \text{mm}$$

i.e., the neutral axis is at a radius of 60 – 5.39 = 54.61 mm

Example 9.7 A crane hook having a trapezoidal horizontal cross-section is 50 mm wide inside and 30 mm wide outside. Thickness of the section is 60 mm. The crane hook carries a vertical load of 20 kN whose line of action is 50 mm from the inside edge of the section. The centre of curvature is 60 mm from the inside edge. Determine the maximum tensile and compressive stresses in the section.

Solution

Given A crane hook of trapezoidal horizontal cross-section and loaded as shown in Fig. 9.15

To find Maximum tensile and compressive stresses

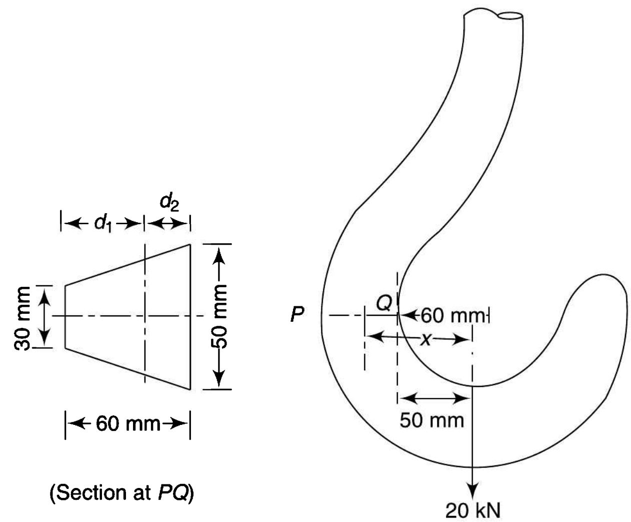


Fig. 9.15

$$A = \frac{50+30}{2} \times 60 = 2400 \text{ mm}^2;\ d = 60 \text{ mm};$$

$$d_2 = \frac{d}{3}\left(\frac{2b_1+b_2}{b_1+b_2}\right) = \frac{60}{3}\left(\frac{2\times 30+50}{30+50}\right) = 27.5 \text{ mm}$$

$$d_1 = 60 - 27.5 = 32.5 \text{ mm};$$

$$R = 60 + 27.5 = 87.5$$

Calculating p^2

$$p^2 = \frac{R^3}{A}\left[\left(b_1 + \frac{(b_2-b_1)}{d}(R+d_1)\right)\ln\frac{R+d_1}{R-d_2} - (b_2-b_1)\right] - R^2 \qquad \ldots\text{(Eq. 9.13)}$$

$$= \frac{87.5^3}{2400}\left[\left(30 + \frac{(50-30)}{60}(87.5+32.5)\right)\ln\frac{87.5+32.5}{87.5-27.5} - (50-30)\right] - 87.5^2$$

$$= 279.1[(30+40)\ln 2 - 20] - 87.5^2 = 303.77 \text{ mm}^2$$

Stress at outside face (P)

$$x = 50 + d_2 = 50 + 27.5 = 77.5 \text{ mm};\ y = d_1 = 32.5 \text{ mm}$$

$$\sigma_o = \frac{-Wx}{AR}\left[1 + \frac{R^2}{p^2}\frac{y}{R+y}\right] = -\frac{20\,000\times 77.5}{2400\times 87.5}\left[1 + \frac{87.5^2}{303.77}\cdot\frac{32.5}{87.5+32.5}\right] = -57.76 \text{ MPa}$$

Stress at inside face (Q)

$y = d_2 = 27.5$ mm

$$\sigma_o = \frac{-Wx}{AR}\left[1 - \frac{R^2}{p^2}\frac{y}{R-y}\right] = -\frac{20\,000 \times 77.5}{2400 \times 87.5}\left[1 - \frac{87.5^2}{303.77}\cdot\frac{27.5}{87.5 - 27.5}\right] = 77.88 \text{ MPa}$$

Direct stress

$$\text{Direct stress} = \frac{20\,000}{2400} = 8.33 \text{ MPa}$$

Maximum stresses

At the outside edge, $\sigma_o = -57.76 + 8.33 = -49.43$ MPa

At the inside edge, $\sigma_i = 77.88 + 8.33 = 86.21$MPa

9.4 STRESSES IN A CIRCULAR RING

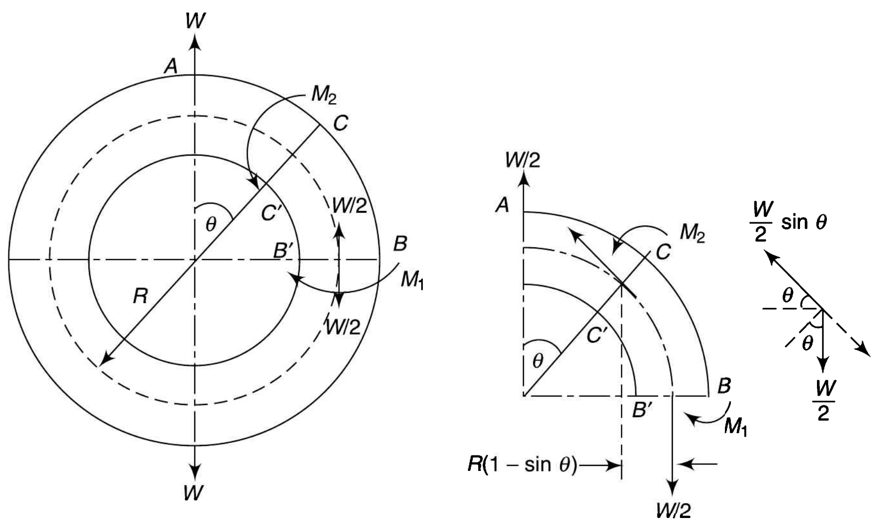


Fig. 9.16

Consider a circular ring acted upon by a tensile load W as shown in Fig. 9.16. At any section C-C' inclined at an angle θ with the load line, let the bending moment be M_2. The portion $BB'C'C$ of the ring is in equilibrium under the action of bending moment M_1 and pull $W/2$ at B-B' and the moment M_2 and pull $(W/2)\sin\theta$ at C-C'.

$$M_2 = M_1 + \frac{W}{2}R(1 - \sin\theta) \qquad \text{(i)}$$

Also, $$M_2 = E(1 + \varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right)Ap^2 \qquad \text{(Eq. 9.6)} \qquad \text{(ii)}$$

From (i) and (ii),

$$E(1 + \varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right)Ap^2 = M_1 + \frac{W}{2}R(1 - \sin\theta) \qquad (9.18)$$

Multiplying both sides by $Rd\theta$ and integrating from 0 to $\pi/2$,

$$E\int_0^{\pi/2}(1+\varepsilon')\left(\frac{1}{R'}-\frac{1}{R}\right)Ap^2\cdot Rd\theta=\int_0^{\pi/2}M_1\cdot Rd\theta+\int_0^{\pi/2}\frac{W}{2}R(1-\sin\theta)Rd\theta$$

or
$$E\left[\int_0^{\pi/2}\frac{(1+\varepsilon')R}{R'}Ap^2d\theta-\int_0^{\pi/2}(1+\varepsilon')Ap^2d\theta\right]$$
$$=\int_0^{\pi/2}M_1\cdot Rd\theta+\int_0^{\pi/2}\frac{WR^2}{2}d\theta-\int_0^{\pi/2}\frac{WR^2}{2}\sin\theta d\theta \qquad \text{(iii)}$$

Now, $\dfrac{\varphi}{\theta}=(1+\varepsilon')\dfrac{R}{R'}$ or $\varphi=(1+\varepsilon')\dfrac{R}{R'}\theta$ (Refer section 9.3)

Differentiating, $d\varphi=(1+\varepsilon')\dfrac{R}{R'}d\theta$

$\therefore$ (iii) becomes,

$$E\left[\int_0^{\pi/2}Ap^2d\varphi-\int_0^{\pi/2}(1+\varepsilon')Ap^2d\theta\right]=\int_0^{\pi/2}M_1\cdot Rd\theta+\int_0^{\pi/2}\frac{WR^2}{2}d\theta-\int_0^{\pi/2}\frac{WR^2}{2}\sin\theta d\theta$$

or
$$\frac{\pi}{2}EAp^2-\frac{\pi}{2}EAp^2(1+\varepsilon')=\frac{\pi}{2}M_1R+\frac{\pi}{2}\cdot\frac{WR^2}{2}-\frac{WR^2}{2}$$

or
$$-\frac{\pi}{2}EAp^2\varepsilon'=\frac{\pi}{2}M_1R+\frac{WR^2}{2}\left(\frac{\pi}{2}-1\right) \qquad (9.19)$$

Longitudinal (tangential) force, $F=\dfrac{W}{2}\sin\theta$

Thus
$$EA\left[\varepsilon'-(1+\varepsilon')\left(\frac{1}{R'}-\frac{1}{R}\right)\frac{p^2}{R}\right]=\frac{W}{2}\sin\theta \qquad \text{(Refer Eq. 9.7a)}$$

or
$$REA\varepsilon'-\left[EAp^2(1+\varepsilon')\left(\frac{1}{R'}-\frac{1}{R}\right)\right]=\frac{WR}{2}\sin\theta$$

or
$$REA\varepsilon'-\left[M_1-\frac{W}{2}R(1-\sin\theta)\right]=\frac{WR}{2}\sin\theta \qquad \text{(using Eq. 9.18)}$$

or
$$REA\varepsilon'-M_1-\frac{W}{2}R=0$$

or
$$\varepsilon'=\frac{W}{2EA}+\frac{M_1}{EAR} \qquad (9.20)$$

This is independent of θ and thus is constant for the ring.

From Eq. 9.19, $-\dfrac{\pi}{2}EAp^2\left(\dfrac{W}{2EA}+\dfrac{M_1}{EAR}\right)=\dfrac{\pi}{2}M_1R+\dfrac{\pi}{2}\cdot\dfrac{WR^2}{2}-\dfrac{WR^2}{2}$

Multiplying throughout by $2R$,

$$-\frac{\pi}{2}p^2WR-\pi p^2M_1=\pi M_1R^2+\frac{\pi WR^3}{2}-WR^3$$

or
$$\pi M_1(R^2+p^2)=-\frac{\pi WR}{2}(R^2+p^2)+WR^2$$

or $$M_1 = \frac{WR}{2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - 1\right) \tag{9.21}$$

From (i), $M_2 = \dfrac{WR}{2}\left(\dfrac{2}{\pi}\cdot\dfrac{R^2}{R^2+p^2} - 1\right) + \dfrac{W}{2}R(1-\sin\theta)$

$$= \frac{WR}{2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - \sin\theta\right) \tag{9.22}$$

It is maximum at $\theta = 0°$, $M_{\max} = \dfrac{WR^2}{\pi(R^2+p^2)}$ (9.23)

It is zero at $\sin\theta = \dfrac{2}{\pi}\cdot\dfrac{R^2}{R^2+p^2}$

Now, $$\varepsilon = \varepsilon' + (1+\varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right)\frac{y}{1+y/R} \qquad \text{(Eq. 9.3)}$$

In this equation, $\varepsilon' = \dfrac{W}{2EA} + \dfrac{M_1}{EAR}$ (Eq. 9.20)

$$= \frac{W}{2EA} + \frac{WR}{2EAR}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - 1\right) \qquad \text{(using Eq.9.21)}$$

$$= \frac{W}{EA}\cdot\frac{R^2}{\pi(R^2+p^2)} \tag{9.24}$$

and $(1+\varepsilon')\left(\dfrac{1}{R'} - \dfrac{1}{R}\right) = \dfrac{M_1}{EAp^2} + \dfrac{W}{2EAp^2}R(1-\sin\theta)$ (from Eq. 9.18)

$$= \frac{WR}{2EAp^2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - 1\right) + \frac{W}{2EAp^2}R(1-\sin\theta) \qquad \text{(using Eq.9.21)}$$

Equation 9.3 can be written as,

$$\varepsilon = \frac{W}{EA}\cdot\frac{R^2}{\pi(R^2+p^2)} + \left[\frac{WR}{2EAp^2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - 1\right) + \frac{W}{2EAp^2}R(1-\sin\theta)\right]\frac{Ry}{R+y}$$

$$= \frac{W}{EA}\cdot\frac{R^2}{\pi(R^2+p^2)} + \frac{WR}{2EAp^2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - \sin\theta\right)\frac{Ry}{R+y} \tag{9.25}$$

$$\sigma = \varepsilon E = \frac{W}{A}\left[\frac{R^2}{\pi(R^2+p^2)} + \frac{R^2}{2p^2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - \sin\theta\right)\frac{y}{R+y}\right] \tag{9.26}$$

Direct stress = $\dfrac{W\sin\theta}{2A}$

i.e., along line of action it is zero and on a section perpendicular to line of action it is $W/2A$.

Total stress, $$\sigma = \frac{W}{A}\left[\frac{R^2}{\pi(R^2+p^2)} + \frac{R^2}{2p^2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - \sin\theta\right)\frac{y}{R+y}\right] + \frac{W\sin\theta}{2A} \tag{9.27}$$

- Stress along line of action of load,

$$\sigma = \frac{W}{A}\left[\frac{R^2}{\pi(R^2+p^2)} + \frac{R^2}{2p^2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2}\right)\frac{y}{R+y}\right]$$

$$= \frac{W}{\pi A}\left[\frac{R^2}{(R^2+p^2)}\left(1+\frac{R^2}{p^2}\cdot\frac{y}{R+y}\right)\right] \tag{9.28}$$

- Stress on a section perpendicular to line of action of load,

$$\sigma = \frac{W}{A}\left[\frac{R^2}{\pi(R^2+p^2)} + \frac{R^2}{2p^2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - 1\right)\frac{y}{R+y}\right] + \frac{W}{2A}$$

$$= \frac{W}{A}\left[\frac{R^2}{\pi(R^2+p^2)} + \frac{R^2}{2p^2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - 1\right)\frac{y}{R+y} + \frac{1}{2}\right] \tag{9.29}$$

Example 9.8 A ring of steel bar has a diameter of 20 mm and carries a pull of 4 kN as shown in Fig. 9.17. Determine the stresses at points *A, B, C* and *D* of the rim. The mean radius of the rim is 100 mm.

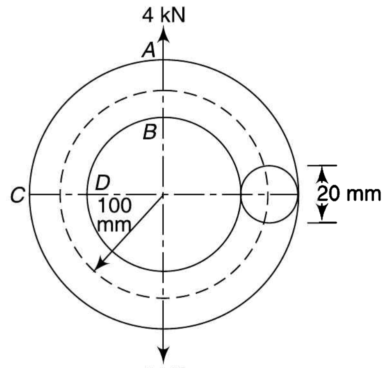


Fig. 9.17

Solution

Given A ring of steel bar subjected to a pull as shown in Fig. 9.17.
To find Stresses at *A, B, C* and *D*

$$A = \frac{\pi}{4}\times 20^2 = 100\pi \text{ mm}^2$$

Calculating p^2

$$p^2 = \frac{d^2}{16} + \frac{d^4}{128R^2} + \ldots = \frac{20^2}{16} + \frac{1}{128}\cdot\frac{20^4}{100^2} = 25.125 \text{ mm}^2$$

$$\sigma = \frac{W}{A}\left[\frac{R^2}{\pi(R^2+p^2)} + \frac{R^2}{2p^2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - \sin\theta\right)\frac{y}{R+y}\right] + \frac{W\sin\theta}{2A} \quad \text{(Eq. 9.27)}$$

Stress at A

At *A*, $\theta = 0°$,

$$\sigma = \frac{W}{\pi A}\left[\frac{R^2}{(R^2+p^2)}\left(1+\frac{R^2}{p^2}\cdot\frac{y}{R+y}\right)\right]$$

$$= \frac{4000}{\pi\times 100\pi}\left[\frac{100^2}{(100^2+25.125)}\left(1+\frac{100^2}{25.125}\cdot\frac{10}{100+10}\right)\right]$$

$= 4.043 \times 37.18 = 150.3$ MPa (tensile)

Stress at B

At *B*, $\theta = 0°$, y is negative.

$$\sigma = \frac{4000}{\pi\times 100\pi}\left[\frac{100^2}{(100^2+25.125)}\left(1-\frac{100^2}{25.125}\cdot\frac{10}{100-10}\right)\right]$$

$= 4.043 \times (-43.22)$
$= -174.7$ MPa (compressive)

Stress at C

At C, $\theta = 90°$,

$$\sigma = \frac{W}{A}\left[\frac{R^2}{\pi(R^2+p^2)} + \frac{R^2}{2p^2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - 1\right)\frac{y}{R+y} + \frac{1}{2}\right]$$

$$= \frac{4000}{100\pi}\left[\frac{100^2}{\pi(100^2+25.125)} + \frac{100^2}{2\times 25.125}\left(\frac{2}{\pi}\cdot\frac{100^2}{100^2+25.125} - 1\right)\frac{10}{100+10} + \frac{1}{2}\right]$$

$$= 12.732\left[0.3175 - 72.632\times\frac{10}{110} + 0.5\right] = -\ 74 \text{ MPa (compressive)}$$

Stress at D

At D, $\theta = 90°$, y is negative.

$$\sigma = \frac{W}{A}\left[\frac{R^2}{\pi(R^2+p^2)} + \frac{R^2}{2p^2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - 1\right)\frac{-y}{R-y}\cdot\frac{1}{2}\right]$$

$$= \frac{4000}{100\pi}\left[\frac{100^2}{\pi(100^2+25.125)} + \frac{100^2}{2\times 25.125}\left(\frac{2}{\pi}\cdot\frac{100^2}{100^2+25.125} - 1\right)\frac{-10}{100-10} + \frac{1}{2}\right]$$

$$= 12.732\left[0.3175 - 72.632\times\frac{-10}{90} + 0.5\right] = 113.16 \text{ MPa (tensile)}$$

Example 9.9 || A steel ring of 240 mm of mean diameter has a rectangular cross-section of 60 mm X 40 mm, the larger section being in the radial direction. Determine the tensile force which the ring can carry safely if the permissible stress is 140 MPa.

Solution

Given A steel ring of a rectangular section as shown in Fig. 9.18

To find Safe tensile force if maximum stress is 140 MPa

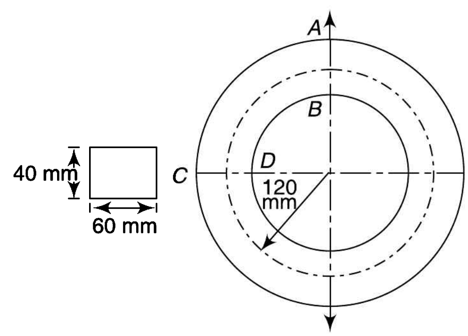


Fig. 9.18

Calculating p^2

$$p^2 = \frac{R^3}{d}\ln\left(\frac{2R+d}{2R-d}\right) - R^2 = \frac{120^3}{60}\ln\left(\frac{2\times120+60}{2\times120-60}\right) - 120^2 = 311.8 \text{ mm}^2 \qquad \text{(Eq. 9.12)}$$

Calculating safe tensile force

The tensile stresses are developed at A and D,

At A, $$\sigma = \frac{W}{\pi A}\left[\frac{R^2}{(R^2+p^2)}\left(1 + \frac{R^2}{p^2}\cdot\frac{y}{R+y}\right)\right] \qquad \text{(Eq. 9.28)}$$

$$140 = \frac{W}{\pi\times 60\times 40}\left[\frac{120^2}{(120^2+311.8)}\left(1 + \frac{120^2}{311.8}\times\frac{30}{120+30}\right)\right]$$

or $$140 = \frac{W}{2400\pi}[0.9788(1+9.2367)] \text{ or } W = 105\ 350 \text{ N or } 105.35 \text{ kN}$$

At D,
$$\sigma = \frac{W}{A}\left[\frac{R^2}{\pi(R^2+p^2)} + \frac{R^2}{2p^2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - 1\right)\frac{-y}{R-y} + \frac{1}{2}\right] \qquad \text{(Eq. 9.29)}$$

$$140 = \frac{W}{60\times 40}\left[\frac{120^2}{\pi(120^2+311.8)} + \frac{120^2}{2\times 311.8}\left(\frac{2}{\pi}\cdot\frac{120^2}{120^2+311.8} - 1\right)\frac{-30}{120-30} + \frac{1}{2}\right]$$

$$140 = \frac{W}{2400}[0.3116 + 23.092(0.1256) + 0.5]$$

$$140 = \frac{3.712W}{2400}, \quad W = 90\,517 \text{ N}$$

Thus the safe load is 90.517 kN

9.5 STRESSES IN A CHAIN LINK

Consider a chain link as shown in Fig. 9.19. Let l be the length of the straight sides and R the mean radius of semicircular ends. Proceeding as in case of circular ring and considering the equilibrium of the portion $BB'C'C$,

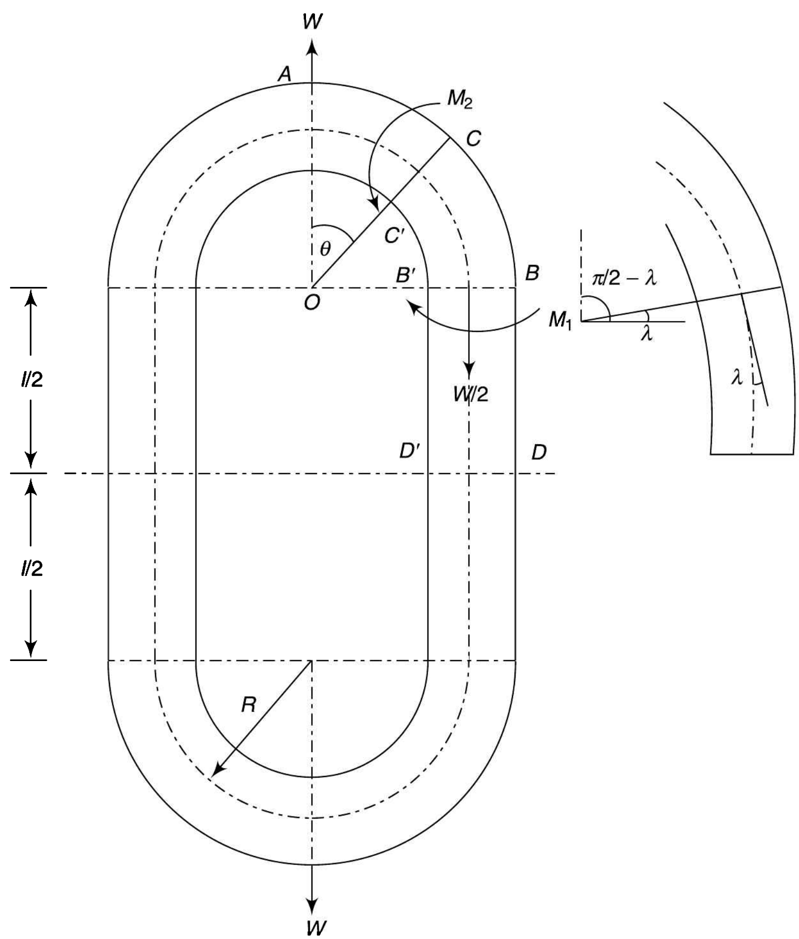


Fig. 9.19

$$M_2 = M_1 + \frac{W}{2}R(1 - \sin\theta) \qquad \text{(i)}$$

and $$M_2 = E(1+\varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right)Ap^2 \qquad \text{...(Eq. 9.6)} \qquad \text{(ii)}$$

From (i) and (ii),

$$E(1+\varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right)Ap^2 = M_1 + \frac{W}{2}R(1 - \sin\theta) \qquad (9.30)$$

Multiplying both sides by $Rd\theta$ and integrating,

$$E\int_0^{\pi/2}(1+\varepsilon')\left(\frac{1}{R'} - \frac{1}{R}\right)Ap^2 \cdot Rd\theta = \int_0^{\pi/2} M_1 \cdot Rd\theta + \int_0^{\pi/2}\frac{W}{2}R(1-\sin\theta)\,Rd\theta$$

$$E\left[\int_0^{\pi/2}\frac{(1+\varepsilon')R}{R'}Ap^2d\theta - \int_0^{\pi/2}(1+\varepsilon')Ap^2d\theta\right]$$

$$= \int_0^{\pi/2} M_1 \cdot Rd\theta + \int_0^{\pi/2}\frac{WR^2}{2}d\theta - \int_0^{\pi/2}\frac{WR^2}{2}\sin\theta\, d\theta$$

Now, $\frac{\varphi}{\theta} = (1+\varepsilon')\frac{R}{R'}$ or $\varphi = (1+\varepsilon')\frac{R}{R'}\theta$ (Refer Section 9.3)

Differentiating, $d\varphi = (1+\varepsilon')\frac{R}{R'}d\theta$

Therefore,

$$E\left[\int_0^{\pi/2-\lambda}Ap^2d\varphi - \int_0^{\pi/2}(1+\varepsilon')Ap^2d\theta\right] = \int_0^{\pi/2}M_1 \cdot Rd\theta + \int_0^{\pi/2}\frac{WR^2}{2}d\theta - \int_0^{\pi/2}\frac{WR^2}{2}\sin\theta d\theta$$

For the first integral, the upper limit is not $(\pi / 2)$ as the section BB' does not remain perpendicular to the load line, i.e., the $\angle AOB$ does not remain 90° but varies slightly due to bending of vertical portion DB as shown in Fig. 9.19. Considering the horizontal portion DD' as fixed, the slope of end B is given by,

$$\lambda = \frac{M_1}{EI}\cdot\frac{l}{2} \quad \text{where } l \text{ is the length of the straight sides} \qquad \text{(Eq. 7.13)}$$

$$EAp^2\left(\frac{\pi}{2} - \frac{M_1 l}{2EI}\right) - \frac{\pi}{2}EAp^2(1+\varepsilon') = \frac{\pi}{2}M_1R + \frac{\pi}{2}\cdot\frac{WR^2}{2} - \frac{WR^2}{2}$$

$$M_1\left(\frac{\pi}{2}\cdot R + \frac{Alp^2}{2I}\right) = \frac{WR^2}{2}\left(1 - \frac{\pi}{2}\right) - \frac{\pi}{2}EAp^2\varepsilon' \qquad (9.31)$$

Also $$\varepsilon' = \frac{W}{2EA} + \frac{M_1}{EAR} \qquad \text{(Eq. 9.20)}$$

$\therefore$ $$M_1\left(\frac{\pi}{2}\cdot R + \frac{Alp^2}{2I}\right) = \frac{WR^2}{2}\left(1 - \frac{\pi}{2}\right) - \frac{\pi}{2}EAp^2\left(\frac{W}{2EA} + \frac{M_1}{EAR}\right)$$

$$M_1\left(\frac{\pi}{2}\cdot R + \frac{Alp^2}{2I} + \frac{\pi}{2}\frac{p^2}{R}\right) = \frac{WR^2}{2}\left(1 - \frac{\pi}{2}\right) - \frac{\pi}{4}Wp^2$$

Now inserting $I = Ak^2$ where k is radius of gyration and multiplying throughout by $2/\pi$.

$$\therefore \qquad M_1\left(R+\frac{lp^2}{\pi k^2}+\frac{p^2}{R}\right)=W\left(\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{p^2}{2}\right)$$

or $$M_1 = W\left(\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{p^2}{2}\right)\bigg/\left(R+\frac{lp^2}{\pi k^2}+\frac{p^2}{R}\right) \tag{9.32}$$

and $$M_2 = W\left(\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{p^2}{2}\right)\bigg/\left(R+\frac{lp^2}{\pi k^2}+\frac{p^2}{R}\right)+\frac{W}{2}R(1-\sin\theta) \tag{9.33}$$

Thus, $$\varepsilon' = \frac{W}{EA}\left[\frac{1}{2}+\frac{1}{R}\left(\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{p^2}{2}\right)\bigg/\left(R+\frac{lp^2}{\pi k^2}+\frac{p^2}{R}\right)\right] \qquad \text{(From Eq. 9.20)} \tag{9.34}$$

Now, $$\varepsilon = \varepsilon' + (1+\varepsilon')\left(\frac{1}{R'}-\frac{1}{R}\right)\frac{y}{1+y/R} \qquad \text{(Eq. 9.3)}$$

and $$\sigma = E\varepsilon' + E(1+\varepsilon')\left(\frac{1}{R'}-\frac{1}{R}\right)\frac{y}{1+y/R}$$

As $$E(1+\varepsilon')\left(\frac{1}{R'}-\frac{1}{R}\right)Ap^2 = M_1+\frac{W}{2}R(1-\sin\theta) \qquad \text{(Eq. 9.30)}$$

$$E(1+\varepsilon')\left(\frac{1}{R'}-\frac{1}{R}\right)=\frac{M_1}{Ap^2}+\frac{W}{2Ap^2}R(1-\sin\theta)$$

$$=\frac{W}{Ap^2}\left(\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{p^2}{2}\right)\bigg/\left(R+\frac{lp^2}{\pi k^2}+\frac{p^2}{R}\right)+\frac{W}{2Ap^2}R(1-\sin\theta)$$

$$\therefore \qquad \sigma = E\varepsilon' + E(1+\varepsilon')\left(\frac{1}{R'}-\frac{1}{R}\right)\frac{y}{1+y/R}$$

$$=\frac{W}{A}\left[\frac{1}{2}+\frac{1}{R}\,\frac{\dfrac{R^2}{\pi}-\dfrac{R^2}{2}-\dfrac{p^2}{2}}{R+\dfrac{lp^2}{\pi k^2}+\dfrac{p^2}{R}}\right]+\frac{W}{Ap^2}\cdot\frac{\dfrac{R^2}{\pi}-\dfrac{R^2}{2}-\dfrac{p^2}{2}}{R+\dfrac{lp^2}{\pi k^2}+\dfrac{p^2}{R}}\cdot\frac{y}{1+y/R}+\frac{W}{2Ap^2}\,\frac{R(1-\sin\theta)\,y}{1+y/R}$$

$$=\frac{W}{AR}\,\frac{\dfrac{R^2}{\pi}-\dfrac{R^2}{2}-\dfrac{p^2}{2}}{R+\dfrac{lp^2}{\pi k^2}+\dfrac{p^2}{R}}+\frac{W}{Ap^2}\cdot\frac{\dfrac{R^2}{\pi}-\dfrac{R^2}{2}-\dfrac{p^2}{2}}{R+\dfrac{lp^2}{\pi k^2}+\dfrac{p^2}{R}}\cdot\frac{yR}{R+y}+\frac{W}{2Ap^2}\,\frac{R(1-\sin\theta)\,yR}{R+y}+\frac{W}{2A}$$

Direct stress $= \dfrac{W\sin\theta}{2A}$

$$\therefore \qquad \sigma = \frac{W}{AR}\left[\frac{\dfrac{R^2}{\pi}-\dfrac{R^2}{2}-\dfrac{p^2}{2}}{R+\dfrac{lp^2}{\pi k^2}+\dfrac{p^2}{R}}\left(1+\frac{R^2}{p^2}\cdot\frac{y}{R+y}\right)\right]+\frac{WR^2}{2Ap^2}\,\frac{y}{R+y}(1-\sin\theta)+\frac{W}{2A}(1+\sin\theta) \tag{9.35}$$

The above equation gives the resultant stress on any section along the curved portion. On the straight portion the bending moment remains constant and the stress can be found by the usual flexural formula and by adding the direct tensile stress.

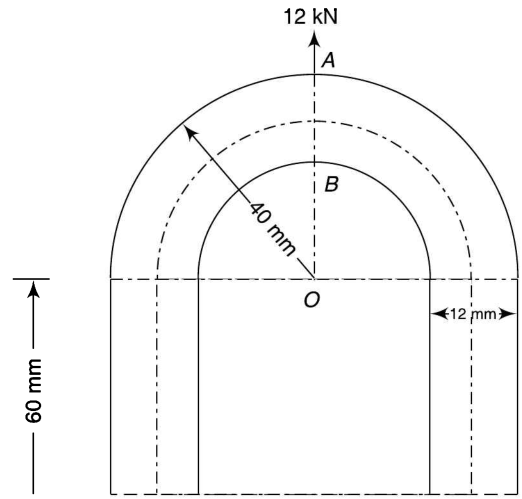


Fig. 9.20

Example 9.10 A chain link is made from a steel rod of circular cross-section of 12-mm diameter. The mean radius of the circular portion is 40 mm and the length of the straight portion is 60 mm. The tensile load applied is 1.2 kN. Determine the maximum compressible stress developed in the link. Also find the value of the maximum tensile stress at the same section.

Solution

Given A chain link of a circular section and loaded as shown in Fig. 9.20

To find Maximum compressive stress and tensile stress at same section

Area, $A = \dfrac{\pi}{4} \times 12^2 = 36\,\pi \text{ mm}^2$

$$I = Ak^2 \quad \text{or} \quad \frac{\pi}{64}d^4 = \frac{\pi}{4}d^2k^2 \quad \text{or} \quad k^2 = \frac{d^2}{16} = \frac{12^2}{16} = 9 \text{ mm}^2$$

Calculating p^2

$$p^2 = \frac{d^2}{16} + \frac{d^4}{128R^2} = \frac{12^2}{16} + \frac{12^4}{128 \times 40^2} = 9.1$$

Maximum compressive stress

Compressive stress is maximum at the inside of the section along the load line, i.e., at B, where $\theta = 0°$ and y is negative. Then Eq. 9.35 reduces to

$$\sigma = \frac{W}{AR}\left[\frac{\dfrac{R^2}{\pi} - \dfrac{R^2}{2} - \dfrac{p^2}{2}}{R + \dfrac{lp^2}{\pi k^2} + \dfrac{p^2}{R}}\left(1 + \frac{R^2}{p^2} \cdot \frac{-y}{R-y}\right)\right] + \frac{WR^2}{2Ap^2} \cdot \frac{-y}{R-y} + \frac{W}{2A}$$

$$= \frac{1200}{36\pi \times 40}\left[\frac{\dfrac{40^2}{\pi} - \dfrac{40^2}{2} - \dfrac{9.1}{2}}{40 + \dfrac{60 \times 9.1}{\pi \times 9} + \dfrac{9.1}{40}}\left(1 + \frac{40^2}{9.1} \cdot \frac{-6}{40-6}\right)\right] + \frac{1200 \times 40^2}{2 \times 36\pi \times 9.1} \cdot \frac{-6}{40-6} + \frac{1200}{2 \times 36\pi}$$

$= 0.2653 \times (-4.959)(-30.02) + 932.78 \times (-0.1765) + 5.3$
$= 39.5 - 164.6 + 5.3 = -119.8 \text{ MPa}$

Maximum tensile stress

Maximum tensile stress at the same section, i.e., at A,

$$\sigma = \frac{W}{AR}\left[\frac{\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{p^2}{2}}{R+\frac{lp^2}{\pi k^2}+\frac{p^2}{R}}\left(1+\frac{R^2}{p^2}\cdot\frac{y}{R+y}\right)\right]+\frac{WR^2}{2Ap^2}\cdot\frac{y}{R+y}+\frac{W}{2A}$$

$$= 0.2653(-4.959)\left(1+\frac{40^2}{9.1}\times\frac{6}{40+6}\right)+\frac{1200\times 40^2}{2\times 36\pi\times 9.1}\cdot\frac{6}{40+6}+5.3$$

$$= 0.2653 \times (-4.959)(23.93) + 932.78 \times (0.1304) + 5.3$$
$$= -31.5 + 121.6 + 5.3 = 95.4 \text{ MPa}$$

9.6 DEFLECTION OF CURVED BARS

Let a length δs of an initially curved beam is acted upon by a bending moment M (Fig. 9.21), Then from Eq. 9.1 (for small initial curvature),

$$M = EI\left(\frac{1}{R}-\frac{1}{R_o}\right)$$

Multiply both sides by δs,

$$\frac{M}{EI}\delta s = \left(\frac{1}{R}-\frac{1}{R_o}\right)\delta s \quad \text{or} \quad \frac{M}{EI}\delta s = \frac{\delta s}{R}-\frac{\delta s}{R_o}$$

However, $\delta s/R - \delta s/R_o$ is the change of angle $\delta\varphi$ which is subtended by δs at the centre of curvature, i.e., it is the angle through which the tangent at one end rotates relative to that at the other end. Thus

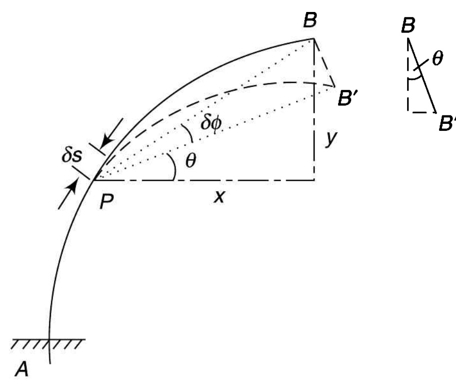


Fig. 9.21

$$\delta\varphi = \frac{M}{EI}\delta s \tag{9.36}$$

Consider a small length δs of a loaded bar AB fixed at A as shown in Fig. 9.21. Due to bending moment at M on the length δs at P, the length PB is rotated through an angle $\delta\varphi$. B moves to B'.

Vertical deflection at $B = BB' \cos\theta = (PB \cdot \delta\phi)\cos\theta$

$$= (PB \cdot \cos\theta)\,\delta\phi = x \cdot \delta\phi$$

Thus net vertical deflection of $B = \int x \cdot d\varphi = \int x \cdot \frac{M}{EI}ds = \int \frac{Mx}{EI}ds$ (9.37)

Horizontal deflection at $B = BB' \sin\theta = y\delta\phi$

Thus net horizontal deflection of $B = \int y \cdot d\varphi = \int \frac{My}{EI}ds$ (9.38)

Closed Ring

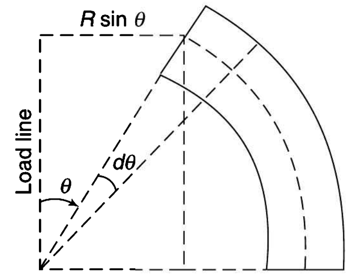


Fig. 9.22

(i) Deflection Along Load Line (Fig. 9.22)

$$M = \frac{WR}{2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - \sin\theta\right) \qquad \text{...(Eq. 9.22)}$$

$$x = R\sin\theta \quad \text{and} \quad ds = Rd\theta$$

Consider a half of the ring, bending moment tends to straighten it, thus it is negative.

$$\delta_v = \int \frac{Mx}{EI}ds = -\frac{1}{EI}\int_0^{\pi}\frac{WR}{2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - \sin\theta\right)\cdot R\sin\theta\cdot R\cdot d\theta$$

$$= -\frac{WR^3}{2EI}\int_0^{\pi}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2}\sin\theta - \sin^2\theta\right)\cdot d\theta$$

$$= -\frac{WR^3}{2EI}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2}\int_0^{\pi}\sin\theta\cdot d\theta - \int_0^{\pi}\frac{1-\cos\theta}{2}d\theta\right)$$

$$= -\frac{WR^3}{2EI}\left[\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2}(-\cos\theta)_0^{\pi} - \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi}\right]$$

$$= -\frac{WR^3}{2EI}\left[\frac{4}{\pi}\cdot\frac{R^2}{R^2+p^2} - \frac{\pi}{2}\right] \qquad (9.39)$$

*The limits can also have been taken for the quarter ring, i.e., from 0 to $\pi/2$ and the deflection will be the twice of the quantity so obtained.

(ii) Deflection Perpendicular to Load Line

$$y = R\cos\theta,\ M = \frac{WR}{2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - \sin\theta\right) \text{ and } ds = Rd\theta$$

Consider a quarter of the ring,

$$\delta_h = -2\int_0^{\pi/2}\frac{R\cos\theta}{EI}\cdot\frac{WR}{2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - \sin\theta\right)\cdot R\cdot d\theta$$

$$= -\frac{WR^3}{EI}\int_0^{\pi/2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2}\cos\theta - \sin\theta\cos\theta\right)\cdot d\theta$$

$$= -\frac{WR^3}{EI}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - \frac{1}{2}\int_0^{\pi/2}\sin 2\theta\, d\theta\right) = -\frac{WR^3}{EI}\left[\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - \frac{1}{4}(-\cos 2\theta)_0^{\pi/2}\right]$$

$$= -\frac{WR^3}{EI}\left[\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - \frac{1}{2}\right] = -\frac{WR^3}{2EI}\left[\frac{4}{\pi}\cdot\frac{R^2}{R^2+p^2} - 1\right] \qquad (9.40)$$

Chain Link

(i) Deflection Along Load Line (Fig. 9.23)

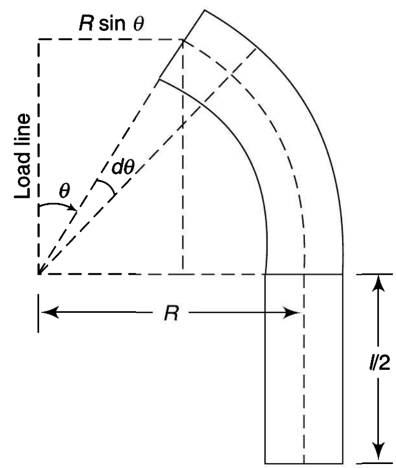


Fig. 9.23

Assume $\left(\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{p^2}{2}\right)\Big/\left(R+\frac{lp^2}{2\pi k^2}+\frac{p^2}{R}\right)=q$

Consider a quarter of the chain, bending moment tends to straighten it, thus it is negative.

On the straight portion, bending moment is constant and is

$$M_1=-W\left(\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{p^2}{2}\right)\Big/\left(R+\frac{lp^2}{2\pi k^2}+\frac{p^2}{R}\right)=-Wq,$$

$x=R$ and $ds=l/2$

$\therefore$ vertical deflection, $\delta_v=\int\frac{Mx}{EI}ds=-2\times\frac{Wq\cdot R}{EI}\cdot\frac{l}{2}$

On the curved portion,

$$M_2=-\left[W\frac{\frac{R^2}{\pi}-\frac{R^2}{2}-\frac{p^2}{2}}{R+\frac{lp^2}{\pi k^2}+\frac{p^2}{R}}+\frac{W}{2}R(1-\sin\theta)\right]=-\left[Wq+\frac{W}{2}R(1-\sin\theta)\right]$$

$x=R\sin\theta,\ ds=Rd\theta$

Vertical deflection, $\delta_v=\int\frac{Mx}{EI}ds=-2\int_0^{\pi/2}\frac{R\sin\theta}{EI}\left[Wq+\frac{WR}{2}(1-\sin\theta)\cdot R\cdot d\theta\right]$

$$\delta_v=\int\frac{Mx}{EI}ds(\text{for the straight portion})+\int\frac{Mx}{EI}ds(\text{for the curved portion})$$

$$=\frac{2R}{EI}\left(Wq\cdot\frac{l}{2}\right)+2\int_0^{\pi/2}\frac{R\sin\theta}{EI}\left[Wq+\frac{WR}{2}(1-\sin\theta)\cdot R\cdot d\theta\right]$$

$$=-\frac{WqlR}{EI}-\frac{2WR^2}{EI}\int_0^{\pi/2}\left[q\sin\theta+\frac{R}{2}(\sin\theta-\sin^2\theta)d\theta\right]$$

$$=-\frac{WqlR}{EI}-\frac{2WR^2}{EI}\left[q\int_0^{\pi/2}\sin\theta\cdot d\theta+\frac{R}{2}\int_0^{\pi/2}\sin\theta\cdot d\theta-\frac{R}{4}\int_0^{\pi/2}(1-\cos 2\theta)d\theta\right]$$

$$=-\frac{WqlR}{EI}-\frac{2WR^2}{EI}\left[q(-\cos\theta)_0^{\pi/2}+\frac{R}{2}(-\cos\theta)_0^{\pi/2}-\frac{R}{4}\left(\theta-\frac{\sin 2\theta}{2}\right)_0^{\pi/2}\right]$$

$$=-\frac{WqlR}{EI}-\frac{WR^2}{EI}\left[2q+R-\frac{\pi R}{4}\right]$$

$$=\frac{WR^2}{EI}\left[\frac{\pi R}{4}-2q-R\right]-\frac{WqlR}{EI}\tag{9.41}$$

Deflection of the straight length $l/2$ under direct load $W/2=\frac{Wl}{2AE}$

$$\therefore \quad \text{total deflection } = \frac{WR^2}{EI}\left[\frac{\pi R}{4} - 2q - R\right] - \frac{WqlR}{EI} + \frac{Wl}{2AE} \tag{9.42}$$

(ii) Deflection Perpendicular to Load Line

$$y = R\cos\theta,\; M_2 = \frac{WR}{2}\left(\frac{2}{\pi}\cdot\frac{R^2}{R^2+p^2} - \sin\theta\right) \text{ and } ds = Rd\theta$$

Consider a quarter of the ring,

$$\delta_h = -\frac{WqlR}{EI} - 2\int_0^{\pi/2}\frac{R\cos\theta}{EI}\left[Wq + \frac{WR}{2}(1-\sin\theta)\cdot R\cdot d\theta\right]$$

$$= -\frac{WqlR}{EI} - \frac{2WR^2}{EI}\int_0^{\pi/2}\left[q\cos\theta + \frac{R}{2}\cos\theta\cdot d\theta - \frac{R}{4}\sin 2\theta\cdot d\theta\right]$$

$$= -\frac{WqlR}{EI} - \frac{2WR^2}{EI}\left[q(\sin\theta)_0^{\pi/2} + \frac{R}{2}(\sin\theta)_0^{\pi/2} + \frac{R}{8}(\cos 2\theta)_0^{\pi/2}\right]$$

$$= -\frac{WqlR}{EI} - \frac{WR^2}{EI}\left[2q + R - \frac{R}{2}\right] = \frac{WqlR}{EI} + \frac{WR^2}{EI}\left[2q + \frac{R}{2}\right] \tag{9.43}$$

Example 9.11 A steel tube of 80-mm outer diameter and 50-mm inner diameter is bent in the form of a quadrant of 1.6-m radius. One end is rigidly fixed to a horizontal base and the free end supports a load of 800 N. Find the vertical and the horizontal deflections of the free end. E = 205 GPa.

Solution

Given A steel tube bent in the form of a quadrant, fixed at one end and loaded as shown in Fig. 9.24. E = 205 GPa

To find Vertical and horizontal deflections of free end

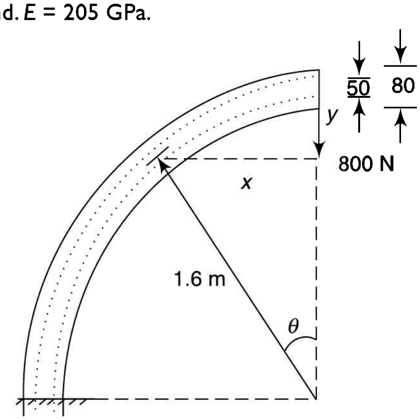


Fig. 9.24

$$I = \frac{\pi}{64}(40^4 - 25^2) = 33\,896\pi \text{ mm}^4$$

Consider a small length δs of the tube.

Bending moment, $M = WR\sin\theta$; $\quad x = R\sin\theta$

Vertical deflection

$$\text{Vertical deflection } = \int\frac{Mx}{EI}ds = \frac{1}{EI}\int_0^{\pi/2}(WR\sin\theta)(R\sin\theta)\cdot Rd\theta$$

$$= \frac{WR^3}{EI}\int_0^{\pi/2}\sin^2\theta\cdot d\theta = \frac{WR^3}{EI}\int_0^{\pi/2}\frac{1-\cos 2\theta}{2}\cdot d\theta$$

$$= \frac{800\times 1600^3}{205\,000\times 33\,896\pi\times 2}\cdot\left(\theta - \frac{\sin 2\theta}{2}\right) = 75.05\times\left(\frac{\pi}{2} - 0\right) = 117.9 \text{ mm}$$

Horizontal deflection

$$\text{Horizontal deflection } = \int\frac{My}{EI}ds = \frac{1}{EI}\int_0^{\pi/2}(WR\sin\theta)R(1-\cos\theta)\cdot Rd\theta$$

$$= \frac{WR^3}{EI}\int_0^{\pi/2}\left(\sin\theta - \frac{\sin 2\theta}{2}\right)\cdot d\theta = \frac{800 \times 1600^3}{205\,000 \times 33\,896\pi}\left(-\cos\theta + \frac{\cos 2\theta}{2}\right)_0^{\pi/2}$$

$$= 150.1(-0 + 1 - 1/4 - 1/4) = 75.05 \text{ mm}$$

Example 9.12 A steel ring made of a circular cross-section of 20-mm diameter is applied a pulling force of 8 kN. The mean diameter of the ring is 160 mm. Determine the increase in the diameter along the load and decrease in diameter in the lateral direction. E = 205 GPa.

Solution

Given A steel ring of circular cross-section and loaded as shown in Fig. 9.25

$$E = 205 \text{ GPa}$$

To find Increase in diameter along load line and decrease in diameter in lateral direction.

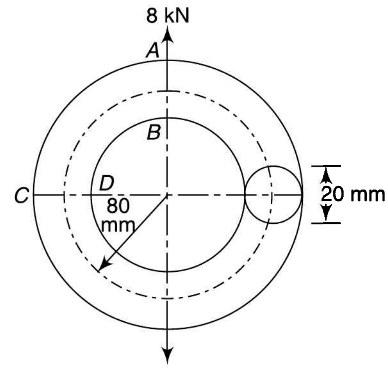


Fig. 9.25

$$I = \frac{\pi \times 20^4}{64} = 7854 \text{ mm}^4$$

Calculating p^2

$$p^2 = \frac{d^2}{16} + \frac{d^4}{128R^2} + \ldots = \frac{20^2}{16} + \frac{1}{128}\cdot\frac{20^4}{80^2} = 25.195 \text{ mm}^2$$

Increase in diameter

Along the load line,

$$\delta = -\frac{WR^3}{2EI}\left[\frac{4}{\pi}\cdot\frac{R^2}{R^2 + p^2} - \frac{\pi}{2}\right] \quad \ldots\text{(Eq. 9.39)}$$

$$= -\frac{8000 \times 80^3}{2 \times 205\,000 \times 7854}\left[\frac{4}{\pi}\cdot\frac{80^2}{80^2 + 25.195} - \frac{\pi}{2}\right]$$

$$= -1.272 \times (1.268 - 1.571) = 0.3854 \text{ mm}$$

Decrease in diameter

In the lateral direction,

$$\delta = -\frac{WR^3}{2EI}\left[\frac{4}{\pi}\cdot\frac{R^2}{R^2 + p^2} - 1\right] \quad \ldots\text{(Eq. 9.40)}$$

$$= -1.272\,[1.268 - 1] = -0.34 \text{ mm}$$

Example 9.13 A steel chain link is made of a 16-mm diameter rod and has ends of 60 mm in radius. The straight portion is 80 mm long. The link is subjected to a load of 20 kN. Determine the deflection of the link along the load line. E = 205 GPa

Solution

Given A steel chain link as shown in Fig. 9.26

$$E = 205 \text{ GPa}$$

To find Deflection of link along load line

Area, $A = \frac{\pi}{4} \times 16^2 = 64\pi \text{ mm}^2$

Calculating p^2

$$p^2 = \frac{d^2}{16} + \frac{d^4}{128R^2} = \frac{16^2}{16} + \frac{16^4}{128 \times 60^2} = 16.14$$

$$I = \frac{\pi}{64} d^4 = \frac{\pi}{64} \times 16^4 = 1024\pi \text{ mm}^4$$

$$I = Ak^2 \quad \text{or} \quad \frac{\pi}{64} d^4 = \frac{\pi}{4} d^2 k^2$$

or $$k^2 = \frac{d^2}{16} = \frac{16^2}{16} = 16 \text{ mm}^2$$

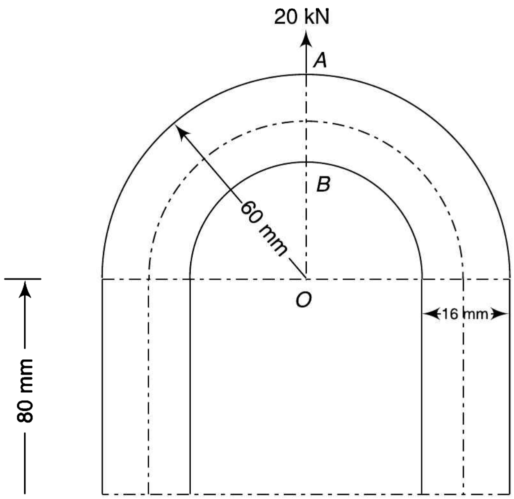


Fig. 9.26

Calculating q

$$q = \left(\frac{R^2}{\pi} - \frac{R^2}{2} - \frac{p^2}{2}\right) \Big/ \left(R + \frac{lp^2}{2\pi k^2} + \frac{p^2}{R}\right)$$

$$= \left(\frac{60^2}{\pi} - \frac{60^2}{2} - \frac{16.14}{2}\right) \Big/ \left(60 + \frac{80 \times 16.14}{2\pi \times 16} + \frac{16.14}{60}\right) = \frac{-662.2}{73.1} = -9.06$$

Deflection

$$\text{Total deflection } = \frac{WR^2}{EI}\left[\frac{\pi R}{4} - 2q - R\right] - \frac{WqlR}{EI} + \frac{Wl}{2AE} \qquad \text{(Eq. 9.42)}$$

$$= \frac{20\,000 \times 60^2}{205\,000 \times 1024\pi}\left[\frac{\pi \times 60}{4} - 2(-9.06) - 60\right] - \frac{20\,000 \times (-9.06) \times 80 \times 60}{205\,000 \times 1024\pi}$$

$$+ \frac{20\,000 \times 80}{2 \times 64\pi \times 205\,000}$$

$$= 0.5726 + 1.3188 + 0.0194 = 1.91 \text{ mm}$$

Example 9.14 A steel ring having a mean diameter of 120 mm and of a circular cross-section of 30-mm diameter has a saw cut along the horizontal diameter. It is subjected to a vertical tensile load of 10 kN such that the load line passes through the centre of the ring. Find the deflection of the ring along the load line. E =203 GPa.

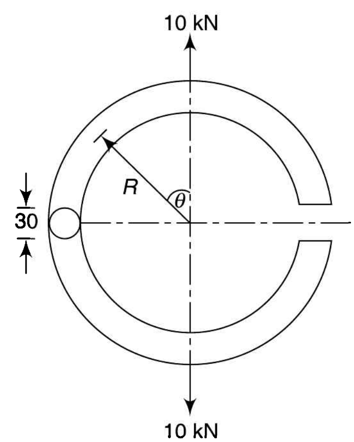


Fig. 9.27

Solution

Given A steel ring of circular section with a horizontal saw cut subjected to a vertical tensile load as shown in Fig. 9.27.

E =203 GPa

To find Deflection of ring along load line

Consider a small length δs of the tube

Vertical deflection

$$\text{Vertical deflection } = \int \frac{Mx}{EI} ds = \frac{1}{EI}\int_0^{\pi} (WR\sin\theta)(R\sin\theta)\cdot Rd\theta$$

$$= \frac{WR^3}{EI}\int_0^{\pi} \sin^2\theta \cdot d\theta$$

$$= \frac{WR^3}{EI}\int_0^{\pi} \left(\frac{1-\cos 2\theta}{2}\right)\cdot d\theta$$

$$= \frac{WR^3}{2EI}\left(\theta - \frac{\sin 2\theta}{2}\right)_0^{\pi}$$

$$= \frac{WR^3}{2EI}(\pi)$$

$$= \frac{\pi}{2}\frac{WR^3}{EI}$$

$$= \frac{\pi}{2}\cdot\frac{10\,000\times 60^3}{203\,000\times(\pi\times 30^4/64)}$$

$$= 0.42 \text{ mm}$$

9.7 DEFLECTION BY STRAIN ENERGY (CASTIGLIANO9S THEOREM)

This method has already been discussed in Section 7.7. The same may be applied to the problems of curved beams as illustrated below.

Example 9.15 A steel rod of 20-mm diameter has its one end fixed to a horizontal base and the other end is bent into the form of three quarters of a circle of 200-mm radius as shown in Fig. 9.28. The free end is constrained to move in a vertical direction only. Find the vertical deflection for a load of 150 N. E =205 GPa.

Solution

Given A steel rod bent into the form of three quarters of a circle and fixed to a horizontal base as shown in Fig. 9.28.

d =20 mm $\qquad$ E =203 GPa

To find Vertical deflection

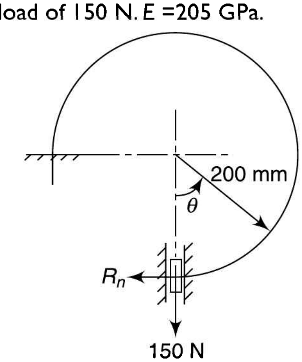


Fig. 9.28

Let $\quad W$ = vertical load

R_n = Normal reaction due to constraint

$M = R_n \times 200(1-\cos\theta) - W\times 200\sin\theta$

$ds = Rd\theta = 200\, d\theta$

Strain energy

$$U = \int\frac{M^2 ds}{2EI} = \frac{1}{2EI}\int_0^{3\pi/2}\left[200R_n(1-\cos\theta) - 200W\sin\theta\right]^2 \times 200\times d\theta$$

$$= \frac{200^3}{2EI}\int_0^{3\pi/2}[R_n(1-\cos\theta) - W\sin\theta]^2 d\theta$$

Applying Castigliano's theorem

- As there is no horizontal component, $\dfrac{\partial U}{\partial R_n} = 0$

$$\int_0^{3\pi/2} 2\,[R_n(1-\cos\theta) - W\sin\theta](1-\cos\theta)\,d\theta = 0$$

$$\int_0^{3\pi/2} [2R_n(1-\cos\theta)^2 - 2W\sin\theta(1-\cos\theta)]\,d\theta = 0$$

$$\int_0^{3\pi/2} [2R_n(1+\cos^2\theta - 2\cos\theta) - 2W\sin\theta + 2W\sin\theta\cos\theta]\,d\theta = 0$$

$$\int_0^{3\pi/2} \left[2R_n + 2R_n\frac{1+\cos 2\theta}{2} - 4R_n\cos\theta - 2W\sin\theta + W\sin 2\theta\right]d\theta = 0$$

$$\left[3R_n\theta + R_n(\sin 2\theta)/2 - 4R_n\sin\theta + 2W\cos\theta - W(\cos 2\theta)/2\right]_0^{3\pi/2} = 0$$

$$3R_n\cdot\frac{3\pi}{2} + 0 + 4R_n + 2W(0-1) - \frac{W}{2}(-1-1) = 0$$

$$\frac{9\pi R_n}{2} + 4R_n - 2W + W = 0 \text{ or } \frac{9\pi R_n}{2} + 4R_n - 2W = 0 \text{ or } R_n = \frac{W}{4.5\pi + 4}$$

$$\therefore \quad R_n = \frac{150}{4.5\pi + 4} = 8.27\text{ N}$$

- Vertical component $= \dfrac{\partial U}{\partial W} = \dfrac{R^3}{2EI}\displaystyle\int_0^{3\pi/2} 2[R_n(1-\cos\theta) - W\sin\theta](-\sin\theta)\,d\theta$

$$= \frac{R^3}{2EI}\int_0^{3\pi/2}[-2R_n\sin\theta + R_n\sin 2\theta + 2W\sin^2\theta]\,d\theta$$

$$= \frac{R^3}{2EI}\int_0^{3\pi/2}\left[-2R_n\sin\theta + R_n\sin 2\theta + W(1-\cos 2\theta)\right]d\theta$$

$$= \frac{R^3}{2EI}\left[2R_n\cos\theta - \frac{R_n\sin 2\theta}{2} + W\theta - \frac{W\sin 2\theta}{2}\right]_0^{3\pi/2}$$

$$= \frac{R^3}{2EI}\left[2R_n(0-1) - \frac{R_n}{2}(-1-1) + \frac{3\pi W}{2} - (0-0)\right]$$

$$= \frac{R^3}{2EI}\left[-2R_n + R_n + \frac{3\pi W}{2}\right]$$

$$= \frac{200^3}{2\times 205\,000\times(\pi\times 20^4/64)}\times\left[-8.27 + \frac{3\pi\times 150}{2}\right]$$

$= 1.736$ mm

Example 9.16 Determine the vertical deflection of the point *P* of the steel beam shown in Fig. 9.29. The cross-section of the beam is 25 mm × 4 mm. Neglect the effect of shear. *E* = 205 GPa.

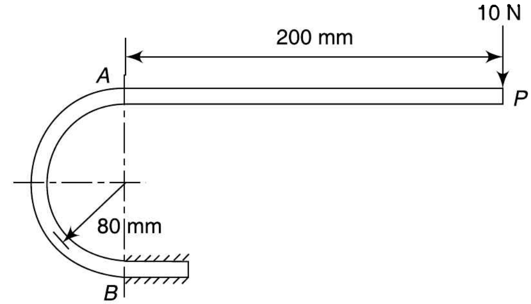


Fig. 9.29

Solution

Given A steel beam carrying a load as shown in Fig. 9.29.

$w = 25$ mm $\qquad t = 4$ mm $\qquad E = 205$ GPa

To find Vertical deflection of point *P*

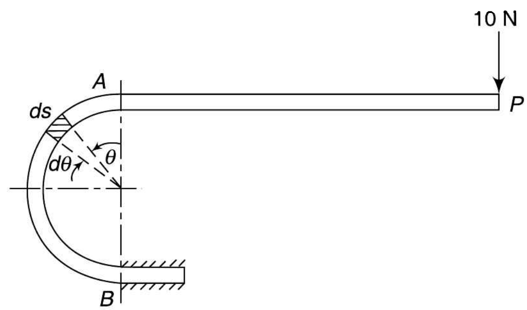


Fig. 9.30

In the curved portion, consider an element of length *ds* on the beam (Fig. 9.30).

$$M = W(200 + 80 \sin \theta)$$

For the portion *AP*, at a distance *x* from *P*, $M = W \cdot x$

Strain energy

$$U = \int \frac{M^2 ds}{2EI} = \frac{1}{2EI}\left[\int_0^{\pi} [W(200 + 80\sin\theta)]^2 R d\theta + \int_0^{200} (Wx)^2 \cdot dx\right]$$

Vertical deflection

$$\delta = \frac{\partial U}{\partial W} = \frac{1}{2EI}\left[\int_0^{\pi} 2W(200 + 80\sin\theta)^2 R d\theta + \int_0^{200} 2Wx^2 \cdot dx\right]$$

$$= \frac{W}{EI}\left[\int_0^{\pi} (200 + 80\sin\theta)^2 R d\theta + \int_0^{200} x^2 \cdot dx\right]$$

$$= \frac{W}{EI}\left[\int_0^{\pi} (40\,000 + 6400\sin^2\theta + 2 \times 200 \times 80 \times \sin\theta) R d\theta + \int_0^{200} x^2 \cdot dx\right]$$

$$= \frac{W}{EI}\left[\int_0^{\pi} 100\left\{(400 + 64\left(\frac{1-\cos\theta}{2}\right) + 320\sin\theta)\right\} 80 d\theta + \left(\frac{x^3}{3}\right)_0^{200}\right]$$

$$= \frac{W}{EI}\left[8000\,\{432\theta - 32\sin\theta - 320\cos\theta\}_0^{\pi} + \frac{200^3}{3}\right]_0^{\pi}$$

$$= \frac{W}{EI}\left[8000\{432\pi - 0 - 320(-1-1)\} + \frac{200^3}{3}\right]$$

$$= \frac{10}{205\,000 \times (25 \times 4^3/12)}\left[8000\{432\pi + 640\} + \frac{200^3}{3}\right]$$

$$= \frac{10 \times 18.644 \times 10^6}{205\,000 \times (25 \times 4^3/12)} = 6.82 \text{ mm}$$

Example 9.17 Find the vertical and horizontal deflections of the end P of a thin curved beam shown in Fig. 9.31. The cross-section of the beam is 12-mm wide and 8 mm thick. E = 205 GPa.

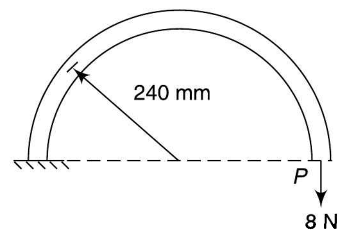


Fig. 9.31

Solution

Given A thin curved beam as shown in Fig. 9.31.

w = 12 mm t = 8 mm E = 205 GPa

To find Vertical deflection of the end P

Consider an element of length ds on the beam (Fig. 9.32a).

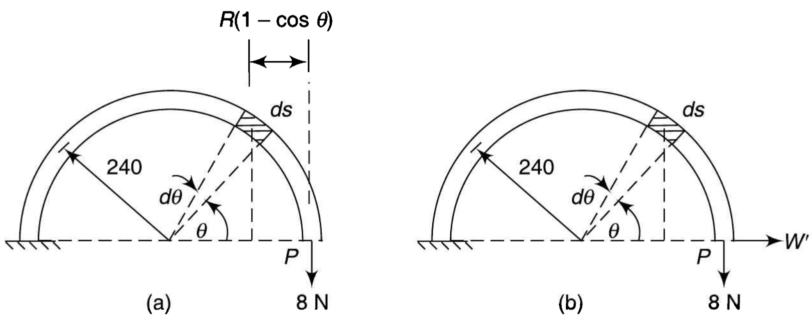


Fig. 9.32

Bending moment at the element, $M = WR(1 - \cos\theta)$

For vertical deflection

$$U = \int \frac{M^2 ds}{2EI} = \frac{1}{2EI}\left[\int_0^\pi [WR(1-\cos\theta)]^2 R d\theta\right]$$

$$\delta = \frac{\partial U}{\partial W} = \frac{1}{2EI}\left[\int_0^\pi 2WR^2(1-\cos\theta)^2 R d\theta\right]$$

$$= \frac{WR^3}{EI}\left[\int_0^\pi (1-\cos\theta)^2 d\theta\right]$$

$$= \frac{WR^3}{EI}\left[\int_0^\pi (1+\cos^2\theta - 2\cos\theta) d\theta\right]$$

$$= \frac{WR^3}{EI}\left[\int_0^\pi (1+\frac{1+\cos 2\theta}{2} - 2\cos\theta) d\theta\right]$$

$$= \frac{WR^3}{EI}\left[\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} - 2\sin\theta\right]_0^{\pi} = \frac{WR^3}{EI}\left[\pi + \frac{\pi}{2} + 0 - 0\right] = \frac{3\pi}{2EI}WR^3$$

$$= \frac{3\pi}{2 \times 205\,000 \times (12 \times 8^3/12)} \times 8 \times 240^3 = 4.97 \text{ mm}$$

For horizontal deflection

To determine the horizontal deflection, introduce a virtual horizontal force W' at P (Fig. 9.32b).
Bending moment at the element, $M = WR(1 - \cos\theta) - W' R \sin\theta$

$$U = \int \frac{M^2 ds}{2EI} = \frac{1}{2EI}\left[\int_0^{\pi} \{WR(1 - \cos\theta) - W' R\sin\theta\}^2 R d\theta\right]$$

Differentiating with respect to W',

$$\delta = \frac{\partial U}{\partial W'} = \frac{1}{2EI}\left[\int_0^{\pi} 2\{WR(1 - \cos\theta) - W' R\sin\theta\}(-R\sin\theta)Rd\theta\right]$$

put $W' = 0$,

$$\delta = -\frac{WR^3}{EI}\left[\int_0^{\pi}(1 - \cos\theta)(\sin\theta)d\theta\right] = -\frac{WR^3}{EI}\left[\int_0^{\pi}\left(\sin\theta - \frac{\sin 2\theta}{2}\right)d\theta\right]$$

$$= -\frac{WR^3}{EI}\left[-\cos\theta + \frac{\cos 2\theta}{4}\right]_0^{\pi} = -\frac{WR^3}{EI}[-(-1-1)] = \frac{2WR^3}{EI}$$

$$= -\frac{2}{205\,000 \times (12 \times 8^3/12)} \times 8 \times 240^3 = -2.107 \text{ mm}$$

Example 9.18 A steel ring of 120-mm mean diameter and of a circular cross-section of 30 mm diameter has a saw cut along the horizontal diameter. It is subjected to tangential separating forces of 10 kN at the cut in the plane of the ring. Find the additional clearance of the ring at the cut. E = 203 GPa.

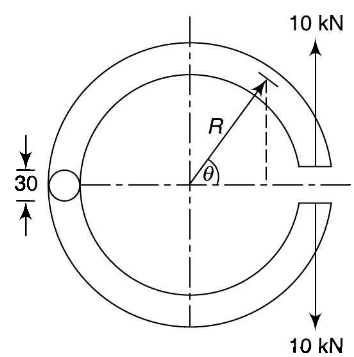


Fig. 9.33

Solution

Given A steel ring of circular cross-section having a cut as shown in Fig. 9.33.

E = 203 GPa

To find Additional clearance of ring at cut

$$M = WR(1 - \cos\theta)$$

Strain energy

$$U = \int \frac{M^2 ds}{2EI} = 2\int_0^{\pi} \frac{1}{2EI}[WR(1 - \cos\theta)]^2 R \cdot d\theta = \frac{W^2R^3}{EI}\int_0^{\pi}[1 - \cos\theta]^2 \cdot d\theta$$

Vertical deflection

$$\text{Vertical deflection} = \frac{\partial U}{\partial W} = \frac{2WR^3}{EI}\int_0^{\pi}[1 - \cos\theta]^2 \cdot d\theta$$

$$= \frac{2WR^3}{EI}\int_0^{\pi}(1 + \cos^2\theta - 2\cos\theta) \cdot d\theta$$

$$= \frac{2WR^3}{EI}\int_0^{\pi}\left(1+\frac{1+\cos 2\theta}{2} - 2\cos\theta\right)\cdot d\theta$$

$$= \frac{2WR^3}{EI}\left(\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} - 2\sin\theta\right)_0^{\pi}$$

$$= \frac{2WR^3}{EI}\left(\pi + \frac{\pi}{2} + 0 - 0\right)$$

$$= \frac{3\pi WR^3}{EI}$$

$$= \frac{3\times\pi\times 10\,000\times 60^3}{203\,000\times(\pi\times 30^4/64)} = 2.52 \text{ mm}$$

Solution by direct method

The problem can be solved by direct method also:

Vertical deflection $= \int \frac{Mx}{EI}ds = 2\cdot\frac{1}{EI}\int_0^{\pi} WR(1-\cos\theta)R(1-\cos\theta)\cdot Rd\theta$

$$= \frac{2WR^3}{EI}\int_0^{\pi}(1-\cos\theta)^2\cdot d\theta$$

$$= \frac{2WR^3}{EI}\int_0^{\pi}(1+\cos^2\theta - 2\cos\theta)\cdot d\theta$$

$$= \frac{2WR^3}{EI}\int_0^{\pi}\left(1+\frac{1+\cos 2\theta}{2} - 2\cos\theta\right)\cdot d\theta$$

$$= \frac{2WR^3}{EI}\left(\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4} - 2\sin\theta\right)_0^{\pi}$$

$$= \frac{2WR^3}{EI}\left(\pi + \frac{\pi}{2} + 0 - 0\right)$$

$$= \frac{3\pi WR^3}{EI}$$

$$= \frac{3\times\pi\times 10\,000\times 60^3}{203\,000\times(\pi\times 30^4/64)} = 2.52 \text{ mm}$$

Example 9.19 A steel ring made of a circular cross-section of 20 mm diameter is applied a pulling force of 8 kN. The mean diameter of the ring is 160 mm. Determine the increase in the diameter along the load. Use the strain energy method. E = 205 GPa.

Solution

Given A steel ring of circular cross-section and loaded as shown in Fig. 9.34.

E = 205 GPa

To find Increase in diameter along load

$$I = \frac{\pi \times 20^4}{64} = 7854 \text{ mm}^4$$

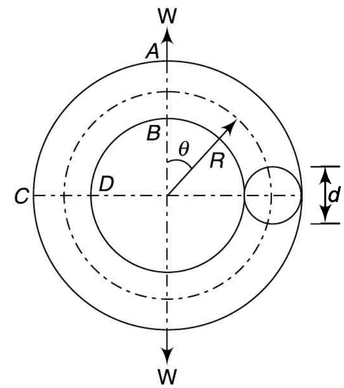


Fig. 9.34

Calculation of p^2

$$p^2 = \frac{d^2}{16} + \frac{d^4}{128R^2} + \ldots = \frac{20^2}{16} + \frac{1}{128} \cdot \frac{20^4}{80^2} = 25.195 \text{ mm}^2$$

$$M = \frac{WR}{2}\left(\frac{2}{\pi} \cdot \frac{R^2}{R^2 + p^2} - \sin\theta\right) \qquad \ldots(\text{Eq. } 9.22)$$

Strain energy

$$U = \int \frac{M^2 ds}{2EI} = \frac{4}{2EI}\left[\int_0^{\pi}\left\{\frac{WR}{2}\left(\frac{2}{\pi} \cdot \frac{R^2}{R^2 + p^2} - \sin\theta\right)\right\}^2 R d\theta\right]$$

Increase in diameter along load line

$$\delta = \frac{\partial U}{\partial W} = \frac{4}{2EI}\left[\int_0^{\pi/2} 2W\left\{\frac{R}{2}\left(\frac{2}{\pi} \cdot \frac{R^2}{R^2 + p^2} - \sin\theta\right)\right\}^2 R d\theta\right]$$

$$= \frac{WR^3}{EI}\left[\int_0^{\pi/2}\left\{\left(\frac{2}{\pi} \cdot \frac{R^2}{R^2 + p^2} - \sin\theta\right)\right\}^2 R d\theta\right]$$

$$= \frac{WR^3}{EI}\left[\int_0^{\pi/2}\left\{\frac{4}{\pi^2}\left(\frac{R^2}{R^2 + p^2}\right)^2 + \sin^2\theta - 2 \cdot \frac{2}{\pi} \cdot \frac{R^2}{R^2 + p^2} \cdot \sin\theta\right\} d\theta\right]$$

$$= \frac{WR^3}{EI}\left[\int_0^{\pi/2}\left\{\frac{4}{\pi^2}\left(\frac{R^2}{R^2 + p^2}\right)^2 + \frac{1 - \cos 2\theta}{2} - \frac{4}{\pi} \cdot \frac{R^2}{R^2 + p^2} \cdot \sin\theta\right\} d\theta\right]$$

$$= \frac{WR^3}{EI}\left[\frac{4}{\pi^2}\left(\frac{R^2}{R^2 + p^2}\right)^2 \theta + \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + \frac{4}{\pi} \cdot \frac{R^2}{R^2 + p^2} \cdot \cos\theta\right]_0^{\pi/2}$$

$$= \frac{WR^3}{EI}\left[\frac{2}{\pi}\left(\frac{R^2}{R^2 + p^2}\right)^2 + \frac{\pi}{4} - \frac{4}{\pi} \cdot \frac{R^2}{R^2 + p^2}\right]$$

$$= \frac{8000 \times 80^3}{205\,000 \times 7854}\left[\frac{2}{\pi}\left(\frac{80^2}{80^2 + 25.195}\right)^2 + \frac{\pi}{4} - \frac{4}{\pi} \cdot \frac{80^2}{80^2 + 25.195}\right]$$

$$= 2.544(0.6316 + 0.7854 - 1.268) = 0.379 \text{ mm}$$

Compare the result with that of Example 9.12.

Summary

1. In case of bars of small initial curvature, $\dfrac{M}{I} = \dfrac{\sigma}{y} = E\left(\dfrac{1}{R'} - \dfrac{1}{R}\right)$
2. Strain energy of a small length δs under the action of bending moment M, $\delta U = (M^2 / 2EI)\, \delta s$
3. In case of large initial curvature (Winkler–Bach theory), $\sigma = \dfrac{M}{AR}\left[1 + \dfrac{R^2}{p^2}\dfrac{y}{R+y}\right]$ where y is the distance of the fibre from the neutral axis and is negative for inner fibres.
4. p^2 is obtained from $p^2 = \dfrac{R^3}{A}\displaystyle\int \dfrac{1}{R+y}dA - R^2$ for different sections.

For rectangular section, $p^2 = \dfrac{R^3}{d}\ln\left(\dfrac{2R+d}{2R-d}\right) - R^2$

For trapezoidal section, $p^2 = \dfrac{R^3}{A}\left[\left(b_1 + \dfrac{(b_2 - b_1)}{d}(R + d_1)\right)\ln\dfrac{R+d_1}{R-d_2} - (b_2 - b_1)\right] - R^2$

For triangular section, $p^2 = \dfrac{2R^3}{h}\left[\left(\dfrac{3R+2h}{3h}\right)\ln\dfrac{(3R+2h)}{(3R-h)} - 1\right] - R^2$

For circular section, $p^2 = \dfrac{d^2}{16} + \dfrac{d^4}{128R^2} + \dots$

For I-section, $p^2 = \dfrac{R^3}{A}\left[b_1 \ln\dfrac{R_2}{R_1} + t\ln\dfrac{R_3}{R_2} + b_2 \ln\dfrac{R_4}{R_3}\right] - R^2$

For T-section, $p^2 = \dfrac{R^3}{A}\left[b_1 \ln\dfrac{R_2}{R_1} + t\ln\dfrac{R_3}{R_2}\right] - R^2$

5. Stress in a circular ring, $\sigma = \dfrac{W}{A}\left[\dfrac{R^2}{\pi(R^2+p^2)} + \dfrac{R^2}{2p^2}\left(\dfrac{2}{\pi}\cdot\dfrac{R^2}{R^2+p^2} - \sin\theta\right)\dfrac{y}{R+y}\right]$
6. Stress in a chain link, $\sigma = E\varepsilon' + \dfrac{W}{A}\left[\dfrac{R^2}{2p^2}\left(\dfrac{2}{\pi}\cdot\dfrac{R^2}{R^2+p^2} - \sin\theta\right)\dfrac{y}{R+y}\right]$
7. Deflection of curved bars, vertical = $\displaystyle\int \dfrac{Mx}{EI}ds$, horizontal $\displaystyle\int \dfrac{My}{EI}ds$
8. Deflection of a closed ring, along load line $= -\dfrac{WR^3}{2EI}\left[\dfrac{4}{\pi}\cdot\dfrac{R^2}{R^2+p^2} - \dfrac{\pi}{2}\right]$

Along perpendicular to load line $= -\dfrac{WR^3}{2EI}\left[\dfrac{4}{\pi}\cdot\dfrac{R^2}{R^2+p^2} - 1\right]$

9. Deflection of a chain link, along the load line $= \dfrac{WqlR}{EI} + \dfrac{WR^2}{EI}\left[2q + R - \dfrac{\pi R}{4}\right] + \dfrac{Wl}{2AE}$

10. Perpendicular to load line $= \dfrac{WqlR}{EI} + \dfrac{WR^2}{EI}\left[2q + \dfrac{R}{2}\right]$

Objective Type Questions

1. The curved beam theory was forwarded by
(a) Bernoulli (b) Winkler–Bach (c) Rankine (d) Maxwell

2. The variation of bending stress in a curved beam is
(a) linear (b) parabolic (c) hyperbolic (d) cubic

3. Usually, the cross-section of a crane hook is
(a) square (b) rectangular (c) circular (d) trapezoidal

4. When a crane hook carries the load, the nature of stress at the inside edge is
(a) tensile (b) compressive (c) shear

5. The maximum stress in a ring under tension occurs at ______ with the line of action of the load.
(a) 0° (b) 30° (c) 45° (d) 90°

6. If a small cut is made at a horizontal diameter of a ring under compression, the maximum stress
(a) decreases (b) increases (c) does not change (d) becomes zero

7. If a small cut is made at a horizontal diameter of a ring under compression, the deflection
(a) decreases (b) increases (c) does not change (d) becomes infinite

Answers

1. (b) 2. (c) 3. (d) 4. (a) 5. (a) 6. (b)
7. (b)

Review Questions

9.1 Establish the relation for bars of small initial curvature, $\dfrac{M}{I} = \dfrac{\sigma}{y} = E\left(\dfrac{1}{R'} - \dfrac{1}{R}\right)$

9.2 Explain the Winkler–Bach theory as applicable to bars of large initial curvature.

9.3 Deduce expression to determine the stress in a circular ring acted upon by a tensile load W.

9.4 Establish a relation to find the stress in a chain link acted upon by a tensile load W.

9.5 Describe the procedure to find the deflection of curved bars. Deduce the expressions to find the deflections of closed links along the load line and perpendicular to load line.

9.6 Establish relations to determine deflections of a chain link along and perpendicular to the load line.

Numerical Problems

9.1 A curved bar of 30-mm square section has a mean radius of curvature of 45 mm. Assuming the bar initially to be unstressed, find the stresses at the inner and the outer faces when a bending moment of 300 N.m is applied to the bar tending to straighten it. (85.9 MPa, tension; 54.1 MPa, compression)

9.2 The inside diameter of a circular ring is 100 mm. The ring has a circular cross-section of 80 mm diameter as shown in Fig. 9.35. Find the stresses at P and Q.
(22.56 MPa, tensile; 58.67 MPa, compressive)

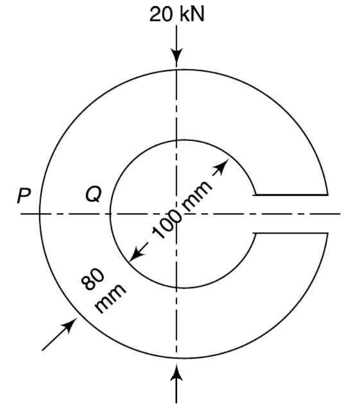


Fig. 9.35

9.3 An open ring with a *T*-section is shown in Fig. 9.36. Determine the stresses at *P* and *Q* if it is subjected to a compressive load of 120 kN. (177.1 MPa tensile; 183.6 MPa compressive)

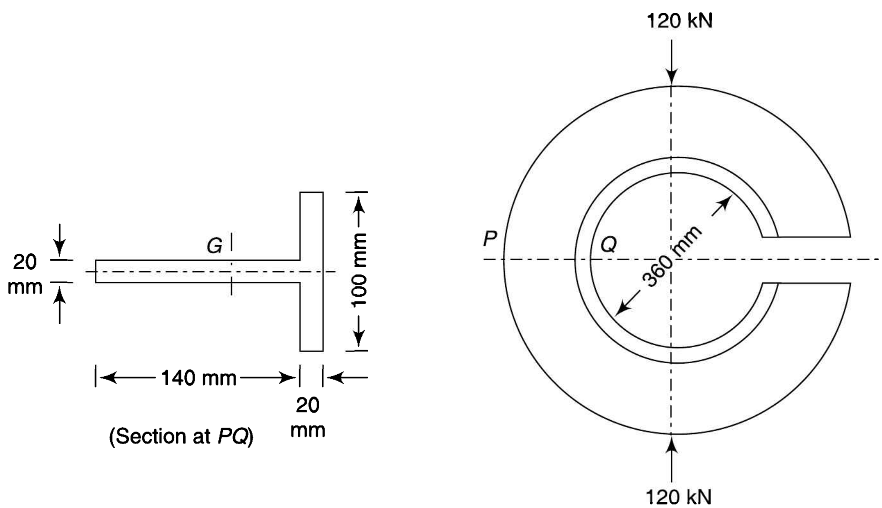


Fig. 9.36

9.4 Find the ratio of numerical values of maximum and minimum stresses for a curved bar of rectangular section in pure bending if the radius of curvature is 90 mm and the depth of beam 60 mm. Also locate the position of neutral axis. (1.397, radius of neutral axis, -3.44 mm)

9.5 A crane hook having a trapezoidal horizontal cross-section is 50 mm wide at inside and 25 mm wide at outside. Thickness of the section is 50 mm. The crane hook carries a vertical load of 10 kN whose line of action is 38 mm from the inside edge of the section. The centre of curvature is 50 mm from the inside edge. Determine the maximum tensile and compressive stresses in the section. (51 MPa, tensile; 30.45 MPa, compressive)

9.6 A ring made of 20-mm diameter steel bar carries a pull of 4.5 kN. The mean radius of the ring is 160 mm. Determine the maximum tensile and compressive stresses in the material of the ring. (133.1 MPa, 160.7 MPa)

9.7 A ring made of a rectangular cross-section of 40 mm $\times$ 30 mm has a mean diameter of 160 mm, the larger section being in the radial direction. Determine the compressive force which the ring can sustain safely if the allowable stress is 160 MPa. (75.24 kN)

9.8 A tube of 80 mm external diameter and 5 mm thick is bent in the form of a circular arc of 150-mm radius. Find the maximum tensile and compressive stresses set up by a bending moment of 250 N.m tending to increase its curvature. (68.7 MPa, tensile; 114.7 MPa, compressive)

9.9 A steel tube of 50-mm outer diameter and 30-mm inner diameter is bent in the form of a quadrant of 2 m radius. One end is rigidly fixed to a horizontal base and the free end supports a load of 2 kN. Find the vertical and the horizontal deflections of the free end. $E = 208$ GPa. (226 mm, 144 mm)

9.10 A ring made of round steel of 15 mm diameter and having a mean diameter of 120 mm is applied a pull of 5 kN. Determine the increase in the diameter along the load line and decrease in diameter along the lateral direction. $E = 200$ GPa. (0.329 mm, 0.291 mm)

9.11 A steel ring is made of rectangular cross-section 12-mm wide and 6-mm thick. The mean diameter of the ring is 300 mm. A narrow radial saw cut is made and at the cut, tangential separating forces are applied in the plane of the ring. Determine the additional separation due to these forces. $E = 200$ GPa. (3.68 mm)

Chapter 10

Torsion

A shaft is an example of a member in torsion. Shafts are widely used in engineering applications to transmit power from one point to another, from a motor to a machine tool, from an engine to the rear axle of an automobile or from a steam or hydraulic turbine to an electric generator. The shafts used for the purpose may be solid or hollow. When a shaft is made to transmit power, a pure twisting couple or torque acts about its polar axis. Shear stresses are set up perpendicular to the radius at all transverse sections. As a result, complementary shear stresses are developed on the longitudinal planes which cause a distortion of filaments. As the cross-section across any transverse plane is symmetrical, the points remain at the same radial distance before and after the twist. The angle of twist is reckoned over a length of the shaft.

The assumptions made in the theory of torsion are as under:

1. The material of the shaft is homogeneous, isotropic and perfectly elastic.
2. The material obeys Hooke's law and the stress remains within limit of proportionality.
3. The twisting couples act in the transverse planes only.
4. All radii remain straight after torsion.
5. Parallel planes normal to the axis do not warp or distort after torsion.
6. A cross-section at any axial length rotates as a rigid plane, i.e., all diameters of a cross-section rotate through the same angle.

10.1 CIRCULAR SHAFTS

Figure 10.1 shows a circular shaft of length l and diameter d acted upon by a torque T. The shear strain φ of elements at a radial distance r from the axis is constant for a particular value of T. A line OA thus twists to OB.

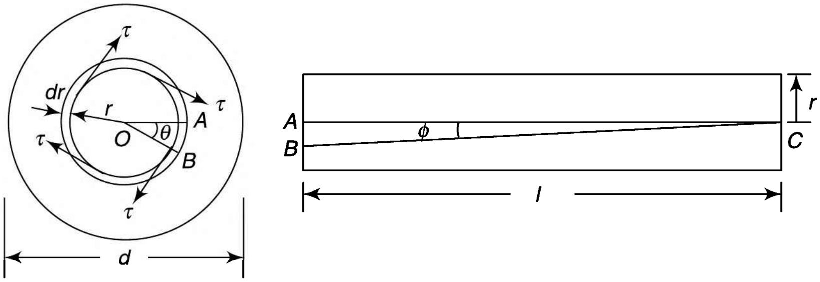


Fig. 10.1

Let θ be the angle of twist of a cross-section relative to another cross-section at distance l apart. Then

$$\text{Arc } AB = r\theta = l\phi = l \cdot \frac{\tau}{G} \qquad \ldots\ldots(G = \tau/\varphi)$$

or

$$\frac{\tau}{r} = \frac{G\theta}{l} \tag{i}$$

Now, tangential force on the element = $\tau(2\pi r \cdot dr)$

Moment of the tangential force on the element = $\tau(2\pi r \cdot dr)r$

Sum of moments for the whole shaft, $T = \int \tau(2\pi r \cdot dr)r$

$$= \int \frac{G\theta}{l} r \cdot (2\pi r \cdot dr)r = \frac{G\theta}{l} \int (2\pi r \cdot dr)r^2$$

But $\int(2\pi r \cdot dr)r^2$ is the *polar moment of inertia* of the shaft,

Thus

$$T = \frac{G\theta}{l} \cdot J \tag{ii}$$

From (i) and (ii),

$$\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{l} \tag{10.1}$$

This shows that for a given shaft, shear stress is proportional to radius. Figures 10.2a shows the shear stress distribution in solid shaft of radius R. Similarly, the stress distribution in a hollow shaft of inner radius R_i and outer radius R_o is shown in Fig. 10.2b.

The Eq. 10.1 is analogous to Eq. 5.4 for bending where

— T/J corresponds to M/I

— τ/r corresponds to σ/y

— $G\theta/l$ corresponds to E/R.

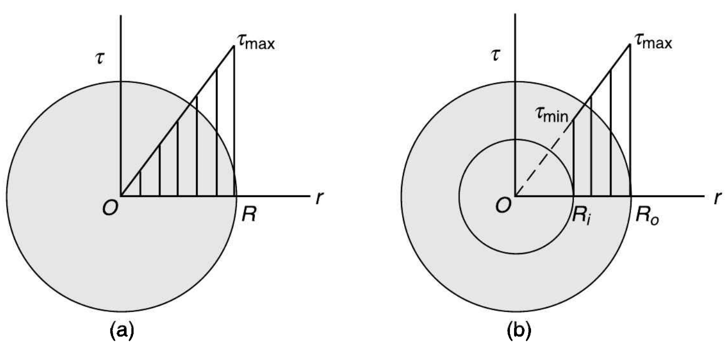


Fig. 10.2

As polar moment of inertia, $J = I_x + I_y$,

For circular shafts, $J = \frac{\pi}{64}d^4 + \frac{\pi}{64}d^4 = \frac{\pi}{32}d^4$

Thus for solid and hollow shafts, the following deductions can be made:

Solid Shaft

Maximum shear stress, $\tau = \frac{T \cdot d/2}{\pi d^4/32} = \frac{16T}{\pi d^3}$ at the outer surface (10.2)

Hollow Shaft

If D and d are the outer and inner diameters respectively of the hollow shaft, then $J = \dfrac{\pi(D^4 - d^4)}{32}$

$$\therefore \quad \text{Maximum stress, } \tau = \frac{T \cdot D/2}{\pi(D^4 - d^4)/32} = \frac{16T \cdot D}{\pi(D^4 - d^4)} \text{ at the outer surface} \qquad (10.3)$$

Torsional Moment of Resistance (Resisting Torque)

The maximum torque which can be carried by a given section of the shaft for a given maximum value of shear stress is known as *torsional moment of resistance* or *resisting torque* (T_r). It is the sum of the moments of tangential shear stress acting on any transverse section.

- J/R is known as the *torsional* or *polar sectional moment.*
- Torque per radian twist is known as the *torsional stiffness* (k), $k = \dfrac{T}{\theta} = \dfrac{JG}{l}$ (10.4)
- The parameter GJ is known as the *torsional rigidity* of the shaft similar to flexural rigidity EI of a beam.

From Eq. 10.1, $GJ = \dfrac{T}{\theta / l}$, thus *torsional rigidity* is also the torque per unit angular twist.

10.2 POWER TRANSMISSION

Shafts are the medium to transmit power from the point of power generation to the point of its application. The power transmitted is the work done per second. Thus

$$P = T \cdot \omega = T \cdot \frac{2\pi N}{60} = \frac{2\pi NT}{60}$$

where T is the applied torque and N the revolutions of the shaft in rpm.

A shaft may transmit power at more than one section to different machines. Also, the shafts may be subjected to different torques in different sections. In such cases the principle of conservation of energy can be applied. Usually, the shafts are designed on the basis of limiting values of stress and the angular deformation.

Example 10.1 A solid steel shaft transmits 100 kW at 150 rpm. Determine the suitable diameter of the shaft if the maximum torque transmitted exceeds the mean by 20% in each revolution. The shear stress is not to exceed 60 MPa. Also find the maximum angle of twist in a length of 4 m of the shaft. G = 80 GPa.

Solution

Given

$P = 100$ kW $\qquad N = 150$ rpm

$l = 4$ m $\qquad \tau = 60$ MPa

$G = 80$ GPa

To find Diameter of shaft if torque increases by 20%

Determination of torque

$$T = \frac{60P}{2\pi N} = \frac{60 \times 100\,000}{2\pi \times 150} = 6366 \text{ N}\cdot\text{m}$$

$$T_{max} = 6366 \times 1.2 = 7639 \text{ N}\cdot\text{m} \quad \text{or} \quad 7.639 \times 10^6 \text{ N}\cdot\text{mm}$$

Shaft diameter

$$\tau = \frac{16T}{\pi d^3}$$

or $$60 = \frac{16 \times 7.639 \times 10^6}{\pi d^3}$$

or $$d = 86.6 \text{ mm}^4$$

Maximum angle of twist

$$\frac{\tau}{r} = \frac{G\theta}{l} \quad \text{or} \quad \frac{60}{86.6/2} = \frac{80\,000 \times \theta}{4000}$$

or $$0.069\,28 \text{ rad} = (0.069\,28 \times 180/\pi)^\circ = 3.97^\circ$$

Example 10.2 A hollow steel shaft transmits 200 kW of power at 150 rpm. The total angle of twist in a length of 5 m of the shaft is 3°. Find the inner and outer diameters of the shaft if the permissible shear stress is 60 MPa. G = 80 GPa.

Solution

Given

$P = 200$ kW	$N = 150$ rpm
$l = 5$ m	$\tau = 60$ MPa
$G = 80$ GPa	$\theta = 3^\circ$

To find Outer and inner diameters of hollow shaft, D and d

Torque

$$T = \frac{60P}{2\pi N} = \frac{60 \times 200\,000}{2\pi \times 180} = 10\,610 \text{ N}\cdot\text{m} \quad \text{or} \quad 10.61 \times 10^6 \text{ N}\cdot\text{mm}$$

From angle of twist limit

$$\theta = 3 \times \frac{\pi}{180} = 0.0524 \text{ rad}$$

$$\frac{T}{J} = \frac{G\theta}{l} \quad \text{or} \quad \frac{10.61 \times 10^6}{\pi(D^4 - d^4)/32} = \frac{80\,000 \times 0.0524}{5000}$$

or $$D^4 - d^4 = 124 \times 10^6 \text{ mm}^4 \qquad \text{(i)}$$

From maximum shear stress limit

Also, $$T = \tau \cdot \frac{\pi}{16}\left(\frac{D^4 - d^4}{D}\right)$$

or $$10.61 \times 10^6 = 60 \times \frac{\pi}{16}\left(\frac{D^4 - d^4}{D}\right)$$

or $$D^4 - d^4 = 900\,605D \qquad \text{(ii)}$$

Inner and outer diameters

From (i) and (ii), $900\,605D = 129 \times 10^6$ or $D = 143.2$ mm

and $143.2^4 - d^4 = 129 \times 10^6$ or $d = 130.7$ mm

Example 10.3 || Compare the weights of equal lengths of a solid and a hollow shaft to transmit a given torque for the same maximum stress if the inside diameter of the shaft is three fourth of the outside.

Solution

Given Solid and hollow shafts of equal length, transmitting same torque, same maximum stress
Inner diameter = $3D/4$

To find To compare weights of two shafts

For solid shaft

$$\frac{T}{\tau} = \frac{\pi d^3}{16} \qquad \text{(i)}$$

For hollow shaft

$$\frac{T}{\tau} = \frac{\pi[D^4 - (3D/4)^4]}{16D} = \frac{\pi D^4(1 - 81/256)}{16D} = 0.6836\frac{\pi D^3}{16} \qquad \text{(ii)}$$

Equating the two

As the shafts transmits a given torque for the same maximum stress, i.e., T/τ is the same,

Equating (i) and (ii), $0.6836\dfrac{\pi D^3}{16} = \dfrac{\pi d^3}{16}$ or $\dfrac{D}{d} = 1.135$

Ratio of weights

$$\text{Ratio of weights of equal lengths} = \frac{\text{Area of hollow shaft}}{\text{Area of solid shaft}}$$

$$= \frac{D^2 - (3D/4)^2}{d^2} = \frac{D^2}{d^2}\cdot\left(1 - \frac{9}{16}\right) = 1.135^2 \times \frac{7}{16} = 0.564$$

Example 10.4 || A shaft transmits 280 kW of power at 160 rpm. Determine

(i) The diameter of a solid shaft to transmit the required power
(ii) The inner and outer diameters of a hollow circular shaft if the ratio of the inner to the outer diameter is 2/3
(iii) The percentage saving in the material on using a hollow shaft instead of a solid shaft

Take the allowable stress as 80 MPa and the density of material 78 kN/m^3.

Solution

Given A solid shaft

$P = 280$ kW $\qquad N = 160$ rpm
$T = 80$ MPa $\qquad \rho = 78$ kN/m^3

To find

— Diameter of solid shaft
— Inner and outer diameters of hollow shaft if ratio is 2/3
— percentage saving in material of hollow shaft over solid shaft

$$P = \frac{2\pi NT}{60} \quad \text{or} \quad 280 = \frac{2\pi \times 160T}{60} \quad \text{or} \quad T = 16.71\ \text{kN}\cdot\text{m} \quad \text{or} \quad 16.71 \times 10^6\ \text{N}\cdot\text{mm}$$

For solid shaft

$$\frac{T}{\tau} = \frac{\pi d^3}{16} \quad \text{or} \quad \frac{16.71 \times 10^6}{80} = \frac{\pi d^3}{16} \quad \text{or} \quad d = 102\ \text{mm}$$

For hollow shaft

$$\frac{T}{\tau} = \frac{\pi[D^4 - (2D/3)^4]}{16D} \quad \text{or} \quad \frac{16.71 \times 10^6}{80} = \frac{65\pi D^3}{81 \times 16} \quad \text{or} \quad D = 109.8 \text{ mm}$$

$$\text{and inner diameter} = \frac{2}{3} \times 109.8 = 73.2 \text{ mm}$$

Saving in material

$$\text{Saving of material} = \frac{(\pi/4)d^2 - (\pi/4)[D^2 - (2D/3)^2)]}{(\pi/4)d^2}$$

$$= 1 - \frac{5D^2}{9d^2} = 1 - \frac{5 \times 109.8^2}{9 \times 102^2} = 0.356 \quad \text{or} \quad 35.6\,\%$$

Example 10.5 Compare the resistance to torsion of a hollow shaft to that of a solid shaft if the inside diameter of the hollow shaft is two-third of the external diameter and the two shafts have the same material and weight and of equal length.

Solution

Given Solid and hollow shafts of same material, weight and of equal length
Inner diameter of hollow shaft = two third of outer diameter
To find To compare resistance to torsion of two shafts

Let d = diameter of solid shaft
D = outer diameter of hollow shaft

Ratio of diameters

As the weight of the two shafts is the same, cross-sectional areas are to be equal.

$$\frac{\pi}{4}\left[D^2 - \left(\frac{2D}{3}\right)^2\right] = \frac{\pi}{4}d^2 \quad \text{or} \quad \frac{5}{9}D^2 = d^2 \quad \text{or} \quad D = 0.1342d$$

For hollow shaft

Resisting torque of a hollow shaft,

$$\frac{T_h}{\tau} = \frac{\pi[D^4 - (2D/3)^4]}{16D} = \frac{\pi}{16} \times \frac{65}{81}D^3 = \frac{\pi}{16} \times \frac{65}{81}(1.342d)^3 = \frac{\pi}{16} \times 1.939d^3$$

For solid shaft

Resisting torque of a solid shaft, $\dfrac{T_s}{\tau} = \dfrac{\pi d^3}{16}$

Ratio of resistance to torsion

As the material of the two shafts is the same, the maximum stress is to be the same.

$\therefore$ ratio of resistance, $\dfrac{T_h}{T_s} = 1.939$

Example 10.6 A shaft transmits 800 kW of power at 210 rpm. Determine the actual working stress and the diameter of the shaft if the shaft twists one degree on a length of 18 diameters and the shear stress is not to exceed 50 MPa. Take G = 81 GPa.

Solution

Given

P = 800 kW $\qquad$ N = 210 rpm

$l = 18\,d$ $\qquad \tau = 50$ MPa

$G = 81$ GPa $\qquad \theta = 1°$

To find

— Diameter of shaft

— Actual working stress

$P = 800$ kW $= 800 \times 10^3$ N·m/s $= 800 \times 10^6$ N·mm/s

From angle of twist limit

$$T = J \cdot \frac{G\theta}{l} = \frac{\pi d^4}{32} \cdot \frac{81\,000}{18d}\left(\frac{1 \times \pi}{180}\right) = 7.711d^3$$

From maximum shear stress limit

$$T = \tau \cdot \frac{\pi d^3}{16} = 50 \times \frac{\pi d^3}{16} = 9.817d^3$$

Diameter of shaft

Considering the smaller value for safe working,

$$P = T \cdot \omega$$

or $\quad 800 \times 10^6 = 7.711d^3 \times \dfrac{2\pi \times 210}{60} \quad$ or $\quad d = 167.7$ mm

Actual working stress

$$\tau = \frac{16T}{\pi d^3} = \frac{16 \times 7.711d^3}{\pi \times d^3} = 39.3 \text{ MPa}$$

Example 10.7 || A 2-m long and of 40-mm diameter aluminium shaft is to be replaced by a hollow steel shaft of the same length and the same outside diameter with the same angle of twist over the total length. Determine the inside diameter of the hollow shaft. Take the modulus of rigidity of the steel thrice of that of the aluminium.

Solution

Given A solid aluminium shaft of 40-mm diameter and of 2-m length

To find Inside diameter of a hollow steel shaft of same length, same angle of twist, same outside diameter and with same torque as the solid shaft

$$G_s = 3\,G_a$$

Let $\quad D$ = diameter of solid shaft = outside diameter of hollow shaft

and $\quad d$ = inside diameter of the hollow shaft

From torsion equation

In the relation, $\dfrac{T}{J} = \dfrac{G\theta}{l}$, as T, l and θ are the same for the solid steel shaft and hollow aluminium shafts,

$\therefore \quad G_s \cdot J_s = G_a \cdot J_a \quad$ or $\quad J_s = \dfrac{G_a}{G_s} J_a$

or $\quad \dfrac{\pi}{32} \times \dfrac{D^4 - d^4}{D} = \dfrac{1}{3} \times \dfrac{\pi}{32} \times D^3$

or $\quad 2D^4 = 3d^4 \quad$ or $\quad \dfrac{d}{D} = \left(\dfrac{2}{3}\right)^{1/4} = 0.9036$

or $\quad d = 0.9036 \times 40 = 36.14$ mm

Example 10.8 || The cross-sectional area of a solid shaft of 160-mm diameter is the same as of a hollow shaft of the same material and of 120-mm inside diameter. Determine

(i) the ratio of power transmitted for the same angular velocity
(ii) the ratio of angles of twist for the equal lengths when stressed to same intensity

Solution

Given A solid shaft of 160-mm diameter and a hollow shaft of same material and same cross-sectional area and of 120-mm inside diameter

To find

— Ratio of power transmitted for the same angular velocity
— ratio of angles of twist when stressed to same intensity

Let D be the outer diameter of the hollow shaft.
Then, as areas of cross-section of both shafts are equal,

$$\frac{\pi}{4}(D^2 - 120^2) = \frac{\pi}{4} \times 160^2 \quad \text{or} \quad D = 200 \text{ mm}$$

Ratio of power transmitted

As the angular velocities of the two shafts are the same, the power transmitted will be in the ratio of their torques.
As the material of the two shafts is the same, the maximum stress is to be the same.

$$\text{Torque of the hollow shaft, } T_h = \tau \cdot \frac{\pi}{16}\left(\frac{200^4 - 120^4}{200}\right)$$

$$\text{Torque of the solid shaft, } T_s = \tau \cdot \frac{\pi}{16} \times 160^4$$

$$\therefore \quad \text{ratio of power transmitted, } \frac{P_h}{P_s} = \frac{T_h}{T_s} = \frac{200^4 - 120^4}{200 \times 160^3} = 1.7$$

Ratio of angles of twist

In the relation $\frac{\tau}{r} = \frac{G\theta}{l}$, as τ, l and G are the same for the solid steel shaft and hollow aluminium shafts,

$$\therefore \quad r_s \cdot \theta_s = r_h \cdot \theta_h \quad \text{or} \quad \frac{\theta_s}{\theta_h} = \frac{r_h}{r_s} = \frac{200}{160} = 1.25$$

Example 10.9 || A 300-mm diameter steel shaft is to be replaced by a hollow shaft of the same outside diameter and made up of an alloy steel to transmit 32% increased power. The allowable shear stress value for the alloy steel is greater by 30%. The speed is also increased by 10%. Find the maximum inside diameter of the hollow shaft.

Solution

Given A solid steel shaft and a hollow alloy shaft
Diameter of solid shaft = outside diameter of hollow shaft = 300 mm

$\tau_h = 1.3\, \tau_s$ $\qquad$ $P_h = 1.32\, P_s$
$\omega_h = 1.1\, \omega_s$

To find Inside diameter of hollow shaft

Let D be the diameter of solid shaft as well as outside diameter of hollow shaft and d be the inside diameter of the hollow shaft with corresponding subscripts.

Using the torsion equation

$$\frac{P_h}{P_s}=\frac{T_h}{T_s}\cdot\frac{\omega_h}{\omega_s}=\frac{J_h\tau_h/r_h}{J_s\tau_s/r_s}\cdot\frac{\omega_h}{\omega_s}=\frac{J_h}{J_s}\cdot\frac{\tau_h}{\tau_s}\cdot\frac{D_s}{D_h}\cdot\frac{\omega_h}{\omega_s}$$

or $1.32=\dfrac{D_h^4-d_h^4}{D_h\cdot D_s^3}\times 1.3\times 1\times 1.1$ (Outside diameters equal)

or $$\frac{300^4-d_h^4}{300^4}=\frac{1.32}{1.3\times 1.1}$$

or $d_h = 158$ mm

Example 10.10 Deduce an expression for the allowable twisting moment of a thin–walled tube. Also find an approximate expression for the strength weight ratio of such a tube.

Solution

Given A thin-walled tube

To find

— expression for allowable twisting moment

— expression for strength weight ratio

Twisting moment

Let t be the thickness of the tube.

$$D=2R \quad \text{and} \quad d=2(R-t)$$

$$J=\frac{\pi}{32}[D^4-d^4]=\frac{\pi}{32}[(2R)^4-16(R-t)^4]$$

$$=\frac{\pi}{2}[R^4-(R-t)^4]$$

$$=\frac{\pi}{2}[R^4-(R^2+t^2-2Rt)^2] \qquad \text{....(neglecting higher powers of } t)$$

$$=\frac{\pi}{2}[R^4-(R^2-2Rt)^2]$$

$$=\frac{\pi}{2}[R^4-(R^4+4R^2t^2+4R^3t)]$$

$$=\frac{\pi}{2}[4R^3t]=2\pi R^3t \qquad \text{....(neglecting higher powers of } t)$$

Now as $T=J\cdot\dfrac{\tau}{R}$ $\quad\therefore\quad T=2\pi R^2t\cdot\tau$

Strength–weight ratio

Area of cross-section, $A=\pi\lfloor R^2-(R-t)^2\rfloor$

$=\pi\lfloor R^2-(R^2+t^2-2Rt\rfloor$

$=2\pi Rt$ ....(neglecting higher powers of t)

Let w be the specific weight of the material of the tube.

Weight of the tube $=Al\cdot w=2\pi Rtlw$

Strength–weight ratio, $\dfrac{T}{w}=\dfrac{2\pi R^2t\cdot\tau}{2\pi Rtlw}=\dfrac{R\cdot\tau}{lw}$

This ratio is important in aircraft designs.

10.3 TORSION OF TAPERED SHAFT

Let a shaft of length l under the action of an axial torque T tapers uniformly from a radius $r + r'$ at one end to $r - r'$ at the other as shown in Fig. 10.3.

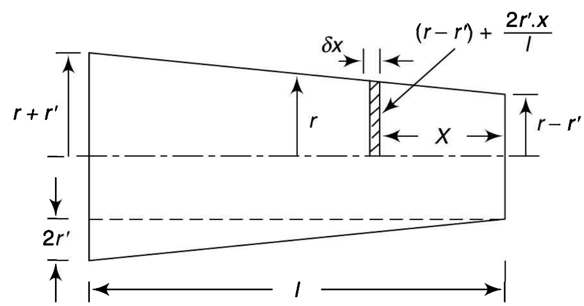


Fig. 10.3

Radius of the shaft at a distance x from the small end $= (r - r') + \dfrac{2r' \cdot x}{l}$

Twist of a small length δx from small end $= \dfrac{T \cdot \delta x}{GJ} = \dfrac{T \cdot \delta x \cdot 2}{G\pi(r - r' + 2r'x/l)^4}$

Total angle of twist,

$$\theta = \frac{2T}{G\pi}\int_0^l \frac{1}{(r - r' + 2r'x/l)^4} \cdot dx = \frac{2T}{G\pi}\int_0^l \left(r - r' + \frac{2r'x}{l}\right)^{-4} \cdot dx = \frac{2T}{G\pi} \cdot \frac{1}{-3} \cdot \frac{l}{2r'}\left[\frac{1}{(r - r' + 2r'x/l)^3}\right]_0^l$$

$$= -\frac{T}{3G\pi} \cdot \frac{l}{r'}\left[\frac{1}{(r + r')^3} - \frac{1}{(r - r')^3}\right]$$

Example 10.11 Determine the percentage error in the angle of twist for a given length of a tapered shaft when calculations are made on the assumption of a constant mean radius. The larger cross-section of the shaft is 12% more and the smaller 12% less than the mean radius.

Solution

Given A tapered shaft

$r' = 0.12\, r$

To find Percentage error in the angle of twist

For tapered shaft

$$\theta = -\frac{T}{3G\pi} \cdot \frac{l}{r'}\left[\frac{1}{(r + r')^3} - \frac{1}{(r - r')^3}\right]$$

$$= -\frac{T}{3G\pi} \cdot \frac{l}{0.12r} \cdot \frac{1}{r^3}\left[\frac{1}{1.12^3} - \frac{1}{0.88^3}\right]$$

$$= 2.099\,\frac{Tl}{G\pi r^4}$$

For uniform shaft

For a shaft of uniform radius r, $\theta = \dfrac{Tl}{GJ} = \dfrac{2Tl}{G\pi r^4}$

$$\text{Percentage error} = \frac{2.099 - 2.0}{2} \times 100 = 4.95$$

Example 10.12 || A tapered shaft under the action of an axial torque T tapers uniformly from a radius R at one end to R' at the other. Derive an expression for the angle of twist in a length l.

Find the angle of twist for a 2-m long shaft when the larger end is fixed and a twisting moment of 8 kN·m is applied to the smaller end, the radii at the two ends being 48 mm and 64 mm respectively. Also determine the percentage error in the angle of twist when calculations are made on the basis of mean radius of 56 mm. G = 84 MPa.

Solution

Given A tapered shaft

$T = 8$ kN·m	$l = 2$ m
$R = 64$ mm	$R' = 48$ mm
$R_{mean} = 56$ mm	$G = 84$ MPa

To find Percentage error in the angle of twist

Expression for the angle of twist

The required expression can be derived in a similar way as in Section 10.3.

Here, $R = r + r'$ (i)

and $R' = r - r'$ (ii)

Subtracting (ii) from (i), $R - R' = 2r'$ or $r' = \dfrac{R - R'}{2}$

Thus,

$$\theta = -\frac{T}{3G\pi} \cdot \frac{l}{r'}\left[\frac{1}{(r + r')^3} - \frac{1}{(r - r')^3}\right] = -\frac{T}{3G\pi} \cdot \frac{2l}{(R - R')}\left[\frac{1}{R^3} - \frac{1}{R'^3}\right]$$

For tapered shaft

$$\theta = -\frac{8 \times 10^6}{3 \times 84\,000\pi} \cdot \frac{2 \times 2000}{(64 - 48)}\left[\frac{1}{64^3} - \frac{1}{48^3}\right]$$

$$= -0.002\,526 \times 10^6 (-5.2275 \times 10^{-6})$$

$$0.0132 \text{ rad} = (0.0132 \times 180/\pi)^\circ = 0.757^\circ$$

For uniform shaft

For a solid shaft of 56-mm mean radius

$$\theta = \frac{Tl}{GJ} = \frac{8 \times 10^6 \times 2000}{84\,000 \times (\pi/32) \times (2 \times 56)^4} = 0.01233 \text{ rad}$$

$$\text{Percentage error} = \frac{0.0132 - 0.01233}{0.0132} \times 100 = 6.59\%$$

10.4 SHAFTS IN SERIES AND PARALLEL

Two shafts may be joined in series or parallel. Let subscripts 1 and 2 denote various parameters for the two shafts.

Shafts in Series When two shafts are joined in series and a single torque is applied, both shafts are subjected to the same torque. Thus

$$T = J_1 \cdot \frac{\tau_1}{R_1} = J_2 \cdot \frac{\tau_2}{R_2}; \text{ Also } T = \frac{G_1 J_1 \theta_1}{l_1} = \frac{G_2 J_2 \theta_2}{l_2}$$

The angle of twist is the sum of tangle of twist of each shaft, i.e.,

$$\theta = \frac{Tl_1}{G_1 J_1} + \frac{Tl_2}{G_2 J_2}$$

Shafts in Parallel When two shafts are joined in parallel, torque applied to the composite shaft is the sum of the torques on the two shafts, i.e.,

$$T = T_1 + T_2 = \frac{G_1 J_1 \theta_1}{l_1} + \frac{G_2 J_2 \theta_2}{l_2}$$

If angular twist and the length are the same, $T = \frac{\theta}{l}(G_1 J_1 + G_2 J_2)$

Thus angular twist, $\theta = \dfrac{Tl}{G_1 J_1 + G_2 J_2}$

Example 10.13 Figure 10.4 shows a stepped steel shaft. It is subjected to a torque T at the free end and a torque $2T$ in the opposite direction at the junction of the two sizes. Determine the total angle of twist if the maximum shear stress is limited to 80 MPa. Take G = 80 GPa.

Solution

Given A stepped steel shaft as shown in Fig. 10.4.

τ = 80 MPa $\quad$ G = 80 GPa

To find Total angle of twist

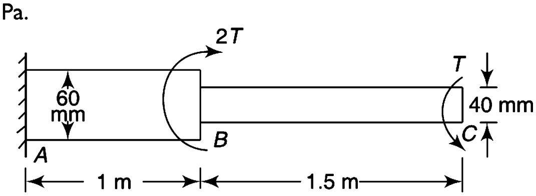


Fig 10.4

Torque in the portion $BC = T$ (counter-clockwise)

Torque in the portion $AB = T - 2T$ (counter-clockwise) or (T clockwise)

Thus the two portions of the shaft are subjected to a torque of same magnitude but in the opposite direction.

Maximum stress will reach the maximum value in the thinner portion first.

$$T = \frac{\pi d^3}{16} \cdot \tau = \frac{\pi \times 40^3}{16} \times 80 = 1\,005\,310 \text{ N} \cdot \text{mm}$$

Angle of twist

$$\theta = \frac{Tl_{bc}}{GJ_{bc}} - \frac{Tl_{ab}}{GJ_{ab}} = \frac{T}{G}\left(\frac{l_{bc}}{J_{bc}} - \frac{l_{ab}}{J_{ab}}\right)$$

$$= \frac{1\,005\,310}{80\,000 \times (\pi/32)}\left(\frac{1500}{40^4} - \frac{1000}{60^4}\right)$$

$$= 0.065 \text{ rad} = \left(0.065 \times \frac{180}{\pi}\right)^\circ = 3.73^\circ$$

Example 10.14 Figure 10.5 shows a hollow shaft. Determine the maximum power transmitted by the shaft at 200 rpm if the shear stress in the shaft is not to exceed 70 MPa. Also find the lengths of the two portions if the twist produced in the two portions of the shaft are equal.

Solution

Given A hollow shaft as shown in Fig. 10.5.

$\tau = 70$ MPa $\qquad N = 200$ rpm

To find

— Maximum power

— lengths of two portions if twists are equal

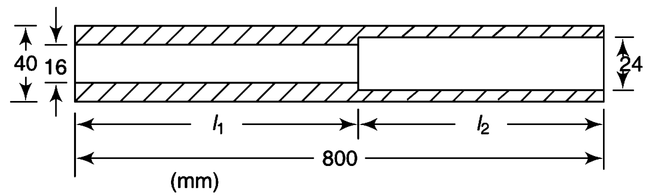


Fig. 10.5

Maximum power

As $\tau = \dfrac{16T}{\pi}\left(\dfrac{D}{D^4 - d^4}\right)$, the maximum value of shear stress will reach in the portion in which d is higher, i.e., in the latter portion.

Thus maximum torque transmitted,

$$T = \frac{\pi}{16}\left(\frac{D^4 - d^4}{D}\right)\tau = \frac{\pi}{16}\left(\frac{40^4 - 24^4}{40}\right) \times 70$$

$$= 765.6 \times 10^3 \text{ N}\cdot\text{mm} \quad \text{or} \quad 765.6 \text{ N}\cdot\text{m}$$

$$P = \frac{2\pi NT}{60} = \frac{2\pi \times 200 \times 765.6}{60} = 16\ 036 \text{ W}$$

Lengths of two portions

If twist is to be same in the two portions,

$$\theta = \frac{Tl_1}{GJ_1} = \frac{Tl_2}{GJ_2} \quad \text{or} \quad \frac{l_1}{l_2} = \frac{J_1}{J_2} = \frac{40^4 - 16^2}{40^4 - 24^2} = 1.1195 \quad \text{or} \quad l_1 = 1.1195 l_2$$

But $\quad l_1 + l_2 = 800 \quad$ or $\quad 1.1195 l_2 + l_2 = 800 \quad$ or $\quad l_2 = 377.4$ mm

and $\quad l_1 = 800 - 377.4 = 422.6$mm

Example 10.15 A stepped shaft fixed at the two ends as shown in Fig. 10.6 is subjected to a torque of 300 N·m at the section C. The larger section is of aluminium and the smaller one is of steel. Determine the maximum stresses in the two materials. G_s = 82 GPa and G_a = 27 GPa.

Solution

Given A stepped shaft fixed at ends and subjected to a couple as shown in Fig. 10.6.

$G_s = 82$ GPa $\quad G_a = 27$ GPa

To find Maximum stresses in two materials

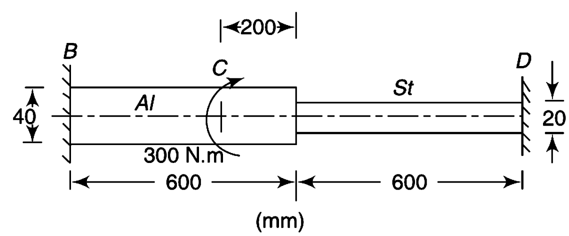


Fig. 10.6

$$J_a = \frac{\pi}{32} \times 40^4 = 80\ 000\pi \text{ mm}^4$$

and $$J_s = \frac{\pi}{32} \times 20^4 = 5\ 000\pi \text{ mm}^4$$

Fixing torques

As the torque is applied at the section C, it rotates through an angle θ. This torque is resisted by opposite torques at the two ends which are known as *fixing torques*. Let the fixing torque at D be T_d.

The fixing torque at $B = (300 - T_d)$ N·m

Thus
$$\theta = \frac{(300 - T_d) \times (600 - 200)}{G_a J_a} = \frac{T_d \times 200}{G_a J_a} + \frac{T_d \times 600}{G_s J_s}$$

or
$$\frac{120\,000}{G_a J_a} = \frac{600 T_d}{G_a J_a} + \frac{600 T_d}{G_s J_s}$$

or
$$\frac{120\,000}{27\,000 \times 80\,000\pi} = 600 T_d \left(\frac{1}{27\,000 \times 80\,000\pi} + \frac{1}{82\,000 \times 5000\pi} \right)$$

$$55.555 = 1.741 T_d \quad \text{or} \quad T_d = 31.9 \text{ N}\cdot\text{m}$$

and
$$T_b = 300 - 31.9 = 268.1 \text{ N}\cdot\text{m}$$

Maximum stresses

Now,
$$\tau_a = \frac{16T}{\pi d^3} = \frac{16 \times 268.1 \times 10^3}{\pi \times 40^3} = 21.3 \text{ MPa}$$

and
$$\tau_s = \frac{16 \times 31.9 \times 10^3}{\pi \times 20^3} = 20.3 \text{ MPa}$$

Example 10.16 A vertical compound shaft is made by securely fixing a 500-mm long brass bar to a 500-mm long aluminium bar so that the total length of the shaft is 1 m. The diameter of each bar is 40 mm. The aluminium bar is rigidly fixed at its upper end. Determine the maximum torque which can be applied at the lower end of the shaft if the maximum angle of twist is not to exceed 3° and the maximum shear stress in aluminium and in the brass 78 MPa and 55 MPa respectively. G_b = 34 GPa and G_a = 28 GPa.

Solution

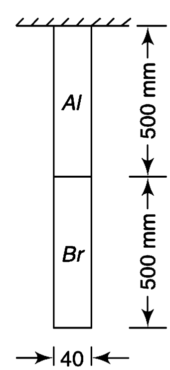


Fig. 10.7

Given A compound shaft fixed at upper end as shown in Fig. 10.7

G_b = 34 GPa $\quad$ G_a = 28 GPa.

T_b = 55 MPa $\quad$ T_a = 78 MPa

$\theta = 3°$

To find Maximum torque at lower end

$$J_a = J_b = \frac{\pi}{32} \times 40^4 = 80\,000\pi \text{ mm}^4$$

From maximum shear stress values

If the shear stress in brass reaches the maximum value,

$$T = \frac{\tau J}{r} = \frac{55 \times 80\,000\pi}{20} = 691\,150 \text{ N}\cdot\text{mm}$$

If the shear stress in aluminium reaches the maximum value,

$$T = \frac{78 \times 80\,000\pi}{20} = 980\,177 \text{ N}\cdot\text{mm}$$

From maximum angle of twist

If the angle of twist reaches the maximum value,

$$\theta = \frac{Tl_a}{G_a J_a} + \frac{Tl_b}{G_b J_b} = \frac{Tl_a}{J_a}\left(\frac{1}{G_a} + \frac{1}{G_b}\right)$$

or
$$3 \times \frac{\pi}{180} = \frac{T \times 500}{80\,000\pi}\left(\frac{1}{28\,000} + \frac{1}{34\,000}\right) \quad \text{or} \quad T = 404\,120 \text{ N}\cdot\text{mm}$$

The maximum allowable torque is to be the minimum of the three, i.e., 404.12 N·m

Example 10.17 || A circular steel shaft of 40-mm diameter is provided with enlarged portions of 80 mm diameter at the two ends as shown in Fig. 10.8. On the enlarged portions, a steel tube of 2-mm thickness is shrunk. During the shrunk process, the 40-mm diameter shaft is held twisted by a couple of magnitude of 800 N·m. When the tube is firmly set on the tube, the couple is removed. Determine the twisting couple left on the shaft if the tube and the shaft are made of the same material.

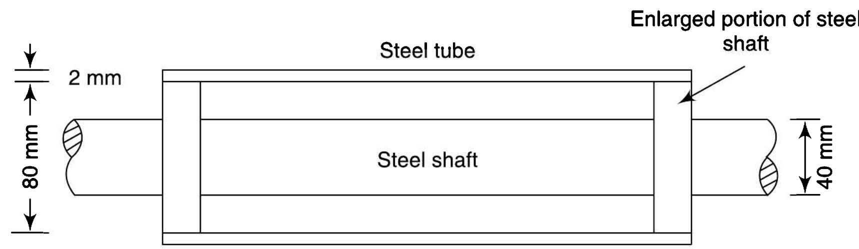


Fig. 10.8

Solution

Given A compound steel shaft as shown in Fig. 10.8. The shaft is twisted by a couple of 800 N·m and then tube is shrunk.

To find Twisting couple left on the shaft

For the shaft, $J_s = \dfrac{\pi}{32} \times 40^4 = 80\ 000\pi \text{ mm}^4$

For the tube, $J_t = \dfrac{\pi}{32} \times (84^4 - 80^4) = 275\ 848\pi \text{ mm}^4$

Initial angle of twist of shaft

Initially let θ be the angle of twist in the shaft, Then $\theta = \dfrac{T'l}{GJ_s} = \dfrac{800\ 000l}{80\ 000\pi G} = \dfrac{3.183l}{G}$

On removing the twisting couple

Let T be the residual twisting couple on the shaft which will also be equal to the couple acting on the tube.

θ_s = residual angle of twist in the shaft

θ_t = angle of twist in the tube

$$\therefore\ T = \frac{J_s G\theta_s}{l} = \frac{J_t G\theta_t}{l} \quad \text{or} \quad \theta_s = \frac{J_t\theta_t}{J_t} = \frac{275.848\theta_t}{80\ 000} = 3.448\theta_t$$

Equating the total angles of twist

As total angle of twist after removal of couple must be equal to the initial angle of twist,

$$\therefore\quad \theta_t + 3.448\theta_t = \frac{3.183l}{G} \quad \text{or} \quad 4.448\frac{Tl}{GJ_t} = \frac{3.183l}{G}$$

$$\text{or}\quad T = \frac{3.183 \times 275\ 848\pi}{4.448} = 620.14 \times 10^3 \text{ N·mm} \quad \text{or} \quad 620.14 \text{ N·m}$$

Example 10.18 || A torque of 2.5 kN·m is applied to a composite shaft made of aluminium, brass and steel as shown in Fig. 10.9 through a thin rigid plate. Determine the maximum shear stress in each material.

G_a = 27 GPa; G_s = 82 GPa and G_b = 40 GPa.

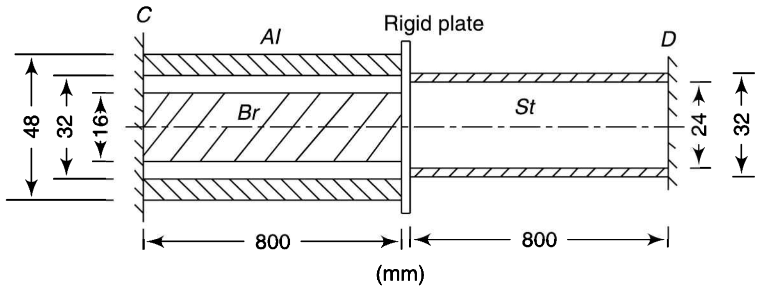


Fig. 10.9

Solution

Given A composite shaft with a thin rigid plate as shown in Fig. 10.9.

$G_a = 27$ GPa $\quad G_s = 82$ GPa $\quad G_b = 40$ GPa

To find Maximum shear stress in each material on applying a torque of 2.5 kN·m to the plate

$$J_b = \frac{\pi}{32} \times 16^4 = 2048\pi \text{ mm}^4;\ J_s = \frac{\pi}{32} \times (32^4 - 24^4) = 22\,400\pi \text{ mm}^4$$

and $$J_a = \frac{\pi}{32} \times (48^4 - 32^4) = 133\,120\pi \text{ mm}^4$$

Fixing torques

As the torque is applied to the rigid plate it rotates through an angle θ. This torque is resisted by opposite torques at the two ends which are known as *fixing torques*. Let the fixing torque at D be T_d.

The fixing torque at $C = (2500 - T_d)$ N·m

Thus $$\theta = \frac{(2500 - T_d)l}{G_a J_a + G_b J_b} = \frac{T_d l}{G_s J_s} \text{ or } \frac{(2500 - T_d)}{T_d} = \frac{G_a J_a + G_b J_b}{G_s J_s}$$

or $$\frac{2500}{T_d} - 1 = \frac{27\,000 \times 133\,120\pi + 40\,000 \times 2048\pi}{82\,000 \times 22\,400\pi} = 2.0014$$

or $$\frac{2500}{T_d} = 2.0014 + 1 \quad \text{or} \quad T_d = 832.9 \text{ N·m and } T_c = 2500 - 832.9 = 1667.1 \text{ N·m}$$

The fixing torque T_c is shared by two shafts. Let T_a be the torque taken by aluminium shaft and T_b by the brass shaft.

Then $T_a = 1667.1 - T_b$

and $$\frac{T_a}{T_b} = \frac{G_a J_a \theta / l}{G_b J_b \theta / l} = \frac{G_a J_a}{G_b J_b} = \frac{27\,000 \times 133\,120}{40\,000 \times 2048} = 43.875$$

or $$\frac{1667.1 - T_b}{T_b} = 43.875 \quad \text{or} \quad T_b = 37.1 \text{ N·m}$$

and $T_a = 1667.1 - 37.1 = 1630$ N·m

Shear stresses

Now, $$\tau_b = \frac{16T_b}{\pi d^3} = \frac{16 \times 37.1 \times 10^3}{\pi \times 16^3} = 46.1 \text{ MPa}$$

$$\tau_a = \frac{16 \times T_a \times D}{\pi(D^4 - d^4)} = \frac{16 \times 1630 \times 10^3 \times 48}{\pi(48^4 - 32^4)} = 93.5 \text{ MPa}$$

$$\tau_s = \frac{16 \times T_d \times D}{\pi(D^4 - d^4)} = \frac{16 \times 832.9 \times 10^3 \times 32}{\pi(32^4 - 24^4)} = 189.4 \text{ MPa}$$

10.5 STRAIN ENERGY IN TORSION

Total strain energy of a shaft under the action of a torque is the work done in twisting. Thus for a gradually applied torque T to a shaft of length l (Fig. 10.10), $U = \frac{1}{2}T\theta$

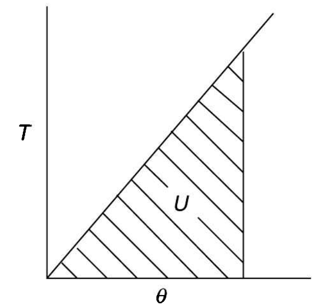


Fig. 10.10

- As $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{l}$

$\therefore$ for a solid shaft, $T = \left(\frac{\pi d^4}{32}\right)\frac{\tau}{d/2} = \frac{\pi d^3 \cdot \tau}{16}$

and $\theta = \frac{\tau \cdot l}{Gd/2} = \frac{2\tau \cdot l}{Gd}$

$\therefore$ $U = \frac{1}{2}T\theta = \frac{1}{2} \cdot \frac{\pi d^3 \cdot \tau}{16}\frac{2\tau \cdot l}{Gd} = \frac{\tau^2}{4G} \cdot \left(\frac{\pi d^2}{4}l\right) = \frac{\tau^2}{4G} \times \text{Volume}$

Strain energy/unit volume $= \frac{\tau^2}{4G}$ (10.5)

This is the mean strain energy over the whole shaft. However, in a shaft the shear stress varies from zero at the axis to maximum at the outer surface which means intensity of strain energy is also not uniform and varies across the cross-section, i.e., zero at the axis to maximum at the outer surface.

Thus total strain energy, $U = \frac{0 + U_{\max}/\text{unit volume}}{2} \times \text{Volume}$

or $$\frac{\tau^2}{4G} \times \text{Volume} = \frac{U_{\max}/\text{unit volume}}{2} \times \text{Volume}$$

or Maximum strain energy /unit volume $= \frac{\tau^2}{2G}$ (10.6)

For a hollow shaft,

$$U = \frac{1}{2}T\theta = \frac{1}{2} \cdot \frac{\pi(D^4 - d^4) \cdot \tau}{16D}\frac{2\tau \cdot l}{GD} = \frac{\tau^2}{4G} \cdot \frac{D^2 + d^2}{D^2}\left(\frac{\pi(D^2 - d^2)}{4}l\right)$$

$$= \frac{\tau^2}{4G} \cdot \frac{D^2 + d^2}{D^2} \times \text{Volume} \tag{10.7}$$

Example 10.19 Determine the ratio of inner to outer diameters of a hollow shaft that attains a maximum shear stress of τ when subjected to a pure torque. Take the strain energy per unit volume to be $\tau^2/3G$. Also find the inner and the outer diameters if the shaft transmits 5 MW at 120 rpm when the energy stored is 35 kN·m/m³. G = 78 GPa.

Solution

Given

A hollow shaft subjected to a pure torque.

Strain energy/unit volume = $\tau^2/3G$

$$P = 5\text{ MW} = 5\times10^6\text{ W} = 5\times10^9\text{ N}\cdot\text{mm/s}$$
$$U = 35\text{ kN}\cdot\text{m/m}^3 = 35\times10^{-3}\text{ N}\cdot\text{mm/mm}^3$$
$$N = 120\text{ rpm}$$
$$G = 78\text{ GPa}$$

To find

— Ratio of inner to outer diameters
— Inner and outer diameters

Equating strain energies

For a hollow shaft, strain energy per unit volume = $\dfrac{\tau^2}{4G}\cdot\dfrac{D^2+d^2}{D^2}$

Strain energy/unit volume (given) = $\tau^2/3G$

$$\therefore \qquad \frac{\tau^2}{3G} = \frac{\tau^2}{4G}\cdot\frac{D^2+d^2}{D^2}$$

or $\quad 3(D^2+d^2) = 4D^2$

or $\quad \dfrac{d}{D} = \dfrac{1}{\sqrt{3}}$

Determination of torque and stress

$$P = T\cdot\omega = T\cdot\frac{2\pi N}{60}$$

or $\quad 5\times10^6 = T\cdot\dfrac{2\pi\times120}{60}$

or $\quad T = 397\,887\text{ N}\cdot\text{m}$

From given condition, $\quad \dfrac{\tau^2}{3G} = 35\times10^{-3}$

or $\quad \tau^2 = 3\times78\,000\times35\times10^{-3}$ or $\tau = 90.5$ MPa

Determination of diameters

Now, $\quad T = \tau\cdot\dfrac{\pi(D^4-d^4)}{16D} = \tau\cdot\dfrac{\pi D^3}{16}\left[1-\left(\dfrac{d}{D}\right)^4\right]$

or $\quad 397\,887\times10^3 = 90.5\times\dfrac{\pi D^3}{16}\left[1-\left(\dfrac{1}{\sqrt{3}}\right)^4\right]$

or $\quad D^3 = 25.19\times10^6$

or $\quad D = 293$ mm and $d = \dfrac{293}{\sqrt{3}} = 169$ mm

Example 10.20 A 14-m long horizontal shaft is securely fixed at each end. It is acted upon by two axial couples; one of 40 kN·m clockwise at a distance of 5 m and another of 50 kN·m counter-clockwise at a distance of 11 m from the left end. Find the end-fixing couples. What will be the diameter of the solid shaft for a maximum shear stress of 45 MPa? In what way will a line on the surface originally parallel to the axis appear after the application of the torque? Also find the position where the angular twist of the shaft is zero.

Solution

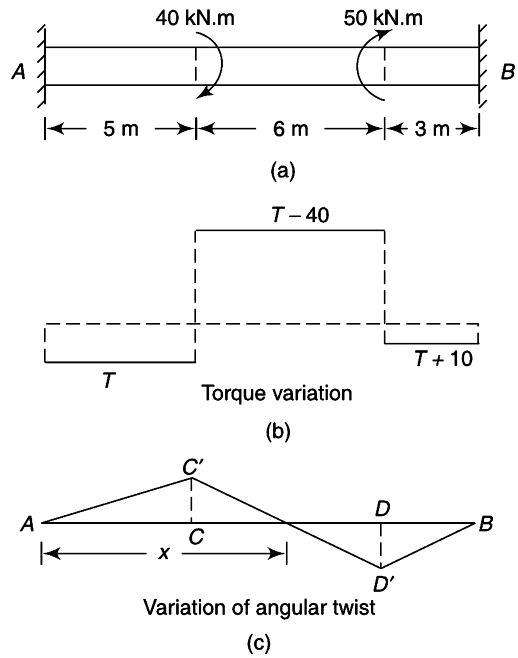


Fig. 10.11

Given A horizontal shaft fixed at ends and acted upon by couples as shown in Fig. 10.11a.

$\tau = 45$ MPa

To find

— Fixing end couples
— diameter of solid shaft
— shape of a line originally parallel to axis
— position where the angular twist of the shaft is zero

Fixing end couples

Let T be the fixing torque at the end A from which distances are considered (Fig. 10.11a).

Torque in the middle portion = $T - 40$ (Fig. 10.11b)

Torque in the end portion = $T - 40 + 50 = T + 10$

$$\theta = \frac{Tl}{GJ} \text{ or } \theta \propto Tl \text{ for a shaft of uniform cross-section}$$

Thus for zero resultant twist = $T \times 5 + (T - 40) \times 6 + (T + 10) \times 3 = 0$

or $T = 15$ kN·m

Fixing couple at the other end = $T + 10 = 15 + 10 = 25$ kN·m

Maximum torque = $T - 40 = 15 - 40 = -25$ kN·m or 25 kN·m numerically

Diameter of solid shaft

$$\text{Maximum shear stress, } \tau = \frac{16T}{\pi d^3} \quad \text{or} \quad 45 = \frac{16 \times 25 \times 10^6}{\pi d^3} \quad \text{or} \quad d = 141.4 \text{ mm}$$

Shape of line

In Fig. 10.11c, $CC' \propto 15 \times 5 = 75$ and $DD' \propto 25 \times 3 = 75$

Thus the shape of original line $ACDB$ will become $AC'D'B$.

Position for zero angular twist

From Fig. 10.11c, it is indicated that the angular twist is zero at the midpoint of CD.

Thus distance from A, $x = 5 + 3 = 8$ m

Therefore, the angular twist is zero at 8 m from the end A.

Example 10.21 || A steel shaft of 650-mm length is made up of three segments as follows:

— The first segment is of a 150-mm long hollow shaft and of 50-mm outside diameter.
— The second segment is of 200-mm long solid shaft and of 50-mm diameter.
— The third segment is also of 300-mm long solid shaft and of 40-mm diameter.

Determine the maximum permissible value of inner diameter of the first segment of the shaft if equal opposite torques are applied to the ends of the shaft and the shear stress in the first segment is not to exceed of that in third segment. Also find the total angle of twist if the torque applied is 1000 N·m. G = 80 GPa.

Solution

Given A steel shaft made up of three segments as shown in Fig. 10.12.

$T = 1000$ N·m $\qquad G = 80$ GPa

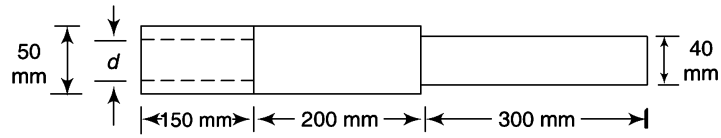


Fig. 10.12

To find

- Inner diameter of hollow section if equal opposite torques are applied at ends, shear stress in the first segment is not to exceed that in the third segment
- total angle of twist

Determination of inner diameter

Shear stress in the first segment, $\tau = \dfrac{16TD}{\pi(D^4 - d^4)} = \dfrac{16T \times 50}{\pi(50^4 - d^4)}$

Shear stress in the third segment, $\tau = \dfrac{16T}{\pi \times 40^3}$

Equating the two,

$$\frac{16T \times 50}{\pi(50^4 - d^4)} = \frac{16T}{\pi \times 40^3}$$

or
$$\frac{50^4 - d^4}{50} = 40^3$$

or
$$50^4 - d^4 = 40^4 \times 50 \quad \text{or} \quad d = 41.79 \text{ mm}$$

Angle of twist

$$\theta = \frac{Tl}{GJ} = \frac{1000 \times 10^3}{80\,000}\left[\frac{32 \times 150}{\pi(50^4 - 41.79^4)} + \frac{32 \times 200}{\pi \times 50^4} + \frac{32 \times 300}{\pi \times 40^4}\right]$$

$$= \frac{1000 \times 10^3 \times 32}{80\,000 \times \pi}\left[\frac{150}{50^4 - 41.79^4} + \frac{200}{50^4} + \frac{300}{40^4}\right]$$

$$= 127.3 \times 196 \times 10^{-6} \text{ rad} = \left(24\,955 \times 10^{-6} \times \frac{180}{\pi}\right)^\circ = 1.43^\circ$$

Example 10.22 A solid alloy shaft of 60-mm diameter is coupled with a hollow steel shaft of the same external diameter in series. If the angle of twist of the steel shaft per unit length is 80% of that of the alloy shaft, find the inner diameter of the steel shaft.

What will be the speed to transmit 300 kW if the limiting stresses in the alloy and the steel are to be 50 MPa and 72 MPa respectively? Take $G_{st} = 2\ G_{alloy}$.

Solution

Given A solid alloy shaft coupled with a hollow steel shaft,

$(\theta/l)_{st} = 0.8(\theta/l)_{alloy}$

$d_s = 60$ mm	$P = 300$ kW
$T = 1000$ N·m	$G_{st} = 2\ G_{alloy}$
$\tau_{amax} = 50$ MPa	$\tau_{smax} = 72$ MPa

To find

— Inner diameter of steel shaft
— speed of shaft

Inner diameter

Angle of twist per unit length,

$$\left(\frac{\theta}{l}\right)_{\text{st}} = 0.8\left(\frac{\theta}{l}\right)_{\text{alloy}}$$

or $$\left(\frac{T}{GJ}\right)_{\text{st}} = 0.8\left(\frac{T}{GJ}\right)_{\text{alloy}}$$

or $$\frac{32T}{2G_{st}\pi(60^4 - d^4)} = 0.8\frac{32T}{G_{\text{alloy}}\pi \times 60^4}$$

or $$1.6(60^4 - \text{d}^4) = 60^4$$

or $$60^4 - \text{d}^4 = 8.1 \times 10^6 \quad \text{or} \quad d = 46.95 \text{ mm}$$

Stresses

Also, as $$\left(\frac{\theta}{l}\right)_{\text{st}} = 0.8\left(\frac{\theta}{l}\right)_{\text{alloy}}, \; \therefore \; \left(\frac{\tau}{Gr}\right)_{\text{st}} = 0.8\left(\frac{\tau}{Gr}\right)_{\text{alloy}}$$

or $$\frac{\tau_{\text{st}}}{\tau_{\text{alloy}}} = 0.8 \times \frac{G_{\text{st}}}{G_{\text{alloy}}} \times \frac{d_{\text{st}}}{d_{\text{alloy}}} = 0.8 \times 2 \times 1 = 1.6$$

If maximum stress in the steel shaft is to be 72 MPa, the maximum stress in the steel will be 72/1.6 = 45 MPa, which is within limits.

Determination of speed

$$T = \tau\frac{\pi d^3}{16} = 45 \times \frac{\pi \times 60^3}{16} = 1908.5 \times 10^3 \text{ N}\cdot\text{mm} \quad \text{or} \quad 1908.5 \text{ N}\cdot\text{m}$$

$$P = T \cdot \frac{2\pi N}{60} \quad \text{or} \quad 300 \times 10^3 = 1908.5 \times \frac{2\pi N}{60} \quad \text{or} \quad N = 1501 \text{ rpm}$$

Example 10.23 || A compound shaft is made by mounting a gunmetal sleeve on a steel shaft. The shaft is subjected to a torque in such a way that the torque on the sleeve is thrice that on the shaft. Determine the ratio of external diameter of the sleeve to that of the shaft.

What will be the torque transmitted by the compound shaft if the steel shaft diameter is 60 mm and the limiting values of the shear stresses in the gun metal and the steel are 48 MPa and 75 MPa respectively. $G_{st} = 2.2\ G_{gm}$

Solution

Given A compound shaft of gun metal sleeve on a steel shaft

$T_{gm} = 3\ T_{st}$ $\quad$ $G_{st} = 2.2\ G_{gm}$

$T_{gmax} = 48$ MPa $\quad$ $T_{smax} = 75$ MPa

$d = 60$ mm

To find

— Ratio of external to internal diameter of sleeve
— Total torque transmitted

Let D and d be the outside and inside diameters of the gun metal sleeve. Thus d is also the diameter of the steel shaft.

Determining ratio of D/d

For the compound shaft, the strain may be assumed proportional to the distance from the axis and thus the strain is the same for each at the common surface. This also indicates that the twist per unit length is the same for the sleeve and the shaft.

Angle of twist per unit length, $\left(\dfrac{\theta}{l}\right)_{\text{st}} = \left(\dfrac{\theta}{l}\right)_{\text{gm}}$ or $\left(\dfrac{T}{GJ}\right)_{\text{st}} = \left(\dfrac{T}{GJ}\right)_{\text{gm}}$

or $$\frac{32T}{2.2G_{\text{g}}\pi \times d^4} = \frac{32 \times 3T}{G_{\text{g}}\pi(D^4 - d^4)}$$

or $D^4 - d^4 = 3 \times 2.2 \times d^4$ or $D^4 = 7.6d^4$ or $D = 1.66\,d$

Determining ratio of $\tau_{\text{st}}/\tau_{\text{g}}$

As $$\left(\frac{\theta}{l}\right)_{\text{st}} = \left(\frac{\theta}{l}\right)_{\text{g}}, \therefore \left(\frac{\tau}{Gr}\right)_{\text{st}} = \left(\frac{\tau}{Gr}\right)_{\text{g}}$$

or $$\frac{\tau_{\text{st}}}{\tau_{\text{g}}} = \frac{G_{\text{st}}}{G_{\text{g}}} \times \frac{d}{D} = 2.2 \times \frac{1}{1.66} = 1.325$$

Total torque

If maximum stress in the gun metal sleeve is to be 48 MPa, the maximum stress in the steel will be 48 × 1.325 = 63.6 MPa which is within limits.

Torque in the steel shaft, $T = \tau\dfrac{\pi d^3}{16} = 63.6 \times \dfrac{\pi \times 60^3}{16} = 2697 \times 10^3\,\text{N·mm}$ or $2697\,\text{N·m}$

Torque in the sleeve, T = 3 × 2697 = 8091 N·m

Total torque = 2697 + 8091 = 10 788 N·m

10.6 COMBINED BENDING AND TORSION

This condition of combined bending and torsion is frequently encountered in shafts transmitting the torque. The shafts are subjected to not only twisting but also to bending moments due to gravity or inertia loads. In such cases stresses are set up due to bending moment, torque and shear force. However, shear stress due to shear force is usually unimportant as its maximum value occurs at the neutral axis where the bending stress is zero.

Let σ_b = maximum bending stress and τ = maximum shear stress due to twisting

Then, for a solid shaft, $\sigma_b = \dfrac{32M}{\pi D^3}$ and $\tau = \dfrac{16T}{\pi D^3}$

For vertical loading on a shaft, the maximum values of bending stress occur at the ends of a vertical diameter whereas maximum value of shear stress occurs at the outer surface. As there is no normal stress on the longitudinal planes of the shaft,

Maximum principal stress, $$\sigma = \frac{1}{2}\sigma_b + \frac{1}{2}\sqrt{\sigma_b^2 + 4\tau^2} = \frac{16M}{\pi D^3} + \frac{1}{2}\sqrt{\left(\frac{32M}{\pi D^3}\right)^2 + 4\left(\frac{16T}{\pi D^3}\right)^2}$$

$$= \frac{16}{\pi D^3}\left(M + \sqrt{M^2 + T^2}\right) \tag{10.8}$$

It can be noted that $\frac{1}{2}(M + \sqrt{M^2 + T^2})$ is the *equivalent bending moment* which would give the same maximum bending stress.

Maximum shear stress,

$$\tau = \frac{1}{2}\sqrt{\sigma_b^2 + 4\tau^2} = \frac{1}{2}\sqrt{\left(\frac{32M}{\pi D^3}\right)^2 + 4\left(\frac{16T}{\pi D^3}\right)^2} = \frac{16}{\pi D^3}\sqrt{M^2+T^2} \tag{10.9}$$

Note that $\sqrt{M^2+T^2}$ is the *equivalent torque* which would give the same shear stress.

Strain energy, $U = \frac{1}{2E}(\sigma_1^2 + \sigma_2^2 - 2v\sigma_1\sigma_2)$ where σ_1 and σ_2 are the principal stresses.

$$\text{Thus, } U = \frac{1}{2E}\cdot\frac{16^2}{\pi^2 D^6}\left[\begin{array}{l}(M+\sqrt{M^2+T^2})^2 + (M-\sqrt{M^2+T^2})^2 \\ -2v(M+\sqrt{M^2+T^2})(M-\sqrt{M^2+T^2})\end{array}\right]$$

$$= \frac{1}{2E}\cdot\frac{16^2}{\pi^2 D^6}\left[\begin{array}{c}(M^2+M^2+T^2+2M\sqrt{M^2+T^2}) + (M^2+M^2+T^2 \\ -2M\sqrt{M^2+T^2}) - 2v\{M^2-(M^2+T^2)\}\end{array}\right]$$

$$= \frac{16^2}{\pi^2 D^6 E}[2M^2 + T^2(1+v)] \tag{10.10}$$

Example 10.24 || An 800-mm long shaft with a diameter of 80 mm carries a flywheel weighing 4 kN at its midway. The shaft transmits 24 kW at a speed of 240 rpm. Determine the principal stresses and the maximum shear stress at the ends of a vertical and horizontal diameter in a plane near the flywheel.

Solution

Given

$L = 800$ mm $\quad d = 80$ mm

$W = 4$ kN $\quad P = 24$ kW

$N = 240$ rpm

To find Principal stresses and maximum shear stresses on vertical and horizontal diameters

Torque

Maximum bending moment, $M = \frac{Wl}{4} = \frac{4000 \times 800}{4} = 800 \times 10^3$ N·mm

$$T = \frac{60P}{2\pi N} = \frac{60 \times 24\,000}{2\pi \times 240} = 955 \text{ N·m} \quad \text{or} \quad 955 \times 10^3 \text{ N·mm}$$

At the ends of a vertical diameter

$$\text{Principal stresses} = \frac{16}{\pi d^3}[M \pm \sqrt{M^2+T^2}]$$

$$= \frac{16}{\pi \times 80^3} \times 10^3[800 \pm \sqrt{800^2 + 955^2}] = 9.947 \times 10^{-3}[800 \pm 1245.8]$$

$$= 20.35 \text{ MPa} \quad \text{and} \quad -4.43 \text{ MPa}$$

- On the tension side of the shaft (lower end), the principal stresses are 20.35 MPa tension and 4.43 MPa compression.
- On the compression side of the shaft (upper end), the principal stresses are 20.35 MPa compression and 4.43 MPa tension.

Maximum shear stress = $\dfrac{16}{\pi d^3}\sqrt{M^2+T^2}$

$$= \frac{16}{\pi \times 80^3} \times 10^3 \sqrt{800^2+955^2} = 9.947 \times 10^{-3} \times 1245.8 = 12.39 \text{ MPa}$$

At the ends of a horizontal diameter

The bending stress is zero.

Thus torsional shear stress = $\dfrac{16T}{\pi d^3} = 9.947 \times 10^{-3} \times 955 = 9.5$ MPa

Example 10.25 The outer and inner diameters of a hollow steel shaft are 120 mm and 60 mm respectively. The shaft transmits 800 kW at a speed of 400 rpm while an end thrust of 70 kN acts on the shaft. Determine the bending moment which can safely be applied to the shaft if the maximum principal stress does not exceed 80 MPa.

Solution

Given A hollow steel shaft

$D = 120$ mm	$d = 60$ mm
$P = 800$ kW	$F = 70$ kN
$N = 400$ rpm	$\sigma = 80$ MPa

To find Bending moment

Maximum shear stress

$$T = \frac{60P}{2\pi N} = \frac{60 \times 800 \times 10^3}{2\pi \times 400} = 19\,100 \text{ N·m or } 19.1 \times 10^6 \text{ N·mm}$$

Maximum shear stress on a transverse plane,

$$\tau = \frac{T}{J} \cdot \frac{d}{2} = \frac{19.1 \times 10^6}{\pi(120^4 - 60^4)/32} \cdot \frac{120}{2} = 60 \text{ MPa}$$

Determination of maximum bending stress

Let σ be the safe compressive stress which can be applied.

Then, maximum principal stress = $\dfrac{\sigma}{2} + \dfrac{1}{2}\sqrt{\sigma^2 + 4\tau^2}$

or $\quad 80 = \dfrac{\sigma}{2} + \dfrac{1}{2}\sqrt{\sigma^2 + 4 \times 60^2}$ $\quad$ or $\quad 160 = \sigma + \sqrt{\sigma^2 + 14\,400}$

or $\quad \sigma^2 + 14\,400 = (160 - \sigma)^2 = 25\,600 + \sigma^2 - 320\sigma$

or $\quad 14\,400 = 25\,600 - 320\sigma$ $\quad$ or $\quad \sigma = 35$ MPa

However, this also includes the end thrust component. The limiting case will be when both the end thrust and the bending stress are compressive and the maximum principal stress is 35 MPa compressive i.e. at the upper end of the shaft.

Normal stress due to end thrust = $\dfrac{F}{A} = \dfrac{70\,000}{\pi(120^2 - 60^2)/4} = 8.25$ MPa

$\therefore$ permissible stress due to bending = 35 – 8.25 = 26.75 MPa

Bending moment

$$\tau = \frac{My}{I} \quad \text{or} \quad 26.75 = \frac{M \times 60}{\pi(120^4 - 60^4)/64} \quad \text{or} \quad M = 4254 \text{ N·m}$$

10.7 THIN TUBULAR SECTIONS (BREDT–BATHO THEORY)

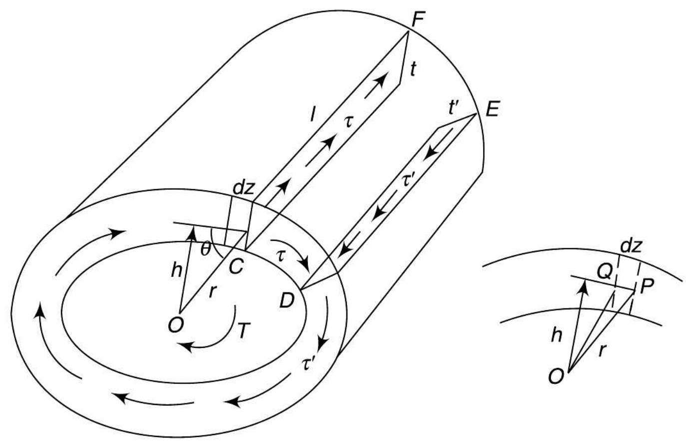


Fig. 10.13

Consider a thin-walled closed tube acted upon by a torque T in a transverse plane as shown in Fig. 10.13. It may be assumed that the shear stress at a point across the thickness of the tube wall is constant.

Let τ be the shear stress at the point C where the thickness is t and τ' at the point D where the thickness is t'. Considering the strip $CDEF$ of the tube, the nature of the complementary shear stresses will be as shown in the figure. For the equilibrium of the strip,

$$\tau \cdot t \cdot l = \tau' t' \, l \quad \text{or} \quad \tau \cdot t = \tau' t' = \text{constant} \tag{10.11}$$

i.e., product of shear stress and thickness is constant at all points on the periphery. This is called shear flow (q) and is constant at any given point.

Let dz be an element round the circumference. Then

Force on the element $= \tau \cdot (t \cdot dz)$

Moment about $O = \tau \cdot t \cdot dz \cdot (r \sin \theta) = \tau \cdot t \cdot dz \cdot h$

where h is the perpendicular distance of shear stress τ from O.

Therefore, $T = \int \tau \cdot t \cdot h \cdot dz = \tau \cdot t \int h \cdot dz = 2\tau \cdot t \int \text{Area } OPQ$ …….(area $OPQ = \frac{1}{2} \cdot h \cdot dz$)

$$= 2\tau \cdot t \cdot A \tag{10.12}$$

where A = Area enclosed by the mean circumference

- Strain energy, $U = \dfrac{1}{2} \cdot T\theta = \dfrac{1}{2} \cdot (2\tau t A)\theta$ (i)

Also strain energy, $U = \int \dfrac{\tau^2}{2G} \times \text{Vol} = \int \dfrac{\tau^2}{2G} \cdot l \cdot t \cdot dz = \dfrac{\tau \cdot tl}{2G} \int \tau \cdot dz$ (ii)

From (i) and (ii),

or $$\frac{1}{2} \cdot (2\tau t A)\theta = \frac{\tau \cdot tl}{2G} \int \tau \cdot dz$$

or $$\theta = \frac{l}{2GA} \int \tau \cdot dz \tag{10.13}$$

If t is constant, $$\theta = \frac{l\tau}{2GA} \int dz = \frac{l\tau}{2GA} z \tag{10.14}$$

Also $$\theta = \frac{lz}{2GA}\tau = \frac{lz}{2GA}\cdot\frac{T}{2tA} = \frac{lTz}{4GA^2t} \quad \ldots.. (\because T = 2\tau \cdot t \cdot A) \qquad (10.15)$$

Example 10.26 A thin-walled 800-mm long member has the cross-section as shown in Fig. 10.14. Determine

(i) the maximum torque if the angle carried by the section is limited to 4°
(ii) the maximum shear stress induced for the maximum torque
G = 82 GPa.

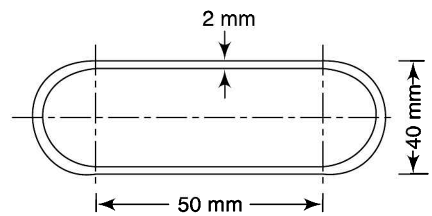


Fig. 10.14

Given A thin-walled member as shown in Fig. 10.14.
$l = 800$ mm $\theta = 3°$ $G = 82$ GPa

To find
— Maximum torque
— maximum shear stress

Solution

$$A = 50 \times 40 + \pi \times 20^2 = 3256.6 \text{ mm}^2; \; z = 50 + 50 + 2\,(\pi \times 40/2) = 225.7 \text{ mm}$$

Maximum torque

$$\theta = \frac{lTz}{4GA^2t} \quad \text{or} \quad 4 \times \frac{\pi}{180} = \frac{800 \times T \times 225.7}{4 \times 82\,000 \times 3256.6^2 \times 2}$$

or $T = 2690 \times 10^3$ N · mm or 2690 N · m

Maximum shear stress

As $T = 2\tau \cdot t \cdot A$,

$\therefore$ $2690 \times 10^3 = 2 \times \tau \times 2 \times 3256.6$

or $\tau = 206.5$ MPa

Example 10.27 Find the value by which the Bredt–Batho theory underestimates the maximum shear stress due to a given torque of a uniform hollow tube which has ratio of inner to outer diameter as 0.9. Also determine the error in the angle of twist.

Solution

Given A uniform hollow tube
$d/D = 0.9$

To find Ratio of shear stress and error in the angle of twist by Bredt–Batho theory and normal theory

Ratio of shear stress

- By Bredt–Batho theory,

$$T = 2\tau \cdot t \cdot A \quad \text{or} \quad T = 2\tau \cdot \left(\frac{D-d}{2}\right)\cdot\frac{\pi}{4}\left(\frac{D+d}{2}\right)^2 = \frac{\pi}{16}\cdot D\left(1-\frac{d}{D}\right)\cdot D^2\left(1+\frac{d}{D}\right)^2$$

$$= 2\tau \cdot \frac{\pi}{16}\cdot D(1-0.9)\cdot D^2(1+0.9)^2 \quad \text{or} \quad \tau = \frac{16T}{1.134D^3}$$

- By normal theory, $\tau' = \dfrac{16TD}{\pi(D^4-d^4)} = \dfrac{16TD}{\pi D^4(1-0.9^4)} = \dfrac{16T}{1.08D^3}$

Thus Ratio $\dfrac{\tau}{\tau'} = \dfrac{1.08}{1.134} = 0.952$

Ratio of angle of twist

- By Bredt–Batho theory,

$$\theta = \frac{lTz}{4GA^2t} = \frac{lT}{4G} \cdot \pi\left(\frac{D+d}{2}\right) \cdot \frac{1}{[(\pi/4)\{(D+d)/2\}^2]^2} \cdot \frac{1}{(D-d)/2}$$

$$= \frac{lT}{4G} \cdot \frac{\pi D}{2}(1+0.9) \cdot \frac{16}{\pi^2} \frac{16}{D^4(1+0.9)^4} \cdot \frac{1}{D} \frac{2}{(1-0.9)} = 29.71\frac{Tl}{GD^4}$$

- By shaft theory, $\theta' = \dfrac{l}{G} \cdot \dfrac{32T}{\pi(D^4 - d^4)} = \dfrac{l}{G} \cdot \dfrac{32T}{\pi D^4(1-0.9^4)} = 29.62\dfrac{Tl}{GD^4}$

Thus, ratio $\dfrac{\theta'}{\theta} = \dfrac{29.62}{29.71} = 0.997$

10.8 THIN-WALLED SECTIONS

Consider a thin-walled twin-celled section shown in Fig. 10.15. Let ADC be of uniform thickness t_1 and stress τ_1, CBA of uniform thickness t_2 and stress τ_2, and AC of uniform thickness t_3 and stress τ_3. Also let A_1 and A_2 be the mean areas of the two cells. It may be assumed that the direction of shear flow in AC is downwards.

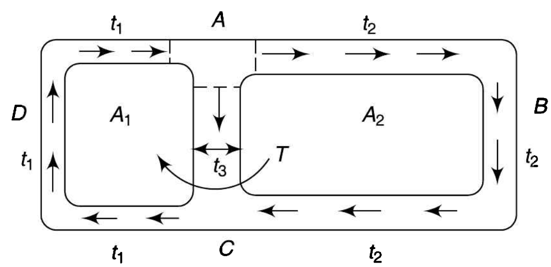


Fig. 10.15

From the equilibrium of complimentary shear stresses on longitudinal sections,

$$\tau_1 \cdot t_1 = \tau_2 t_2 + \tau_3 t_3 \tag{10.16}$$

As $T = 2\tau \cdot t \cdot A$ from the previous section,

Total torque, $T = 2(\tau_1 t_1 A_1 + \tau_2 t_2 A_2)$ (10.17)

From Eq. 10.14, $\theta = \dfrac{l}{2G} \dfrac{\tau_1 z_1 + \tau_3 z_3}{A_1} = \dfrac{l}{2G} \dfrac{\tau_2 z_2 - \tau_3 z_3}{A_2}$ (10.18)

where z_1, z_2 and z_3 are the mean perimeters ADC, ABC and AC respectively. Negative sign indicates a traverse against the direction of the stress.

Example 10.28 || The mean dimensions of the two cells of a thin-walled twin-celled section (Fig. 10.16) are 60 mm × 30 mm and 30 mm square. Thicknesses are t_1 = 3 mm, t_2 = 6 mm and t_3 = 4 mm. For an applied torque of 540 N·m, determine the shear stress in each section and the angle of twist per metre length. G = 80 GPa.

Solution

Given Two cells of a thin-walled twin-celled section

T = 540 N·m G = 80 GPa

To find

— Shear stress in each section
— angle of twist per metre length

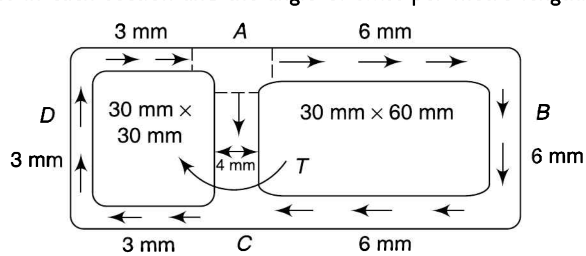


Fig. 10.16

$A_1 = 60 \times 30 = 1800$ mm^2; $A_2 = 30 \times 30 = 900$ mm^2

$z_1 = 60 + 30 + 60 = 150$ mm; $z_2 = 30 \times 3 = 90$ mm; $z_3 = 30$ mm

Shear stresses

- $\tau_1 t_1 = \tau_2 t_2 + \tau_3 t_3$ or $3\tau_1 = 6\tau_2 + 4\tau_3$ (i)
- $\dfrac{\tau_1 \cdot z_1 + \tau_3 z_3}{A_1} = \dfrac{\tau_2 \cdot z_2 - \tau_3 z_3}{A_1}$

  or $\dfrac{\tau_1 \times 150 + \tau_3 \times 30}{1800} = \dfrac{\tau_2 \times 90 - \tau_3 \times 30}{900}$ or $5\tau_1 = 6\tau_2 - 3\tau_3$ (ii)

  Multiply (i) by 3 and (ii) by 4 and add, $29\tau_1 = 42\tau_2$ (iii)
- $T = 2(\tau_1 t_1 A_1 + \tau_2 t_2 A_2)$ or $540\,000 = 2(\tau_1 \times 3 \times 1800 + \tau_2 \times 6 \times 900)$

  or $\tau_1 + \tau_2 = 25$ (iv)

  From (iii) and (iv), $42\tau_2/29 + \tau_2 = 25$

  or $71/29\tau_2 = 25$ or $\tau_2 = 10.2$ MPa

  $\tau_1 = 25 - 10.2 = 14.8$ MPa

  and $5 \times 14.8 = 6 \times 10.2 - 3\tau_3$ or $\tau_3 = -4.2$ MPa

Angle of twist

$$\theta = \frac{l}{2G} \frac{\tau_1 \cdot z_1 + \tau_3 z_3}{A_1} = \frac{1000}{2 \times 80\,000} \cdot \frac{14.8 \times 150 - 4.2 \times 30}{1800} = 7.27 \times 10^{-3} \text{ rad}$$

or
$$\theta = 7.27 \times 10^{-3} \times \frac{180}{\pi} = 0.417°$$

Example 10.29 The cross-section of an aluminum elevator 6 m long is shown in Fig. 10.17. Determine the total angle of twist of the section and the shear stress in each part for an applied torque of 100 N·m. G = 82 GPa.

Solution

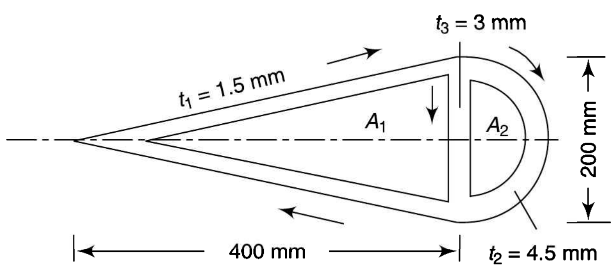


Fig.10.17

Given Two cells of a thin-walled twin-celled section

T = 100 N·m l = 6 m G = 82 GPa

To find

— Shear stress in each section

— Total angle of twist

$$A_1 = \frac{400 \times 200}{2} = 40\,000 \text{ mm}^2;\ A_2 = \frac{\pi \times 100^2}{2} = 5000\pi \text{ mm}^2;$$

$$z_1 = 2\sqrt{400^2 + 100^2} = 824.6 \text{ mm};\ z_2 = 100\pi;\ z_3 = 200 \text{ mm}$$

Determination of shear stresses

$\tau_1 \cdot t_1 = \tau_2 t_2 + \tau_3 t_3$ or $\tau_1 \times 1.5 = \tau_2 \times 4.5 + \tau_3 \times 3$ or $\tau_3 = 0.5\tau_1 - 1.5\tau_2$ (i)

$T = 2(\tau_1 \cdot t_1 A_1 + \tau_2 t_2 A_2)$ or $400\,000 = 2(\tau_1 \times 1.5 \times 40\,000 + \tau_2 \times 4.5 \times 5000\pi)$

or $20\,000 = 60\,000\,\tau_1 + 22\,500\,\pi\tau_2$ (ii)

$$\theta = \frac{l}{2G}\frac{\tau_1 \cdot z_1 + \tau_3 z_3}{A_1} = \frac{6000}{2 \times 82\,000}\left[\frac{\tau_1 \times (2 \times \sqrt{400^2 + 100^2}) + z_3 \times 200}{40\,000}\right]$$

$$= 36.586 \times 10^{-3}\,[0.0206\,\tau_1 + 0.005\tau_3] \quad \text{(iii)}$$

Also, $$\theta = \frac{l}{2G}\frac{\tau_2 \cdot z_2 - \tau_3 z_3}{A_1} = \frac{6000}{2 \times 82\,000}\left[\frac{\tau_2 \times 100\pi - z_3 \times 200}{5000\pi}\right]$$

$$= 36.586 \times 10^{-3}\,[0.02\tau_2 - 0.0127\,\tau_3] \quad \text{(iv)}$$

Comparing (iii) and (iv),

$0.0206\tau_1 + 0.005\tau_3 = 0.02\tau_2 - 0.0127\tau_3$

or $\tau_3 = 1.13\tau_2 - 1.164\tau_1$ (v)

From (i) and (v), $0.5\tau_1 - 1.5\tau_2 = 1.13\tau_2 - 1.164\tau_1$ or $\tau_1 = 1.581\tau_2$

Thus from (ii), $200\,000 = 60\,000 \times 1.581\tau_2 + 22\,500\pi\tau_2$ or $\tau_2 = 1.208$ MPa

$\therefore$ $\tau_1 = 1.581 \times 1.208 = 1.91$ MPa

From (v), $\tau_3 = 1.13 \times 1.208 - 1.164 \times 1.91 = -0.858$ MPa

Total angle of twist

$\theta = 36.586 \times 10^{-3}\,[0.02 \times 1.208 - 0.0127 \times (-0.858)] = 1.283 \times 10^{-3}$ rad

or $1.283 \times 10^{-3} \times 180/\pi = 73.5 \times 10^{-3}$ deg

10.9 THIN RECTANGULAR MEMBERS

Consider a rectangular thin section having large and short edges as h and t respectively. In case of rectangular thin sections, the flow of shear stress will be as shown in Fig. 10.18. The maximum value of shear stress will be along the long edge h. Let it be τ. Also let the shear stress along the short edge t to be τ'. It may be assumed that the stress variation in the x– and y–directions is linear, being zero at the centre by symmetry.

Experimentally, it has been shown that $\dfrac{\tau}{\tau'} = \dfrac{h}{t}$ (i)

Considering a small strip along the short edge, shear stress = $\tau \cdot \dfrac{x}{t/2}$

Shear force = $\tau \cdot \dfrac{x}{t/2} \cdot h \cdot dx$

Moment about vertical axis = $\tau \cdot \dfrac{x}{t/2} \cdot h \cdot dx \cdot x$

Similarly, moment about horizontal axis = $\tau' \cdot \dfrac{y}{h/2} \cdot t \cdot dy \cdot y$

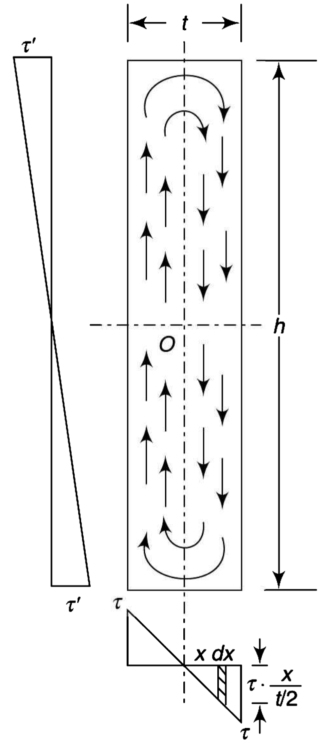


Fig.10.18

Total torque, $T = 2\int_0^{t/2} \dfrac{2\tau x}{t} \cdot hx \cdot dx + 2\int_0^{t/2'} \cdot \dfrac{2\tau' y}{h/2} \cdot t \cdot y \cdot dy$

$$= \frac{4\tau h}{t}\int_0^{t/2} x^2 \cdot dx + \frac{4\tau' t}{h}\int_0^{t/2} y^2 \cdot dy$$

$$= \frac{\tau h t^2}{6} + \frac{\tau' t h^2}{6} = \frac{\tau h t^2}{3} \quad \text{...[using (i)]} \qquad (10.19)$$

Let θ be the angle of twist of a length l,

$$U = \int \frac{\tau^2}{2G} \times \text{Vol}$$

or $$\frac{1}{2}T\theta = 2 \cdot \frac{1}{2G}\left[\int_0^{t/2}\left(\tau \cdot \frac{x}{t/2}\right)^2 \cdot h \cdot l \cdot dx + \int_0^{h/2}\left(\tau' \cdot \frac{y}{h/2}\right)^2 \cdot t \cdot l \cdot dy\right]$$

$$= \frac{1}{G}\left[\frac{4\tau hl}{t^2}\int_0^{t/2} x^2 \cdot dx + \frac{4\tau' tl}{h^2}\int_0^{h/2} y^2 . dy\right]$$

$$= \frac{1}{G}\left[\frac{\tau^2 htl}{6} + \frac{\tau'^2 htl}{6}\right] = \frac{1}{G}\left[\frac{\tau^2 htl}{6} + \frac{4(\tau \cdot t/h)^2 htl}{6}\right] \quad \text{...[using (i)]}$$

$$= \frac{\tau^2 htl}{6G}\left(1 + \frac{t^2}{h^2}\right)$$

For a long thin rectangular section, t^2/h^2 can be neglected,

Then $\dfrac{1}{2}T\theta = \dfrac{\tau^2 htl}{6G}$

- As, $T = \dfrac{\tau h t^2}{3}$, $\tau = \dfrac{3T}{ht^2}$ and thus $\dfrac{1}{2}T\theta = \left(\dfrac{3T}{ht^2}\right)^2 \dfrac{htl}{6G}$ or $\theta = \dfrac{3Tl}{ht^3 \cdot G}$ (10.20)
- As $T = \dfrac{\tau h t^2}{3}$, $\dfrac{1}{2} \cdot \dfrac{\tau h t^2}{3} \cdot \theta = \dfrac{\tau^2 htl}{6G}$ or $\theta = \dfrac{\tau l}{tG}$ (10.21)

Example 10.30 An *I*-section having dimensions of the web as 100 mm × 3 mm and of the two flanges as 60 mm × 4 mm is subjected to a torque *T*. If the limited shear stress in the section is to be 30 MPa and the twist per metre length 6.5°, determine the maximum value of the torque *T*. *G* = 80 GPa.

Solution

Given An *I*-section: web 100 mm × 3 mm, each flange: 60 mm × 4 mm

τ = 30 MPa G = 80 GPa

θ = 6.5°/m length

To find Maximum torque

From shear stress limit

$$\Sigma ht^2 = 100 \times 3^2 + 2 \times 60 \times 4^2 = 2820 \text{ mm}^3$$

∴ Torque, $T = \dfrac{\tau h t^2}{3} = \dfrac{30 \times 2820}{3} = 28\,200$ N·mm or 28.2 N·m

From angle of twist limit

$$\Sigma ht^3 = 100 \times 3^3 + 2 \times 60 \times 4^3 = 10\,380 \text{ mm}^4$$

Now, twist /m length, $\theta = \dfrac{3Tl}{ht^3 \cdot G}$ or $6.5 \times \dfrac{\pi}{180} = \dfrac{3T \times 1000}{10\,380 \times 80\,000}$

or $T = 31\,400$ N·mm or 31.4 N·m

Thus the permissible torque is to be 28.2 N·m

Example 10.31 || The thickness of all walls of a thin-walled cellular section is 2 mm as shown in Fig. 10.19 has. It is subjected to a torque of 20 N·m. Determine

(i) the torques shared by the two cells
(ii) the maximum shear stress in each cell
(iii) angular twist per m length
Take G = 102 GPa.

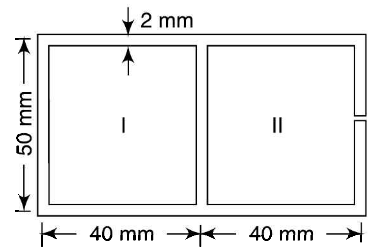


Fig. 10.19

Solution

Given A thin-walled cellular section shown in Fig. 10.17

T = 20 N·m G = 102 GPa

To find

— Torques shared by two cells
— maximum shear stress in each cell
— angular twist per m length

Torques shared by two cells

- *Cell I*: It is a thin-walled closed section Let T_1 N·mm be the torque in this cell.

$A = 50 \times 40 = 2000$ mm²; $z = 2(50 + 40) = 180$ mm; $l = 1000$ mm

$$\theta = \frac{lT_1 z}{4GA^2 t} = \frac{1000 \times T_1 \times 180}{4 \times 102\,000 \times 2000^2 \times 2} = 55.15 \times 10^{-9}\, T_1 \text{ rad} \qquad \text{...(Eq. 10.15)}$$

- *Cell II*: Here the length (40 + 25) mm can be considered as a thin rectangular section.

$h = 40 + 25 = 65$ mm; $l = 1000$ mm

Let T_2 N·mm be the torque in cell II,

$$\theta = \frac{3T_2 l}{ht^3 \cdot G} = \frac{3T_2 \times 1000}{2(40+25)2^3 \times 102\,000} = 28.28 \times 10^{-6}\, T_2 \text{ rad} \qquad \text{...(Eq. 10.20)}$$

For continuity, θ is to be the same for the two cells,

i.e., $55.15 \times 10^{-9}\, T_1 = 28.28 \times 10^{-6}\, T_2$ or $\dfrac{T_1}{T_2} = 512.8$

$T_1 + T_2 = 20\,000$ N·mm or $512.8T_2 + T_2 = 20\,000$ or $T_2 = 38.9$ N·mm

$T_1 = 512.8 \times 38.9 = 19\,961.1$ N·mm

Shear stresses

Also, $T_1 = 2\,\tau_1 . t_1\, A_1$ ...(Eq. 10.12)

or $19\,961.1 = 2\,\tau_1 \times 2 \times 2000$ or $\tau_1 = 2.495$ MPa

$$T_2 = \frac{\tau_2 h t^2}{3} \qquad \text{...(Eq. 10.19)}$$

or $$38.9 = \frac{\tau_2 \times (2 \times 65) \times 2^2}{3} \quad \text{or} \quad \tau_2 = 0.224 \text{ MPa}$$

Angular twist

$$\theta = 55.15 \times 10^{-9} \times 19\,961.1 = 1.1 \times 10^{-3} \text{ rad/m}$$

Summary

1. A shaft transmitting power is acted upon by a pure torque T about its polar axis and shear stresses are set up perpendicular to the radius at all transverse sections.
2. In circular shafts, $\frac{T}{J} = \frac{\tau}{r} = \frac{G\theta}{l}$
3. In solid circular shafts, maximum stress, $\tau = 16T/(\pi d^3)$
4. In hollow shafts, maximum stress, $\tau = \frac{T \cdot D/2}{\pi(D^4 - d^4)/32} = \frac{16T \cdot D}{\pi(D^4 - d^4)}$
5. Torque per radian twist is known as the *torsional stiffness*, $k = \frac{T}{\theta} = \frac{JG}{l}$
6. Total strain energy of a shaft under the action of a torque is the work done in twisting
7. In a solid shaft, $U = (\tau^2/4G)$ Volume
8. In a solid shaft, maximum strain energy /unit volume = $\tau^2/2G$
9. In a hollow shaft, $U = \frac{\tau^2}{4G} \cdot \frac{D^2 + d^2}{D^2} \times$ Volume
10. In combined bending and torsion,

    Maximum principal stress, $\sigma = \frac{16}{\pi D^3}(M + \sqrt{M^2 + T^2})$

    Maximum shear stress, $\tau = \frac{16}{\pi D^3}\sqrt{M^2 + T^2}$
11. $(\sqrt{M^2 + T^2})$ is known as the *equivalent torque.*
12. In thin tubular sections, (Bredt–Batho theory), $\tau \cdot t = \tau' \, t'$ = constant i.e. product of shear stress and thickness is constant at all points on the periphery. This is called *shear flow* (q) and is constant at any given point. Also, $T = 2\tau.t.A$ and $\theta = \frac{lTz}{4GA^2t}$
13. In thin-walled sections, $\tau_1 \cdot t_1 = \tau_2 t_2 + \tau_3 t_3$, $T = 2(\tau_1 \cdot t_1 A_1 + \tau_2 t_2 A_2)$,

    $$\theta = \frac{l}{2G} \frac{\tau_1 \cdot z_1 + \tau_3 z_3}{A_1} = \frac{l}{2G} \frac{\tau_2 \cdot z_2 - z_3 z_3}{A_1}$$
14. In thin rectangular members, $T = \frac{\tau h t^2}{3}$, $\theta = \frac{3Tl}{ht^3 \cdot G}$

Objective Type Questions

1. Magnitude of shear stress induced in a shaft due to applied torque varies from
   (a) maximum at centre to zero at circumference
   (b) maximum at centre to minimum (not–zero) at circumference
   (c) zero at centre to maximum at circumference
   (d) Minimum (not zero) at centre to maximum at circumference
2. The variation of shear stress in a circular shaft subjected to torsion is
   (a) linear (b) parabolic (c) hyperbolic (d) uniform
3. The relation governing the torsional torque in circular shafts is
   (a) $\frac{T}{r}=\frac{\tau}{l}=\frac{G\theta}{J}$ (b) $\frac{T}{J}=\frac{\tau}{r}=\frac{G\theta}{l}$ (c) $\frac{T}{J}=\frac{\tau}{l}=\frac{G\theta}{r}$ (d) $\frac{T}{l}=\frac{\tau}{r}=\frac{G\theta}{J}$
4. Torsional rigidity of a shaft is defined as
   (a) G/J (b) GJ (c) TJ (d) T/J
5. Torsional rigidity of a shaft is given by
   (a) Gl/θ (b) $T\theta$ (c) Tl/θ (d) T/l
6. A solid shaft of same cross-sectional area and of same material as that of a hollow shaft can resist
   (a) less torque (b) more torque (c) equal torque
7. Angle of twist of a circular shaft is given by
   (a) $\frac{GJ}{Tl}$ (b) $\frac{Tl}{GJ}$ (c) $\frac{TJ}{Gl}$ (d) $\frac{TG}{Jl}$
8. Maximum shear stress of a solid shaft is given by
   (a) $\frac{16T}{\pi d}$ (b) $\frac{16T}{\pi d^2}$ (c) $\frac{16T}{\pi d^3}$ (d) $\frac{16T}{\pi d^4}$
9. The ratio of maximum bending stress to maximum shear stress on the cross–section when a shaft is simultaneously subjected to a torque T and bending moment M,
   (a) T/M (b) M/T (c) $2T/M$ (d) $2M/T$
10. Maximum shear stress in a hollow shaft subjected to a torsional moment is at the
    (a) middle of thickness (b) at the inner surface of the shaft
    (c) at the outer surface of the shaft (d) none of the above
11. The ratio of strength of a hollow shaft to that of a solid shaft subjected to torsion if both are of the same material and of the same outer diameters, the inner diameter of hollow shaft being half of the outer diameter, is
    (a) 15/16 (b) 16/15 (c) 7/8 (d) 8/7
12. Ratio of diameters of two shafts joined in series is 2. If the two shafts have the same material and the same length, the ratio of their angles of twist is
    (a) 2 (b) 4 (c) 8 (d) 16
13. Ratio of diameters of two shafts joined in series is 2. If the two shafts have the same material and the same length, the ratio of their shear stresses will be
    (a) 2 (b) 4 (c) 8 (d) 16
14. For two shafts joined in series, the ____ in each shaft is the same.
    (a) shear stress (b) angle of twist (c) torque
15. For two shafts joined in parallel, the ____ in each shaft is the same.
    (a) shear stress (b) angle of twist (c) torque

Answers

1. (c)	2. (a)	3. (b)	4. (b)	5. (c)	6. (a)
7. (b)	8. (c)	9. (d)	10. (c)	11. (a)	12. (d)
13. (c)	14. (c)	15. (b)			

Review Questions

10.1 Deduce the torsion equation stating the assumptions made. Deduce the expressions for maximum stresses in solid and hollow shafts.

10.2 Develop an expression for strain energy in a shaft subjected to torsion. Show that the maximum strain energy per unit volume is twice the total strain energy per unit volume.

10.3 Obtain a relation for maximum principal stress and maximum shear stress for a shaft under the action of combined bending and torsion.

10.4 What do you mean by equivalent torque?

10.5 Find the expression for strain energy of a shaft acted upon by bending and torsional stresses.

10.6 Discuss the Bredt–Batho theory as applicable to thin tubular sections.

10.7 In what way the shear stress and the angle of twist can be found in case of a thin-walled twin-celled section when a torque T is applied.

10.8 Give the procedure to analyse a thin rectangular member subjected to a torque T.

Numerical Problems

10.1 A hollow circular shaft of 180-mm outer diameter and 100-mm inner diameter is used to transmit power. Determine the maximum torque which can be safely transmitted if the shear stress is not to exceed 50 MPa. (93.24 kN·m)

10.2 Two shafts of the same material and of the same lengths are applied equal torques. The first shaft is a solid one whereas the second shaft is a hollow shaft with inner diameter two thirds of the external diameter. If the maximum stress developed in each shaft is the same, compare the weights of the two shafts. (0.643)

10.3 A solid shaft is replaced by a hollow shaft having internal diameter as 60% of the external diameter to transmit same power at the same speed. Determine the percentage saving in material for the same material of both the shafts. (29.8%)

10.4 A hollow shaft transmits 280 kW of power at 180 rpm. If the outside diameter of the shaft is 120 mm and the maximum stress developed is 70 MPa, determine the inside diameter of the shaft. (93.8 mm)

10.5 A 1.5-m long solid aluminium shaft with a 60-mm diameter is to be replaced by a steel hollow shaft of the same length and same external diameter to transmit same torque with same angle of twist over the same length. Determine the diameter of the hollow shaft. G_s = 82 GPa and G_a = 27 GPa. (54.3 mm)

10.6 A hollow shaft with external and internal diameters of 120 mm and 80 mm respectively is to be replaced by a solid shaft of the same weight. Find the torques transmitted by the shafts if the permissible shear stress is 100 MPa. If the solid shaft is replaced by a hollow shaft of 160-mm external diameter, what is the torque transmitted for the same weight of the shafts? (27.2 kN·m, 14 kN·m; 42.4 kN·m)

10.7 A solid shaft transmits 200 kW of power at 80 rpm. Determine the diameter of the shaft if the shear stress is not to exceed 75 MPa. If this shaft is replaced by a hollow shaft whose internal diameter is 0.6 of the external diameter while the length, material and the maximum shear stress are the same, find the percentage saving in weight. (117.5 mm; 29.8%)

10.8 Two shafts are made of the same material. Each shaft transmits the same power. The first shaft rotates at 50 rpm while the second at 5000 rpm. Determine the ratio of diameters of the two shafts for the same maximum shear stress in each shaft. (4.64)

10.9 A compound shaft is made by mounting a gunmetal sleeve on a steel shaft. The shaft is subjected to a torque in such a way that the torque on the sleeve is twice that on the shaft. Determine the ratio of external diameter of the sleeve to that of the shaft. What will be the torque transmitted by the compound shaft if the steel shaft diameter is 56 mm and the limiting values of the shear stresses in the gun metal and the steel are 50 MPa and 85 MPa respectively? $G_{st} = 2.4\ G_{gm}$. (1.55; 8.007 kN·m)

10.10 A steel shaft *ABC* is of 150-mm diameter over a length of $AB = 3$ m and of 100 mm diameter over a length of $BC = 1.5$ m. At *B*, a clockwise torque of 45 kN·m and at *C* a counter-clockwise torque of 15 kN·m is applied. If the shaft is in equilibrium, determine (i) the maximum shear stress in the shaft, (ii) angle of twist of *B* with respect to *A*, and (iii) angle of twist of *C* with respect to *A*. $G = 80$ GPa. (76.4 MPa; 1.95°; 1.64°)

10.11 A shaft, 960 mm in length and of 96-mm diameter is bored for a part of its length to 48-mm diameter and for the remaining part, to 72-mm diameter. Determine the maximum torque to be transmitted by the shaft at 220 rpm if the shear stress is not to exceed 75 MPa. If the angle of twist in the two portions of different diameters is the same, determine the length of the shaft that has been bored to 48-mm diameter. (205.3 kW; 510.8 mm)

10.12 A solid alloy shaft of 60-mm diameter is coupled in series with a hollow steel shaft of the same outside diameter. Determine the inside diameter of the steel shaft if the angle of twist per unit length is 7% of that of the alloy shaft. What is the speed at which the shafts transmit 420 kW of power? The permissible values of shear stresses in the alloy and steel shafts are 65 MPa and 80 MPa respectively. $G_s = 2\ G_{alloy}$. (50.5 mm; 1656 rpm)

10.13 A 10-m long composite shaft is made up of a steel shaft of 240-mm diameter surrounded by a closely fitted 30-mm thick bronze tube. If the shear stress in the steel shaft is not to exceed 20 MPa, determine the maximum power transmitted by the shaft at 180 rpm. $G_s = 84$ GPa and $G_b = 42$ GPa. (1760 kW)

10.14 A 1-m long shaft tapers uniformly from a diameter of 80 mm to a diameter of 120 mm. Determine the angle of twist and the maximum shear stress induced if the shaft transmits a torque of 10 kN·m. $G = 82$ GPa. (0.98°; 99.5 MPa)

10.15 A shaft of 72-mm diameter supported on bearings 640 mm apart transmits 32 kW of power at 400 rpm. The shaft carries a flywheel weighing 5 kN midway of the shaft. Find the principal stresses and the maximum shear stress at the ends of a vertical and horizontal diameter in a plane close to the flywheel. (26.01 MPa, –4.18 MPa, 15.09 MPa; 10.42 MPa)

10.16 Determine the inside and outside diameters of a hollow steel shaft whose internal diameter is 0.6 of the external diameter and transmits 120 kW at 210 rpm and the allowable stress is limited to 75 MPa. If a bending moment of 2800 N·m is applied to the shaft, find the speed at which the shaft must be rotated to transmit the same power for the same value of the maximum shear stress. (45.1 mm, 75.2 mm; 245 rpm)

10.17 The mean dimensions of the two cells of a thin-walled twin-celled section (Fig. 10.16) are 50 mm × 20 mm and 20 mm square. Thicknesses are $t_1 = 2$ mm, $t_2 = 4$ mm and $t_3 = 3$ mm. For an applied torque of 120 N·m, determine the shear stress in each section and the angle of twist per metre length. $G = 84$ GPa. (19.38 MPa, 13.27 MPa, – 4.79 MPa; 0.76°)

Chapter 11

Springs

Any elastic member which can deform under a force can act as a spring. The main function of a spring is to deflect under a load and to recover the original shape when the load is released. However, the term is normally used for those members that deform considerably under the action of forces without exceeding the safe limit of stresses, e.g., a steel helical spring may be expanded to twice its length without losing its elasticity. Springs can be made to act under tension, compression, torsion, bending or a combination of these loads. Usually, the springs are made from conventional metals though sometimes they can be of nonmetallic materials.

In general, the springs are used to

— absorb energy and to release the same according to the desired function to be performed such as a spring of a clock
— absorb shocks as in case of automobiles
— deflect under external forces to provide the desired motion to a machine member such as springs used in weighing machines, safety valves, clutches governors, etc.

- *Stiffness* or *spring constant* of a spring is defined as the force required for unit deflection.
- *Solid length* of a spring is the length of a spring in the fully compressed state when the coils touch each other.

11.1 CLOSE-COILED HELICAL SPRINGS

Close-coiled helical springs are those in which the angle of the helix is so small that if the axis of the spring is vertical, the coils may be assumed to be in a horizontal plane. Such a spring may be acted upon by an axial load or an axial torque. When they are acted upon by an axial load, there is axial extension and when there is an axial torque, there is a change in the radius of curvature of the spring coils. In the latter case, there is an angular rotation of the free end and the action is known as *wind-up*.

(i) Under Axial Load

Let W = Axial load
D = Mean coil diameter
R = Mean coil radius
d = wire diameter
θ = total angle of twist along the wire

δ = deflection of W along the axis of the coils
n = number of coils
l = length of wire

As shown in Fig. 11.1, the action of the load W on any cross-section is to twist it like a shaft with a pure torque WR. Bending and shear effects may be neglected.

Equating the work done by the axial force to the torsional strain energy

i.e., $$\frac{1}{2}W\delta = \frac{1}{2}\cdot T\theta = \frac{1}{2}\cdot WR\cdot\theta$$

or $$\delta = R\cdot\theta$$

Now, $$\theta = \frac{Tl}{GJ} = \frac{WRl}{G(\pi d^4/32)} = \frac{32WRl}{G\pi d^4} \tag{11.1}$$

Also as $l = 2\pi Rn$,

$$\therefore \quad \theta = \frac{32WR(2\pi Rn)}{G\pi d^4} = \frac{64WR^2 n}{Gd^4} \tag{11.2}$$

Deflection of the spring, $$\delta = R\theta = \frac{32WR^2 l}{G\pi d^4} \tag{11.3}$$

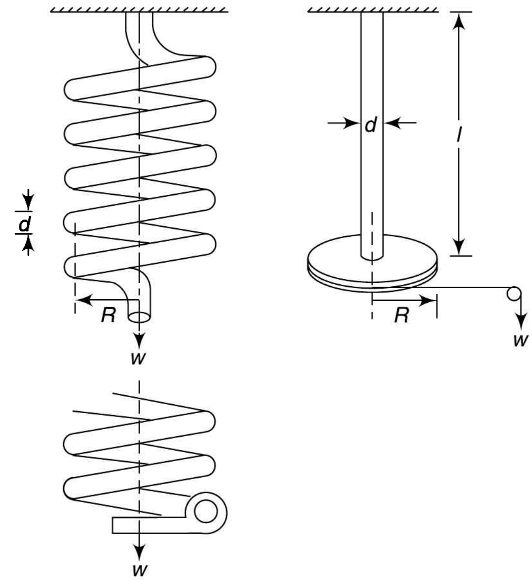


Fig. 11.1

Also by using Eq. 11.2, $$\delta = R\theta = \frac{64WR^3 n}{Gd^4} = \frac{8WD^3 n}{Gd^4} \tag{11.4}$$

Stresses Direct shear stress = $\dfrac{W}{A} = \dfrac{W}{(\pi/4)d^2}$ (assuming uniform distribution)

Maximum torsional stress, $\tau = \dfrac{T}{J}\cdot r = \dfrac{WR}{(\pi d^4/32)}\cdot\dfrac{d}{2} = \dfrac{16WR}{\pi d^3} = \dfrac{8WD}{\pi d^3}$

$$\therefore \text{ maximum shear stress} = \frac{W}{(\pi/4)d^2} + \frac{8WD}{\pi d^3} = \frac{8WD}{\pi d^3}\left(\frac{d}{2D}+1\right) \tag{11.5}$$

For most springs, $\dfrac{d}{2D} << 1$ and thus can be ignored.

$$\text{Thus shear stress} = \frac{8WD}{\pi d^3} \tag{11.5a}$$

Equation 11.5a can be applied to springs of thin wires where D/d ratio is more than 20

Wahl's correction This is a method to incorporate the effects of direct shear and the curvature and is based on the experimental investigations of A H Wahl. According to him, the maximum stress in the spring wire can be expressed as

$$\tau_{\max} = \frac{8WD}{\pi d^3}\cdot K \tag{11.5b}$$

K is known as the *Wahl's constant* and is given by

$$K = \frac{4S_i - 1}{4S_i - 4} + \frac{0.615}{S_i} \qquad \text{(for inner surface)}$$

and $$K = \frac{4S_i - 1}{4S_i - 4} - \frac{0.615}{S_i}$$ (for outer surface)

where S_i is the *spring index* and is the ratio of the mean diameter of the coil to the diameter of the wire.

Equation 11.5b is suitable for heavy springs used for railway wagons.

(ii) Under Axial Torque

The axial torque T tends to wind up the spring by producing approximately a pure bending moment at all cross-sections (Fig. 11.2).

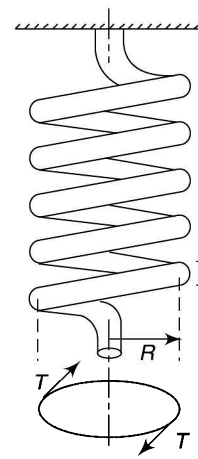


Fig. 11.2

When bending moment is applied to curved bars with small curvature (T is the applied bending moment),

$$T = EI\left(\frac{1}{R'} - \frac{1}{R}\right) \qquad \text{(Eq. 9.1)}$$

where R and R' are the radii of curvature before and after applying the bending moment.

Let n and n' be the number of turns of the spring before and after applying the bending moment.

Then length of spring, $l = 2\pi nR = 2\pi n'R'$ or $n' = n(R/R')$

If φ is the angle of rotation of one end of the spring relative to the other end about the axis,

$$\varphi = 2\pi \times \text{increase in number of turns of the spring}$$

$$= 2\pi(n' - n) = 2\pi[n(R/R') - n] = 2\pi nR\left(\frac{1}{R'} - \frac{1}{R}\right) = \frac{Tl}{EI}$$

$$= \frac{Tl}{E(\pi d^4/64)} = \frac{64Tl}{E\pi d^4} = \frac{64T(\pi Dn)}{E\pi d^4} = \frac{64TDn}{Ed^4} \qquad (11.6)$$

- The relation can also be obtained by considering the total strain energy due to bending moment of magnitude T,

  $$U = \frac{T^2 l}{2EI} \qquad \text{(Eq. 7.34)}$$

  Then strain energy due to torsional effect,

  $$U = \frac{1}{2}T\varphi \qquad \text{(Refer Section 10.6)}$$

  Equating the two, $\varphi = \dfrac{Tl}{EI} = \dfrac{64TDn}{Ed^4}$ as above

Example 11.1 A close-coiled helical spring having 24 turns is made of 8-mm diameter wire. The mean diameter of the spring is 80 mm and it carries a load of 250 N. Determine the shear stress developed, the deflection and the stiffness of the spring. Take G = 84 GPa.

Solution

Given A close-coiled helical spring

W = 250 N $\qquad$ D = 80 mm

n = 24 $\qquad$ d = 8 mm

G = 84 GPa

To find

Shear stress, deflection and stiffness

Shear stress

$$\tau = \frac{8WD}{\pi d^3} = \frac{8 \times 250 \times 80}{\pi \times 8^3} = 99.5 \text{ MPa}$$

Deflection

$$\delta = \frac{8WD^3 n}{Gd^4} = \frac{8 \times 250 \times 80^3 \times 24}{84\,000 \times 8^4} = 71.4 \text{ mm}$$

Stiffness

$$\text{Stiffness} = \frac{W}{\delta} = \frac{250}{71.4} = 3.5 \text{ N/mm}$$

Example 11.2 A close-coiled helical spring carries a load of 400 N. Its mean coil diameter is 10 times the wire diameter. Determine these diameters if the maximum value of shear stress in the spring is not to exceed 75 MPa.

Solution

Given A close-coiled helical spring

$W = 400$ N $\qquad D = 10\,d$

$\tau = 75$ MPa

To find

— Wire diameter

— mean diameter

Wire diameter

$$\tau = \frac{8WD}{\pi d^3} \quad \text{or} \quad 75 = \frac{8 \times 400 \times (10d)}{\pi \times d^3} \quad \text{or} \quad d = 11.65 \text{ mm}$$

Mean diameter

$$D = 10 \times 11.65 = 116.5 \text{ mm}$$

Example 11.3 A safety valve of 120-mm diameter is designed to blow off at a gauge pressure of 1 MPa. A close-coiled helical spring of 160 mm mean diameter is used to hold the valve in position. Determine the diameter of the coils of the spring and the number of turns required if the initial compression of the spring is 60 mm and the maximum value of shear stress is 70 MPa. G = 84 GPa.

Solution

Given A close-coiled helical spring for a safety valve of 120-mm diameter with a blow-off pressure of 1 MPa.

$\delta = 60$ mm $\qquad D = 160$ mm

$\tau = 70$ MPa $\qquad G = 84$ GPa

To find

— Wire diameter

— Number of coils

Determination of load

$$\text{Force on the spring, } W = \frac{\pi}{4} \times (120)^2 \times 1 = 11\,310 \text{ N}$$

Wire diameter

$$\tau = \frac{8WD}{\pi d^3} \quad \text{or} \quad 70 = \frac{8 \times 11\,310 \times 200}{\pi \times d^3} \quad \text{or} \quad d = 43.5 \text{ mm}$$

Number of coils

$$\delta = \frac{8WD^3 n}{Gd^4}$$

or

$$60 = \frac{8 \times 11\,310 \times 200^3 \times n}{84\,000 \times 43.5^4}; \qquad n = 24.9 \approx 25 \text{ turns}$$

Example 11.4 || A close-coiled helical spring absorbs 72 N·m of energy when compressed through 60 mm. There are 8 coils in the spring. The coil diameter is 10 times the wire diameter. Find the diameters of the coil and wire and the maximum shear stress. G = 82 GPa.

Solution

Given A close-coiled helical spring

U = 72 N·m $\qquad$ δ = 60 mm
D = 10 d $\qquad$ n = 8
G = 82 GPa

To find

— Wire and coil diameters
— Maximum shear stress

Determination of load

$$U = \frac{1}{2} W\delta$$

or

$$72\,000 = \frac{1}{2} W \times 60 \qquad \text{or} \qquad W = 2400 \text{ N}$$

Wire and coil diameters

$$\delta = \frac{8WD^3 n}{Gd^4}$$

or

$$60 = \frac{8 \times 2400 \times (10d)^3 \times 8}{82\,000 d^4} \qquad \text{or} \qquad d = 31.2 \text{ mm}$$

and

$$D = 312 \text{ mm}$$

Maximum shear stress

$$\tau = \frac{8WD}{\pi d^3} = \frac{8 \times 2400 \times 312}{\pi \times 31.2^3} = 62.8 \text{ MPa}$$

Example 11.5 || A wagon weighing 35 kN moves at a speed of 3.6 km/h. Find the number of springs required in a buffer stop to absorb the energy of motion during a compression of 180 mm? The mean diameter of coils is 220 mm and the diameter of the steel rod of the spring is 24 mm. Each spring consists of 30 coils. G = 90 GPa.

Solution

Given A close-coiled helical spring

$v = 3.6$ km/h	$m = 35$ kN
$\delta = 180$ mm	$D = 220$ mm
$n = 30$	$d = 24$ mm
$G = 90$ GPa	

To find Number of springs

Determination of load

$$v = \frac{3.6 \times 1000}{3600} = 1 \text{ m/s}$$

$$\text{Kinetic energy } = \frac{1}{2}mv^2 = \frac{1}{2} \times \frac{35\,000}{9.81} \times 1^2$$

$$= 1784 \text{ N}\cdot\text{m or } 1784\,000 \text{ N}\cdot\text{mm}$$

Now,
$$\delta = \frac{8WD^3n}{Gd^4}$$

or
$$180 = \frac{8W \times 220^3 \times 30}{90\,000 \times (24)^4} \quad \text{or} \quad W = 2103 \text{ N}$$

Number of springs

$$\text{Energy stored by one spring} = \frac{1}{2}W\delta = \frac{1}{2} \times 2103 \times 180 = 189\,270 \text{ N}\cdot\text{mm}$$

$$\text{Number of springs required } = \frac{1\,784\,000}{189\,270} = 9.43\text{, i.e., 10 (say)}$$

Example 11.6 || A weight of 240 N is dropped on to a close-coiled helical spring made of 18 mm steel wire. The spring consists of 22 coils wound to a diameter of 180 mm. If the instantaneous compression is 120 mm, determine the height of drop of the weight and the instantaneous stress produced. $G = 88$ GPa.

Solution

Given A close-coiled helical spring

$P = 240$ N	$d = 18$ mm
$n = 22$	$D = 180$ mm
$\delta = 120$ mm	$G = 88$ GPa

To find
- — Height of weight drop
- — Stress

Determination of gradual load

Let W be the equivalent gradually applied load in N

$$\delta = \frac{8WD^3n}{Gd^4} \quad \text{or} \quad 120 = \frac{8W \times 180^3 \times 22}{88\,000 \times (18)^4} \quad \text{or} \quad W = 1080 \text{ N}$$

Height of drop

Now, Energy loss due to weight = Work done on the spring

$$P(h + \delta) = \frac{1}{2} \cdot W\delta \quad \text{or} \quad 240(h + 120) = \frac{1}{2} \times 1080 \times 120 \quad \text{or} \quad h = 150 \text{ mm}$$

Stress

$$\tau = \frac{8WD}{\pi d^3} = \frac{8 \times 1080 \times 180}{\pi \times 18^3} = 84.9 \text{ MPa}$$

Example 11.7 A close-coiled helical spring has a maximum load of 40 N and maximum shear stress induced is 100 MPa. The spring constant is 600 N/m in compression. If the solid length (coils touching) of the spring is 60 mm, determine the wire diameter, number of coils and the coil diameter. G = 32 GPa.

Solution

Given A close-coiled helical spring

$W = 40$ N $\qquad \tau = 100$ MPa

$nd = 60$ mm $\qquad G = 32$ GPa

s = 600 N/m = 0.6 N/mm

To find

— Wire and coil diameters
— number of coils

Relation between d and D

Relation between d and D can be obtained from the relation, $\tau = \dfrac{8WD}{\pi d^3}$

i.e., $$100 = \frac{8 \times 40 \times D}{\pi \times d^3} \quad \text{or} \quad D = 0.982\, d^3$$

Determination of diameters

As $$\delta = \frac{8WD^3 n}{Gd^4}$$

$\therefore$ $$\text{Stiffness, } s = \frac{W}{\delta} = \frac{Gd^4}{8D^3 n}$$

or $$0.6 = \frac{32\,000 \times d^4}{8(0.982 d^3)^3 \times 60/d}$$

$$d^4 = 117.3 \quad \text{or} \quad d = 3.29 \text{ mm}$$

$\therefore$ $$D = 0.982 \times 3.29^3 = 35 \text{ mm}$$

Number of coils

Number of coils, $n = \dfrac{60}{3.29} = 18.24$

Example 11.8 A close-coiled helical spring is built-in at both ends. A force of 800 N is applied at an intermediate point such that the number of coils on one side is 10 and on the other 6. Determine the distribution of the force on two sides.

Solution

Given A close-coiled helical spring as shown in Fig. 11.3.

$W = 800$ N $\qquad n_2/n_1 = 6/10 = 0.6$

To find Distribution of force on two sides

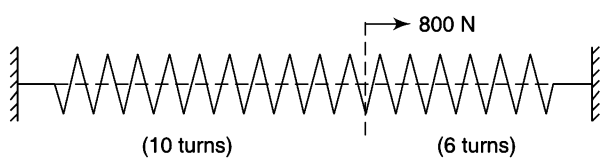


Fig.11.3

Determination of ratio of forces

As the deflection of both sides will be equal,

$$\delta = \frac{8W_1D^3n_1}{Gd^4} = \frac{8W_2D^3n_2}{Gd^4} \quad \text{or} \quad W_1n_1 = W_2n_2 \quad \text{or} \quad \frac{W_1}{W_2} = \frac{n_2}{n_1}$$

i.e. the force on each side will be in the inverse ratio of number of coils.

or $$\frac{W_1}{W_2} = 0.6 \quad \text{or} \quad W_1 = 0.6W_2$$

Distribution of forces

$$W_1 + W_2 = W = 800$$

$\therefore$ $$0.6\,W_2 + W_2 = 800 \quad \text{or} \quad W_2 = 500 \text{ N} \quad \text{and} \quad W_1 = 300 \text{ N}$$

Example 11.9 A close-coiled helical spring made of a 10-mm diameter steel bar has 8 coils of 150-mm mean diameter. Calculate the elongation, torsional stress and the strain energy per unit volume when the spring is subjected to an axial load of 130 N. G = 80 GPa.

If instead of the axial load, an axial torque of 9 N · m is applied, find the axial twist, bending stress and the strain energy per unit volume. E = 205 GPa

Solution

Given A close-coiled helical spring

W = 130 N $\qquad d$ = 10 mm

n = 8 $\qquad D$ = 150 mm

T = 9 N · m $\qquad G$ = 80 GPa

E = 205 GPa

To find

— Elongation, torsional stress and strain energy under axial load

— Axial twist, bending stress strain energy under axial torque

Under axial load

$$\delta = \frac{8WD^3n}{Gd^4} = \frac{8 \times 130 \times 150^3 \times 8}{80\,000 \times 10^4} = 35.1 \text{ mm}$$

$$\tau = \frac{8WD}{\pi d^3} = \frac{8 \times 130 \times 150}{\pi \times 10^3} = 49.66 \text{ MPa}$$

Strain energy/unit volume $$= \frac{\tau^2}{4G} \qquad \text{...(Eq. 10.5)}$$

$$= \frac{49.66^2}{4 \times 80\,000} = 0.007\,71 \text{ N} \cdot \text{mm/mm}^3$$

or $$7710 \text{ N} \cdot \text{m/m}^3 \quad \text{or} \quad 7.71 \text{ kN} \cdot \text{m/m}^3$$

Under axial torque

$$\varphi = \frac{64TDn}{Ed^4} = \frac{64 \times 9000 \times 150 \times 8}{205\,000 \times 10^4} = 0.3372 \text{ rad}$$

Bending stress, $$\sigma_b = \frac{32T}{\pi d^3} = \frac{32 \times 9000}{\pi \times 10^3} = 91.7 \text{ MPa}$$

$$\text{Strain energy/unit volume } = \frac{T\varphi/2}{\text{Volume}}$$

$$= \frac{9000 \times 0.3372/2}{(\pi \times 10^2/4)\,\pi \times 150 \times 8} = 0.005\,124 \text{ mm/mm}^3 \text{ or } 5.124 \text{ N}\cdot\text{m/m}^3$$

Example 11.10 Show that for a close-coiled spring of circular section of mean coil diameter D, if W is the axial load applied per unit deflection and T the torque applied per unit angular rotation independent of W, then $T/W = D^2(1 + \nu)/4$ where ν is the Poisson's ratio.

Also determine the Poisson's ratio if a load of 120 N extends the spring by 80 mm and a torque of 500 N·mm produces an angular rotation of 90°. The mean coil diameter of the spring is 25 mm.

Solution

Given A close-coiled helical spring

$W = 120$ N $\quad\quad \delta = 80$ mm

$T = 500$ N $\quad\quad D = 25$ mm

$\varphi = 90°$

To find

— $T/W = D^2(1 + \nu)/4$

— Poisson's ratio

Ratio of T/D

As W is the axial load applied per unit deflection, $\therefore\ 1 = \dfrac{8WD^3n}{Gd^4}$

As T is the torque applied per unit angular rotation independent of W,

$$\therefore \quad 1 = \frac{64TDn}{Ed^4}$$

Thus
$$\frac{8WD^3n}{Gd^4} = \frac{64TDn}{Ed^4}$$

or
$$\frac{T}{W} = \frac{D^2E}{8G} = \frac{D^2 \times 2G(1+\nu)}{8G} = \frac{D^2(1+\nu)}{4} \tag{i}$$

Numerical

$$\text{Torque/unit angular rotation, } T = \frac{500}{90 \times (\pi/180)} = 318.3 \text{ N}\cdot\text{mm/rad}$$

$$\text{Axial load applied per unit deflection, } W = \frac{120}{80} = 1.5 \text{ N}$$

Thus from (i),
$$\frac{318.3}{1.5} = \frac{25^2(1+\nu)}{4}$$

or
$$1 + \nu = \frac{848.8}{25^2} = 1.358 \quad \text{or} \quad \nu = 0.358$$

Example 11.11 A close-coiled helical spring has its free length as 120 mm. It absorbs 40 N·m of energy when fully compressed and the coils are in contact. The mean coil diameter is 80 mm. Determine the diameter of the steel wire required and the number of coils if the maximum shear stress is to be 120 MPa. G = 82 GPa.

Solution

Given A close-coiled helical spring

$U = 40$ N·m $\qquad D = 80$ mm

$l = 120$ N $\qquad \tau = 120$ MPa

$G = 82$ GPa

To find

— Wire diameter

— number of coils

Relation for deflection

Deflection = Free length – solid length = 120 – *nd*,

We have $\delta = \dfrac{8WD^3n}{Gd^4}$ or $120 - nd = \dfrac{8W \times 80^3 \times n}{82\,000 \times d^4}$

or $\qquad (120 - nd)d^4 = 49.95Wn \qquad$ (i)

Relation for shear stress

W and *n* can be expressed in terms of *d* as follows:

$$\tau = \frac{8WD}{\pi d^3} \quad \text{or} \quad W = \frac{\pi\tau d^3}{8D} = \frac{\pi \times 120 \times d^3}{8 \times 80} = 0.589d^3$$

Relation for strain energy

$$U = \frac{\tau^2}{4G} \times \text{Volume} = \frac{\tau^2}{4G} \times \text{Area of cross-section} \times \text{length}$$

$$40\,000 = \frac{120^2}{4 \times 82\,000} \times \frac{\pi}{4} \times d^2 \times \pi \times 80 \times n \quad \text{or} \quad n = \frac{4616}{d^2}$$

Determination of d and n

From (i), $\left(120 - \dfrac{4616}{d^2}d\right)d^4 = 49.95 \times 0.589d^3 \times \dfrac{4616}{d^2}$

or $\quad 120d^3 - 4616d^2 = 135\,805 \quad$ or $\quad d^3 - 38.47d^2 = 1132$

Solving by trial and error, $d = 39.2$ m and $n = \dfrac{4616}{39.2^2} = 3$

11.2 SPRINGS IN SERIES AND PARALLEL

- When two springs of different stiffnesses are joined end to end, they are said to be connected in *series*. For springs in series, the load is the same for both the springs whereas the deflection is the sum of deflection of each,

  i.e., $\delta = \delta_1 + \delta_2$ or $\dfrac{W}{s} = \dfrac{W}{s_1} + \dfrac{W}{s_2}$ or $\dfrac{1}{s} = \dfrac{1}{s_1} + \dfrac{1}{s_2}$ or $s = \dfrac{s_1 s_2}{s_1 + s_2}$

  where s_1 and s_2 are the stiffness of each joined spring and *s* is the equivalent stiffness of the composite spring.

- When two springs of different stiffnesses are joined in *parallel*, they have the common deflection and the load is the sum of load taken by each,

  i.e., common deflection $\delta = \dfrac{W}{s} = \dfrac{W_1}{s_1} = \dfrac{W_2}{s_2}$

and $W = W_1 + W_2$ or $s\delta = s_1\delta + s_2\delta$ or $s = s_1 + s_2$

where W_1 and W_2 are the loads taken by each spring.

Example 11.12 A composite spring consists of two close-coiled helical springs connected in series. Each spring has 14 coils at a mean diameter of 20 mm. The stiffness of the composite spring is 800 N/m. If the wire diameter of one spring is 2.5 mm, find the wire diameter of the other.

What will be the maximum load which can be carried by the composite spring and the corresponding deflection for a maximum shear stress of 150 MPa? G = 78 GPa.

Solution

Given A composite spring having two close-coiled helical springs in series

$n = 14$ $\quad D = 20$ mm

$d_1 = 2.5$ mm $\quad \tau = 150$ MPa

$G = 78$ GPa $\quad s = 800$ N/m $= 0.8$ N/mm

To find

— Wire diameter d_2

— Maximum load and deflection

Determination of wire diameter

For springs in series, $\dfrac{1}{s} = \dfrac{1}{s_1} + \dfrac{1}{s_2}$

$$s_1 = \frac{W}{\delta} = \frac{Gd_1^4}{8D^3n} = \frac{78\,000 \times 2.5^4}{8X\,20^3 \times 14} = 3.4$$

and
$$s_2 = \frac{78\,000 \times d_2^4}{8 \times 20^3 \times 14} = 0.087\,05\, d_2^4$$

$\therefore$
$$\frac{1}{0.8} = \frac{1}{3.4} + \frac{1}{0.087\,05\, d_2^4}$$

or
$$\frac{1}{0.087\,05\, d_2^4} = \frac{1}{0.8} - \frac{1}{3.4} = 0.956$$

or $\quad d_2^4 = 12.02 \quad$ or $\quad d_2 = 1.86$ mm

Load and deflection

$$\tau = \frac{8WD}{\pi d_2^3} \quad \text{or} \quad 150 = \frac{8W \times 20}{\pi \times 1.86^3} \quad \text{or} \quad W = 18.95 \text{ N}$$

$$\text{Total deflection} = \frac{W}{s} = \frac{18.95}{0.8} = 23.7 \text{ mm}$$

Example 11.13 A load W is transmitted to a combination of three springs through a weightless bar as shown in Fig. 11.4. All the springs are made from the same bar of steel and are of the same length initially. The number of coils in the three springs are 20, 24 and 30 and the mean diameters are in the ratio 5:6:7 respectively. Determine the value of x so that the bar remains horizontal after applying the load.

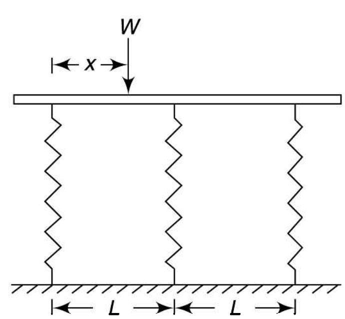


Fig. 11.4

Solution

Given A composite spring having two close-coiled helical springs in series as shown in Fig. 11.4.

$n_1 = 20$ $n_2 = 24$ $n_3 = 30$ $D_1 : D_2 : D_3 = 5:6:7 = 1:1.2:1.4$

To find Value of x

The bar remains horizontal after applying the load implying that the deflection is equal for all the three springs. Let W_1, W_2 and W_3 be the loads shared by the three springs.

Sharing of load by wires

We have $\delta = \dfrac{8W_1D_1^3n_1}{Gd^4} = \dfrac{8W_2D_2^3n_2}{Gd^4} = \dfrac{8W_3D_3^3n_3}{Gd^4}$

From first and second terms, $W_2 = W_1\left(\dfrac{D_1}{D_2}\right)^3\dfrac{n_1}{n_2} = W_1\left(\dfrac{1}{1.2}\right)^3 \times \dfrac{20}{24} = 0.4823W_1$

From first and third terms, $W_3 = W_1\left(\dfrac{D_1}{D_3}\right)^3\dfrac{n_1}{n_3} = W_1\left(\dfrac{1}{1.4}\right)^3 \times \dfrac{20}{30} = 0.243W_1$

Determination of x

$$W = W_1 + W_2 + W_3 = W_1 + 0.4823W_1 + 0.243W_1 = 1.7253W_1$$

Taking moments about the line of force of the first load,

$$W \cdot x = W_2L + W_3 \times 2L$$

or $\quad 1.7253W_1 \cdot x = 0.4823W_1L + 0.243W_1 \times 2L \quad$ or $\quad x = 0.5612\,L$

11.3 CONCENTRIC (CLUSTER) SPRINGS

Concentric or cluster springs are those which are close-coiled helical springs and one placed inside the other as shown in Fig. 11.5.

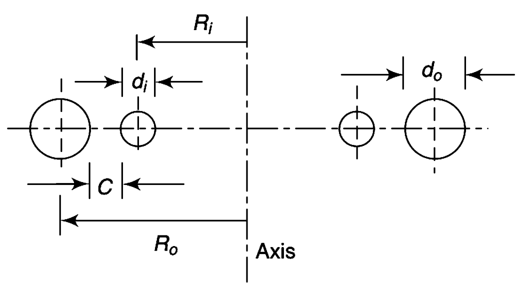


Fig. 11.5

Let W_i, W_o = Load shared by the inner and outer springs respectively
D_i, D_o = Mean coil diameters of the inner and outer springs respectively
d_i, d_o = Wire diameters of the inner and outer springs respectively
n_i, n_o = Number of coils in the outer and inner springs respectively
c = Clearance between the inner and the outer springs

From Fig. 11.5, $\quad R_o = R_i + \dfrac{d_i}{2} + c + \dfrac{d_o}{2} \quad$ or $\quad D_o = D_i + d_i + 2c + d_o$

- If initially the free length of the springs is the same, the deflection will be the same.

and $$\delta = \frac{8W_iD_i^3n_i}{G_id_i^4} = \frac{8W_oD_o^3n_o}{G_od_o^4} \qquad \text{(i)}$$

- If the maximum shear stress is equal in the two springs,

$$\tau = \frac{8W_iD_i}{\pi d_i^3} = \frac{8W_oD_o}{\pi d_o^3} \tag{ii}$$

Dividing (ii) by (i), $\dfrac{G_i d_i}{n_i D_i^2} = \dfrac{G_o d_o}{n_o D_o^2}$ or $\left(\dfrac{D_o}{D_i}\right)^2 = \dfrac{n_i}{n_o}\dfrac{G_o}{G_i}\dfrac{d_o}{d_i}$

If the material and the solid length is the same i.e. $G_i = G_o$ and $n_i d_i = n_o d_o$

$$\therefore \quad \left(\frac{D_o}{D_i}\right)^2 = \left(\frac{d_o}{d_i}\right)^2 \quad \text{or} \quad \frac{D_o}{D_i} = \frac{d_o}{d_i} \tag{iii}$$

From (ii), $\dfrac{W_o}{W_i} = \dfrac{D_i}{D_o}\dfrac{d_o^3}{d_i^3} = \dfrac{d_i}{d_o}\left(\dfrac{d_o}{d_i}\right)^3 = \left(\dfrac{d_o}{d_i}\right)^2$

Also using (iii), $\dfrac{W_o}{W_i} = \left(\dfrac{D_o}{D_i}\right)^2$

Example 11.14 || In a compound helical spring, the inner ring is concentric with the outer one. The maximum permissible deflection of the springs is 30 mm and the solid length is 40 mm. The springs are subjected to an axial load of 5 kN. Both the springs are made of the same material having G = 82 GPa. The diameter of the inner spring is 120 mm and the radial clearance is 2 mm. What will be the wire diameters and the load shared by each spring if the allowable shear stress is 180 MPa?

Solution

Given A compound helical spring

$\delta = 30$ mm	$n_i d_i = n_o d_o = 40$ mm
$W = 5$ kN	D_i = 120 mm
$c = 2$ mm	$\tau = 180$ MPa
G = 82 GPa	

To find

— Wire diameters

— Load shared by each spring

Determination of wire diameters

- When the springs are fully compressed, both the springs attain the maximum value of stress.

Then $$\tau = \frac{8W_iD_i}{\pi d_i^3} = \frac{8W_oD_o}{\pi d_o^3} = 180 \tag{i}$$

- Deflection is same in both springs, $$\delta = \frac{8W_iD_i^3 n_i}{Gd_i^4} = \frac{8W_oD_o^3 n_o}{Gd_o^4} = 30 \tag{ii}$$

Dividing (i) by (ii), $$\frac{Gd_i}{n_i \pi D_i^2} = \frac{180}{30} = 6$$

or $$\frac{82\,000 d_i}{(40/d_i)\pi D_i^2} = 6 \quad \text{..... (solid length is same, } n_i d_i = n_o d_o = 40)$$

or $$\left(\frac{D_i}{d_i}\right)^2 = 108.76 \quad \text{or} \quad \frac{D_i}{d_i} = 10.43 \tag{iii}$$

or $$d_i = \frac{120}{10.43} = 11.5 \text{ mm}$$

In a similar way, it can be shown that $\dfrac{D_o}{d_o} = 10.43$ (iv)

$$D_o = D_i + d_i + 2c + d_o$$

or $10.43 d_o = 120 + 11.5 + 2 \times 2 + d_o$ ...(Refer Fig. 11.6)

or $d_o = 14.37$ mm

and $D_o = 150$ mm

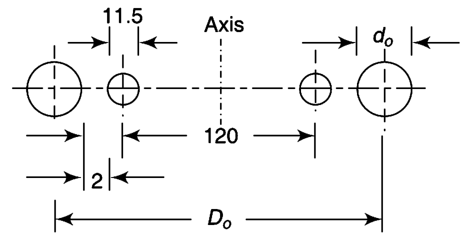


Fig. 11.6

Load sharing

From (iii) and (iv), $\dfrac{D_o}{d_o} = \dfrac{D_i}{d_i}$ or $\dfrac{D_o}{D_i} = \dfrac{d_o}{d_i}$

From (i), $\dfrac{W_i}{W_o} = \dfrac{D_o}{D_o} \cdot \dfrac{d_i^3}{d_o^3} = \dfrac{d_i^2}{d_o^2} = \left(\dfrac{11.5}{14.37}\right)^2 = 0.64$

Also, $W_i + W_o = 5$ or $0.65 W_o + W_o = 5$

or $W_o = 3.05$ kN and $W_i = 1.95$ kN

Example 11.15 || A compound spring consists of two co-axial close-coiled springs with the same initial length. The outer spring has 20 turns of 15-mm diameter wire coiled to a mean diameter of 140 mm and the inner spring has 28 turns with a mean diameter of 100 mm. Determine the diameter of the wire for the inner spring and the stiffness of the compound spring if the working stress in each spring is the same. G = 84 GPa.

Solution

Given A compound helical spring

$n_o = 20$	$d_o = 15$ mm
$D_o = 140$ mm	$n_i = 28$
$D_i = 100$ mm	$G = 84$ GPa

To find

— Wire diameter of inner spring
— Stiffness of compound spring

Determination of inner diameter

- The working stress in each spring is the same,

$$\therefore \quad \frac{8W_i D_i}{\pi d_i^3} = \frac{8W_o D_o}{\pi d_o^3} \tag{i}$$

- As the initial length of both the springs is the same, deflection is to be the same in both springs,

$$\frac{8W_i D_i^3 n_i}{G d_i^4} = \frac{8W_o D_o^3 n_o}{G d_o^4} \tag{ii}$$

Dividing (i) by (ii), $\dfrac{d_i}{n_i D_i^2} = \dfrac{d_o}{n_o D_o^2}$ or $n_i D_i^2 d_o = n_o D_o^2 d_i$

or $28 \times 100^2 \times 15 = 20 \times 140^2 \times d_i$ or $d_i = 10.7$ mm

Determination of stiffness

Stiffness of inner spring, $\dfrac{W_i}{\delta_i} = \dfrac{W_i}{8W_iD_i^3n_i/Gd_i^4}$

$$= \frac{Gd_i^4}{8D_i^3n_i} = \frac{84\,000 \times 10.7^4}{8 \times 100^3 \times 28} = 4.92 \text{ N/mm}$$

Stiffness of outer spring $= \dfrac{W_o}{\delta_o} = \dfrac{Gd_o^4}{8D_o^3n_o} = \dfrac{84\,000 \times 15^4}{8 \times 140^3 \times 20} = 9.69$ N/mm

As the springs are in parallel, the combined stiffness = 4.92 + 9.69 = 14.61 N/mm

Example 11.16 Two close-coiled concentric helical springs of equal length are made out of the same wire of 12-mm diameter and support a compressive load of 1.2 kN. The outer spring consists of 20 turns of 240-mm mean diameter and the inner spring has 24 turns of 200-mm mean diameter. Determine the maximum stress produced in each spring.

Solution

Given A compound helical spring

$d_i = d_o = 12$ mm $\qquad W = 1.2$ kN

$n_o = 20 \qquad D_o = 240$ mm

$n_i = 24 \qquad D_i = 200$ mm

To find Maximum shear stress in each spring

Determination of load distribution

Deflection is to be the same in both springs,

$\therefore \qquad \delta = \dfrac{8W_iD_i^3n_i}{G_id_i^4} = \dfrac{8W_oD_o^3n_o}{G_od_o^4}$

or $\qquad W_iD_i^3n_i = W_oD_o^3n_o \qquad$ (For same material, $G_i = G_o$ and $d_i = d_o = 12$ mm)

$W_i \times 200^3 \times 24 = W_o \times 240^3 \times 20$

or $\qquad W_i = 1.44W_o$

But $\qquad W_i + W_o = 1200 \qquad$ or $\qquad 1.4\,W_o + W_o = 1200$

or $\qquad W_o = 491.8$ N

and $\qquad W_i = 1200 - 491.8 = 708.2$ N

Maximum shear stresses

Maximum shear stress in outer spring $= \dfrac{8W_oD_o}{\pi d_o^3} = \dfrac{8 \times 491.8 \times 240}{\pi \times 12^3} = 174$ MPa

Maximum shear stress in inner spring $= \dfrac{8W_iD_i}{\pi d_i^3} = \dfrac{8 \times 708.8 \times 200}{\pi \times 12^3} = 209$ MPa

Example 11.17 The inner spring of a compound helical spring is concentric with the outer one but is 8 mm shorter than that. The outer spring is of 30-mm mean diameter, and has 12 coils with a wire diameter of 4 mm. If an axial load of 250 N compresses the outer spring by 20 mm, determine the stiffness of the inner spring.

What will be the wire diameter of the inner spring if it has 10 coils and the radial clearance between the springs is 2 mm? G = 78 GPa.

Solution

Given A compound helical spring

$\delta_o - \delta_i = 8$ mm	$D_o = 30$ mm
$n_o = 12$	$d_o = 4$ mm
$W = 250$ N	$\delta_o = 20$ mm
$n_i = 10$	$c = 2$ mm
$G = 78$ GPa	

To find

— Stiffness of inner spring
— Wire diameter of inner spring

Stiffness of inner spring

For outer spring, $$\delta_o = \frac{8WD_o^3 n}{Gd_o^4}$$

or $$20 = \frac{8 \times W \times 30^3 \times 12}{78\,000 \times 4^4} \quad \text{or} \quad W = 154.1 \text{ N}$$

Load carried by the inner spring $= 250 - 154.1 = 95.9$ N
Compression of the outer spring $= 20$ mm
$\therefore$ compression of the inner spring $= 20 - 8 = 12$ mm

Thus stiffness of the inner spring, $\dfrac{95.9}{12} = 8$ N/mm

Wire diameter of inner spring

$$D_o = D_i + d_i + 2c + d_o \quad \text{or} \quad 30 = D_i + d_i + 2 \times 2 + 4 \qquad \text{...(Refer Fig. 11.7)}$$

or $$D_i = 22 - d_i$$

For inner spring,

$$\delta = \frac{8WD_i^3 n}{Gd_i^4} \quad \text{or} \quad 12 = \frac{8 \times 95.9 \times (22 - d_i)^3 \times 10}{78\,000 \times d_i^4}$$

or $$122d_i^4 = (22 - d_i)^3 \quad \text{or} \quad d_i = [(22 - d_i)^3/122]^{0.25}$$

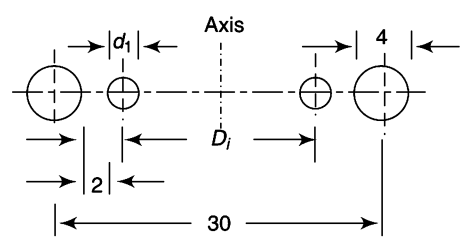


Fig. 11.7

The equation can be solved by trial and error or use the following technique:
As d_i is small compared to 22,
For first approximation, take $d_i = 0$, $d_i = [(22 - 0)^3/122]^{0.25}$ or $d_i = 3.06$
For second approximation, take $d_i = 3.06$, $d_i = [(22 - 3.06)^3/122]^{0.25}$ or $d_i = 2.73$
For third approximation, take $d_i = 2.73$, $d_i = [(22 - 2.73)^3/122]^{0.25}$ or $d_i = 2.76$
For final approximation, take $d_i = 3.76$, $d_i = [(22 - 2.76)^3/122]^{0.25}$ or $d_i = 2.763$

Thus $d_i = 2.763$ mm

11.4 OPEN-COILED HELICAL SPRINGS

Let α be the angle of helix and let the following axes be in a vertical plane tangential to the helix at O (Fig. 11.6).

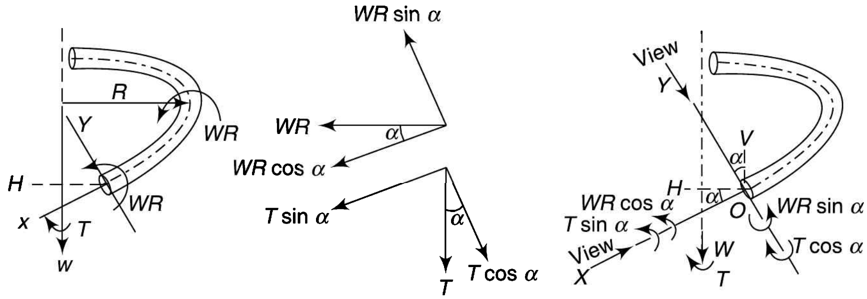


Fig. 11.8

OH = Horizontal axis
OV = Vertical axis
OX= Polar axis or axis of twist at any normal cross-section inclined at angle α to OH
OY = Bending axis about which bending takes place inclined at angle α to OV

l = Length of wire = $\dfrac{2\pi Rn}{\cos\alpha} = \dfrac{\pi Dn}{\cos\alpha}$

Apply an axial load W and an axial torque T to the spring.

The axial load W produces a couple WR about OH which can be represented by a vector by taking a vector along the axis of the couple and using the right-hand screw rule for the direction, i.e., for a clockwise couple, a vector away from the viewer. The components of the vector WR along X and Y are

- $WR \cos\alpha$ that tends to twist the wire
- $WR \sin\alpha$ that tends to open up the coil or to reduce the curvature

The axial torque T about OV has the following components along X and Y

- $T \sin\alpha$ that tends to twist the wire
- $T \cos\alpha$ that tends to close up the coil or to increase the curvature

Thus combined twisting couple, $T' = WR\cos\alpha - WR\sin\alpha$
and combined bending couple tending to increase the curvature

$$M = T\cos\alpha - WR\sin\alpha$$

Total strain energy, $$U = \frac{(WR\cos\alpha + T\sin\alpha)^2 l}{2GJ} + \frac{(T\cos\alpha - WR\sin\alpha)^2 l}{2EI} \tag{11.7}$$

Axial deflection and axial rotation may be obtained using Castigliano's theorem.

Axial Deflection

The axial deflection, $$\delta = \frac{\partial U}{\partial W}$$

$$= \frac{2l(WR\cos\alpha + T\sin\alpha)R\cos\alpha}{2GJ} + \frac{2l(T\cos\alpha - WR\sin\alpha)(-R\sin\alpha)}{2EI}$$

$$= WR^2 l\left(\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{EI}\right) + TRl\left(\frac{1}{GJ} - \frac{1}{EI}\right)\sin\alpha\cos\alpha$$

(i) For axial load only ($T = 0$),

$$\delta = WR^2 l\left(\frac{\cos^2\alpha}{GJ} + \frac{\sin^2\alpha}{GJ}\right)$$

$$= W \cdot \frac{D^2}{4} \cdot \frac{\pi D n}{\cos\alpha}\left(\frac{\cos^2\alpha}{G(\pi d^4/32)} + \frac{\sin^2\alpha}{E(\pi d^4/64)}\right)$$

$$= W \cdot \frac{D^2}{4} \cdot \frac{\pi D n}{\cos\alpha} \cdot \frac{32}{\pi d^4}\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)$$

$$= \frac{8WD^3 n}{d^4 \cdot \cos\alpha}\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)$$

(ii) For axial torque only ($W = 0$),

$$\delta = TRl\left(\frac{1}{GJ} - \frac{1}{EI}\right)\sin\alpha\cos\alpha$$

$$= T \cdot \frac{D}{2} \cdot \frac{\pi D n}{\cos\alpha}\left(\frac{1}{G(\pi d^4/32)} + \frac{1}{E(\pi d^4/64)}\right)\sin\alpha\cos\alpha$$

$$= T \cdot \frac{D}{2} \cdot \frac{\pi D n}{\cos\alpha} \cdot \frac{32}{\pi d^4}\left(\frac{1}{G} - \frac{2}{E}\right)\sin\alpha\cos\alpha$$

$$= \frac{16TD^3 n \sin\alpha}{d^4 \cdot \cos\alpha}\left(\frac{1}{G} - \frac{2}{E}\right)$$

Axial Rotation

The axial rotation, $\varphi = \dfrac{\partial U}{\partial T}$

$$= \frac{2l(WR\cos\alpha + T\sin\alpha)\sin\alpha}{2GJ} + \frac{2l(T\cos\alpha - WR\sin\alpha)\cos\alpha}{2EI}$$

$$= WRl\sin\alpha\cos\alpha\left(\frac{1}{GJ} - \frac{1}{EI}\right) + Tl\left(\frac{\sin^2\alpha}{GJ} - \frac{\cos^2\alpha}{EI}\right)$$

(i) For axial load only ($T = 0$),

$$\varphi = WRl\sin\alpha\cos\alpha\left(\frac{1}{GJ} - \frac{1}{EI}\right)$$

$$= W\frac{D}{2}\frac{\pi D n}{\cos\alpha} \cdot \sin\alpha\cos\alpha\left(\frac{1}{G(\pi d^4/32)} - \frac{1}{E(\pi d^4/64)}\right)$$

$$= W\frac{D}{2} \cdot \frac{\pi D n}{\cos\alpha} \cdot \frac{32}{\pi d^4} \cdot \sin\alpha\cos\alpha\left(\frac{1}{G} - \frac{2}{E}\right)$$

$$= \frac{16WD^2 n \cdot \sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right) \tag{11.8}$$

(ii) For axial torque only ($W = 0$),

$$\varphi = Tl\left(\frac{\sin^2\alpha}{GJ} - \frac{\cos^2\alpha}{EI}\right)$$

$$= T \cdot \frac{\pi Dn}{\cos\alpha}\left(\frac{\sin^2\alpha}{G(\pi d^4/32)} - \frac{\cos^2\alpha}{E(\pi d^4/64)}\right)$$

$$= T \cdot \frac{\pi Dn}{\cos\alpha} \cdot \frac{32}{\pi d^4}\left(\frac{\sin^2\alpha}{G} - \frac{2\cos^2\alpha}{E}\right)$$

$$= \frac{32TDn}{d^4 \cdot \cos\alpha}\left(\frac{\sin^2\alpha}{G} + \frac{2\cos^2\alpha}{E}\right) \tag{11.9}$$

Stresses

Combined twisting couple, $T' = WR\cos\alpha + T\sin\alpha$

Combined bending couple, $M = \mathrm{T}\cos\alpha - WR\sin\alpha$

$$\tau = \frac{16T'}{\pi d^3} \quad \text{and} \quad \sigma_b = \frac{32M}{\pi d^3}$$

(i) Axial load only, $\quad T' = WR\cos\alpha \quad$ or $\quad \tau = \dfrac{16WR\cos\alpha}{\pi d^3}$

and $\quad M = WR\sin\alpha \quad$ or $\quad \sigma_b = \dfrac{32WR\sin\alpha}{\pi d^3}$

(ii) Axial torque only, $\quad T' = T\sin\alpha \quad$ or $\quad \tau = \dfrac{16T\sin\alpha}{\pi d^3}$

and $\quad M = T\cos\alpha \quad$ or $\quad \sigma_b = \dfrac{32T\cos\alpha}{\pi d^3}$

$$\text{Principal stresses} = \frac{\sigma_b}{2} \pm \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2}$$

$$\text{Maximum shear stress} = \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2} \tag{11.10}$$

Example 11.18 An open-coiled helical spring has 12 turns wound to a mean diameter of 100 mm. The angle of the coils with a plane perpendicular to the axis of the coil is 30°. The wire diameter is 8 mm. Determine

(i) the axial extension with a load of 80 N

(ii) the angle turned by the free end if free to rotate

E = 205 MPa and G = 80 GPa.

Solution

Given A open-coiled helical spring

$D = 100$ mm	$n = 12$
$d = 8$ mm	$\alpha = 30^\circ$
$W = 80$ N	$G = 80$ GPa
$E = 205$ MPa	

To find

— Axial extension

— Angle turned by free end

Axial extension

$$\delta = \frac{8WD^3n}{d^4 \cdot \cos\alpha}\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)$$

$$= \frac{8 \times 80 \times 100^3 \times 12}{8^4 \cdot \cos 30^o}\left(\frac{\cos^2 30°}{80\,000} + \frac{2\sin^2 30°}{205\,000}\right)$$

$$= 2.165 \times 10^6\,(9.375 \times 10^{-6} + 2.439 \times 10^{-6}) = 25.88 \text{ mm}$$

Angle turned

$$\varphi = \frac{16WD^2n\sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right)$$

$$= \frac{16 \times 80 \times 100^2 \times 12 \sin 30°}{8^4}\left(\frac{1}{80\,000} - \frac{2}{205\,000}\right)$$

$$= 18\,750 \times 2.744 \times 10^{-6} = 0.0515 \text{ rad} = \left(0.0515 \times \frac{180}{\pi}\right)^{\circ} = 2.95°$$

Example 11.19 An open-coiled helical spring has 10 coils and is made out of a 12 mm diameter steel rod. The mean diameter of the coils is 80 mm and the helix angle 15°. Find the deflection under an axial load of 250 N. What are the maximum intensities of direct and shear stresses induced in the section of the wire?

If the above axial load is replaced by an axial torque of 60 N · m, determine the axial deflection and the angle of rotation about the axis of the coil. G = 80 MPa and E = 204 GPa.

Solution

Given A open-coiled helical spring

$D = 80$ mm $\quad n = 10$

$d = 12$ mm $\quad \alpha = 15°$

$W = 250$ N

~~~~~~

$T = 60$ N · m $\quad G = 80$ GPa $\quad E = 204$ MPa

To find

— Deflection and direct and shear stresses

— Deflection and angle of rotation

For axial load

$$\text{(i)}\quad \delta = \frac{8WD^3n}{d^4 \cdot \cos\alpha}\left(\frac{\cos^2\alpha}{G} - \frac{2\sin^2\alpha}{E}\right)$$

$$= \frac{8 \times 250 \times 80^3 \times 10}{12^4 \cdot \cos 15°}\left(\frac{\cos^2 15°}{80\,000} + \frac{2\sin^2 15°}{204\,000}\right) = 6.3 \text{ mm}$$

(ii) $T = WR\cos\alpha = 250 \times 40 \times \cos 15° = 9659$ N · mm

$M = WR\sin\alpha = 250 \times 40 \times \sin 15° = 2588$ N · mm
~~~~~~

$$\tau = \frac{16T}{\pi d^3} = \frac{16 \times 9659}{\pi \times 12^3} = 28.47 \text{ MPa}$$

$$\sigma_b = \frac{32M}{\pi d^3} = \frac{32 \times 2588}{\pi \times 12^3} = 15.26 \text{ MPa}$$

$$\text{Principal stresses} = \frac{\sigma_b}{2} \pm \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2} = \frac{15.26}{2} \pm \sqrt{\left(\frac{15.26}{2}\right)^2 + 28.47^2} = 7.63 \pm 29.47$$

$$= 37.1 \text{ N/mm}^2 \text{ and } -21.84 \text{ MPa}$$

$$\text{Maximum shear stress} = \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2} = \sqrt{\left(\frac{15.26}{2}\right)^2 + 28.47^2} = 29.47 \text{ MPa}$$

For axial torque

If axial load is replaced by an axial torque of 60 N · m,

$$\delta = \frac{16TD^2 n \sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right)$$

$$= \frac{16 \times 60\,000 \times 80^2 \times 10 \sin 15°}{12^4}\left(\frac{1}{80\,000} - \frac{2}{204\,000}\right) = 2.07 \text{ mm}$$

$$\varphi = \frac{32TDn}{d^4 \cdot \cos\alpha}\left(\frac{\sin^2\alpha}{G} + \frac{2\cos^2\alpha}{E}\right)$$

$$= \frac{32 \times 60\,000 \times 80 \times 10}{12^4 \cdot \cos 15°}\left(\frac{\sin^2 15°}{80\,000} + \frac{2\cos^2 15°}{204\,000}\right)$$

$$= 0.765 \text{ rad or } \left(0.765 \times \frac{180}{\pi}\right)^{\circ} = 43.9°$$

Example 11.20 In an open-coiled helical spring, the stresses due to twisting and bending are 120 MPa and 90 MPa respectively when the spring is loaded axially. The springs consists of 8 coils and the mean diameter of the coils is 10 times the diameter of the wire. Determine the maximum permissible load and the diameter of wire for a maximum deflection of 30 mm. G = 80 GPa and E = 204 GPa.

Solution

Given A open-coiled helical spring

τ = 120 MPa	σ_b = 90 MPa
n = 8	D = 10d
δ = 30 mm	G = 80 GPa
E = 204 MPa	

To find

— Maximum load

— Wire diameter

$$D = 10d \quad \text{or} \quad R = 5d$$

Bending and shear stress constraints

$$\tau = \frac{16WR\cos\alpha}{\pi d^3}$$

or
$$120 = \frac{16W(5d)\cos\alpha}{\pi d^3} = \frac{80W\cos\alpha}{\pi d^2} \qquad \text{(i)}$$

and
$$\sigma_b = \frac{32WR\sin\alpha}{\pi d^3}$$

or
$$90 = \frac{16W(5d)\sin\alpha}{\pi d^3} = \frac{160W\sin\alpha}{\pi d^2} \qquad \text{(ii)}$$

Dividing (ii) by (i), $\tan\alpha = 0.375$ or $\alpha = 20.56°$

From (i), $120 = \dfrac{80W\cos 20.56°}{\pi d^2}$ or $W = 5.033\ d^2$

Deflection constraints

$$\delta = \frac{8WD^3n}{d^4\cdot\cos\alpha}\left(\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right)$$

$$30 = \frac{8\times 5.033(10d)^3\times 8}{d^4\cdot\cos 20.56°}\left(\frac{\cos^2 20.56°}{80\,000} + \frac{2\sin^2 20.56°}{204\,000}\right)$$

or
$$30 = 344\,024(10.958\times 10^{-6} + 1.203\times 10^{-6})$$

or
$$d = 7.17 \text{ mm}$$

$$W = 5.033\times 7.17^2 = 258.7 \text{ N}$$

Example 11.21 The mean coil diameter of an open-coiled helical spring is 96 mm and the pitch of the coils is 100 mm. The diameter of the wire is 12 mm and the number of coils is 10. Find the axial torque on the spring which will produce a shear stress of 210 MPa. Also determine the angular twist due to this torque. E = 204 GPa and G = 82 GPa.

Solution

Given A open-coiled helical spring

$D = 96$ mm	$p = 100$ mm
$d = 12$ mm	$n = 10$
$\tau = 210$ MPa	$G = 82$ GPa
$E = 204$ MPa	

To find

— Axial torque
— Angular twist

$$\tan\alpha = \frac{p}{2\pi R} = \frac{100}{2\pi\times 48} = 0.3316 \quad \text{or} \quad \alpha = 18.34°$$

Axial torque

Let the axial torque be T. Then

$$\tau = \frac{16T\sin\alpha}{\pi d^3} \quad \text{and} \quad \sigma_b = \frac{32T\cos\alpha}{\pi d^3}$$

$$\text{Maximum shear stress} = \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau^2}$$

$$= \sqrt{\left(\frac{32T\cos\alpha}{2\pi d^3}\right)^2 + \left(\frac{16T\sin\alpha}{\pi d^3}\right)^2}$$

or $$210 = \frac{16T}{\pi \times 12^3} \quad \text{or} \quad T = 71\,251\ \text{N}$$

Angular twist

$$\varphi = \frac{32TDn}{d^4 \cdot \cos\alpha}\left(\frac{\sin^2\alpha}{G} + \frac{2\cos^2\alpha}{E}\right)$$

$$= \frac{32 \times 71\,251 \times 96 \times 10}{12^4 \times \cos 18.34°}\left(\frac{\sin^2 18.34°}{82\,000} + \frac{2\cos^2 18.34°}{204\,000}\right)$$

$$= 111\,206(1.2074 \times 10^{-6} + 8.8333 \times 10^{-6})$$

$$= 1.117\ \text{rad} = \left(1.117 \times \frac{180}{\pi}\right)^{\circ} = 64°$$

Example 11.22 In case the formula for the deflection of a close-coiled helical spring is used instead of that for an open-coiled spring, find the maximum angle of helix for which the error does not exceed by 2%, given that $E = 2G$.

Solution

Given $\dfrac{\delta_o - \delta_c}{\delta_o} = 0.02 \qquad E = 2G$

To find Maximum helix angle

Deflection formulae

For a closed-coiled spring, $\delta_c = \dfrac{8WD^3n}{Gd^4}$

and for an open-coiled spring, $\delta_o = \dfrac{8WD^2n}{d^4 \cdot \cos\alpha}\left(\dfrac{\cos^2\alpha}{G} + \dfrac{2\sin^2\alpha}{E}\right)$

Maximum helix angle

Now, $$\frac{\delta_o - \delta_c}{\delta_o} = 0.02 \quad \text{or} \quad 1 - \frac{\delta_c}{\delta_o} = 0.02$$

or $$1 - \frac{\dfrac{8WD^2n}{Gd^4}}{\dfrac{8WD^2n}{d^4 \cdot \cos\alpha}\left(\dfrac{\cos^2\alpha}{G} + \dfrac{2\sin^2\alpha}{E}\right)} = 0.02$$

$$1 - \frac{E\cos\alpha}{E\cos^2\alpha + 2G\sin^2\alpha} = 0.02$$

or $$1 - \frac{2G\cos\alpha}{2G\cos^2\alpha + 2G\sin^2\alpha} = 0.02$$

or $\cos\alpha = 0.98$ or $\alpha = 11.48°$

11.5 FLAT SPIRAL SPRINGS

A flat spiral type of spring consists of a uniform thin strip wound into a spiral in one plane and pinned at the outer end (Fig. 11.9). It is wound by applying a torque with a spindle which turns the inner end at the centre. The wound spring is released slowly over a period of time. Such type of springs finds their use in clocks.

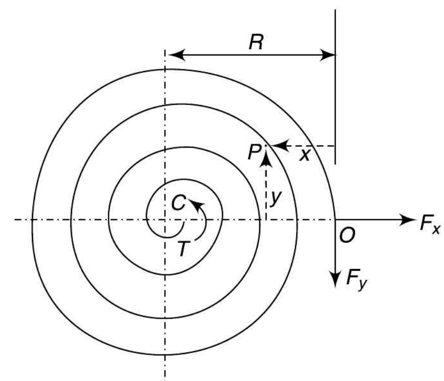


Fig. 11.9

Let T = Torque tending to wind up the spring
R = Maximum radius of the spiral
F_x = Reaction at the outer end in x-direction
F_y = Reaction at the outer end in y-direction

Taking moments about the spindle axis, $T = F_y \cdot R$

Assuming the centre to be at fixed end O,

Bending moment at any point P (x,y) on the spiral,

$$M = F_y \cdot x - F_y \cdot y$$

(tending to increase the radius)

Strain energy of the spring may be found by considering an infinitesimal length ds at P,

$$U = \int \frac{M^2}{2EI} \cdot ds = \frac{1}{2EI}\int (F_y \cdot x - F_x \cdot y)^2 \cdot ds \qquad \text{(i)}$$

As O is the fixed end of the spiral strip, deflection in x-direction is zero. Applying Castigliano's theorem,

$$\frac{\partial U}{\partial F_x} = 0$$

or $$\int \frac{(F_y \cdot x - F_x \cdot y)(-y)}{EI} \cdot ds = 0$$

or $$\int (F_y \cdot x - F_x \cdot y)(-y) ds = 0$$

or $$F_x \int y^2 \cdot ds - F_y \int xy \cdot ds = 0$$

or $$F_x = \frac{F_x \int xy \cdot ds}{\int y^2 \cdot ds} = \frac{F_x \int xy \cdot ds}{\int y^2 . ds}$$

Due to symmetry of the spiral curve about the x-axis, $F_x \int xy \cdot ds = 0$

$\therefore$ $F_x = 0$

Thus $$U = \frac{1}{2EI}\int (F_y \cdot x)^2 \cdot ds = \int \frac{(T \cdot x/R)^2}{2EI} \cdot ds \qquad (\text{as } T = F_y \cdot R)$$

If θ is the rotation of the spiral,

$$\theta = \frac{\partial U}{\partial T} = \frac{T}{EIR^2}\int x^2 \cdot ds$$

The integral $\int x^2 \cdot ds$ can approximately be found by assuming the spiral to be a disc.

Let l, t denote the length and thickness of the spiral. Also let A be the area of the disc.

Area of the infinitesimal length, $dA = ds \cdot t$

Total area of the disc = $l \cdot t$ (approximately).

Now, $\int x^2 \cdot ds$ = Second moment of area about vertical axis through O
= Second moment of area about y-axis through $C + AR^2$

Thus, $\int x^2 \cdot ds = \frac{1}{t}\int x^2 \cdot dA = \frac{1}{t}\left(\frac{\pi}{64}D^4 + AR^2\right) = \frac{1}{t}\left(A \cdot \frac{R^2}{4} + AR^2\right)$

$$= \frac{1.25 l \cdot t \cdot R^2}{t} = 1.25R^2 t$$

Therefore, $$\theta = \frac{1.25R^2 lT}{EIR^2} = \frac{1.25Tl}{EI} \qquad (11.11)$$

Maximum bending moment = $F_y \cdot 2R = 2T$ at the left-hand edge of the spiral

If b is the width and t the thickness of the spiral material,

$$\text{Maximum stress } \sigma_{max} = \frac{2T}{Z} = \frac{2T}{bt^2/6} = \frac{12T}{bt^2} \qquad (11.12)$$

Example 11.23 A 5-mm wide and 0.3-mm thick flat spiral spring is 2 m long. The maximum stress is 600 MPa at the point of greatest bending moment. Determine the torque, the work stored and the number of turns to wind up the spring. E = 208 GPa.

Solution

Given

b = 5 mm $\quad t$ = 0.3 mm
l = 2 m $\quad \sigma_{max}$ = 600 MPa
E = 208 GPa

To find

— Torque
— Work stored
— Number of turns

Torque

Maximum stress, $\sigma_{max} = \frac{12T}{bt^2}$

or $$600 = \frac{12T}{5 \times 0.3^2} \quad \text{or} \quad T = 22.5 \text{ N} \cdot \text{mm}$$

Work stored

$$\theta = \frac{1.25Tl}{EI} = \frac{1.25 \times 22.5 \times 2000}{208\,000 \times (5 \times 0.3^3/12)} = 24 \text{ rad}$$

$$\text{Work stored} = \frac{1}{2}T\theta = \frac{1}{2} \times 22.5 \times 24 = 270 \text{ N} \cdot \text{mm}$$

Number of turns

Number of turns of the spindle = $\frac{24}{2\pi}$ = 3.82 turns

11.6 LEAF OR LAMINATED SPRINGS

Leaf or laminated springs are made up of a number of leaves of varying lengths but of equal width and thickness placed in laminations and loaded as a beam. These are widely used in carriages such as cars and railway wagons. There are two main types of leaf springs: semi-elliptic and quarter-elliptic or cantilever type.

Semi-elliptic Type

Let W = Central load
l = Length of span
b = Width of leaves
t = Thickness of leaves
y = Rise of crown (midpoint) above the level of ends

The leaves are put in the form of laminations one above the other. The contact will be throughout the surface if the radius of the upper leaf is increased by the thickness of the leaf underneath. However, the leaves are bent to the same radius R_o and thus the contact between the leaves takes place at their edges only as shown in Fig. 11.10a. In this way, each leaf is regarded as simply supported at the ends on the leaf below it, the load at the overhanging end being $W/2$ (Fig. 11.10b). Figure 11.10c shows the bending moment diagram for such loading. Over the central portion it is constant but at the ends it is linear.

From Eq. 9.1 for curved bars, $$\frac{M}{EI} = \frac{1}{R'} - \frac{1}{R} \tag{11.13}$$

where R and R' are the radii of curvature before and after bending.

As R is assumed to be constant initially, the radius of curvature R' in the strained condition must remain same for all leaves if it is to be ensured that the contact between them continues to remain at the ends only.

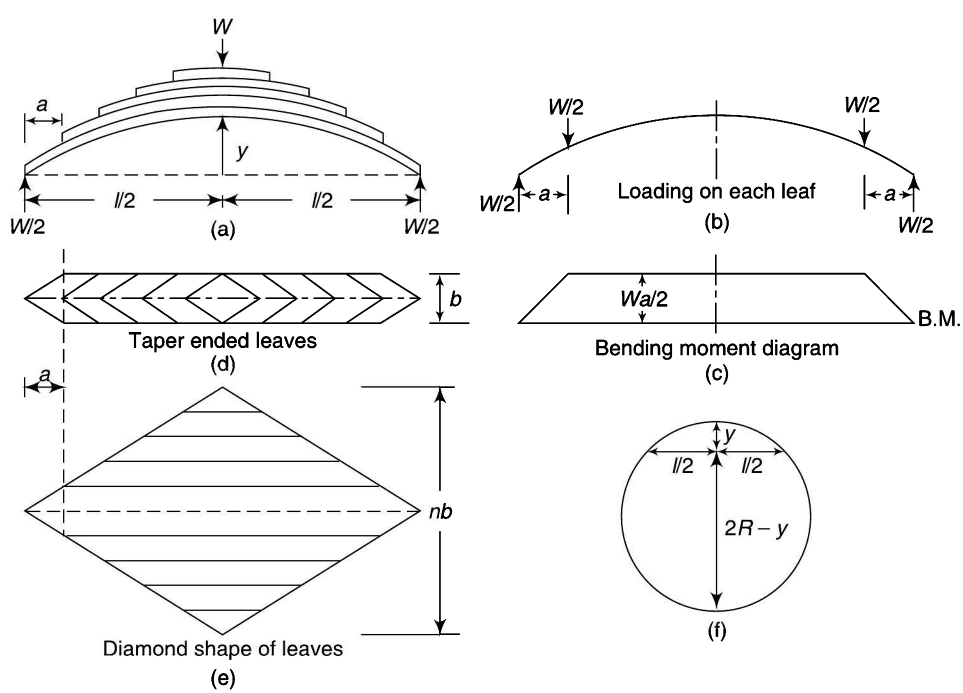


Fig. 11.10

This is possible if M/EI value in the above equation remains constant for all the leaves throughout the section. As M varies linearly at the ends, I must vary accordingly to fulfill the condition. This can be done by making the ends tapered uniformly to a point as shown in Fig. 11.10d. Thus to complete the set, the shortest leaf is made of a diamond shape. However, in practice the main leaf is not tapered to maintain full width where it is supported.

As each leaf is free to slide over the lower leaf and since the radius of curvature is the same for all the leaves, they can be assumed to be made of diamond shape by imagining that all the leaves except the main are cut longitudinally through the centre and placed as shown in Fig. 11.10e.

If n is the number of leaves, the overhang, $a = l/2n$

Width of the equivalent leaf = nb

Now, bending moment as well as the moment of inertia for the equivalent section are proportional to the distance from either end, the spring is equivalent to a beam of uniform strength, i.e., having the same stress throughout. Now, the analysis can be made by considering any section. Making use of the central section,

Bending moment, $M = -\dfrac{Wl}{4}$ tending to decrease the curvature

Moment of inertia, $I = \dfrac{nbt^3}{12}$

From the geometry of a circle (Fig. 11.10f), $y(2R - y) = \dfrac{l}{2} \cdot \dfrac{l}{2} = \dfrac{l^2}{4}$

Treating y to be small compared to R, $\quad 2yR = \dfrac{l^2}{4} \quad$ or $\quad \dfrac{1}{R} = \dfrac{8y}{l^2}$

Similarly, $\dfrac{1}{R'} = \dfrac{8y'}{l^2}$

Thus from Eq. 11.13, $\dfrac{-Wl/4}{E \cdot nbt^3/12} = \dfrac{8}{l^2}(y' - y)$

or deflection $$\delta = y - y' = \frac{3}{8}\frac{Wl^3}{nbt^3E} \tag{11.14}$$

The load which can straighten the spring ($y' = 0$) is called *Proof load.*

$$\text{Proof load, } W = \frac{8nbt^3Ey}{3l^3} \tag{11.15}$$

$$\text{The maximum bending stress, } \sigma = \frac{M}{I} \cdot \frac{t}{2} = \frac{Wl/4}{nbt^3/12} \cdot \frac{t}{2} = \frac{3}{2}\frac{Wl}{nbt^2} \tag{11.16}$$

Quarter-elliptic Type

The equivalent plan section is as shown in Fig. 11.11. The width varies from zero to nb at the fixed end. The treatment is similar as for semi-elliptic type. The results can be obtained directly also from the above results by replacing $2W$ for W and $2l$ for l.

$$\delta = y - y' = \frac{3}{8}\frac{(2W)(2l)^3}{nbt^3E} = \frac{6Wl^3}{nbt^3E} \tag{11.17}$$

and $$\sigma = \frac{3}{2}\frac{(2W)(2l)}{nbt^2} = \frac{6Wl}{nbt^2} \tag{11.18}$$

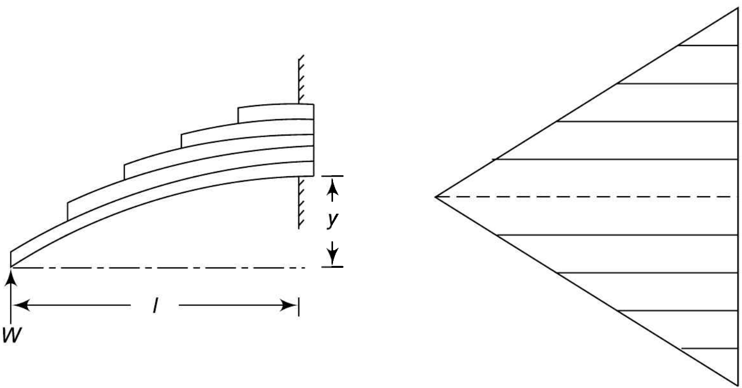


Fig. 11.11

Example 11.24 A leaf spring of semi-elliptical type having 8 plates is 1 m long. The metallic plates have a proof stress in bending of 600 MPa. Each plate is 60 mm wide and 12 mm thick. Find the initial radius of the plates. Also find the height from which a load of 400 N may fall on the centre of the spring if the maximum stress so produced is one half of the proof stress. E = 204 GPa.

Solution

Given Semi-elliptical type leaf spring

$\sigma_p = 600$ MPa	$l = 1$ m
$n = 8$	$b = 60$ mm
$t = 12$ mm	$W = 400$ N
$E = 204$ GPa	

To find

— Initial radius

— Height of drop

Initial radius

$$\frac{\sigma}{y} = \frac{M}{I} = \frac{E}{R} - \frac{E}{R'} \qquad \text{...(Eq. 11.13)}$$

At proof load, R' is infinite and thus $E/R' = 0$

$$\therefore \qquad \frac{\sigma}{y} = \frac{E}{R}$$

or $$\frac{600}{12/2} = \frac{204\,000}{R} \quad \text{or} \quad R = 2040 \text{ mm} \quad \text{or} \quad 2.04 \text{ m}$$

Determination of equivalent static load

Let W_e be the equivalent static load which produces the same maximum stress and deflection as the impact load.

Also, $$\sigma = \frac{3}{2}\frac{W_e l}{nbt^2}$$

As the maximum stress is one half of the proof stress,

$$\frac{600}{2} = \frac{3}{2} \cdot \frac{W_e \times 1000}{8 \times 60 \times 12^2} \quad \text{or} \quad W_e = 13\,824 \text{ N}$$

Height of drop

$$\delta = \frac{3}{8}\frac{W_e l^3}{nbt^3 E} = \frac{3}{8} \times \frac{13\,824 \times 1000^3}{8 \times 60 \times 12^3 \times 204\,000} = 30.64 \text{ mm}$$

Now, $W(h + \delta) = \frac{1}{2} W_e \cdot \delta$

or $\quad 400(h + 30.64) = \frac{1}{2} \times 13\,824 \times 30.64 \quad$ or $\quad h = 498.8$ mm

Example 11.25 The length of the largest plate of a semi-elliptical laminated spring is 800 mm. The central load is 5.5 kN and the central deflection is 20 mm. The allowable bending stress is 200 MPa and the width of the plates is 10 times the thickness. Determine

(i) thickness and width of plates
(ii) number of plates
(iii) radius of curvature
(iv) overlap

Take E = 205 GPa.

Solution

Given Semi-elliptical type leaf spring

l = 800 mm	W = 5.5 kN
δ = 20 mm	b = 10 t
σ = 200 MPa	E = 205 GPa

To find

— Thickness, width and number of plates
— radius of curvature
— overlap

Thickness and width

$$\delta = \frac{3}{8}\frac{Wl^3}{nbt^3 E} = \frac{3}{2}\frac{Wl}{nbt^2} \cdot \frac{l^2}{4tE} = \sigma \cdot \frac{l^2}{4tE}$$

or

$$20 = 200 \times \frac{800^2}{4t \times 205\,000}$$

or $\quad t = 7.8$ mm and $b = 10 \times 7.8 = 78$ mm

Number of plates

$$\sigma = \frac{3}{2}\frac{Wl}{nbt^2}$$

or $\quad 200 = \frac{3}{2} \cdot \frac{5500 \times 800}{n \times 78 \times 7.8^2} \quad$ or $\quad n = 6.95$, say 7

Radius of curvature

Actual bending stress $= 200 \times \frac{6.95}{7} = 198.6$ MPa

$$\frac{\sigma}{y} = \frac{E}{R} \quad \text{or} \quad R = \frac{205\,000 \times (7.8/2)}{198.6} = 4026 \text{ mm} \quad \text{or} \quad 4.026 \text{ m}$$

Overlap

$$a = \frac{l}{2n} = \frac{800}{2 \times 7} = 57.1 \text{ mm}$$

Example 11.26 The span of a simply supported and centrally loaded laminated steel spring is 650 mm. The central deflection of the spring does not exceed 40 mm for a proof load of 6 kN. The bending stress also does not exceed 360 MPa. Find the suitable values of width, thickness and the number of plates if they are available in multiples of 1 mm for thickness and 5 mm for width. Also determine the radius to which the plates should be formed. Assume the width to be ten times the thickness. E =205 GPa.

Solution

Given Laminated steel spring

$l = 650$ mm $\delta = 40$ mm

$W = 6$ kN $\sigma_b = 360$ MPa

$b = 10\,t$ E =205 GPa

Plates available in multiples of 1 mm for thickness and 5 mm for width.

To find

— thickness, width and number of plates
— radius of curvature

Thickness and width

$$\delta = \frac{3}{8}\frac{Wl^3}{nbt^3E} \quad \text{or} \quad 40 = \frac{3}{8}\cdot\frac{6000\times 650^3}{n(10t)t^3\times 205\,000} \quad \text{or} \quad nt^4 = 7535 \qquad \text{(i)}$$

$$\text{Maximum stress, } \sigma = \frac{3}{2}\frac{Wl}{nbt^3} \quad \text{or} \quad 360 = \frac{3}{2}\cdot\frac{6000\times 650}{n(10t)t^3} \quad \text{or} \quad nt^3 = 1625 \qquad \text{(ii)}$$

From (i) and (ii), $t = \dfrac{7535}{1625} = 4.64$ mm

As the plates are available in multiples of 1 mm, let us take t =5 mm

Then width $b = 10 \times 5 = 50$ mm …(50 is multiple of 5)

Number of plates

$$n = \frac{1625}{5^3} = 13$$

Radius of curvature

The actual deflection for a proof load of 6 kN,

$$\delta = \frac{3}{8}\cdot\frac{6000\times 650^3}{13\times 50\times 5^3\times 205\,000} = 37.1 \text{ mm}$$

At proof load, the spring is straight, therefore, the initial radius of curvature is

$$R = \frac{l^2}{8\delta} = \frac{650^2}{8\times 37.1} = 1424 \text{ mm} \quad \text{or} \quad 1.424 \text{ m}$$

Example 11.27 A quarter-elliptic type of laminated spring is 800-mm long. The static deflection of the spring under an end load of 3 kN is 100 mm. Determine the number of leaves required and the maximum stress if the leaf is 75-mm wide and 8-mm thick. Also find the height from which a load can be dropped on to the undeflected spring to induce a maximum stress of 750 MPa. E = 208 GPa.

Solution

Given A quarter-elliptic type of laminated spring

$l = 800$ mm $W = 3$ kN

$\delta = 100$ mm b =75 mm

$t = 8$ mm $\sigma = 750$ MPa

E =208 GPa

To find

— Number of leaves
— maximum stress
— height of drop

Number of leaves

$$\delta = \frac{6Wl^3}{nbt^3E} \quad \text{or} \quad 100 = \frac{6 \times 3000 \times 800^3}{n \times 75 \times 8^3 \times 208\,000}; n = 11.54 \text{ say 12 leaves}$$

Maximum stress

$$\sigma = \frac{6Wl}{nbt^2} = \frac{6 \times 3000 \times 800}{12 \times 75 \times 8^2} = 250 \text{ N/mm}^2$$

Height of drop

Equivalent gradually applied load to induce a maximum stress of 750 MPa = $3000 \times \frac{750}{250} = 9000$ N

The corresponding deflection, $\delta = \frac{6 \times 9000 \times 800^3}{12 \times 75 \times 8^3 \times 208\,000} = 288.5$ mm

Loss of potential energy = Gain of strain energy

$$3000(h + 288.5) = \frac{1}{2} \times 9000 \times 288.5$$

Height from which the load can be dropped, $h = 432.8 - 288.5 = 144.3$ mm

Summary

1. Springs are used to absorb energy and to release the same according to the desired function to be performed or to absorb shocks or to deflect under external forces to provide the desired motion to a machine member.
2. Any member which can deform under a force can act as a spring. However, the term is normally used for those members that deform considerably under the action of forces without exceeding the safe limit of stresses.
3. *Stiffness* of a spring is defined as the force required for unit deflection.
4. Close-coiled helical springs are those in which the angle of helix is so small that the coils may be assumed to be in a horizontal plane if the axis of the spring is vertical.
5. When a close-coiled helical spring is acted upon by an axial load, there is axial extension and when an axial torque is applied, there is a change in the radius of curvature of the spring coils and there is angular rotation of the free end (wind-up).
6. In a close-coiled helical spring, deflection under an axial load, $\delta = \frac{8WD^3n}{Gd^4}$ and rotation about the axis under the action of an axial torque, $\varphi = \frac{64TDn}{Ed^4}$.
7. In an open-coiled helical spring acted upon by an axial load only,

$$\delta = \frac{8WD^2n}{d^4 \cdot \cos\alpha}\left(\frac{\cos^2\alpha}{G} - \frac{2\sin^2\alpha}{E}\right) \quad \text{and} \quad \varphi = \frac{16WD^2n\sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right)$$

Axial torque only,

$$\delta = \frac{32TDn}{d^4 \cdot \cos\alpha}\left(\frac{\sin^2\alpha}{G} - \frac{2\cos^2\alpha}{E}\right) \quad \text{and} \quad \varphi = \frac{16TD^2 n \sin\alpha}{d^4}\left(\frac{1}{G} - \frac{2}{E}\right)$$

8. A flat spiral spring consists of a uniform thin strip wound into a spiral in one plane and pinned at the outer end. It is wound up by applying a torque by turning the inner end at the centre of the spiral with a spindle. The wound spring is released slowly over a period of time.

   Rotation of the spiral spring, $\theta = \dfrac{1.25Tl}{EI}$ and Maximum stress $\sigma_{max} = \dfrac{12T}{bt^2}$

9. A leaf spring is made up of a number of leaves of varying length but of equal width and thickness placed in laminations and loaded as a beam. The leaves are bent to the same radius initially and the contact between the leaves takes place at their edges only.

10. In a semi-elliptic-type leaf spring, $\delta = y - y' = \dfrac{3}{8}\dfrac{Wl^3}{nbt^3E}$ and maximum bending stress, $\sigma = \dfrac{3}{2}\dfrac{Wl}{nbt^2}$

11. In a quarter-elliptic-type leaf spring, $\delta = \dfrac{6Wl^3}{nbt^3E}$ and $\sigma = \dfrac{6Wl}{nbt^2}$.

Objective Type Questions

1. The deflection of a closely coiled helical spring under an axial load is given by,

(a) $\dfrac{WR^3n}{Gr^4}$ (b) $\dfrac{2WR^3n}{Gr^4}$ (c) $\dfrac{4WR^3n}{Gr^4}$ (d) $\dfrac{8WR^3n}{Gr^4}$

2. Shear stress in a closed-coiled helical spring under an axial load is

(a) $\dfrac{8WD}{\pi d^3}$ (b) $\dfrac{4WD}{\pi d^3}$ (c) $\dfrac{8WD}{\pi d^2}$ (d) $\dfrac{16WD}{\pi d^3}$

3. The predominant effect of an axial tensile force on a helical spring is

(a) bending (b) tension (c) compression (d) twisting

4. The equivalent stiffness of two springs joined in series is

(a) $s = \dfrac{s_1 s_2}{s_1 + s_2}$ (b) $s = \dfrac{s_1 / s_2}{s_1 + s_2}$ (c) $s = s_1 + s_2$ (d) $s = s_1 \cdot s_2$

5. The equivalent stiffness of two springs joined in parallel is

(a) $s = \dfrac{s_1 s_2}{s_1 + s_2}$ (b) $s = \dfrac{s_1 / s_2}{s_1 + s_2}$ (c) $s = s_1 + s_2$ (d) $s = s_1 \cdot s_2$

6. The angle of twist of a closely coiled helical spring under an axial torque is

(a) $\dfrac{32TDn}{Ed^4}$ (b) $\dfrac{64TDn}{Ed^4}$ (c) $\dfrac{64Tdn}{ED^4}$ (d) $\dfrac{32Tdn}{ED^4}$

7. Rotation of a flat spiral spring is given by

(a) $\dfrac{Tl}{EI}$ (b) $\dfrac{1.5Tl}{EI}$ (c) $\dfrac{1.75Tl}{EI}$ (d) $\dfrac{1.25Tl}{EI}$

8. Maximum stress is a flat spiral spring is given by

(a) $\dfrac{12T}{bt^3}$ (b) $\dfrac{12T}{b^2t}$ (c) $\dfrac{12T}{bt^2}$ (d) $\dfrac{T}{12bt^2}$

9. Widely used springs in automobile industry are

(a) flat spiral springs (b) leaf springs
(c) closely-coiled helical springs (d) open-coiled helical springs

10. Proof load in a leaf spring is

(a) $\dfrac{8nbt^3Ey}{3l^3}$ (b) $\dfrac{3nbtEy}{8l^3}$ (c) $\dfrac{3nbt^3Ey}{8l}$ (d) $\dfrac{8nbt^2Ey}{3l^2}$

11. The central deflection of a leaf of leaf springs is

(a) $\dfrac{3}{8}\dfrac{Wl^2}{nbt^2E}$ (b) $\dfrac{8}{3}\dfrac{Wl^3}{nbt^3E}$ (c) $\dfrac{8}{3}\dfrac{Wl^2}{nbt^2E}$ (d) $\dfrac{3}{8}\dfrac{Wl^3}{nbt^3E}$

12. Leaf springs are subjected to

(a) tensile stress (b) compressive stress (c) shear stress (d) bending stress

Answers

1. (c)	2. (a)	3. (d)	4. (a)	5. (c)	6. (b)
7. (d)	8. (c)	9. (b)	10. (a)	11. (d)	12. (d)

Review Questions

11.1 What is a spring? What is its use?

11.2 How are the springs classified? Mention the use of each type.

11.3 What do you mean by a close-coiled helical spring? Deduce an expression for its deflection under the action of an axial load.

11.4 Show that a close-coiled helical spring does not rotate under axial load and does not deflect under torque.

11.5 Deduce an expression for the angle of rotation of one end relative to the other of a close-coiled helical spring under the action of an axial torque.

11.6 Deduce expression for the deformation of an open-coiled helical spring under the action of an axial load. Show that for small angle of helix, the expression reduces to that of a close-coiled spring.

11.7 Deduce expressions for the axial deflection and axial rotation of an open-coiled helical spring under the action of a torque.

11.8 What is a flat spiral spring? Deduce an expression for the rotation of the spiral.

11.9 What are leaf springs? What are their types? Develop relations to find the proof load and the maximum bending stress in case of semi-elliptic type leaf springs.

Numerical Problems

11.1 Determine the mean and the coil diameters of a close-coiled helical spring which has its mean diameter 10 times the wire diameter. The spring is to carry a load of 600 N and the shear stress in the material is to be 100 MPa. (124 mm, 12.4 mm)

11.2 A close-coiled spring consists of 12 coils of 12-mm diameter steel wire. The mean diameter of the spring is 140 mm. Find the spring constant. Also find the force required to elongate the spring by 50 mm. $G = 80$ GPa. (6.3 N/mm, 315 N)

11.3 The coil diameter of a close-coiled helical spring having 10 coils is eight times the wire diameter. The spring absorbs 60 N·m of energy when compressed by 40 mm. Find the coil and the wire diameters and the maximum shear stress. $G = 85$ GPa. (289 mm, 36.1 mm; 46.9 MPa)

11.4 A close-coiled helical spring having 12 coils of 16-mm wire diameter is subjected to an axial load of 320 N. The mean diameter of the coils is 240 mm. Determine the axial deflection, strain energy stored, maximum torsional shear stress and the maximum shear stress using Wahl's correction factor. $G =$ 82 GPa. (79 mm, 12.64 N·m, 47.7 MPa, 52.2 MPa)

11.5 A close-coiled helical spring having eight coils has mean diameter of 75 mm and spring constant of 80 kN/m. Find the diameter of the spring wire if the maximum shear stress is not to exceed 260 MPa. Also find the maximum axial load which the spring can carry. $G = 80$ GPa. (12.82 mm; 2.868 N)

11.6 The mean coil diameter of a helical spring is 8 times the wire diameter. It is made to absorb 200 N·m of energy with a deflection of 100 mm. If the maximum shear stress is not to exceed 125 MPa, find mean diameter of the coils, wire diameter and the number of turns. $G = 77$ GPa.

(204 mm, 25.5 mm, 12)

11.7 A close-coiled helical spring is made of 12-mm diameter steel wire. The spring has 12 turns and the mean diameter is 100 mm. Find the increase in the number of turns and the bending stress produced when the spring is subjected to an axial twist of 20 N · m. Also find the torsional stiffness of the spring. $E = 200$ GPa. (0.059, 117.9 MPa; 54 N·m/rad).

11.8 A close-coiled helical spring has a stiffness of 900 N/m in compression with a maximum load of 45 N and a maximum shear stress of 120 MPa. The solid length of the spring is 42 mm. Determine the mean coil diameter, wire diameter and the number of coils. $G = 40$ GPa. (43 mm; 3.45 mm; 12.17)

11.9 Two concentric springs have a solid length of 50 mm. When they are subjected to an axial load of 6 kN, the maximum deflection is found to be 40 mm. The springs are made of the same material. The maximum permissible shear stress is 160 MPa. Determine the load shared by each spring and the wire diameters if the inner spring diameter is 120 mm and the radial clearance is 2.5 mm. $G = 84$ GPa.

(Outer: 3.373 kN, 2.627 kN; Inner: 5.89 mm, 5.2 mm)

11.10 The inner spring of a compound helical spring is arranged within and concentric with the outer one and is shorter by 9 mm. The outer spring has 10 coils with a mean diameter of 24 mm and a wire diameter of 3 mm. Determine the stiffness of the inner spring when an axial load of 150 N causes the outer spring to compress by 16 mm. If the radial clearance between the springs is 1.5 mm and the inner spring has 8 coils, find its wire diameter. $G = 78$ GPa. (6.51 N/mm; 2.15 mm)

11.11 A composite spring consists of two close-coiled springs in series and the stiffness of the composite spring is 1.3 N/mm. The mean diameter of each spring is eight times its wire diameter. One spring is of 2.5 mm diameter and has 20 coils. Determine the wire diameter of the other spring if it has 15 coils. Also find the maximum axial load which can be applied and the corresponding deflection for a maximum shear stress of 240 MPa. $G = 80$ GPa. (2.136 mm, 53.8 N, 41.3 mm)

11.12 An open-coiled helical spring is made from a 6-mm diameter wire and has a mean diameter of 50 mm. If the spring is assumed to be closely coiled, it would induce an error of 2.5 % in axial deflection when loaded with axial load. Find the pitch of the open-coiled spring. Poisson's ratio is 0.3.

(48.6 mm)

11.13 A 2.5-m long flat spiral spring is made up of 2-mm thick and 8-mm wide wire. Assuming the permissible stress to be 540 MPa, find the torque, the work stored and the number of turns to wind up the spring. $E = 210$ GPa. (1.44 N·m, 2.89 N·m, 0.64)

11.14 A 600-mm long semi-elliptic spring is made of 10-mm thick and 42-mm wide steel plates. Find the number of plates required to carry a load of 5 kN if the induced stress is limited to 220 MPa. Also determine the central deflection and the initial radius of curvature under the load if the plates are straightened. $E = 200$ GPa. (5, 9.6 mm, 4.545 m)

11.15 A leaf spring of semi-elliptic type has 10 plates. Each plate is 10 mm thick and 80 mm wide. The length of the spring is 1.4 m. The material of the plates is steel having a proof stress in bending of 630 MPa. Find the initial radius to which the plates should be bent. Also find the height from which a load of 500 N can be dropped so that the maximum stress produced is half of the proof stress. $E = 208$ GPa.

(1.65 m; 816 mm)

Chapter 12

Columns and Struts

A bar or member of a structure in any position acted upon by a compressive load is known as a *strut*. However, when the compressive member is in a vertical position and is liable to fail by bending or buckling, it may be referred as a *column* or *strut*. Otherwise, they are known as *short columns* which fail by compression.

The equilibrium of a column or strut is similar to that of a rigid body, i.e., *stable, unstable* and *neutral*. If a small axial load is applied to a column and the column returns to its original position, it is said to be in stable equilibrium. If the load is increased to a value that on its removal the deflection remains, it is the neutral equilibrium. This load is usually known as the *critical* or *crippling* or *buckling load*. Any load beyond this load further deflects the column and it is in unstable equilibrium.

The resistance of a member to bending is due to its flexure rigidity EI or EAk^2, where A is the area of cross-section and k, the radius of gyration. For a member there can be two moments of inertia and if the least of the two is considered, then the ratio l/k is known as the *slenderness ratio*.

12.1 EULER'S THEORY

Struts and columns which fail by buckling may be analysed by Euler's theory. The analysis is made on the assumptions that

- the column is initially straight
- the cross-section is uniform throughout
- the line of thrust coincides exactly with the axis of the column
- the material is homogeneous and isotropic
- the shortening of column due to axial compression is negligible.

The following cases may be considered:

(i) Both Ends Hinged (Fig. 12.1)

Consider a strut initially straight acted upon by an axial load P through the centroid. The deflected shape of the strut is shown in Fig. 12.1. The axis OX is taken through the centroids of the end sections and the origin is assumed at O. At a section at a distance x from O, let y be the deflection from the central line.

From equation of bending of beams, $EI\,\dfrac{d^2y}{dx^2} = M = -\,Py$

To write the above equation with the proper sign is important. As the left-hand side indicates the bending moment from a positive value of y, the Y-axis is to be taken in such a way that the deflection is positive, i.e., towards left in this case. Then viewing from the right side so that OX is horizontal and OY is upwards and treating it as a beam, consider the bending moment at a point where the deflection is assumed to be y. As P provides counter-clockwise moment on the left portion, it is negative. Had we taken deflection of the strut towards right, Y-axis would have been towards right and then viewing from left end would have given the same result.

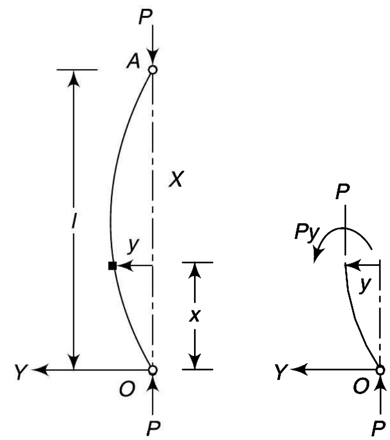


Fig.12.1

The equation can be written as $\dfrac{d^2y}{dx^2} + \alpha^2 y = 0$ where $\alpha^2 = \dfrac{P}{EI}$

The solution is $y = A \sin \alpha x + B \cos \alpha x$

At $x = 0, y = 0, \therefore B = 0$ and at $x = l, y = 0$ and thus $A \sin \alpha l = 0$

If $A = 0$, y is zero for all values of load and there is no bending.

$\therefore \quad \sin \alpha l = 0 \quad$ or $\quad \alpha l = \pi \qquad$ (considering the least value)

or $\quad \alpha = \pi / l$

$$\therefore \quad \text{Euler crippling load, } P_e = \alpha^2 EI = \frac{\pi^2 EI}{l^2} \tag{12.1}$$

The higher solutions of $\sin \alpha l = 0$ leads to higher harmonics of the deflected column and practically are not important.

(ii) One End Fixed, Other Free (Fig. 12.2)

Take Y-axis towards right for positive value of y. Viewing from the left end, P provides a clockwise bending moment $P(a-y)$ on the left portion and is thus positive.

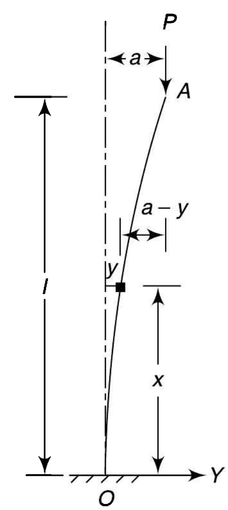


Fig. 12.2

Then $\qquad EI\dfrac{d^2y}{dx^2} = M = P(a-y) = Pa - Py$

or $\qquad \dfrac{d^2y}{dx^2} + \alpha^2 y = \dfrac{P \cdot a}{EI} \quad$ where $\quad \alpha^2 = \dfrac{P}{EI}$

The solution is $y = A \sin \alpha x + B \cos \alpha x + \dfrac{P.a}{EI\alpha^2} = A \sin \alpha x + B \cos \alpha x + a$

At $\qquad x = 0, y = 0, \therefore B = -a;$

At $\qquad x = 0, \dfrac{dy}{dx} = 0 \quad$ or $\quad A\alpha \cos \alpha x - B\alpha \sin \alpha x = 0 \quad$ or $\quad A = 0$

$\therefore \qquad y = -\alpha \cos \alpha x + a = a(1 - \cos \alpha x)$

At $\qquad x = l, y = a, \therefore a = a(1 - \cos \alpha l)$

or $\qquad \cos \alpha l = 0 \quad$ or $\quad \alpha l = \dfrac{\pi}{2}$ (considering the least value)

or $\qquad \alpha = \pi / 2l$

$$\therefore \text{Euler crippling load, } P_e = \alpha^2 EI = \frac{\pi^2 EI}{4l^2} \tag{12.2}$$

(iii) Fixed at Both Ends (Fig. 12.3)

Let M be the end fixing moments, then the governing equation becomes,

$$EI\frac{d^2y}{dx^2} = -Py + M \quad \text{or} \quad \frac{d^2y}{dx^2} + \alpha^2 y = \frac{M}{EI} \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

The solution is $y = A\sin\alpha x + B\cos\alpha x + \dfrac{M}{EI\alpha^2} = A\sin\alpha x + B\cos\alpha x + \dfrac{M}{P}$

At $\quad x = 0, y = 0, \therefore\ B = -\dfrac{M}{P}$;

At $\quad x = 0, \dfrac{dy}{dx} = 0 \quad$ or $\quad A\alpha\cos\alpha x - B\alpha\sin\alpha x = 0 \quad$ or $\quad A = 0$

$\therefore \quad y = -\dfrac{M}{P}\cos\alpha x + \dfrac{M}{P} = \dfrac{M}{P}(1 - \cos\alpha x)$

At $\quad x = l, y = 0, \therefore\ 0 = \dfrac{M}{P}(1 - \cos\alpha l)$ or $\quad \cos\alpha l = 1$

or $\quad \alpha l = 2\pi \quad$ (considering the least value)

or $\quad \alpha = 2\pi/l$

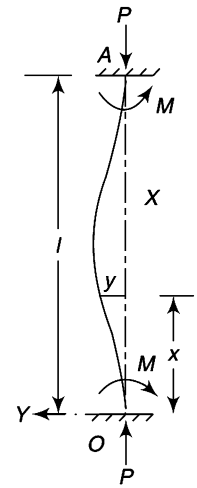


Fig. 12.3

$\therefore$ Euler crippling load, $P_e = \alpha^2 EI = \dfrac{4\pi^2 EI}{l^2}$ (12.3)

(iv) One end Fixed, Other Hinged (Fig. 12.4)

Let M be the fixing moment at end O. For equilibrium of strut, a horizontal force R will act at the free end.

Then $\quad EI\dfrac{d^2y}{dx^2} = -Py + R(l-x) \quad$ or $\quad \dfrac{d^2y}{dx^2} + \alpha^2 y = \dfrac{R(l-x)}{EI} \quad$ where $\quad \alpha^2 = \dfrac{P}{EI}$

The solution is $y = A\sin\alpha x + B\cos\alpha x + \dfrac{R(l-x)}{EI\alpha^2} = A\sin\alpha x + B\cos\alpha x + \dfrac{R}{P}(l-x)$

At $x = 0, y = 0, \therefore\ B = -\dfrac{Rl}{P}$;

At $\quad x = 0, \dfrac{dy}{dx} = 0 \quad$ or $\quad A\alpha\cos\alpha x - B\alpha\sin\alpha x - \dfrac{R}{P} = 0 \quad$ or $\quad A = \dfrac{R}{P\alpha}$

$\therefore \quad y = \dfrac{R}{P\alpha}\sin\alpha x - \dfrac{Rl}{P}\cos\alpha x + \dfrac{R}{P}(l-x)$

At $\quad x = l, y = 0, \therefore\ 0 = \dfrac{R}{P\alpha}\sin\alpha l - \dfrac{Rl}{P}\cos\alpha l$ or $\quad \tan\alpha l = \alpha l$

or $\quad \alpha l = 4.49$ rad $\quad$ (considering the least value)

or $\quad \alpha = 4.49 / l$

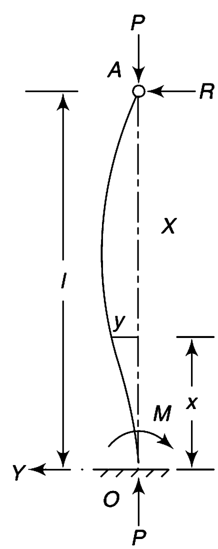


Fig. 12.4

$\therefore$ Euler crippling load, $P_e = \alpha^2 EI = \dfrac{4.49^2 EI}{l^2} = \dfrac{20.2EI}{l^2} \approx \dfrac{2\pi^2 EI}{l^2}$ (12.4)

12.2 EQUIVALENT LENGTH

In all the cases discussed in the previous section, Euler's load can be expressed as

$$P_e = \frac{\pi^2 EI}{l_e^2} \tag{12.5}$$

where l_e^2 is referred as *equivalent length* of the column which takes into account the type of fixing of the ends. The equivalent lengths for different types of end conditions are

(i) both ends hinged, $l_e = l$
(ii) one end fixed and the other free, $l_e = 2l$
(iii) both ends fixed, $l_e = l / 2$
(iv) one end fixed, other hinged, $l_e = l/\sqrt{2}$

The equivalent lengths for different types of end conditions are shown in Fig. 12.5.

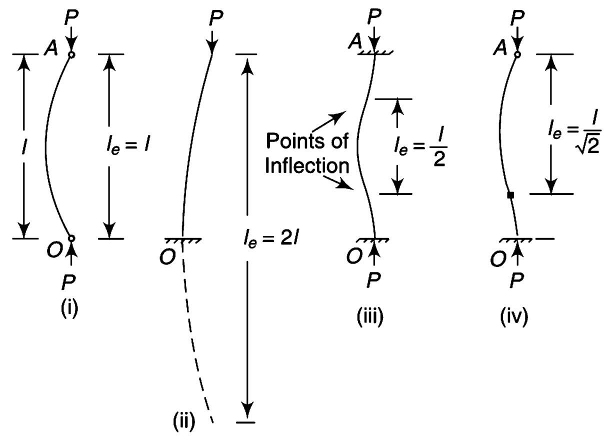


Fig. 12.5

12.3 LIMITATIONS OF EULER'S FORMULA

Euler's formula is derived on the assumptions that the struts are initially perfectly straight and the load is exactly axial. However, in practice these assumptions are never realised. There is always some eccentricity and initial curvature present. Also, it is to be noted that no strength property of the material of the strut exists in Euler's formulae. The only property involved is E which represents the stiffness characteristics of the material.

Thus due to imperfections, in practice a strut suffers a deflection before the crippling load which increases with the load. As a result, a bending moment acts that causes the failure before the Euler load is reached and the failure is by stress rather than buckling.

The *critical stress* σ_c which is defined as an average stress over the cross–section for a standard case is

$$\sigma_c = \frac{P_e}{A} = \frac{\pi^2 EI}{Al_e^2} = \frac{\pi^2 EAk^2}{Al_e^2} = \frac{\pi^2 E}{(l_e/k)^2} \tag{12.6}$$

In this expression, k is the radius of gyration and l/k is known as the *slenderness ratio.*

Figure 12.6 presents the graphical representation of the above equation in which the curve ABC is plotted for *critical stress* versus the slenderness ratio and is known as *Euler's curve*. The curve is entirely defined by the magnitude of E. At higher values of l/k or for long columns, the value of critical stress falls rapidly. The condition cannot be improved even by taking a steel of higher strength as the modulus of elasticity E does not vary much with alloy and heat treatment.

If OD represents the yield stress of the material, obviously the Euler formula cannot be applied if l/k is less than OE as below this value the material will become plastic and will not follow the Hooke's law.

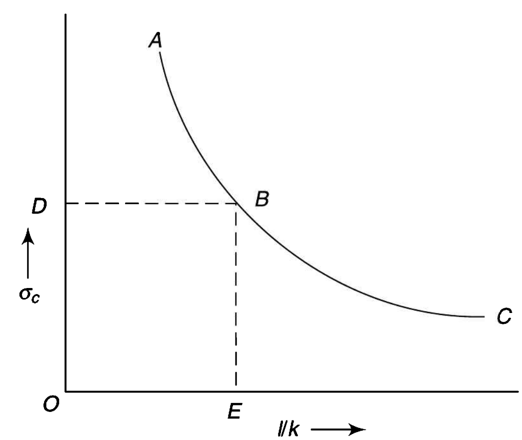


Fig. 12.6

For structural steels, let $\sigma_c = 320$ MPa and $E = 205$ GPa, then OD will correspond to

$$\frac{l}{k} = \pi\sqrt{\frac{E}{\sigma_c}} = \pi\sqrt{\frac{205\,000}{320}} = 79.5 \approx 80$$

Thus the Euler formula is applicable for slenderness ratio greater than 80 for steel hinged at both the ends.

Example 12.1 Using Euler's formula, determine the critical stresses for a strut of slenderness ratio 80, 120 and 160 and 200 under the condition of (a) both ends hinged, and (b) both ends fixed. E = 205 GPa.

Solution

Given A strut with slenderness ratio 80, 120, 160 and 200 and with both ends hinged and both ends fixed. $E = 205$ GPa

To find Critical stresses

Both ends hinged

For both ends hinged, $l_e = l$,

(i) $\sigma_c = \dfrac{\pi^2 E}{(l_e/k)^2} = \dfrac{\pi^2 \times 205\,000}{(80)^2} = \dfrac{2.023 \times 10^6}{80^2} = 316.1\,\text{MPa}$

(ii) $\sigma_c = 2.023 \times 10^6/(120)^2 = 149.5$ MPa

(iii) $\sigma_c = 2.023 \times 10^6/(160)^2 = 79$ MPa

(iv) $\sigma_c = 2.023 \times 10^6/(200)^2 = 50.6$ MPa

Both ends fixed

For both ends fixed, $l_e = l/2$

(i) $\sigma_c = \dfrac{\pi^2 E}{(l_e/k)^2} = \dfrac{\pi^2 \times 205\,000}{(80)^2/4} = \dfrac{8.092 \times 10^6}{80^2} = 1264.4\,\text{MPa}$

(ii) $\sigma_c = 8.092 \times 10^6/(120)^2 = 561.9$ MPa

(iii) $\sigma_c = 8.092 \times 10^6/(160)^2 = 316.1$ MPa

(iv) $\sigma_c = 8.092 \times 10^6/(200)^2 = 202.3$ MPa

Example 12.2 Determine the shortest length for a pin-jointed steel column of cross-section 75 mm × 48 mm using Euler's formula. Take critical stress value as 220 MPa and E = 205 GPa.

Solution

Given A pin-jointed steel column of cross-section 75 mm × 48 mm.

$\sigma_c = 220$ MPa $\qquad E = 205$ GPa

To find Shortest length

Least moment of inertia of the section = $\dfrac{75 \times 48^3}{12} = 691.2 \times 10^3\ \text{mm}^4$

Determination of shortest length

$$\sigma_c = \frac{P_e}{A} = \frac{\pi^2 EI}{Al^2}$$

or $\quad 220 = \dfrac{\pi^2 \times 205\,000 \times 691.2}{75 \times 48 \times l^2} \quad$ or $\quad l = 1329$ mm or 1.329 m

Example 12.3 || A 4-m long circular bar deflects 20 mm at the centre when used as simply supported beam under a 200-N load at the centre. Determine critical load for the same bar when used as a strut which is firmly fixed at one end and pin-jointed at the other.

Solution

Given A circular simply supported beam

$W = 200$ N $\quad\quad l = 4$ m

$\delta = 20$ MPa

To find Critical load if used as a strut with one end fixed and other pin-jointed

Bar as a beam

Deflection due to central load, $\delta = \dfrac{Wl^3}{48EI}$ or $\dfrac{EI}{l^2} = \dfrac{Wl}{48\delta}$

Bar as a strut

$$\text{Euler load, } P_e = \frac{\pi^2 EI}{l_e^2} = \frac{\pi^2 EI}{(l/\sqrt{2})^2} = \frac{2\pi^2 EI}{l^2} = \frac{2\pi^2 Wl}{48\delta}$$

$$= \frac{2\pi^2 \times 200 \times 4000}{48 \times 20} = 16\,450 \text{ N} \quad \text{or} \quad 16.45 \text{ kN}$$

Example 12.4 || A 4-m long hollow alloy tube with inside and outside diameters as 36 mm and 48 mm elongates by 3 mm under a tensile force of 50 kN. Determine the buckling load for the tube when it is used as a column with both ends pinned and with a factor of safety of 5.

Solution

Given A hollow alloy tube

$P = 50$ kN $\quad\quad l = 4$ m

$D = 48$ mm $\quad\quad d = 36$ mm

$\delta = 3$ mm $\quad\quad FOS = 5$

To find Buckling load when used as column with both ends pinned

$$A = \frac{\pi}{4}(48^2 - 36^2) = 791.7 \text{ mm}^2\, ;\, I = \frac{\pi}{64}(48^4 - 36^4) = 178\,128 \text{ mm}^4$$

When used as a bar

$$\Delta = \frac{PL}{AE} \quad \text{or} \quad 3 = \frac{50\,000 \times 4000}{791.7 \times E} \quad \text{or} \quad E = 84\,207 \text{ MPa}$$

When used as a column

$$\therefore \text{ Buckling load, } P_e = \frac{\pi^2 EI}{l_e^2} = \frac{\pi^2 \times 842\,07 \times 178\,128}{4000^2} = 9252.5 \text{ N}$$

$$\text{Safe load} = \frac{9252.5}{5} = 1850.5 \text{ N}$$

Example 12.5 || An 800-mm long straight bar of alloy steel and of 10 mm × 4 mm section is mounted on a strut testing machine and loaded axially. The load is increased till the bar buckles. Determine the maximum central deflection before the material attains the yield point of 300 MPa. Assume the Euler formula for pinned ends. $E = 75$ GPa.

Solution

Given An alloy steel bar

$l = 800$ mm $\qquad b = 10$ mm

$d = 4$ mm $\qquad \sigma = 300$ MPa

$E = 75$ GPa

To find Maximum central deflection

Least moment of inertia of the section $= \dfrac{10 \times 4^3}{12} = 53.3\ \text{mm}^4$

Euler load

There is no deflection until the Euler load is reached. Thus

$$P_e = \frac{\pi^2 EI}{l_e^2} = \frac{\pi^2 \times 75\,000 \times 53.3}{800^2} = 61.7\ \text{N}$$

Bending stress

Maximum bending moment occurs at the centre, $M_{\text{max}} = P_e \cdot \delta = 61.7\ \delta$ where δ is the central deflection.

Bending stress at the centre $= \dfrac{M}{Z} = \dfrac{61.7\delta}{10 \times 4^2/6}$

Determination of central deflection

Maximum stress = Direct stress + bending stress

$$300 = \frac{61.7}{10 \times 4} + \frac{61.7\delta}{10 \times 4^2/6} = 1.543 + 2.314\delta \quad \text{or} \quad \delta = 129\ \text{mm}$$

Example 12.6 Two steel struts have the same cross-sectional areas. One is a solid one and the other is a hollow with internal diameter three-fourth of the external diameter. Compare the ratio of the strength of the solid steel strut to that of the hollow one.

Solution

Given Two steel struts of same cross-sectional area: one solid, other hollow with inner diameter three-fourth of the external diameter

To find Ratio of strengths

Let suffix s represent the solid strut and h the hollow strut. Also let d be the diameter of the solid strut and D the outer diameter of the hollow strut.

Ratio of diameters

As the areas of two struts are equal, $\dfrac{\pi}{4}d^2 = \dfrac{\pi}{4}\left[D^2 - \left(\dfrac{3D}{4}\right)^2\right]$

or $$d^2 = D^2 - \left(\frac{3D}{4}\right)^2 \quad \text{or} \quad \left(\frac{d}{D}\right)^2 = \frac{7}{16}$$

Ratio of strengths

As the cross-sectional areas of two struts are equal, the ratio of their strengths depends upon critical stress values.

$$\frac{\sigma_s}{\sigma_h} = \frac{\pi^2 E/(l/k_s)^2}{\pi^2 E/(l/k_h)^2} = \left(\frac{k_s}{k_h}\right)^2 = \frac{I_s}{I_h}$$

$$= \frac{\frac{\pi}{64} d^4}{\frac{\pi}{64}\left[D^4 - \left(\frac{3D}{4}\right)^4\right]} = \frac{d^4}{D^4 - \left(\frac{3D}{4}\right)^4}$$

$$= \frac{256 d^4}{174 D^4} = \frac{256}{175} \times \frac{49}{256} = \frac{7}{25}$$

Example 12.7 A simply supported beam of *I*-section as shown in Fig. 12.7 deflects 12 mm when subjected to a uniformly distributed load of 50 kN/m length. Determine the safe load if the beam is used as a column with both ends fixed. Use Euler's formula with a factor of safety 5. E = 205 GPa.

Solution

Given A simply supported beam of *I*-section as shown in Fig. 12.7

$\Delta = 12$ mm $\qquad w = 50$ kN/m = 50 N/mm

$E = 205$ GPa

To find Safe load when beam is used as a column, factor of safety of 5

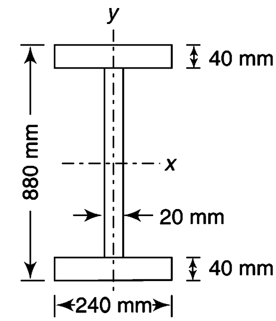


Fig. 12.7

$$I_x = \frac{1}{12}(240 \times 880^3 - 220 \times 880^3) = 4242.8 \times 10^6$$

$$I_y = \frac{1}{12}(800 \times 20^3 + 2 \times 40 \times 240^3) = 92.7 \times 10^6 \text{ mm}^4$$

As a beam

$$\delta = \frac{5}{384}\frac{wl^4}{EI} \quad \text{or} \quad 12 = \frac{5}{384} \times \frac{50 \times l^4}{205\,000 \times 4242.8 \times 10^6} \quad \text{or} \quad l = 11\,252 \text{ mm}$$

As a column

$$P_e = \frac{\pi^2 EI}{l_e^2} = \frac{\pi^2 \times 205\,000 \times 92.7 \times 10^6}{(11\,252/2)^2} = 5925.6 \times 10^3 \text{ N}$$

$$\text{Safe load} = \frac{5925.6}{5} = 1185.1 \text{ kN}$$

Example 12.8 A strut of length l fixed at both ends in an elastic material exerts a restraining moment of magnitude k per unit angular displacement. Show that the critical load P is given by $\tan\frac{\alpha l}{2} + \frac{P}{k\alpha} = 0$ where $\alpha^2 = \frac{P}{EI}$.

If actual length of such a strut is 2.2 m and carries a critical load of 20 kN on the assumption of pinned ends, find the percentage increase in the critical load if the constraint offered at the ends is 200 N.m per degree of rotation.

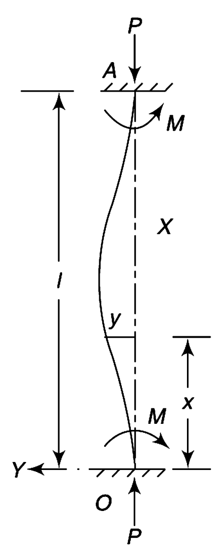


Fig. 12.8

Solution

Given A strut fixed at both ends with a restraining moment as shown in Fig. 12.8.

$W = 20$ kN $\qquad l = 2.2$ m

$k = 200$ N·m per degree of rotation

To find To show that the critical load is $\tan\frac{\alpha l}{2} + \frac{P}{k\alpha} = 0$ where $\alpha^2 = \frac{P}{EI}$.

Expression for load

Let M to be the restraining or fixing moment at each end (Fig. 12.8).

Then $EI\dfrac{d^2y}{dx^2} = -Py + M$ or $\dfrac{d^2y}{dx^2} + \alpha^2 y = \dfrac{M}{EI}$ where $\alpha^2 = \dfrac{P}{EI}$

The solution is $y = A\sin\alpha x + B\cos\alpha x + \dfrac{M}{EI\alpha^2} = A\sin\alpha x + B\cos\alpha x + \dfrac{M}{P}$

At $x = 0, y = 0,\ B = -\dfrac{M}{P}$;

At $x = 0,\ M = k\left(\dfrac{dy}{dx}\right)_0$

or $M = k(A\alpha \cos\alpha(0) - B\alpha \sin\alpha\ (0) = kA\alpha$ or $A = M/k\alpha$

$\therefore$ $y = \dfrac{M}{k\alpha}\sin\alpha x + \dfrac{M}{P}(1 - \cos\alpha x)$

At $x = l/2,\ \dfrac{dy}{dx} = 0$, $\therefore$ $0 = \dfrac{M}{k\alpha}\alpha\cos\dfrac{\alpha l}{2} + \dfrac{M}{P}\sin\dfrac{\alpha l}{2}$

or $\dfrac{M}{k\alpha}\alpha + \dfrac{M}{P}\tan\dfrac{\alpha l}{2} = 0$ or $\tan\dfrac{\alpha l}{2} + \dfrac{P}{k\alpha} = 0$ (i)

Numerical problem

Now, Euler load, $P_e = \dfrac{\pi^2 EI}{l^2}$ or $20\,000 = \dfrac{\pi^2 EI}{2.2^2}$ or $EI = 9808$

$\therefore$ $\alpha = \sqrt{\dfrac{P}{EI}} = \sqrt{\dfrac{P}{9808}}$

$k = 200$ N.m per degree of rotation $= 200 \times \dfrac{180}{\pi} = 11\,459$ N.m per rad of rotation

$\therefore$ from (i), $\tan\left(\dfrac{2.2}{2}\sqrt{\dfrac{P}{9808}}\right) + \dfrac{P}{11\,459}\sqrt{\dfrac{9808}{P}} = 0$

Solving by trial and error, $P = 36\,290$ N

Percentage increase in the critical load $= \dfrac{36\,290 - 20\,000}{20\,000} = 0.815$ or 81.5%

Example 12.9 A uniform bar of steel is heated such that its temperature varies from 0.5t at one end to t at the other. The bar is of uniform cross-sectional area A and flexural stiffness EI. One end is pin-jointed to a rigid foundation and the other is pin-jointed so that the bar can slide along its axis, the thermal expansion of which is resisted by the compression of a spring of stiffness s. Assuming that there is no load on the bar before heating, deduce an expression for the stress in the bar when it is heated. Also show that the bar buckles in flexture when

$t = \dfrac{\pi^2 I}{0.75\alpha l^2 A}\left(1 + \dfrac{AE}{sl}\right)$ where α is the coefficient of linear thermal expansion.

Solution

Given A uniform bar of steel heated under given conditions

To find

— An expression for stress in bar when it is heated

— To show that bar buckles in flexture when $t = \dfrac{\pi^2 I}{0.75\alpha l^2 A}\left(1 + \dfrac{AE}{sl}\right)$

Expression for stress

The average temperature along the bar = $\dfrac{0.5t + t}{2} = 0.75t$

Thermal expansion of the bar = $0.75\alpha lt$

Let P be the force exerted by the spring on the bar on heating of the bar.

Compression of the bar under force $\dfrac{Pl}{AE}$

Net expansion of bar = Compression of spring

or $$0.75\alpha lt - \frac{Pl}{AE} = \frac{P}{s} \quad \text{or} \quad 0.75\alpha lt = P\left(\frac{l}{AE} + \frac{1}{s}\right)$$

Thus $$P = \frac{0.75\alpha lt}{(l/AE + 1/s)}$$

and stress in the bar, $\sigma = \dfrac{0.75\alpha lt}{(l/E + A/s)}$

Buckling of bar

The bar will buckle when the load reaches to Euler's value, i.e.,

$$\frac{\pi^2 EI}{l^2} = \frac{0.75\alpha lt}{(l/AE + 1/s)}$$

or $$\pi^2 EI\left(\frac{l}{AE} + \frac{1}{s}\right) = 0.75\alpha l^3 t$$

or $$t = \frac{\pi^2 EI}{0.75\alpha l^3}\left(\frac{l}{AE} + \frac{1}{s}\right)$$

$$= \frac{\pi^2 I}{0.75\alpha l^2 A}\left(1 + \frac{EA}{sl}\right)$$

Example 12.10 A straight cylindrical bar of 15-mm diameter and 1.2-m length is freely supported at its two ends in a horizontal position. It is loaded with a concentrated load of 100 N at the centre when the centre deflection is observed to be 5 mm. If placed in the vertical position and loaded vertically, what load would cause it to buckle? Also find the ratio of the maximum stress in the two cases.

Solution

Given A cylindrical bar

$W = 100$ N	$l = 1.2$ m
$\delta = 5$ mm	$d = 15$ mm

To find

— Euler's load

— Ratio of maximum stress

Bar as a beam

Deflection due to central load, $\delta = \dfrac{Wl^3}{48EI}$ or $\dfrac{EI}{l^2} = \dfrac{Wl}{48\delta}$

Bar as a column

$\therefore$ Euler load, $P_e = \dfrac{\pi^2 EI}{l^2} = \pi^2 \dfrac{Wl}{48\delta} = \pi^2 \dfrac{100 \times 1200}{48 \times 5} = 4935 \text{ N}$

Maximum stress when used as beam

As a beam, maximum bending moment $= \dfrac{Wl}{4} = \dfrac{100 \times 1200}{4} = 30\,000 \text{ N}\cdot\text{mm}$

and bending stress, $\sigma_b = \dfrac{M}{Z} = \dfrac{30\,000}{\pi \times 15^3/32} = 90.54 \text{ MPa}$

Maximum stress when used as strut

As a strut maximum stress, $\sigma_e = \dfrac{P_e}{A} = \dfrac{4935}{\pi \times 15^2/4} = 27.93 \text{ MPa}$

Ratio of stresses

$$\therefore \qquad \frac{\sigma_b}{\sigma_e} = \frac{90.54}{27.93} = 3.24$$

12.4 RANKINE'S FORMULA

It was discussed in the previous section that Euler's formula is applicable to long columns only in which the *l/k* ratio is larger than a certain value for a particular material. Also it does not take into account the direct compressive stress and therefore, it gives correct results for very long columns only. Thus for columns of medium length, it does not provide accurate results. To take into account this effect, Rankine forwarded an empirical relation for columns which covered all cases from very short to very long columns. The relation is

$$\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e} \tag{12.7}$$

where P = Rankine's crippling load
P_c = ultimate load for a strut = $\sigma_u \cdot A$, constant for a material
P_e = Eulerial load for a strut = $\pi^2 EI/l^2$

- For short columns, P_e is very large and therefore $1/P_e$ is small in comparison to $1/P_c$. Thus the crippling load P is practically equal to P_c
- For long columns, P_e is very small and therefore $1/P_e$ is quite large in comparison to $1/P_c$. Thus the crippling load P is practically equal to P_e

This shows that the value of P obtained from the above relation can cover all cases of columns. The relation is also known as *Rankine–Gorden formula*. The relation can be rearranged as follows:

$$\frac{1}{P} = \frac{P_e + P_c}{P_c P_e}$$

or

$$P = \frac{P_c P_e}{P_e + P_c} = \frac{P_c}{1 + P_c/P_e} = \frac{P_c}{1 + \dfrac{\sigma_c A \cdot l^2}{\pi^2 EI}} = \frac{\sigma_c \cdot A}{1 + \dfrac{\sigma_c \cdot A \cdot l^2}{\pi^2 EAk^2}} = \frac{\sigma_e \cdot A}{1 + a\left(\dfrac{l}{k}\right)^2} \tag{12.8}$$

where σ_c is the crushing stress and a is the Rankine's constant $(\sigma_c/\pi^2 E)$ for the material. Generally the value of constant a is found experimentally. For some of the commonly used materials the values are as under:

No.	Material	σ_c (MPa)	a (for hinged ends)
1.	Cast iron	550	1/1600
2.	Wrought iron	250	1/9000
3.	Mild steel	320	1/7500
4.	Aluminium	120	1/5000
5.	Timber	40	1/2500

A factor of safety may be considered in the value of σ_c in the Rankine formula. The above Rankine formula is for the standard case columns with hinged ends. For columns with other end conditions, the value of the constant a may be modified accordingly or for convenience, the Rankine's formula may be put in the form

$$P = \frac{\sigma_c \cdot A}{1 + a\left(\frac{l_e}{k}\right)^2} \tag{12.9}$$

in which l_e is the equivalent length of the column which has values as defined earlier i.e. both ends hinged, $l_e = 1$; one end fixed and the other free, $l_e = 2l$; both ends fixed, $l_e = l / 2$ and one end fixed, other free, $l_e = l/\sqrt{2}$.

12.5 OTHER FORMULAE

- Johnson's parabolic formula: $P = \sigma_c \cdot A\left[1 - b\left(\frac{l_e}{k}\right)^2\right]$ (12.10)

  For mild steel, the values are, $\sigma_c = 290$ MPa, and $b = 0.000\,03$ for pinned ends.

- Straight-line formula: $P = \sigma_c \cdot A\left[1 - c\frac{l_e}{k}\right]$ (12.11)

  where c is a constant depending upon the material. For mild steel struts, $\sigma_c = 110$ MPa, $c = 1/200$ for pinned ends.

- Gorden's formula: $P = \dfrac{\sigma_c \cdot A}{1 + a'(l/b)^2}$ (12.12)

  in which b is the least diameter or the width of the strut and a' is the Gorden's constant and is given by $\dfrac{\sigma_c \cdot Ab^2}{EI\pi^2}$.

Example 12.11 || A 3.2 m long fixed-end hollow cast-iron column has its internal and external diameters as 60 mm and 80 respectively. Determine Rankine's crippling load using the value of crushing stress to be 500 MPa and the value of the Rankine's constant 1/1600.

Solution

Given A fixed-end hollow cast-iron column

$D = 80$ N $\quad l = 3.2$ m

$d = 60$ mm $\quad \sigma_c = 500$ MPa

$a = 1/1600$

To find Rankine's crippling load

Determination of k^2

$$I = \frac{\pi}{64}(80^4 - 60^4) = 1374.4 \times 10^3 \text{ mm}^4; A = \frac{\pi}{4}(80^2 - 60^2) = 2199.1 \text{ mm}^2;$$

$$k^2 = \frac{I}{A} = \frac{1374.4 \times 10^3}{2199.1} = 625 \text{ mm}^2 \quad \text{or} \quad k^2 = \frac{D^2 + d^2}{16} = \frac{80^2 + 60^2}{16} = 625 \text{ mm}^2$$

Rankine's crippling load

$$P = \frac{\sigma_c \cdot A}{1 + a\left(\frac{l_e}{k}\right)^2} \quad \text{where } l_e = \frac{l}{2} = \frac{3.2}{2} = 1.6 \text{ m for both ends fixed}$$

or
$$P = \frac{500 \times 2199.1}{1 + \frac{1}{1600} \times \frac{1600^2}{625}} = 308.9 \times 10^3 \text{ N} = 308.9 \text{ kN}$$

Example 12.12 A 4-m long cast-iron hollow column with both ends firmly fixed supports an axial load of 400 kN. The inside diameter of the column is 0.6 times the external diameter. Determine the section of the column. Assume factor of safety to be 5, crushing stress 560 MPa and Rankine's constant 1/1600.

Solution

Given A cast-iron hollow column with ends firmly fixed

$W = 400$ kN $\quad l = 4$ m

$d = 0.6\,D$ $\quad \sigma_c = 560$ MPa

$a = 1/1600$ $\quad$ Factor of safety = 5

To find Section of the column

Factor of safety = 5, $\therefore$ Working stress = 560/5 = 112 MPa

Determination of k^2

$$A = \frac{\pi}{4}[D^2 - (0.6D)^2] = 0.16\pi D^2 \text{ mm}^2;$$

$$k^2 = \frac{D^2 + d^2}{16} = \frac{D^2 + (0.6D)^2}{16} = 0.085 \text{ mm}^2$$

Using Rankine's formula

$$P = \frac{\sigma_c \cdot A}{1 + a\left(\frac{l_e}{k}\right)^2} \quad \text{where } l_e = \frac{l}{2} = \frac{3}{2} = 1.5 \text{ m for both ends fixed}$$

or $$400\,000 = \frac{112 \times 0.16\pi D^2}{1 + \dfrac{1}{1600}\left(\dfrac{1500^2}{0.085D^2}\right)} \quad \text{or} \quad 400\,000 = \frac{112 \times 16\pi D^4}{D^2 + 16\,544}$$

or $$D^4 - 7105D^2 - 117.547 \times 10^6 = 0$$

or $$D^2 = 14\,961.6 \quad \text{or} \quad D = 122.3 \text{ mm}$$

$$d = 122.3 \times 0.6 = 73.4 \text{ mm}$$

Example 12.13 An 8-m long steel column consists of two channels of 200 mm × 75 mm size. How far apart should these channels be placed back to back so that the column carries the maximum load? The properties of the channels used in the column are

Area = 2620 mm²

Minimum moment of inertia = 1.47 × 10⁶ mm⁴

Maximum moment of inertia = 17.25 × 10⁶ mm⁴

Distance of centre of gravity from back of channel = 23.5 mm

Also determine the safe load that the column can carry with both of its ends fixed and the channels are spaced 120 mm apart back to back. Rankine's constant is 1/7500 and yield point stress 320 MPa. Also assume a factor of safety of 3.

Solution

Given A column consisting of two channels as shown in Fig. 12.9.
Properties of the channels:

$A = 2620$ mm^2

$I_x = 17.25 \times 10^6$ mm^4

$I_y = 1.47 \times 10^6$ mm^4

Distance of centre of gravity from back of channel = 23.5 mm

$2x = 120$ mm $\qquad a = 1/7500$

$\sigma_c = 320$ MPa $\qquad$ Factor of safety = 3

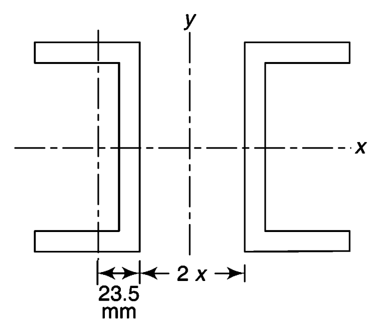


Fig. 12.9

To find

— Spacing to carry maximum load
— safe load of column with both ends fixed

Spacing to carry maximum load

For a column to carry maximum load, its moment of inertia about the x–axis must be equal to moment of inertia about the y-axis.

If the distance between the channels back to back is $2x$,

$$I_x = I_y$$

or $$2 \times 17.25 \times 10^6 = 2[1.47 \times 10^6 + 2620(x + 23.5)^2]$$

or $$(x + 23.5)^2 = 6020.9 \quad \text{or} \quad x = 54.1 \text{ mm}$$

Thus distance between channels = $2x$ = 108.2 mm

Spacing of 120 mm

As the distance between the channels is more than the distance to carry maximum load, I_y will be more than I_x, the safe load will be given by the lesser value which is I_x as above

$$2x = 120 \text{ mm}, \; k^2 = \frac{I}{A} = \frac{2 \times 17.25 \times 10^6}{2 \times 2620} = 65.8 \text{ mm}^2$$

Now $$P = \frac{\sigma_c \cdot A}{1 + a\left(\frac{l_e}{k}\right)^2} \quad \text{where } l_e = \frac{l}{2} = \frac{8}{2} = 4 \text{ m for both ends fixed}$$

or $$= \frac{320 \times 2 \times 2620}{1 + \frac{1}{7500} \cdot \frac{4000^2}{65.8}} = 50\,172 \text{ N} \quad \text{or} \quad 50.172 \text{ kN}$$

$$\text{Safe load} = \frac{50.172}{3} = 16.72 \text{ kN}$$

Example 12.14 A 4-m long fixed-end hollow cast-iron column supports an axial load of 1 MN. The external diameter of the column is 200 mm. Determine the thickness of the column by using Rankine formula taking a constant of 1/6400 and working stress as 78 MN/m^2.

Solution

Given Fixed-end hollow cast-iron column

$P = 1$ MN $\qquad a = 1/6400$

$L = 4$ m $\qquad D = 200$ mm

$\sigma_c = 78$ MN/m^2 = 78 MPa

To find Thickness of column

Determination of k^2

$$k^2 = \frac{I}{A} = \frac{\frac{\pi}{64}(D^4 - d^4)}{\frac{\pi}{4}(D^2 - d^2)} = \frac{D^2 + d^2}{16} = \frac{0.2^2 + d^2}{16}$$

Applying Rankine's formula

$$P = \frac{\sigma_c \cdot A}{1 + a\left(\frac{l_e}{k}\right)^2}$$ where l_e is the equivalent length, the value of which is $l/2$ for columns having both ends fixed. However, the value of Rankine's constant for cast iron is 1/1600 which indicates that the effect of the end conditions is already incorporated in the value of Rankine's constant and thus the equivalent length can be considered as equal to the given length.

Thus,

$$1 \times 10^6 = \frac{78 \times (\pi/4)(200^2 - d^2)}{1 + \frac{1}{6400} \times \frac{4000^2}{(200^2 + d^2)/16}}$$

$$1 \times 10^6 = \frac{61.26(200^2 - d^2)}{1 + 40\,000/(200^2 + d^2)}$$

$$= \frac{61.26(200^2 - d^2)(200^2 + d^2)}{(200^2 + d^2) + 40\,000}$$

or $$10^6\,(200^2 + d^2 + 40\,000) = 61.26(200^4 - d^4)$$

or $$61.26(200^4 - d^4) = 10^6\,(d^2 + 80\,000)$$

$$200^4 - d^4 = 16\,324(d^2 + 80\,000)$$

or $$d^4 + 16\,324d^2 - 294 \times 10^6 = 0$$

or $$d^2 = \frac{-16\,324 + \sqrt{16\,324^2 + 4 \times 294 \times 10^6}}{2} = 10\,828$$

Thus $$d = 104 \text{ mm} \quad \text{and} \quad t = \frac{200 - 104}{2} = 48 \text{ mm}$$

Example 12.15 A tubular strut pin-jointed at both ends has outer and inner diameters as 40 mm and 36 mm respectively and is 2.4 m long. Compare the crippling loads given by Euler's and Rankine's formulae. E = 204 GPa; Yield stress = 310 MPa; a = 1/7500. If the elastic limit stress is taken as 220 MPa, find the length below which the Euler's formula ceases to apply.

Solution

Given A tubular strut pin-jointed at both ends

$D = 40$ mm	$d = 36$ mm
$l = 2.4$ m	$a = 1/7500$
$\sigma_c = 310$ MPa	$E = 204$ GPa

To find

— To compare crippling loads by Euler's and Rankine's formulae

— Minimum length to apply Euler's formula

Determination of k^2

$$I = \frac{\pi}{64}(40^4 - 36^4) = 43\,216 \text{ mm}^4;\; A = \frac{\pi}{4}(40^2 - 36^2) = 238.76 \text{ mm}^2;$$

$$k^2 = \frac{I}{A} = \frac{43\,216}{238.76} = 181 \text{ mm}^2$$

Euler's load

$$P_e = \frac{\pi^2 EI}{l_e^2} = \frac{\pi^2 \times 204\,000 \times 43\,216}{2400^2} = 15\,106 \text{ N}$$

By Rankine's formula

$$P = \frac{\sigma_c \cdot A}{1 + a\left(\dfrac{l}{k}\right)^2} = \frac{310 \times 238.76}{1 + \dfrac{1}{7500} \cdot \dfrac{2400^2}{181}} = 14\,054 \text{ N}$$

$$\frac{P_e}{P} = \frac{15\,106}{14\,117} = 1.07$$

Limiting length for Euler's formula

Euler load, $$P_e = \frac{\pi^2 EI}{l^2}$$

or $$\sigma_c \times A = \frac{\pi^2 EAk^2}{l^2} \quad \text{or} \quad \sigma_c = \frac{\pi^2 Ek^2}{l^2}$$

or $$220 = \frac{\pi^2 \times 204\,000 \times 181}{l^2} \quad \text{or} \quad l = 1287 \text{ mm} \quad \text{or} \quad 1.287 \text{ m}$$

12.6 STRUT WITH ECCENTRIC LOAD (SECANT FORMULA)

Let e be the eccentricity of the applied end load as shown in Fig. 12.10. Deflection y is measured from the line of action of the load.

Then $EI\dfrac{d^2y}{dx^2} = -Py$ or $\dfrac{d^2y}{dx^2} + \alpha^2 y = 0$ where $\alpha^2 = \dfrac{P}{EI}$

The solution is $y = A \sin \alpha x + B \cos \alpha x$

At $x = 0, y = e, \therefore B = e$

At $x = l/2;\ \dfrac{dy}{dx} = 0$ or $A\alpha \cos\dfrac{\alpha l}{2} - B\alpha \sin\dfrac{\alpha l}{2} = 0$ or $A = e\tan\dfrac{\alpha l}{2}$

$$\therefore \quad y = e\left(\tan\frac{\alpha l}{2}\sin \alpha x + \cos \alpha x)\right)$$

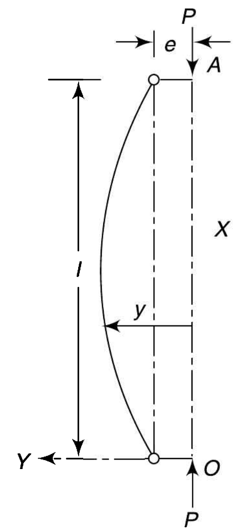


Fig. 12.10

The expression shows that deflection is always there with an eccentric load. It is infinite when $\tan(\alpha l/2) = \infty$ or $\alpha l = \pi$ or $\sqrt{\dfrac{P}{EI}}\cdot l = \pi$ or $P = \dfrac{\pi^2 EI}{l^2}$ which is the same value as for crippling load. However, due to additional bending moment which is set up due to deflection, the strut always fails by compressive stress before the crippling load.

$$y_{\max(x=l/2)} = e\left(\tan\frac{\alpha l}{2}\sin\frac{\alpha l}{2} + \cos\frac{\alpha l}{2}\right) = e\left(\frac{\sin^2\frac{\alpha l}{2} + \cos^2\frac{\alpha l}{2}}{\cos\frac{\alpha l}{2}}\right) = e\cdot\frac{1}{\cos\frac{\alpha\, l}{2}} = e\sec\frac{\alpha l}{2}$$

Maximum bending moment, $M = P\cdot y_{\max} = P\cdot e\sec\dfrac{\alpha l}{2}$

Maximum stress (compressive),

$$\sigma_{\max} = \frac{P}{A} + \frac{M\cdot y_c}{I} = \frac{P}{A} + \frac{P\cdot e\cdot y_c \sec\frac{\alpha l}{2}}{Ak^2} = \frac{P}{A}\left(1 + \frac{e\cdot y_c}{k^2}\cdot\sec\frac{\alpha l}{2}\right) \tag{12.13}$$

This is known as *secant formula* for eccentric loads. In this formula, y_c is the distance of the extreme compressive fibre from the neutral axis.

Perry's Formula

The secant formula is quite convenient to calculate the maximum value of stress if P and e are known. But if it is required to find the load for a given stress and eccentricity, the formula does not provide an easy solution as α also involves P. Perry put the secant formula in a more workable form.

Now, $\sec\dfrac{\alpha l}{2} = \sec\dfrac{l}{2}\sqrt{\dfrac{P}{EI}} = \sec\dfrac{\sqrt{P}}{2}\cdot\dfrac{l}{\sqrt{EI}} = \sec\dfrac{\sqrt{P}}{2}\cdot\dfrac{\pi}{\sqrt{\pi^2 EI/l^2}} = \sec\dfrac{\pi}{2}\sqrt{\dfrac{P}{P_e}}$

Prof. Perry found that $\sec\dfrac{\pi}{2}\sqrt{\dfrac{P}{P_e}} = \dfrac{1.2P_e}{P_e - P}$ and thus $\sec\dfrac{\alpha l}{2} = \dfrac{1.2P_e}{P_e - P}$

Inserting this in the secant formula of Eq. 12.14, $\sigma_{\max} = \dfrac{P}{A}\left(1 + \dfrac{e.y_c}{k^2}\cdot\dfrac{1.2P_e}{P_e - P}\right)$

Let $\sigma_o = \dfrac{P}{A}$ and $\sigma_e = \dfrac{P_e}{A}$;

Then $$\sigma_{\max} = \sigma_o\left(1 + \frac{e \cdot y_c}{k^2} \cdot \frac{1.2\sigma_e}{\sigma_e - \sigma_o}\right)$$

or $$\frac{\sigma_{\max}}{\sigma_o} = 1 + \frac{e \cdot y_c}{k^2} \cdot \frac{1.2}{1 - \sigma_o / \sigma_e}$$

or $$\left(\frac{\sigma_{\max}}{\sigma_o} - 1\right)\left(1 - \frac{\sigma_o}{\sigma_e}\right) = \frac{1.2e \cdot y_c}{k^2} \qquad (12.14)$$

This is Perry's approximate formula from which σ_o and hence P can easily be calculated.

Example 12.16 A column built-in at one end is applied an eccentric load at the free end. Deduce an expression for the maximum length of column such that the deflection of the free end does not exceed the eccentricity of the loading.

Solution

Given A column built-in at one end with eccentric load at free end as shown in Fig. 12.11.
To find Maximum length so that deflection does not exceed eccentricity

Let δ be the deflection of the free end and e the eccentricity of the applied load as shown in Fig. 12.11. Take x-axis through the built-in end.

Equation of bending of beams

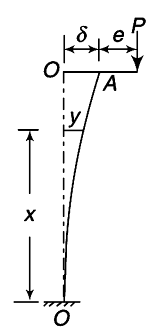


Fig. 12.11

Then $$EI\frac{d^2y}{dx^2} = P(\delta + e - y)$$

or $$\frac{d^2y}{dx^2} = \frac{P}{EI}(\delta + e - y)$$

$$\frac{d^2y}{dx^2} + \alpha^2 y = \alpha^2(\delta + e) \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

The solution is $y = A \sin \alpha x + B \cos \alpha x + \delta + e$

- At $x = 0, y = 0$, $\therefore$ $B = -(\delta + e)$
- At $x = 0$; $\dfrac{dy}{dx} = 0$

  or $A\alpha \cos \alpha x - B\alpha \sin \alpha x = 0$ or $A = 0$

  $\therefore$ $y = -(\delta + e)\alpha \cos \alpha x + \delta + e$
- At $x = l, y = \delta$,

  $\therefore$ $\delta = -(\delta + e) \cos \alpha l + \delta + e$

  or $$\cos \alpha l = \frac{e}{\delta + e}$$

Applying the given condition

According to given problem, the deflection of the free end does not exceed the eccentricity of the loading,

$\therefore$ $\delta = e$

Thus from above, $\cos \alpha l = \dfrac{e}{e + e} = \dfrac{1}{2}$

or $\alpha l = \dfrac{\pi}{3}$ or $l = \dfrac{\pi}{3}\sqrt{\dfrac{EI}{P}}$

Example 12.17 An initially straight steel tube of 48-mm external diameter and 40-mm internal diameter is 2.4 m long and has hinged ends. It carries a compressive load of 25 kN parallel to the axis at an eccentricity of 2 mm. Determine the maximum and the minimum intensities of stresses in the tube. Also find the maximum permissible eccentricity so that no tension exists anywhere in the section. E = 205 GPa.

Solution

Given An initially straight steel tube

$D = 48$ mm	$d = 40$ mm
$l = 2.4$ m	$P = 25$ kN
$e = 2$ mm	$E = 205$ GPa

To find

— Maximum and minimum intensities of stresses
— Maximum eccentricity to have no tension in section

$$I = \frac{\pi}{64}(48^4 - 40^4) = 42\,944\pi \text{ mm}^4;\ A = \frac{\pi}{4}(48^2 - 40^2) = 176\pi \text{ mm}^2$$

$$k^2 = \frac{I}{A} = \frac{42\,944}{176} = 244 \text{ mm}^2$$

Applying secant formula

$$\sec\frac{\alpha l}{2} = \sec\frac{l}{2}\sqrt{\frac{P}{EI}} = \sec\frac{2400}{2}\sqrt{\frac{25\,000}{205\,000 \times 42\,944\pi}} = \sec 1.141 = \sec 65.4° = 2.4$$

$$\sigma_{max} = \frac{P}{A}\left(1 + \frac{e \cdot y_c}{k^2} \cdot \sec\frac{\alpha l}{2}\right) = \frac{25\,000}{176\pi}\left(1 + \frac{2 \times 24}{244} \times 2.4\right) \text{MPa}$$

$$= 45.22(1 + 0.472) = 66.56 \text{ MPa} \qquad \text{(compressive)}$$

$$\sigma_{min} = \frac{P}{A}\left(1 + \frac{e \cdot (-y_c)}{k^2} \cdot \sec\frac{\alpha l}{2}\right) = 45.22(1 - 0.472) = 23.88 \text{ MPa} \qquad \text{(compressive)}$$

Maximum eccentricity

For maximum permissible eccentricity so that no tension exists anywhere in the section

$$\sigma_{min} = \frac{25\,000}{176\pi}\left(1 + \frac{e \times (-24)}{244} \times 2.4\right) = 0 \quad \text{or} \quad 1 - 0.236e = 0 \quad \text{or} \quad e = 4.24 \text{ mm}$$

Example 12.18 A hollow steel strut hinged at both ends has an outside diameter of 64 mm, an inside diameter of 52 mm and is 2.4 m long. The load is parallel to the axis but is eccentric. Determine the maximum value of eccentricity if the crippling load is 70% of Euler value. The yield stress is 300 MPa and E = 205 GPa.

Solution

Given A hollow steel strut hinged at both ends

$D = 64$ mm	$d = 52$ mm
$l = 2.4$ m	$\sigma_{max} = 300$ MPa
$E = 205$ GPa	

To find Maximum eccentricity if crippling load is 70% of Euler load

$$I = \frac{\pi}{64}(64^4 - 52^4) = 147\ 900\pi \text{ mm}^4;\ A = \frac{\pi}{4}(64^4 - 52^4) = 348\pi \text{ mm}^2$$

$$k^2 = \frac{I}{A} = \frac{147\ 900}{348} = 425 \text{ mm}^2$$

Euler load

Euler load, $P_e = \dfrac{\pi^2 EI}{l^2} = \dfrac{\pi^2 \times 205\ 000 \times 147\ 900\pi}{2400^2} = 163\ 211$ N

Actual load for failure = 163 211 × 0.7 = 114 237 N

Applying secant formula

$$\sec\frac{\alpha l}{2} = \sec\frac{l}{2}\sqrt{\frac{P}{EI}} = \sec\frac{2400}{2}\sqrt{\frac{114\ 237}{205\ 000 \times 147\ 900\pi}} = \sec 1.314 = \sec 75.3° = 3.94$$

$$\sigma_{\max} = \frac{P}{A}\left(1 + \frac{e \cdot y_c}{k^2}\cdot\sec\frac{\alpha l}{2}\right) = \frac{114\ 237}{348\pi}\left(1 + \frac{e \times 32}{425} \times 3.94\right) \text{ MPa}$$

or $\quad 300 = 104.49(1 + 0.297e)$

or $\quad 1 + 0.297e = 2.871 \quad$ or $\quad e = 6.3$ mm

Example 12.19 A hollow mild steel column hinged at both ends is of circular cross-section with an outside diameter of 100 mm and an inside diameter of 80 mm. The length of the column is 2.5 m. Determine the maximum permissible load with an eccentricity of 15 mm if the maximum compressive stress is limited to 75 MPa. E = 205 GPa.

Solution

Given A hollow steel column hinged at both ends

D = 100 mm $\qquad d$ = 80 mm

l = 2.5 m $\qquad e$ = 15 mm

$\sigma_{\max}$ = 75 MPa $\qquad E$ = 205 GPa

To find Maximum permissible load

$$I = \frac{\pi}{64}(100^4 - 80^4) = 922\ 500\ \pi \text{ mm}^4;\ A = \frac{\pi}{4}(100^2 - 80^2) = 900\ \pi \text{ mm}^2$$

$$k^2 = \frac{I}{A} = \frac{922\ 500}{900} = 1025 \text{ mm}^2$$

As load is to be found, it is convenient to use Perry's formula.

Euler load

Euler load, $P_e = \dfrac{\pi^2 EI}{l^2} = \dfrac{\pi^2 \times 205\ 000 \times 922\ 500\pi}{2500^2} = 938\ 188$ N

Now, σ_o = 75 MPa (given); $\sigma_e = \dfrac{P_e}{A} = \dfrac{938\ 188}{900\pi} = 331.8$ MPa

Applying Perry's formula

$$\left(\frac{\sigma_{\max}}{\sigma_o} - 1\right)\left(1 - \frac{\sigma_o}{\sigma_e}\right) = \frac{1.2e \cdot y_c}{k^2}$$

or $$\left(\frac{75}{\sigma_o}-1\right)\left(1-\frac{\sigma_o}{331.8}\right)=\frac{1.2\times 15\times 50}{1025}=0.878$$

or $$(75-\sigma_o)(331.8-\sigma_o)=\sigma_o\times 331.8\times 0.878$$

or $$\sigma_o^2-698.1\sigma_o+24\,885=0$$

or $$\sigma_o=\frac{698.1\pm\sqrt{698.1^2-4\times 24\,885}}{2}=37.7\text{ MPa}$$

(considering negative sign as other value is very high and not feasible)

$\therefore$ maximum permissible load = $37.7\times 900\,\pi = 106.6\times 10^3$ N or 106.6 kN

12.7 STRUT WITH INITIAL CURVATURE

Straight columns do not exist in reality. There are always some imperfections in the form of some initial deviations from the column axes. These deviations can be expressed in the form of sinusoidal, parabolic or circular curves. In the following analysis a strut hinged at both ends is considered along with a sinusoidal initial shape of the deflected beam. Thus if c is the maximum deviation of the curve, then it may be assumed that the curve $y'=c\sin\frac{\pi x}{l}$ satisfies the end conditions (Fig. 12.12).

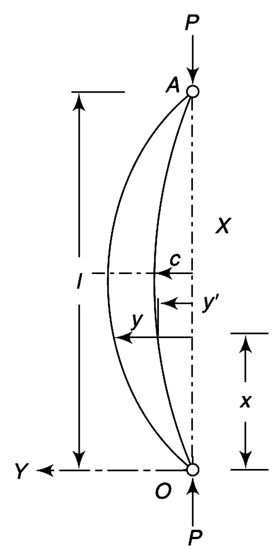


Fig. 12.12

Thus $$\frac{dy'}{dx}=\frac{c\pi}{l}\cos\frac{\pi x}{l}\quad\text{and}\quad\frac{d^2y'}{dx^2}=-\frac{c\pi^2}{l^2}\sin\frac{\pi x}{l}$$

Now, $$EI\frac{d^2(y-y')}{dx^2}=-Py$$

or $$\frac{d^2y}{dx^2}+\alpha^2y=\frac{d^2y'}{dx^2}\quad\text{where}\quad\alpha^2=\frac{P}{EI}$$

or $$\frac{d^2y}{dx^2}+\alpha^2y=-\frac{c\pi^2}{l^2}\sin\frac{\pi x}{l}$$

The solution of this equation will consist of the complimentary function (CF) and the particular integral (PI). CF is the solution of the equation $\frac{d^2y}{dx^2}+\alpha^2y=0$ and is

$$CF=A\sin\alpha x+B\cos\alpha x$$

PI can be obtained by using the D-operator, $(D^2+\alpha)=-\frac{c\pi^2}{l^2}\sin\frac{\pi x}{l}$

$$PI=\frac{-\frac{c\pi^2}{l^2}\sin\frac{\pi x}{l}}{D^2+\alpha^2}=-\frac{\frac{c\pi^2}{l^2}\sin\frac{\pi x}{l}}{-\frac{\pi^2}{l^2}+\alpha^2}=\frac{\frac{c\pi^2}{l^2}}{\frac{\pi^2}{l^2}-\alpha^2}\sin\frac{\pi x}{l}$$

$$y=A\sin\alpha x+B\cos\alpha x+\frac{\frac{c\pi^2}{l^2}}{\frac{\pi^2}{l^2}-\alpha^2}\sin\frac{\pi x}{l}$$

At $\quad x = 0, y = 0, \quad \therefore \quad B = 0;$

At $\quad x = l/2; \ \dfrac{dy}{dx} = 0$

or
$$A\alpha \cos\frac{\alpha l}{2} + \frac{c\pi^2/l^2}{\pi^2/l^2 - \alpha^2} \cdot \frac{\pi}{l}\cos\frac{\pi}{2} = 0 \qquad \text{or} \quad A = 0$$

$$\therefore \quad y = \frac{\dfrac{c\pi^2}{l^2}}{\left(\dfrac{\pi^2}{l^2} - \alpha^2\right)} \sin\frac{\pi x}{l} = \frac{\dfrac{c\pi^2}{l^2}}{\left(\dfrac{\pi^2}{l^2} - \dfrac{P}{EI}\right)} \sin\frac{\pi x}{l} = \frac{\dfrac{cEI\pi^2}{l^2}}{\left(\dfrac{EI\pi^2}{l^2} - P\right)} \sin\frac{\pi x}{l}$$

$$= \frac{cP_e}{P_e - P} \sin\frac{\pi x}{l}$$

This can be written as

$$\frac{P_e - P}{P_e} = \frac{c}{y}\sin\frac{\pi x}{l}$$

and
$$y_{\max(x = l/2)} = \frac{cP_e}{P_e - P} \tag{12.15}$$

Maximum bending moment, $M = P \cdot y_{\max} = P \cdot \dfrac{cP_e}{P_e - P}$

Maximum stress (compressive), $\sigma_{\max} = \dfrac{P}{A} + \dfrac{M_{\max} \cdot y_c}{I}$

$$\sigma_{\max} = \frac{P}{A} + \left(P \cdot \frac{cP_e}{P_e - P}\right)\frac{y_c}{Ak^2} = \frac{P}{A}\left[1 + \frac{cy_c}{k^2}\left(\frac{P_e}{P_e - P}\right)\right] = \sigma_o\left[1 + \frac{cy_c}{k^2}\left(\frac{\sigma_e}{\sigma_e - \sigma_o}\right)\right]$$

$$\frac{\sigma_{\max}}{\sigma_o} - 1 = \frac{cy_c}{k^2}\left(\frac{\sigma_e}{\sigma_e - \sigma_o}\right) \quad \text{or} \quad \left(\frac{\sigma_{\max}}{\sigma_o} - 1\right)\left(1 - \frac{\sigma_o}{\sigma_e}\right) = \frac{c.y_c}{k^2} \tag{12.16}$$

- Note that Eqs. 12.14 and 12.16 for the eccentric loading and for the strut with initial curvature respectively are similar. Thus effect of both can be taken into account easily in a combined way by taking the equivalent of one to the other. Thus if a column is initially bent and is also eccentricity loaded, the combined initial curvature may be taken as $c + 1.2\ e$ or initial eccentricity may be taken as $e + c/1.2$.

Example 12.20 A tubular steel strut with hinged ends is 2 m long and has external and internal diameters as 80 mm and 72 mm respectively. Before loading, the strut is bent with a maximum deviation of 4 mm. Assuming a sinusoidal initial shape of the deflected beam, find the maximum stress due to central compressive end load of 8 kN. *E* = 205 GPa.

Solution

Given A tubular steel strut with hinged ends

$D = 80$ mm	$d = 72$ mm
$l = 2$ m	$c = 4$ mm
$P = 8$ kN	$E = 205$ GPa

To find Maximum stress

$$I = \frac{\pi}{64}(80^4 - 72^4) = 220\,096\pi \text{ mm}^4;\ A = \frac{\pi}{4}(80^4 - 72^4) = 304\pi \text{ mm}^2$$

$$k^2 = \frac{I}{A} = \frac{220\,096}{304} = 724 \text{ mm}^2$$

Euler load

Euler load, $P_e = \dfrac{\pi^2 EI}{l^2} = \dfrac{\pi^2 \times 205\,000 \times 220\,096\pi}{2000^2} = 349\,748$ N

$$\sigma_o = \frac{P}{A} = \frac{8000}{304\pi} = 8.377 \text{ MPa};$$

$$\sigma_e = \frac{P_e}{A} = \frac{349\,748}{304\pi} = 366.2 \text{ MPa}$$

For strut with initial curvature

$$\left(\frac{\sigma_{\max}}{\sigma_o} - 1\right)\left(1 - \frac{\sigma_o}{\sigma_e}\right) = \frac{c \cdot y_c}{k^2}$$

or $$\left(\frac{\sigma_{\max}}{8.377} - 1\right)\left(1 - \frac{8.377}{366.2}\right) = \frac{4 \times 40}{724}$$

or $$\frac{\sigma_{\max}}{8.377} - 1 = 0.2262 \quad \text{or} \quad \sigma_{\max} = 10.27 \text{ MPa}$$

Example 12.21 An initially curved column of section 36 mm × 48 mm is 1.2 m long. It is fixed at both ends and has the equation of curve as $y = 0.006 \sin(\pi x / 2)$ m. A compressive load of 20 kN is applied parallel to the axis at an eccentricity of 20 mm. Find the maximum stress induced. E = 204 GPa.

Solution

Given An initially curved column

b = 48 mm	d = 36 mm
l = 1.2 m	c = 0.006 m = 6 mm
e = 20 mm	P = 20 kN
E = 204 GPa	

To find Maximum stress

Equivalent eccentricity = $e + c/1.2 = 20 + 6/1.2 = 25$ mm

$$I = \frac{48 \times 36^3}{12} = 186\,624 \text{ mm}^4;\ A = 48 \times 36 = 1728 \text{ mm}^2$$

$$k^2 = \frac{I}{A} = \frac{186\,624}{1728} = 108 \text{ mm}^2$$

Euler load

∴ Euler load, $P_e = \dfrac{4\pi^2 EI}{l^2} = \dfrac{4\pi^2 \times 204\,000 \times 186\,624}{1200^2} = 1.044 \times 10^6$ N

$$\sigma_o = \frac{P}{A} = \frac{20\,000}{1728} = 11.57 \text{ N/mm}^2;$$

$$\sigma_e = \frac{P_e}{A} = \frac{1.044 \times 10^6}{1728} = 604 \text{ N/mm}^2$$

Applying Perry's formula

$$\left(\frac{\sigma_{\max}}{\sigma_o} - 1\right)\left(1 - \frac{\sigma_o}{\sigma_e}\right) = \frac{1.2c \cdot y_c}{k^2} \qquad \ldots\text{(Eq. 12.14)}$$

or $$\left(\frac{\sigma_{\max}}{11.57} - 1\right)\left(1 - \frac{11.57}{604}\right) = \frac{1.2 \times 25 \times 18}{108}$$

or $$\frac{\sigma_{\max}}{11.57} - 1 = 5.098 \quad \text{or} \quad \sigma_{\max} = 70.55 \text{ MPa}$$

- Considering equivalent initial curvature, $e = 1.2e + c = 1.2 \times 20 + 6 = 30$ mm and then applying Eq. 12.16, i.e., $\left(\frac{\sigma_{\max}}{\sigma_o} - 1\right)\left(1 - \frac{\sigma_o}{\sigma_e}\right) = \frac{c \cdot y_c}{k^2}$ will give the same result.

Example 12.22 A strut of length l is rigidly fixed at its lower end. The upper end is elastically supported against lateral deflection such that the resisting force is k times the end deflection. Prove that the critical load P is given by $\frac{P}{kl} = 1 - \frac{\tan \alpha l}{\alpha l}$ where $\alpha = P/EI$.

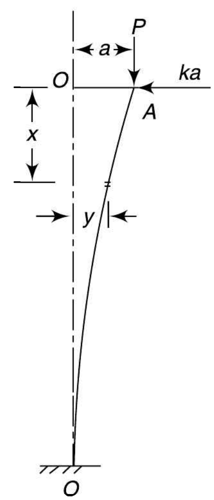


Fig. 12.13

Solution

Given A strut rigidly fixed at its lower end as shown in Fig. 12.13.

To find To show that $\frac{P}{kl} = 1 - \frac{\tan \alpha l}{\alpha l}$

Take the axes as shown in Fig. 12.13

Equation of bending of beams

Now, $EI\frac{d^2y}{dx^2} = P(a - y) - ka \cdot x$ where a is the end deflection.

or $$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{Pa}{EI} - \frac{kax}{EI}$$

or $$\frac{d^2y}{dx^2} + \alpha^2 y = \alpha^2 \cdot a - \frac{ka\alpha^2 x}{P} \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

Solution is $y = A \sin \alpha x + B \cos \alpha x + a - \frac{kax}{P}$

When $x = 0$, $y = a$; $\therefore B = 0$

When $x = l$, y = 0; Thus $0 = A \sin \alpha l + a - \frac{kal}{P}$ (i)

When $x = l$, $dy/dx = 0$; $\therefore A\alpha \cos \alpha l - \frac{ka}{P} = 0$ or $A = \frac{ka}{P\alpha \cos \alpha l}$

And thus from (i), $\frac{ka}{P\alpha \cos \alpha l} \sin \alpha l + a - \frac{kal}{P} = 0$ or $\frac{ka}{P\alpha} \tan \alpha l + a - \frac{kal}{P} = 0$

Multiplying by P/kal, $\frac{P}{kl} = 1 - \frac{\tan \alpha l}{\alpha l}$

12.8 STRUT WITH LATERAL LOADING

The lateral load on a strut may be centric or eccentric point load or it may be uniformly distributed load. The various cases are discussed below:

(i) Concentric Load at the Centre

Consider a pin-jointed strut with an axial compressive load P along with a lateral concentrated load W at the centre as shown in Fig. 12.14.

To write the equation for the bending moment, assume a point at a distance x from A on the dotted elastic curve as y is positive upwards.

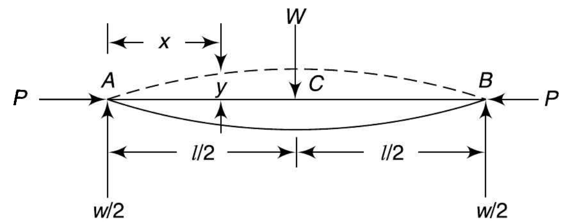


Fig. 12.14

Bending moment is, $M = \dfrac{Wx}{2} + P(-y) = \dfrac{Wx}{2} - Py$

$$\therefore \qquad EI\frac{d^2y}{dx^2} = M = \frac{Wx}{2} - Py \quad \text{or} \quad \frac{d^2y}{dx^2} + \alpha^2 y = \frac{Wx}{2EI} \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

The solution is $y = A\sin\alpha x + B\cos\alpha x + \dfrac{Wx}{2EI\alpha^2} = A\sin\alpha x + B\cos\alpha x + \dfrac{Wx}{2P}$

At $\qquad x = 0, y = 0, \therefore B = 0;$

At $\qquad x = l/2; \dfrac{dy}{dx} = 0 \quad \text{or} \quad A\alpha\cos\dfrac{\alpha l}{2} + \dfrac{W}{2P} = 0 \quad \text{or} \quad A = -\dfrac{W}{2\alpha P\cos(\alpha l/2)}$

$$\therefore \qquad y = -\frac{W}{2\alpha P\cos(\varepsilon l/2)}\sin\alpha x + \frac{Wx}{2P}$$

$$y_{\max(x = l/2)} = -\frac{W}{2\alpha P\cos(\alpha l/2)}\sin\frac{\alpha l}{2} + \frac{Wl}{4P} = -\frac{W}{2\alpha P}\tan\frac{\alpha l}{2} + \frac{Wl}{4P} = -\frac{W}{2P}\left(\frac{1}{\alpha}\tan\frac{\alpha l}{2} - \frac{l}{2}\right) \qquad (12.17)$$

Maximum bending moment,

$$M_{(x = l/2)} = \frac{Wx}{2} - Py_{\max} = \frac{Wl}{4} + P\frac{W}{2P}\left(\frac{1}{\alpha}\tan\frac{\alpha l}{2} - \frac{l}{2}\right) = \frac{W}{2\alpha}\tan\frac{\alpha l}{2} \qquad (12.18)$$

Maximum stress (compressive), $\sigma_{\max} = \dfrac{P}{A} + \dfrac{M \cdot y_c}{I} = \dfrac{P}{A} + \dfrac{W}{2\alpha}\tan\dfrac{\alpha l}{2}\dfrac{y_c}{Ak^2} \qquad (12.19)$

(ii) Eccentric Concentric Load

Consider a pin-jointed strut with an axial compressive load P along with a lateral concentrated load W at a distance a from the end A as shown in Fig. 12.15.

- At a distance x from A, the bending moment is,

$$M = \frac{Wb}{l}x - Py$$

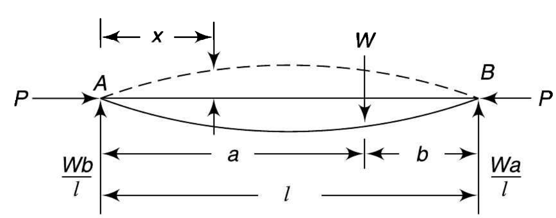


Fig. 12.15

$$EI\frac{d^2y}{dx^2} = M = \frac{Wb}{l}x - Py \quad \text{or} \quad \frac{d^2y}{dx^2} + \alpha^2 y = \frac{Wbx}{EIl} \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

The solution is $y = A\sin\alpha x + B\cos\alpha x + \dfrac{Wbx}{EIl\alpha^2} = A\sin\alpha x + B\cos\alpha x + \dfrac{Wbx}{Pl}$

At $\quad x = 0, y = 0, \quad \therefore \quad B = 0; \quad$ or $\quad y = A\sin\alpha x + \dfrac{Wbx}{Pl}$

At $\quad x = a; y = A\sin\alpha a + \dfrac{Wab}{Pl}$ (i)

- By taking origin at B, in a similar way it can be shown that deflection y' at x from B,

$$y' = A'\sin\alpha x + \frac{Wax}{Pl}$$

At $\quad x = b$ from B, $y' = A'\sin ab + \dfrac{Wab}{Pl}$ (ii)

As deflection at the load point is to be the same whether the origin is taken at A or B,

$\therefore$ from (i) and (ii), $A\sin\alpha a = A'\sin\alpha b \quad$ or $\quad A' = \dfrac{A\sin\alpha a}{\sin\alpha b}$ (iii)

As $\quad \dfrac{dy}{dx} = -\dfrac{dy'}{dx}$

$$\therefore \quad A\alpha\cos\alpha a + \frac{Wb}{Pl} = -A'\alpha\cos\alpha b - \frac{Wa}{Pl}$$

$$A\alpha\cos\alpha a + \frac{A\sin\alpha a}{\sin\alpha b}\alpha\cos\alpha b = -\frac{W(a+b)}{Pl} \quad \text{[from (iii)]}$$

$$A\alpha\left(\frac{\sin\alpha a\cos\alpha b + \cos\alpha b\sin\alpha a}{\sin\alpha b}\right) = -\frac{W}{P}$$

$$A\alpha\sin\alpha(a+b) = -\frac{W}{P}\sin\alpha b$$

$$A = -\frac{W}{\alpha P}\frac{\sin\alpha b}{\sin\alpha l}$$

Thus $\quad y = -\dfrac{W}{\alpha P}\dfrac{\sin\alpha b}{\sin\alpha l}\sin\alpha x + \dfrac{Wbx}{Pl}$

and $\quad M = \dfrac{Wb}{l}x + P\dfrac{W}{\alpha}\dfrac{\sin\alpha b}{\sin\alpha l}\sin\alpha x - \dfrac{Wbx}{l} = P\dfrac{W}{\alpha}\dfrac{\sin\alpha b}{\sin\alpha l}\sin\alpha x$ (12.20)

At $\quad x = a, \quad M = P\dfrac{W}{\alpha}\dfrac{\sin\alpha b}{\sin\alpha l}\sin\alpha a$

For y to be maximum, $\dfrac{dy}{dx} = 0$ or $-\dfrac{W}{\alpha P}\dfrac{\sin\alpha b}{\sin\alpha l}\cdot\alpha\cos\alpha x + \dfrac{Wb}{Pl} = 0$

or $\quad \dfrac{\sin\alpha b}{\sin\alpha l}\cdot\cos\alpha x = \dfrac{b}{l} \quad$ or $\quad \cos\alpha x = \dfrac{b}{l}\cdot\dfrac{\sin\alpha l}{\sin\alpha b} \quad$ or $\quad x = \dfrac{1}{\alpha}\cdot\cos^{-1}\left(\dfrac{b}{l}\cdot\dfrac{\sin\alpha l}{\sin\alpha b}\right)$ (12.21)

(iii) Uniformly Distributed Load

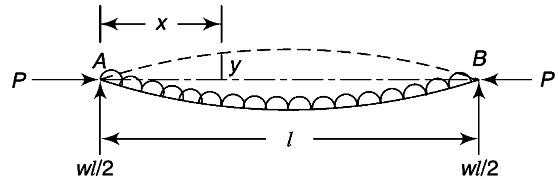


Fig. 12.16

Consider a pin-jointed strut with an axial compressive load P along with a uniformly distributed load as shown in Fig. 12.16.

At a distance x from A, the bending moment is,

$$M = -Py + \frac{wl}{2}x - \frac{wx^2}{2}$$

$$EI\frac{d^2y}{dx^2} = M = -Py + \frac{wl}{2}x - \frac{wx^2}{2} \quad \text{or} \quad \frac{d^2y}{dx^2} + \alpha^2 y = \frac{wl}{2EI}x - \frac{wx^2}{2EI} \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

Solution is of two parts.

$$\text{CF} = A\sin\alpha x + B\cos\alpha x$$

$$\text{PI} = \frac{\frac{wl}{2EI}x}{D^2+\alpha^2} - \frac{\frac{w}{2EI}x^2}{D^2+\alpha^2} = \frac{\frac{wl}{2EI}x}{0+\alpha^2} - \frac{wx^2}{2EI}\frac{1}{\alpha^2}\left(1+\frac{D^2}{\alpha^2}\right)^{-1}$$

$$= \frac{wl}{2EI\alpha^2}x - \frac{wx^2}{2EI}\frac{1}{\alpha^2}\left(1-\frac{D^2}{\alpha^2}\right) = \frac{wl}{2P}x - \frac{wx^2}{2P} + \frac{wEI}{P^2}$$

Complete solution is $y = A\sin\alpha x + B\cos\alpha x - \dfrac{wx^2}{2P} + \dfrac{wl}{2P}x + \dfrac{wEI}{P^2}$

At $\quad x = 0, y = 0, \quad \text{or} \quad B = -\dfrac{wEI}{P^2}$

At $\quad x = l; y = 0, y = A\sin\alpha l - \dfrac{wEI}{P^2}\cos\alpha l - \dfrac{wl^2}{2P} + \dfrac{wl}{2P}\cdot l + \dfrac{wEI}{P^2} = 0$

$$A\sin\alpha l = -\frac{wEI}{P^2}(1-\cos\alpha l) \quad \text{or} \quad A\cdot 2\sin\frac{\alpha l}{2}\cos\frac{\alpha l}{2} = -\frac{wEI}{P^2}\cdot 2\sin^2\frac{\alpha l}{2}$$

or $\quad A = -\dfrac{wEI}{P^2}\tan\dfrac{\alpha l}{2}$

$$y = -\frac{wEI}{P^2}\tan\frac{\alpha l}{2}\sin\alpha x - \frac{wEI}{P^2}\cos\alpha x - \frac{wx^2}{2P} + \frac{wl}{2P}\cdot x + \frac{wEI}{P^2}$$

$$= -\frac{wEI}{P^2}\left(\frac{\sin\frac{\alpha l}{2}\sin\alpha x + \cos\frac{\alpha l}{2}\cos\alpha x}{\cos\frac{\alpha l}{2}}\right) - \frac{wx^2}{2P} + \frac{wl}{2P}\cdot x + \frac{wEI}{P^2}$$

$$= -\frac{wEI}{P^2}\cos\alpha\left(\frac{l}{2}-x\right)\sec\frac{\alpha l}{2} - \frac{wx^2}{2P} + \frac{wl}{2P}\cdot x + \frac{wEI}{P^2}$$

At $\quad x = l/2, y$ is maximum

$$y_{\max} = -\frac{wEI}{P^2}\left(\sec\frac{\alpha l}{2}-1\right) - \frac{wl^2}{8P} + \frac{wl^2}{4P} = -\frac{wEI}{P^2}\left(\sec\frac{\alpha l}{2}-1\right) + \frac{wl^2}{8P} \tag{12.22}$$

$$M_{\max} = P\left[\frac{wEI}{P^2}\left(\sec\frac{\alpha l}{2}-1\right) - \frac{wl^2}{8P}\right] + \frac{wl}{2}\cdot\frac{l}{2} - \frac{wl^2}{8} = \frac{wEI}{P}\left(\sec\frac{\alpha l}{2}-1\right) \tag{12.23}$$

Alternative Solution The bending moment is, $M = -Py + \dfrac{wl}{2}x - \dfrac{wx^2}{2}$

Differentiating twice, $\dfrac{d^2M}{dx^2} = -P \cdot \dfrac{d^2y}{dx^2} - w = -P \cdot \dfrac{M}{EI} - w$ .... $\left(\because EI\dfrac{d^2y}{dx^2} = M\right)$

or $$\frac{d^2M}{dx^2} + \alpha^2 M = -w \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

Solution is, $M = A\sin\alpha x + B\cos\alpha x - \dfrac{w}{\alpha^2}$

At $x = 0, M = 0,$ or $B = \dfrac{w}{\alpha^2}$

At $x = l/2; y = dM/dx = 0,$ (zero shear),

Thus $\dfrac{dM}{dx} = A\alpha\cos\alpha x - B\alpha\sin\alpha x = 0$ or $A = \dfrac{w}{\alpha^2}\tan\dfrac{\alpha l}{2}$

$\therefore$ $$M = A\sin\alpha x + B\cos\alpha x - \frac{w}{\alpha^2} = \frac{w}{\alpha^2}\left(\tan\frac{\alpha l}{2}\sin\alpha x + \cos\alpha x - 1\right)$$

At $$x = l/2;\ M = \frac{w}{\alpha^2}\left(\frac{\sin^2\frac{\alpha l}{2}}{\cos\frac{\alpha l}{2}} + \cos\frac{\alpha l}{2} - 1\right) = -\frac{w}{\alpha^2}\left(\frac{1}{\cos\frac{\alpha l}{2}} - 1\right) = \frac{wEI}{P}\left(\sec\frac{\alpha l}{2} - 1\right)$$

Maximum deflection y_{max} at $x = l/2$ can be obtained from $M = Py + \dfrac{wl^2}{4} - \dfrac{w\alpha^2}{2}$

or $$\frac{wEI}{P^2}\left(\sec\frac{\alpha l}{2} - 1\right) = P_y + \frac{Wl}{4} - \frac{wl^2}{8} \quad \text{or} \quad y_{max} = \frac{wEI}{P^2}\left(\sec\frac{\alpha l}{2} - 1\right) - \frac{wl^2}{8}$$

(iv) Uniformly Distributed Load with Eccentric Axial Load

Consider a pin-jointed strut with an eccentric compressive load P along with a uniformly distributed load as shown in Fig. 12.17.

At a distance x from A, the bending moment is, $M = -P(y + e) + \dfrac{wl}{2}x - \dfrac{wx^2}{2}$ (12.24)

Differentiating twice, $\dfrac{d^2M}{dx^2} = -P \cdot \dfrac{d^2y}{dx^2} - w = -P \cdot \dfrac{M}{EI} - w$ .... $\left(\because EI\dfrac{d^2y}{dx^2} = M\right)$

or $$\frac{d^2M}{dx^2} + \alpha^2 M = -w \quad \text{where} \quad \alpha^2 = \frac{P}{EI}$$

Solution is $M = A\sin\alpha x + B\cos\alpha x - \dfrac{w}{\alpha^2}$

At $x = 0, M = -P \cdot e, \therefore B = \dfrac{w}{\alpha^2} - P \cdot e$

At $x = l/2; y = dM/dx = 0,$ (zero shear),

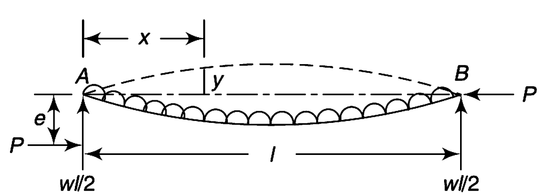


Fig. 12.17

Thus $$\frac{dM}{dx} = A\alpha \cos \alpha x - B\,\alpha \sin \alpha x = 0$$

or $$A \cos\frac{\alpha l}{2} = \left(\frac{w}{\alpha^2} - P\cdot e\right)\sin\frac{\alpha l}{2} \text{ or } A = \left(\frac{w}{\alpha^2} - P\cdot e\right)\tan\frac{\alpha l}{2}$$

$$M = A \sin \alpha x + B \cos \alpha x - \frac{w}{\alpha^2} = \left(\frac{w}{\alpha^2} - P\cdot e\right)\tan\frac{\alpha l}{2}\cdot \sin \alpha x + \left(\frac{w}{\alpha^2} - P\cdot e\right)\cos \alpha x - \frac{w}{\alpha^2}$$

$$= \left(\frac{w}{\alpha^2} - P\cdot e\right)\left(\tan\frac{\alpha l}{2}\cdot \sin \alpha x + \cos \alpha x\right) - \frac{w}{\alpha^2} \tag{12.25}$$

At $$x = l/2;\ M_{max} = \left(\frac{w}{\alpha^2} - P\cdot e\right)\left(\tan\frac{\alpha l}{2}\cdot \sin\frac{\alpha l}{2} + \cos\frac{\alpha l}{2}\right) - \frac{w}{\alpha^2}$$

$$= \left(\frac{w}{\alpha^2} - P\cdot e\right)\sec\frac{\alpha l}{2} - \frac{w}{\alpha^2} \tag{12.26}$$

The maximum bending moment will be the least if it is equal to bending moment at the origin, i.e.,

$$Pe = \left(\frac{w}{\alpha^2} - P\cdot e\right)\sec\frac{\alpha l}{2} - \frac{w}{\alpha^2} \quad \text{or} \quad Pe\left(\sec\frac{\alpha l}{2} + 1\right) = \frac{w}{\alpha^2}\left(\sec\frac{\alpha l}{2} - 1\right)$$

or $$e = \frac{w}{P\alpha^2}\left(\frac{\sec\frac{\alpha l}{2} - 1}{\sec\frac{\alpha l}{2} + 1}\right) \tag{12.27}$$

- Deflection can be obtained from $M = -P(y - e) + \frac{wl}{2}x - \frac{wx^2}{2}$

or $$\left(\frac{w}{\alpha^2} - P\cdot e\right)\left(\tan\frac{\alpha l}{2}\cdot \sin \alpha x + \cos \alpha x\right) - \frac{w}{\alpha^2} = -Py + P\cdot e + \frac{wl}{2}x - \frac{wx^2}{2}$$

or $$y = \left(\frac{wEI}{P} - e\right)\left(\tan\frac{\alpha l}{2}\cdot \sin \alpha x + \cos \alpha x - 1\right) - \frac{w}{2P}(lx - x^2) \tag{12.28}$$

and $$y_{max(y = l/2)} = \left(\frac{wEI}{P} - e\right)\left[\frac{\sin^2\frac{\alpha l}{2} + \cos^2\frac{\alpha l}{2}}{\cos\frac{\alpha l}{2}} - 1\right] - \frac{wl^2}{8P}$$

$$= \left(\frac{wEI}{P} - e\right)\left[\sec\frac{\alpha l}{2} - 1\right] - \frac{wl^2}{8P} \tag{12.29}$$

Example 12.23 A rectangular horizontal strut is 30 mm wide, 80 mm deep and 2 m long. It has pin joints at its ends and carries an axial thrust of 60 kN along with a vertical uniformly distributed load of 3 kN/m length. Determine the maximum stress induced. What is the percentage error if the additional bending moment due to eccentricity of the thrust is neglected? E = 205 GPa.

Solution

Given

$b = 30$ mm	$d = 80$ mm
$l = 2$ m	$P = 60$ kn
$w = 3$ kN/m = 3 N/mm	$E = 205$ GPa

To find

— Maximum stress

— percentage error if additional bending moment due to eccentricity neglected

$$I = \frac{30 \times 80^3}{12} = 1.28 \times 10^6 \text{ mm}^4$$

Maximum bending moment

$$\frac{\alpha l}{2} = \frac{l}{2}\sqrt{\frac{P}{EI}} = \frac{2000}{2}\sqrt{\frac{60\,000}{205\,000 \times 1.28 \times 10^6}} = 0.478 \text{ rad } (=27.39°)$$

$$M = \frac{wEI}{P}\left(\sec\frac{\alpha l}{2} - 1\right) = \frac{3 \times 205\,000 \times 1.28 \times 10^6}{60\,000}(\sec 27.39° - 1)$$

$$= 13.12 \times 10^6 \times 0.1263 = 1.657 \times 10^6 \text{ N}\cdot\text{mm}$$

Maximum stress

$$\sigma_{max} = \frac{P}{A} + \frac{M.y_c}{I} = \frac{60\,000}{30 \times 80} + \frac{1.657 \times 10^6 \times 40}{1.28 \times 10^6} = 25 + 51.8 = 76.8 \text{ MPa (compressive)}$$

Neglecting additional bending moment

If the eccentricity of the thrust, i.e., central deflection is neglected,

$$M = \frac{wl^2}{8} = \frac{3 \times 2000^2}{8} = 1.5 \times 10^6 \text{ N}\cdot\text{mm}$$

$$\sigma_{max} = \frac{P}{A} + \frac{M \cdot y_c}{I} = 25 + \frac{1.5 \times 10^6 \times 40}{1.28 \times 10^6} = 25 + 46.875 = 71.875 \text{ N}\cdot\text{mm}$$

$$\text{Percentage error} = \frac{76.8 - 71.875}{76.8} = 6.4\ \%$$

Example 12.24 A 2-m long horizontal bar of uniform rectangular section is 60 mm wide and 20 mm deep. It is simply supported at its ends and weighs 100 N/m. The bar is also subjected to a longitudinal end thrust 10 kN acting at points on the vertical centre lines of the end sections at a distance e below the centres. Determine the value of e if the resultant maximum bending moment in the beam is to be the least. E = 210 GPa. Also find the corresponding maximum deflection.

Solution

Given $b = 60$ mm $\quad d = 20$ mm

$l = 2$ m $\quad P = 10$ kN

$w = 100$ N/m $= 0.1$ N/mm $\quad E = 210$ GPa

To find

— Eccentricity if resultant maximum bending moment is least

— Maximum deflection

Eccentricity

$$\alpha = \sqrt{\frac{P}{EI}} = \sqrt{\frac{10\,000}{210\,000 \times (60 \times 20^3/12)}} = 0.001\,091$$

$$\frac{\alpha l}{2} = \frac{0.001\,091 \times 2000}{2} = 1.091 \text{ rad}$$

$$e = \frac{w}{P\alpha^2}\left(\frac{\sec\frac{\alpha l}{2} - 1}{\sec\frac{\alpha l}{2} + 1}\right) \qquad \text{(Eq. 12.27)}$$

$$= \frac{0.1}{10\,000 \times 1.19 \times 10^{-6}}\left(\frac{\sec 62.5° - 1}{\sec 62.5° + 1}\right) = 3.094 \text{ mm}$$

Maximum deflection

$$M = -Py + \frac{wl}{2}x - \frac{wx^2}{2} \qquad \text{...(From Eq. 12.24)}$$

Maximum bending moment (at the centre, $y = l/2$)

$$M = -P\,(y + e) + \frac{wl^2}{4} - \frac{wl^2}{8} = -P(y + e) + \frac{wl^2}{8} = -10\,000(y + 3.094) + \frac{0.1 \times 2000^2}{8}$$

$$= -10\,000y - 30\,940 + 50\,000$$

Also $\quad M = P \cdot e = 10\,000 \times 3.094 = 30\,940$

$\therefore \quad -10\,000y - 30\,940 + 50\,000 = 30\,940$

$y = -1.188$ mm

The deflected shape is shown in Fig. 12.18.

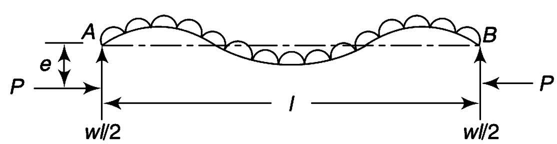


Fig. 12.18

12.9 TIE WITH LATERAL LOADING

Consider a pin-jointed strut with an axial tensile load P along with a uniformly distributed load as shown in Fig. 12.19.

At a distance x from A, the bending moment is, $M = Py + \frac{wl}{2}x - \frac{wx^2}{2}$

Differentiating twice, $\frac{d^2M}{dx^2} = P \cdot \frac{d^2y}{dx^2} - w = P \cdot \frac{M}{EI} - w$ .... $\left(\because EI\frac{d^2y}{dx^2} = M\right)$

or $\quad \frac{d^2M}{dx^2} - \alpha^2 M = -w$ where $\alpha^2 = \frac{P}{EI}$

Solution is $M = A \sin \alpha x + B \cosh \alpha x + \frac{w}{\alpha^2}$

At $\quad x = 0, M = 0, \quad \therefore \quad B = -\frac{w}{\alpha^2}$

At $\quad x = l/2; y = dM/dx = 0$, (zero shear),

$$\frac{dM}{dx} = A\alpha \cosh \alpha x + B\,\alpha \sinh \alpha x = 0$$

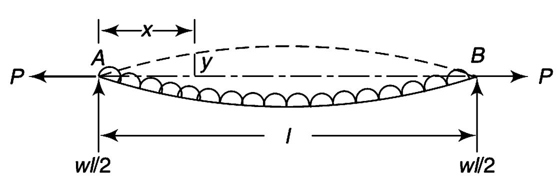


Fig. 12.19

or $\quad A = -B \tanh \frac{\alpha l}{2} = \frac{w}{\alpha^2} \tanh \frac{\alpha l}{2}$

$$\therefore \quad M = \frac{w}{\alpha^2}\left(\tanh\frac{\alpha l}{2}\sin \alpha x - \cosh \alpha x + 1\right)$$

At $$x = l/2;\ M = \frac{w}{\alpha^2}\left(\frac{\sinh^2\frac{\alpha l}{2}}{\cosh\frac{\alpha l}{2}} - \cosh\frac{\alpha l}{2} + 1\right)$$

$$= \frac{w}{\alpha^2}\left(\frac{-1}{\cosh\frac{\alpha l}{2}} + 1\right)$$

$$= \frac{wEI}{P}\left(1 - \operatorname{sech}\frac{\alpha l}{2}\right) \tag{12.30}$$

Maximum deflection y_{max} at $x = l/2$ can be obtained from $M = Py + \frac{wl}{2}x - \frac{wx^2}{2}$

or $$\frac{wEI}{P}\left(1 - \operatorname{sech}\frac{\alpha l}{2}\right) = Py + \frac{wl^2}{4} - \frac{wl^2}{8}$$

or $$\frac{wEI}{P^2}\left(1 - \operatorname{sech}\frac{\alpha l}{2}\right) - \frac{wl^2}{8} \tag{12.31}$$

Example 12.25 A 4-m long steel tie bar with a diameter of 32 mm is supported horizontally through pin joints. Determine the maximum tensile stress in the bar if it sustains an axial pull of 15 kN. Density of steel is 7600 kg/m^3 and E is 205 GPa.

Solution

Given

$d = 32$ mm $\qquad l = 4$ m

$P = 15$ kN $\qquad \rho = 7600$ kg/m^3 $= 7600 \times 10^{-9}$ kg/mm^3

$E = 210$ GPa

To find Maximum tensile stress

Maximum bending moment

$$\alpha = \sqrt{\frac{P}{EI}} = \sqrt{\frac{15\,000}{205\,000 \times (\pi \times 32^4/64)}} = 0.001\ 19$$

$$\frac{\alpha l}{2} = \frac{0.001\,19 \times 4000}{2} = 2.38$$

$$w = \rho(a \cdot l)\, g = \frac{7600}{10^9} \times \left(\frac{\pi}{4} \times 32^2 \times 1\right) \times 9.81 = 0.059\ 96 \text{ N/mm of length}$$

$$M = \frac{wEI}{P}\left(1 - \operatorname{sech}\frac{\alpha l}{2}\right)$$

$$= \frac{0.059\ 96 \times 205\ 000}{15\ 000} \times \frac{\pi \times 32^4}{64}(1 - \operatorname{sech} 2.38)$$

$$= 34\ 439 \text{ N} \cdot \text{mm}$$

Maximum tensile stress

$$\sigma_{max} = \frac{P}{A} + \frac{M}{Z} = \frac{15\,000}{(\pi/4)\times 32^2} + \frac{34\,439}{\pi \times 32^3/32} = 18.65 + 10.7 = 29.35 \text{ MPa}$$

12.10 STRUTS OF VARYING CROSS-SECTION

Under a compressive axial load there is shortening of the length of a strut. If Δ is the axial movement of an end of the strut under the action of a crippling load P (Fig. 12.20), it may be assumed that for any such small axial movement, there is no change in the crippling load and it remains constant. As the strut remains stable for all loads less than P, the approximate strain energy of the strut will be $P \cdot \Delta$. However, when the strut deflects under the action of crippling load P, the strain energy will mainly be due to the bending moment.

Take the X-axis through the built-in end. Let x and s be the distances measured along the axes of the undeflected and deflected strut respectively.

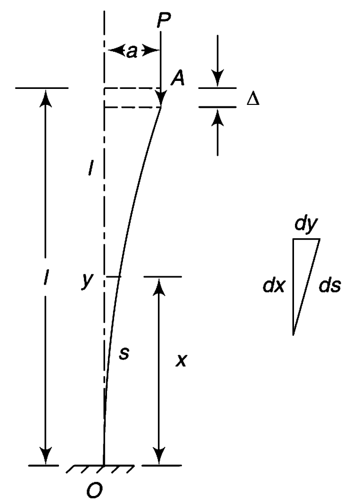


Fig. 12.20

Then
$$\Delta = \int_0^l (ds - dx)$$
$$= \int_0^l \sqrt{(dx)^2 + (dy)^2} - \int_0^l dx$$
$$= \int_0^l \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx - \int_0^l dx$$
$$= \int_0^l \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2} \cdot dx - \int_0^l dx$$
$$= \int_0^l \left[1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right] \cdot dx - \int_0^l dx$$
$$= \frac{1}{2}\int_0^l \left(\frac{dy}{dx}\right)^2 \cdot dx$$

Thus strain energy $= P \cdot \Delta = \dfrac{1}{2} P \displaystyle\int_0^l \left(\frac{dy}{dx}\right)^2 \cdot dx$

Also strain energy due to bending moment $= \displaystyle\int_0^l \frac{M^2}{2EI} dx$

Equating the two,
$$\frac{1}{2} P \int_0^l \left(\frac{dy}{dx}\right)^2 \cdot dx = \frac{1}{2}\int_0^l \frac{M^2}{EI} dx$$

or
$$P = \frac{\displaystyle\int_0^l \frac{M^2}{EI} dx}{\displaystyle\int_0^l \left(\frac{dy}{dx}\right)^2 \cdot dx} \tag{12.32}$$

This expression can be evaluated taking into account variations in l. The following forms of the deflected strut can be suitably assumed:

For a pin ended strut, $y = a\sin\dfrac{\pi x}{l}$ where l is the length of strut

For a strut fixed at one end, $y = a\left(1-\cos\dfrac{\pi x}{2l}\right)$

Example 12.26 A 180-mm long steel strut is made up of two lengths, one is 90-mm long and 6 mm in diameter whereas the other is of the same length but is 8 mm in diameter. It is rigidly fixed at the larger end and carries an axial load at the smaller end. Determine the magnitude of the crippling load. E for steel is 210 GPa.

Solution

Given

$d_1 = 8$ mm $\quad d_2 = 6$ mm

$l = 180$ mm $\quad E = 210$ GPa

To find Crippling load

Refer Fig.12.19.

Assuming the deflected form under the action of the crippling load be

$$y = a\left(1-\cos\frac{\pi x}{2l}\right) = a\left(1-\cos\frac{\pi x}{360}\right)$$

and thus,
$$\frac{dy}{dx} = \frac{a\pi}{360}\sin\frac{\pi x}{360}$$

Also,
$$M = P(a-y) = Pa - Pa\left(1-\cos\frac{\pi x}{360}\right) = Pa\cos\frac{\pi x}{360}$$

The crippling load is given by,
$$P = \frac{\displaystyle\int_0^l \frac{M^2}{EI}dx}{\displaystyle\int_0^l \left(\frac{dy}{dx}\right)^2\cdot dx}$$

Numerator $\displaystyle\int_0^l \frac{M^2}{2EI}dx$

$$\int_0^l \frac{M^2}{2EI}dx = \frac{P^2a^2}{2E}\left[\int_0^{90}\frac{\cos^2\frac{\pi x}{360}}{I_1}dx + \int_{90}^{180}\frac{\cos^2\frac{\pi x}{360}}{I_2}dx\right]$$

$$= \frac{P^2a^2}{2E}\left[\int_0^{90}\frac{1+\cos\frac{\pi x}{180}}{\frac{\pi}{64}8^4}dx + \int_{90}^{180}\frac{\cos\frac{\pi x}{180}}{\frac{\pi}{64}6^4}dx\right]$$

$$= \frac{P^2a^2}{2\cdot\frac{\pi}{64}E}\left[\frac{1}{8^4}\left(x+\frac{180}{\pi}\sin\frac{\pi x}{180}\right)_0^{90} + \frac{1}{6^4}\left(x+\frac{180}{\pi}\sin\frac{\pi x}{180}\right)_0^{90}\right]$$

$$= \frac{64P^2a^2}{2.\pi E}\left[\frac{1}{8^4}\left(90+\frac{180}{\pi}\right)+\frac{1}{6^4}\left(180-90+0-\frac{180}{\pi}\right)_0^{90}\right]$$

$$= 10.19 \times \frac{P^2 a^2}{E}[0.036 + 0.0252]$$

$$= 0.623\frac{P^2 a^2}{E}$$

Denominator $\int_0^l \left(\frac{dy}{dx}\right)^2 \cdot dx$

$$\int_0^l \left(\frac{dy}{dx}\right)^2 \cdot dx = \left(\frac{a\pi}{360}\right)^2 \int_0^{180} \left(\sin^2 \frac{\pi x}{360}\right) \cdot dx$$

$$= \left(\frac{a\pi}{360}\right)^2 \int_0^{180} \frac{1 - \cos\dfrac{\pi x}{180}}{2} \cdot dx$$

$$= \frac{1}{2}\left(\frac{a\pi}{360}\right)^2 \left(x - \frac{\pi}{180}\sin\frac{\pi x}{180}\right)_0^{180}$$

$$= \frac{1}{2}\left(\frac{a\pi}{360}\right)^2 \times 180 = 0.006\,85a^2$$

Crippling load

$$P = \frac{0.623P^2 a^2}{E \times 0.006\,85a^2} = \frac{0.623P^2}{210\,000 \times 0.006\,85} = 433.1 \times 10^{-6} P^2$$

or $\quad 1 = 433.1 \times 10^{-6} P$

or $\quad P = 2309$ N

Summary

1. A bar or member of a structure in any position acted upon by a compressive load is known as a *strut.* However, when the compressive member is in a vertical position and is liable to fail by bending or buckling, it may be referred as a *column* or *strut.*
2. If the load on a strut is increased to a value that on its removal, the deflection remains, the load is usually known as the *critical* or *crippling* or *buckling load.* Any load beyond this load further deflects the column.
3. The ratio l/k is known as the *slenderness ratio*.
4. Euler's load can be expressed as $P_e = \pi^2 EI/l_e^2$ where l_e^2 is term referred as *equivalent length* of the column. For
   - both ends hinged, $l_e = 1$
   - one end fixed and the other free, $l_e = 2l$
   - both ends fixed, $l_e = l/2$
   - one end fixed, other hinged, $l_e = l/\sqrt{2}$
5. Euler's formula is derived on the assumptions that the struts are initially perfectly straight and the load is exactly axial. However, in practice, these assumptions are never realized. There is always some eccentricity and initial curvature present.

6. Euler formula is applicable for slenderness ratio greater than 80 for steel struts hinged at both the ends.
7. The critical stress σ_c, i.e., the average stress over the cross–section for a standard case is $\sigma_e = \dfrac{\pi^2 E}{(l_e/k)^2}$
8. Rankine's empirical relation for columns covering all cases from very short to very long columns is $\dfrac{1}{P} = \dfrac{1}{P_c} + \dfrac{1}{P_e}$ or $P = \dfrac{\sigma_c \cdot A}{1 + a(l/k)^2}$.
9. *Secant formula* for eccentric loading, $\sigma_{\max} = \dfrac{P}{A} + \dfrac{P \cdot e \cdot y_c \sec(\alpha l/2)}{Ak^2}$
10. Perry's formula, $\left(\dfrac{\sigma_{\max}}{\sigma_o} - 1\right)\left(1 - \dfrac{\sigma_e}{\sigma_e}\right) = \dfrac{1 \cdot 2e \cdot y_c}{k^2}$
11. In a strut with initial curvature, $\left(\dfrac{\sigma_{\max}}{\sigma_o} - 1\right)\left(1 - \dfrac{\sigma_o}{\sigma_e}\right) = \dfrac{c \cdot y_c}{k^2}$ and $y_{\max(x = l/2)} = \dfrac{cP_e}{P_e - P}$
12. In a strut with lateral loading:

With central loading, $\sigma_{max} = \dfrac{P}{A} + \dfrac{W}{2\alpha} \tan \dfrac{\alpha l}{2} \dfrac{y_c}{Ak^2}$

For eccentric loading, $y = -\dfrac{W}{\alpha P} \dfrac{\sin \alpha b}{\sin \alpha l} \sin \alpha x + \dfrac{Wbx}{Pl}$,

y is maximum at $x = \dfrac{1}{\alpha} \cdot \cos^{-1}\left(\dfrac{b}{l} \cdot \dfrac{\sin \alpha l}{\sin \alpha b}\right)$

For uniformly distributed load, $y_{max} = \dfrac{wEI}{P^2}\left(\sec \dfrac{\alpha l}{2} - 1\right) - \dfrac{wl^2}{8}$

For uniformly distributed load with eccentric axial load,

$$y_{max(y = l/2)} = \left(\frac{wEI}{P} - e\right)\left[\sec \frac{\alpha l}{2} - 1\right] - \frac{wl^2}{8P}$$

13. In a tie with lateral loading, $y_{max} = \dfrac{wEI}{P^2}\left(1 - \operatorname{sech} \dfrac{\alpha l}{2}\right) - \dfrac{wl^2}{8}$
14. For a strut of varying cross-section, $P = \dfrac{\int_0^l (M^2/EI)dx}{\int_0^l (dy/dx)^2 \cdot dx}$

Objective Type Questions

1. If the load on a column is increased to a value that on its removal the deflection remains, the load is known as

(a) critical load (b) crippling load (c) buckling load (d) all of these

2. The equivalent length of a column fixed at one end with other free is

(a) l (b) $2l$ (c) $l/\sqrt{2}$ (d) $l/2$

3. The equivalent length of a column fixed at both ends is
(a) l (b) $2l$ (c) $l/\sqrt{2}$ (d) $l/2$

4. Euler crippling load for both ends hinged is given by,
(a) $\dfrac{\pi^2 EI}{l^2}$ (b) $\dfrac{2\pi^2 EI}{l^2}$ (c) $\dfrac{4\pi^2 EI}{l^2}$ (d) $\dfrac{\pi^2 EI}{4l^2}$

5. Euler buckling load for one end fixed and the other hinged is given by,
(a) $\dfrac{\pi^2 EI}{l^2}$ (b) $\dfrac{2\pi^2 EI}{l^2}$ (c) $\dfrac{4\pi^2 EI}{l^2}$ (d) $\dfrac{\pi^2 EI}{4l^2}$

6. Euler buckling load for one end fixed and the other free is given by,
(a) $\dfrac{\pi^2 EI}{l^2}$ (b) $\dfrac{2\pi^2 EI}{l^2}$ (c) $\dfrac{4\pi^2 EI}{l^2}$ (d) $\dfrac{\pi^2 EI}{4l^2}$

7. Euler buckling load for both ends fixed is given by,
(a) $\dfrac{\pi^2 EI}{l^2}$ (b) $\dfrac{2\pi^2 EI}{l^2}$ (c) $\dfrac{4\pi^2 EI}{l^2}$ (d) $\dfrac{\pi^2 EI}{4l^2}$

8. In a long column with one end fixed and the other free, if the slenderness ratio increases, the critical stress
(a) increases (b) decreases (c) remains same

9. The buckling load for a given material depends upon
(a) Poisson's ratio and slenderness ratio
(b) Poisson's ratio and modulus of elasticity
(c) Slenderness ratio and cross-sectional area
(d) Slenderness ratio and modulus of elasticity

10. Rankine's formula is applicable to
(a) long columns (b) short columns (c) both of these (d) none of these

11. Secant formula is applicable for
(a) short columns under axial loading
(b) long columns under axial loading
(c) short columns under eccentric loading
(d) long columns under eccentric loading

12. Perry's approximate formula is
(a) $\left(\dfrac{\sigma_{\max}}{\sigma_o} - 1\right)\left(1 - \dfrac{\sigma_o}{\sigma_e}\right) = \dfrac{e \cdot y_c}{1.2k^2}$ (b) $\left(\dfrac{\sigma_e}{\sigma_o} - 1\right)\left(1 - \dfrac{\sigma_o}{\sigma_{\max}}\right) = \dfrac{1.2e \cdot y_c}{k^2}$

(c) $\left(\dfrac{\sigma_{\max}}{\sigma_o} - 1\right)\left(1 - \dfrac{\sigma_o}{\sigma_e}\right) = \dfrac{1.2e \cdot y_c}{k^2}$ (d) $\left(\dfrac{\sigma_{\max}}{\sigma_o} - 1\right)\left(\dfrac{\sigma_o}{\sigma_e} - 1\right) = \dfrac{1.2e \cdot y_c}{k^2}$

Answers

1. (d)	2. (b)	3. (d)	4. (a)	5. (b)	6. (d)
7. (c)	8. (b)	9. (d)	10. (c)	11. (d)	12. (c)

Review Questions

12.1 What is a strut? How does it differ from a column?
12.2 What is the difference between crushing and buckling failures of a column?

12.3 What is meant by crippling or buckling load?
12.4 Define slenderness ratio of a column? What is its importance?
12.5 What assumptions are made in the analysis of struts and columns by Euler's buckling theory? What are its limitations?
12.6 What is Euler's curve? What is its importance?
12.7 What is meant by equivalent length of columns? What are its values for different end conditions of columns?
12.8 What is critical stress of a column? What is its value?
12.9 What is Rankine empirical relation for columns?
12.10 Develop the secant formula. Discuss its importance.

Numerical Problems

12.1 Determine the crippling load for a 5-m long fixed ends timbre column of 150 mm × 200 mm section. Young's modulus for timbre is 17 GPa. (1510 kN)

12.2 A 5-m long simply supported beam is applied a uniformly distributed load of 40 kN/m over the entire span. The deflection at the midspan is observed to be 20 mm. Find the crippling load when this beam is used as a column with one end fixed and the other hinged. Also find the crippling load if both the ends are pin joined. (17.134 kN, 8.567 kN)

12.3 Calculate the crippling load for a 4 m long *T*–section strut of dimensions 100 mm × 20 mm assuming both of its ends hinged. $E = 210$ GPa. (222.806 kN)

12.4 A hollow alloy tube having internal and external diameters as 36 mm and 52 mm respectively is 6 m long. It extends by 3 mm under a tensile load of 50 kN. Determine the crippling load for the tube when used as a strut with both ends pinned. (6.854 kN)

12.5 A 1.5-m long straight steel bar which is 20 mm × 5 mm in section is compressed longitudinally until it buckles. Applying the Euler's formula for pinned ends, determine the maximum central deflection before the steel attains the yield point stress of 320 MPa. $E = 210$ GPa. (138 mm)

12.6 A 5-m long hollow cast iron column with fixed ends supports an axial load of 800 kN. The external diameter of the column is 240 mm. Determine the thickness of the column using the Rankine formula with a constant of 1/6400. The working stress is 80 MPa. (24.6 mm)

12.7 A 4-m long hollow cast iron column of cylindrical section is used to carry a safe load of 250 kN with both of its ends fixed. The external diameter of the column is 1.25 times the internal diameter. Find these diameters by taking a factor of safety of 5 and $\sigma_c = 550$ MPa and Rankine's constant 1/1600. (127 mm, 101.6 mm)

12.8 Compare the crippling loads given by Euler's and Rankine's formulae for a 3 m long hollow steel strut having inner and outer diameters as 52 mm and 48 mm respectively. The strut is pin-jointed at the ends. The yield stress is 320 MPa, Rankine's constant 1/7500 and $E = 200$ GPa. (1.037)

12.9 A short length of a tube of external and internal diameters of 50 mm and 40 mm respectively failed in compression at a load of 250 kN. When the same tube of 2 m length is tested as a strut with fixed ends, it failed at a load of 150 kN. Assuming that in the Rankine's formula F_c used is the same as from the first test, determine the value of *a* in the same formula. Also calculate the buckling load of the tube if it is used as strut of 3 m length with one end fixed and the other hinged. (1/5853, 62.5 kN)

12.10 A 200-mm long steel strut is made up of two lengths of 100 mm each. The diameter of the first length is 4.5 mm and that of the other is 7 mm. It has built-in larger section and at the smaller end it supports an axial load. Determine the crippling load. $E = 210$ GPa. (811 N)

12.11 An initially straight tube of 40–mm external diameter and 36–mm internal diameter carries a compressive load of 15 kN. The load is parallel to the axis at an eccentricity of 1.5 mm. Find the maximum and minimum stresses in the tube. E = 210 GPa. (99.8 MPa, 25.8 MPa)

12.12 A tubular steel strut with a 44-mm internal diameter and 56-mm external diameter is 2.2 m long and has hinged ends. The load is parallel to the axis but eccentric. Calculate the maximum eccentricity for a crippling load of 75% of the Euler value. The yield stress is 290 MPa and E = 207 GPa. (4.49 mm)

12.13 A circular column of cast iron 20-mm thick and of 200-mm external diameter is used with both ends fixed. For a length of 4 m, the column carries a load of 150 kN at an eccentricity of 30 mm from the axis of the column. Determine the maximum stresses on the column section. Also calculate the permissible value of eccentricity in order to avoid tension anywhere on the section. E = 94 GPa. (23.14 MPa, 40.3 mm)

12.14 A 3.4-m long hollow circular steel strut has hinged ends. The internal and external diameters are 80 mm and 96 mm respectively. Initially, the strut is bent with a maximum deviation of 5 mm. Assuming the central line of the strut as sinusoidal, find the maximum stress due to a central compressive load of 10 kN. E =210 GPa. (5.65 MPa)

12.15 A strut of 40-mm diameter and 3.2-m long carries a compressive load of 36 kN alongwith a transverse wind load of 800 N/m. Find the maximum stress developed in the strut. E =200 GPa.
(381.4 MPa (compressive) and 324 MPa (compressive))

12.16 A coupling rod of I-section is 2.5-m long and is 125-mm deep overall. The two flanges are 60-mm wide and 5-mm thick whereas the web is 2.5-mm thick. The axial load on the rod is 180 kN and the lateral load 3.75 kN uniformly distributed. Find the maximum value of the stress developed. E = 200 GPa.
(306 MPa)

12.17 A 2.7-m long column of rectangular section measuring 150 mm × 50 mm carries an axial load of 60 kN and a lateral distributed load of 2.7 kN. Find the minimum stress induced if both ends of the column are hinged. Also calculate the percentage error in the minimum stress if axial load effects on bending are neglected. E = 80 GPa. (22.6 MPa, 26.6%)

Chapter **13**

Cylinders and Spheres

Cylinders and spheres subjected to fluid pressure are common in engineering practices such as steam boilers, tanks, chambers of engines, reservoirs, etc. These are also known as *pressure vessels* or *shells*. Due to the internal or the external pressure on the walls of a pressure vessel, stresses are induced in them from which the thickness can be determined while designing. In a cylindrical vessel, there are three principal stresses: *circumferential* or *hoop* stress, *longitudinal* stress and *radial* stress.

A shell may be termed as *thin* or *thick* depending upon the ratio of the thickness of the wall to the diameter of the shell. If the ratio is less than about 1 to 15, a shell is considered as thin, otherwise it is thick. In a thin shell, it can be assumed that the hoop stresses are constant over the thickness and the radial stress is small and may be neglected.

13.1 THIN CYLINDER

Let

d = internal diameter of the shell
t = thickness of wall of the shell
p = internal pressure
σ_c = circumferential or hoop stress
σ_l = longitudinal stress

Consider half of the cylinder of length l, sectioned through a diametral plane as shown in Fig. 13.1a. As the internal pressure in a thin cylinder is low, the radial stresses are small and are ignored.

For equilibrium of forces in the vertical direction,

Resisting force = resultant vertical pressure force

Circumferential stress × Resisting area = Pressure × Projected area

$$\sigma_c \cdot 2tl = p \cdot dl \quad \text{or} \quad \sigma_c = \frac{pd}{2t} \tag{13.1}$$

Now, consider a section cut by a transverse plane (Fig. 13.1b). For equilibrium of forces in the longitudinal direction,

Longitudinal stress × Resisting area = Pressure × Projected area

$$\sigma_l \cdot \pi dt = p \cdot \frac{\pi}{4} d^2 \quad \text{or} \quad \sigma_l = \frac{pd}{4t} \tag{13.2}$$

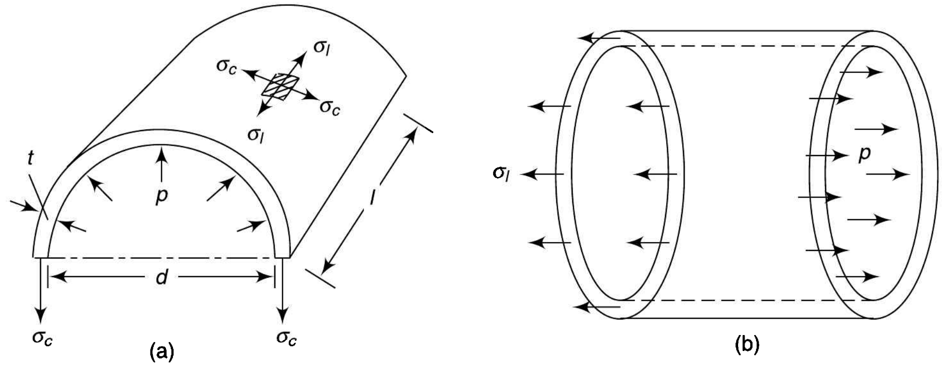


Fig. 13.1

Note that πdt is the approximate area. Actually, d here should be taken as the mean diameter.

Maximum Shear Stress

- If p is neglected, the state of stress in the wall of shell simplifies to a simple two-dimensional system with principal stresses σ_c and σ_l. Then maximum shear stress in the plane of σ_c and σ_l,

$$\tau_{\max} = \frac{1}{2}(\sigma_c - \sigma_l) = \frac{1}{2}\left(\frac{pd}{2t} - \frac{pd}{4t}\right) = \frac{pd}{8t}$$

- Similarly, the maximum shear stress in the plane of σ_c and p,

$$\tau_{\max} = \frac{1}{2}[\sigma_c - (-p)] = \frac{1}{2}\cdot(\sigma_c + p) \approx \frac{pd}{4t} \quad \text{(if } p \text{ is neglected as compared to } \sigma_c\text{)}$$

- The maximum shear stress in the plane of σ_l and p,

$$\tau_{\max} = \frac{1}{2}[\sigma_l - (-p)] = \frac{1}{2}\cdot(\sigma_l + p) \approx \frac{pd}{8t} \quad \text{(if } p \text{ is neglected as compared to } \sigma_c\text{)}$$

The greatest of these maximum shear stresses $= \dfrac{1}{2}\cdot(\sigma_c + p) \approx \dfrac{pd}{4t}$ (13.3)

Circumferential and Longitudinal Strains Let ε_c and ε_l be the circumferential and longitudinal strains respectively. Then (p is compressive),

$$\varepsilon_c = \frac{1}{E}[\sigma_c - \nu\sigma_l - (-\nu p)] = \frac{1}{E}(\sigma_c - \nu\sigma_l + \nu p) = \frac{1}{E}\left(\frac{pd}{2t} - \nu\frac{pd}{4t} + \nu p\right) = \frac{pd}{2tE}(1 - 2\nu) + \frac{\nu p}{m}$$

$$\varepsilon_l = \frac{1}{E}[\sigma_l - \nu\sigma_c + \nu p)] = \frac{1}{E}\left(\frac{pd}{4t} - \nu\frac{pd}{2t} + \nu p\right) = \frac{pd}{2tE}\left(\frac{1}{2} - \nu\right) + \frac{\nu p}{m}$$

As the effect of pressure on the strains is very small, sometimes it is ignored. Then,

$$\varepsilon_c = \frac{1}{E}[\sigma_c - \nu\sigma_l] = \frac{1}{E}\left(\frac{pd}{2t} - \nu\frac{pd}{4t}\right) = \frac{pd}{2tE}(1 - 2\nu)$$

$$\varepsilon_l = \frac{1}{E}[\sigma_l - \nu\sigma_c] = \frac{1}{E}\left(\frac{pd}{4t} - \nu\frac{pd}{2t}\right) = \frac{pd}{2tE}\left(\frac{1}{2} - \nu\right)$$

Riveted Boiler Shells If joints in the plates of a pressure vessel are to be considered, the strength of the plates is reduced owing to decrease in area due to holes in the plates. For example, in case of a boiler, there are two types of joints: longitudinal and circumferential (Fig. 13.2). If the *efficiency* of a joint is defined as the strength of the jointed plate to that of the solid plate, then the hoop and the longitudinal stresses will be given by

$$\sigma_c = \frac{pd}{2t\eta_l} \quad \text{where } \eta_l \text{ is the efficiency of longitudinal joint} \tag{13.4}$$

$$\sigma_l = \frac{pd}{4t\eta_t} \quad \text{where } \eta_t \text{ is the efficiency of circumferential joint} \tag{13.5}$$

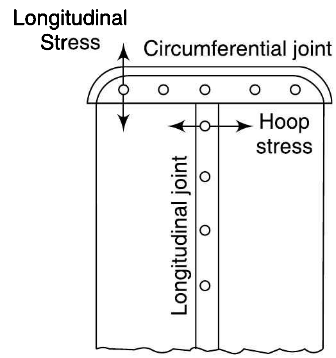


Fig. 13.2

13.2 THIN SPHERICAL SHELL

Again neglecting the radial stresses, the two principal stresses are equal due to symmetry and may be termed as σ (Fig. 13.3).

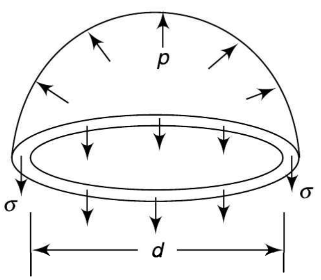


Fig. 13.3

$$\sigma \cdot \pi dt = p \cdot \frac{\pi}{4} d^2 \quad \text{or} \quad \sigma = \frac{pd}{4t} \tag{13.6}$$

13.3 THIN CYLINDER WITH SPHERICAL ENDS

Let t be the thickness of the cylindrical portion and t' of the hemispheric portion of the shell. The internal diameter may be taken as d both for the cylinder and for the spherical ends (Fig. 13.4).

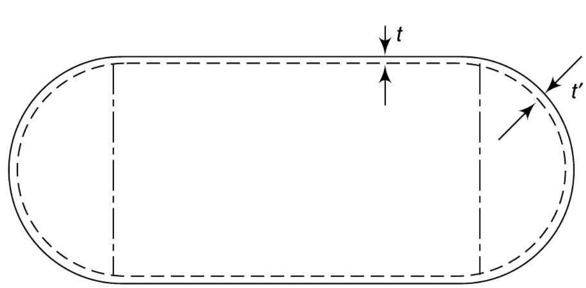


Fig. 13.4

Stresses in the Cylinder

Hoop stress, $\sigma_c = \dfrac{pd}{2t}$ and longitudinal stress, $\sigma_l = \dfrac{pd}{4t'}$

Hoop strain, $\varepsilon_c = \dfrac{1}{E}(\sigma_c - \nu\sigma_l) = \dfrac{1}{E}\left(\dfrac{pd}{2t} - \nu\dfrac{pd}{4t}\right) = \dfrac{pd}{4tE}(2-\nu)$

Stresses in the Hemispherical Portion

Hoop stress, $\sigma = \dfrac{pd}{4t'}$

Hoop strain, $\varepsilon = \dfrac{1}{E}(\sigma - \nu\sigma) = \dfrac{pd}{4t'E}(1-\nu)$

If there is no distortion of the junction under pressure,

$$\varepsilon_c = \varepsilon \quad \text{or} \quad \frac{pd}{4tE}(2-\nu) = \frac{pd}{4t'E}(1-\nu)$$

or
$$\frac{2-\nu}{t} = \frac{1-\nu}{t'} \quad \text{or} \quad \frac{t'}{t} = \frac{1-\nu}{2-\nu} \tag{13.7}$$

If ν is taken as 0.3, $\dfrac{t'}{t} = \dfrac{7}{17}$

Then $\sigma = \sigma = \dfrac{pd}{4t'} = \dfrac{pd}{4(7/17)t} = \dfrac{17pd}{28t}$ which is greater than the hoop stress in the cylinder, $\dfrac{pd}{2t}$. For maximum stress to be equal, $\dfrac{t'}{t} = \dfrac{1}{2}$

13.4 VOLUMETRIC STRAIN

The capacity of a cylinder, $V = \dfrac{\pi}{4}d^2 \cdot l$

Let the dimensions be increased by δd and δl due to internal pressure,

The volumetric strain, $\varepsilon_v = \dfrac{(d+\delta d)^2(l+\delta l) - d^2 l}{d^2 l}$

$$= \frac{(d^2 + 2d\cdot\delta d)(l+\delta l) - d^2 l}{d^2 l} = \frac{(d^2 l + d^2\cdot\delta l + 2dl\cdot\delta d) - d^2 l}{d^2 l}$$

(neglecting products of small quantities)

$$= \frac{d^2\cdot\delta l + 2dl\cdot\delta d}{d^2 l} = \frac{\delta l}{l} + \frac{2\delta d}{d}$$

= longitudinal strain + 2 × diametral strain

= longitudinal strain + 2 × hoop strain (13.7a)

(As circumference = a constant × diameter; ∴ Diametral strain = Hoop strain)

$= \varepsilon_l = 2\,\varepsilon_c$

- Volumetric strain can also be obtained by differentiating the expression, $V = \dfrac{\pi}{4}d^2 \cdot l$,

$$\delta V = \frac{\pi}{4}[d^2\cdot\delta l + 2dl(\delta d)]$$

Dividing by $\frac{\pi}{4}d^2 \cdot l$ (or V) throughout,

$$\frac{\delta V}{V} = \left(\frac{\delta l}{l} + 2 \cdot \frac{\delta d}{d}\right)$$

or $$\varepsilon_{v} = \varepsilon_l + 2\varepsilon_c$$

Neglecting the Effect of Internal Pressure

$$\text{Volumetric strain, } \varepsilon = \varepsilon_l + 2\varepsilon_c = \frac{1}{E}(\sigma_l - \nu\sigma_c) + \frac{2}{E}(\sigma_c - \nu\sigma_l)$$

$$= \frac{1}{E}\left(\frac{pd}{4t} - \nu\frac{pd}{2t}\right) + \frac{2}{E}\left(\frac{pd}{2t} - \nu\frac{pd}{4t}\right)$$

$$= \frac{pd}{2tE}(2.5 - 2\nu)$$

$$= \frac{\sigma_c}{E}(2.5 - 2\nu)$$

Considering the Effect of Internal Pressure

$$\text{Volumetric strain, } \varepsilon = \varepsilon_l + 2\varepsilon_c = \frac{1}{E}(\sigma_l - \nu\sigma_c + \nu p) + \frac{2}{E}(\sigma_c - \nu\sigma_l + \nu p)$$

$$= \frac{1}{E}\left(\frac{pd}{4t} - \nu\frac{pd}{2t} + \nu p\right) + \frac{2}{E}\left(\frac{pd}{2t} - \nu\frac{pd}{4t} + \nu p\right) = \frac{pd}{2tE}(2.5 - 2\nu) + \frac{3\nu p}{E} \qquad (13.8)$$

Similarly, for a spherical shell,

Neglecting the Effect of Internal Pressure

$$\text{Volumetric strain} = 3 \times \text{hoop strain} = \frac{3}{E}(\sigma - \nu\sigma) = \frac{3\sigma}{E}(1 - \nu) = \frac{3pd}{4tE}(1 - \nu) \qquad (13.9)$$

Considering the Effect of Internal Pressure

$$\text{Volumetric strain} = 3 \times \text{hoop strain} = \frac{3}{E}(\sigma - \nu\sigma + \nu p) = \frac{3\sigma}{E}(1 - \nu) + \frac{3\nu p}{E} = \frac{3pd}{4tE}(1 - \nu) + \frac{3\nu p}{E} \qquad (13.10)$$

Example 13.1 || A cast-iron pipe of 750-mm diameter is used to carry water under a head of 60 m. Determine the thickness of the pipe if the permissible stress is to be 20 MPa.

Solution

Given A cast-iron pipe

$d = 750$ mm $\quad h = 60$ m $\quad \sigma_c = 20$ MPa

To find Thickness of pipe

Specific weight of water, $w = 9.81$ kN/m^3

Internal pressure, $p = wh = 9.81 \times 60 = 588.6$ kN/m^2 $= 0.5886$ MPa

Determination of thickness

$$\text{Hoop stress, } \sigma_c = \frac{pd}{2t}$$

or $$20 = \frac{0.5886 \times 750}{2t} \quad \text{or} \quad t = 11.04 \text{ mm}$$

Example 13.2 A boiler of 1.6-m diameter is made of 20-mm thick steel plates. Determine the permissible steam pressure in the boiler if the efficiency of the longitudinal joint of the boiler is 80% and maximum tensile stress in the steel plates is not to exceed 80 MPa. What will be the circumferential stress in the solid plate section at this pressure? Also calculate the longitudinal stress in the plate section through the rivets of the circumferential joint if the efficiency of the joint is 70%.

Solution

Given A boiler shell

$d = 1.6$ m $\quad t = 20$ mm

$\sigma_c = 80$ MPa $\quad \eta_l = 0.8$

$\eta_c = 0.7$

To find Circumferential and longitudinal stresses

$$\sigma_c = \frac{pd}{2t\eta_l} \quad \text{or} \quad 80 = \frac{p \times 1600}{2 \times 20 \times 0.8} \quad \text{or} \quad p = 1.6 \text{ MPa}$$

Determination of stresses

At this pressure, circumferential stress in the solid plate $= \dfrac{1.6 \times 1600}{2 \times 20} = 64$ MPa

At the circumferential joint, $\sigma_l = \dfrac{pd}{4t\eta_c} = \dfrac{1.6 \times 1600}{4 \times 20 \times 0.7} = 45.7$ MPa

Example 13.3 Wall thickness of a cylindrical shell of 800-mm internal diameter and 2-m long is 10 mm. If the shell is subjected to an internal pressure of 1.5 MPa, find the following:

(i) The maximum intensity of shear stress induced
(ii) The change in dimensions of the shell

Take E = 205 GPa and ν = 0.3

Solution

Given A cylindrical shell

$d = 800$ mm $\quad t = 10$ mm

$l = 2$ m $\quad p = 1.5$ MPa

$\nu = 0.3$ $\quad E = 205$ GPa

To find

— Maximum shear stress
— change in dimensions

Maximum shear stress

$$\sigma_c = \frac{pd}{2t} = \frac{1.5 \times 800}{2 \times 10} = 60 \text{ MPa (tensile)}$$

$$\sigma_l = \frac{pd}{4t} = \frac{1.5 \times 800}{4 \times 10} = 30 \text{ MPa (tensile)}$$

Maximum shear stress at any point in the thickness of metal

$$= \frac{\sigma_c + p}{2} = \frac{60 + 1.5}{2} = 30.75 \text{ MPa} \qquad \ldots\text{(Eq. 13.3)}$$

or $$\frac{\sigma_c}{2} = \frac{60}{2} = 30 \text{ MPa if } p \text{ is neglected.}$$

Change in dimensions

$$\delta d = \varepsilon_c \cdot d = \frac{\sigma_c - \nu\sigma_l}{E} \cdot d = \frac{60 - 0.3 \times 30}{205\,000} \times 800 = \frac{51}{205\,000} \times 800 = 0.199 \text{ mm (increase)}$$

$$\delta l = \varepsilon_l \cdot l = \frac{\sigma_l - \nu\sigma_c}{E} \cdot l = \frac{30 - 0.3 \times 60}{205\,000} \times 2000 = \frac{12}{205\,000} \times 2000 = 0.117 \text{ mm (increase)}$$

$$\delta \nu = (\varepsilon_l + 2\varepsilon_c) \cdot V \qquad \text{(Eq. 13.7a)}$$

$$= \left(\frac{12 + 2 \times 51}{205\,000}\right) \times \frac{\pi}{4} \times 800^2 \times 2000 = 559\,050 \text{ mm}^3 \text{ (increase)}$$

Example 13.4 || A closed-end copper tube of 72-mm internal diameter, 800-mm long and 2 mm thick is filled with water under pressure. Find the change in pressure if additional volume of 4000 mm³ of water is pumped into the tube. Neglect any distortion of the end plates. Take E = 102 GPa, K = 2200 MPa and Poisson's ratio = 0.3.

Solution

Given A closed-end copper tube

$d = 72$ mm	$t = 2$ mm
$l = 800$ mm	$K = 2200$ MPa
$\nu = 0.3$	$E = 102$ GPa

To find Change in pressure on addition of 4000 mm³ of water

When additional volume of water is pumped, the volume of tube is increased due to increase in pressure, whereas the volume of water in the tube is decreased due to compression. Let p be the increase in pressure.

Internal volume of cylinder $= \frac{\pi}{4} \times 72^2 \times 800 = 3.257 \times 10^6$ mm³

For tube

$$\sigma_c = \frac{pd}{2t} = \frac{p \times 72}{2 \times 2} = 18\,p \text{ MPa (tensile)}$$

and $$\sigma_l = \frac{pd}{4t} = 9\,p \text{ MPa (tensile)}$$

Increase in volume, $\delta\nu = (\varepsilon_l + 2\varepsilon_c) \cdot V$

$$= \frac{V}{E}[(\sigma_l - \nu \cdot \sigma_c) + 2(\sigma_c - \nu \cdot \sigma_l)]$$

$$= \frac{V}{E}[(9 - 0.3 \times 18) + 2(18 - 0.3 \times 9)]p$$

$$= \frac{34.2pV}{E}$$

For water

Decrease in volume, $\delta v = \varepsilon_w \cdot V = \frac{p}{K} \cdot V$

Thus additional volume of water $= \frac{34.2pV}{E} + \frac{p}{K}V$

or $$4000 = p\left(\frac{34.2}{102\,000} + \frac{1}{2200}\right) \times 3.257 \times 10^6 \quad \text{or} \quad p = 1.555 \text{ MPa}$$

Example 13.5 A cylindrical tank is 3 m in length, 2.4 m in diameter and 15 mm in thickness. Its flat ends are joined by 12 equally spaced tie bars, each tie bar being 50 mm in diameter. Initially, when the tank is filled with water, the tie bars are stressed to 48 MPa. Determine the increase in the capacity of the tank and the resultant stress in the tie bars when the pressure is raised to 1.5 MPa. E = 208 GPa (both for tank and tie bars) and Poisson's ratio = 0.3.

Solution

Given A cylindrical tank as shown in Fig. 13.5.

$d = 2.4$ m $\quad t = 15$ mm

$l = 3$ m $\quad p = 1.5$ MPa

$\nu = 0.3 \quad d_{\text{tie}} = 50$ mm

$E = 208$ GPa

To find

— Increase in capacity of tank

— resultant stress in tie bars

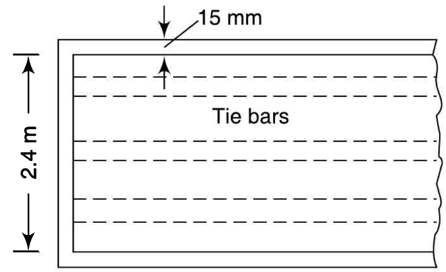


Fig. 13.5

Before the rise in water pressure

Let σ_l = longitudinal stress in the cylinder walls

σ_c = hoop or circumferential stress in the cylinder walls = 0

σ_t = tensile stress in the tie bars = 48 MPa

Equilibrium equation

Force in tie bars = Force in the tank wall

$\sigma_l (\pi \times \text{circumference} \times \text{thickness}) = \sigma_t \times \text{area of tank wall}$

$$\sigma_l (\pi \times 2400 \times 15) = 48 \times 12\left(\frac{\pi}{4} \times 50^2\right) \quad \text{or} \quad \sigma_l = 10 \text{ MPa}$$

As the tie bars are in tension, the walls are in compression.

After the rise in water pressure

Let σ_l' = longitudinal stress in the cylinder walls

σ_c' = hoop or circumferential stress in the cylinder walls

σ_t' = tensile stress in the tie bars

Now the tank walls will be in tension.

Equilibrium equation

Force in tie bars + Force in the tank wall = Force due to pressure rise

= Pressure rise (total area – area of tie bars)

$$\sigma_t' \times 12\left(\frac{\pi}{4} \times 50^2\right) + \sigma_l'(\pi \times 2400 \times 15) = 1.5\left(\frac{\pi}{4} \times 2400^2 - 12 \times \frac{\pi}{4} \times 50^2\right)$$

or $$\sigma_t' + 4.8\sigma_l' = 286.5 \qquad \text{(i)}$$

Hoop stress in the cylinder, $\sigma_c' = \dfrac{pd}{2t} = \dfrac{1.5 \times 2400}{2 \times 15} = 120$ MPa

Increase in capacity of tank

Increase in longitudinal strain is to be the same for the tie bars and the cylinder.

$$\frac{\sigma_t' - \sigma_t}{E} = \frac{(\sigma_l' - \nu\sigma_c') - (\sigma_l - \nu\sigma_c)}{E}$$

$$\sigma_t' - 48 = (\sigma_l' - 0.3 \times 120) - (-10 - 0.3 \times 0)$$

$$\sigma_t' - \sigma_l' = 22 \tag{ii}$$

Subtracting (ii) from (i), $5.8\sigma_l' = 264.5$ or $\sigma_l' = 45.6$ MPa

and $\sigma_t' = 45.6 + 22 = 67.6$ MPa

Volumetric strain = Increase of longitudinal strain + 2 × increase of hoop strain

$$= \frac{1}{E}[(67.6 - 48) + 2\{120 - 0.3 \times 45.6 - 0.3 \times 10\}] = \frac{226.24}{E}$$

$$\text{Increase in capacity } = \frac{226.24}{208\,000} \times \frac{\pi \times 2400^2}{4} \times 3000 = 14.76 \times 10^6 \text{ mm}^3$$

Example 13.6 || A steel tube of inside and outside diameters as 36 mm and 40 mm respectively is firmly plugged at both ends. Internal length between the flat ends of the plugs is 220 mm. The plugs are so designed that water can be admitted into the tube through them as well as an axial pull can be applied to the tube. Initially, the tube is filled with water at a pressure of 2 MPa and also an axial pull of 22 kN is applied to the tube through the plugs. Determine the volume of water that will escape from the tube if the axial pull is removed and the pressure inside the tube is made to fall upto atmospheric pressure. Take E = 205 GPa, K = 2100 MPa and Poisson's ratio = 0.28.

Solution

Given A steel tube

$d_i = 36$ mm	$d_o = 40$ mm
$l = 220$ mm	$p = 2$ MPa
$\nu = 0.28$	$F = 22$ kN
$E = 205$ GPa	$K = 2100$ MPa

To find Volume of water to escape from tube on removing the axial pull

$$t = \frac{40 - 36}{2} = 2 \text{ mm}$$

$$\text{Internal volume of cylinder} = \frac{\pi}{4} \times 36^2 \times 220 = 223.9 \times 10^3 \text{ mm}^3$$

Initial stresses

$$\sigma_l = \frac{(\pi/4) \times 36^2 \times 2 + 22\,000}{(\pi/4)(40^2 - 36^2)} = 100.7 \text{ MPa}$$

$$\sigma_c = \frac{pd}{2t} = \frac{2 \times 36}{2 \times 2} = 18 \text{ MPa}$$

On removing the axial pull

On removing the axial pull, there is decrease in the volume of tube,

$$\delta v = (\varepsilon_l + 2\varepsilon_c) \cdot V = \frac{V}{E}[(\sigma_l - \nu \cdot \sigma_c - \nu \cdot p) + 2(\sigma_c - \nu \cdot \sigma_l - \nu \cdot p)]$$

$$= \frac{V}{E}[(100.7 - 0.28 \times 18 + 0.28 \times 2) + 2(18 - 0.28 \times 100.7 + 0.28 \times 2)]$$

$$= \frac{223.9 \times 10^3}{205\,000}[96.22 - 19.27] = 84 \text{ mm}^3$$

Increase in the volume of water, $\delta v = \varepsilon_w \cdot V = \frac{p}{K} \cdot V = \frac{2}{2100} \times 223.9 \times 10^3 = 213.2 \text{ mm}^3$

On removing the axial pull, as the volume of water increases and the volume of tube decreases, the volume of water that will escape = 213.2 + 84 = 297.2 mm^3.

Example 13.7 A 600-mm long steel cylinder is made up of 4-mm thick plates. The inside diameter of the cylinder is 120 mm. When it is subjected to an internal pressure of 5 MPa, the increase in its volume is found to be 5000 mm^3. Determine the value of Poisson's ratio and the modulus of rigidity. E = 205 GPa.

Solution

Given A steel cylinder

d = 120 mm	t = 4 mm
l = 600 mm	p = 5 MPa
dV = 5000 mm^3	E = 205 GPa

To find Poisson's ratio and modulus of rigidity

Determination of Poisson's ratio

Volumetric strain,

$$\frac{dV}{V} = \frac{pd}{2tE}(2.5 - 2\nu) + \frac{3\nu p}{E}$$

or

$$dV = \frac{V}{E}\left[\frac{pd}{2t}(2.5 - 2\nu) + 3\nu p\right]$$

$$5000 = \frac{(\pi/4) \times 120^2 \times 600}{205\,000}\left[\frac{5 \times 120}{2 \times 4}(2.5 - 2\nu) + 3\nu \times 5\right]$$

$$= 33.1017[187.5 - 150\nu + 15\nu] = 6206.6 - 4468 \cdot 7\,\nu \quad \text{or} \quad \nu = 0.27$$

Modulus of rigidity

$$G = \frac{E}{2(1+\nu)} = \frac{205\,000}{2(1+0.27)} = 80\,709 \text{ MPa} \quad \text{or} \quad 80.709 \text{ GPa}$$

Neglecting effect of internal pressure

If the effect of internal pressure on strain is ignored,

$$dV = V \cdot \frac{pd}{2tE}(2.5 - 2\nu)$$

$$5000 = 33.1017[187.5 - 150\nu]$$

$$= 2482.6(2.5 - 2\nu) = 6206.6 - 4965.3\nu \quad \text{or} \quad \nu = 0.243$$

$$G = \frac{E}{2(1+\nu)} = \frac{205\,000}{2(1+0.243)} = 82\,462 \text{ MPa} \quad \text{or} \quad 82.462 \text{ GPa}$$

Example 13.8 A thin cylinder of 200-mm inside diameter is 4-mm thick. The ends of the cylinder are closed by rigid plates and then it is filled with water under pressure. If an external axial pull of 75 kN is applied to the ends, the water pressure falls by 0.12 MPa. Find the value of the Poisson's ratio. K = 2100 MPa and E = 150 GPa.

Solution

Given A thin cylinder as shown in Fig. 13.6.

$d = 200$ mm $\qquad t = 4$ mm

$F = 75$ kN $\qquad p = 0.12$ MPa

$K = 2100$ MPa $\qquad E = 105$ GPa

To find Poisson's ratio

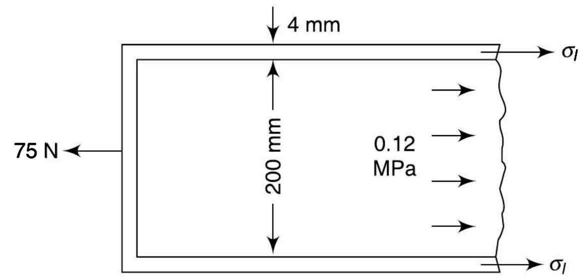


Fig. 13.6

It may be assumed that the cylinder remains full of water.

Considering the change of stresses.

Reduction in the hoop stress

$$\sigma_c = \frac{pd}{2t} \quad \text{where } p \text{ is the reduction in the pressure}$$

or
$$\sigma_c = \frac{pd}{2t} = \frac{0.12 \times 200}{2 \times 4} = 3 \text{ MPa} \qquad \text{(compressive)}$$

Increase in the longitudinal stress

$$\sigma_l(\pi dt) + p \cdot \frac{\pi}{4} d^2 = F$$

or
$$\sigma_l(\pi \times 200 \times 4) + 0.12 \times \frac{\pi}{4} \times 200^2 = 75\,000$$

$$\sigma_l = 28.3 \text{ MPa}$$

Determination of Poisson's ratio

Increase in volume of water = increase in the volume of cylinder

or volumetric strain = longitudinal strain + 2 × hoop strain

or
$$\frac{p}{K} = \frac{\sigma_l - \nu(-\sigma_c)}{E} + \frac{2[-\sigma_c - \nu(\sigma_l)]}{E}$$

or
$$\frac{0.12}{2100} = \frac{1}{150\,000}[28.3 + 3\nu + 2(-3 - 28.3\nu)]$$

or
$$53.6\nu = 22.3 - 8.57 \qquad \text{or} \qquad \nu = 0.256$$

Example 13.9 || A spherical shell of 1.2-m internal diameter and 6-mm thickness is filled with water under pressure until the volume is increased by 400×10^3 mm^3. Find the pressure exerted by water on the shell. Take E = 204 GPa and ν = 0.3.

Solution

Given A spherical shell

$d = 1.2$ m $\qquad t = 6$ mm

$\nu = 0.3$ $\qquad \delta V = 400 \times 10^3$ mm^3

$E = 204$ GPa

To find Pressure on the shell surface

$$\text{Internal volume of shell} = \frac{\pi}{6} \times 1.2^3 = 0.9048 \text{ m}^3 = 904.8 \times 10^6 \text{ mm}^3$$

Hoop strain

Volumetric strain = 3 × hoop strain $\qquad$ ...(Eq. 13.9) or $$3\varepsilon = \frac{\delta V}{V} = \frac{400 \times 10^3}{904.8 \times 10^6} = 442.1 \times 10^{-6}$$

or $$\varepsilon = 147.4 \times 10^{-6}$$

Hoop stress

Neglecting the radial stress, $\varepsilon = \dfrac{\sigma - \nu\sigma}{E} = \dfrac{\sigma(1-\nu)}{E}$

or $$147.4 \times 10^{-6} = \frac{\sigma(1-0.3)}{204\,000} \quad \text{or} \quad \sigma = 42.96 \text{ MPa}$$

Radial pressure

$$\sigma_c = \frac{pd}{4t} \quad \text{or} \quad 42.96 = \frac{p \times 1200}{4 \times 6} \quad \text{or} \quad p = 0.86 \text{ MPa}$$

Example 13.10 Compare the maximum tensile stresses of a thin cylinder to that of a thin spherical shell having the same internal pressure and the *d*/*t* ratio where *d* and *t* are the internal diameter and the thickness respectively. Also find the ratio of their proportional increases in volumes. Take Poisson's ratio as 0.3.

Solution

Given A thin cylinder and a thin spherical shell with same *d*/*t* ratio

$\nu = 0.3$

To find

— To compare maximum tensile stresses

— ratio of proportional increase in volumes

Tensile stresses

For cylinder, $\sigma_c = \dfrac{pd}{2t}$

For spherical shell, $\sigma_c' = \dfrac{pd}{4t}$

$\therefore$ Ratio, $\dfrac{\sigma_c}{\sigma_c'} = 2$

Increase in volumes

For cylinder,
$$\delta v = (\varepsilon_l + 2\varepsilon_c) \cdot V$$
$$= \frac{V}{E} \cdot \frac{pd}{4t}[(1 - 2 \times 0.3) + 2(2 - 1 \times 0.3)]$$
$$= \frac{V}{E} \cdot \frac{3.8pd}{4t}$$

For spherical shell,
$$\delta v' = \frac{V}{E} \cdot \frac{pd}{4t}[(1 - 0.3) + 2(1 - 0.3)]$$
$$= \frac{V}{E} \cdot \frac{2.1pd}{4t}$$

$\therefore$ ratio, $$\frac{\delta v}{\delta v'} = \frac{3.8}{2.1} = 1.81$$

Example 13.11 || A cylindrical boiler drum has hemispherical ends. The cylindrical portion is 1.6-m long, 800 mm in diameter and 20-mm thick. After filling it with water at atmospheric pressure, it is put on a hydraulic test and the pressure is raised to 12 MPa. Find the additional volume of water required to be filled in the drum at this pressure. Assume the hoop strain at the junction of cylinder and the hemisphere to be the same for both. E = 205 GPa, K = 2080 MPa and Poisson's ratio = 0.3.

Solution

Given A cylindrical boiler drum having hemispherical ends

$d = 800$ mm $t = 20$ mm

$l = 1.6$ m $p = 12$ MPa

$\nu = 0.3$ $E = 205$ GPa

$K = 2080$ MPa

To find Additional volume of water

For cylinder

Hoop stress, $\sigma_c = \dfrac{pd}{2t} = \dfrac{12 \times 800}{2 \times 20} = 240$ MPa

Longitudinal stress, $\sigma_l = \dfrac{pd}{4t} = \dfrac{12 \times 800}{4 \times 20} = 120$ MPa

Hoop strain, $\varepsilon_c = \dfrac{1}{E}(\sigma_c - \nu\sigma_l) = \dfrac{1}{E}(240 - 0.3 \times 120) = \dfrac{204}{E}$

Longitudinal strain, $\varepsilon_l = \dfrac{1}{E}(\sigma_l - \nu\sigma_c) = \dfrac{1}{E}(120 - 0.3 \times 240) = \dfrac{48}{E}$

Increase in capacity, $= (2\varepsilon_c + \varepsilon_l)V = \dfrac{2 \times 204 + 48}{205\,000} \times \dfrac{\pi}{4} \times (800)^2 \times 1600$

$= 2.224 \times 10^{-3} \times 804.3 \times 10^6 = 1.789 \times 10^6$ mm^3

For two hemispherical ends

Hoop strain, $\varepsilon_c = \dfrac{204}{E}$ (same as for cylinder)

Increase in capacity, $= 3\varepsilon \cdot V = \dfrac{3 \times 204}{205\,000} \times \dfrac{\pi}{6} \times (800)^3$

$= 2.985 \times 10^{-3} \times 268.1 \times 10^6 = 0.8 \times 10^6$ mm^3

Decrease in volume of water

$$\delta_\nu = \frac{p}{K} \times \text{Volume}$$

$$= \frac{12}{2080} \times \left[\frac{\pi}{4} \times (800)^2 \times 1600 + \frac{\pi}{6} \times (800)^3\right]$$

$$= \frac{12}{2080} \times [804.3 \times 10^6 + 268.1 \times 106]$$

$$= 6.187 \times 10^6 \text{ mm}^3$$

Additional volume required

Additional volume of water required in the drum =

$1.789 \times 10^6 + 0.8 \times 10^6 + 6.187 \times 10^6 = 8.776 \times 10^6$ mm^3 at atmospheric pressure

Example 13.12 A thin cylindrical tube with closed ends is subjected to an internal pressure of 8 MPa. The tube is of 90-mm internal diameter and 6-mm thickness. Determine the maximum and minimum principal stresses and the maximum shear stress if a torque of 4000 N · m is also applied to the tube.

Solution

Given A thin cylindrical tube

$d = 90$ mm $\qquad t = 6$ mm

$p = 8$ MPa $\qquad T = 4000 \times 10^3$ N·mm

To find

— Principal stresses

— Maximum shear stress

Determination of σ_c, σ_l and v

Mean diameter = 90 + 6 = 96 mm

Hoop stress, $\sigma_c = \dfrac{pd}{2t} = \dfrac{8 \times 90}{2 \times 6} = 60$ MPa

Longitudinal stress, $\sigma_l = \dfrac{pd}{4t} = \dfrac{8 \times 90}{4 \times 6} = 30$ MPa

Torque applied = shear force on transverse plane × mean radius

= shear stress × Area × mean radius

$4000 \times 10^3 = \tau \times (\pi \times 96 \times 6) \times 48$

or $\tau = 46$ MPa

Principal stresses

Principal stresses, $\sigma = \dfrac{1}{2}(\sigma_x + \sigma_y) \pm \dfrac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$

$= \dfrac{1}{2}(60 + 30) \pm \dfrac{1}{2}\sqrt{(60 - 30)^2 + 4 \times 46^2} = 45 \pm 48.4$

$= 93.4$ MPa and -3.4 MPa

Maximum shear stress

Maximum shear stress, $\sigma = \dfrac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$

$= \dfrac{1}{2}\sqrt{(60 - 30)^2 + 4 \times 46^2} = 48.4$ MPa

or Maximum shear stress, $\sigma = \dfrac{\sigma_1 - \sigma_2}{2} = \dfrac{93.4 - (-3.4)}{2} = 48.4$ MPa

13.5 WIRE WINDING OF THIN CYLINDERS

A tube can be strengthened against the internal pressure by winding it with wire under tension and putting the tube wall in compression. As the pressure is applied, the resultant hoop stress produced is much less as it would have been in the absence of wire. The maximum stress will be in the wire which is made of a high-tensile material.

The analysis of wire wound cylinders is made on the assumption that one layer of wire of diameter d is closely wound on the tube with an initial tension T. The procedure is as follows:

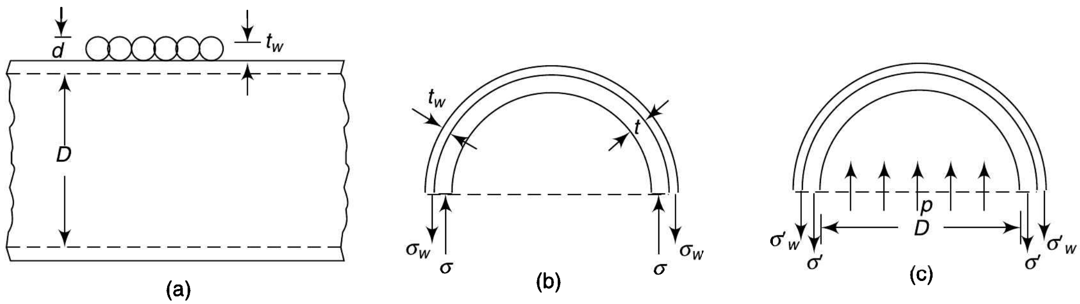


Fig. 13.7

- Initial tensile stress in the wire, $\sigma_w = \dfrac{T}{(\pi/4)d^2} = \dfrac{4T}{\pi d^2}$
- Replace the wire by a wire of rectangular cross-section of thickness t_w and width d having the same cross-sectional area as of circular wire (Fig. 13.7a). Thus

$$t_w d = \frac{\pi}{4} d^2 \quad \text{or} \quad t_w = \frac{\pi d}{4}$$

  Thus now the cylinder is assumed to be wound with a rectangular wire of width d and thickness t.
- *For unit axial length of the cylinder*
  The initial compressive hoop stress σ in the cylinder can be found by equating the compressive circumferential force in the cylinder to tensile force in the wire for a unit axial length (Fig. 13.7b), i.e.,

$$(t \cdot 1)\,\sigma = (t_w \cdot 1)\,\sigma_w \quad \text{or} \quad \sigma = \frac{t_w \sigma_w}{t} \tag{13.11}$$

- *Stresses due to fluid pressure alone*
  On applying an internal pressure p, let the stresses be σ' tensile (hoop) in the cylinder and σ'_w tensile in the wire due to fluid pressure alone (Fig. 13.7c). Then for equilibrium.
  Resisting force in the cylinder and wire = Fluid force on projected area

$$(2t \cdot 1)\sigma' + (2t_w \cdot 1)\sigma'_w = pD$$

or

$$2t\sigma' + 2t_w\sigma'_w = pD \tag{13.12}$$

- Equating the circumferential strains of the wire and the cylinder,

$$\frac{(\sigma' - \nu\sigma_l)}{E} = \frac{\sigma'_w}{E_w} \tag{13.13}$$

  On solving Eqs. 13.12 and 13.13 simultaneously, σ' and σ'_w can be determined.
  Final stresses are calculated by taking the algebraic sum of the initial stresses and stresses due to fluid pressure, i.e.,
  Final stresses: In the pipe = $\sigma' - \sigma$
  In the wire = $\sigma'_w + \sigma_w$

Example 13.13 A cast-iron pipe of 240-mm inside diameter and 15-mm thickness is closely wound with a layer of 6-mm diameter steel wire under a stress of 35 MPa. Find the stresses developed in the pipe and the steel wire when water is admitted into the pipe at a pressure of 3 MPa.

For steel E = 204 GPa; For cast iron E = 102 GPa; Poisson's ratio = 0.3

Solution

Given A cast-iron pipe

$D = 240$ mm	$t = 15$ mm
$d = 6$ mm	$\sigma_w = 35$ MPa
$p = 3$ MPa	$E_s = 204$ GPa
$E_{ci} = 102$ GPa	$\nu = 0.3$

To find Stresses in the pipe and the steel wire

$$E_s = 2\,E_{ci}$$

Initial stresses

Equivalent wire thickness, $t_w = \dfrac{\pi d}{4} = \dfrac{\pi \times 6}{4} = 4.712$ mm

Initial compressive hoop stress in the tube, $\sigma = \dfrac{t_w}{t}\sigma_w = \dfrac{4.712}{15} \times 35 = 11$ MPa

Stresses due to fluid pressure alone

On applying an internal pressure p, let the stresses be σ' tensile (hoop) in the tube and σ'_w tensile in the wire due to pressure alone. Then for equilibrium,

$$2t\sigma' + 2t_w\sigma'_w = pD \qquad \text{...(Eq.13.12)}$$

or $\quad 2 \times 15\sigma' + 2 \times 4.712\,\sigma'_w = 3 \times 240$

or $\quad 30\sigma' + 9.424\,\sigma'_w = 720 \quad$ or $\quad \sigma' = -0.314\,\sigma'_w + 24 \qquad$ (i)

Equating the circumferential strains of the wire and the tube,

$$\frac{(\sigma' - \nu\sigma_l)}{E_{ci}} = \frac{\sigma'_w}{E_s} \qquad \text{...(Eq.13.13)}$$

where $\quad \sigma_l = \dfrac{pD}{4t} = \dfrac{3 \times 240}{4 \times 15} = 12$ MPa

$\therefore \quad \dfrac{(\sigma' - 0.3 \times 12)}{E_{ci}} = \dfrac{\sigma'_w}{2E_{ci}} \quad$ or $\quad \sigma' = 0.5\,\sigma'_w + 3.6 \qquad$ (ii)

From (i) and (ii), $0.5\,\sigma'_w + 3.6 = -0.314\,\sigma'_w + 24$

or $\quad \sigma'_w = 25.1$ MPa

and $\quad \sigma' = 0.5 \times 25.1 + 3.6 = 16.1$ MPa

Final stresses

In the pipe $= 16.1 - 11 = 5.1$ MPa

In the wire $= 25.1 + 35 = 60.1$ MPa

Example 13.14 The internal and external diameters of a copper tube are 42 mm and 45 mm respectively. It is closely wound with a steel wire of 1 mm diameter. Determine the required tension in the wire so that an internal pressure of 2 MPa produces a tensile circumferential stress of 8 MPa in the tube. $E_s = 1.65\,E_c$.

Solution

Given A cast-iron pipe

$D_i = 42$ mm	$D_o = 45$ mm
$d = 1$ mm	$p = 2$ MPa
$\sigma_c = 8$ MPa	$E_s = 1.65E_c$

To find Wire tension

Thickness of tube $= t = \dfrac{45-42}{2} = 1.5$ mm

Initial stresses

Equivalent wire thickness, $t_w = \dfrac{\pi d}{4} = \dfrac{\pi \times 1}{4} = 0.785$ mm

Initial compressive hoop stress in the tube, $\sigma = \dfrac{t_w}{t}\sigma_w = \dfrac{0.785}{1.5}\sigma_w = 0.524\sigma_w$

Stresses due to fluid pressure alone

On applying an internal pressure p, let the stresses be σ' tensile (hoop) in the tube and σ'_w tensile in the wire due to pressure alone. Then for equilibrium,

$$2t\sigma' + 2t_w\sigma'_w = pD_i \quad \text{...(Eq.13.12)}$$

or $\quad 2 \times 1.5\sigma' + 2 \times 0.785\sigma'_w = 2 \times 42$

or $\quad 3\sigma' + 1.57\sigma'_w = 84 \quad$ (i)

Equating the circumferential strains of the wire and the tube,

$$\frac{(\sigma' - \nu\sigma_l)}{E} = \frac{\sigma'_w}{E_w} \quad \text{...(Eq.13.13)}$$

Neglecting longitudinal stress in the tube as Poisson's ratio is not given,

$$\frac{\sigma'}{E_c} = \frac{\sigma'_w}{1.65E_c} \quad \text{or} \quad \sigma'_w = 1.65\sigma'$$

$\therefore$ from (i), $\quad 3\sigma' + 1.57 \times 1.65\sigma' = 84$

or $\quad \sigma' = 15.03$ MPa

Determination of wire tension

Final stress in the tube = 15.03 – initial stress

or $\quad 8 = 15.03 + (-0.524\sigma_w)$

or $\quad \sigma_w = 13.41$ MPa

Winding tension $= 13.41 \times \dfrac{\pi}{4} \times 1^2 = 10.53$ N

13.6 THICK CYLINDERS (LAME'S THEORY)

The analysis of thick cylinders is usually based on Lame's theory and for which assumptions made are as follows:

- The material of the cylinder is homogeneous and isotropic.
- The material is stressed within elastic limits.
- Plane sections normal to the longitudinal axis of the cylinder remain plane after the application of pressure.
- Young's modulus is the same in tension and compression.
- All fibres are free to expand or contract under the action of forces irrespective of action of adjacent fibres.

When a thick cylinder is acted upon by internal radial pressures at the surfaces, three principal stresses are

(i) radial (p) (compressive)

(ii) circumferential (σ_c) (tensile)
(iii) longitudinal (σ_l) (tensile)

In a thick cylinder, the stresses vary over any cross-section. Longitudinal strain (ε) may be considered constant implying that the cross-sections remain plane after straining. However, this will be true for the sections far away from the end sections.

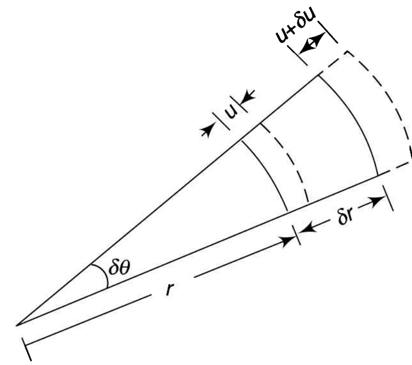


Fig. 13.8

Let u be the radial shift at an unstrained radius r, i.e., r becomes $r+u$ after straining. Similarly, $u+\delta u$ be the radial shift at an unstrained radius $r+\delta r$ (Fig. 13.8).

- Longitudinal strain, $\varepsilon = \dfrac{\sigma_l - (\nu\sigma_c - \nu p)}{E}$

  $\therefore$ longitudinal stress, $E\varepsilon = \sigma_l - \nu(\sigma_c - p)$ ...(tensile)

  Differentiating it (E and ε is constant),

$$0 = \frac{d\sigma_l}{dr} - \nu\left(\frac{d\sigma_c}{dr} - \frac{dp}{dr}\right)$$

or
$$\frac{d\sigma_l}{dr} = \nu\left(\frac{d\sigma_c}{dr} - \frac{dp}{dr}\right) \qquad \text{(i)}$$

- Radial strain $= \dfrac{\text{Increase in } \delta r}{\delta r} = \dfrac{u + \delta u - u}{dr} = \dfrac{du}{dr}$ in the limit

  Also radial strain $= \dfrac{p + \nu(\sigma_c + \sigma_l)}{E}$ ... (p is compressive and σ_c and σ_l are tensile)

  $\therefore$ radial stress, $E \cdot \dfrac{du}{dr} = p + \nu(\sigma_c + \sigma_l)$ ...(compressive) (ii)

- Circumferential (hoop) strain $= \dfrac{\text{Increase in circumference}}{\text{original circumference}} = \dfrac{2\pi(r+u) - 2\pi r}{2\pi r} = \dfrac{u}{r}$

  $\therefore$ circumferential stress $= E \cdot \dfrac{u}{r} = \sigma_c - \nu\sigma_l + \nu p$ (tensile)

  or $E \cdot u = r(\sigma_c - \nu\sigma_l + \nu p)$

  Differentiating it,

$$E \cdot \frac{du}{dr} = \sigma_c - \nu(\sigma_l - p) + r\left(\frac{d\sigma_c}{dr} - \nu\frac{d\sigma_l}{dr} + \nu\frac{dp}{dr}\right) \qquad \text{(iii)}$$

From (ii) and (iii),

$$-[p + \nu(\sigma_c + \sigma_l)] = \sigma_c - \nu(\sigma_l - p) + r\left(\frac{d\sigma_c}{dr} - \nu\frac{d\sigma_l}{dr} + \nu\frac{dp}{dr}\right)$$

$$(p + \sigma_c)(1 + \nu) + r\left(\frac{d\sigma_c}{dr} - \nu\frac{d\sigma_l}{dr} + \nu\frac{dp}{dr}\right) = 0 \qquad \text{(iv)}$$

Substituting the value of $\dfrac{d\sigma_l}{dr}$ from (i) in (iv),

$$(p + \sigma_c)(1 + \nu) + r\left[\frac{d\sigma_c}{dr} - \nu^2\left(\frac{d\sigma_c}{dr} - \frac{dp}{dr}\right) + \nu\frac{dp}{dr}\right] = 0$$

or $$(p+\sigma_c)(1+\nu)+r\left[(1-\nu^2)\frac{d\sigma_c}{dr}+\nu(1+\nu)\frac{dp}{dr}\right]=0$$

Cancelling $(1+\nu)$ throughout,

$$(p+\sigma_c)+r\left[(1-\nu)\frac{d\sigma_c}{dr}+\nu\frac{dp}{dr}\right]=0 \qquad \text{(v)}$$

- Radial equilibrium equation

For a unit length of element (Fig. 13.9),
Outward radial force on inner face $= p(r \cdot \delta\theta)$
Inward radial force on outer face $= (p + \delta p)(r + \delta r) \cdot \delta\theta$
Inward radial components of circumferential force

$$= 2\sigma_c \cdot (\delta r \cdot 1)\sin(\delta\theta/2) \approx \sigma_c \cdot \delta r \cdot \delta\theta$$

For equilibrium,

$$(p+\delta p)(r+\delta r)\cdot\delta\theta - p(r\cdot\delta\theta)+\sigma_c\cdot\delta r\cdot\delta\theta=0$$

or $$(p+\delta p)(r+\delta r)-p\cdot r+\sigma_c\cdot\delta r=0$$

Simplifying and taking limits,

$$\sigma_c+p+r\cdot\frac{dp}{dr}=0$$

or $$\sigma_c+p=-r\cdot\frac{dp}{dr} \qquad \text{(vi)}$$

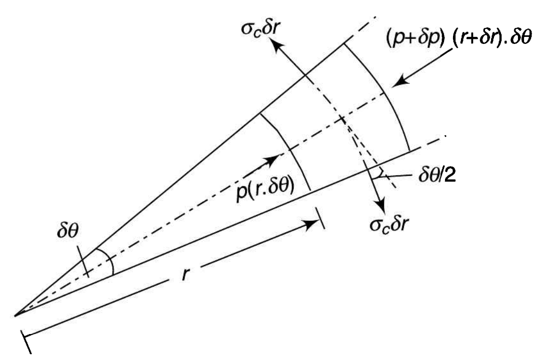


Fig. 13.9

Substituting the value of $(\sigma_c + p)$ from (vi) in (v),

$$-r\cdot\frac{dp}{dr}+r\left[(1-\nu)\frac{d\sigma_c}{dr}+\nu\frac{dp}{dr}\right]=0$$

or $$-\frac{dp}{dr}+(1-\nu)\frac{d\sigma_c}{dr}+\nu\frac{dp}{dr}=0$$

or $$\frac{d\sigma_c}{dr}-\frac{dp}{dr}=0$$

Integrating, $\sigma_c - p = \text{constant} = 2a$ (say) or $\sigma_c = p + 2a$ (vii)

From (vi), $p+2a+p=-r\cdot\dfrac{dp}{dr}$ or $2p+r\cdot\dfrac{dp}{dr}=-2a$

or $$\frac{1}{r}\cdot\frac{d}{dr}(pr^2)=-2a \quad \text{or} \quad \frac{d}{dr}(pr^2)=-2ar$$

Integrating, $pr^2 = -ar^2 + B$

or $$p=-a+\frac{B}{r^2} \quad \text{or} \quad p=-a+\frac{b}{d^2} \text{ where } b=4B \qquad \text{(viii)}$$

From (vii),

$$\sigma_c=-a+\frac{b}{d^2}+2a=a+\frac{b}{d^2} \qquad \text{(ix)}$$

a and b are the constants which depend upon the dimensions and loading conditions.

If the cylinder is acted upon by an internal pressure p_i at diameter d_i and an external pressure p_o at d_o. The radial stresses at these surfaces must be equal to the applied pressures. Then

$$p_i=-a+\frac{b}{d_i^2} \text{ and } p_o=-a+\frac{b}{d_o^2} \qquad (13.14)$$

Subtracting, $p_i - p_o = b\left(\frac{1}{d_i^2} - \frac{1}{d_o^2}\right) = b\left(\frac{d_o^2 - d_i^2}{d_i^2 d_o^2}\right)$

or $$b = \frac{d_i^2 d_o^2 (p_i - p_o)}{d_o^2 - d_i^2} \quad \text{and} \quad a = -p_i + \frac{d_o^2 (p_i - p_o)}{d_o^2 - d_i^2} = \frac{p_i d_i^2 - p_o d_o^2}{d_o^2 - d_i^2}$$

$\therefore$ from (viii), $$p = -\frac{p_i d_i^2 - p_o d_o^2}{d_o^2 - d_i^2} + \frac{d_i^2 d_o^2 (p_i - p_o)}{(d_o^2 - d_i^2) d^2}$$

$$= \frac{p_o d_o^2 - p_i d_i^2 + (d_i^2 d_o^2 / d^2)(p_i - p_o)}{d_o^2 - d_i^2} \tag{13.15}$$

and from (ix), $$\sigma_c = \frac{p_i d_i^2 - p_o d_o^2}{d_o^2 - d_i^2} + \frac{d_i^2 d_o^2 (p_i - p_o)}{(d_o^2 - d_i^2) d^2}$$

$$= \frac{p_i d_i^2 - p_o d_o^2 + (d_i^2 d_o^2 / d^2)(p_i - p_o)}{d_o^2 - d_i^2} \tag{13.16}$$

$$\text{Maximum shear stress} = \frac{1}{2}(\sigma_c + p) = \frac{d_i^2 d_o^2 (p_i - p_o)}{(d_o^2 - d_i^2) d_i^2} \text{ (at inner diameter)} \tag{13.17}$$

Internal pressure only

$$p = \frac{-p_i d_i^2 d^2 + p_i d_i^2 d_o^2}{(d_o^2 - d_i^2) d^2} = \frac{d_o^2 - d^2}{d_o^2 - d_i^2} \cdot \frac{d_i^2}{d^2} p_i \tag{13.18}$$

and $$\sigma_c = \frac{p_i d_i^2 d^2 + p_i d_i^2 d_o^2}{(d_o^2 - d_i^2) d^2} = \frac{d_o^2 + d^2}{d_o^2 - d_i^2} \cdot \frac{d_i^2}{d^2} p_i \tag{13.19}$$

The maximum hoop stress is at $d = d_i$,

$$\sigma_c = \frac{p_i d_i^2 (d_o^2 + d_i^2)}{(d_o^2 - d_i^2) d_i^2} = \frac{p_i (d_o^2 + d_i^2)}{(d_o^2 - d_i^2)} = \frac{p_i (r_o^2 + r_i^2)}{(r_o^2 - r_i^2)} \tag{13.20}$$

$$\text{Maximum shear stress is (at } d = d_i) = \frac{1}{2}(\sigma_c + p_i) = \frac{p_i d_o^2}{(d_o^2 - d_i^2)} = \frac{p_i r_o^2}{(r_o^2 - r_i^2)} \tag{13.21}$$

Variation of radial and circumferential (hoop) stresses is shown in Fig. 13.10. Note that $(\sigma_c - p) = 2a$. Longitudinal stress for a cylinder with closed ends can be obtained from the equilibrium equation for any transverse section,

$$\sigma_l \left(\frac{\pi}{4}\right)(d_o^2 - d_i^2) = p_i \cdot \left(\frac{\pi}{4}\right) d_i^2 \quad \text{or} \quad \sigma_l = \frac{p_i d_i^2}{(d_o^2 - d_i^2)} = \frac{p_i r_i^2}{(r_o^2 - r_i^2)} \tag{13.22}$$

External pressure only

$$p = \frac{p_o d_o^2 - (d_i^2 d_o^2 / d^2) p_o}{d_o^2 - d_i^2} = \frac{d^2 - d_i^2}{d_o^2 - d_i^2} \cdot \frac{d_o^2}{d^2} p_o = \frac{r^2 - r_i^2}{r_o^2 - r_i^2} \cdot \frac{r_o^2}{r^2} p_o \tag{13.23}$$

and $$\sigma_c = \frac{-p_o d_o^2 - (d_i^2 d_o^2 / d^2) p_o}{d_o^2 - d_i^2} = -\frac{d^2 + d_i^2}{d_o^2 - d_i^2} \cdot \frac{d_o^2}{d^2} p_o = -\frac{r^2 + r_i^2}{r_o^2 - r_i^2} \cdot \frac{r_o^2}{r^2} p_o \tag{13.24}$$

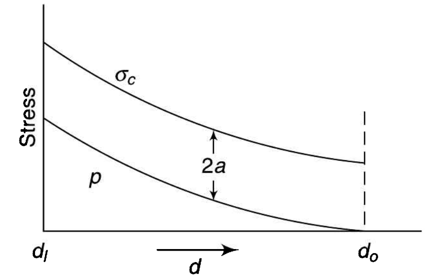


Fig. 13.10

- *All derivations are done by assuming p to be compressive and σ to be tensile. So, when calculations are done using the derived relations directly, a positive value of p indicates a compressive stress and that of σ a tensile stress.*

The summary of above results is tabulated in tables 13.1 and 13.2:

Table-13.1

(Internal pressure only)

At diameter	Radial stress (compressive)	Circumferential stress (tensile)
d	$\frac{d_o^2-d^2}{d_o^2-d_i^2}\cdot\frac{d_i^2}{d^2}p_i$	$\frac{d_o^2+d^2}{d_o^2-d_i^2}\cdot\frac{d_i^2}{d^2}p_i$
d_i	p_i	$\frac{d_o^2+d_i^2}{d_o^2-d_i^2}\cdot p_i$
d_o	0	$\frac{2d_i^2}{d_o^2-d_i^2}\cdot p_i$

Table-13.2

(External pressure only)

At diameter	Radial stress (compressive)	Circumferential stress (tensile)
d	$\frac{d^2-d_i^2}{d_o^2-d_i^2}\cdot\frac{d_o^2}{d^2}p_o$	$-\frac{d^2+d_i^2}{d_o^2-d_i^2}\cdot\frac{d_o^2}{d^2}p_o$
d_i	0	$-\frac{2d_o^2}{d_o^2-d_i^2}\cdot p_o$
d_o	p_o	$-\frac{d_o^2+d_i^2}{d_o^2-d_i^2}\cdot p_o$

In the above relations, diameter d can be replaced with radius r or R also.

Many times, it is required to find the circumferential stresses at inner or outer surfaces of a cylinder when it is subjected to inner or outer radial pressure. The relations for the same can easily be remembered by noting that

- if the pressure is internal, circumferential stresses are tensile and if it is external, the stresses are compressive
- the denominator is same in all cases, i.e., $d_o^2 - d_i^2$
- the numerator is $d_o^2 + d_i^2$ to find the stress at the surface where pressure is applied
- the numerator is $2d^2$ to find the stress at the surface where pressure is not applied (d is the diameter of the surface where pressure is applied)

Error in thin cylinder formula (subjected to internal pressure only)

If t is the thickness then, $d_o = d_i + 2t$

Hoop stress at the inner surface,

$$\sigma_c = \frac{p_i[(d_i + 2t)^2 + d_i^2]}{(d_i + 2t)^2 - d_i^2} = \frac{p_i(d_i^2 + 4t^2 + 4td_i + d_i^2)}{d_i + 4t^2 + 4td_i - d_i^2} = \frac{p_i(2d_i^2 + 4t^2 + 4td_i)}{4t^2 + 4td_i}$$

$$= \frac{2(d_i/t)^2 + 4 + 4(d_i/t)}{4 + 4(d_i/t)} p_i$$

If $d_i/t = 10$, $\sigma_c = \dfrac{244}{44} p_i = 5.55\, p_i$

which is about 11% higher than the value given by $\sigma_c = \dfrac{p_i d_i}{2t}$

If $d_i/t = 20$, $\sigma_c = \dfrac{884}{84} p_i = 10.5\, p_i$

which is about 5.2% higher than the value given by $\sigma_c = \dfrac{p_i d_i}{2t}$

It can be noted that if the mean value of the diameter is used in the thin cylinder formula, the error is considerably reduced.

Example 13.15 A pipe of 100-mm external diameter and 20-mm thickness carries water at a pressure of 20 MPa. Determine the maximum and minimum intensities of hoop stresses in the section of pipe. Also, plot the variation of hoop and radial stresses across the thickness of pipe.

Solution

Given

$t = 20$ mm $\quad\quad p_i = 20$ MPa

$d_o = 100$ mm $\quad\quad d_i = 100 - 2 \times 20 = 60$ mm

To find

— Hoop stresses

— To plot variation of hoop and radial stresses

Hoop stresses

From table 1

Maximum hoop stress is at internal diameter,

$$\sigma_{ci} = \frac{d_o^2 + d_i^2}{d_o^2 - d_i^2} \cdot p_i = \frac{100^2 + 60^2}{100^2 - 60^2} \times 20 = 42.5 \text{ MPa}$$

Minimum hoop stress is at external diameter,

$$\sigma_{co} = \frac{2d_i^2}{d_o^2 - d_i^2} \cdot p_i = \frac{2 \times 60^2}{100^2 - 60^2} \times 20 = 22.5 \text{ MPa}$$

Plotting the variation of stresses

To plot the variation across the section, the following table may be completed using the expression, $\sigma_c = \frac{d_o^2 + d^2}{d_o^2 - d_i^2} \cdot \frac{d_i^2}{d^2} p_i$

Diameter (mm)→	60	70	80	90	100
Hoop stress (MPa)	42.5	34.2	28.8	25.1	22.5

As the radial stress at external diameter is zero, the difference between hoop and radial stresses = 22.5 MPa

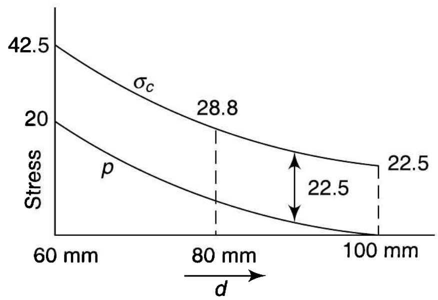


Fig. 13.11

Figure 13.11 shows the variation of hoop and radial stresses across the thickness of pipe.

Example 13.16 || Find the thickness of the cylinder of a hydraulic ram of 50-mm internal diameter to withstand an internal pressure of 30 MPa. The allowable tensile stress is limited to 45 MPa and the allowable shear stress to 40 MPa.

Solution

Given

$d_i = 50$ mm $\quad p = 30$ MPa

$\sigma_c = 45$ MPa $\quad \tau = 40$ MPa

To find Thickness

Hoop stress constraint

The maximum hoop stress is at $d = d_i$, $\sigma_c = \frac{d_o^2 + d_i^2}{d_o^2 - d_i^2} \cdot p_i$

or $$45 = \frac{d_o^2 + 50^2}{d_o^2 - 50^2} \times 30$$

or $$d_o^2 - 50^2 = \frac{30}{45}(d_o^2 + 50^2)$$

or $$3d_o^2 - 3 \times 50^2 = 2(d_o^2 + 50^2)$$

or $$d_o^2 = 12\,500 \quad \text{or} \quad d_o = 111.8 \text{ mm}$$

Shear stress constraint

Maximum shear stress, $\tau = \dfrac{p_i d_o^2}{d_o^2 - d_i^2}$

or $\quad 40 = \dfrac{30 d_o^2}{d_o^2 - 50^2}$

or $\quad 4d_o^2 - 10\,000 = 3d_o^2$

or $\quad d_o^2 = 10\,000 \quad \text{or} \quad d_o = 100 \text{ mm}$

Thickness

$$\text{Thickness} = \frac{111.8 - 50}{2} = 30.9 \text{ mm}$$

Example 13.17 A thick cylinder of 200-mm outside diameter and 140-mm inside diameter is subjected to internal pressure of 40 MPa and external pressure of 24 MPa. Determine the maximum shear stress in the material of the cylinder at the inside diameter.

Solution

Given

$d_i = 140$ mm $\quad d_o = 200$ mm

$p_o = 24$ MPa $\quad p_i = 40$ MPa

To find Maximum shear stress

Equations for thick cylinders

When a cylinder is acted upon by internal and external pressures, the radial and hoop stresses in terms of constants a and b are

$$p = -a + \frac{b}{d^2} \quad \text{and} \quad \sigma_c = a + \frac{b}{d^2} \qquad \ldots\text{(Section 13.7)}$$

where the stresses correspond to the diameter d considered.

$$\text{At inside diameter, } 40 = -a + \frac{b}{140^2}$$

$$\text{At outside diameter, } 24 = -a + \frac{b}{200^2}$$

Subtracting (ii) from (i), $\quad 16 = b\left(\dfrac{1}{140^2} - \dfrac{1}{200^2}\right) \quad \text{or} \quad b = 614\,902$

and $\quad a = \dfrac{614\,902}{140^2} - 40 \quad \text{or} \quad a = -8.63$

Determination of stresses

Hoop stress at inside diameter, $\sigma_c = a + \dfrac{b}{d^2} = -8.63 + \dfrac{614\,902}{140^2} = 22.74$ MPa

Shear stress, $= \dfrac{1}{2}(\sigma_c + p_i) = \dfrac{1}{2}(22.74 + 40) = 31.37$ MPa

- Equation 3.19 can also be used to find hoop stress at inside diameter,

$$\sigma_c = \frac{p_i d_i^2 - p_o d_o^2 + (d_i^2 d_o^2 / d^2)(p_i - p_o)}{d_o^2 - d_i^2}$$

$$= \frac{40 \times 140^2 - 24 \times 200^2 + 200^2(40 - 24)}{200^2 - 140^2} = 22.74 \text{ MPa}$$

Example 13.18 A thick cylinder has inner and outer diameters as 120 mm and 180 mm respectively. It is subjected to an external pressure of 9 MPa. Find the value of the internal pressure which can be applied if the maximum stress is not to exceed 30 MPa. Draw the curves showing the variation of hoop and radial stresses through the material of the cylinder.

Solution

Given

d_i = 120 mm d_o = 180 mm

p_o = 9 MPa σ_c = 30 MPa

To find

— Internal pressure

— Plot curves for variation of hoop and radial stresses

Calculations for internal pressure

When a cylinder is acted upon by internal and external pressures, the radial and hoop stresses in terms of constants a and b are

$$p = -a + \frac{b}{d^2} \text{ and } \sigma_c = a + \frac{b}{d^2}$$

where the stresses correspond to the diameter d considered.

- At outer diameter, the radial stress is 9 MPa

$$\therefore \quad 9 = -a + \frac{b}{180^2} \quad \text{(i)}$$

- From the expression $\sigma_c = a + \frac{b}{d^2}$, it can be observed that the maximum value of hoop stress occurs at inner diameter,

$$\therefore \quad 30 = a + \frac{b}{120^2} \quad \text{(ii)}$$

Adding (i) and (ii), $39 = \frac{b}{100}\left(\frac{144 + 324}{324 \times 144}\right)$

or $\quad b = 388\,800 \quad$ and $\quad a = \frac{388\,800}{180^2} - 9 = 3$

Internal pressure which can be applied for the maximum hoop stress can be found from the expression $p = -a + \frac{b}{d^2}$ at the inner diameter.

i.e. internal pressure $= -3 + \frac{388\,800}{120^2} = -3 + 27 = 24$ MPa

Plotting of curves

To plot the curves the radial and hoop stresses at various diameters using the above expressions can be tabulated as under:

Diameter (mm)	120	140	150	160	180
Radial stress (MPa)	24	16.8	14.3	12.2	9
Hoop stress (MPa)	30	22.8	20.3	18.2	15

Difference between hoop and radial stresses = $2a = 2 \times 3 = 6$ MPa

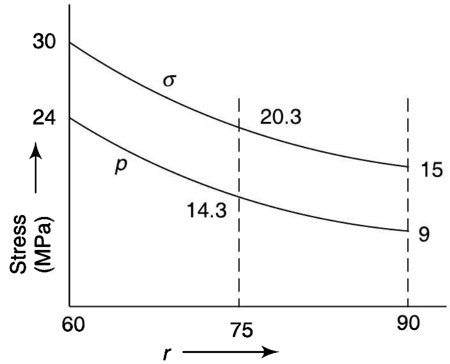


Fig. 13.12

Variation of hoop and radial stresses through the material of the cylinder is shown in Fig. 13.12. Note that $\sigma_c = p + 2a$

Example 13.19 A thick steel cylinder closed at the ends has its outer diameter 1.5 times the inner diameter and is subjected to internal pressure only. Another cylinder having the same dimensions is subjected to an external pressure only. Determine the ratio of these pressures if

(i) the maximum hoop stress has the same numerical value

(ii) the maximum hoop strain has the same numerical value

Poisson's ratio is 0.3.

Solution

Given Two cylinders: one subjected to internal pressure and other to external pressure

$d_o = 1.5\, d_i$ $\qquad \nu = 0.3$

To find

Ratio of pressures if

— maximum hoop stress is numerically same

— maximum hoop strain is numerically same

***For cylinder* 1**

The maximum hoop stress, $\sigma_c = \dfrac{d_o^2 + d_i^2}{d_o^2 - d_i^2} \cdot p_i = \dfrac{(1.5d_i)^2 + d_i^2}{(1.5d_i)^2 - d_i^2} p_i = 2.6\, p_i$

Longitudinal stress, $\sigma_l = \dfrac{d_i^2}{d_o^2 - d_i^2} p_i = \dfrac{d_i^2}{(1.5d_i)^2 - d_i^2} p_i = 0.8\, p_i$

The maximum hoop strain,

$$\varepsilon_c = \frac{1}{E}(\sigma_c + \nu p_i - \nu\sigma_l) = \frac{p_i}{E}(2.6 + 0.3\times 1 - 0.3\times 0.8) = 2.66\frac{p_i}{E}$$

For cylinder 2

The maximum hoop stress, $\sigma_c = -\dfrac{p_o \times 2d_o^2}{(d_o^2 - d_i^2)} = -\dfrac{2\times 2.25}{1.25}p_o = -3.6p_o$

Longitudinal stress, $\sigma_l = \dfrac{p_o d_o^2}{(d_o^2 - d_i^2)} = \dfrac{2.25}{1.25}p_o = 1.8p_o$

The maximum hoop strain, $\varepsilon_c = \dfrac{p_o}{E}(-3.6 + 0.3\times 1.8) = 3.06\dfrac{p_o}{E}$

Maximum hoop stress numerically same

$$2.6\,p_i = 3.6\,p_o \qquad \text{or} \qquad \frac{p_i}{p_o} = 1.385$$

Maximum hoop strain numerically same

$$\frac{2.66\,p_i}{E} = \frac{3.06\,p_o}{E} \qquad \text{or} \qquad \frac{p_i}{p_o} = 1.15$$

Example 13.20 Determine the ratio of thickness to inner diameter of a tube subjected to internal pressure if the ratio of the internal pressure to the maximum circumferential stress is 0.5.

For such a tube of 250-mm inside diameter, find the alteration of thickness of metal when the internal pressure is 80 MPa. E = 205 GPa.

Solution

Given A tube subjected to internal pressure, ratio $p/\sigma_c = 0.5$

$d_i = 250$ mm $\qquad$ $E = 80$ MPa

$E = 205$ GPa

To find

— Ratio of thickness to inner diameter

— alteration of thickness when internal pressure is 80 MPa

Ratio of thickness to inner diameter

Maximum circumferential stress in a tube subjected to internal pressure p is at the inner surface and is given by

$$\sigma_c = \frac{d_o^2 + d_i^2}{d_o^2 - d_i^2}\cdot p$$

or $\qquad \dfrac{\sigma_c}{p} = \dfrac{d_o^2 + d_i^2}{d_o^2 - d_i^2} = 0.5$

or $\qquad 2(d_o^2 - d_i^2) = d_o^2 + d_i^2$

or $\qquad d_o^2 = 3d_i^2 \qquad$ or $\qquad d_o = \sqrt{3}d_i$

Now $\qquad d_o - d_i = 2t$

or $\qquad \sqrt{3}d_i - d_i = 2t$

or $\qquad (\sqrt{3} - 1)d_i = 2t \qquad$ or $\qquad d_i = 2.732\,t$

or $$\frac{t}{d_i} = 0.366$$

Change in thickness

Hoop stress at inner radius = $2p$;

Hoop stress at outer radius = $\dfrac{2d_i^2}{d_o^2 - d_i^2}p = \dfrac{2d_i^2}{3d_i^2 - d_i^2}p = p$

Longitudinal stress = $\dfrac{\pi r_i^2 \cdot p}{\pi(r_o^2 - r_i^2)} = \dfrac{125^2 \times p}{216.5^2 - 125^2} = 0.5p$

Radial stress at inner radius = p

Radial stress at outer radius = 0

$$d_o = \sqrt{3} \times 250 = 433 \text{ mm} \quad \text{or} \quad r_o = 216.5 \text{ mm}$$

Increase in internal radius = $\dfrac{r_i}{E}(\sigma_c - \nu\sigma_l + \nu p)$

$$= \frac{125}{E}(2p - 0.3 \times 0.5p + 0.3p) = \frac{268.8p}{E}$$

Increase in external radius = $\dfrac{r_o}{E}(\sigma_c - \nu\sigma_l + \nu p)$

$$= \frac{216.5}{E}(p - 0.3 \times 0.5p + 0) = \frac{184p}{E}$$

Change in thickness = $\dfrac{268.8 - 184}{E} \cdot p = \dfrac{84.8 \times 80}{205\,000} = 0.033 \text{ mm}$

13.7 COMPOUND TUBES

As the hoop stress is maximum at the inner radius of a thick cylindrical tube, the material at the outside of the tube is not stressed to its limit. To even out the stresses, a tube may be made of two parts, one shrunk on to the other by heating it to the required temperature. This makes the inner tube in compression and the outer tube in tension (Fig. 13.13). On applying the internal pressure, a tensile hoop stress is superimposed on the compressive or shrinkage stresses of the inner tube.

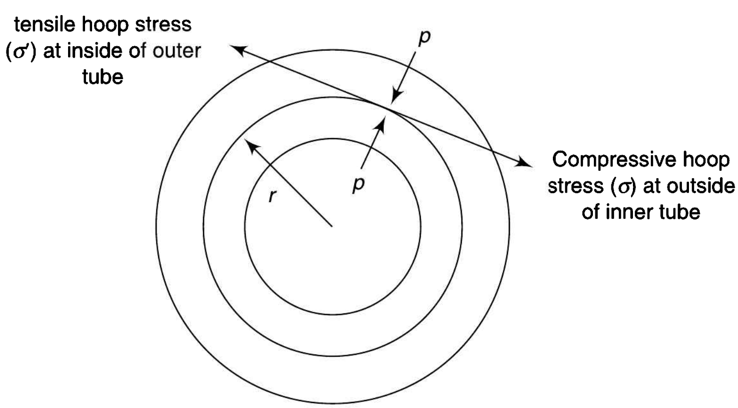


Fig. 13.13

Thus initially, the inside diameter of the outer tube is smaller than the outside diameter of the inner tube. On heating the outer tube, this is increased and made equal to that. On cooling, both of these diameters are reduced by the same amount. The analysis is made first by calculating the stresses due to shrinkage in each component. Thus, the final diameter of the outer tube increases from the original and that of inner tube decreases. On applying the internal pressure, the stresses are calculated by assuming the two tubes as a single tube if they are made of the same material.

Due to shrinkage, let

d = diameter of the common surface of the inner and outer tubes

p = radial pressure at the common surface

σ = compressive hoop stress at the outside of the inner tube

σ' = tensile hoop stress at the inside of the outer tube

E = Young's modulus of the material of the inner tube

E' = Young's modulus of the material of the outer tube

ν = Poisson's ratio of the material of the inner tube

ν' = Poisson's ratio of the material of the outer tube

Decrease in inner tube diameter = strain × diameter = $\dfrac{\sigma - \nu p}{E} \cdot d$

(σ and p both are compressive and remembering that diametral strain is equal to hoop strain)

Increase in outer tube diameter = $\dfrac{\sigma' + \nu' p}{E'} \cdot d$ (σ is tensile and p is compressive)

Difference of diameters before shrinking

= Decrease in Inner tube diameter + Increase in outer tube diameter

$$= \frac{\sigma - \nu p}{E} \cdot d + \frac{\sigma' + \nu' p}{E'} \cdot d = d\left(\frac{\sigma - \nu p}{E} + \frac{\sigma' + \nu' p}{E'}\right) \tag{13.25}$$

If both tubes are of the same material,

Difference of diameters before shrinking

$$= \frac{\sigma - \nu p}{E} \cdot d + \frac{\sigma + \nu p}{E} \cdot d = d\left(\frac{\sigma + \sigma'}{E}\right) \tag{13.25a}$$

Example 13.21 A compound cylinder is formed by shrinking one tube to another, the inside and outside diameters of the outer tube being 120 mm and 180 mm respectively and of the inner tube being 60 mm and 120 mm respectively. After shrinking, the radial pressure at the common surface is 30 MPa. If the cylinder is subjected to an internal pressure of 80 MPa, determine the final stresses set up at various surfaces of the cylinder. What is the resultant radial pressure at the common surface?

Solution

Given A compound cylinder:

Outer tube:	$d_i = 120$ mm	$d_o = 180$ mm
Inner tube:	$d_i = 60$ mm	$d_o = 120$ mm
	$p = 30$ MPa	$p_i = 80$ MPa

To find

— Final stresses

— resultant radial pressure at the common surface

Shrinkage stresses

Outer tube: pressure is internal, (Refer Tables 13.1 and 13.2)

$$\therefore \qquad \sigma_{c120} = \frac{180^2 + 120^2}{180^2 - 120^2} \times 30 = 78 \text{ MPa};$$

$$\text{and} \qquad \sigma_{c180} = \frac{2 \times 120^2}{180^2 - 120^2} \times 30 = 48 \text{ MPa}$$

Inner tube: pressure is external,

$$\sigma_{c120} = -\frac{120^2 + 60^2}{120^2 - 60^2} \times 30 = -50 \text{ MPa};$$

$$\text{and} \qquad \sigma_{c60} = \frac{2 \times 120^2}{120^2 - 60^2} \times 30 = -80 \text{ MPa}$$

Stresses due to pressure

Cylinder is assumed as a single tube (pressure is internal),

$$\sigma_{c60} = -\frac{180^2 + 60^2}{180^2 - 60^2} \times 80 = 100 \text{ MPa}$$

$$\sigma_{c180} = -\frac{2 \times 60^2}{180^2 - 60^2} \times 80 = 20 \text{ MPa}$$

$$\sigma_{c60} = -\frac{180^2 + 120^2}{180^2 - 60^2} \times \frac{60^2}{120^2} \times 80 = 32.5 \text{ MPa}$$

Final stresses

The final stresses are summed up in Table 13.3.

Table-13.3

	Inner tube		Outer tube	
	60 mm	120 mm	120 mm	180 mm
Shrinkage stresses (MPa)	–80	–50	78	48
Stresses due to pressure (MPa)	100	32.5	32.5	20
Final stresses (MPa)	20	–17.5	110.5	68

Radial pressure at common surface

Radial pressure at the common surface due to internal pressure,

$$\sigma_{r120} = -\frac{180^2 - 120^2}{180^2 - 60^2} \times \frac{60^2}{120^2} \times 80 = 10 \text{ MPa} \qquad \ldots\text{(Eq. 13.21)}$$

Total radial pressure = 30 + 10 = 40 MPa

Example 13.22 A steel tube of 120-mm external diameter is shrunk on another steel tube of 48-mm internal diameter. After shrinking, the diameter at the junction is 80 mm. Initial difference of diameters at the junction before shrinking was 0.04 mm. Determine

(i) radial pressure at the junction

(ii) hoop stresses developed in the two tubes after the shrinking

Take E = 210 GPa

Solution

Given A compound cylinder

Outer tube:	$d_i = 80$ mm	$d_o = 120$ mm
Inner tube:	$d_i = 48$ mm	$d_o = 80$ mm
	$\delta_d = 0.04$ mm	$E = 210$ GPa

To find

— Radial pressure at the junction

— hoop stresses

Radial pressure at the junction

Let the radial pressure at the junction be p.

$$\text{Difference of diameters before shrinking, } \delta d = d\left(\frac{\sigma + \sigma'}{E}\right) \qquad \text{...(Eq. 13.25a)}$$

Inner tube: pressure is external,

$$\sigma = \frac{80^2 + 48^2}{80^2 - 48^2} \cdot p = 2.125\,p \qquad \text{(compressive)}$$

Outer tube: pressure is internal,

$$\sigma' = \frac{120^2 + 80^2}{120^2 - 80^2} \cdot p = 2.6\,p \qquad \text{(tensile)}$$

$$\therefore \qquad 0.04 = \frac{80p}{210\,000}(2.125 + 2.6) \quad \text{or} \quad p = 22.22 \text{ MPa}$$

Hoop stresses in inner tube

At outer radius, $\sigma = 2.125 \times 22.22 = 47.22$ MPa (compressive)

$$\text{At inner radius, } \sigma = \frac{2d_o^2}{d_o^2 - d_i^2} \cdot p = \frac{2 \times 80^2}{80^2 - 48^2} \times 22.22 = 69.44 \text{ MPa} \qquad \text{(compressive)}$$

Hoop stresses in outer tube

$$\text{At outer radius, } \sigma = \frac{2d_i^2}{d_o^2 - d_i^2} \cdot p = \frac{2 \times 80^2}{120^2 - 80^2} \times 22.22 = 35.55 \text{ MPa} \qquad \text{(tensile)}$$

At inner radius, $\sigma = 2.6 \times 22.22 = 57.77$ MPa (tensile)

Example 13.23 A compound cylinder is formed by shrinking a cylinder of 240-mm internal diameter and 320-mm external diameter on a cylinder of 160-mm internal diameter and 240-mm external diameter. The shrinking pressure is 12 MPa. Determine the minimum temperature to which the outer cylinder must be heated to slip it over the inner cylinder. $E = 205$ GPa, $\alpha = 6 \times 10^{-6}/°C$.

Solution

Given A compound cylinder

Outer tube:	$d_i = 240$ mm	$d_o = 320$ mm
Inner tube:	$d_i = 160$ mm	$d_o = 240$ mm
	$p = 12$ MPa	$E = 205$ GPa
	$\alpha = 6 \times 10^{-6}/°C$	

To find Minimum temperature to which outer cylinder is heated

Shrinkage stresses

Outer tube: pressure is internal,

$$\therefore \qquad \sigma_{c240} = \frac{320^2 + 240^2}{320^2 - 240^2} \times 12 = 42.85 \text{ MPa}$$

Inner tube: pressure is external,

$$\sigma_{c240} = -\frac{240^2 + 160^2}{240^2 - 160^2} \times 12 = -31.2 \text{ MPa i.e. compressive}$$

Difference of diameters

$$\delta d = d\left(\frac{\sigma + \sigma'}{E}\right) = 240 \times \frac{42.85 + 31.2}{205\,000} = 0.0867 \text{ mm}$$

Calculation of temperature

Let the temperature to which the outer tube is to be heated is δt.

Then $\qquad \delta d = d \cdot \delta t \cdot \alpha$

or $\qquad 0.0867 = 240 \times \delta t \times 6 \times 10^{-6} \qquad$ or $\qquad \delta t = 60.2°$

Example 13.24 A 20-mm thick steel tyre is shrunk on a cast-iron rim of 360-mm outer diameter and 60-mm thick. Determine the inner diameter of the steel tyre if after shrinking on the tyre it exerts a radial pressure of 80 MPa on the cast-iron rim. Take E for steel 205 GPa, for cast iron 102 GPa and Poisson's ratio 0.28 for both.

Solution

Given

Tyre:	$d_i = 360$ mm	$t = 20$ mm
Rim:	$d_o = 360$ mm	$t = 60$ mm
	$p = 80$ MPa	$E_s = 205$ GPa
	$E_{ci} = 102$ GPa	$\nu = 0.28$

To find Inner diameter of steel tyre

For tyre: $d_o = 360 + 40 = 400$ mm; $d_i = 360$ mm

For rim: $d_o = 360$ mm; $d_i = 360 - 120 = 240$ mm

Let p be the radial pressure (compressive) at the common surface.

Circumferential stress in the inner tube is to be compressive.

Difference of diameters

Difference of diameters before shrinking

= Decrease in cast-iron rim diameter + Increase in outer-tube diameter

$$\delta d = d\left(\frac{\sigma - \nu p}{E} + \frac{\sigma' + \nu' p}{E'}\right) \qquad \text{...(Eq.13.25)}$$

$$= d\left[\frac{1}{E_{ci}}\left(\frac{360^2 + 240^2}{360^2 - 240^2}p - 0.28p\right) + \frac{1}{E_s}\left(\frac{400^2 + 360^2}{400^2 - 360^2}p + 0.28p\right)\right]$$

$$= pd\left[\frac{1}{102\,000}\left(\frac{360^2 + 240^2}{360^2 - 240^2} - 0.28\right) + \frac{1}{205\,000}\left(\frac{400^2 + 360^2}{400^2 - 360^2} + 0.28\right)\right]$$

$$= 80 \times 360[22.75 \times 10^{-6} + 47.84 \times 10^{-6}] = 2.03 \text{ mm}$$

Inner diameter of tyre

As initial difference of diameters of the tyre and the rim is to be 2.03 mm in order to have the required radial pressure after shrinking, the inner diameter of the tyre must be less than the outer diameter of rim, i.e.,

Inside diameter of tyre = 360 – 2.03 = 357.97 mm

Example 13.25 A tube 96 mm in diameter is used to reinforce a tube of 48-mm internal diameter and 72-mm outer diameter. The compound tube is made to withstand an internal pressure of 60 MPa. The shrinkage allowance is such that the final maximum stress in each tube is the same. Determine this stress and plot a diagram to show the variation of hoop stress in the two tubes. Also, calculate the initial difference of diameters before the shrinkage. E = 208 GPa.

Solution

Given A compound tube

Outer tube:	$d_i = 72$ mm	$d_o = 96$ mm
Inner tube:	$d_i = 48$ mm	$d_o = 72$ mm
	$p = 60$ MPa	$E = 208$ GPa

To find

— Maximum stress
— To plot variation of hoop stress
— Initial difference of diameters

Shrinkage stresses

Let p be the common radial pressure due to shrinkage.

- *Inner tube*: Pressure p is external,

$$\therefore \quad \text{hoop stress at inner radius, } \sigma_c = -\frac{2d_o^2}{d_o^2 - d_i^2}\cdot p = -\frac{2\times 72^2}{72^2 - 48^2}\cdot p = -3.6\,p$$

$$\text{Hoop stress at the common surface, } \sigma_c = -\frac{d_o^2 + d_i^2}{d_o^2 - d_i^2}\cdot p = -\frac{72^2 + 48^2}{72^2 - 48^2}\cdot p = -2.6\,p$$

- *Outer tube*: Pressure p is internal,

$$\therefore \quad \text{hoop stress at outer radius, } \sigma_c = \frac{2d_i^2}{d_o^2 - d_i^2}\cdot p = \frac{2\times 72^2}{96^2 - 72^2}\cdot p = 2.572\,p$$

$$\text{Hoop stress at the common surface, } \sigma_c = \frac{d_o^2 + d_i^2}{d_o^2 - d_i^2}\cdot p = \frac{96^2 + 72^2}{96^2 - 72^2}\cdot p = 3.572\,p$$

On applying internal pressure

When internal pressure of 60 MPa is applied, the whole tube may be assumed as a single tube.

Hoop (circumferential) stresses

$$\text{At 48-mm diameter, } \sigma_c = \frac{d_o^2 + d_i^2}{d_o^2 - d_i^2}\cdot p = \frac{96^2 + 48^2}{96^2 - 48^2}\times 60 = 100 \text{ MPa}$$

$$\text{At 72-mm diameter, } \sigma_c = \frac{d_o^2 + d^2}{d_o^2 - d_i^2}\cdot\frac{d_i^2}{d^2}p_i = \frac{96^2 + 72^2}{96^2 - 48^2}\cdot\frac{48^2}{72^2}\times 60 = 55.6 \text{ MPa}$$

$$\text{At 96-mm diameter, } \sigma_c = \frac{2d_i^2}{d_o^2 - d_i^2}\cdot p = \frac{2\times 48^2}{96^2 - 48^2}\times 60 = 40 \text{ MPa}$$

Applying the given condition

According to the statement,

Maximum stress of the inner tube = Maximum stress of the outer tube

$$-3.6\,p + 100 = 3.572\,p + 55.6$$

or $$7.172\,p = 44.4 \quad \text{or} \quad p = 6.19 \text{ MPa}$$

Maximum hoop stress = 3.572 ×6.19 +55.6 = 77.71 MPa

- *Shrinkage stresses*

  In inner tube:

  At inner diameter: $-3.6 \times 6.19 = -22.3$ MPa

  At common diameter: $-2.6 \times 6.19 = -16.1$ MPa

  In outer tube:

  At common diameter: $3.572 \times 6.19 = 22.1$ MPa

  At outer diameter: $2.572 \times 6.19 = 15.9$ MPa

- *Resultant stresses*

  In inner tube:

  At inner diameter: $100 - 22.3 = 77.7$ MPa

  At common diameter: $55.6 - 16.1 = 39.5$ MPa

  In outer tube:

  At common diameter: $55.6 + 22.1 = 77.7$ MPa

  At outer diameter: $40 + 15.9 = 55.9$ MPa

Initial difference of diameters

Initial difference of diameters at common surface,

$$\delta d = d\left(\frac{\sigma + \sigma'}{E}\right) = 72 \times \frac{16.1 + 22.1}{208\,000} = 0.0132 \text{ mm}$$

Variation of hoop stresses in the two tubes is shown in Fig. 13.14.

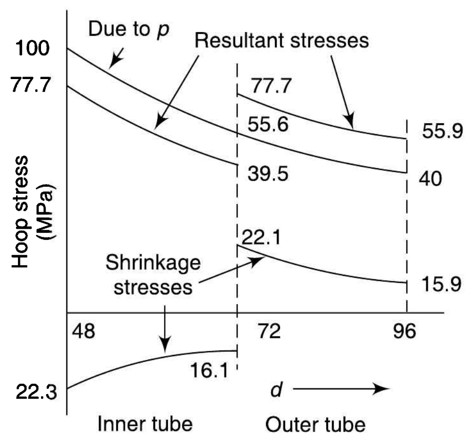


Fig. 13.14

Example 13.26 A compound cylinder is made by shrinking an outer tube of 400-mm external diameter on to an inner tube of 200-mm internal diameter. Find the common diameter at the junction if the maximum circumferential stress in the inner tube is to be two thirds of the maximum circumferential stress in the outer tube.

Solution

Given A compound cylinder

Outer tube: $d_o = 400$ mm

Inner tube: $d_i = 200$ mm

Maximum circumferential stress in inner tube to be two thirds of that in outer tube

To find Common diameter

Maximum circumferential stress in inner tube

Let d be the common diameter and p the radial pressure at the junction.

For inner tube, the radial pressure p is external; the maximum circumferential stress is compressive and is at the inner diameter.

$$\sigma_c = -\frac{2d^2}{d^2 - d_i^2}p_i = -\frac{2d^2}{d^2 - 200^2}p = -\frac{2}{1-(200/d)^2}p = -\frac{2}{1-K}p$$

where $K = (200/d)^2$

Maximum circumferential stress in outer tube

For outer tube, the radial pressure p is internal; the maximum circumferential stress is tensile and is at the inner diameter.

$$\sigma_c = \frac{d_o^2 + d_i^2}{d_o^2 - d_i^2} p_i = \frac{400^2 + d^2}{400^2 - d^2} p = \frac{(400/d)^2 + 1}{(400/d)^2 - 1} p = \frac{4K+1}{4K-1} p$$

Applying the given condition

According to the given condition,

$$\frac{2}{1-K} p = \frac{2}{3} \cdot \frac{4K+1}{4K-1} p$$

or $\quad 3(4K - 1) = (4K + 1)(1 - K)$

or $\quad 12K - 3 = 4K - 4K^2 + 1 - K$

or $\quad 4K^2 + 9K - 4 = 0$

or $\quad K^2 + 2.25K - 1 = 0$

or $\quad K = \dfrac{-2.25 \pm \sqrt{2.25^2 + 4}}{2} = 0.38$

or $\quad \dfrac{200^2}{d^2} = 0.38 \quad$ or $\quad d = 324.4$ mm

13.8 HUB ON SOLID SHAFT

- The shaft is subjected to an external pressure. Let σ_c and p be the hoop and radial stresses at a radius r.

Obtain the following relation in the same way as for thick cylinders (Section 13.6):

$$p = -a + \frac{b}{r^2} \quad \text{and} \quad \sigma_c = a + \frac{b}{r^2}$$

The stresses cannot be infinite at the centre of the solid shaft, therefore b has to be zero or

$$\sigma_c = a = -p \tag{13.26}$$

which shows that the hoop stress is numerically equal to the radial stress and both are compressive.

Let

d = diameter of the common surface of the shaft and the hub
= Inner diameter of the hub = outer diameter of shaft
p = radial pressure at the common surface
σ = compressive hoop stress at the outside of shaft
σ' = tensile hoop stress at the inside of the hub
E = Young's modulus of the material of the shaft
E' = Young's modulus of the material of the hub
ν = Poisson's ratio of the material of the shaft
ν' = Poisson's ratio of the material of the hub

Decrease in shaft diameter = strain $\times$ diameter = $\dfrac{\sigma - \nu p}{E} \cdot d$

(σ and p both are compressive and remembering that diametral strain is equal to hoop strain)

Increase in hub diameter $= \dfrac{\sigma' + \nu' p}{E'} \cdot d$ (σ is tensile and p is compressive)

Difference of diameters before shrinking (shrinkage allowance)

= Decrease in shaft diameter + Increase in hub diameter

$$= \frac{\sigma - \nu p}{E} \cdot d + \frac{\sigma' + \nu' p}{E'} \cdot d = d\left[\frac{1}{E}(\sigma - \nu p) + \frac{1}{E'}(\sigma' + \nu' p)\right]$$

$$= d_i\left[\frac{1}{E}(p - \nu p) + \frac{1}{E'}\left(\frac{d_o^2 + d_i^2}{d_o^2 - d_i^2} \cdot p + \nu' p\right)\right]$$

where d_i and d_o are the internal and external diameters of the hub.

If the shaft and the hub are of the same material,

Difference of diameters before shrinking (shrinkage allowance)

$$= d_i\left(\frac{\sigma + \sigma'}{E}\right) = \frac{d_i p}{E}\left(1 + \frac{d_o^2 + d_i^2}{d_o^2 - d_i^2}\right) = \frac{2pd_i d_o^2}{E(d_o^2 - d_i^2)} \tag{13.27}$$

Shrinkage allowance can also be expressed in terms of difference in radii,

i.e., Difference of radii before shrinking $= \dfrac{1}{2E}\left(\dfrac{2pd_i d_o^2}{d_o^2 - d_i^2}\right) = \dfrac{2pR_i R_o^2}{E(R_o^2 - R_i^2)}$ (13.28)

Example 13.27 A bronze bush of 15-mm thickness is shrunk on a steel shaft of 80 mm diameter. If the shrinkage pressure is 60 MPa find the tight allowance between the bush and the shaft.

For steel: E = 204 GPa; Poisson's ratio = 0.28

For bronze: E = 102 GPa; Poisson's ratio = 0.32

Solution

Given A bronze bush shrunk on a steel shaft

$d_s = 80$ mm $t = 15$ mm

$p = 60$ MPa $E_s = 204$ GPa

$\nu_s = 0.28$ $E_b = 102$ GPa

$\nu_b = 0.32$

To find Tight allowance

For bush: $d_i = 80$ mm $d_o = 80 + 15 \times 2 = 110$ mm

Let p be the radial pressure at the common surface. σ_s (= p) ...(compressive).

Tight allowance

$$\delta d = \varepsilon_c \cdot d = d\left[\frac{1}{E_b}(\sigma_b + \nu_b p) - \frac{1}{E_s}(-\sigma_s + \nu_s p)\right]$$

$$= d\left[\frac{1}{E_b}\left(\frac{110^2 + 80^2}{110^2 - 80^2} p + 0.32p\right) - \frac{1}{E_s}(-p + 0.28p)\right]$$

$$= 80 \times 60\left[\frac{1}{102\,000}\left(\frac{110^2 + 80^2}{110^2 - 80^2} + 0.32\right) - \frac{1}{204\,000}(-1 + 0.28)\right]$$

$$= 80 \times 60[34.96 \times 10^{-6} + 3.53 \times 10^{-6}] = 0.185 \text{ mm}$$

Example 13.28 A steel ring of 180-mm outside diameter and 50-mm width is mounted on a steel plug of 120-mm diameter. An electric resistance strain gauge fitted on the external surface of the ring in the circumferential direction measures the strain to be 180×10^{-6} mm per mm. Determine the force required to push the plug out of the ring. What is the maximum hoop stress in the ring? E = 204 GPa, coefficient of friction is 0.22.

Solution

Given A steel ring mounted on a steel plug

$d_o = 180$ mm $b = 50$ mm

$d_i = 120$ mm $E = 204$ GPa

$\mu = 0.22$ $\varepsilon = 180 \times 10^{-6}$

To find

— Force required to push the plug out of ring

— Maximum hoop stress

Radial pressure at common surface

Radial pressure at the outer surface of the ring = 0

Let p be the radial pressure at the common surface.

Hoop strain at the outer surface of the ring,

$$\varepsilon = \frac{\sigma_c - \nu\sigma_r}{E} = \frac{1}{E}\left[\frac{2 \times 120^2}{180^2 - 120^2} \cdot p - 0\right] = \frac{1.6p}{E}$$

or $$180 \times 10^{-6} = \frac{1.6p}{204\ 000} \quad \text{or} \quad p = 22.95 \text{ MPa}$$

Force required

Total radial force = $22.95 \times (\pi \times 120 \times 50) = 432.6 \times 10^3$ N or 432.6 kN

Tangential force = $0.22 \times 432.6 = 95.17$ kN

Thus the force required to push the plug out of the ring = 95.17 kN

Maximum hoop stress

Maximum hoop stress is at the inner surface of the ring = $\dfrac{180^2 + 120^2}{180^2 - 120^2} \times 22.95 = 59.67$ MPa

Example 13.29 The outer diameter of a steel collar is 250 mm and the internal diameter increases by 0.15 mm when shrunk on to a solid steel shaft of 150-mm diameter. Determine

(i) the pressure between the collar and the shaft

(ii) the reduction in diameter of the shaft

(iii) hoop stress at the inner surface of the collar

E = 205 GPa and Poisson's ratio = 0.3.

Solution

Given A steel collar on a solid steel shaft

$d_o = 250$ mm $\delta d = 0.15$ mm

$d_i = 150$ mm $E = 205$ GPa

$\nu = 0.3$

To find

— Pressure between collar and shaft

— Reduction in diameter of shaft

— Hoop stress at inner surface of collar

Determination of pressure

Let p be radial pressure at the common surface.

Increase of inner diameter of the collar

$$\delta d = \frac{\sigma_c + \nu\sigma_r}{E} \cdot d = \frac{d}{E}\left[\frac{250^2 + 150^2}{250^2 - 150^2} \cdot p + \nu p\right] = \frac{pd}{E}[2.125 + 0.3] = \frac{2.425pd}{E}$$

or
$$0.15 = \frac{2.425 \times 150p}{204\,000} \quad \text{or} \quad p = 84.5 \text{ MPa}$$

Reduction in diameter of shaft

Hoop as well as radial stresses, both are compressive

$$\delta d = \frac{\sigma_c + \nu\sigma_r}{E} \cdot d = \frac{d}{E}[-p + \nu p] = \frac{pd}{E}[-1 + 0.3] = -\frac{0.7pd}{E}$$

$$= -\frac{0.7 \times 84.5 \times 150}{204\,000} = -0.0435 \text{ mm}$$

Hoop stress

Hoop stress at the inner surface of collar,

$$\sigma_c = \frac{250^2 + 150^2}{250^2 - 150^2} \times 84.5 = 179.6 \text{ MPa}$$

Example 13.30 A cast-iron hub of 120-mm external diameter and 80 mm long is to be tightly pressed on to a steel shaft of 40-mm diameter so that there is no relative slip between the two under a torque of 4 kN·m. Determine the maximum hoop stress in the hub and the necessary fit allowance. Take E_s = 208 GPa and E_{ct} = 104 GPa, coefficient of friction = 0.3 and Poisson's ratio = 0.28.

If after the assembly, an axial compressive stress of 60 MPa is applied to the shaft, find the resultant increase in the maximum hoop stress in the hub.

Solution

Given A cast-iron hub on a solid steel shaft

d_o = 120 mm $\quad$ $T = 4$ kN·m $= 4 \times 10^6$ N·mm

d_s = 40 mm $\quad$ l = 80 mm

ν = 0.28 $\quad$ μ = 0.3

E_s = 208 GPa $\quad$ E_{ct} = 104 GPa

To find

— Maximum hoop stress in hub

— fit allowance

— Increase in maximum hoop stress in the hub on applying compressive stress of 60 MPa

Radial pressure

Let p be the radial pressure at the common surface

Then torque = (μ × Radial force) × Radius

= μ × Radial pressure × Area × Radius

or $\quad 4 \times 10^6 = 0.3\,p(\pi \times 40 \times 80) \times 20 \quad$ or $\quad p = 66.3$ MPa

Maximum hoop stress in hub

Maximum hoop stress (at inside)

$$\sigma_c = \frac{120^2 + 40^2}{120^2 - 40^2} \times 66.3 = 82.875 \text{ MPa}$$

Fit allowance

- *For the hub*

  Increase of inside diameter $= \dfrac{82.875 + 0.28 \times 66.3}{104\,000} \times 40 = 0.039$ mm
- *For the shaft*

  Hoop stress = Radial pressure = 66.3 MPa (compressive)

  Decrease in outside diameter = Hoop strain × diameter

  $$= \frac{66.3 - 0.28 \times 66.3}{208\,000} \times 40 = 0.0092 \text{ mm}$$

  Force fit allowance = 0.039 + 0.0092 = 0.0482 mm

On applying axial compressive stress

- When an axial compressive stress of 60 MPa is applied to the shaft , let σ be the increase in maximum hoop stress and p the corresponding increase in the radial pressure at the common surface,

  For the hub

  Increase in maximum hoop stress = σ

  Also as $\sigma = \dfrac{120^2 + 40^2}{120^2 - 40^2} \times p$;

  ∴ increase in radial stress, $p = 0.8\,\sigma$

  For the shaft

  As the radial and hoop stresses both are equal and compressive, the increase in their values must be equal, i.e.,

  Increase in hoop stress = Increase in radial stress = 0.8 σ

  Now,

  Increase in hoop strain at outside of shaft = Increase in hoop strain at inside of hub

  $$\frac{-0.8\sigma - (-0.28 \times 0.8\sigma) - (-0.28 \times 60)}{E_s} = \frac{\sigma - (-0.28 \times 0.8\sigma)}{E_{ci}}$$

  or $$\frac{-0.576\sigma + 16.8}{2} = \frac{1.224\sigma}{1} \quad \text{or} \quad \sigma = 5.556 \text{ MPa}$$

13.9 THICK SPHERICAL SHELLS

At any radius r in a sphere of inner and outer radius r_i and r_o respectively (Fig. 13.15),

Let σ = circumferential or hoop stress (tensile)

p = radial stress (compressive)

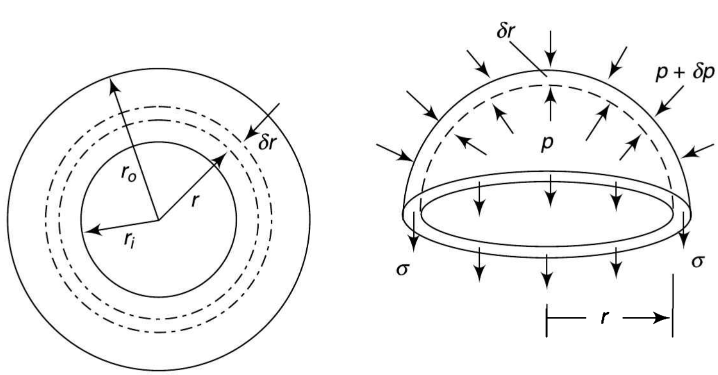


Fig. 13.15

Let u be the radial shift at an unstrained radius r, i.e., r becomes $r+u$ after straining. Similarly, $u+\delta u$ be the radial shift at an unstrained radius $r+\delta r$.

- Radial strain $= \dfrac{\text{Increase in } \delta r}{\delta r} = \dfrac{du}{dr}$ in the limit

$$\therefore \quad E \cdot \frac{du}{dr} = -p - \nu\sigma - \nu\sigma = -p - 2\nu\sigma \tag{i}$$

- Circumferential (hoop) strain $= \dfrac{\text{Increase of circumference}}{\text{original circumference}} = \dfrac{2\pi(r+u) - 2\pi r}{2\pi r} = \dfrac{u}{r}$

$$\therefore \quad E \cdot \frac{u}{r} = \sigma - \nu\sigma + \nu p$$

or $\quad E \cdot u = r(\sigma - \nu\sigma + \nu p)$

Differentiating it, $$E \cdot \frac{du}{dr} = \sigma - \nu\sigma + \nu p + r\left(\frac{d\sigma}{dr} - \nu\frac{d\sigma}{dr} + \nu\frac{dp}{dr}\right) \tag{ii}$$

From (i) and (ii),

$$-p - 2\nu\sigma = \sigma - \nu\sigma + \nu p + r\left(\frac{d\sigma}{dr} - \nu\frac{d\sigma}{dr} + \nu\frac{dp}{dr}\right)$$

$$(1+\nu)(\sigma + p) + r(1-\nu)\frac{d\sigma}{dr} + \nu r\frac{dp}{dr} = 0 \tag{iii}$$

- *Equilibrium equation,*

Consider the equilibrium of a hemisphere,

$$\sigma \cdot 2\pi r \cdot \delta r + (p + \delta p)\,\pi(r + \delta r)^2 = p\,\pi r^2$$

Taking the limits and neglecting small quantities,

$$\sigma \cdot 2r \cdot dr + pr^2 + 2rp \cdot dr + r^2 \cdot dp = pr^2$$

or $$2\sigma \cdot dr + 2p \cdot dr + r \cdot dp = 0$$

or $$\sigma + p = -\frac{r}{2}\frac{dp}{dr} \tag{iv}$$

(iii) becomes, $$(1+\nu)\left(-\frac{r}{2}\frac{dp}{dr}\right) + r(1-\nu)\frac{d\sigma}{dr} + \nu r\frac{dp}{dr} = 0$$

or $$(1-\nu)\frac{d\sigma}{dr} - \frac{1}{2}\frac{dp}{dr}(1-\nu) = 0$$

or $$\frac{d\sigma}{dr} - \frac{1}{2}\frac{dp}{dr} = 0$$

Integrating, $$\sigma - \frac{p}{2} = \text{constant} = A \text{ (say)} \quad \text{or} \quad \sigma = \frac{p}{2} + A \tag{v}$$

(iv) can be written as $$\frac{p}{2} + A + p = -\frac{r}{2}\frac{dp}{dr}$$

or $$\frac{3p}{2} + A = -\frac{r}{2}\frac{dp}{dr} \qquad \text{or} \qquad 3p + r\frac{dp}{dr} = -2A$$

or $$\frac{1}{r^2}\frac{d(pr^3)}{dr} = -2A \quad \text{or} \quad \frac{d(pr^3)}{dr} = -2Ar^2$$

Integrating, $$pr^3 = -\frac{2}{3}Ar^3 + B$$

or $$p = -\frac{2}{3}A + \frac{B}{r^3} = -a + \frac{b}{d^3} \text{ where } a = \frac{2A}{3} \text{ and } b = 8B$$

and from (v), $$\sigma = \frac{1}{2}\left(-a + \frac{b}{d^3}\right) + \frac{3a}{2} = a + \frac{b}{2d^3}$$

Let d_i and d_o be the inner and outer diameters and the pressures on these surfaces be p_i and p_o.

$$p_i = -a + \frac{b}{d_i^3} \quad \text{and} \quad p_o = -a + \frac{b}{d_o^3}$$

Subtracting, $$p_i - p_o = b\left(\frac{1}{d_i^3} - \frac{1}{d_o^3}\right) = b\left(\frac{d_o^3 - d_i^3}{d_i^3 d_o^3}\right)$$

or $$b = \frac{d_i^3 d_o^3 (p_i - p_o)}{d_o^3 - d_i^3}$$

and $$a = -p_i + \frac{d_o^3(p_i - p_o)}{d_o^3 - d_i^3} = \frac{p_i d_i^3 - p_o d_o^3}{d_o^3 - d_i^3}$$

Now, $$p = -\frac{p_i d_i^3 - p_o d_o^3}{d_o^3 - d_i^3} + \frac{d_i^3 d_o^3 (p_i - p_o)}{(d_o^3 - d_i^3)d^3}$$

$$= \frac{p_o d_o^3 - p_i d_i^3 + (d_i^3 d_o^3 / d^3)(p_i - p_o)}{d_o^3 - d_i^3} \tag{13.29}$$

and $$\sigma_c = \frac{p_i d_i^3 - p_o d_o^3 + (d_i^3 d_o^3 / 2d^3)(p_i - p_o)}{d_o^3 - d_i^3} \tag{13.30}$$

Internal pressure only

$$p = \frac{-p_i d_i^3 d^3 + p_i d_i^3 d_o^3}{(d_o^3 - d_i^3)d^3} = \frac{p_i d_i^3 (d_o^3 - d^3)}{(d_o^3 - d_i^3)d^3} \tag{13.31}$$

and $$\sigma = \frac{2p_i d_i^3 d^3 + p_i d_i^3 d_o^3}{(d_o^3 - d_i^3)2d^3} = \frac{p_i d_i^3 (d_o^3 + 2d^3)}{(d_o^3 - d_i^3)2d^3} \tag{13.32}$$

Maximum hoop stress is at $d = d_i$, $$\sigma = \frac{p_i (d_o^3 + 2d_i^3)}{2(d_o^3 - d_i^3)} \tag{13.33}$$

Maximum shear stress is at $d = d_i$, $$\tau = \frac{1}{2}(\sigma_c + p_i) = \frac{3}{4}\frac{p_i d_o^3}{(d_o^3 - d_i^3)} \tag{13.34}$$

Example 13.31 A spherical shell of 60-mm inside diameter has to withstand an internal pressure of 25 MPa. Find the thickness of the shell if the maximum tensile stress is to be 75 MPa.

Solution

Given A spherical shell

$d_i = 60$ mm $\quad p = 25$ MPa $\quad \sigma = 75$ MPa

To find Thickness

Thickness of shell

$$\sigma = \frac{p_i(d_o^3 + 2d_i^3)}{2(d_o^3 - d_i^3)}$$

or
$$75 = \frac{d_o^3 + 2 \times 60^3}{2(d_o^3 - 60^3)} \times 25$$

or
$$d_o^3 + 2 \times 60^3 = 6(d_o^3 - 60^3)$$

or
$$5d_o^3 = 8 \times 60^3 \quad \text{or} \quad d_o = 70.18 \text{ mm}$$

Summary

1. Cylinders and spheres subjected to fluid pressure are also known as *pressure vessels* or *shells*.
2. There are three principal stresses: *circumferential* or *hoop* stress, *longitudinal* stress and *radial* stress.
3. A shell may be termed as *thin* or *thick* depending upon the ratio of the thickness of the wall to the diameter of the shell. If the ratio is less than 1 to 20, a shell is considered as thin, otherwise it is thick.
4. In a thin shell, it can be assumed that the hoop stresses are constant over the thickness and the radial stress is small and may be neglected.
5. Circumferential or hoop stress in a thin cylinder, $\sigma_c = pd/2t$
6. Longitudinal stress in a thin cylinder, $\sigma_l = pd/4t$
7. Hoop stress in a thin spherical shell, $\sigma = pd/4t$
8. Volumetric strain of a thin cylinder = Longitudinal strain + 2 × Hoop strain
9. Volumetric strain of a spherical shell = 3 × Hoop strain
10. A tube can be strengthened against the internal pressure by winding it with wire under tension and putting the tube wall in compression.
11. The governing equations for thick cylinders are $p = -a + \dfrac{b}{d^2}$ and $\sigma_c = a + \dfrac{b}{d^2}$
12. For thick cylinders with internal pressure only, stresses at any diameter d,

$$p = \frac{p_i d_i^2 (d_o^2 - d^2)}{(d_o^2 - d_i^2)d^2} \quad \text{and} \quad \sigma_c = \frac{p_i d_i^2 (d_o^2 + d^2)}{(d_o^2 - d_i^2)d^2}$$

The maximum hoop stress is at $d = d_i$, $\sigma_c = \dfrac{p_i(d_o^2 + d_i^2)}{(d_o^2 - d_i^2)}$

Maximum shear stress (at $d = d_i$) $= \dfrac{1}{2}(\sigma_c + p_i) = \dfrac{p_i d_o^2}{(d_o^2 - d_i^2)}$

Longitudinal stress for a closed end cylinder, $\sigma_l = \dfrac{p_i d_i^2}{(d_o^2 - d_i^2)}$

13. For thick cylinders with external pressure only, stresses at any diameter d,

$$p = \frac{p_o d_o^2}{d_o^2 - d_i^2}\left(1 - \frac{d_i^2}{d^2}\right) \text{ and } \sigma_c = \frac{-p_o d_o^2}{d_o^2 - d_i^2}\left(1 - \frac{d_i^2}{d^2}\right)$$

14. To even out the stresses in a thick tube, it may be made of two parts, one shrunk on to the other by heating it to the required temperature. This makes the inner tube in compression and the outer tube in tension. On applying the internal pressure, a tensile hoop stress is superimposed on the compressive or shrinkage stresses of the internal tube.
15. For thick spherical shells with internal pressure only, stresses at any diameter d,

$$p = \frac{p_i d_i^3 (d_o^3 - d^3)}{(d_o^3 - d_i^3) d^3} \text{ and } \sigma = \frac{p_i d_i^3 (d_o^3 - 2d^3)}{(d_o^3 - d_i^3) 2d^3}$$

Maximum hoop stress is at $d = d_i$, $\sigma_c = \dfrac{p_i (d_o^3 + 2d_i^3)}{2(d_o^3 - d_i^3)}$

Maximum shear stress is at $d = d_i$, $\tau = \dfrac{1}{2}(\sigma_c + p_i) = \dfrac{3}{4}\dfrac{p_i d_o^3}{(d_o^3 - d_i^3)}$

Objective Type Questions

1. A shell may be termed as *thin* if the ratio of thickness of the wall to the diameter of the shell is less than one to
(a) 5 (b) 10 (c) 15 (d) 20

2. In a thin cylinder, the hoop stress is given by
(a) $\dfrac{pd}{4t}$ (b) $\dfrac{pd}{t}$ (c) $\dfrac{pd}{2t}$ (d) $\dfrac{2pd}{t}$

3. In a thin cylinder, the longitudinal stress is given by
(a) $\dfrac{2pd}{t}$ (b) $\dfrac{pd}{t}$ (c) $\dfrac{pd}{2t}$ (d) $\dfrac{pd}{4t}$

4. In a thin cylinder, the ratio of hoop stress to longitudinal stress is
(a) 1/4 (b) 1/2 (c) 2 (d) 4

5. In a thin spherical shell, the hoop stress is given by
(a) $\dfrac{pd}{4t}$ (b) $\dfrac{pd}{2t}$ (c) $\dfrac{pd}{t}$ (d) $\dfrac{2pd}{t}$

6. The volumetric strain in a thin spherical shell is
(a) $\dfrac{3pd}{4tE}(1-\nu)$ (b) $\dfrac{3pd}{4tE}(1-2\nu)$ (c) $\dfrac{pd}{4tE}(1-\nu)$ (d) $\dfrac{pd}{4tE}(1-2\nu)$

7. The initial hoop stress in a thin cylinder when it is wound with a wire under tension is
(a) zero (b) tensile (c) compressive (d) bending

8. In a thick-walled cylinder subjected to internal pressure, maximum hoop stress occurs at
(a) outer wall (b) inner wall (c) midpoint of thickness

9. In thick cylindrical pressure vessels, the variation of the hoop stress is
(a) parabolic (b) uniform (c) linear (d) cubic

10. In thick cylindrical pressure vessels, the variation of the radial stress is

(a) parabolic (b) uniform (c) linear (d) cubic

11. In a thick-walled cylinder subjected to external pressure, the hoop stresses are
(a) tensile (b) compressive (c) bending

12. The use of compound tubes subjected to internal pressure are made to
(a) even out the stresses (b) increase the thickness
(c) increase the diameter of the tube (d) increase the strength

13. In compound tubes, initially, the inside diameter of the outer tube is ______ the outside diameter of the inner tube.
(a) smaller than (b) larger than (c) equal to (d) 1.2 times

14. The maximum stress in thick cylinders is
(a) Radial stress (b) hoop stress (c) longitudinal stress

15. In thick spherical pressure vessels, the variation of the stresses is
(a) linear (b) uniform (c) parabolic (d) cubic

Answers

1. (c)	2. (c)	3. (d)	4. (c)	5. (b)	6. (a)
7. (c)	8. (b)	9. (a)	10. (a)	11. (b)	12. (a)
13. (a)	14. (b)	15. (d)			

Review Questions

13.1 What do you mean by pressure vessels or shells? What type of stresses act upon them?

13.2 How do you distinguish between thin and thick pressure shells?

13.3 What assumptions are taken in the analysis of thin cylinders? Deduce expressions for the circumferential and longitudinal stresses developed in them. What are the stresses developed in a spherical shell?

13.4 Show that the volumetric strain of a cylindrical shell is the sum of longitudinal strain and twice of hoop strain.

13.5 What is the importance of wire winding of thin cylinders? Explain the procedure to analyze them? What are the assumptions taken?

13.6 Deduce the general equations for circumferential and radial stresses developed in thick cylinders. What are the assumptions made?

13.7 Deduce the simplified expressions for the maximum values of circumferential and radial stresses in a thick cylinder when acted upon by (a) internal pressure only and (b) external pressure only.

13.8 What do you mean by compound tubes used as pressure vessels? Explain.

13.9 Deduce the general equations for circumferential and radial stresses developed in a thick spherical shell acted upon by internal and external pressures. Deduce the simplified expressions for their maximum values when acted upon by internal pressure only.

Numerical Problems

13.1 A 6-m long thin cylindrical shell is 800 mm in diameter and 10 mm thick. It is subjected to an internal pressure of 4 MPa. Determine the change in diameter, change in length and change in volume of the shell. E = 205 GPa and Poisson's ratio 0.3. (0.664 mm, 0.937 mm, 4.472×10^6 mm^2)

13.2 A mild steel water pipeline of 2-m diameter and of 10-mm thickness sustains an allowable stress of 140 MPa. Find the maximum pressure in the pipe. What will be the change in the volume of the pipe per metre length under the maximum pressure? E = 200 GPa and ν =0.3. (1.4 MPa; 4.18 mm^3)

13.3 A 600-mm long, 8-mm thick and 160-mm in diameter cylindrical shell is filled with water at atmospheric pressure. Determine the pressure exerted by the fluid on the cylinder and the hoop stress developed if an additional volume of 18 000 mm^3 of water is pumped into the cylinder. $E = 210$ GPa, $K = 2250$ MPa and $\mu = 0.3$. (2.204 MPa, 22.04 MPa)

13.4 A hollow cylinder of 800-mm diameter and 10-mm thickness is 1.5 m long. When it is subjected to an internal pressure of 4.5 MPa, the volume is increased by 1250 ml. Calculate the Poisson's ratio of the material of the cylinder. $E = 205$ GPa. (0.306)

13.5 A steel tube of 100-mm internal diameter and 1.2 mm thick is 250 mm long. It is completely filled with oil and is subjected to a compressive load of 45 kN. Determine the stress produced in the oil and the resulting hoop stress in the tube wall. Take $E = 200$ GPa, Poisson's ratio 0.3 and $K = 2600$ MPa for oil. (0.272 MPa; 11.33 MPa)

13.6 A thin cylinder of 4-mm thickness and of 60-mm internal diameter is subjected to an internal pressure of 2 MPa along with a torque of 96 N · m, the axis of which coincides with that of the cylinder. Determine the principal stresses and the maximum shear stress at a point on the surface of the cylinder. (16.54 MPa, 5.96 MPa; 5.29 MPa)

13.7 A thin spherical steel vessel having a diameter of 1.5 m is of uniform thickness. After filling it with water at a pressure of 2 MPa, a relief valve attached to the vessel is opened and water is allowed to escape until the pressure falls to atmospheric. If the volume of water escaped is 4000 cc, find thickness of the plates of the vessel. $K = 2000$ MPa, $E = 200$ GPa and Poisson's ratio is 0.3. (6.23 mm)

13.8 A cast iron pipe of 260-mm diameter and 20-mm thick is wound with a layer of 6-mm diameter steel wire under a tensile stress of 45 MPa. Determine the stresses developed in the pipe and in the wire when water is admitted under a pressure of 3.5 MPa into the pipe. $E_s = 204$ GPa, $E_{ci} = 102$ GPa and Poisson's ratio 0.3. (In pipe: $\sigma_c = 5.185$ MPa, $\sigma_l = 16.56$ MPa; In wire: 71.3 MPa)

13.9 Determine the thickness of metal required for a cylindrical shell of steel of 180-mm internal diameter to withstand an internal pressure of 30 MPa. The circumferential stress in the section must not exceed 150 MPa. (47.7 mm)

13.10 The maximum permissible stress in a cylinder of 500 mm diameter and of 100 mm thickness is 15 MPa. Find the maximum allowable internal and external pressures on the cylinder, when applied separately. (4.87 MPa, 3.67 MPa)

13.11 A cylinder of 150-mm inside diameter and 200-mm outside diameter is subjected to a liquid pressure from inside apart from a compressive load of 220 kN applied at the ends of the cylinder. What is the greatest pressure of the liquid so that the maximum stress in the material reaches to its maximum value of 44 MPa? (12.43 MPa)

13.12 A thick cylinder of 160-mm internal diameter and 240-mm external diameter is subjected to an external pressure of 12 MPa. Determine the maximum value of the internal pressure that can be applied if the maximum allowable stress is to be 36 MPa. Plot the variation of radial and circumferential stresses developed in the material of the cylinder. (23.5 MPa)

13.13 The cylinder of a hydraulic ram of 60-mm diameter is to withstand an internal pressure of 35 MPa. Determine the thickness of the metal if the maximum tensile stress and the maximum shear stress in the material are limited to 55 MPa and 50 MPa respectively. (24.75 mm)

13.14 A thick hollow cylinder of 200-mm internal diameter and 300-mm external diameter is subjected to an internal pressure of 50 MPa and external pressure of 25 MPa. Find the maximum shear stress developed in the material at the inner radius. (45 MPa)

13.15 Determine the ratio of thickness to the internal diameter of a hollow cylinder subjected to internal pressure if the pressure is three-fourth of the value of the maximum allowable hoop stress. (0.823)

13.16 A thick cylindrical sleeve of 200-mm external diameter and 120-mm internal diameter is shrunk on a solid shaft. The internal diameter of the sleeve increases by 0.15 mm due to force fitting. Calculate the force fit pressure between the sleeve and the shaft. $E = 205$ GPa, Poisson's ratio is 0.28. (106.5 MPa)

13.17 A steel hub of 75-mm inside radius and 125-mm outside radius is shrunk on a steel shaft with initial diameter of 150.1 mm. Determine the maximum hoop stress, contact pressure and the final diameter of the surface in contact. $E = 200$ GPa, $\nu = 0.3$. (116.9 MPa; 55 MPa; 150.071 mm)

13.18 A steel tube of 50-mm outer diameter is shrunk on another tube of 30-mm inner diameter and 40-mm outer diameter. The compound tube is made to withstand an internal pressure of 50 MPa. The shrinkage allowance is such that the final maximum stress in each tube is the same. Find this value of the stress and show on a diagram the variation of circumferential stress in the tube. Also find the initial difference of diameters before shrinking on? $E = 208$ GPa. (89.2 MPa; 0.0059 mm)

13.19 A thick spherical shell of 80-mm inside diameter is subjected to an internal fluid pressure of 30 MPa. Determine the thickness of the shell if the permissible tensile stress is 75 MPa. (8.2 mm)

13.20 A thick spherical shell of 250-mm inside diameter and 50-mm thickness is subjected to an internal pressure of 15 MPa. Find the variation of circumferential and radial stresses in the shell and plot the same. What is increase in the inside and outside diameters of the shell? $E = 200$ GPa and $\nu = 0.3$. (0.0235 mm and 0.0158 mm)

Chapter **14**

Rotating Discs and Cylinders

There are machine elements which rotate while performing the required function. These include flywheels, thin rings, circular discs, pulley rims, cylinders and spherical shells. Due to rotation, centrifugal stresses are developed in these elements. As these stresses are equivalent to outward radial pressure on vessels, these give rise to radial and circumferential or hoop stresses as in case of cylinders and shells. Thus, the analysis of the rotating elements will be almost similar to those. In this chapter, a study of stresses on rotating rings and discs of solid and hollow sections is made.

14.1 THIN ROTATING RING

A ring may be considered thin in the radial direction if the variation of stresses along the thickness is negligible and can be ignored. Consider a thin ring rotating about its centre of mass as shown in Fig. 14.1.

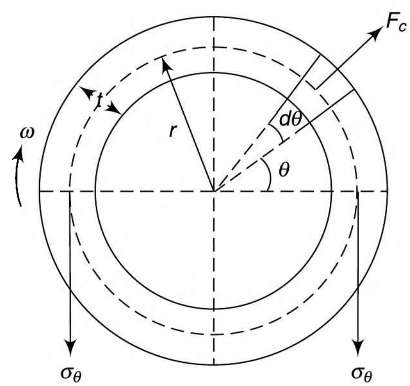


Fig. 14.1

Let ω = angular velocity

r = mean radius

t = thickness of the ring

ρ = density of the material of the ring

Consider an element of the ring subtending an angle $d\theta$ at the centre at an angle θ with the x-axis.

Centrifugal force on the element/unit length

$$= [\rho(r \cdot d\,\theta)\, t \cdot 1] \cdot r\, \omega^2$$

Vertical component of the force $= \rho r^2 \cdot d\theta \cdot t \cdot \omega^2 \sin\theta$

$$\text{Total vertical force/unit length} = \int_0^\pi \rho \cdot r^2 \cdot d\theta \cdot t\,.\,\omega^2 \sin\theta = \rho \cdot r^2 \cdot t \cdot \omega^2 \int_0^\pi \sin\theta \cdot d\theta$$

$$= \rho \cdot r^2 \cdot t \cdot \omega^2 (-\cos\theta)_0^\pi = 2\rho \cdot r^2 \cdot t \cdot \omega^2$$

Let σ_θ = Hoop stress induced in the ring

Then for equilibrium, $\quad \sigma_\theta\,(2t) \cdot 1 = 2\,\rho \cdot r^2 \cdot t \cdot \omega^2$

$$\sigma_\theta = \rho \cdot r^2 \omega^2 = \rho \cdot v^2 \quad (14.1)$$

where v is the mean tangential velocity of the ring.

Example 14.1 The rim of a rotating wheel is 1.2 m in diameter. Determine the limiting speed of the wheel and the change in diameter if the maximum stress is not to exceed 130 MPa. Density of the material is 7700 kg/m³ and E = 205 GPa. Neglect the effect of spokes of the wheel. Treat the rim to be thin.

Solution

Given Rim of a rotating wheel

$d = 1.2$ m $\sigma_\theta = 130$ N/mm² $= 130 \times 10^6$ N/m²

$\rho = 7700$ kg/m³ $E = 205$ GPa

To find

— Limiting speed of wheel

— Change in diameter

$$r = 1.2/2 = 0.6 \text{ m}$$

Limiting speed

$$\sigma_\theta = \rho \cdot r^2 \omega^2$$

or $$130 \times 10^6 = 7700 \times 0.6^2 \times \omega^2$$

or $$\omega = 216.6 \text{ rad/s} \quad \text{or} \quad N = \frac{216.6 \times 60}{2\pi} = 2068 \text{ rpm}$$

Change in diameter

Hoop strain, $$\varepsilon = \frac{\delta d}{d} = \frac{\sigma_\theta}{E}$$

or $$\delta d = \frac{\sigma_\theta}{E} \cdot d = \frac{130}{205\,000} \times 1200 = 0.76 \text{ mm}$$

(Note the consistency of units in the two relations used)

Example 14.2 A flywheel with a moment of inertia of 300 kg·m² rotates at 300 rpm. If the maximum stress is not to exceed 6 MPa, find the thickness of the rim. Take the width of the rim as 150 mm and the density of the material 7400 kg/m³. Neglect the effect of inertia of spokes.

Solution

Given A flywheel rim

$I = 300$ kg·m² $\sigma_\theta = 6$ N/mm² $= 6 \times 10^6$ N/m²

$\rho = 7400$ kg/m³ $w = 0.15$ m

$N = 300$ rpm

To find Thickness of rim

$$\omega = \frac{2\pi \times 300}{60} = 10\pi \text{ rad/s}$$

Determination of outer radius

The stress is maximum at the outer radius r_o,

$$\sigma_\theta = \rho \cdot r_o^2 \omega^2$$

or $$6 \times 10^6 = 7400\, r_o^2 \times (10\pi)^2$$

or $$r_o^2 = 0.82 \quad \text{or} \quad r_o = 0.9 \text{ m}$$

Determination of thickness

As an approximation, initially assume some value of the mean radius and the radius of gyration. Let it be 0.85 m (a little less than 0.9 m). Then if t is the thickness,

$$\text{Moment of inertia} = [(2\pi r \cdot w \cdot t)\,\rho]\,r^2$$

or $$300 = (2\pi \times 0.85 \times 0.15 \times t \times 7400) \times 0.85^2$$

or $$t = 0.07 \text{ m}$$

Inner radius, $r_i = 0.9 - 0.07 = 0.83$ m and mean radius $= \dfrac{0.9 + 0.83}{2} = 0.865$ m

If k is the radius of gyration, $k^2 = \dfrac{0.9^2 + 0.83^2}{2} = 0.749 \text{ m}^2$

$\therefore$ moment of inertia, $I = mk^2$ or $300 = (2\pi \times 0.865 \times 0.15 \times t \times 7400) \times 0.749$

or $t = 0.066$ m or 66 mm

which approximately satisfies the assumption of mean radius (865 mm) and outer radius (900 mm).

Example 14.3 A built-up ring is made up of two materials.The outer ring is of steel and the inner one of copper. The diameter of the common surfaces is 800 mm. Each ring has a width of 30 mm and a thickness of 20 mm in the radial direction.The ring rotates at 1800 rpm. Find the stresses set up in the steel and the copper. $E_s = 2E_c$; Density of steel = 7300 kg/m^3; Density of copper = 9000 kg/m^3.

Solution

Given A built-up ring is made up of two materials

$d = 800$ mm	$t = 20$ mm
$w = 30$ mm	$N = 1800$ rpm
$\rho_s = 7300$ kg/m^3	$\rho_c = 9000$ kg/m^3
$E_s = 2E_c$	

To find Stresses in steel and copper

Refer Fig. 14.2.

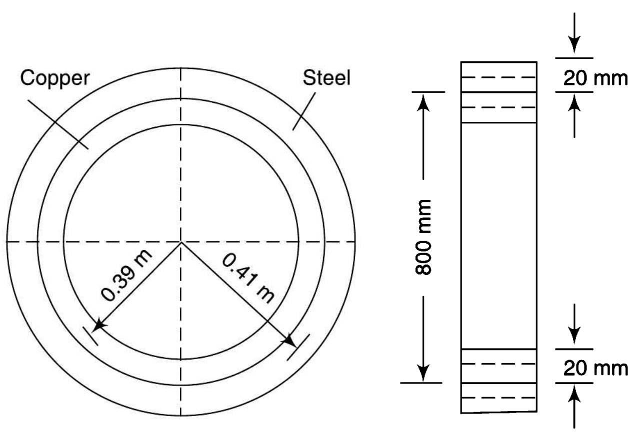


Fig. 14.2

$$r = 400 \text{ mm} = 0.4 \text{ m};\ \omega = \frac{2\pi \times 1800}{60} = 60\pi \text{ rad/s}$$

Let p be the shrinkage pressure at the common surface at stand still.

Hoop stress due to shrinkage

In the steel ring $= \dfrac{pd}{2t} = \dfrac{p \times 800}{2 \times 20} = 20p$ (tensile)

(as in case of thin cylinder with internal pressure)

In the copper ring $= \dfrac{p \times 800}{2 \times 20} = 20p$ (compressive)

(as in case of thin cylinder with external pressure)

Hoop stress due to rotation

In steel, $\sigma_\theta = \rho \cdot r^2 \omega^2 = 7300(0.4 + 0.01)^2 \times (60\pi)^2$

$= 43.6 \times 10^6$ N/m^2 or 43.6 MPa (tensile)

In copper, $\sigma_\theta = 9000(0.4 - 0.01)^2 \times (60\pi)^2$

$= 48.6 \times 10^6$ N/m^2 or 48.6 MPa (tensile)

Equating the net strains

As the net strains of the two must be equal,

$$\frac{20p + 43.6}{E_s} = \frac{-20p + 48.6}{E_c}$$

or
$$\frac{20p + 43.6}{2E_c} = \frac{-20p + 48.6}{E_c}$$

or $20p + 43.6 = 2(48.6 - 20p)$ or $p = 0.893$ MPa

Total stress in steel $= 20 \times 0.893 + 43.6 = 61.5$ MPa

Total stress in copper $= -20 \times 0.893 + 48.6 = 30.7$ MPa

14.2 DISC OF UNIFORM THICKNESS

Consider a flat rotating disc of uniform thickness t and having R_o and R_i as the outer and the inner radii respectively. Let the angular speed of the disc be ω.

For a disc of small axial width, it may be assumed that the stress in the axial direction is zero.

For an element of the disc of unit thickness (Fig. 14.3), let

σ_r = radial stress at the inner face at radius r of the element

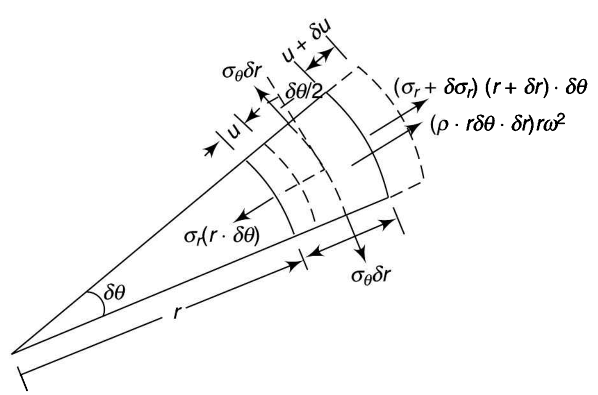


Fig. 14.3

$\sigma_r + \delta\sigma_r$ = radial stress at the outer face at radius $(r + \delta r)$ of the element

σ_θ = circumferential (hoop) stress on the radial faces

As the disc rotates, let u be the radial shift at an unstrained radius r i.e. r becomes $r+u$ after straining. Similarly, $u+\delta u$ be the radial shift at an unstrained radius $r + \delta r$.

- Radial strain $= \dfrac{\text{Increase in } \delta r}{\delta r} = \dfrac{u + \delta u - u}{dr} = \dfrac{du}{dr}$ in the limit

$$\therefore \text{ radial stress} = E \cdot \frac{du}{dr} = \sigma_r - \nu\sigma_\theta \tag{14.2}$$

- Circumferential (hoop) strain $= \dfrac{\text{Increase in circumference}}{\text{original circumference}} = \dfrac{2\pi(r+u) - 2\pi r}{2\pi r} = \dfrac{u}{r}$

$$\therefore \text{ circumferential stress} = E \cdot \frac{u}{r} = \sigma_\theta - \nu\sigma_r$$

or $$E \cdot u = r(\sigma_\theta - \nu\sigma_r)$$

Differentiating it, $$E \cdot \frac{du}{dr} = r\left(\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr}\right) + \sigma_\theta - \nu\sigma_r \tag{14.3}$$

From Eqs. 14.2 and 14.3,

$$\sigma_r - \nu\sigma_\theta = r\left(\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr}\right) + \sigma_\theta - \nu\sigma_r$$

or $$(1+\nu)(\sigma_r - \sigma_\theta) = r\left(\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr}\right)$$

or $$\sigma_r - \sigma_\theta = \frac{r}{1+\nu}\left(\frac{d\sigma_\theta}{dr} - \nu\frac{d\sigma_r}{dr}\right) \tag{14.4}$$

- For equilibrium of forces in radial direction, considering a unit length of element,

Centrifugal force $= mr\omega^2 = [\rho(r\,\delta\,\theta)\cdot\delta r]\,r\,\omega^2$ (outward)

Radial force on inner face $= \sigma_r\,(r\cdot\delta\theta)$ (inward)

Radial force on outer face $= (\sigma_r + \delta\sigma_r)(r + \delta r)\cdot\delta\theta$ (outward)

Radial components of tangential force $= 2\sigma_\theta\cdot(\delta r\cdot 1)\sin\frac{1}{2}\delta\theta \approx \sigma_\theta\cdot\delta r\cdot\delta\theta$ (inward)

For equilibrium, Net inward force = Net outward force

$$\sigma_\theta\cdot\delta r\cdot\delta\theta + \sigma_r r\cdot\delta\theta - (\sigma_r + \delta\sigma_r)(r + \delta r)\cdot\delta\theta = \rho(r\delta\theta)\cdot\delta r\cdot r\omega^2$$

or $$\sigma_\theta\cdot\delta r + \sigma_r r - (\sigma_r + \delta\sigma_r)(r + \delta r) = \rho r^2\cdot\delta r\cdot\omega^2$$

Simplifying and taking limits,

$$\sigma_\theta\cdot dr + \sigma_r r - (\sigma_r r + \sigma_r dr + r\cdot d\sigma_r) = \rho r^2\omega^2\cdot dr$$

$$\sigma_\theta\cdot dr - \sigma_r dr - r\cdot d\sigma_r) = \rho r^2\omega^2\cdot dr$$

Dividing by dr, the equilibrium equation is obtained,

$$\sigma_\theta - \sigma_r - \frac{r\cdot d\sigma_r}{dr} = \rho r^2\omega^2$$

or $$\sigma_r - \sigma_\theta = -\frac{r\cdot d\sigma_r}{dr} - \rho r^2\omega^2 \tag{14.5}$$

From Eqs. 14.4 and 14.5,

$$\frac{r}{1+\nu}\left(\frac{d\sigma_\theta}{dr}-\nu\frac{d\sigma_r}{dr}\right)=-\frac{r\cdot d\sigma_r}{dr}-\rho r^2\omega^2$$

or $$\frac{d\sigma_\theta}{dr}-\nu\frac{d\sigma_r}{dr}+(1+\nu)\frac{d\sigma_r}{dr}=-\rho r^2\omega^2(1+\nu)$$

or $$\frac{d\sigma_\theta}{dr}-\frac{d\sigma_r}{dr}(\nu-1-\nu)=-\rho r\omega^2(1+\nu)$$

or $$\frac{d}{dr}(\sigma_\theta+\sigma_r)=-(1+\nu)\rho r^2\omega^2$$

Integrating, $$\sigma_r+\sigma_\theta=-\frac{1}{2}(1+\nu)\rho r^2\omega^2+2A \quad (14.6)$$

Adding Eqs. 14.5 and 14.6,

$$2\sigma_r+r\cdot\frac{d\sigma_r}{dr}+\rho r^2\omega^2=-\frac{1}{2}(1+\nu)\rho r^2\omega^2+2A$$

or $$\left(2\sigma_r+r\cdot\frac{d\sigma_r}{dr}\right)=-\frac{3+\nu}{2}\rho r^2\omega^2+2A$$

or $$\frac{1}{r}\cdot\frac{d(r^2\cdot\sigma_r)}{dr}=-\frac{3+\nu}{2}\rho r^2\omega^2+2A$$

or $$\frac{d(r^2\cdot\sigma_r)}{dr}=-\frac{3+\nu}{2}\rho r^3\omega^2+2Ar$$

Integrating, $$r^2\cdot\sigma_r=-\frac{3+\nu}{8}\rho r^4\omega^2+Ar^2-B$$

or $$\sigma_r=A-\frac{B}{r^2}-\frac{3+\nu}{8}\rho r^2\omega^2 \quad (14.7)$$

Inserting this value in Eq. 14.6,

$$A-\frac{B}{r^2}-\frac{3+\nu}{8}\rho r^2\omega^2+\sigma_\theta=-\frac{1}{2}(1+\nu)\rho r^2\omega^2+2A$$

$$\sigma_\theta=-\frac{1}{2}(1+\nu)\rho r^2\omega^2+A+\frac{B}{r^2}+\frac{3+\nu}{8}\rho r^2\omega^2=A+\frac{B}{r^2}-\frac{1+3\nu}{8}\rho r^2\omega^2 \quad (14.8)$$

Equations 14.7 and 14.8 are the governing equations of a rotating disc of uniform thickness.

Solid Disc

In a solid disc, the stress at the centre, $\sigma_r=A-\frac{B}{0}$, i.e., infinite which is not possible,

$\therefore$ B has to be zero.

Let R be the outside diameter. Then at the outer surface, $\sigma_R=0$,

Therefore $0=A-\frac{3+\nu}{8}\rho R^2\omega^2$ or $A=\frac{3+\nu}{8}\rho R^2\omega^2$

Thus $\sigma_r = \dfrac{3+\nu}{8}\rho\omega^2(R^2 - r^2)$ (14.9)

and $\sigma_\theta = \dfrac{\rho\omega^2}{8}[(3+\nu)R^2 - (1+3\nu)r^2]$ (14.10)

- At the centre, $r = 0$, thus $\sigma_r = \sigma_\theta = \dfrac{3+\nu}{8}\rho\omega^2 R^2$ (maximum value of stress) (14.11)
- At the outer surface, $\sigma_\theta = \dfrac{1-\nu}{4}\rho\omega^2 R^2$ and $\sigma_r = 0$ (14.12)
- If $\nu = 0.3$, $\sigma_r = \sigma_\theta = \dfrac{3+0.3}{8}\rho\omega^2 R^2 = 0.413\rho\omega^2 R^2$ at the centre (maximum)

and at the outer surface, $\sigma_\theta = \dfrac{1-0.3}{4}\rho\omega^2 R^2 = 0.175\rho\omega^2 R^2$

$$= \frac{0.175}{0.413}\sigma_{\theta(\max)} = 0.424\sigma_{\theta(\max)}$$

The variation of stresses with radius is shown in Fig. 14.4.

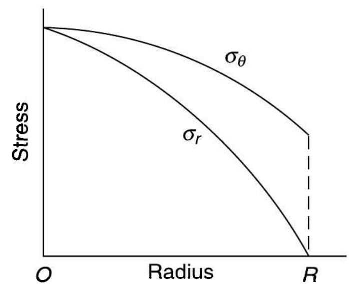


Fig. 14.4

Hollow Disc

Let R_i and R_o be the inside and the outside radii of a hollow disc respectively. Radial stresses are to be zero at these values.

$$\sigma_r = 0 = A - \frac{B}{R_i^2} - \frac{3+\nu}{8}\rho R_i^2\omega^2 \quad \text{(i)}$$

$$\sigma_r = 0 = A - \frac{B}{R_o^2} - \frac{3+\nu}{8}\rho R_o^2\omega^2 \quad \text{(ii)}$$

From (i) from (ii), $-\dfrac{B}{R_i^2} - \dfrac{3+\nu}{8}\rho R_i^2\omega^2 = -\dfrac{B}{R_o^2} - \dfrac{3+\nu}{8}\rho R_o^2\omega$

Multiplying throughout by $R_i^2 R_o^2$,

or $$-BR_o^2 - \frac{3+\nu}{8}\rho R_i^4 R_o^2\omega^2 + BR_i^2 + \frac{3+\nu}{8}\rho R_i^2 R_o^4\omega = 0$$

$$B(R_i^2 - R_o^2) - \frac{3+\nu}{8}\rho R_i^2 R_o^2\omega^2(R_i^2 - R_o^2) = 0$$

or $$B = \frac{3+\nu}{8}\rho R_i^2 R_o^2\omega^2$$

and thus $\sigma_r = 0 = A - \dfrac{3+\nu}{8}\rho R_o^2\omega^2 - \dfrac{3+\nu}{8}\rho R_i^2\omega^2$ [from (i)]

or $$A = \frac{3+\nu}{8}\rho\omega^2(R_i^2 + R_o^2)$$

Inserting the values of constants A and B in Eq. 14.7,

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2(R_i^2 + R_o^2) - \frac{3+\nu}{8r^2}\rho R_i^2 R_o^2\omega^2 - \frac{3+\nu}{8}\rho r^2\omega^2$$

$$= \frac{\rho\omega^2}{8}(3+\nu)\left(R_i^2 + R_o^2 - \frac{R_i^2 R_o^2}{r^2} - r^2\right) \tag{14.13}$$

Inserting the values of constants A and B in Eq. 14.8,

$$\sigma_\theta = \frac{\rho\omega^2}{8}\left[(3+\nu)\left(R_i^2 + R_o^2 + \frac{R_i^2 R_o^2}{r^2}\right) - (1+3\nu)r^2\right] \tag{14.14}$$

- σ_r is maximum when $\frac{d\sigma_r}{dr} = 0$ or $\frac{2R_i^2 R_o^2}{r^3} - 2r = 0$

or $R_i^2 R_o^2 - r^4 = 0$ or at $r = \sqrt{R_i R_o}$ (14.15)

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2\left(R_i^2 + R_o^2 - \frac{R_i^2 R_o^2}{R_i R_o} - R_i R_o\right)$$

$$= \frac{3+\nu}{8}\rho\omega^2(R_i^2 + R_o^2 - 2R_i R_o) = \frac{3+\nu}{8}\rho\omega^2(R_o - R_i)^2 \tag{14.16}$$

- σ_θ is maximum at inside ($r = R_i$),

$$\sigma_\theta = \frac{\rho\omega^2}{8}\left[(3+\nu)\left(R_i^2 + R_o^2 + \frac{R_i^2 R_o^2}{R_i^2}\right) - (1+3\nu)R_i^2\right]$$

$$= \frac{\rho\omega^2}{4}[(1-\nu)R_i^2 + (3+\nu)R_o^2] \tag{14.17}$$

- If R_i is very small, $\sigma_\theta = \frac{\rho\omega^2}{4}[(3+\nu)R_o^2]$, i.e., twice that for a solid disc. (14.18)

- At outside, $\sigma_\theta = \frac{\rho\omega^2}{8}\left[(3+\nu)\left(R_i^2 + R_o^2 + \frac{R_i^2 R_o^2}{R_o^2}\right) - (1+3\nu)R_o^2\right]$

$$= \frac{\rho\omega^2}{8}[(3+\nu)(2R_i^2 + R_o^2) - (1+3\nu)R_o^2]$$

$$= \frac{\rho\omega^2}{8}[(3+\nu)2R_i^2 + (3+\nu)R_o^2 - (1+3\nu)R_o^2]$$

$$= \frac{\rho\omega^2}{4}[(3+\nu)R_i^2 + (1-\nu)R_o^2] \tag{14.19}$$

- In a thin rotating ring, $R_i \to R_o = R$

$$\sigma_\theta = \frac{\rho\omega^2}{8}\left[(3+\nu)\left(R^2 + R^2 + \frac{R^2R^2}{R^2}\right) - (1+3\nu)R^2\right] = \rho\omega^2 R^2$$

the same result as obtained in Section 14.2.

The variation of stresses is shown in Fig. 14.5.

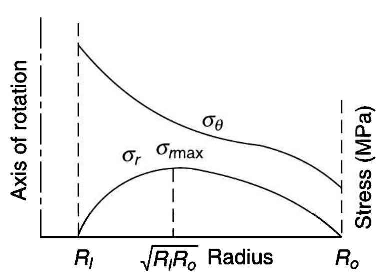


Fig. 14.5

Example 14.4 A disc of uniform thickness and of 600-mm diameter rotates at 1800 rpm. Find the maximum stress developed in the disc. If a hole of 100-mm diameter is made at the centre of the disc, find the maximum values of radial and hoop stresses. Density of the material of the disc = 7700 kg/m^3 and ν = 0.3.

Solution

Given A solid disc of uniform thickness

d = 600 mm $\qquad N$ = 1800 rpm

ρ = 7700 kg/m^3 $\qquad \nu = 0.3$

To find

— Maximum stress

— Maximum radial and hoop stresses when a hole of 100 mm is made

$$R = 0.3\text{ m}; \qquad \omega = \frac{2\pi \times 1800}{60} = 60\pi \text{ rad/s}$$

For solid disc

Maximum radial stress and hoop stress are at the centre and are equal,

$$\sigma_r = \sigma_\theta = \frac{3+\nu}{8}\rho\omega^2 R^2 = \frac{3+0.3}{8} \times 7700 \times (60\pi)^2 \times 0.3^2 \qquad \text{...(Eq. 14.11)}$$

$$= 112.85 \times 10^6 \times 0.3^2 = 10.16 \times 10^6 \text{ N/m}^2 \quad \text{or} \quad 10.16 \text{ MPa}$$

When a hole is made

R_i = 0.05 m and R_o = 0.3 m

Maximum radial stress is at $\sqrt{R_i R_o}$ radius, i.e., at radius $\sqrt{50 \times 300} = 122.5$ mm

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2 (R_o - R_i)^2 \qquad \text{...(Eq. 14.15)}$$

$$= \frac{3+0.3}{8} \times 7700 \times (60\pi)^2 (0.3 - 0.05)^2 \qquad \text{...(Eq. 14.16)}$$

$$= 122.85 \times 10^6 \times 0.0625 = 7.05 \times 10^6 \text{ N/m}^2 \text{ or } 7.05 \text{ MPa}$$

Maximum hoop stress is at the inner radius,

$$\sigma_\theta = \frac{\rho\omega^2}{4}[(1-\nu)R_i^2 + (3+\nu)R_o^2] \qquad \text{...(Eq. 14.17)}$$

$$= \frac{7700 \times (60\pi)^2}{4}[(1-0.3) \times 0.05^2 + (3+0.3) \times 0.3^2]$$

$$= 68.396 \times 10^6 \times 0.298\,75 = 20.43 \times 10^6 \text{ N/m}^2$$

or 20.43 MPa

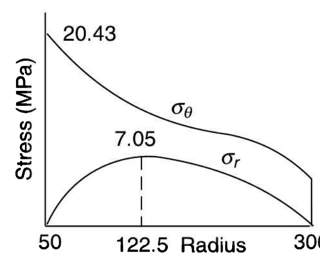


Fig. 14.6

Figure 14.6 shows the maximum values of hoop and radial stresses.

Example 14.5 A solid disc of uniform thickness and having a diameter of 400 mm rotates at 7500 rpm. Determine the radial and the hoop stresses at radii of 0, 50 mm, 100 mm, 150 mm and 200 mm. Density of the material is 7500 kg/m^2. What are the maximum values of the radial, hoop and shear stresses?

Solution

Given A solid disc of uniform thickness

d = 400 mm $\qquad N$ = 7500 rpm

ρ = 7500 kg/m^3

To find

— Radial and hoop stresses at 0, 50 mm, 100 mm, 150 mm and 200 mm
— maximum values of radial, hoop and shear stresses

$$R = 200 \text{ mm} = 0.2 \text{ m};\ \omega = \frac{2\pi \times 7500}{60} = 250\pi \text{ rad/s}$$

Radial stresses

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2(R^2 - r^2) \quad \text{...(Eq. 14.9)}$$

$$= \frac{3+0.25}{8} \times 7500 \times (250\pi)^2(0.2^2 - r^2)$$

$$= 1879.5 \times 10^6 (0.04 - r^2) \text{ N/m}^2$$

$$= 1879.5(0.04 - r^2) \text{ MPa}$$

R (m)	0	0.05	0.1	0.15	0.2
σ_r (MPa)	75.2	70.5	56.4	32.9	0

Hoop stresses

and $$\sigma_\theta = \frac{\rho\omega^2}{8}[(3+\nu)R^2 - (1+3\nu)r^2] \quad \text{...(Eq. 14.10)}$$

$$= \frac{7500 \times (250\pi)^2}{8}[(3+0.25) \times 0.2^2 - (1+3\times 0.25)r^2]$$

$$= 578.3 \times 10^6 (0.13 - 1.75r^2) \text{ N/m}^2$$

or $$= 578.3(0.13 - 1.75r^2) \text{ MPa}$$

R (m)	0	0.05	0.1	0.15	0.2
σ_θ (MPa)	75.2	72.6	65.1	52.4	34.7

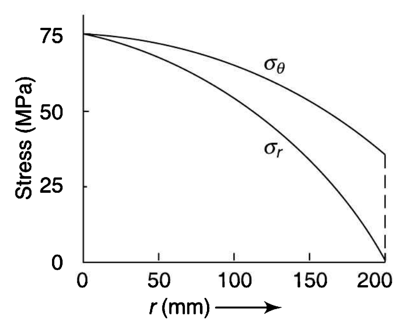


Fig.14.7

Maximum stresses

Maximum radial stress = maximum hoop stress = 75.2 MPa

The principal stresses at any point are σ_r, σ_θ and zero (along axial direction).

$$\therefore \text{ maximum shear stress } = \frac{75.2 - 0}{2} = 37.6 \text{ MPa}$$

The variation of stresses is shown in Fig. 14.7.

Example 14.6 A thin disc of uniform thickness is of 800-mm outer diameter and 50-mm inner diameter. It rotates at 3000 rpm. Determine the radial and the hoop stresses at radii of 0, 25 mm, 50 mm, 100 mm, 150 mm, 200 mm, 300 mm and 400 mm. Density of the material is 7800 kg/m^2. ν = 0.25. What are the maximum values of the radial, hoop and shear stresses?

Solution

Given A thin disc of uniform thickness

$R_i = 0.025$ m $\quad R_o = 0.4$ m $\quad N = 3000$ rpm

$\rho = 7800$ kg/m^3 $\quad \nu = 0.25$

To find

— Radial and hoop stresses at 0, 25 mm, 50 mm, 100 mm, 150 mm, 200 mm, 300 mm and 400 mm
— Maximum values of radial, hoop and shear stresses

$$\omega = \frac{2\pi \times 3000}{60} = 100\pi \text{ rad/s}$$

Radial stresses

$$\sigma_r = \frac{\rho\omega^2}{8}(3+\nu)(R_i^2 + R_o^2 - \frac{R_i^2 R_o^2}{r^2} - r^2) \qquad \text{...(Eq. 14.13)}$$

$$= \frac{7800 \times (100\pi)^2}{8}(3+0.25)\left(0.025^2 + 0.4^2 - \frac{0.025^2 \times 0.4^2}{r^2} - r^2\right)$$

$$= 312.750 \times 10^6 \left(0.1606 - \frac{0.0001}{r^2} - r^2\right) \text{N/m}^2$$

$$= 312.75\left(0.1606 - \frac{0.0001}{r^2} - r^2\right) \text{MPa}$$

r (m)	0.025	0.05	0.1	0.15	0.2	0.3	0.4
σ_r (MPa)	0	36.94	43.97	41.8	36.94	21.73	0

Hoop stresses

$$\sigma_\theta = \frac{\rho\omega^2}{8}\left[(3+\nu)\left(R_i^2 + R_o^2 + \frac{R_i^2 R_o^2}{r^2}\right) - (1+3\nu)r^2\right] \qquad \text{...(Eq. 14.14)}$$

$$= \frac{7800 \times (100\pi)^2}{8}\left[(3+0.25)\left(0.025^2 + 0.4^2 + \frac{0.025^2 \times 0.4^2}{r^2}\right) - (1+3\times 0.25)r^2\right]$$

$$= 96.23 \times 10^6 \left[3.25\left(0.1606 + \frac{0.0001}{r^2}\right) - 1.75r^2\right]$$

R (m)	0.025	0.05	0.1	0.15	0.2	0.3	0.4
σ_r (MPa)	100.17	62.32	51.68	47.83	44.28	35.42	23.48

Maximum stresses

Maximum radial stress is at radius $\sqrt{R_i R_o} = \sqrt{25 \times 400} = 100$ mm and is 43.97 MPa.

Maximum hoop stress = 100.17 MPa

The principal stresses at the inner surface are 100.17 MPa, 0 and 0 (along axial direction)

$\therefore$ maximum shear stress $= \dfrac{100.17 - 0}{2} = 50.09$ MPa

The variation of stresses is shown in Fig. 14.8.

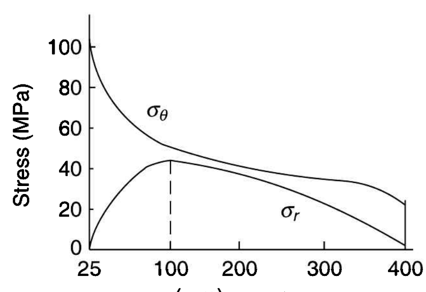


Fig. 14.8

Example 14.7 A hollow steel disc of 400-mm outer diameter and 100-mm inside diameter is shrunk on a steel shaft. The pressure between the disc and the shaft is 60 MPa. Determine the speed of the disc at which it will loosen from the shaft neglecting the change in the dimensions of the shaft due to rotation. ρ = 7700 kg/m^3 and ν = 0.3.

Solution

Given A hollow steel disc shrunk on a solid steel shaft

$R_i = 0.05$ m $\quad R_o = 0.2$ m

$p = 60$ MPa $\quad \rho = 7700$ kg/m^3

$\nu = 0.3$

To find Speed of disc at which it will loosen from the shaft

At stand still

At stand still, the hollow disc on a shaft is similar to a hub on a solid shaft (Section 13.8) and thus the hollow steel disc is equivalent to a thick cylinder subjected to an internal pressure and thus the results of the same may be used.

Maximum stress is at the inner radius and is given by Eq. 13.23.

$$\sigma_\theta = \frac{R_o^2 + R_i^2}{R_o^2 - R_i^2} \cdot p_i = \frac{0.2^2 + 0.05^2}{0.2^2 - 0.05^2} \times 60 = 68 \text{ MPa}$$

Hoop strain at the inner radius = $\dfrac{\sigma_\theta - \nu\sigma_r}{E} = \dfrac{68 + 0.3 \times 60}{E} = \dfrac{86}{E}$ MPa

When the disc rotates

Maximum hoop stress is at the inner radius,

$$\sigma_\theta = \frac{\rho\omega^2}{4}[(1-\nu)R_i^2 + (3+\nu)R_o^2] \qquad \text{...(Eq. 14.17)}$$

$$= \frac{7700 \times \omega^2}{4}[(1-0.3) \times 0.05^2 + (3+0.3) \times 0.2^2]$$

$$= 1925\omega^2\,(0.133\,75) = 257.47\omega^2 \text{ N/m}^2 \quad \text{or} \quad 257.47 \times 10^{-6}\,\omega^2 \text{ N/mm}^2$$

Radial stress is zero at the inner radius for the given condition.

Hoop strain at the inner radius = $\dfrac{\sigma_\theta - \nu\sigma_r}{E} = \dfrac{\sigma_\theta - 0}{E} = \dfrac{257.47 \times 10^{-6}\omega^2}{E}$

If disc is to be loosened

If the disc is to become loosened, the two hoop strains must be the same,

i.e., $$\frac{257.47 \times 10^{-6}\omega^2}{E} = \frac{86}{E}$$

or $\omega = 577.9$ rad/s or $N = \dfrac{577.9 \times 60}{2\pi} = 5519$ rpm

Example 14.8 A hollow steel disc of 600-mm outer diameter and 200-mm inside diameter is shrunk on a solid steel shaft. The shrinkage is 1 in 1000. Determine

(i) the stresses at stand still
(ii) the speed at which shrink fit will loosen
(iii) the maximum stress in the disc at that speed
(iv) the hoop stress in the disc at half the speed found in (ii)

ρ = 7600 kg/m^3 and ν = 0.3, E = 205 GPa

Solution

Given A hollow steel disc shrunk on a solid steel shaft

$R_i = 0.1$ m $\qquad R_o = 0.3$ m

$\rho = 7600$ kg/m^3 $\qquad \nu = 0.3$

$\delta/R = 1/1000$ $\qquad E = 205$ GPa

To find

— stresses at stand still
— speed at which shrink fit loosen
— maximum stress at that speed
— hoop stress at half speed

At stand still

Let the shrinkage pressure between the disc and the shaft at stand still be p.

$$\text{Shrink allowance} = 100 \times \frac{1}{1000} = 0.1 \text{ mm}$$

At stand still, the hollow disc acts similar to a thick cylinder subjected to internal pressure (Refer Section 13.8).

$$\text{Shrinkage allowance (initial difference in radii)} = \frac{2pR_iR_o^2}{E(R_o^2 - R_i^2)} \qquad \text{...(Eq. 13.28)}$$

or
$$0.1 = \frac{2p \times 100 \times 300^2}{205\,000(300^2 - 100^2)}$$

or
$$p = 91.1 \text{ MPa}$$

Maximum hoop stress is at the inner radius and is given by Eq. 13.20.

$$\sigma_\theta = \frac{R_o^2 + R_i^2}{R_o^2 - R_i^2} \cdot p_i$$

$$= \frac{0.3^2 + 0.1^2}{0.3^2 - 0.1^2} \cdot p = 1.25p = 1.25 \times 91.1 = 113.9 \text{ MPa}$$

When shrink fit loosens

When the disc rotates and the shrink fit loosens, radial pressure is zero and thus the radial stress is also zero.

$$\sigma_\theta = \frac{\rho\omega^2}{4}[(1-\nu)R_i^2 + (3+\nu)R_o^2] \qquad \text{...(Eq. 14.17)}$$

$$= \frac{7600 \times \omega^2}{4}\left[(1-0.3) \times 0.1^2 + (3+0.3) \times 0.3^2\right] = 577.6\omega^2 \text{ N/m}^2$$

and for the solid shaft,

$$\sigma_\theta = \frac{1-\nu}{4}\rho\omega^2R^2 \qquad \text{...(Eq. 14.12)}$$

$$= \frac{1-0.3}{4} \times 7600 \times \omega^2 \times 0.1^2 = 13.3\omega^2 \text{ N/m}^2$$

$$\text{Hoop strain at the inner radius} = \frac{(577.6 - 13.3) \times 10^{-6}\omega^2}{E}$$

$$\text{Shrink allowance} = \frac{(577.6 - 13.3)\times 10^{-6}\omega^2}{205\,000}\times 100 = 0.1$$

$$\omega^2 = 363\,282$$

$$\omega = 602.7 \text{ rad/s} \quad \text{or} \quad N = \frac{602.7\times 60}{2\pi} = 5756 \text{ rpm}$$

Maximum stresses

$$\sigma_\theta = 577.6\omega^2 = 577.6 \times 363\,282 = 209.8 \times 10^6 \text{ N/m}^2$$

$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2(R_o - R_i)^2 \qquad \text{...(Eq. 14.16)}$$

$$= \frac{3+0.3}{8}\times 7600\times 363\,282(0.3-0.1)^2$$

$$= 45.56 \times 10^6 \text{ N/m}^2 \quad \text{or} \quad 45.56 \text{ MPa}$$

Hoop stress at half speed

The hoop stress in the disc varies as square of speed.
Hoop stress at zero speed = 113.9 MPa
Hoop stress at 602.7 rad/s = 209.8 MPa
∴ hoop stress in the disc at half the speed

$$= 113.9 + (209.8 - 113.9)\cdot\left(\frac{1}{2}\right)^2 = 137.9 \text{ MPa}$$

Example 14.9 A hollow steel disc of 300-mm outer diameter and 100-mm inside diameter is shrunk on a hollow cast iron disc of 40-mm internal diameter. Determine the change in the shrink fit pressure when the assembly rotates at 4800 rpm.

ρ_s = 7700 kg/m³ ; ρ_{ci} = 7000 kg/m³ and ν = 0.3 for both, $E_s = 2E_{ci}$

Solution

Given A hollow steel disc on a hollow cast-iron disc

Steel disc:	$R_i = 0.05$ m	$R_o = 0.15$ m
Cast iron disc:	$R_i = 0.02$ m	$R_o = 0.05$ m
	$\rho_s = 7700$ kg/m³	$\rho_{ci} = 7000$ kg/m³
	$\nu = 0.3$	$E_s = 2E_{ci}$
	$N = 4800$ rpm	

To find Change in shrink fit pressure

$$\omega = \frac{2\pi\times 4800}{60} = 160\pi$$

At stand still

Let the shrinkage pressure between the disc and the shaft at stand still be p.

At stand still, the hollow disc acts similar to a thick cylinder subjected to internal pressure only and thus the results of the same may be used.

- For the outer disc, hoop stress at the inner radius (50 mm),

$R_i = 0.05$ m and $R_o = 0.15$ m

$$\sigma_\theta = \frac{R_o^2 + R_i^2}{R_o^2 - R_i^2}\cdot p_i = \frac{0.15^2 + 0.05^2}{0.15^2 - 0.05^2}\cdot p = 1.25p \text{ MPa}$$

$$\text{Hoop strain} = \frac{1.25p + 0.3p}{E_s} = \frac{1.55p}{E_s}$$

- For the inner disc, hoop stress is at the outer radius (50 mm),

$R_i = 0.02$ m and $R_o = 0.05$ m

$$\sigma_\theta = -\frac{R_o^2 + R_i^2}{R_o^2 - R_i^2} p_o = -\frac{0.05^2 + 0.02^2}{0.05^2 - 0.02^2} \cdot p = -1.381p \text{ MPa}$$

$$\text{Hoop strain} = \frac{-1.381p + 0.3p}{E_{ci}} = \frac{-1.081p}{E_{ci}}$$

When rotating at 4800 rpm

Let the shrinkage pressure between the disc and the shaft be p'.

For the outer disc, hoop strain at 50 mm radius = $1.25\,p'$ (as above)

For the inner disc, hoop strain at 50 mm radius = $-1.381p'$ (as above)

Due to rotation,

For outer disc at 50-mm radius, $\sigma_\theta = \frac{\rho\omega^2}{4}[(1-\nu)R_i^2 + (3+\nu)R_o^2]$ ...(Eq. 14.17)

$$= \frac{7700 \times (160\pi)^2}{4}[(1-0.3)\times 0.05^2 + (3+0.3)\times 0.15^2]$$

$$= 36.964 \times 10^6 \text{ N/m}^2 \quad \text{or} \quad 36.964 \text{ MPa}$$

For the inner disc at 50-mm radius,

$$\sigma_\theta = \frac{\rho\omega^2}{4}[(3+\nu)R_i^2 + (1-\nu)R_o^2] \qquad \text{...(Eq. 14.19)}$$

$$\sigma_\theta = \frac{7000 \times (160\pi)^2}{4}[(3+0.3)\times 0.02^2 + (1-0.3)\times 0.05^2]$$

$$= 1.357 \times 10^6 \text{ N/m}^2 \quad \text{or} \quad 1.357 \text{ MPa}$$

$$\text{Hoop strain in outer disc} = \frac{(1.25p' + 36.964\times 10^6) + 0.3p'}{E_s} = \frac{1.55p' + 36.964\times 10^6}{E_s}$$

$$\text{Hoop strain in inner disc} = \frac{(-1.381p' + 1.357\times 10^6) + 0.3p'}{E_{ci}} = \frac{-1.081p' + 1.357\times 10^6}{E_{ci}}$$

For equilibrium

$$\frac{1.55p}{E_s} - \frac{-1.081p}{E_{ci}} = \frac{1.55p' + 36.964\times 10^6}{E_s} - \frac{-1.081p' + 1.357\times 10^6}{E_{ci}}$$

or
$$\frac{1.55p}{2E_{ci}} - \frac{-1.081p}{E_{ci}} = \frac{1.55p' + 36.964\times 10^6}{2E_{ci}} - \frac{-1.081p' + 1.357\times 10^6}{E_{ci}}$$

or $1.55p + 2(1.081p) = 1.55p' + 36.964 + 2(1.081p' - 1.357)$

or $3.712(p - p') = 34.25$

or $p - p' = 9.22$ MPa

Example 14.10 A flat circular disc of outer radius R is fitted tightly on a rigid shaft of radius r. Show that when the shaft rotates at angular velocity ω, the reaction between the two neglecting the strain in the shaft is reduced by $\dfrac{\rho\omega^2}{4}\cdot\dfrac{(1-\nu)r^2+(3+\nu)R^2}{(1-\nu)r^2+(1+\nu)R^2}\cdot(R^2-r^2)$

Solution

Given A flat circular disc fitted tightly on a rigid shaft

To find To show that reaction between the two reduces by $\dfrac{\rho\omega^2}{4}\cdot\dfrac{(1-\nu)r^2+(3+\nu)R^2}{(1-\nu)r^2+(1+\nu)R^2}\cdot(R^2-r^2)$

At stand still

Let the shrinkage pressure between the disc and the shaft at stand still be p'.

At stand still, the disc is similar to a thick cylinder subjected to internal pressure only and thus the results of the same may be used.

$$\text{Hoop stress, } \sigma_\theta = \frac{R^2+r^2}{R^2-r^2}\cdot p$$

$$\text{Hoop strain, } \varepsilon_\theta = \frac{\sigma_\theta - \nu\sigma_r}{E} = \frac{1}{E}\left(\frac{R^2+r^2}{R^2-r^2}\cdot p + \nu p\right) = \frac{p}{E}\left(\frac{R^2+r^2}{R^2-r^2}+\nu\right)$$

Rotation at angular velocity ω

Let the shrinkage pressure between the disc and the shaft at stand still be p'.

$$\text{Hoop stress due to pressure, } \sigma_\theta = \frac{R^2+r^2}{R^2-r^2}\cdot p'$$

$$\text{Hoop stress due to rotation, } \sigma_\theta = \frac{\rho\omega^2}{4}[(1-\nu)r^2+(3+\nu)R^2] \qquad \text{...(Eq. 14.17)}$$

$$\text{Total hoop stress, } \sigma_\theta = \frac{R^2+r^2}{R^2-r^2}\cdot p' + \frac{\rho\omega^2}{4}[(1-\nu)r^2+(3+\nu)R^2]$$

$$\text{Hoop strain, } \varepsilon_\theta = \frac{\sigma_\theta-\nu\sigma_r}{E} = \frac{1}{E}\left[\frac{R^2+r^2}{R^2-r^2}\cdot p' + \frac{\rho\omega^2}{4}\{(1-\nu)r^2+(3+\nu)R^2\}+\nu p'\right]$$

$$= \frac{1}{E}\left[p'\left(\frac{R^2+r^2}{R^2-r^2}+\nu\right)+\frac{\rho\omega^2}{4}\{(1-\nu)r^2+(3+\nu)R^2\}\right]$$

For equilibrium

As the two hoop strains are to be equal,

$$\frac{p}{E}\left(\frac{R^2+r^2}{R^2-r^2}+\nu\right) = \frac{1}{E}\left[p'\left(\frac{R^2+r^2}{R^2-r^2}+\nu\right)+\frac{\rho\omega^2}{4}\{(1-\nu)r^2+(3+\nu)R^2\}\right]$$

or $$(p-p')\left(\frac{R^2+r^2}{R^2-r^2}+\nu\right) = \frac{\rho\omega^2}{4}[(1-\nu)r^2+(3+\nu)R^2]$$

or $$(p-p')[(R^2+r^2)+\nu(R^2-r^2)] = \frac{\rho\omega^2}{4}[(1-\nu)r^2+(3+\nu)R^2](R^2-r^2)$$

or $$p-p' = \frac{\rho\omega^2}{4}\cdot\frac{(1-\nu)r^2+(3+\nu)R^2}{(1-\nu)r^2+(1+\nu)R^2}\cdot(R^2-r^2)$$

Example 14.11 A flat circular disc of outer radius R and thickness t is fitted tightly on a rigid shaft of radius r producing a radial pressure p at the common surface. If it is then rotated at an angular velocity of ω rad/s, show that the maximum power is transmitted when $\omega = \omega_o / \sqrt{3}$ where ω_o is the angular velocity at which the pressure at the common surface falls to zero. Also, show that the value of the maximum power is 2.418 $\mu pt\omega_o r^2$ where μ is the coefficient of friction between the shaft and the disc.

Solution

Given A flat circular disc fitted tightly on a rigid shaft

To find To show that maximum power transmitted is when $\omega = \omega_o / \sqrt{3}$ and value of maximum power is 2.418 $\mu pt\omega_o r^2$

At stand still

Shrinkage allowance (initial difference in radii) between the shaft and the disc

$$= \frac{2prR^2}{E(R^2 - r^2)} \qquad \text{...(Eq. 13.28)}$$

When rotating at speed ω_o

When the disc rotates at speed ω_o and the shrink fit loosens, radial pressure is zero and thus the radial stress is also zero.

Hoop stresses

For the disc, $$\sigma_\theta = \frac{\rho\omega_o^2}{4}[(1-\nu)r^2 + (3+\nu)R^2] \qquad \text{...(Eq. 14.17)}$$

and for the shaft, $$\sigma_\theta = \frac{1-\nu}{4}\rho\omega_o^2 r^2 \qquad \text{...(Eq. 14.12)}$$

Hoop strain at the inner radius, $$\varepsilon_\theta = \frac{1}{E}(\sigma_{\theta\text{ disc}} - \sigma_{\theta\text{ shaft}})$$

$$= \frac{\rho\omega_o^2}{4E}[(1-\nu)r^2 + (3+\nu)R^2 - (1-\nu)r^2] = \frac{\rho\omega_o^2}{4E}(3+\nu)R^2$$

Net displacement of shaft and disc due to rotation $$= \frac{\rho r\omega_o^2}{4}[(3+\nu)R^2] \qquad \text{(i)}$$

This must be equal to initial shrinkage allowance, i.e.,

$$\frac{2prR^2}{E(R^2 - r^2)} = \frac{\rho r\omega_o^2}{4}(3+\nu)R^2$$

or $$p = \frac{\rho\omega_o^2}{8}\cdot(3+\nu)(R^2 - r^2)$$

When rotating at speed ω

When the disc rotates at speed ω, let the radial pressure be p',

Shrinkage allowance between the shaft and the disc $$= \frac{2p'rR^2}{E(R^2 - r^2)} \qquad \text{...(Eq. 13.28)}$$

Reduction in shrinkage allowance $$= \frac{2prR^2}{E(R^2 - r^2)} - \frac{2p'rR^2}{E(R^2 - r^2)}$$

$$= \frac{2rR^2}{E(R^2 - r^2)}(p - p')$$

Net displacement of shaft and disc is obtained in the same way as (i) above,

$$= \frac{\rho r \omega^2}{4E}(3+\nu)R^2$$

This must be equal to reduction in the shrinkage allowance,

or $$\frac{2rR^2}{E(R^2-r^2)}(p-p') = \frac{\rho r \omega^2}{4E}(3+\nu)R^2$$

or $$(p-p') = \frac{\rho \omega^2}{8}(3+\nu)(R^2-r^2)$$

or $$\frac{\rho \omega_o{}^2}{8} \cdot (3+\nu)(R^2-r^2) - p' = \frac{\rho \omega^2}{8} \cdot (3+\nu)(R^2-r^2)$$

or $$p' = \frac{\rho}{8} \cdot (3+\nu)(R^2-r^2)(\omega_o^2-\omega^2)$$

Power transmitted

$$P = \text{Torque} \times \text{Angular velocity} = (\text{Tangential force} \times \text{Radius}) \times \omega$$
$$= \mu \times \text{Radial force} \times \text{Radius} \times \omega$$
$$= \mu \times \text{Radial pressure} \times \text{Area} \times \text{Radius} \times \omega$$
$$= \mu p'(2\pi r \cdot t)\, r \cdot \omega = 2\pi \mu t r^2 \omega \cdot \frac{\rho}{8} \cdot (3+\nu)(R^2-r^2)(\omega_o^2-\omega^2)$$

$$P = \pi \mu t r^2 \cdot \frac{\rho}{4} \cdot (3+\nu)(R^2-r^2)(\omega_o^2 \omega - \omega^3)$$

For maximum power, $\dfrac{dP}{d\omega} = 0$

or $\omega_o^2 - 3\omega^2 = 0$ or $\omega = \dfrac{\omega_o}{\sqrt{3}}$

Maximum power transmitted

$$P_{\max} = \pi \mu t r^2 \cdot \frac{\rho}{4} \cdot (3+\nu)(R^2-r^2)\left(\omega_o^2 \frac{\omega_o}{\sqrt{3}} - \frac{\omega_o^3}{3\sqrt{3}}\right)$$
$$= \frac{4}{3\sqrt{3}} \pi \mu t r^2 \omega_o \cdot \frac{\rho \omega_o^2}{8} \cdot (3+\nu)(R^2-r^2)$$
$$= 2.418\, \mu t r^2\, \omega_o \cdot p = 2.418\, \mu p t \omega_o\, r^2$$

14.3 LONG CYLINDER

Assuming the longitudinal strain ε to be constant (i.e., cross-sections remain plane which is true away from the ends),

- Longitudinal strain, $\varepsilon = \dfrac{\sigma_z - \nu(\sigma_\theta + \sigma_r)}{E}$

  or $$E\varepsilon = \sigma_z - \nu(\sigma_\theta + \sigma_r)$$

  Differentiating (ε is constant),

$$0 = \frac{d\sigma_z}{dr} - \nu\left(\frac{d\sigma_\theta}{dr} + \frac{d\sigma_r}{dr}\right)$$

or $$\frac{d\sigma_z}{dr} = \nu\left(\frac{d\sigma_\theta}{dr} + \frac{d\sigma_r}{dr}\right) \tag{i}$$

- Radial strain = $\dfrac{\text{Increase in } \delta r}{\delta r} = \dfrac{u + \delta u - u}{dr} = \dfrac{du}{dr}$ in the limit

$\therefore$ radial stress = $$E \cdot \frac{du}{dr} = \sigma_r - \nu(\sigma_\theta + \sigma_z) \tag{ii}$$

- Circumferential (hoop) strain = $\dfrac{\text{Increase in circumference}}{\text{Original circumference}} = \dfrac{2\pi(r+u) - 2\pi r}{2\pi r} = \dfrac{u}{r}$

$\therefore$ circumferential stress = $E \cdot \dfrac{u}{r} = \sigma_\theta - \nu(\sigma_r + \sigma_z)$

or $$E \cdot u = r[\sigma_\theta - \nu(\sigma_r + \sigma_z)] \tag{iii}$$

Differentiating (iii), $$E \cdot \frac{du}{dr} = r\left[\frac{d\sigma_\theta}{dr} - \nu\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_z}{dr}\right)\right] + \sigma_\theta - \nu(\sigma_r + \sigma_z) \tag{iv}$$

From (ii) and (iv),

$$r\left[\frac{d\sigma_\theta}{dr} - \nu\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_z}{dr}\right)\right] + \sigma_\theta - \nu(\sigma_r + \sigma_z) = \sigma_r - \nu(\sigma_\theta + \sigma_z)$$

or $$r\left[\frac{d\sigma_\theta}{dr} - \nu\left(\frac{d\sigma_r}{dr} + \frac{d\sigma_z}{dr}\right)\right] + (\sigma_\theta - \sigma_r)(1+\nu) = 0 \tag{v}$$

Inserting the value of $\dfrac{d\sigma_z}{dr}$ from (i),

or $$r\frac{d\sigma_\theta}{dr} - r\nu\frac{d\sigma_r}{dr} - r\nu^2\frac{d\sigma_\theta}{dr} - r\nu^2\frac{d\sigma_r}{dr} + (\sigma_\theta - \sigma_r)(1+\nu) = 0$$

or $$r\frac{d\sigma_\theta}{dr}(1-\nu^2) - r\nu\frac{d\sigma_r}{dr}(1+\nu) + (\sigma_\theta - \sigma_r)(1+\nu) = 0$$

or $$\sigma_r - \sigma_\theta = r(1-\nu)\frac{d\sigma_\theta}{dr} - r\nu\frac{d\sigma_r}{dr} \tag{vi}$$

Equilibrium equation is the same as for the disc, i.e.,

$$\sigma_r - \sigma_\theta = -\frac{r \cdot d\sigma_r}{dr} - \rho r^2\omega^2 \quad \text{...(Eq. 14.5)} \tag{vii}$$

From (vi) and (vii),

$$r(1-\nu)\frac{d\sigma_\theta}{dr} + r(1-\nu)\frac{d\sigma_r}{dr} = -\rho r^2\omega^2$$

$$\frac{d\sigma_\theta}{dr} + \frac{d\sigma_r}{dr} = -\frac{\rho r\omega^2}{1-\nu}$$

Integrating, $$\sigma_r + \sigma_\theta = -\frac{\rho r^2\omega^2}{2(1-\nu)} + 2A \tag{viii}$$

Adding (vii) and (viii)

$$2\sigma_r + r\cdot\frac{d\sigma_r}{dr} + \rho r^2\omega^2 = -\frac{\rho r^2\omega^2}{2(1-\nu)} + 2A$$

$$\left(2\sigma_r + r\cdot\frac{d\sigma_r}{dr}\right) = -\frac{[1+2(1-\nu)]\rho r^2\omega^2}{2(1-\nu)} + 2A$$

$$\frac{1}{r}\cdot\frac{d(r^2\cdot\sigma_r)}{dr} = -\frac{(3-2\nu)\rho r^2\omega^2}{2(1-\nu)} + 2A$$

$$\frac{d(r^2\cdot\sigma_r)}{dr} = -\frac{(3-2\nu)\rho r^3\omega^2}{2(1-\nu)} + 2Ar$$

Integrating, $r^2\cdot\sigma_r = -\dfrac{(3-2\nu)\rho r^4\omega^2}{8(1-\nu)} + Ar^2 - B$

$$\sigma_r = A - \frac{B}{r^2} - \frac{(3-2\nu)\rho r^2\omega^2}{8(1-\nu)} \tag{14.20}$$

Inserting this value in (viii),

$$A - \frac{B}{r^2} - \frac{(3-2\nu)\rho r^2\omega^2}{8(1-\nu)} + \sigma_\theta = -\frac{\rho r^2\omega^2}{2(1-\nu)} + 2A$$

$$\sigma_\theta = -\frac{\rho r^2\omega^2}{2(1-\nu)} + A + \frac{B}{r^2} + \frac{(3-2\nu)\rho r^2\omega^2}{8(1-\nu)} = A + \frac{B}{r^2} - \frac{1+2\nu}{8(1-\nu)}\rho r^2\omega^2 \tag{14.21}$$

Solid Cylinder

As the stresses cannot be infinite at the centre, therefore,

$$\sigma_r = 0 = A - \frac{B}{0} \text{ or } B = 0$$

If R is the outside diameter, then $\sigma_r = 0$,

$$\therefore \qquad 0 = A - \frac{(3-2\nu)\rho R^2\omega^2}{8(1-\nu)} \quad \text{or} \quad A = \frac{(3-2\nu)\rho R^2\omega^2}{8(1-\nu)}$$

Thus from Eq. 14.21, $\sigma_r = \dfrac{(3-2\nu)}{8(1-\nu)}\rho\omega^2(R^2 - r^2)$ (14.22)

Maximum value is at the centre, $\sigma_r = \dfrac{(3-2\nu)}{8(1-\nu)}\rho\omega^2 R^2$ (14.23)

For $\nu = 0.3$, $\sigma_r = \dfrac{(3-2\times 0.3)}{8(1-0.3)}\rho\omega^2 R^2 = 0.429\rho\omega^2 R^2$

From Eq. 14.22, $\sigma_\theta = \dfrac{(3-2\nu)\rho R^2\omega^2}{8(1-\nu)} - \dfrac{1+2\nu}{8(1-\nu)}\rho r^2\omega^2$

$$= \frac{\rho\omega^2}{8(1-\nu)}[(3-2\nu)R^2 - (1+2\nu)r^2] \tag{14.24}$$

Hollow Cylinder

Let R_i and R_o be the inside and the outside radii respectively. Radial stresses are to be zero at these values.

$$\sigma_r = 0 = A - \frac{B}{R_i^2} - \frac{(3-2\nu)\rho R_i^2 \omega^2}{8(1-\nu)} \tag{i}$$

$$\sigma_r = 0 = A - \frac{B}{R_o^2} - \frac{(3-2\nu)\rho R_o^2 \omega^2}{8(1-\nu)} \tag{ii}$$

From (i) from (ii), $-\dfrac{B}{R_i^2} - \dfrac{(3-2\nu)\rho R_i^2\omega^2}{8(1-\nu)} + \dfrac{B}{R_o^2} + \dfrac{(3-2\nu)\rho R_o^2\omega^2}{8(1-\nu)} = 0$

or $$-BR_o^2 - \frac{3-2\nu}{8(1-\nu)}\rho R_i^4 R_o^2 \omega^2 + BR_i^2 + \frac{3-2\nu}{8(1-\nu)}\rho R_i^2 R_o^4 \omega = 0$$

or $$B(R_i^2 - R_o^2) - \frac{3-2\nu}{8(1-\nu)}\rho R_i^2 R_o^2 \omega^2 (R_i^2 - R_o^2) = 0$$

or $$B = \frac{3-2\nu}{8(1-\nu)}\rho R_i^2 R_o^2 \omega^2$$

Then from (i), $\sigma_r = 0 = A - \dfrac{3-2\nu}{8(1-\nu)}\rho R_o^2 \omega^2 - \dfrac{(3-2\nu)\rho R_i^2 \omega^2}{8(1-\nu)}$

or $$A = \frac{3-2\nu}{8(1-\nu)}\rho\omega^2 (R_i^2 + R_o^2)$$

Thus from Eq. 14.20,

$$\sigma_r = \frac{3-2\nu}{8(1-\nu)}\rho\omega^2(R_i^2 + R_o^2) - \frac{3-2\nu}{8(1-\nu)r^2}\rho R_i^2 R_o^2 \omega^2 - \frac{(3-2\nu)\rho r^2 \omega^2}{8(1-\nu)}$$

$$= \frac{\rho\omega^2}{8} \cdot \frac{3-2\nu}{(1-\nu)}\left(R_i^2 + R_o^2 - \frac{R_i^2 R_o^2}{r^2} - r^2\right) \tag{14.25}$$

- σ_r is maximum when $\dfrac{d\sigma_r}{dr} = 0$ or $2\dfrac{R_i^2 R_o^2}{r^3} - 2r = 0$

or $R_i^2 R_o^2 - r^4 = 0$ or $r = \sqrt{R_i R_o}$ (14.26)

or $$\sigma_r = \frac{3-2\nu}{8(1-\nu)}\rho\omega^2\left(R_i^2 + R_o^2 - \frac{R_i^2 R_o^2}{R_i R_o} - R_i R_o\right)$$

$$= \frac{3-2\nu}{8(1-\nu)}\rho\omega^2(R_i^2 + R_o^2 - 2R_i R_o) = \frac{3-2\nu}{8(1-\nu)}\rho\omega^2(R_o - R_i)^2 \tag{14.27}$$

From Eq. 14.21,

$$\sigma_\theta = \frac{3-2\nu}{8(1-\nu)}\rho\omega^2(R_i^2 + R_o^2) + \frac{3-2\nu}{8(1-\nu)r^2}\rho R_i^2 R_o^2 \omega^2 - \frac{1+2\nu}{8(1-\nu)}\rho r^2 \omega^2$$

$$= \frac{\rho\omega^2}{8(1-\nu)}\left[(3-2\nu)\left(R_i^2 + R_o^2 + \frac{R_i^2 R_o^2}{r^2}\right) - (1+2\nu)r^2\right] \tag{14.28}$$

- σ_θ is maximum at inside ($r = R_i$),

$$\sigma_\theta = \frac{\rho\omega^2}{8(1-\nu)}\left[(3-2\nu)\left(R_i^2 + R_o^2 + \frac{R_i^2 R_o^2}{R_i^2}\right) - (1+2\nu)R_i^2\right]$$

$$= \frac{\rho\omega^2}{8(1-\nu)}[(3-2\nu)R_i^2 + 2(3-2\nu)R_o^2 - (1+2\nu)R_i^2]$$

$$= \frac{\rho\omega^2}{4(1-\nu)}[(1-2\nu)R_i^2 + (3-2\nu)R_o^2] \qquad (14.29)$$

The results for a long cylinder can also be obtained from those for a thin disc by substituting $\nu / (1-\nu)$ for ν.

Example 14.12 A solid cylinder with a 400-mm diameter rotates at 2100 rpm. Plot the variation of radial and hoop stresses in the cylinder. What is the maximum hoop stress? Density of the cylinder material is 7700 kg/m^3. Poisson's ratio is 0.3.

Solution

Given A solid cylinder

$R = 0.2$ m $\qquad N = 2100$ rpm

$\rho = 7700$ kg/m^3 $\qquad \nu = 0.3$

To find To plot variation of radial and hoop stresses

$$\omega = \frac{2\pi \times 2100}{60} = 70\pi \text{ rad/s}$$

Radial stresses

$$\sigma_r = \frac{\rho\omega^2}{8}\cdot\frac{(3-2\nu)}{(1-\nu)}(R^2 - r^2) \qquad \text{...(Eq. 14.22)}$$

$$= \frac{7700(70\pi)^2}{8}\cdot\frac{(3-2\times 0.3)}{(1-0.3)}(0.2^2 - r^2)$$

$$= 46.568 \times 10^6 \times 3.429(0.2^2 - r^2) = 159.59 \times 10^6 (0.2^2 - r^2) \text{ N/m}^2$$

r (m)	0.0	0.04	0.08	0.12	0.16	0.2
σ_r (MPa)	6.38	6.13	5.36	4.09	2.3	0

Hoop stresses

$$\sigma_\theta = \frac{\rho\omega^2}{8(1-\nu)}[(3-2\nu)R^2 - (1+2\nu)r^2] \qquad \text{...(Eq. 14.24)}$$

$$= \frac{7700(70\pi)^2}{8(1-0.3)}[(3-2\times 0.3)0.2^2 - (1+2\times 0.3)r^2]$$

$$= 66.526 \times 10^6 [0.096 - 1.6r^2] \text{ N/m}^2$$

r (m)	0.0	0.04	0.08	0.12	0.16	0.2
σ_θ (MPa)	6.38	6.22	5.71	4.85	3.66	2.13

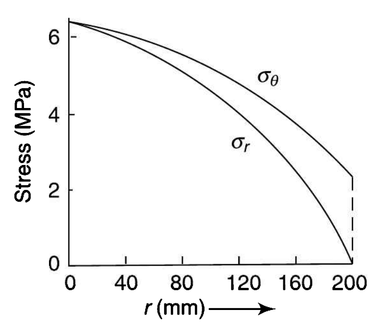


Fig. 14.9

Maximum values of hoop and radial stresses = 6.38 MPa

The variation of stresses is shown in Fig. 14.9.

Example 14.13 A hollow cylinder having an outside diameter of 600 mm and an inside diameter of 300 mm rotates at 2400 rpm. Plot the variation of radial and hoop stresses in the cylinder. Density of the cylinder material is 7500 kg/m³. Poisson's ratio is 0.28.

Solution

Given A hollow cylinder

$R_i = 0.15$ m $\qquad R_o = 0.3$ m

$N = 2400$ rpm $\qquad \rho = 7500$ kg/m³

$\nu = 0.28$

To find To plot variation of radial and hoop stresses

$$\omega = \frac{2\pi \times 2400}{60} = 80\pi \text{ rad/s}$$

Radial stresses

$$\sigma_r = \frac{\rho\omega^2}{8}\cdot\frac{3-2\nu}{(1-\nu)}\left(R_i^2 + R_o^2 - \frac{R_i^2 R_o^2}{r^2} - r^2\right) \qquad \ldots\text{(Eq. 14.25)}$$

$$= \frac{7500\times(80\pi)^2}{8}\cdot\frac{3-2\times 0.28}{(1-0.28)}\left(0.15^2 + 0.3^2 - \frac{0.15^2\times 0.3^2}{r^2} - r^2\right)$$

$$= 200.68\times 10^2\left(0.1125 - \frac{0.002\,025}{r^2} - r^2\right)\text{ N/m}^2$$

r (m)	0.15	0.18	0.21	0.24	0.27	0.3
σ_r (MPa)	0	3.53	4.51	3.96	2.37	0

Radial stress is maximum at a radius $= \sqrt{R_1R_2} = \sqrt{0.15\times 0.3} = 0.212$ mm $\qquad \ldots$(Eq. 14.26)

Its value = 4.52 MPa

Hoop stresses

$$\sigma_\theta = \frac{\rho\omega^2}{8(1-\nu)}\left[(3-2\nu)\left(R_i^2 + R_o^2 + \frac{R_i^2R_o^2}{r^2}\right) - (1+2\nu)r^2\right] \qquad \ldots\text{(Eq. 14.28)}$$

$$= \frac{7500\times(80\pi)^2}{8(1-0.28)}\left[(3-2\times 0.28)\left(0.15^2 + 0.3^2 + \frac{0.15^2\times 0.3^2}{r^2}\right) - (1+2\times 0.28)r^2\right]$$

$$= 82.25\times 10^6\left[\left(0.2745 + \frac{0.004\,94}{r^2}\right) - 1.56r^2\right]\text{ N/m}^2$$

r (m)	0.15	0.18	0.21	0.24	0.27	0.3
σ_θ (MPa)	37.75	30.96	26.13	22.24	18.8	15.54

Maximum stress = 37.75 MPa

The variation of stresses is shown in Fig. 14.10.

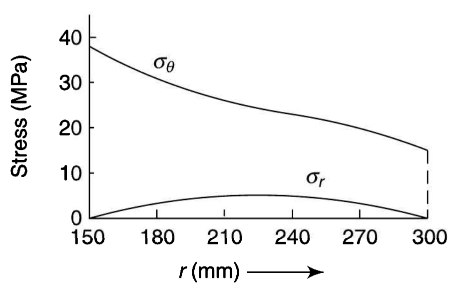


Fig 14.10

14.4 DISC OF UNIFORM STRENGTH

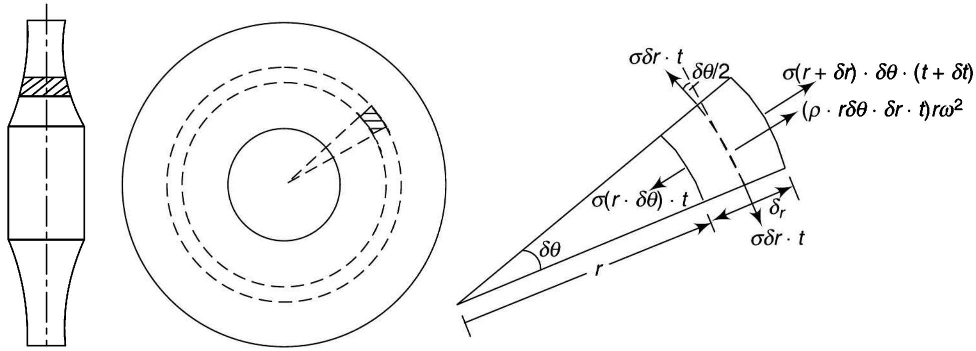


Fig. 14.11

Consider a flat rotating disc of uniform strength σ at all radii.
Taking $\sigma_1 = \sigma_2 = \sigma$
Let t be the thickness at a radius r and $t + \delta t$ at a radius $r + \delta r$ (Fig.14.10),
Mass of element = $\rho \cdot r\delta\theta \cdot \delta r \cdot t$
Centrifugal force = $(\rho \cdot r\delta\theta \cdot \delta r \cdot t)r\omega^2$ (Outward)
Radial force on inner face = $\sigma_r (r \cdot \delta\theta)t$ (Inward)
Radial force on outer face = $\sigma(r + \delta r) \cdot \delta\theta \cdot (t + \delta t)$ (Outward)
Radial components of tangential force = $2\sigma \cdot \delta r \cdot t \sin \frac{1}{2}\delta\theta \approx \sigma \cdot \delta r \cdot t \cdot \delta\theta$ (Inward)

For equilibrium,

$$\sigma \cdot \delta r \cdot t \cdot \delta\theta + \sigma \cdot r \cdot \delta\theta \cdot t - \sigma(r + \delta r) \cdot \delta\theta(t + \delta t) = \rho \cdot r^2 \cdot \delta\theta \cdot \delta r \cdot t \cdot \omega^2$$

$$\sigma \cdot \delta r \cdot t + \sigma \cdot r \cdot t - \sigma(r + \delta r)(t + \delta t) = \rho r^2 \cdot \delta r \cdot t \cdot \omega^2$$

Simplifying and taking limits,

$$\sigma \cdot t \cdot dr + \sigma \cdot r \cdot t - \sigma \cdot r \cdot t - \sigma \cdot r \cdot dt - \sigma \cdot t \cdot dr = \rho r^2 \omega^2 \cdot t \cdot dr$$

$$\sigma \cdot \frac{dt}{t} = -\rho \cdot \omega^2 \cdot r \cdot dr$$

Integrating, $$\log t = -\frac{\rho \cdot \omega^2}{\sigma} \cdot \frac{r^2}{2} + \text{constant}$$

$$t = A \cdot e^{-\rho \cdot \omega^2 \cdot r^2 / 2\sigma} \qquad (14.30)$$

Let $t = t_o$ at $r = 0$, $\therefore A = t_o$

Thus $$t = t_o \cdot e^{-\rho \cdot \omega^2 \cdot r^2 / 2\sigma} \qquad (14.31)$$

Example 14.14 A steam turbine rotor designed for uniform strength of 70 MPa rotates at 3600 rpm. The thickness of the rotor at the centre is 20 mm. Determine the thickness of the rotor at a radius of 400 mm. Density of the material of the rotor is 7700 kg/m^3.

Solution

Given A turbine rotor

$t_o = 20$ mm $\quad \sigma = 70$ MPa
$N = 3600$ rpm $\quad r = 400$ mm
$\rho = 7700$ kg/m^3

To find Thickness of rotor

Using Eq. 14.31,

$$t = t_o \cdot e^{-\rho \cdot \omega^2 \cdot r^2 / 2\sigma}$$

where $$\frac{\rho\omega^2 r^2}{2\sigma} = 7700 \times \left(\frac{2\pi \times 3600}{60}\right)^2 \times \frac{0.4^2}{2 \times 70 \times 10^6} = 1.2507$$

Thickness of rotor

$$t = t_o \cdot e^{-1.2507} = 20 \times e^{-1.2507} = 5.726 \text{ mm}$$

Example 14.15 The rotor disc of a turbine is of 800-mm diameter at the blade ring and is fixed to a 60-mm diameter shaft. If the minimum thickness of the disc is to be 8 mm, find the thickness at the shaft for a uniform stress of 210 MPa at 7500 rpm. Density of the disc material is 7800 kg/m^3.

Solution

Given A turbine rotor

$\sigma = 210$ MPa $\quad N = 7500$ rpm

$\rho = 7800$ kg/m^3 $\quad t = 8$ mm at 0.4 m radius

To find Thickness of rotor at 0.03 m

The minimum thickness of the disc is at the tip at 0.4 m radius. Let t be the thickness of the disc at the shaft at 0.03 m radius.

Now as $$t = A \cdot e^{-\rho \cdot \omega^2 \cdot r^2 / 2\sigma}$$

At* 0.4 *m radius

$$8 = A \cdot e^{-\rho \cdot \omega^2 \cdot 0.4^2 / 2\sigma} \quad \text{(i)}$$

At* 0.03 *m radius

$$t = A \cdot e^{-\rho \cdot \omega^2 \cdot 0.03^2 / 2\sigma} \quad \text{(ii)}$$

Dividing (ii) and (i),

$$t = 8 \times \frac{A \cdot e^{-\rho \cdot \omega^2 \cdot 0.03^2 / 2\sigma}}{A \cdot e^{-\rho \cdot \omega^2 \cdot 0.4^2 / 2\sigma}} = 8 \times e^{\rho \cdot \omega^2 (0.16 - 0.0009)/2\sigma} = 8 \times e^{0.1591 \rho \cdot \omega^2 / 2\sigma}$$

where $$\frac{0.1591 \rho\omega^2}{2\sigma} = 7800 \times \left(\frac{2\pi \times 7500}{60}\right)^2 \times \frac{0.1591}{2 \times 210 \times 10^6} = 1.8226$$

$\therefore$ $$t = 8e^{1.8226} = 49.5 \text{ mm}$$

14.5 COLLAPSE SPEED

As seen in the earlier analysis, the centrifugal forces in a rotating disc induce two types of stresses in the disc, circumferential or hoop stress and the radial stress. At any radius, the hoop stress is either greater than or equal to the radial stress. The maximum value of the hoop stress occurs at the inner radius. Thus as the speed will increase, the yield will first be observed in the circumferential direction when the hoop stress becomes equal to yield stress in tension. If the material is ideal elastic-plastic, the collapse or failure of the disc will occur when plastic stress condition extend to outer surface of the disc.

From Eq. 14.5, (replacing σ_θ by yield stress σ_y)

$$\sigma_r - \sigma_y = -\frac{r \cdot d\sigma_r}{dr} - \rho r^2 \omega^2 \quad \text{or} \quad \sigma_y - \left(\sigma_r + r\frac{d\sigma_r}{dr}\right) = \rho r^2 \omega^2$$

Integrating, $\sigma_y \cdot r - \sigma_r \cdot r = \dfrac{\rho r^3 \omega^2}{3} + A$ (σ_y is constant)

$$\sigma_r = \sigma_y - \frac{\rho r^2 \omega^2}{3} + \frac{A}{r}$$

Solid Disc

At $r = 0$, $\sigma_r = \sigma_y - \dfrac{\rho r^2 \omega^2}{3} + \dfrac{A}{0} = \infty$ which is not possible; $\therefore A = 0$,

Thus $$\sigma_r = \sigma_y - \frac{\rho r^2 \omega^2}{3}$$

At $r = R$, $\sigma_r = 0$, $\therefore$ $\sigma_r = 0 = \sigma_y - \dfrac{\rho R^2 \omega^2}{3}$

or $$\sigma_y = \frac{\rho R^2 \omega^2}{3} \quad \text{or} \quad \omega = \frac{1}{R}\sqrt{\frac{3\sigma_y}{\rho}} \tag{14.32}$$

Hollow Disc

At $r = R_i$, $\sigma_r = 0$; $\therefore$ $\sigma_r = 0 = \sigma_y - \dfrac{\rho R_i^2 \omega^2}{3} + \dfrac{A}{R_1}$ or $A = R_i\left(\dfrac{\rho R_i^2 \omega^2}{3} - \sigma_y\right)$

At $r = R_o$, $\sigma_r = 0 = \sigma_y - \dfrac{\rho R_o^2 \omega^2}{3} + \dfrac{R_i}{R_o}\left(\dfrac{\rho R_i^2 \omega^2}{3} - \sigma_y\right)$

or $$\sigma_y = \frac{\rho R_o^2 \omega^2}{3} - \frac{R_i}{R_o}\left(\frac{\rho R_i^2 \omega^2}{3} - \sigma_y\right) = \frac{\rho R_o^3 \omega^2 - \rho R_i^3 \omega^2 + 3R_i \sigma_y}{3R_o}$$

or $$3R_o \sigma_y = \rho R_o^3 \omega^2 - \rho R_i^3 \omega^2 + 3R_i \sigma_y$$

or $$3\sigma_y (R_o - R_i) = \rho \omega^2 (R_o^3 - R_i^3)$$

or $$\omega = \sqrt{\frac{3\sigma_y (R_o - R_i)}{\rho (R_o^3 - R_i^3)}} \tag{14.33}$$

Example 14.16 A thin uniform steel disc of 200-mm diameter having a central hole of 40 mm diameter is rotating at 8000 rpm. Determine the collapse speed taking the yield stress to be 270 MPa. Density of steel is 7700 kg/m^3.

Solution

Given A uniform steel disc

$R_i = 0.02$ m $R_o = 0.1$ m

$\rho = 7700$ kg/m^3 N = 8000 rpm

$\sigma_y = 270$ N/mm^2 $= 270 \times 10^6$ N/m^2

To find Collapse speed

Collapse speed

$$\omega = \sqrt{\frac{3\sigma_y(R_o - R_i)}{\rho(R_o{}^3 - R_i{}^3)}}$$

$$= \sqrt{\frac{3 \times 270 \times 10^6(0.1 - 0.02)}{7700 \times (0.1^3 - 0.02^3)}} = 2913 \text{ rad/s}$$

or $$N = \frac{2913 \times 60}{2\pi} = 27\,814 \text{ rpm}$$

Summary

1. Hoop stress induced in a rotating ring is $\sigma_\theta = \rho \cdot r^2 \omega^2 = \rho \cdot v^2$
2. General equations for radial and hoop stresses of a rotating disc,
$$\sigma_r = A - \frac{B}{r^2} - \frac{3+\nu}{8}\rho r^2\omega^2 \quad \text{and} \quad \sigma_\theta = A + \frac{B}{r^2} - \frac{1+3\nu}{8}\rho r^2\omega^2$$
3. In a solid rotating disc
$$\sigma_r = \frac{3+\nu}{8}\rho\omega^2(R^2 - r^2) \quad \text{and} \quad \sigma_\theta = \frac{\rho\omega^2}{8}[(3+\nu)R^2 - (1+3\nu)r^2]$$
Maximum values are at the centre of the disc, $\sigma_r = \sigma_\theta = \dfrac{3+\nu}{8}\rho\omega^2 R^2$

At the outer surface, $\sigma_\theta = \dfrac{1-\nu}{4}\rho\omega^2 R^2$ and $\sigma_r = 0$
4. In a hollow rotating disc,
$$\sigma_r = \frac{\rho\omega^2}{8}(3+\nu)(R_i^2 + R_o^2 - \frac{R_i^2 R_o^2}{r^2} - r^2)$$
and $$\sigma_\theta = \frac{\rho\omega^2}{8}\left[(3+\nu)\left(R_i^2 + R_o^2 + \frac{R_i^2 R_o^2}{r^2}\right) - (1+3\nu)r^2\right]$$
— Radial stresses are zero at inside and outside radii.

— σ_r is maximum at $r = \sqrt{R_i R_o}$ and is $\dfrac{3+\nu}{8}\rho\omega^2(R_o - R_i)^2$

— σ_θ is maximum at inner radius and is $\dfrac{\rho\omega^2}{4}[(1-\nu)R_i^2 + (3+\nu)R_o^2]$

— At outer radius, $\sigma_\theta = \dfrac{\rho\omega^2}{4}[(3+\nu)R_i^2 - (1-\nu)R_o^2]$
5. In a long solid rotating cylinder,
$$\sigma_r = \frac{(3-2\nu)}{8(1-\nu)}\rho\omega^2(R^2 - r^2); \quad \sigma_r = \frac{(3-2\nu)}{8(1-\nu)}\rho\omega^2 R^2$$
$$\sigma_\theta = \frac{\rho\omega^2}{8(1-\nu)}[(3-2\nu)R^2 - (1+2\nu)r^2]$$
Maximum values are at the centre, $\sigma_r = \sigma_\theta = \dfrac{(3-2\nu)}{8(1-\nu)}\rho\omega^2 R^2$

6. In a long hollow rotating cylinder,

$$\sigma_r = \frac{\rho\omega^2}{8}\cdot\frac{3-2\nu}{(1-\nu)}\left(R_i^2 + R_i^2 - \frac{R_i^2 R_o^2}{r^2} - r^2\right)$$

— σ_r is maximum at $r = \sqrt{R_i R_o}$ and is $\dfrac{3-2\nu}{8(1-\nu)}\rho\omega^2(R_o - R_i)^2$

— $\sigma_\theta = \dfrac{\rho\omega^2}{8(1-\nu)}\left[(3-2\nu)\left(R_i^2 + R_o^2 + \dfrac{R_i^2 R_o^2}{r^2}\right) - (1+2\nu)r^2\right]$

— Maximum values of hoop stress is at the centre and is,

$$\frac{\rho\omega^2}{4(1-\nu)}[(1-2\nu)R_i^2 + (3-2\nu)R_o^2]$$

— The results for a long cylinder can be obtained from those for a thin disc by substituting $\nu/(1-\nu)$ for ν.

7. For a flat rotating disc of uniform strength σ at all radii, $t = t_o \cdot e^{-\rho\cdot\omega^2\cdot r^2/2\sigma}$

8. Collapse speed of a solid disc, $\omega = \dfrac{1}{R}\sqrt{\dfrac{3\sigma_y}{\rho}}$

9. Collapse speed of a hollow disc, $\omega = \sqrt{\dfrac{3\sigma_y(R_o - R_i)}{\rho(R_o^3 - R_i^3)}}$

Objective Type Questions

1. Hoop stress σ_θ induced in a rotating ring is
(a) $\rho \cdot r^3\,\omega^2$ (b) $\rho \cdot r^2\,\omega^2$ (c) $\rho \cdot r^2\,\omega^3$(d) $\rho \cdot r^3\,\omega^3$

2. In a solid rotating disc, the hoop stress is maximum at the
(a) center (b) outer surface (c) mean radial section (d) none of these

3. In a solid rotating disc, the radial stress is maximum at the
(a) center (b) outer surface (c) mean radial section (d) none of these

4. In a solid rotating disc, the value of the hoop stress at the centre is ______ the radial stress.
(a) greater than (b) less than (c) equal to

5. In a solid rotating disc, the radial stress at the centre is
(a) $\dfrac{1-\nu}{4}\rho\omega^2 R^2$ (b) $\dfrac{3+\nu}{8}\rho\omega^2 R^2$ (c) $\dfrac{3-\nu}{8}\rho\omega^2 R^2$ (d) zero

6. In a solid rotating disc, the radial stress at the outer surface is
(a) $\dfrac{1-\nu}{4}\rho\omega^2 R^2$ (b) $\dfrac{3+\nu}{8}\rho\omega^2 R^2$ (c) $\dfrac{3-\nu}{8}\rho\omega^2 R^2$ (d) zero

7. In a solid rotating disc, the hoop stress at the outer surface is
(a) $\dfrac{1-\nu}{4}\rho\omega^2 R^2$ (b) $\dfrac{3+\nu}{8}\rho\omega^2 R^2$ (c) $\dfrac{3-\nu}{8}\rho\omega^2 R^2$ (d) zero

8. In a hollow rotating disc, radial stress is maximum at
(a) $r = \sqrt{R_i R_o}$ (b) $r = \sqrt{R_i^2 R_o}$ (c) $r = \sqrt{R_i R_o^2}$ (d) $r = \sqrt{R_i / R_o}$

9. In a hollow rotating disc, hoop stress is maximum at

(a) inner surface (b) outer surface (c) $r = \sqrt{R_i R_o^2}$ (d) none of these

10. In a hollow rotating disc if the inner radius is very small, the hoop stress at the inner surface is ______ that for a solid shaft.

(a) same as (b) twice (c) thrice (d) 1.5 times

11. In a long rotating solid cylinder, the maximum value of radial stress is at

(a) = center (b) outer surface (c) mean radial section (d) none of these

12. In a long rotating solid cylinder, the radial stress at the centre is given by

(a) $\frac{(3-2\nu)}{2(1-\nu)}\rho\omega^2 R^2$ (b) $\frac{(3-2\nu)}{4(1-\nu)}\rho\omega^2 R^2$ (c) $\frac{(3-2\nu)}{8(1-\nu)}\rho\omega^2 R^2$ (d) $\frac{(3-2\nu)}{1-\nu}\rho\omega^2 R^2$

13. In a flat rotating disc of uniform strength σ at all radii, the thickness at a radius r is given by

(a) $t_o \cdot e^{-\rho\cdot\omega^2\cdot r^2/\sigma}$ (b) $t_o \cdot e^{\rho\cdot\omega^2\cdot r^2/2\sigma}$ (c) $t_o \cdot e^{-\rho\cdot\omega\cdot r^2/2\sigma}$ (d) $t_o \cdot e^{-\rho\cdot\omega^2\cdot r^2/2\sigma}$

14. The collapse speed of a rotating solid disc is given by

(a) $\omega = \frac{1}{R}\sqrt{\frac{3\sigma_y}{\rho}}$ (b) $\omega = \frac{1}{R}\sqrt{\frac{\sigma_y}{3\rho}}$ (c) $\omega = \frac{1}{R}\sqrt{\frac{2\sigma_y}{\rho}}$ (d) $\omega = \frac{1}{R}\sqrt{\frac{\sigma_y}{2\rho}}$

15. The collapse speed of a rotating hollow disc is given by

(a) $\sqrt{\frac{3\sigma_y(R_o - R_i)}{\rho(R_o^2 - R_i^2)}}$ (b) $\sqrt{\frac{3\sigma_y(R_o - R_i)}{\rho(R_o^3 - R_i^3)}}$ (c) $\sqrt{\frac{3\sigma_y(R_o^3 - R_i^3)}{\rho(R_o - R_i)}}$ (d) $\sqrt{\frac{3\sigma_y(R_o - R_i)}{\rho(R_o - R_i)}}$

Answers

1. (b)	2. (a)	3. (a)	4. (c)	5. (b)	6. (d)
7. (a)	8. (a)	9. (a)	10. (b)	11. (a)	12. (c)
13. (d)	14. (a)	15. (b)			

Review Questions

14.1 Deduce an expression for the hoop stress induced in a thin rotating ring.

14.2 Develop governing equations for radial and hoop stresses induced in a flat rotating disc of uniform thickness. Deduce the expressions for a solid disc. What are the maximum values and at what radii?

14.3 Deduce expressions for radial and hoop stresses induced in a flat rotating hollow disc of uniform thickness. What are the maximum values and at what radii?

14.4 Develop the governing equations for radial and hoop stresses induced in a long rotating cylinder. Deduce the expressions for a solid cylinder. What are the maximum values and at what radii?

14.5 Deduce expressions for radial and hoop stresses induced in a long rotating hollow cylinder. What are the maximum values and at what radii?

14.6 Show that the results for a long cylinder can be obtained from those for a thin disc by substituting $\nu/(1-\nu)$ for ν.

Numerical Problems

14.1 A flywheel rim with a mean diameter of 6 m rotates at a speed such that the hoop stress in the material is 10 MPa. The density of the material of the rim is 7000 kg/m^3. Determine the speed ignoring the effect of arms. (120.3 rpm)

14.2 A thin uniform steel disc with a 260-mm diameter has a central hole of 100-mm diameter. Determine the maximum principal stress and the maximum shear stress in the disc when the disc rotates at 9000 rpm. Density of steel 7700 kg/m^3 and Poisson's ratio 0.3. (98.4 MPa; 49.2 MPa)

14.3 A flywheel having a moment of inertia of 260 kg.m^2 is to rotate at a speed of 270 rpm. The maximum stress is to be 5 MPa. Assuming a width of 150 mm of the flywheel and neglecting the inertia of the spokes, find the thickness of the rim. Density of the material of the flywheel is 7100 kg/m^3. (50 mm)

14.4 A solid disc with a 600-mm diameter rotates at a speed of 3000 rpm. Plot the distribution of the circumferential and radial stresses in the disc. Density of the material is 7800 kg/m^3 and Poisson's ratio is 0.28.

14.5 A steel disc of 250-mm external diameter and 50-mm internal diameter is shrunk on a steel shaft so that the pressure between the shaft and the disc at stand still is 50 MPa. Neglecting the change in dimensions of the shaft, determine the speed at which the disc will loosen from the shaft. Density of steel is 7500 kg/m^3 and Poisson's ratio is 0.3. (8045 rpm)

14.6 A flat steel disc of 800-mm external diameter with a 180-mm diameter hole is shrunk fitted onto a solid steel shaft. The shrinkage allowance is 0.08 mm on the radius. Determine the speed at which the shrink fit will loosen and the maximum hoop stress in the disc at this speed. Also, find the maximum hoop stress in the disc at stand still. $E = 206$ MPa. Density 7600 kg/m^3 and Poisson's ratio 0.3. (4080 rpm; 185 MPa; 96.2 MPa)

14.7 A steel disc of 160-mm inside diameter and 480-mm outside diameter is shrunk on a cast-iron disc of 60-mm inside diameter. Find the change in the shrink-fit pressure produced by the inertia forces when the disc rotates at 3000 rpm. $E_s = 200$ MPa, $E_{ci} = 100$ MPa, density of steel 7800 kg/m^3, density of cast iron = 7200 kg/m^3, Poisson's ratio for both is 0.3. (9.65 MPa)

14.8 A rotor disc of 600-mm external diameter at the blade ring is keyed to a 50-mm diameter shaft. For a uniform stress of 220 MPa at 9000 rpm, find the thickness of the rotor at the shaft if the minimum thickness is 9 mm. Density of the rotor material is 7600 kg/m^3. (35.5 mm)

14.9 A solid rotor of constant strength and having a thickness of 180 mm at the central axis rotates at a speed of 3000 rpm. Determine the thickness at a radius of 200 mm from the axis when the allowable stress is 120 MPa. Density of rotor material is 7500 kg/m^3. (159 mm)

Chapter **15**

Theories of Failure

A material is considered failed when a permanent or non-recoverable deformation occurs. There is either direct separation of particles as in case of brittle materials or slipping of particles as for ductile materials where plastic deformations also take place. Machine components and structural members are generally designed on the hypothesis that the material will not yield during the expected loading conditions. For example, in a uniaxial loading, a machine component will be safe as long as the stress produced by the load is less than the yield stress of the material. In a biaxial loading system, it is not possible to predict directly by the above criterion. In such cases, it may require to find the principal stresses at any given point. Thus different criteria may be required to consider the safety of a component. The criteria used under various load conditions and type of materials are known as *theories of failure*.

15.1 MAIN THEORIES OF FAILURE

As mentioned in the introduction, in simple systems with only one kind of stress, it is easy to anticipate the failure, but in complex stress systems in which direct as well as shear stresses act, it is not easy to do so.

The main theories of failure are discussed below. σ_1, σ_2 and σ_3 denote the principal stresses in any complex system and σ the tensile stress at the elastic limit in simple tension.

Maximum Principal Stress Theory (Rankine's Theory)

According to this theory, the failure of material will occur when the maximum principal stress in the complex stress system attains the value of the maximum stress at the elastic limit in simple tension or the minimum principal stress (i.e., maximum compressive stress) attains the elastic limit in simple compression. Thus, in this theory, maximum or minimum principal stress is the criterion of failure.

Let σ_x, σ_y, τ = Direct and shear stresses on given planes in the complex system

and σ_1 = Maximum principal stress

Then $$\sigma_1 = \frac{1}{2}(\sigma_y + \sigma_x) + \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2}$$

It should not be more than maximum stress in simple tension σ, or in the limit, $\sigma_1 = \sigma$

In a similar way, the minimum principal stress (numerical value) should not be more than maximum stress in simple compression.

This theory is found to give good results when applied to brittle materials such as cast iron.

Maximum Shear Stress Theory (Guest's and Tresca's Theory)

This theory states that the failure occurs when the maximum shear stress in the complex system attains the value of the maximum shear stress at the elastic limit in simple tension.

The maximum value of shear stress in terms of principal stresses σ_1 and σ_2,

$$\tau = \frac{1}{2}(\sigma_1 - \sigma_2) = \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2}$$

In a simple direct stress system, the maximum shear stress = σ /2

Thus $(\sigma_1 - \sigma_2)/2$ must not be more than σ /2,

or in the limit, $\quad \sigma_1 - \sigma_2 = \sigma$

However, this applies to values of principal stresses of opposite type, i.e., if one is tensile, the other is compressive. When the principal stresses are alike, either both are tensile or compressive, then according to the above relation, if σ_1 is the maximum principal stress, it can be more than the limiting value of stress σ which is not possible. In that case, the higher value of principal stress must be less than the limiting value σ.

In the limit, $\sigma_1 = \sigma$ or $-\sigma$

This theory is preferred in case of ductile materials such as mild steel.

Maximum Principal Strain Theory (St. Venant's Theory)

According to this theory, the maximum principal strain in the complex stress system must be less than the elastic limit in simple tension if there is to be no failure.

In the limit,

$$\varepsilon_1 = \left(\frac{\sigma_1 - \nu\sigma_2 - \nu\sigma_3}{E}\right) = \frac{\sigma}{E} \tag{15.1}$$

or

$$\sigma_1 - \nu\sigma_2 - \nu\sigma_3 = \sigma \tag{15.2}$$

This theory is not used in general as it is found to give satisfactory results in particular cases only.

Maximum Strain Energy Theory (Haigh's Theory)

This theory is based on the principle that the work done in bringing a body to a particular state is independent of the method applied to bring the body to that state. According to this theory, the failure takes place when the strain energy per unit volume of a body reaches the value of strain energy at elastic limit in simple tension.

In the limit,

$$\frac{1}{2E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)) = \frac{\sigma^2}{2E} \qquad \text{(Eq. 3.5)}$$

or

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma^2 \tag{15.3}$$

This theory is found satisfactory for ductile materials.

Maximum Shear Strain Energy Theory (Mises' and Henkey's Theory)

This theory states that the failure takes place when the shear strain energy in a complex system becomes equal to that in simple tension.

Shear strain energy in a complex system

$$\frac{1}{12G}[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2] \qquad \text{(Eq. 3.12)}$$

Shear strain energy in simple tension is found by inserting $\sigma_1 = \sigma$, $\sigma_2 = 0$ and $\sigma_3 = 0$ in the above expression, i.e.,

Shear strain energy in simple tension

$$=\frac{1}{12G}[(\sigma-0)^2+(0-0)^2+(0-\sigma)^2]=\frac{2\sigma^2}{12G}=\frac{\sigma^2}{6G}$$

Therefore, in the limit

$$[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2]=2\sigma^2 \qquad (15.4)$$

For a two-dimensional stress system, the above relation may be reduced to

$$[(\sigma_1-\sigma_2)^2+(\sigma^2)^2+(-\sigma_1)^2]=2\sigma^2$$

$$\sigma_1^2+\sigma_2^2-\sigma_1\sigma_2=\sigma^2 \qquad (15.5)$$

This theory gives good results when applied to ductile materials.

As shear stress and shear strain energy theories depend upon the stress differences, a material has no chance of failure if the principal stresses are of the same nature and magnitude since the difference will be negligible. Thus these theories should not be applied in such cases.

Example 15.1 Principal stresses in a cast-iron body are 40 MPa tensile and 90 MPa compressive, third principal stress being zero. Determine the factor of safety based on elastic limit, if the criterion of failure is principal stress theory. The elastic limit in simple tension is 80 MPa and in simple compression 450 MPa for cast iron.

Solution

Given

Principal stresses:	$\sigma_t = 40$ MPa	$\sigma_c = 90$ MPa
Elastic limits:	$\sigma_t = 80$ MPa	$\sigma_c = 450$ MPa

To find Factor of safety

Maximum principal stress criterion

Failure takes place when the maximum principal tensile stress reaches the value of maximum stress at the elastic limit.

Maximum principal stress = 40 MPa
Elastic limit = 80 MPa
Factor of safety = 80/40 = 2

Minimum principal stress criterion

Failure takes place when the minimum principal stress (maximum compressive stress) reaches the value of maximum compressive stress at the elastic limit.

Minimum principal stress, σ = 90 MPa
Elastic limit = 450 MPa
Factor of safety = 450/90 = 5

Thus the failure of the material will occur due to tensile principal stress.

$\therefore$ factor of safety = 2

Example 15.2 Principal stresses in a mild steel body are 50 MPa tensile and 95 MPa compressive, third principal stress being zero. Determine the factor of safety based on elastic limit, if the criterion of failure is maximum shear stress theory. The elastic limit in simple tension as well as in compression is 210 MPa. What will be the factor safety if the stress with 95 MPa is also tensile instead of compressive?

Solution

Given

Principal stresses: $\sigma_t = 50$ MPa $\quad \sigma_c = 95$ MPa

Elastic limits: $\sigma_t = \sigma_c = 210$ MPa

To find

— Factor of safety

— Factor of safety if 95 MPa stress is also tensile

With stresses of opposite sign

In maximum shear stress theory with stresses of opposite sign,

$$\text{Factor of safety} = \frac{\sigma}{\sigma_1 - \sigma_2} = \frac{210}{50-(-90)} = \frac{210}{140} = 1.5$$

With stresses of same sign

In maximum shear stress theory with stresses of same sign,

$$\text{Factor of safety} = \frac{\sigma}{\sigma_1} \quad \text{or} \quad \frac{-\sigma}{-\sigma_2}$$

i.e.
$$\frac{210}{50} \quad \text{or} \quad \frac{-210}{-90} \quad \text{i.e. } 4.2 \quad \text{or} \quad 2.33$$

Thus Factor of safety is 2.33 in this case.

Example 15.3 A steel tube of 40-mm mean diameter and 2-mm thickness is under simple tension. Determine the torque that can be transmitted by the tube if the criterion of failure is

(i) Maximum shear stress

(ii) maximum strain energy

(iii) maximum shear strain energy

Take factor of safety as 3, elastic limit of steel 240 MPa and Poisson's ratio is 0.3.

Solution

Given A steel tube:

$d_{mean} = 40$ mm $\quad t = 2$ mm

$\sigma_e = 240$ MPa $\quad \nu = 0.3$

$FOS = 3$

To find Torque transmitted

Let the torque be T N·m.

Outer diameter = Mean diameter + thickness = 40 + 2 = 42 mm

Inner diameter = Mean diameter – thickness = 40 – 2 = 38 mm

Shear stress for a hollow shaft, $\tau = \dfrac{16T\cdot D}{\pi(D^4-d^4)} = \dfrac{16T\times 10^3\times 42}{\pi(42^4-38^4)} = 0.2084T$ MPa

Limiting simple tensile stress = 240/3 = 80 MPa

Maximum shear stress theory

Maximum shear stress in simple tension = 80/2 = 40 MPa

$\therefore \quad 0.2084\,T = 40 \quad \text{or} \quad T = 192\ \text{N}\cdot\text{m}$

Maximum strain energy theory

Maximum strain energy/unit volume in tension = $\dfrac{\tau^2}{2G} = \dfrac{\sigma^2}{2E}$

Also $E = 2G(1 + \nu)$

$$\therefore \quad \frac{(0.2084T)^2}{2G} = \frac{80^2}{2\times 2G(1+0.3)}$$

or $\quad 2.6(0.2084T)^2 = 80^2 \quad \text{or} \quad T = 238\ \text{N}\cdot\text{m}$

or use the relation $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma^2$

in which $\sigma_1 = \tau$; $\sigma_2 = -\tau$ and $\sigma_3 = 0$

$\therefore \quad \tau^2 + (-\tau)^2 - 2\nu(\tau)(-\tau) = \sigma^2 \quad \text{or} \quad 2\tau^2 + 2\times 0.3\tau^2 = \sigma^2$

or $\quad 2.6(0.2084T)^2 = 80^2 \quad \text{or} \quad T = 238\ \text{N}\cdot\text{m}$

Maximum shear strain energy theory

According to this theory,

$$[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = 2\sigma^2 \qquad \text{...(Eq. 15.4)}$$

or $\quad [(\tau - (-\tau)]^2 + (-\tau)^2 + (\tau)^2 = 2\sigma^2$

or $\quad 6(0.2084T)^2 = 2\times 80^2$

or $\quad T = 221.6\ \text{N}\cdot\text{m}$

Example 15.4 A mild steel shaft of 100-mm diameter is subjected to a maximum torque of 12 kN·m and a maximum bending moment of 8 kN·m at a particular section. Determine the factor of safety according to maximum shear stress theory if the elastic limit of mild steel in simple tension is 240 MPa.

Solution

Given A steel shaft:

$T = 12\ \text{kN}\cdot\text{m} = 12\times 10^6\ \text{N}\cdot\text{mm} \qquad M = 8\ \text{kN}\cdot\text{m} = 8\times 10^6\ \text{N}\cdot\text{mm}$

$\sigma_e = 240\ \text{MPa} \qquad d = 0.1\ \text{m}$

To find Factor of safety

Principal stresses

$$\text{Principal stresses} = \frac{16}{\pi d^3}\left[M \pm \sqrt{M^2+T^2}\right]$$

$$= \frac{16}{\pi\times 100^3}\times 10^6\left[8 \pm \sqrt{8^2+12^2}\right] = 5.093[8 \pm 14.42]$$

$$= 114.2\ \text{MPa} \quad \text{and} \quad -32.7\ \text{MPa}$$

Factor of safety

In maximum shear stress theory with stresses of opposite sign,

$$\text{Factor of safety} = \frac{\sigma}{\sigma_1 - \sigma_2} = \frac{240}{114.2-(-32.7)} = \frac{210}{146.9} = 1.63$$

Example 15.5 A shaft is subjected to a maximum torque of 14 kN·m and a maximum bending moment of 10 kN·m at a particular section. Determine the diameter of the shaft according to maximum shear stress theory if the elastic limit in simple tension is 180 MPa.

Solution

Given A shaft subjected to torque:

$T = 14\text{ kN}\cdot\text{m} = 14 \times 10^6\text{ N}\cdot\text{mm}$ $\qquad$ $M = 10\text{ kN}\cdot\text{m} = 10 \times 10^6\text{ N}\cdot\text{mm}$

$\sigma_e = 180\text{ MPa}$

To find Diameter of shaft

Maximum principal stresses, $\sigma_1 = \dfrac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right]$

Minimum principal stresses, $\sigma_2 = \dfrac{16}{\pi d^3}\left[M - \sqrt{M^2 + T^2}\right]$

Maximum shear stress theory

According to maximum shear stress theory,

$$\sigma_1 - \sigma_2 = \sigma_e$$

or
$$\frac{16}{\pi d^3}\left[M + \sqrt{M^2 + T^2}\right] - \frac{16}{\pi d^3}\left[M - \sqrt{M^2 + T^2}\right] = \sigma_e$$

or
$$\frac{32}{\pi d^3}\sqrt{M^2 + T^2} = \sigma_e$$

or
$$\frac{32 \times 10^6}{\pi d^3}\sqrt{10^2 + 14^2} = 180 \quad \text{or} \quad d = 99.1\text{ mm}$$

Example 15.6 Principal stresses at a point in an elastic material are 100 MPa tensile, 50 MPa tensile and 25 MPa compressive. Determine the factor of safety against failure based on various theories. The elastic limit in simple tension is 220 MPa and Poisson's ratio 0.3.

Solution

Given

$\sigma_1 = 100\text{ MPa}$ $\qquad$ $\sigma_2 = 50\text{ MPa}$

$\sigma_3 = 25\text{ MPa}$ $\qquad$ $\sigma_e = 220\text{ MPa}$

$\nu = 0.3$

To find Factor of safety

Maximum principal stress theory

Failure takes place when the maximum principal stress reaches the value of maximum stress at the elastic limit.

Thus maximum principal stress, $\sigma = 100\text{ MPa}$

Factor of safety = 220/100 = 2.2

Maximum shear stress theory

$$\sigma = 100 - (-25) = 125\text{ MPa}$$

Factor of safety = 220/125 = 1.76

Maximum principal strain theory

$$\sigma_1 - \nu\sigma_2 - \nu\sigma_3 = 100 - 0.3 \times 50 - 0.3 \times (-25) = 92.5\text{ MPa}$$

Factor of safety = 220/92.5 = 2.37

Maximum strain energy theory

$$\sigma^2 = \sigma_1^{\;2} + \sigma_2^{\;2} + \sigma_3^{\;2} - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$
$$= 100^2 + 50^2 + (-25)^2 - 2 \times 0.3 \times [100 \times 50 + 50 \times (-25) + (-25) \times 100]$$
$$= 13\ 125$$
$$\sigma = 114.6 \text{ MPa}$$

Factor of safety = 220/114.6 = 1.92

Maximum shear strain energy theory

$$2\sigma^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2$$
$$= (100 - 50)^2 + (50 + 25)^2 + (-25 - 100)^2$$
$$= 23\ 750$$
$$\sigma^2 = 11\ 875$$
$$\sigma = 108.97$$

Factor of safety = 220/108.97= 2.02

Example 15.7 A bolt is acted upon by an axial pull of 16 kN along with a transverse shear force of 10 kN. Determine the diameter of the bolt required according to different theories. Elastic limit of the bolt material is 250 MPa and a factor of safety 2.5 is to be taken. Poisson's ratio is 0.3.

Solution

Given

$F = 16$ kN	$SF = 10$ MPa
$\sigma_e = 250$ MPa	$FOS = 2.5$
$\nu = 0.3$	

To find Diameter of bolt

The permissible stress in simple tension = 250/2.5 = 100 MPa

Let the required area of cross-section and the diameter of the bolt be a and d respectively under different theories.

The applied tensile stress = 16 000/a

The applied shear stress = 10 000/a

Maximum and minimum principal stresses

Maximum principal stress, $$\sigma_1 = \frac{1}{2}(\sigma_y + \sigma_x) + \frac{1}{2}\sqrt{(\sigma_y - \sigma_x)^2 + 4\tau^2}$$
$$= \frac{1}{2a}(16\ 000) + \frac{1}{2a}\sqrt{16\ 000^2 + 4 \times 10\ 000^2} \quad \ldots(\sigma_y = 0)$$
$$= (8000 + 12\ 806)/a$$
$$= 20\ 806/a \text{ (tensile)}$$

Minimum principal stress, $$\sigma_2 = (8000 - 12\ 806)/a$$
$$= 4806/a \text{ (compressive)}$$

Maximum principal stress theory

Maximum principal stress, $\sigma_1 = 20\ 806/a$

Thus $20\ 806/a = 100$

$$\frac{\pi}{4}d^2 = 208.06 \quad \text{or} \quad d = 16.28 \text{ mm}$$

Maximum shear stress theory

$$\text{Maximum shear stress} = [20\,806 - (-4806)]/2a = 12\,806/a$$

Maximum shear stress in simple tension = 100/2 = 50 MPa

$$\therefore \quad 12\,806/a = 50$$

$$\frac{\pi}{4}d^2 = 256.12 \quad \text{or} \quad d = 18.05 \text{ mm}$$

Maximum principal strain theory

$$\sigma_1 - \nu\sigma_2 - \nu\sigma_3 = [20\,806 - 0.3 \times (-4806)]/a = 22\,247.8/a \quad \ldots(\sigma_3 = 0)$$

$$22\,247.8/a = 100$$

$$\frac{\pi}{4}d^2 = 222.48 \quad \text{or} \quad d = 16.83 \text{ mm}$$

Maximum strain energy theory

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma^2$$

$$[20\,806^2 + (-4806)^2 - 2 \times 0.3 \times 20\,806 \times (-4806)]/a = 100^2$$

$$a = 227.16 \text{ mm}^2$$

$$\frac{\pi}{4}d^2 = 227.16 \quad \text{or} \quad d = 17.0 \text{ mm}$$

Maximum shear strain energy theory

$$[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = 2\sigma^2$$

$$[(20\,806 + 4806)^2 + (-4806)^2 + (-20\,806)^2]/a^2 = 2 \times 100^2$$

$$a^2 = 55\,598 \quad \text{or} \quad a = 235.8 \text{ mm}^2$$

or

$$\frac{\pi}{4}d^2 = 235.8 \quad \text{or} \quad d = 17.32 \text{ mm}$$

15.2 DESIGN OF THICK CYLINDRICAL SHELL

Thin cylinders are designed on the criterion of circumferential or hoop stress which is assumed constant over the thickness. However, in case of thick cylinders, stresses in all the three directions exist. Considering the case of a thick cylinder subjected to internal pressure only, maximum values of three principal stresses at the inner radius are

- Radial stress or pressure, p_i (compressive)
- Hoop stress, $\sigma_c = \dfrac{d_o^2 + d_i^2}{d_o^2 - d_i^2} \cdot p_i = \dfrac{k^2+1}{k^2-1} \cdot p_i$ where $k = d_o/d_i$ (tensile)
- Longitudinal stress, $\sigma_l = \dfrac{p_i d_i^2}{(d_o^2 - d_i^2)} = \dfrac{p_i}{(k^2-1)}$ (tensile)

Out of these three principal stresses, the hoop stress is the biggest. The designing involves the determination of k or the ratio of inner to outer diameter and thus the thickness. The safe ratio can be found on the basis of different theories of elastic failure (Section 3.8). Assuming the allowable stress is σ in tension.

1. Maximum Principal Stress Theory Failure occurs when the hoop stress exceeds the allowable tensile stress for the material. Thus for safe design,

$$\frac{k^2+1}{k^2-1} \cdot p_i \le \sigma \tag{15.6}$$

2. *Maximum Shear Stress Theory* Maximum shear stress at inner radius

$$= \frac{1}{2}(\sigma_c + p_i) = \frac{1}{2}\left(\frac{k^2+1}{k^2-1}\cdot p_i + p_i\right)$$

$\therefore$ for safe design, $\dfrac{2k^2}{k^2-1}\cdot p_i \le \sigma$ (15.7)

3. *Maximum Principal Strain Theory* Maximum principal strain in the complex stress system must be less than the elastic limit in simple tension. In the limit,

$$\varepsilon_1 = \left(\frac{\sigma_c - \nu\sigma_r - \nu\sigma_l}{E}\right) = \frac{\sigma}{E}$$

$$\sigma_c - \nu\sigma_r - \nu\sigma_l = \sigma$$

For safe design, $\dfrac{k^2+1}{k^2-1}\cdot p_i + \nu\cdot p_i - \nu\dfrac{p_i}{k^2-1} \le \sigma$

or $\left(\dfrac{k^2+1}{k^2-1} + \nu - \nu\dfrac{1}{k^2-1}\right)p_i \le \sigma$

or $\dfrac{p_i}{k^2-1}[k^2+1+\nu(k^2-2)] \le \sigma$ (15.8)

4. *Maximum Strain Energy Theory* When the strain energy per unit volume of a body reaches the value of strain energy at elastic limit in simple tension.

$$\frac{1}{2E}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)) = \frac{\sigma^2}{2E}$$

or $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma^2$

Therefore, for safe design,

$$\left(\frac{k^2+1}{k^2-1}\cdot p_i\right)^2 + (-p_i)^2 + \left(\frac{p_i}{k^2-1}\right)^2$$

$$-2\nu p_i^2\left[\left(\frac{k^2+1}{k^2-1}\right)(-1) + (-1)\left(\frac{p_i}{k^2-1}\right) + \left(\frac{p_i}{k^2-1}\right)\left(\frac{k^2+1}{k^2-1}\right)\right] \le \sigma^2$$

or $\dfrac{p_i^2}{(k^2-1)^2}[(2k^4+3) - 2\nu(-k^4+1-k^2+1+k^2+1)] \le \sigma^2$

or $\dfrac{p_i^2}{(k^2-1)^2}[2k^4(1+\nu) + 3(1-2\nu)] \le \sigma^2$ (15.9)

5. *Maximum Shear Strain Energy Theory* Failure takes place when the shear strain energy in a complex system becomes equal to that in simple tension.

For safe design,

$$[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] \le 2\sigma^2$$

$$p_i^2\left[\left(\frac{k^2+1}{k^2-1} + 1\right)^2 + \left(-1 - \frac{1}{k^2-1}\right)^2 + \left(\frac{1}{k^2-1} - \frac{k^2+1}{k^2-1}\right)^2\right] \le 2\sigma^2$$

or $$\frac{p_i^2}{(k^2-1)^2}[(2k^2)^2+(-k^2)^2+(-k^2)^2] \le 2\sigma^2$$

or $$\frac{3k^4 p_i^2}{(k^2-1)^2} \le \sigma^2 \tag{15.10}$$

Example 15.8 A cylinder of 600-mm internal diameter is to be designed to sustain an internal pressure of 30 MPa. Assuming an allowable stress of 180 MPa and Poisson's ratio of 0.25, find the wall thickness by applying the different theories of failure.

Solution

Given A cylinder:

$d_i = 600$ mm $\qquad p_i = 30$ MPa

$\sigma_e = 180$ MPa $\qquad \nu = 0.25$

To find Wall thickness

Let $k = d_o/d_i$

Maximum principal stress theory

$$\frac{k^2+1}{k^2-1}\cdot p_i = \sigma \qquad \text{...(Eq. 15.6)}$$

or $$\frac{k^2+1}{k^2-1}\times 30 = 180 \quad \text{or} \quad k = 1.183$$

$\therefore$ $$d_o = 600 \times 1.183 = 710 \text{ mm}$$

and thus $$t = \frac{710-600}{2} = 55 \text{ mm}$$

Maximum shear stress theory

$$\frac{2k^2}{k^2-1}\cdot p_i = \sigma \qquad \text{...(Eq. 15.7)}$$

or $$\frac{2k^2}{k^2-1}\times 30 = 180$$

or $$k = 1.225 \quad \text{or} \quad d_o = 735 \text{ mm}$$

and $$t = \frac{735-600}{2} = 67.5 \text{ mm}$$

Maximum principal strain theory

$$\frac{p_i}{k^2-1}[k^2+1+\nu(k^2-2)] = \sigma \qquad \text{...(Eq. 15.8)}$$

or $$\frac{30}{k^2-1}[k^2+1+0.25(k^2-2)] = 180$$

or $$k = 1.17 \quad \text{or} \quad d_o = 702 \text{ mm}$$

and $$t = 51 \text{ mm}$$

Maximum strain energy theory

$$\frac{p_i^2}{(k^2-1)^2}[2k^4(1+\nu)+3(1-2\nu)]=\sigma^2 \qquad \text{...(Eq. 15.9)}$$

or $$\frac{30^2}{(k^2-1)^2}[2k^4(1+0.25)+3(1-2\times 0.25)]=180^2$$

$$\frac{1}{(k^2-1)^2}[2.5k^4+1.5]=36$$

on solving $k = 1.195$ or $d_o = 717$ mm

and $t = 58.5$ mm

Maximum shear strain energy theory

$$\frac{3k^4 p_i^2}{(k^2-1)^2}\le\sigma^2 \qquad \text{...(Eq. 15.10)}$$

or $$\frac{3k^4\times 30^2}{(k^2-1)^2}\le 180^2$$

On solving $k = 1.185$ or $d_o = 711$ mm or $t = 55.5$ mm

Thus thickness by various theories is as under:

Maximum principal stress theory	= 55 mm
Maximum shear stress theory	= 67.5 mm
Maximum principal strain theory	= 51 mm
Maximum strain energy theory	= 58.5 mm
Maximum shear strain energy theory	= 55.5 mm

15.3 GRAPHICAL REPRESENTATIONS OF THEORIES OF FAILURE

In a two-dimensional stress system, the limits of principal stresses according to different theories can be shown graphically as under:

In a two-dimensional system, σ_3 is taken to be zero and the values of principal stresses σ_1 and σ_2 are taken along x- and y-axes respectively. Positive values of σ_1 are taken towards right of the y-axis and negative towards left. Similarly, positive values of σ_2 are taken upwards and negative downwards of the x-axis. The elastic limit σ may be taken to be the same both in tension as well as in compression.

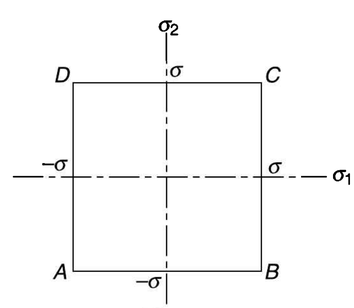


Fig. 15.1

1. Maximum Principal Stress Theory According to maximum principal stress theory, the maximum principal stress σ_1 (or σ_2) must not exceed the elastic limit σ. Thus maximum value of σ_1 and σ_2 can be

$$\sigma_1=\sigma,\ \sigma_2=\sigma,\ \sigma_1=-\sigma \text{ and } \sigma_2=-\sigma$$

This provides a square boundary *ABCD* as shown in Fig. 15.1.

2. Maximum Shear Stress Theory For alike type of stresses when both are tensile or compressive, i.e., both lie in first or third quadrant and the stress in the third perpendicular plane is assumed zero,

$$\sigma_1=(\sigma-0)=\sigma,$$

Similarly, $$\sigma_2=\sigma,\ \sigma_1=-\sigma \quad \text{or} \quad \sigma_2=-\sigma$$

These values generate the boundary lines *FC, CG, HA* and *AE* in the first and third quadrants (Fig. 15.2).

When the principal stresses are of opposite type, i.e., if one is tensile, the other is compressive, then

$$\sigma_1 - \sigma_2 = \pm\,\sigma$$

If σ_2 is assumed negative, it can be written as

$$\sigma_1 - (-\sigma_2) = +\,\sigma$$

or

$$\sigma_1 + \sigma_2 = +\,\sigma$$

This provides a straight boundary *EF* in the fourth quadrant.

When σ_1 is assumed negative, the equation will be,

$$(-\sigma_1) - \sigma_2 = -\,\sigma$$

or

$$-\sigma_1 - \sigma_2 = -\,\sigma$$

This provides the boundary *GH* in the second quadrant.

Thus the boundary for this criterion is *AEFCGHA*

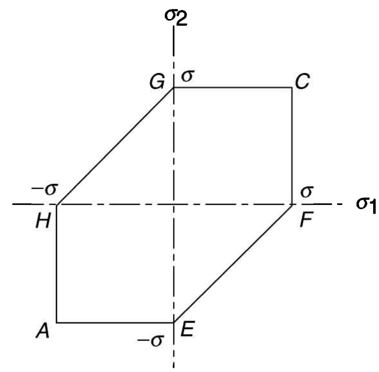


Fig. 15.2

3. Maximum Principal Strain Theory In case of two-dimensional principal strain theory, we have

$$\sigma_1 - \nu\sigma_2 = \pm\,\sigma$$

For like principal stresses, the limits are provided by

$$\sigma_1 - \nu\sigma_2 = \sigma,\ \sigma_2 - \nu\sigma_1 = \sigma,$$

$$\sigma_1 - \nu\sigma_2 = -\,\sigma \text{ and } \sigma_2 - \nu\sigma_1 = -\,\sigma$$

For like principal stresses, the lines generated are *FP, PG, HR* and *RE* (Fig. 15.3).

For unlike stresses, the lines are *GQ, QH, ES* and *SF* respectively in a similar way.

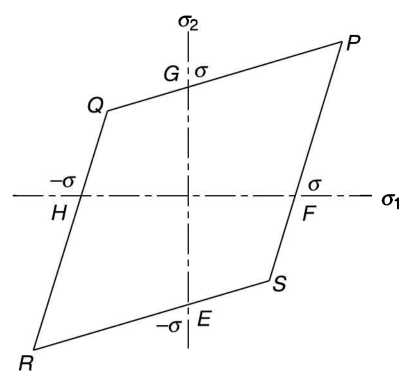


Fig. 15.3

4. Maximum Strain Energy Theory In maximum strain energy theory, the equation in two-dimensional system is

$$\sigma_1^2 + \sigma_2^2 - 2\nu\sigma_1\sigma_2 = \sigma$$

which is the equation of an ellipse with axes at 45° to the axes. It passes through the points *EFGH* (Fig. 15.4).

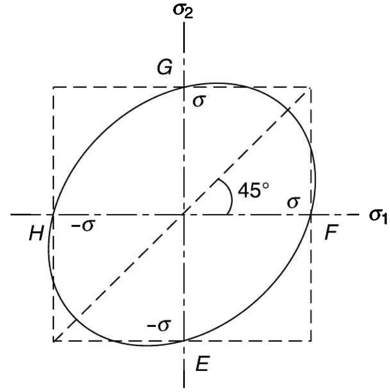


Fig. 15.4

5. Maximum Shear Strain Energy Theory In this, the equation in the two-dimensional system is

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma^2$$

which is again an equation of ellipse and is plotted in Fig. 15.5.

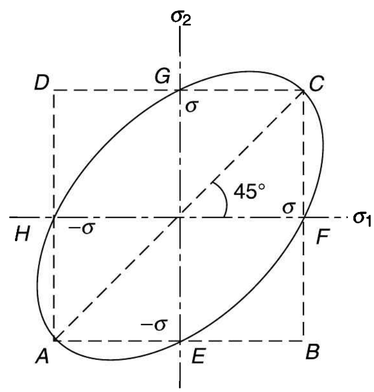


Fig. 15.5

Figure 15.6 shows graphical representation of the above theories in a combined diagram and the plots due to various theories can easily be compared.

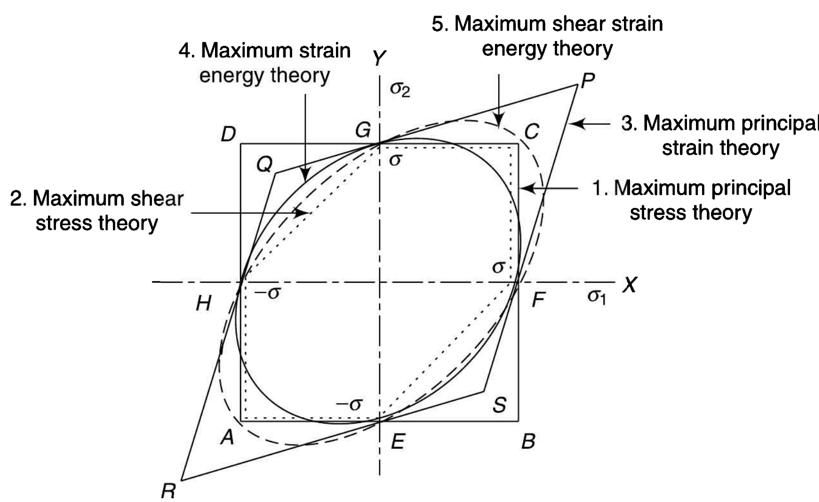


Fig. 15.6

Summary

1. Various theories of failure are
   (i) Maximum principal stress theory (Rankine's theory)
   (ii) Maximum shear stress theory (Guest's and Tresca's theory)
   (iii) Maximum principal strain theory (St. Venant's theory)
   (iv) Maximum strain energy theory (Haigh's theory)
   (v) Maximum shear strain energy theory (Mises' and Henkey's theory)

2. According to maximum principal stress theory, maximum principal stress in a complex stress system should not exceed the value of the maximum stress at the elastic limit in simple tension.

   In the limit, $\sigma_1 = \sigma$

3. Maximum shear stress theory states that the failure occurs when the maximum shear stress in the complex system attains the value of the maximum shear stress at the elastic limit in simple tension.

   In the limit, $\sigma_1 - \sigma_2 = \sigma$

4. According to maximum principal strain theory, the maximum principal strain in the complex stress system must be less than the elastic limit in simple tension if there is to be no failure.

   In the limit, $\sigma_1 - \nu\sigma_2 - \nu\sigma_3 = \sigma$

5. Maximum strain energy theory is based on the principle that the work done in bringing a body to a particular state is independent of the method applied to bring the body to that state. According to this theory, the failure takes place when the strain energy per unit volume of a body reaches the value of strain energy at elastic limit in simple tension.

   In the limit, $\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$

6. Maximum shear strain energy theory states that the failure takes place when the shear strain energy in a complex system becomes equal to that in simple tension.

   In the limit, $[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2] = 2\sigma^2$

   For a 2-D stress system, $\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 = \sigma^2$

7. Maximum principal stress theory gives good results when applied to brittle materials such as cast iron. Maximum shear stress theory, maximum strain energy theory and maximum shear stress energy theories are suitable to ductile materials such as mild steel whereas maximum principal strain theory gives satisfactory results in particular cases only.

8. For safe design of thick cylinders according to

   (i) Maximum principal stress theory, $\dfrac{k^2+1}{k^2-1} \cdot p_i \le \sigma$

   (ii) Maximum shear stress theory, $\dfrac{2k^2}{k^2-1} \cdot p_i \le \sigma$

   (iii) Maximum principal strain theory, $\dfrac{p_i}{k^2-1}[k^2 + 1 + \nu(k^2 - 2)] \le \sigma$

   (iv) Maximum strain energy theory, $\dfrac{p_i^2}{(k^2-1)^2}[2k^4(1+\nu) + 3(1-2\nu)] \le \sigma^2$

   (v) Maximum shear stress energy theory, $\dfrac{3k^4 p_i^2}{(k^2-1)^2} \le \sigma^2$

   In the above relations, $k = d_o/d_i$

Objective Type Questions

1. Maximum normal stress theory is used for

(a) brittle materials (b) ductile materials (c) both of these (d) none of these

2. Maximum principal stress theory of failure was postulated by

(a) Castigliano (b) Rankine (c) Tresca (d) St. Venant

3. Maximum normal stress theory is also known as
   (a) Rankine's theory (b) St. Venant's theory
   (c) Guest's and Tresca's theory (d) Mises' and Henkey's theory
4. Maximum principal strain theory
   (a) Rankine's theory (b) St. Venant's theory
   (c) Guest's and Tresca's theory (d) Mises' and Henkey's theory
5. Maximum shear stress theory
   (a) Rankine's theory (b) St. Venant's theory
   (c) Guest's and Tresca's theory (d) Mises' and Henkey's theory

Answers

1. (a) 2. (b) 3. (a) 4. (b) 5. (c)

Review Questions

15.1 What do you mean by theories of failure? What is their importance?

15.2 What are the main theories of failure for a material? Explain their relative use.

15.3 Give an account of graphical representation of various theories of failure.

15.4 Discuss the design of thick cylindrical shells based on the criteria of various theories of failure.

Numerical Problems

15.1 At a point in a bar of cast iron, the principal stresses are 60 MPa tensile and 120 MPa compressive whereas the third principal stress is zero. Find the factor of safety based on the elastic limit when the criterion of failure is principal stress theory. Assume the elastic limit of cast iron to be 90 MPa in tension and 440 MPa in compression. (1.5)

15.2 In a mild steel body the three principal stresses are 40 MPa tensile and 70 MPa compressive and zero. The elastic limit of the material both in tension and compression is 200 MPa. Determine the factor of safety based on elastic limit if the criterion of failure is shear stress theory. (2.86)

15.3 Find the maximum torque to be transmitted by a hollow mild steel shaft of 40 mm inside diameter and 5 mm thick with a factor of safety of 2.5 using the criterion of (i) maximum shear stress, (ii) maximum strain energy, and (iii) maximum shear strain energy. Assume the elastic limit in tension to be 250 MPa. (724.6 N · m, 898.8 N · m, 836.7 N · m)

15.4 A mild steel shaft of 120 mm diameter is to sustain a maximum torque of 20 kN · m and maximum bending moment of 12 kN · m at a point in the material. Determine the factor of safety according to maximum shear stress theory when the elastic limit in simple tension is 220 MPa. (1.6)

15.5 A thick cylindrical shell of 300-mm inside diameter is to withstand an internal pressure of 30 MPa. The allowable tensile stress for the material of the shell is 150 MPa. Determine the thickness of the shell on the basis of following theories of failure neglecting the longitudinal direct stress:
(a) Maximum principal stress theory
(b) Maximum shear stress theory
(c) Maximum principal strain theory
(d) Maximum strain energy theory
Poisson' ratio is 0.3. (33.75 mm, 43.5 mm, 36 mm, 33 mm)

15.6 An axial pull of 20 kN along with a shear force of 15 kN is applied to a circular bar of 20-mm diameter. The elastic limit of the bar material is 230 MPa and the Poisson's ratio, $\nu = 0.3$. Determine the factor of safety against failure based on

(a) Maximum shear stress theory
(b) Maximum strain energy theory
(c) Maximum principal strain energy theory
(d) Maximum shear strain energy theory (2; 2.3; 2.37; 2.2)

Chapter 16

Circular Plates

Many times, there are cases of loaded circular plates. The load may be symmetric or asymmetric about the central vertical axis of a horizontal plate. In symmetric loading, there can be one point load at the centre or a number of point loads or uniformly distributed loads arranged symmetrically. The plates can be solid or annular. Also, there can be simply supported or rigidly fixed edges of the plates. In all the cases, stresses are developed in the plates and deflection takes place. To develop the theory for asymmetric loading is very complex and thus only some simple cases of symmetric loading are being taken in this chapter. In case of beams, bending is about one horizontal axis. In plates, it is about two horizontal axes and is symmetric if the load is symmetric.

16.1 SYMMETRICALLY LOADED CIRCULAR PLATES

Consider a symmetrically loaded circular plate. Figure 16.1 shows its diametral section. O is the centre. Ox and Oy are the principal axes. Oz is the axis perpendicular to the plane of the figure. Consider a section AB at a distance x from O. Let C be the centre of curvature of this section.

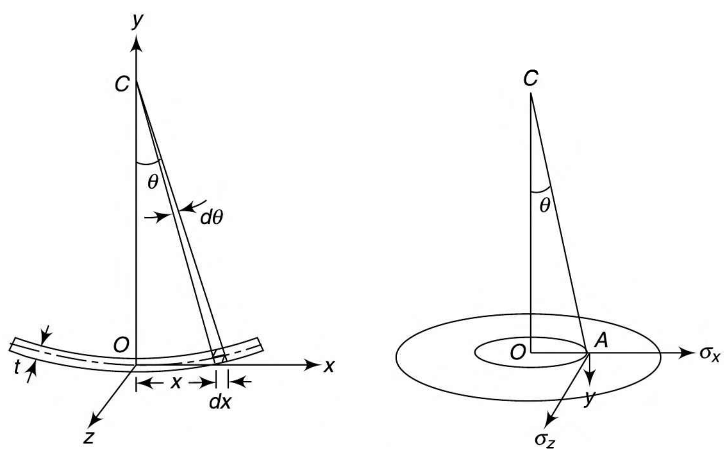


Fig. 16.1

In the deformed shape of the plate, the slope of the elastic curve would change continuously and thus the radius of curvature at any section will depend upon the rate of change of slope. Consider a small length dx at a distance x from O along the radial x-direction. Let C be the centre of curvature of this section. If dx subtends an angle $d\theta$ at C and R is the radius of curvature of the section. Then

$$R_{xy} \cdot d\theta = dx \quad \text{or} \quad \frac{1}{R_{xy}} = \frac{d\theta}{dx} \tag{16.1}$$

Also, from theory of bending, $\dfrac{1}{R_{xy}} = \dfrac{d^2y}{dx^2}$

$$\therefore \qquad \frac{d\theta}{dx} = \frac{d^2y}{dx^2} \quad \text{and} \quad \theta = \frac{dy}{dx}$$

In the tangential z-direction, there is no change in the slope at the same radial distance round the circular path. Thus, radius of curvature is to be the same at a particular radial distance. Sections like AB can be assumed to be part of a cone with C as the apex. So, C is the centre of curvature in the plane y-z and therefore,

$$R_{yz} \cdot \theta = x \text{ or } \frac{1}{R_{yz}} = \frac{\theta}{x} \tag{16.2}$$

Let u be the distance of any fibre from the neutral or central surface of the plate, then the linear strains in the planes xy and yz from the theory of simple bending,

$$\frac{\sigma}{y} = \frac{E}{R} \text{ or } \frac{\sigma}{E} = \frac{y}{R}$$

$$\therefore \qquad e_x = \frac{u}{R_{xy}} = \frac{1}{E}(\sigma_x - \nu\sigma_z) \quad \text{...(replacing } y \text{ with } u\text{)}$$

or

$$\sigma_x - \nu\sigma_z = \frac{uE}{R_{xy}} \tag{i}$$

Similarly,

$$e_z = \frac{u}{R_{yz}} = \frac{1}{E}(\sigma_z - \nu\sigma_x) \quad \text{or} \quad \sigma_z - \nu\sigma_x = -\frac{uE}{R_{yz}} \tag{ii}$$

Multiply (ii) by ν and adding to (i),

$$\sigma_x - \nu^2\sigma_x = uE\left(\frac{1}{R_{xy}} + \frac{\nu}{R_{yz}}\right)$$

or

$$\sigma_x = \frac{uE}{1-\nu^2}\left(\frac{1}{R_{xy}} + \frac{\nu}{R_{yz}}\right) = \frac{uE}{1-\nu^2}\left(\frac{d\theta}{dx} + \frac{\nu\theta}{x}\right) \tag{16.3}$$

Similarly,

$$\sigma_z = \frac{uE}{1-\nu^2}\left(\frac{\nu}{R_{xy}} + \frac{1}{R_{yz}}\right) = \frac{uE}{1-\nu^2}\left(\nu\frac{d\theta}{dx} + \frac{\theta}{x}\right) \tag{16.4}$$

Bending moment per unit length in the z-direction,

$$M_{xy} = \int_{-t/2}^{t/2} \text{(stress} \times \text{area} \times \text{distance from origin)}$$

$$= \int_{-t/2}^{t/2} \sigma_x (du \cdot 1) \cdot u$$

$$= \int_{-t/2}^{t/2} \frac{uE}{1-\nu^2}\left(\frac{d\theta}{dx} + \frac{\nu\theta}{x}\right) du \cdot u = \int_{-t/2}^{t/2} \frac{E}{1-\nu^2}\left(\frac{d\theta}{dx} + \frac{\nu\theta}{x}\right) u^2 \cdot du$$

$$= \frac{E}{1-\nu^2}\left(\frac{d\theta}{dx} + \frac{\nu\theta}{x}\right)\left(\frac{u^3}{3}\right)_{-t/2}^{t/2} = \frac{Et^3}{12(1-\nu^2)}\left(\frac{d\theta}{dx} + \frac{\nu\theta}{x}\right)$$

$$= C\left(\frac{d\theta}{dx} + \frac{\nu\theta}{x}\right) \tag{16.5}$$

where $$C = \frac{Et^3}{12(1-\nu^2)} \tag{16.6}$$

Similarly, $$M_{yz} = C\left(\nu\frac{d\theta}{dx} + \frac{\theta}{x}\right) \tag{16.7}$$

Note that as $$M_{xy} = \frac{Et^3}{12(1-\nu^2)}\left(\frac{d\theta}{dx} + \frac{\nu\theta}{x}\right) = \frac{t^3}{12u}\cdot\frac{Eu}{(1-\nu^2)}\left(\frac{d\theta}{dx} + \frac{\nu\theta}{x}\right) = \frac{t^3}{12u}\cdot\sigma_x$$

∴ $$\sigma_x = \frac{12u}{t^3}M_{xy} \tag{16.8}$$

Similarly, $$\sigma_z = \frac{12u}{t^3}M_{yz} \tag{16.9}$$

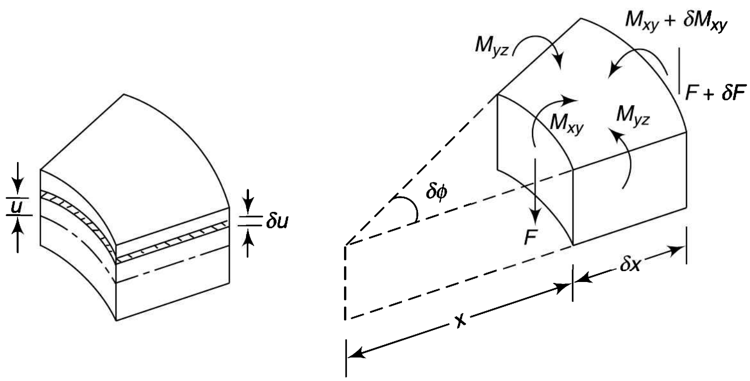


Fig. 16.2

Figure 16.2 shows various forces and moments per unit length acting on an element subtending an angle $\delta\varphi$ at the centre.

Let F = shear force per unit length in the z-direction

Considering the equilibrium of the couples in the central radial plane,

$$(M_{xy} + \delta M_{xy})[(x+\delta x)\delta\phi] - M_{xy}\cdot x\cdot\delta\phi - 2M_{yz}\cdot\delta x\cdot\tfrac{1}{2}\sin\delta\phi + (F + \delta F/2)\delta x\cdot(x\cdot\delta\phi) = 0$$

(Couple due to shear force on the basis of average force)

or $$(M_{xy} + \delta M_{xy})[(x+\delta x)\delta\phi] - M_{xy}\cdot x\cdot\delta\phi - M_{yz}\cdot\delta x\cdot\delta\phi + F\cdot(x\cdot\delta\phi)\cdot\delta x = 0$$

(For small angles, $\sin\delta\varphi \approx \delta\varphi$ and neglecting the product of $\delta F\cdot\delta x\cdot\delta\varphi$

or $$(M_{xy} + \delta M_{xy})(x+\delta x) - M_{xy}\cdot x - M_{yz}\cdot\delta x + F\cdot x\cdot\delta x = 0$$

In the limit, $$M_{xy}\cdot dx + dM_{xy}\cdot x - M_{yz}\cdot dx + F\cdot x\cdot dx = 0$$

Dividing by $x\cdot dx$, $$\frac{1}{x}\cdot M_{xy} + \frac{dM_{xy}}{dx} - \frac{1}{x}\cdot M_{yz} + F = 0$$

or $$\frac{C}{x}\left(\frac{d\theta}{dx}+\frac{\nu\theta}{x}\right)+C\frac{d}{dx}\left(\frac{d\theta}{dx}+\frac{\nu\theta}{x}\right)-\frac{1}{x}C\left(\nu\frac{d\theta}{dx}+\frac{\theta}{x}\right)+F=0$$

or $$\frac{C}{x}\cdot\frac{d\theta}{dx}+\frac{C}{x}\cdot\frac{\nu\theta}{x}+C\frac{d^2\theta}{dx^2}+C\nu\left(\frac{1}{x}\cdot\frac{d\theta}{dx}-\frac{\theta}{x^2}\right)-\frac{C\nu}{x}\frac{d\theta}{dx}-\frac{C\theta}{x^2}+F=0$$

or $$\frac{d^2\theta}{dx^2}+\frac{1}{x}\cdot\frac{d\theta}{dx}-\frac{\theta}{x^2}=-\frac{F}{C}\quad\text{or}\quad\frac{d}{dx}\left(\frac{d\theta}{dx}+\frac{\theta}{x}\right)=-\frac{F}{C}$$

or $$\frac{d}{dx}\left[\frac{1}{x}\cdot\frac{d}{dx}(x\cdot\theta)\right]=-\frac{F}{C}$$

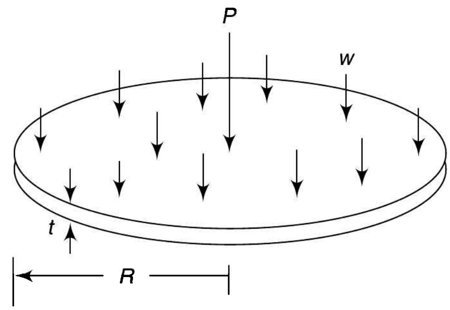


Fig. 16.3

For a plate loaded with uniformly distributed load w per unit area and a point load P at the centre (Fig. 16.3),

$2\pi x\cdot F=\pi x^2 w+P$ or $F=\frac{wx}{2}+\frac{P}{2\pi x}$ per unit length circumferentially (except at centre)

Thus $\frac{d}{dx}\left[\frac{1}{x}\cdot\frac{d}{dx}(x\cdot\theta)\right]=-\frac{wx}{2C}-\frac{P}{2\pi xC}$

Integrating,

$$\frac{1}{x}\cdot\frac{d}{dx}(x\cdot\theta)=-\frac{wx^2}{4C}-\frac{P}{2\pi C}\log x+C_1\quad\text{or}\quad\frac{d}{dx}(x\cdot\theta)=-\frac{wx^3}{4C}-\frac{P}{2\pi C}\log x\cdot x+C_1\cdot x$$

Integrating again, $$x\cdot\theta=-\frac{wx^4}{16C}-\frac{P}{2\pi C}\left(\frac{x^2}{2}\log x-\int\frac{1}{x}\cdot\frac{x^2}{2}dx\right)+C_1\cdot\frac{x^2}{2}+C_2$$

$$x\cdot\theta=-\frac{wx^4}{16C}-\frac{P}{2\pi C}\left(\frac{x^2}{2}\log x-\frac{x^2}{4}\right)+C_1\cdot\frac{x^2}{2}+C_2$$

or $$\theta=-\frac{wx^3}{16C}-\frac{Px}{4\pi C}\log x+\frac{Px}{8\pi C}+C_1\cdot\frac{x}{2}+\frac{C_2}{x}\tag{16.10}$$

Deflection, $y=\int\theta\cdot dx+C_3$

$$=\int\left(-\frac{wx^3}{16C}-\frac{Px}{4\pi C}\log x+\frac{Px}{8\pi C}+C_1\cdot\frac{x}{2}+\frac{C_2}{x}\right)dx+C_3$$

$$=-\frac{wx^4}{64C}-\frac{P}{4\pi C}\left(\frac{x^2}{2}\log x-\frac{x^2}{4}\right)+\frac{Px^2}{16\pi C}+C_1\cdot\frac{x^2}{4}+C_2\cdot\log x+C_3$$

$$=-\frac{wx^4}{64C}-\frac{Px^2}{8\pi C}\log x+\frac{Px^2}{16\pi C}+\frac{Px^2}{16\pi C}+C_1\cdot\frac{x^2}{4}+C_2\cdot\log x+C_3$$

$$=-\frac{wx^4}{64C}-\frac{Px^2}{8\pi C}(\log x-1)+C_1\cdot\frac{x^2}{4}+C_2\cdot\log x+C_3\tag{16.11}$$

16.2 UNIFORMLY DISTRIBUTED LOAD ON A SOLID PLATE

Let R and t be the radius and the thickness respectively of the plate.

(i) Edges Freely Supported A plate simply supported at the edges and having a uniformly distributed load is shown in Fig. 16.4.

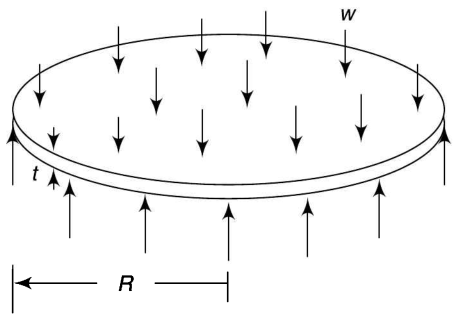


Fig. 16.4

As $P = 0$; $\therefore$ from Eq. 16.10,

$$\theta = -\frac{wx^3}{16C} + C_1 \cdot \frac{x}{2} + \frac{C_2}{x}$$

θ cannot be infinite at the centre, $\therefore\ C_2 = 0$

or $$\theta = -\frac{wx^3}{16C} + C_1 \cdot \frac{x}{2}$$

$$M_{xy} = 0 = C\left(\frac{d\theta}{dx} + \frac{v\theta}{x}\right) \quad \text{or}$$

$$\frac{d\theta}{dx} + \frac{v\theta}{x} = 0 \qquad \text{...(Eq. 16.5)}$$

or $$\frac{d}{dx}\left(-\frac{wx^3}{16C} + C_1 \cdot \frac{x}{2}\right) + \frac{v}{x}\left(-\frac{wx^3}{16C} + C_1 \cdot \frac{x}{2}\right) = 0$$

or $$-\frac{3wx^2}{16C} + \frac{C_1}{2} - \frac{vwx^2}{16C} + \frac{v}{2} \cdot C_1 = 0$$

At $x = R$, $M_{xy} = 0$

$$\frac{C_1}{2}(1 + v) = \frac{wR^2}{16C}(3 + v) \quad \text{or} \quad C_1 = \frac{wR^2}{8C}\,\frac{3 + v}{1 + v}$$

Central deflection, $$y = -\frac{wx^4}{64C} + \frac{wR^2}{8C}\,\frac{3+v}{1+v} \cdot \frac{x^2}{4} + C_2 \cdot \log x + C_3 \qquad \text{....(Eq. 16.11)}$$

y cannot be infinite at the centre, $\therefore\ C_2 = 0$ or $y = -\dfrac{wx^4}{64C} + \dfrac{wR^2x^2}{32C}\,\dfrac{3+v}{1+v} + C_3$

At $x = 0$, $y = 0$, $\therefore C_3 = 0$,

$$y = -\frac{wx^4}{64C} + \frac{wR^2x^2}{32C}\,\frac{3+v}{1+v} = \frac{wx^2}{32C}\left(\frac{3+v}{1+v} \cdot R^2 - \frac{x^2}{2}\right) \qquad (16.12)$$

Central deflection, at $x = R$,

$$y = \frac{wR^4}{64C}\left(\frac{5+v}{1+v}\right) = \frac{wR^4}{64}\left(\frac{5+v}{1+v}\right) \Big/ \frac{Et^3}{12(1-v^2)} = \frac{3wR^4}{16Et^3}(5+v)(1-v) \qquad (16.13)$$

(Inserting the value of C from Eq. 16.6)

$$\sigma_x = \frac{uE}{1-v^2}\left(\frac{d\theta}{dx} + \frac{v\theta}{x}\right) = \frac{uE}{1-v^2}\left(-\frac{3wx^2}{16C} + C_1 \cdot \frac{1}{2} - \frac{vwx^2}{16C} + \frac{v}{2} \cdot C_1\right) \qquad \text{...(Eq. 16.3)}$$

$$= \frac{uE}{1-v^2}\left(-\frac{3wx^2}{16C} + \frac{wR^2}{8C}\,\frac{3+v}{1+v} \cdot \frac{1}{2} - \frac{vwx^2}{16C} + \frac{v}{2} \cdot \frac{wR^2}{8C}\,\frac{3+v}{1+v}\right)$$

$$= \frac{uE}{1-v^2}\left[-\frac{wx^2}{16C}(3+v) + \frac{wR^2}{16C}\left(\frac{(3+v) + v(3+v)}{1+v}\right)\right]$$

$$= \frac{uE}{1-\nu^2}\left[-\frac{wx^2}{16C}(3+\nu)+\frac{wR^2}{16C}\left(\frac{(3+\nu)(1+\nu)}{1+\nu}\right)\right]$$

$$= \frac{uE}{1-\nu^2}\cdot\frac{w(3+\nu)}{16C}(R^2-x^2) \tag{16.14}$$

Maximum stress at $x = 0$, $(u = t/2)$,

$$\sigma_x = \frac{(t/2)E}{1-\nu^2}\left[\frac{wR^2}{16}(3+\nu)\Big/\frac{Et^3}{12(1-\nu^2)}\right] = \frac{3wR^2}{8t^2}(3+\nu) \tag{16.15}$$

$$\sigma_z = \frac{uE}{1-\nu^2}\left(\nu\frac{d\theta}{dx}+\frac{\theta}{x}\right) = \frac{uE}{1-\nu^2}\left[-\frac{wx^2}{16C}(3\nu+1)+\frac{wR^2}{16C}(3+\nu)\right]$$

$$= \frac{uE}{1-\nu^2}\cdot\frac{w}{16C}[(3+\nu)R^2-(3\nu+1)x^2] \tag{16.16}$$

Maximum stress is again at $x = 0$, $(u = t/2)$,

$$\sigma_z = \frac{uE}{1-\nu^2}\left[\frac{wR^2}{16C}(3+\nu)\right] = \frac{3wR^2}{8t^2}(3+\nu) \tag{16.17}$$

Thus maximum values for both stresses are equal and occur at the centre.

(ii) Edges Clamped A plate firmly clamped at the edges and having a uniformly distributed load is shown in Fig. 16.5.

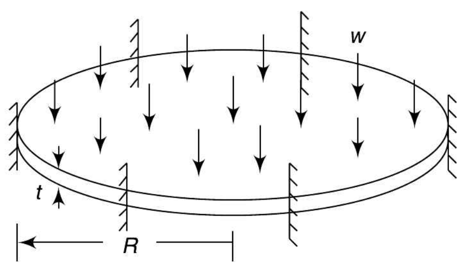


Fig. 16.5

$$P = 0;\ C_2 = 0 \quad \text{or} \quad \theta = -\frac{wx^3}{16C}+C_1\cdot\frac{x}{2} \quad \text{as before.}$$

At $x = R$, $dy/dx = \theta = 0$,

$$\theta = 0 = -\frac{wx^3}{16C}+C_1\cdot\frac{x}{2} \quad \text{or} \quad C_1 = \frac{wR^2}{8C}$$

$$\therefore \qquad \theta = -\frac{wx^3}{16C}+\frac{wR^2}{8C}\cdot\frac{x}{2}$$

$$y = -\frac{wx^4}{64C}+C_1\cdot\frac{x^2}{4}+C_2\cdot\log x+C_3$$

At $x = 0$, $y = 0$, $\therefore C_3 = 0$ and thus

$$y = -\frac{wx^4}{64C}+\frac{wR^2}{8C}\cdot\frac{x^2}{4} = \frac{wR^2}{32C}\left(R^2-\frac{x^2}{2}\right) \tag{16.18}$$

Central deflection, at $x = R$,

$$y = \frac{wR^4}{64C} = \frac{wR^4}{64}\Big/\frac{Et^3}{12(1-\nu^2)} = \frac{3wR^4}{16Et^3}(1-\nu^2) \tag{16.19}$$

$$\sigma_x = \frac{uE}{1-\nu^2}\left(\frac{d\theta}{dx}+\frac{\nu\theta}{x}\right) = \frac{uE}{1-\nu^2}\left(-\frac{3wx^2}{16C}+C_1\cdot\frac{1}{2}-\frac{\nu wx^2}{16C}+\frac{\nu}{2}\cdot C_1\right)$$

$$= \frac{uE}{1-\nu^2}\left(-\frac{3wx^2}{16C}+\frac{wR^2}{8C}\cdot\frac{1}{2}-\frac{\nu wx^2}{16C}+\frac{\nu}{2}\cdot\frac{wR^2}{8C}\right)$$

$$= \frac{uE}{1-\nu^2}\left[-\frac{wx^2}{16C}(3+\nu)+\frac{wR^2}{16C}(1+\nu)\right]$$

$$= \frac{uE}{1-\nu^2}\cdot\frac{w}{16C}[(1+\nu)R^2-(3+\nu)x^2] \qquad (16.20)$$

Maximum value of radial stress occurs at the clamped edge where $x = R$, ($u = t/2$),

$$\sigma_x = \frac{uE}{1-\nu^2}\cdot\frac{w}{16C}[(1+\nu)R^2-(3+\nu)R^2] \qquad (16.20a)$$

$$= \frac{(t/2)E}{1-\nu^2}\cdot\frac{12(1-\nu^2)}{Et^3}\frac{wR^2}{16}(-2) = \frac{3}{4}\frac{wR^2}{t^2} \text{ (numerical value)} \qquad (16.20b)$$

$$\sigma_z = \frac{uE}{1-\nu^2}\left(\nu\frac{d\theta}{dx}+\frac{\theta}{x}\right) = \frac{uE}{1-\nu^2}\left[-\frac{wx^2}{16C}(3\nu+1)+\frac{wR^2}{16C}(1+\nu)\right]$$

$$= \frac{uE}{1-\nu^2}\cdot\frac{w}{16C}[(1+\nu)R^2-(3\nu+1)x^2] \qquad (16.21)$$

Maximum value occurs at the centre, $x = 0$, ($u = t/2$),

$$\sigma_z = \frac{(t/2)E}{1-\nu^2}\cdot\frac{wR^2}{16C}(1+\nu) = \frac{(t/2)E}{1-\nu^2}\cdot\frac{wR^2}{16}(1+\nu)\bigg/\frac{Et^3}{12(1-\nu^2)} = \frac{3wR^2}{8t^2}(1+\nu) \qquad (16.22)$$

Example 16.1 || A solid plate of 400-mm diameter and 20-mm thick is acted upon by a uniformly distributed load of 1000 kN/m^2. Calculate the central deflection and the values of the maximum stresses in the radial and tangential directions when

(i) the edges are freely supported
(ii) the edges are firmly clamped
(iii) E = 205 GPa and Poisson's ratio = 0.3.

Solution

Given

$R = 200$ mm $\qquad t = 20$ mm

$E = 205$ GPa $\qquad \nu = 0.3$

$w = 1000 \text{ kN/m}^2 = 1 \times 10^6 \text{ N/m}^2 = 1 \text{ N/mm}^2$

To find Central deflection and maximum stresses when edges are

— freely supported
— firmly clamped

Edges freely supported

$$y = \frac{3wR^4}{16Et^3}(5+\nu)(1-\nu) \qquad \text{...(Eq. 16.13)}$$

$$= \frac{3\times1\times200^4}{16\times205\,000\times20^3}(5+0.3)(1-0.3) = 0.68 \text{ mm}$$

Maximum values of radial and tangential stresses occur at the centre of plate and are equal,

$$\sigma_x = \sigma_z = \frac{3wR^2}{8t^2}(3+\nu) \qquad \text{...(Eq. 16.15 and 16.17)}$$

$$= \frac{3\times 1\times 200^2}{8\times 20^2}(3+0.3) = 123.8 \text{ MPa}$$

Edges are clamped

$$y = \frac{3wR^4}{16Et^3}(1-\nu^2) \qquad \text{...(Eq. 16.19)}$$

$$= \frac{3\times 1\times 200^4}{16\times 205\,000\times 20^3}(1-0.3^2) = 0.166 \text{ mm}$$

Maximum value of radial stress is at the clamped edge,

$$\sigma_x = \frac{3}{4}\frac{wR^2}{t^2} \qquad \text{...(Eq. 16.20)}$$

$$= \frac{3\times 1\times 200}{4\times 20} = 75 \text{ MPa}$$

Maximum value of tangential stress is at the centre,

$$\sigma_z = \frac{3wR^2}{8t^2}(1+\nu) \text{ MPa} \qquad \text{...(Eq. 16.22)}$$

$$= \frac{3\times 1\times 200^2}{8\times 20^2}(1+0.3) = 48.75 \text{ MPa}$$

Example 16.2 A solid plate of 400-mm diameter and 10-mm thickness is acted upon by a uniformly distributed load of 200 kN/m^2. Sketch the distribution of deflection of the plate under the load and the radial and tangential stresses. The plate is freely supported at the edges. E = 202 GPa and Poisson's ratio = 0.3.

Solution

Given

$R = 200$ mm $\qquad t = 10$ mm

$E = 202$ GPa $\qquad \nu = 0.3$

$w = 200 \text{ kN/m}^2 = 0.2 \times 10^6 \text{ N/m}^2 = 0.2 \text{ N/mm}^2$

To find

— To sketch distribution of deflection

— radial and tangential stresses

Deflection of plate

$$C = \frac{Et^3}{12(1-\nu^2)} \qquad \text{...(Eq. 16.6)}$$

$$= \frac{202\,000\times 10^3}{12(1-0.3^2)} = 18.5\times 10^6$$

$$y = \frac{wx^2}{32C}\left(\frac{3+\nu}{1+\nu}\cdot R^2 - \frac{x^2}{2}\right) \qquad \text{...(Eq. 16.12)}$$

$$= \frac{0.2x^2}{32\times 18.5\times 10^6}\left(\frac{3+0.3}{1+0.3}\times 200^2 - \frac{x^2}{2}\right)$$

$$= \frac{x^2}{10^{12}}(34.3 \times 10^3 - 168.9x^2)$$

In this expression radial distance is from the centre of the plate and by inserting the values of x, the following table is obtained:

x (mm)	0	40	80	120	160	200
y (mm)	0	0.05	0.21	0.46	0.77	1.1

These values obtained are the rise of sections relative to the plate centre, i.e., the edges are 1.1 mm above the central point. If deflections of various sections relative to the edges are desired, the value obtained may be deducted from this value.

Thus deflections relative to the edges are as under:

x (mm)	0	40	80	120	160	200
y (mm)	1.10	1.05	0.89	0.64	0.33	0

Figure 16.6 shows the profile of the deflected plate.

Radial and tangential stresses

$$\sigma_x = \frac{(t/2)E}{1-\nu^2} \cdot \frac{w(3+\nu)}{16} \cdot \frac{12(1-\nu^2)}{Et^3}(R^2 - x^2) \qquad \ldots\text{(Eq. 16.14)}$$

$$= \frac{3}{8} \cdot \frac{w}{t^2}(3+\nu)(R^2 - x^2)$$

$$= \frac{3}{8} \cdot \frac{0.2}{10^2}(3+0.3)(200^2 - x^2)$$

$$= 0.002\,475(40\,000 - x^2)$$

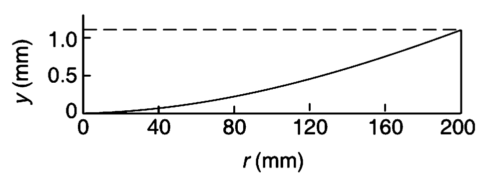


Fig. 16.6

$$\sigma_z = \frac{(t/2)E}{1-\nu^2} \cdot \frac{w}{16} \cdot \frac{12(1-\nu^2)}{Et^3}[(3+\nu)R^2 - (3\nu+1)x^2] \qquad \ldots\text{(Eq. 16.16)}$$

$$= \frac{3w}{8t^2}[(3+\nu)R^2 - (3\nu+1)x^2]$$

$$= \frac{3 \times 0.2}{8 \times 10^2}[(3+0.3)200^2 - (3 \times 0.3 + 1)x^2]$$

$$= 0.000\,75[132\,000 - 1.9\,x^2]$$

Stress values at different cross-sections are tabulated below:

x (mm)	0	40	80	120	160	200
σ_x (N/mm^2)	99	95.04	83.16	63.36	35.64	0
σ_z (N/mm^2)	99	96.72	89.88	78.48	62.52	42

Figure 16.7 shows the variation of stresses with radii at the bottom of the plate. At the upper surface, y is negative and the stresses are of opposite sign, i.e., compressive in place of tensile and vice-versa.

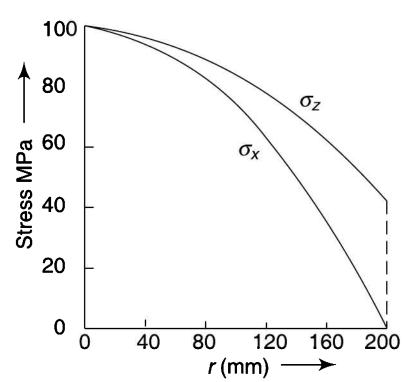


Fig 16.7

Example 16.3 A solid plate of 400-mm diameter and 10-mm thickness is acted upon by a uniformly distributed load of 200 kN/m^2. Sketch the distribution of deflection of the plate under the load and the radial and tangential stresses. The plate has clamped edges. E = 202 GPa and Poisson's ratio = 0.3.

Solution

Given

$R = 200$ mm $\qquad t = 10$ mm

$E = 205$ GPa $\qquad \nu = 0.3$

$w = 200\ \text{kN/m}^2 = 0.2 \times 10^6\ \text{N/m}^2 = 0.2\ \text{N/mm}^2$

To find

— To sketch distribution of deflection

— radial and tangential stresses

Deflection of plate

$$C = \frac{Et^3}{12(1-\nu^2)} = \frac{202\,000 \times 10^3}{12(1-0.3^2)} = 18.5 \times 10^6$$

$$y = \frac{wx^2}{32C}\left(R^2 - \frac{x^2}{2}\right) = \frac{0.2x^2}{32 \times 18.5 \times 10^6}\left(40\,000 - \frac{x^2}{2}\right)$$

Deflections at different cross-sections are tabulated below:

x (mm)	0	40	80	120	160	200
y (mm)	0	0.03	0.11	0.19	0.25	0.27

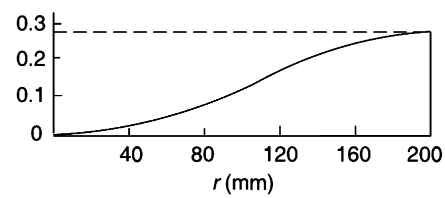


Fig. 16.8

Figure 16.8 shows the profile of the deflected plate.

Radial and tangential stresses

$$\sigma_x = \frac{(t/2)E}{1-\nu^2}\cdot\frac{w}{16}\cdot\frac{12(1-\nu^2)}{Et^3}[(1+\nu)R^2 - (3+\nu)x^2] \qquad \ldots\text{(Eq. 16.20a)}$$

$$= \frac{3}{8}\cdot\frac{w}{t^2}[(1+\nu)R^2 - (3+\nu)x^2]$$

$$= \frac{3}{8}\cdot\frac{0.2}{10^2}[(1+0.3)\times 200^2 - (3+0.3)x^2] = 0.000\,75[52\,000 - 3.3x^2]$$

$$\sigma_z = \frac{(t/2)E}{1-\nu^2}\cdot\frac{w}{16}\cdot\frac{12(1-\nu^2)}{Et^3}[(1+\nu)R^2 - (3\nu+1)x^2] \qquad \ldots\text{(Eq. 16.21)}$$

$$= \frac{3w}{8t^2}[(1+\nu)R^2 - (3\nu+1)x^2]$$

$$= \frac{3 \times 0.2}{8 \times 10^2}[(1+0.3)200^2 - (3\times 0.3+1)x^2] = 0.000\,75[52\,000 - 1.9x^2]$$

Stress values at different cross-sections are tabulated below.

x (mm)	0	40	80	120	160	200
σ_x (N/mm^2)	39	35.04	23.16	3.36	–24.36	–60
σ_z (N/mm^2)	39	36.72	29.88	18.48	2.52	–18

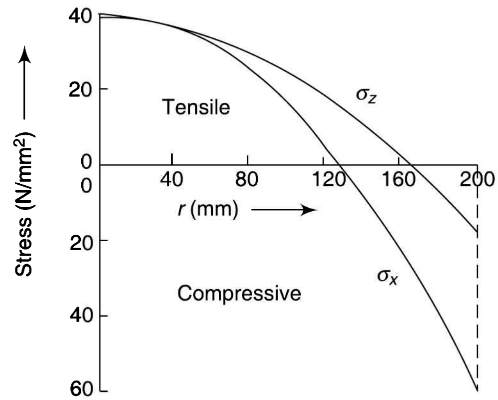


Fig. 16.9

Figure 16.9 shows the variation of stresses with radii at the bottom of the plate.

16.3 CENTRAL POINT LOAD ON SOLID PLATE

(i) Edges Freely Supported A plate simply supported at the edges and having a central point load is shown in Fig. 16.10.

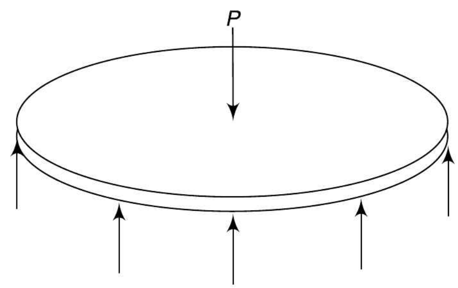


Fig. 16.10

$w = 0$; ∴ $$\theta = -\frac{Px}{4\pi C}\log x + \frac{Px}{8\pi C} + C_1 \cdot \frac{x}{2} + \frac{C_2}{x} \quad \text{...(Eq. 16.10)}$$

θ cannot be infinite at the centre,

$$\therefore \quad C_2 = 0 \text{ or } \theta = -\frac{Px}{4\pi C}\log x + \frac{Px}{8\pi C} + C_1 \cdot \frac{x}{2}$$

$$M_{xy} = C\left(\frac{d\theta}{dx} + \frac{v\theta}{x}\right)$$

$$= C\left[\frac{d}{dx}\left(-\frac{Px}{4\pi C}\log x + \frac{Px}{8\pi C} + C_1 \cdot \frac{x}{2}\right) + \frac{v}{x}\left(-\frac{Px}{4\pi C}\log x + \frac{Px}{8\pi C} + C_1 \cdot \frac{x}{2}\right)\right]$$

$$= C\left[-\frac{P}{4\pi C}(\log x + 1) + \frac{P}{8\pi C} + \frac{C_1}{2} - \frac{Pv}{4\pi C}\log x + \frac{Pv}{8\pi C} + \frac{vC_1}{2}\right]$$

At $x = R$, $M_{xy} = 0$

$$\therefore \quad -\frac{P}{4\pi C}\log R - \frac{P}{4\pi C} + \frac{P}{8\pi C} + \frac{C_1}{2} - \frac{Pv}{4\pi C}\log R + \frac{Pv}{8\pi C} + \frac{vC_1}{2} = 0$$

or $$\frac{C_1}{2}(1+v) - \frac{P}{8\pi C}2\log R(1+v) - \frac{P}{8\pi C}(1-v) = 0$$

$$C_1 = \frac{P}{4\pi C}\left(2\log R + \frac{1-v}{1+v}\right)$$

Deflection, $y = -\dfrac{Px^2}{8\pi C}(\log x - 1) + C_1 \cdot \dfrac{x^2}{4} + C_3$ ...(Eq. 16.11)

At $x = 0, y = 0, \therefore C_3 = 0$

Thus, $y = \dfrac{Px^2}{8\pi C}\left[\dfrac{1}{2}\left(2\log R + \dfrac{1-v}{1+v}\right) - (\log x - 1)\right]$

Central deflection, $y = \dfrac{PR^2}{8\pi C}\left[\dfrac{1}{2}\left(2\log R + \dfrac{1-v}{1+v}\right) - (\log R - 1)\right] = \dfrac{PR^2}{16\pi C}\left[\dfrac{1-v}{1+v} + 2\right]$

$$= \frac{PR^2}{16\pi C}\cdot\left(\frac{3+v}{1+v}\right) = \frac{PR^2}{16\pi}\cdot\left(\frac{3+v}{1+v}\right)\Big/\frac{Et^3}{12(1-v^2)} = \frac{3PR^2}{4\pi Et^3}\cdot(3+v)(1-v) \qquad (16.23)$$

$$\sigma_x = \frac{uE}{1-v^2}\left(\frac{d\theta}{dx} + \frac{v\theta}{x}\right) = \frac{uE}{1-v^2}\begin{bmatrix} -\dfrac{P}{4\pi C}(\log x + 1) + \dfrac{P}{8\pi C} + \dfrac{1}{2}\dfrac{P}{4\pi C}\left(2\log R + \dfrac{1-v}{1+v}\right) \\ -\dfrac{Pv}{4\pi C}\log x + \dfrac{Pv}{8\pi C} + \dfrac{v}{2}\dfrac{P}{4\pi C}\left(2\log R + \dfrac{1-v}{1+v}\right) \end{bmatrix}$$

$$= \frac{uE}{1-v^2}\left[\frac{P}{4\pi C}(\log R - \log x + v\log R - v\log x) + \frac{P}{8\pi C}\left(-2 + 1 + \frac{1-v}{1+v} + v + v\cdot\frac{1-v}{1+v}\right)\right]$$

$$= \frac{uE}{1-v^2}\left[\frac{2P}{8\pi C}\left(\log\frac{R}{x}\cdot(1+v)\right) + \frac{P}{8\pi C}\left\{-1 + \frac{1-v}{1+v} + v\left(1 + \frac{1-v}{1+v}\right)\right\}\right]$$

$$= \frac{uE}{1-v^2}\left[\frac{2P}{8\pi C}\left(\log\frac{R}{x}\cdot(1+v)\right) + \frac{P}{8\pi C}\left\{\frac{-2v}{1+v} + \frac{2v}{1+v}\right\}\right]$$

$$= \frac{uE}{1-v^2}\cdot\frac{2P}{8\pi C}\cdot(1+v)\cdot\log\frac{R}{x}$$

$$= \frac{uE}{1-v^2}\cdot\frac{2P}{8\pi}\cdot(1+v)\cdot\log\frac{R}{x}\Big/\frac{Et^3}{12(1-v^2)}$$

$$= \frac{3}{2}\cdot\frac{P}{\pi t^2}\cdot(1+v)\cdot\log\frac{R}{x} \qquad (16.24)$$

Similarly, $\sigma_z = \dfrac{3}{2}\cdot\dfrac{P}{\pi t^2}\left[(1+v)\cdot\log\dfrac{R}{x} + (1-v)\right]$ (16.25)

At the centre, these stresses become infinite theoretically. However, as the load cannot be a point load in the true sense, i.e., it must extend over a finite area, the maximum stresses will be finite in practice.

(ii) Edges Clamped A plate firmly clamped at the edges and having a central point load is shown in Fig. 16.11.

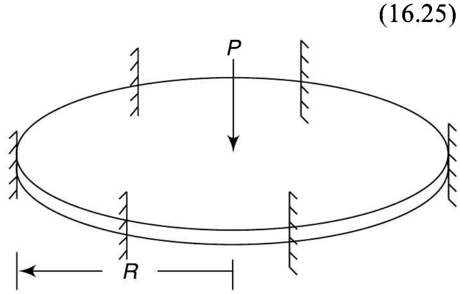


Fig. 16.11

As $w = 0$; $\therefore\ \theta = -\dfrac{Px}{4\pi C}\log x + \dfrac{Px}{8\pi C} + C_1 \cdot \dfrac{x}{2} + \dfrac{C_2}{x}$

θ cannot be infinite at the centre, $\therefore\ C_2 = 0$ or $\theta = -\dfrac{Px}{4\pi C}\log x + \dfrac{Px}{8\pi C} + C_1 \cdot \dfrac{x}{2}$

Now $\quad y = -\dfrac{Px^2}{8\pi C}(\log x - 1) + C_1 \cdot \dfrac{x^2}{4} + C_3$

At $x = 0, y = ,\quad \therefore\ C_3 = 0$

$\therefore\quad y = -\dfrac{Px^2}{8\pi C}(\log R - 1) + C_1 \cdot \dfrac{x^2}{4}$

At $x = R$, $dy/dx = \theta = 0$, $0 = -\dfrac{PR}{4\pi C}\log R + \dfrac{PR}{8\pi C} + C_1 \cdot \dfrac{R}{2}$

or $\qquad C_1 = \dfrac{P}{4\pi C}(2\log R - 1)$

$$\therefore \qquad y = -\frac{Px^2}{8\pi C}(\log R - 1) + \frac{P}{4\pi C}(2\log R - 1)\cdot\frac{x^2}{4} = \frac{Px^2}{16\pi C}$$

$$= \frac{Px^2}{16\pi}\cdot\frac{12(1-\nu^2)}{Et^3} = \frac{3}{4}\cdot\frac{Px^2}{\pi Et^3}(1-\nu^2) \tag{16.26}$$

$$\text{Central deflection} = \frac{3}{4}\cdot\frac{PR^2}{\pi Et^3}(1-\nu^2) \tag{16.27}$$

$$\sigma_x = \frac{uE}{1-\nu^2}\left(\frac{d\theta}{dx} + \frac{\nu\theta}{x}\right)$$

$$= \frac{uE}{1-\nu^2}\left[\frac{d}{dx}\left(-\frac{Px}{4\pi C}\log x + \frac{Px}{8\pi C} + C_1\cdot\frac{x}{2}\right) + \frac{\nu}{x}\left(-\frac{Px}{4\pi C}\log x + \frac{Px}{8\pi C} + C_1\cdot\frac{x}{2}\right)\right]$$

$$= \frac{uE}{1-\nu^2}\left[\begin{array}{l} -\dfrac{P}{4\pi C}(\log x + 1) + \dfrac{P}{8\pi C} + \dfrac{P}{8\pi C}(2\log R - 1) - \dfrac{P\nu}{4\pi C}\log x \\ \qquad + \dfrac{P\nu}{8\pi C} + \dfrac{P\nu}{8\pi C}(2\log R - 1) \end{array}\right]$$

$$= \frac{uE}{1-\nu^2}\left[-\frac{P}{4\pi C}(\log x + 1) + \frac{P}{4\pi C}\log R - \frac{P\nu}{4\pi C}\log x + \frac{P\nu}{4\pi C}\log R\right]$$

$$= \frac{uE}{1-\nu^2}\cdot\frac{P}{4\pi C}\left[(1+\nu)\log\frac{R}{x} - 1\right]$$

Maximum stress is at $x = R$,

$$= -\frac{uE}{1-\nu^2}\cdot\frac{P}{4\pi C}$$

$$= -\frac{(t/2)E}{1-\nu^2}\cdot\frac{P}{4\pi}\cdot\frac{12(1-\nu^2)}{Et^3} = \frac{3}{2}\cdot\frac{P}{\pi t^2} \quad \text{(numerically)} \tag{16.28}$$

$$\sigma_z = \frac{uE}{1-\nu^2}\left(\nu\,\frac{d\theta}{dx} + \frac{\theta}{x}\right)$$

$$= \frac{uE}{1-\nu^2}\left[\nu\,\frac{d}{dx}\left(-\frac{Px}{4\pi C}\log x + \frac{Px}{8\pi C} + C_1\cdot\frac{x}{2}\right) + \frac{1}{x}\left(-\frac{Px}{4\pi C}\log x + \frac{Px}{8\pi C} + C_1\cdot\frac{x}{2}\right)\right]$$

$$= \frac{uE}{1-\nu^2}\left[\begin{array}{c} -\dfrac{P\nu}{4\pi C}(\log x + 1) + \dfrac{P\nu}{8\pi C} + \dfrac{P\nu}{8\pi C}(2\log R - 1) - \dfrac{P}{4\pi C}\log x \\ + \dfrac{P}{8\pi C} + \dfrac{P}{8\pi C}(2\log R - 1) \end{array}\right]$$

$$= \frac{uE}{1-\nu^2}\left[-\frac{P\nu}{4\pi C}(\log x + 1) + \frac{P\nu}{4\pi C}\log R - \frac{P}{4\pi C}\log x + \frac{P}{8\pi C}2\log R\right]$$

$$= \frac{uE}{1-\nu^2}\cdot\frac{P}{4\pi C}\left[(1+\nu)\log\frac{R}{x} - \nu\right] \tag{16.29}$$

Maximum stress is at $x = R$,

$$= -\frac{uE}{1-\nu^2}\cdot\frac{P\nu}{4\pi C}$$

$$= -\frac{(t/2)E}{1-\nu^2}\cdot\frac{P\nu}{4\pi}\cdot\frac{12(1-\nu^2)}{Et^3}$$

$$= \frac{3}{2}\cdot\frac{P\nu}{\pi t^2} \quad \text{(numerically)} \tag{16.30}$$

Example 16.4 A solid plate of 600-mm diameter and 10-mm thickness is acted upon by a concentrated load of 5 kN at the centre of the plate. Calculate the central deflection and the values of radial and tangential stresses at a radial distance of 5 mm from the centre when the plate is simply supported at the edges.

E = 205 GPa and Poisson's ratio = 0.3.

Solution

Given

$R = 300$ mm	$t = 10$ mm
$E = 205$ GPa	$\nu = 0.3$
$P = 5$ kN	

To find

— Central deflection

— Radial and tangential stresses at 5 mm from centre

Deflection of plate

$$y = \frac{3PR^2}{4\pi Et^3}\cdot(3+\nu)(1-\nu) \qquad \text{...(Eq. 16.23)}$$

$$= \frac{3\times 5000\times 300^2}{4\pi\times 205\,000\times 10^3}\times(3+0.3)(1-0.3) = 0.524 \text{ mm}$$

Radial and tangential stresses

$$\sigma_x = \frac{3}{2}\cdot\frac{P}{\pi t^2}\cdot(1+\nu)\cdot\log\frac{R}{x} \qquad \text{...(Eq. 16.24)}$$

$$= \frac{3}{2}\cdot\frac{5000}{\pi\times 10^2}\cdot(1+0.3)\cdot\log\frac{300}{5} = 127 \text{ MPa}$$

$$\sigma_z = \frac{3}{2}\cdot\frac{P}{\pi t^2}\left[(1+\nu)\cdot\log\frac{R}{x}+(1-\nu)\right] \qquad \text{...(Eq. 16.25)}$$

$$= \frac{3}{2}\cdot\frac{5000}{\pi\times 10^2}\left[(1+0.3)\cdot\log\frac{300}{5}+(1-0.3)\right] = 186.9 \text{ MPa}$$

Example 16.5 || A solid flat circular plate of 800-mm diameter and 15-mm thickness is acted upon by a concentrated load of 40 kN at the centre of the plate. Calculate the central deflection and the maximum radial stress at the edge when the plate is clamped at the edges. E = 205 GPa and Poisson's ratio = 0.3.

Solution

Given

$R = 400$ mm $\qquad t = 15$ mm

$E = 202$ GPa $\qquad \nu = 0.3$

$P = 40$ kN

To find

— Central deflection

— Maximum radial stress at edge

Deflection of plate

$$y = \frac{3}{4}\cdot\frac{PR^2}{16\pi E t^3}(1-\nu^2) \qquad \text{...(Eq. 16.26)}$$

$$= \frac{3}{4}\cdot\frac{40\,000\times 400^2}{16\pi\times 202\,000\times 15^3}(1-0.28^2) = 0.129 \text{ mm}$$

Maximum radial stress

$$\sigma_x = \frac{3}{2}\cdot\frac{P}{\pi t^2} \qquad \text{...(Eq. 16.30)}$$

$$= \frac{3}{2}\cdot\frac{40\,000}{\pi\times 15^2} = 84.9 \text{ mm}$$

Example 16.6 || A solid plate of 400-mm diameter and 10-mm thickness is acted upon by a concentrated load of 5 kN at the centre of the plate. Sketch the distribution of deflection of the plate under the load and the radial and tangential stresses when the plate is simply supported at the edges. E = 205 GPa and Poisson's ratio = 0.3.

Solution

Given

$R = 200$ mm $\qquad t = 10$ mm

$E = 205$ GPa $\qquad \nu = 0.3$

$P = 5$ kN

To find

— To sketch distribution of deflection
— Radial and tangential stresses

Deflection of plate

$$C = \frac{Et^3}{12(1-\nu^2)} = \frac{202\,000 \times 10^3}{12(1-0.3^2)} = 18.5 \times 10^6$$

$$y = \frac{Px^2}{8\pi C}\left[\frac{1}{2}\left(2\log R + \frac{1-\nu}{1+\nu}\right) - (\log x - 1)\right]$$

$$= \frac{5000x^2}{8\pi \times 18.5 \times 10^6}\left[\frac{1}{2}\left(2\log 200 + \frac{1-0.3}{1+0.3}\right) - (\log x - 1)\right]$$

$$= 10.75 \times 10^{-6} x^2 [6.568 - \log x]$$

Deflections at different cross-sections are tabulated below:

x (mm)	0	40	80	120	160	200
y (mm)	0	0.05	0.15	0.28	0.41	0.55

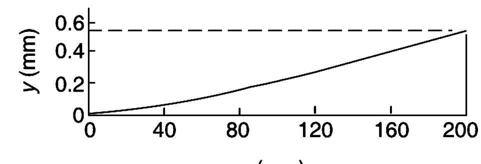


Fig. 16.12

Figure 16.12 shows the profile of the deflected plate.

Radial and tangential stresses

$$\sigma_x = \frac{3}{2}\cdot\frac{P}{\pi t^2}\cdot(1+\nu)\cdot\log\frac{R}{x} = \frac{3}{2}\cdot\frac{5000}{\pi \times 10^2}\cdot(1+0.3)\cdot\log\frac{200}{x} = 31.04 \times \log\frac{200}{x}$$

$$\sigma_z = \frac{3}{2}\cdot\frac{P}{\pi t^2}\left[(1+\nu)\cdot\log\frac{R}{x} + (1-\nu)\right]$$

$$= \frac{3}{2}\cdot\frac{5000}{\pi \times 10^2}\left[(1+0.3)\cdot\log\frac{200}{x} + (1-0.3)\right]$$

$$= 23.87\left[1.3\log\frac{200}{x} + 0.7\right]$$

Stress values at different cross-sections are tabulated below:

x (mm)	5	40	80	120	160	200
σ_x (N/mm^2)	114.5	49.96	28.44	15.9	6.93	0
σ_z (N/mm^2)	131.2	66.7	45.14	32.56	23.63	16.7

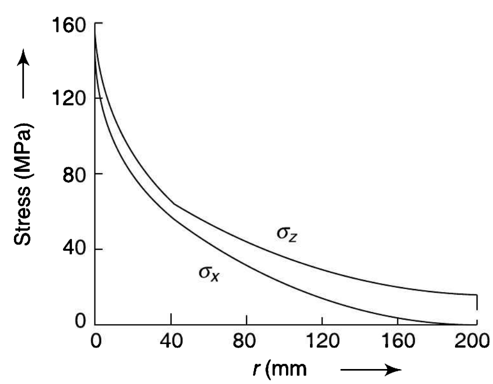


Fig. 16.13

Figure 16.13 shows the variation of stresses with radii at the bottom of the plate.

Example 16.7 A solid plate of 400-mm diameter and 10-mm thickness is acted upon by a concentrated load of 5 kN at the centre of the plate. Sketch the distribution of deflection of the plate under the load and the radial and tangential stresses when the plate is rigidly clamped at the edges. E = 205 GPa and Poisson's ratio = 0.3.

Solution

Given

$R = 200$ mm	$t = 10$ mm	$E = 205$ GPa
$\nu = 0.3$	$P = 5$ kN	

To find

— To sketch distribution of deflection
— Radial and tangential stresses

Deflection of plate

$$C = \frac{Et^3}{12(1-v^2)} = \frac{202\ 000 \times 10^3}{12(1-0.3^2)} = 18.5 \times 10^6$$

$$y = \frac{3}{4} \cdot \frac{Px^2}{\pi Et^3}(1-v^2) = \frac{3}{4} \cdot \frac{5000x^2}{\pi \times 202\ 000 \times 10^3}(1-0.3^2) = 5.377 \times 10^{-6}x^2$$

Deflections at different cross-sections are tabulated below:

x (mm)	0	40	80	120	160	200
y (mm)	0	0.008	0.015	0.077	0.138	0.215

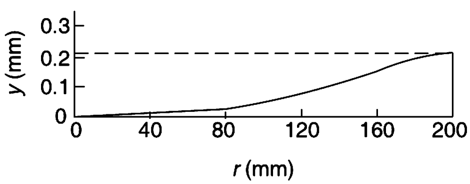


Fig. 16.14

The profile of the deflected plate is shown in Fig. 16.14.

Radial and tangential stresses

$$\sigma_x = \frac{uE}{1-v^2} \cdot \frac{P}{4\pi C}\left[(1+v)\log\frac{R}{x} - 1\right]$$

$$= \frac{(t/2)E}{1-v^2} \cdot \frac{P}{4\pi}\frac{12(1-v^2)}{Et^3}\left[(1+v)\log\frac{R}{x} - 1\right]$$

$$= \frac{3P}{2\pi t^2}\left[(1+v)\log\frac{R}{x} - 1\right]$$

$$= \frac{3 \times 5000}{2\pi \times 10^2}\left[(1+0.3)\log\frac{200}{x} - 1\right]$$

$$= 31.04\log\frac{200}{x} - 23.88$$

$$\sigma_z = \frac{uE}{1-v^2} \cdot \frac{P}{4\pi C}\left[(1+v)\log\frac{R}{x} - v\right]$$

$$= \frac{(t/2)E}{1-v^2} \cdot \frac{P}{4\pi}\frac{12(1-v^2)}{Et^3}\left[(1+v)\log\frac{R}{x} - v\right]$$

$$= \frac{3 \times 5000}{2\pi t^2}\left[(1+0.3)\log\frac{200}{x} - 0.3\right]$$

$$= 23.87\left[1.3\log\frac{200}{x} - 0.3\right]$$

Stress values at different cross-sections are tabulated below:

x (mm)	5	40	80	120	160	200
σ_x (N/mm^2)	90.62	26.08	4.56	–7.98	–16.95	–23.88
σ_z (N/mm^2)	107.32	42.82	21.26	8.68	–0.25	–7.15

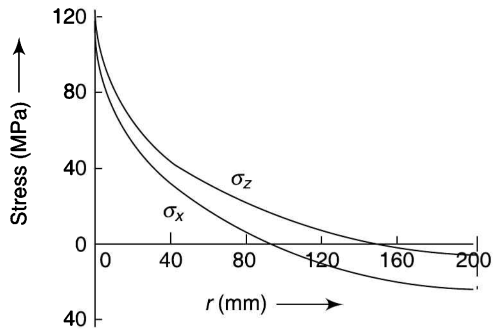


Fig. 16.15

Figure 16.15 shows the variation of stresses with radii at the bottom of the plate.

16.4 LOAD ROUND A CIRCLE ON A SOLID PLATE

Consider a total load P distributed round a circle or radius r on a solid plate simply supported at the edges (Fig. 16.16). The plate may be divided into two regions: one for $x < r$ and the other $x > r$.

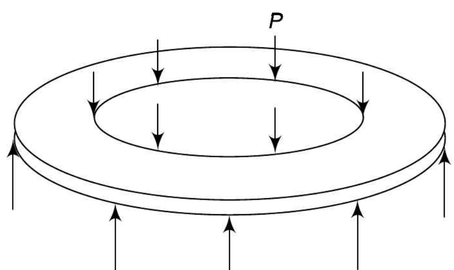


Fig. 16.16

- $x < r$, $w = 0$ and $P = 0$, $\theta = C_1 \cdot \frac{x}{2} + \frac{C_2}{x}$

  and $y = C_1 \cdot \frac{x^2}{4} + C_2 \cdot \log x + C_3$

  As θ is not infinite at $x = 0$, $C_2 = 0$
  As $y = 0$ at $x = 0$, $C_3 = 0$

$$\therefore \qquad \theta = C_1 \cdot \frac{x}{2} \tag{16.31}$$

$$\text{Also,} \qquad y = C_1 \cdot \frac{x^2}{4} \tag{16.32}$$

From Eq. 16.3, $$\sigma_x = \frac{uE}{1-\nu^2}\left(\frac{d\theta}{dx} + \frac{\nu\theta}{x}\right) = \frac{uE}{1-\nu^2}\left(\frac{C_1}{2} + \frac{\nu C_1}{2}\right) = \frac{uE}{1-\nu} \cdot \frac{C_1}{2} \tag{16.33}$$

Similarly From Eq. 16.4, $$\sigma_z = \frac{uE}{1-\nu^2}\left(\nu\,\frac{d\theta}{dx} + \frac{\theta}{x}\right) = \frac{uE}{1-\nu} \cdot \frac{C_1}{2} \tag{16.34}$$

i.e., same expression as for σ_x

- $x > r$, $w = 0$, $\theta = -\frac{Px}{4\pi C}\log x + \frac{Px}{8\pi C} + C_1' \cdot \frac{x}{2} + \frac{C_2'}{x}$

  $$\frac{d\theta}{dx} = -\frac{P}{4\pi C}(\log x + 1) + \frac{P}{8\pi C} + \frac{C_1'}{2} - \frac{C_2'}{x^2}$$

  and $y = -\frac{Px^2}{8\pi C}(\log x - 1) + C_1' \cdot \frac{x^2}{4} + C_2' \cdot \log x + C_3'$

  At $y = r$, the values of θ and y in the two cases must be equal.

Thus equating the values of θ and y at $x = r$,

$$-\frac{P\cdot r}{4\pi C}\log r+\frac{P\cdot r}{8\pi C}+C_1'\cdot\frac{r}{2}+\frac{C_2'}{r}=C_1\cdot\frac{r}{2} \qquad \text{(i)}$$

$$-\frac{P\cdot r^2}{8\pi C}(\log r-1)+C_1'\cdot\frac{r^2}{4}+C_2'\cdot\log r+C_3'=C_1\cdot\frac{r^2}{4} \qquad \text{(ii)}$$

Equating the values of M_{xy} at $x = r$,

$$-\frac{P}{4\pi C}\log r-\frac{P}{4\pi C}+\frac{P}{8\pi C}+\frac{C_1'}{2}-\frac{C_2'}{r^2}-\frac{Pv}{4\pi C}\log r+\frac{Pv}{8\pi C}+\frac{vC_1'}{2}-\frac{vC_2'}{r^2}=\frac{C_1}{2}+\frac{vC_1}{2}$$

$$-\frac{P}{8\pi C}[1+v)2\log r+1-v]+\frac{C_1'}{2}(1+v)-\frac{C_2'}{r^2}(1-v)=\frac{C_1}{2}(1+v) \qquad \text{(iii)}$$

Also at $x = R$, $M_{xy} = 0$,

$$-\frac{P}{8\pi C}[(1+v)2\log R+1-v]+\frac{C_1'}{2}(1+v)-\frac{C_2'}{R^2}(1-v)=0 \qquad \text{(iv)}$$

- Multiplying (i) by $(1 + v)$

$$-\frac{P\cdot r}{4\pi C}(1+v)\log r+\frac{P\cdot r}{8\pi C}(1+v)+C_1'\cdot\frac{r}{2}(1+v)+\frac{C_2'}{r}(1+v)=C_1\cdot\frac{r}{2}(1+v) \qquad \text{(v)}$$

Multiplying (iii) by r,

$$-\frac{Pr}{8\pi C}[1+v)2\log r+1-v]+\frac{C_1'r}{2}(1+v)-\frac{C_2'}{r}(1-v)=\frac{C_1 r}{2}(1+v) \qquad \text{(vi)}$$

Subtracting (vi) from (v),

$$-\frac{P\cdot r}{8\pi C}-\frac{C_2'}{r}=0 \text{ or } C_2'=-\frac{P\cdot r^2}{8\pi C}$$

- Multiplying (i) by

$$-\frac{P\cdot r^2}{8\pi C}\log r+\frac{P\cdot r^2}{16\pi C}+C_1'\cdot\frac{r^2}{4}+\frac{C_2'}{2}=C_1\cdot\frac{r^2}{4}$$

and equation (ii) is, $-\frac{P\cdot r^2}{8\pi C}(\log r-1)+C_1'\cdot\frac{r^2}{4}+C_2'\cdot\log r+C_3'=C_1\cdot\frac{r^2}{4}$

Subtracting first from the second,

$$C_3'+\frac{P\cdot r^2}{16\pi C}-\frac{P\cdot r^2}{8\pi C}\log r+\frac{P\cdot r^2}{16\pi C}=0 \text{ or } C_3'=\frac{P\cdot r^2}{8\pi C}(\log r-1)$$

- Equation (iv) is, $-\frac{P}{8\pi C}[(1+v)2\log R+1-v]+\frac{C_1'}{2}(1+v)-\frac{C_2'}{R^2}(1-v)=0$

or $$-\frac{P}{8\pi C}[(1+v)2\log R+1-v]+\frac{C_1'}{2}(1+v)+\frac{P\cdot r^2}{8\pi CR^2}(1-v)=0$$

or $$-\frac{P}{4\pi C}\left[2\log R+\frac{1-v}{1+v}\right]+C_1'+\frac{P\cdot r^2}{4\pi CR^2}\frac{(1-v)}{(1+v)}=0$$

$$C_1'=\frac{P}{4\pi C}\left[2\log R+\frac{1-v}{1+v}\left(1-\frac{r^2}{R^2}\right)\right] \text{ or } C_1'=\frac{P}{4\pi C}\left[2\log R+\frac{1-v}{1+v}\left(\frac{R^2-r^2}{R^2}\right)\right]$$

Thus deflection is given by,

$$y = -\frac{Px^2}{8\pi C}(\log x - 1) + C_1' \cdot \frac{x^2}{4} + C_2' \cdot \log x + C_3'$$

$$y = -\frac{Px^2}{8\pi C}(\log x - 1) + \frac{P}{4\pi C}\left[2\log R + \frac{1-\nu}{1+\nu}\left(\frac{R^2 - r^2}{R^2}\right)\right] \cdot \frac{x^2}{4} - \frac{P \cdot r^2}{8\pi C} \cdot \log x + \frac{P \cdot r^2}{8\pi C}(\log r - 1)$$

$$= \frac{P}{8\pi C}\left[x^2\left\{\log R + \frac{(1-\nu)}{1+\nu}\left(\frac{R^2 - r^2}{2R^2}\right) - (\log x - 1)\right\} - r^2 \log x + r^2(\log r - 1)\right] \tag{16.35}$$

The central deflection can be found by inserting $x = R$,

$$y = \frac{P}{8\pi C}\left[R^2\left\{\log R + \frac{(1-\nu)}{1+\nu}\left(\frac{R^2 - r^2}{2R^2}\right) - (\log R - 1)\right\} - r^2 \log R + r^2(\log r - 1)\right]$$

$$= \frac{P}{8\pi C}\left[R^2 \log R + \frac{1-\nu}{1+\nu}\left(\frac{R^2 - r^2}{2}\right) - R^2(\log R - 1) - r^2 \log R + r^2(\log r - 1)\right]$$

$$= \frac{P}{8\pi C}\left[\frac{1-\nu}{1+\nu}\left(\frac{R^2 - r^2}{2}\right) + R^2 - r^2 - r^2(\log R - \log r)\right]$$

$$= \frac{P}{8\pi C}\left[(R^2 - r^2)\left\{1 + \frac{1-\nu}{2(1+\nu)}\right\} - r^2 \log \frac{R}{r}\right]$$

$$= \frac{P}{8\pi C}\left[(R^2 - r^2)\frac{(3+\nu)}{2(1+\nu)} - r^2 \cdot \log \frac{R}{r}\right]$$

$$= \frac{P \times 12(1-\nu^2)}{8\pi Et^3}\left[(R^2 - r^2)\frac{(3+\nu)}{2(1+\nu)} - r^2 \cdot \log \frac{R}{r}\right]$$

$$= \frac{3P(1-\nu^2)}{2\pi Et^3}\left[(R^2 - r^2)\frac{(3+\nu)}{2(1+\nu)} - r^2 \cdot \log \frac{R}{r}\right] \tag{16.36}$$

$$M_{xy} = C\left[-\frac{P}{8\pi C}[1+\nu)2\log x + 1 - \nu] + \frac{C_1'}{2}(1+\nu) - \frac{C_2'}{x^2}(1-\nu)\right]$$

$$= -\frac{P}{8\pi}[(1+\nu)2\log x + 1 - \nu] + \frac{P(1+\nu)}{8\pi}\left[2\log R + \frac{1-\nu}{1+\nu}\left(\frac{R^2 - r^2}{R^2}\right)\right] + \frac{P}{8\pi} \cdot \frac{r^2}{x^2}(1-\nu)$$

$$= -\frac{P}{8\pi}[1+\nu)2\log x + (1-\nu)] + 2\log R\frac{P(1+\nu)}{8\pi} + \frac{P}{8\pi}(1-\nu)\left(1 - \frac{r^2}{R^2}\right) + \frac{P}{8\pi} \cdot \frac{r^2}{x^2}(1-\nu)$$

$$= -\frac{P}{8\pi}[(1+\nu)2\log x] + 2\log R\frac{P(1+\nu)}{8\pi} - \frac{P}{8\pi}(1-\nu)\frac{r^2}{R^2} + \frac{P}{8\pi} \cdot \frac{r^2}{x^2}(1-\nu)$$

$$= \frac{P}{8\pi}\left[(1+\nu)2\log \frac{R}{x} + (1-\nu)r^2\left(\frac{1}{x^2} - \frac{1}{R^2}\right)\right] \tag{16.37}$$

It has maximum value at $x = r$

Radial stress $\sigma_x = \dfrac{6P}{8\pi t^2}\left[(1+\nu)2\log \dfrac{R}{x} + (1-\nu)r^2\left(\dfrac{1}{x^2} - \dfrac{1}{R^2}\right)\right]$ (16.38)

$$\text{Maximum radial stress} = \frac{6}{t^2} M_{xy}$$

$$= \frac{6P}{8\pi t^2}\left[(1+\nu)2\log\frac{R}{r} + (1-\nu)r^2\left(\frac{1}{r^2} - \frac{1}{R^2}\right)\right]$$

$$= \frac{3P}{4\pi t^2}\left[(1+\nu)2\log\frac{R}{r} + (1-\nu)\left(1 - \frac{r^2}{R^2}\right)\right] \quad (16.39)$$

Similarly,

$$\sigma_z = \frac{6P}{8\pi t^2}\left[(1+\nu)2\log\frac{R}{x} + (1-\nu)\left(\frac{2R^2 - r^2}{R^2} - \frac{r^2}{x^2}\right)\right] \quad (16.40)$$

$$\text{and maximum tangential stress} = \frac{3P}{4\pi t^2}\left[(1+\nu)2\log\frac{R}{r} + (1-\nu)\left(1 - \frac{r^2}{R^2}\right)\right] \quad (16.41)$$

i.e., it is equal to the radial stress.

Example 16.8 || A solid plate of 400-mm diameter is 12-mm thick. It is freely supported at the edges and has a load of 10 kN distributed round a circle of 100-mm diameter. Determine the central deflection and the maximum values of the radial and tangential stresses. E = 205 GPa and Poisson's ratio = 0.3.

Solution

Given

$R = 200$ mm	$t = 12$ mm
$E = 205$ GPa	$\nu = 0.3$
$P = 10$ kN	$r = 50$ mm

To find

— Central deflection
— Maximum radial and tangential stresses

Deflection of plate

$$y = \frac{3P(1-\nu^2)}{2\pi E t^3}\left[(R^2 - r^2)\frac{(3+\nu)}{2(1+\nu)} - r^2 \cdot \frac{\log R}{\log r}\right] \quad \ldots\text{(Eq. 16.36)}$$

$$= \frac{3 \times 10\,000(1 - 0.3^2)}{2\pi \times 205\,000 \times 12^3}\left[(200^2 - 50^2)\frac{(3+0.3)}{2(1+0.3)} - 50^2 \cdot \log\frac{200}{50}\right]$$

$$= 12.265 \times 10^{-6}\,(47\,596 - 3466) = 0.54 \text{ mm}$$

Maximum stresses

The maximum values of stresses are equal and are at $x = r$.

$$\sigma_x = \sigma_z = \frac{3P}{4\pi t^2}\left[(1+\nu)2\log\frac{R}{r} + (1-\nu)\left(1 - \frac{r^2}{R^2}\right)\right] \quad \ldots\text{(Eq. 16.39 and 16.41)}$$

$$= \frac{3 \times 10\,000}{4\pi \times 12^2}\left[(1+0.3)2\log\frac{200}{50} + (1-0.3)\left(1 - \frac{50^2}{200^2}\right)\right]$$

$$= 70.6 \text{ MPa}$$

Example 16.9 || A solid plate of 400-mm diameter and 10-mm thickness is acted upon by a load of 5 kN round a circle of 50-mm radius. Sketch the distribution of deflection of the plate under the load and the radial and tangential stresses when the plate is simply supported at the edges. E = 205 GPa and Poisson's ratio = 0.3.

Solution

Given

$R = 200$ mm $\quad t = 10$ mm

$E = 205$ GPa $\quad \nu = 0.3$

$P = 5$ kN $\quad r = 50$ mm

To find

— Deflection of plate

— Radial and tangential stresses

Deflection of plate

$$C = \frac{Et^3}{12(1-\nu^2)} = \frac{202\,000 \times 10^3}{12(1-0.3^2)} = 18.5 \times 10^6 \qquad \text{...(Eq. 16.6)}$$

For $x > 50$ mm,

$$y = \frac{P}{8\pi C}\left[x^2\left\{\log R + \frac{(1-\nu)}{1+\nu}\left(\frac{R^2-r^2}{2R^2}\right) - (\log x - 1)\right\} - r^2 \log x + r^2(\log r - 1)\right] \text{...(Eq. 16.35)}$$

$$= \frac{5000}{8\pi \times 18.5 \times 10^6}\left[\begin{array}{c} x^2\left\{\log 200 + \dfrac{(1-0.3)}{1+0.3}\left(\dfrac{200^2-50^2}{2\times 200^2}\right) - (\log x - 1)\right\} \\ -50^2 \log x + 50^2(\log 50 - 1)\end{array}\right]$$

$$= 10.75 \times 10^{-6}[x^2\{5.551 - \log x + 1)\} + 2500(\log 50 - 1 - \log x)]$$

$$= 10.75 \times 10^{-6}[x^2(6.551 - \log x) + 2500(2.912 - \log x]$$

At $x = 50$ mm,

$$y = 10.75 \times 10^{-6}[50^2(6.551 - \log 50) + 2500(2.912 - \log 50] = 0.044 \text{ mm}$$

For $x < 50$ mm, $y = C_1 \cdot \dfrac{x^2}{4}$ ...(Eq. 16.32)

At $x = 50$ mm, the values of y in the two cases must be equal.

Thus $C_1 \cdot \dfrac{50^2}{4} = 0.044$ or $C_1 = 70.4 \times 10^{-6}$

$\therefore$ For $x < 50$ mm, $y = 17.6 \times 10^{-6} \cdot x^2$

Deflections at different cross-sections are tabulated below:

x (mm)	0	25	50	80	120	160	200
y (mm)	0	0.011	0.044	0.11	0.223	0.348	0.475

The profile of the deflected plate in shown in Fig. 16.17.

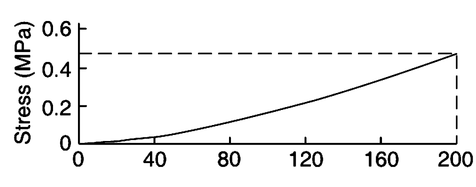


Fig. 16.17

Radial and tangential stresses

$$\sigma_x = \frac{6P}{8\pi t^2}\left[(1+\nu)2\log\frac{R}{x} + (1-\nu)r^2\left(\frac{1}{x^2} - \frac{1}{R^2}\right)\right] \qquad \text{...(Eq. 16.38)}$$

$$= \frac{6 \times 5000}{8\pi \times 10^2}\left[(1+0.3)2\log\frac{200}{x} + (1-0.3)\times 50^2 \times\left(\frac{1}{x^2} - \frac{1}{200^2}\right)\right]$$

$$= 11.937\left[2.6\log\frac{200}{x} + \frac{1750}{x^2} - 0.0438\right]$$

$$\sigma_z = \frac{6P}{8\pi t^2}\left[(1+\nu)2\log\frac{R}{x} + (1-\nu)\left(\frac{2R^2-r^2}{R^2} - \frac{r^2}{x^2}\right)\right] \quad \text{...(Eq. 16.39)}$$

$$= 11.937\left[2.6\log\frac{200}{x} + (1-0.3)\left(\frac{2\times 200^2 - 50^2}{200^2} - \frac{50^2}{x^2}\right)\right]$$

$$= 11.937\left[2.6\log\frac{200}{x} + 1.356 - \frac{1750}{x^2}\right]$$

For $x < 50$ mm, $\sigma_x = \sigma_z = \frac{uE}{1-\nu}\cdot\frac{C_1}{2}$ ...(Eq. 16.33 and 16.34)

As the right hand side has only constant terms, it means the stress is constant inside the 50-mm radius and is equal to 50.87 MPa.

Stress values at different cross-sections are tabulated below:

x (mm)	0–50	50	80	120	160	200
σ_x (N/mm²)	50.87	50.87	31.18	16.78	7.22	0
σ_z (N/mm²)	50.87	50.87	41.36	30.59	22.3	15.66

Figure 16.18 shows the variation of stresses with radii at the bottom of the plate.

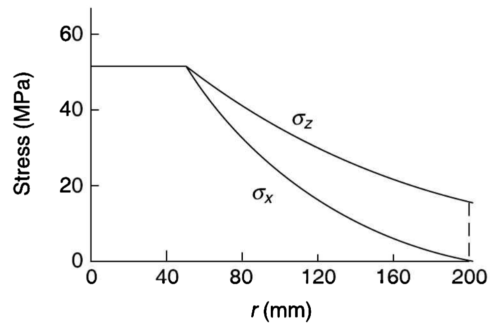


Fig. 16.18

16.5 ANNULAR RING LOAD ROUND AN INNER EDGE

Consider an annular ring acted upon by a total load P round the inner edge and simply supported at the outer edge as shown in Fig. 16.19.

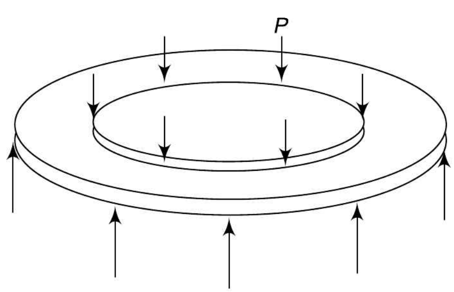


Fig. 16.19

At $x = R$, $M_{xy} = 0$,

$$-\frac{P}{8\pi C}[(1+\nu)2\log R + (1-\nu)] + \frac{C_1}{2}(1+\nu) - \frac{C_2}{R^2}(1-\nu) = 0 \quad \text{(i)}$$

Also at $x = r$, $M_{xy} = 0$,

$$-\frac{P}{8\pi C}[(1+\nu)2\log r + (1-\nu)] + \frac{C_1}{2}(1+\nu) - \frac{C_2}{r^2}(1-\nu) = 0 \quad \text{(ii)}$$

Subtracting (ii) from (i),

$$-\frac{P}{8\pi C}[(1+\nu)2\log R - (1+\nu)2\log R] + C_2(1-\nu)\left(\frac{1}{r^2} - \frac{1}{R^2}\right) = 0$$

or
$$-\frac{P}{8\pi C}\left[\frac{1+\nu}{1-\nu}2\log\frac{R}{r}\right] + C_2\left(\frac{R^2-r^2}{r^2R^2}\right) = 0 \text{ or } C_2 = \frac{P}{4\pi C}\cdot\frac{1+\nu}{1-\nu}\cdot\log\frac{R}{r}\cdot\frac{R^2r^2}{R^2-r^2}$$

From (i),

$$-\frac{P}{8\pi C}[(1+\nu)2\log R + (1-\nu)] + \frac{C_1}{2}(1+\nu) + \frac{P}{4\pi CR^2}\cdot\frac{1+\nu}{1-\nu}\cdot\log\frac{R}{r}\cdot\frac{r^2R^2}{R^2-r^2}(1-\nu) = 0$$

or
$$-\frac{P}{8\pi C}[(1+\nu)2\log R + (1-\nu)] + \frac{C_1}{2}(1+\nu) + \frac{P}{4\pi C}\cdot(1+\nu)\cdot\log\frac{R}{r}\cdot\frac{r^2}{R^2-r^2} = 0$$

or $$-\frac{P}{4\pi C}\left[2\log R+\frac{1-\nu}{1+\nu}\right]+C_1+\frac{P}{2\pi C}\cdot\frac{r^2}{R^2-r^2}(\log R-\log r)=0$$

$$C_1=\frac{P}{4\pi C}\left[2\log R+\frac{1-\nu}{1+\nu}+\frac{2r^2}{R^2-r^2}(\log R-\log r)\right]$$

$$=\frac{P}{4\pi C}\left[2\log R+\frac{1-\nu}{1+\nu}+\frac{2r^2}{R^2-r^2}\log R-\frac{2r^2}{R^2-r^2}\log r\right]$$

$$=\frac{P}{4\pi C}\left[\log R\left(2+\frac{2r^2}{R^2-r^2}\right)-\frac{2r^2}{R^2-r^2}\log r+\frac{1-\nu}{1+\nu}\right]$$

$$=\frac{P}{4\pi C}\left[\log R\cdot\left(\frac{2R^2-2r^2+2r^2}{R^2-r^2}\right)-\frac{2r^2}{R^2-r^2}\log r+\frac{1-\nu}{1+\nu}\right]$$

$$=\frac{P}{4\pi C}\left[\frac{2(R^2\log R-r^2\log r)}{R^2-r^2}+\frac{1-\nu}{1+\nu}\right]$$

Then $M_{xy}/C=-\dfrac{P}{8\pi C}[(1+\nu)2\log x+(1-\nu)]+\dfrac{C_1}{2}(1+\nu)-\dfrac{C_2}{x^2}(1-\nu)$

$$M_{yz}/C=-\frac{P}{8\pi C}[(1+\nu)2\log x-(1-\nu)]+\frac{C_1}{2}(1+\nu)+\frac{C_2}{x^2}(1-\nu)$$

Maximum bending moment is M_{yz} at $x = r$,

Maximum bending stress $=\dfrac{6}{t^2}M_{yz}$ at $x = r$

$$\sigma_z=\frac{6C}{t^2}\left[-\frac{P}{8\pi C}\{(1+\nu)2\log r-(1-\nu)\}+\frac{C_1}{2}(1+\nu)+\frac{C_2}{r^2}(1-\nu)\right]$$

$$=\frac{6C}{t^2}\left[\begin{array}{l}-\dfrac{P}{8\pi C}\{(1+\nu)2\log r-(1-\nu)\}+\dfrac{P}{8\pi C}\left(\dfrac{2(R^2\log R-r^2\log r)}{R^2-r^2}+\dfrac{1-\nu}{1+\nu}\right)(1+\nu)\\ \qquad+\dfrac{1-\nu}{r^2}\left(\dfrac{P}{4\pi C}\cdot\dfrac{1+\nu}{1-\nu}\cdot\log\dfrac{R}{r}\cdot\dfrac{R^2r^2}{R^2-r^2}\right)\end{array}\right]$$

$$=\frac{P}{4\pi}\cdot\frac{3}{t^2}\left[\begin{array}{l}-\{(1+\nu)2\log r-(1-\nu)\}+\dfrac{2(R^2\log R-r^2\log r)}{R^2-r^2}(1+\nu)+(1-\nu)\\ \qquad+2(1+\nu)\left(\log\dfrac{R}{r}\cdot\dfrac{R^2}{R^2-r^2}\right)\end{array}\right]$$

$$=\frac{P}{4\pi}\cdot\frac{3}{t^2}\left[-\{2(1+\nu)\log r\}+2(1+\nu)\frac{(R^2\log R-r^2\log r)+R^2\log(R/r)}{R^2-r^2}+2(1-\nu)\right]$$

$$=\frac{P}{2\pi}\cdot\frac{3}{t^2}\left[(1+\nu)\left\{\frac{(R^2\log R-r^2\log r)}{R^2-r^2}-\log r\right\}+\frac{(1+\nu)R^2\log(R/r)}{R^2-r^2}+(1-\nu)\right]$$

$$= \frac{P}{2\pi}\cdot\frac{3}{t^2}\left[(1+\nu)\left(\frac{R^2\log R - r^2\log r - R^2\log r + r^2\log R}{R^2-r^2}\right) + \frac{(1+\nu)R^2\log(R/r)}{R^2-r^2} + (1-\nu)\right]$$

$$= \frac{P}{2\pi}\cdot\frac{3}{t^2}\left[(1+\nu)\left(\frac{R^2\log(R/r)}{R^2-r^2}\right) + \frac{(1+\nu)R^2\log(R/r)}{R^2-r^2} + (1-\nu)\right]$$

$$= \frac{3P}{\pi t^2}\left[\frac{(1+\nu)R^2\log(R/r)}{R^2-r^2} + \frac{1-\nu}{2}\right] \tag{16.42}$$

Example 16.10 A 12-m thick annular plate freely supported at the outer edges has 400-mm outer diameter and 100-mm inner diameter. It has a load of 10 kN distributed round the inner edge. Determine the maximum stress in the plate. E = 205 GPa and Poisson's ratio = 0.3.

Solution

Given

R = 200 mm	r = 50 mm
t = 12 mm	P = 10 kN
E = 205 GPa	ν = 0.3

To find Maximum stress

Maximum stress

$$\sigma_z = \frac{3P}{\pi t^2}\left[\frac{(1+\nu)R^2\log(R/r)}{R^2-r^2} + \frac{1-\nu}{2}\right] \quad \text{...(Eq. 16.42)}$$

$$= \frac{3\times 10\,000}{\pi\times 12^2}\left[\frac{(1+0.3)200^2\log(200/50)}{200^2-50^2} + \frac{1-0.3}{2}\right]$$

$$= 150.7 \text{ MPa}$$

Summary

1. In circular plates, freely supported at the edges and with uniformly distributed load,

   Deflection at the centre $= \frac{3wR^4}{16Et^3}(5+\nu)(1-\nu)$

   Radial and tangential stress at the centre $= \frac{3wR^2}{8t^2}(3+\nu)$ (maximum)

2. In circular plates having clamped edges, and with uniformly distributed load,

   Deflection at the centre $= \frac{3wR^4}{16Et^3}(1-\nu^2)$

   Maximum radial stress $= \frac{3}{4}\frac{wR^2}{t^2}$ at clamped edge

   Maximum tangential stress $= \frac{3wR^2}{8t^2}(1+\nu)$ at centre

3. In circular plates, freely supported at the edges and with a point load at the centre,

$$\text{Deflection at the centre} = \frac{3PR^2}{4\pi Et^3}\cdot(3+\nu)(1-\nu)$$

4. In circular plates, freely supported at the edges and with a point load at the centre, the radial and tangential stresses become infinite at the centre theoretically. However, as the load cannot be a point load in the true sense, i.e., it must extend over a finite area, the maximum stresses will be finite in practice.
5. In circular plates having clamped edges and with central point load,

$$\text{Deflection at the centre} = \frac{3}{4}\cdot\frac{PR^2}{\pi Et^3}(1-\nu^2)$$

$$\text{Maximum radial stress} = \frac{3}{2}\cdot\frac{P}{\pi t^2} \quad \text{(at the edges)}$$

$$\text{Maximum tangential stress} = \frac{3}{2}\cdot\frac{P\nu}{\pi t^2} \quad \text{(at the edges)}$$

6. In circular plates, freely supported at edges and with load round a circle of radius r,

$$\text{Central deflection, } y = \frac{3P(1-\nu^2)}{2\pi Et^3}\left[(R^2-r^2)\frac{(3+\nu)}{2(1+\nu)} - r^2\cdot\log\frac{R}{r}\right]$$

Maximum values of radial and tangential stresses

$$= \frac{3P}{4\pi t^2}\left[(1+\nu)2\log\frac{R}{r} + (1-\nu)\left(1-\frac{r^2}{R^2}\right)\right] \text{ at } x = r$$

Objective Type Questions

1. In circular plates, freely supported at the edges with uniformly distributed load, the deflection at the centre is

(a) $\frac{3wR^4}{16Et^3}(5+\nu)(1-\nu)$ (b) $\frac{3wR^4}{16Et^3}\frac{(5+\nu)}{(1-\nu)}$

(c) $\frac{3wR^4}{8Et^3}\frac{(5+\nu)}{(1-\nu)}$ (d) $\frac{3wR^4}{8Et^3}(5+\nu)(1-\nu)$

2. In circular plates, freely supported at the edges with uniformly distributed load, the radial stress at the centre is given by

(a) $\frac{3wR^2}{16t^2}(3+\nu)$ (b) $\frac{3wR^2}{8Et^2}(3+\nu)$ (c) $\frac{3wR^2}{16Et^2}(3+\nu)$ (d) $\frac{3wR^2}{8t^2}(3+\nu)$

3. In circular plates with edges clamped and with uniformly distributed load, central deflection is given by

(a) $\frac{3wR^4}{8Et^3}(1-\nu^2)$ (b) $\frac{3wR^4}{16Et^3}(1-\nu^2)$ (c) $\frac{3wR^4}{16Et^3}(1-\nu)$ (d) $\frac{3wR^4}{8Et^3}(1-\nu)$

4. In circular plates with edges clamped and with uniformly distributed load, the maximum radial stress occurs at the

(a) clamped edge (b) centre (c) mean radius (d) none of these

5. In circular plates with edges clamped and with uniformly distributed load, the maximum tangential stress occurs at the

(a) clamped edge (b) centre (c) mean radius (d) none of these

6. In circular plates with central point load and edges freely supported, central deflection is given by

(a) $\frac{3PR^2}{4Et^3}\cdot(3+\nu)(1-\nu)$ (b) $\frac{3PR^2}{16Et^3}\cdot(3+\nu)(1-\nu)$

(c) $\frac{3PR^2}{4\pi Et^3}\cdot(3+\nu)(1-\nu)$ (d) $\frac{3PR^2}{16\pi Et^3}\cdot(3+\nu)(1-\nu)$

7. In circular plates having clamped edges and with central point load, the maximum radial stress is at the

(a) edges (b) centre (c) mean radius (d) none of these

8. In circular plates having clamped edges and with central point load, the maximum tangential stress is at the

(a) edges (b) centre (c) mean radius (d) none of these

9. In circular plates having clamped edges and with central point load, the radial stress at the edges is

(a) $\frac{2}{3}\cdot\frac{P}{\pi t^2}$ (b) $\frac{2}{3}\cdot\frac{P\nu}{\pi t^2}$ (c) $\frac{3}{2}\cdot\frac{P\nu}{\pi t^2}$ (d) $\frac{3}{2}\cdot\frac{P}{\pi t^2}$

10. In circular plates having clamped edges and with central point load, the maximum tangential stress is at the

(a) $\frac{2}{3}\cdot\frac{P}{\pi t^2}$ (b) $\frac{2}{3}\cdot\frac{P\nu}{\pi t^2}$ (c) $\frac{3}{2}\cdot\frac{P\nu}{\pi t^2}$ (d) $\frac{3}{2}\cdot\frac{P}{\pi t^2}$

Answers

1. (a)	2. (d)	3. (b)	4. (a)	5. (b)	6. (c)
7. (a)	8. (a)	9. (d)	10. (c)		

Review Questions

16.1 A flat circular plate freely supported at the edges is acted upon by a uniformly distributed load w. Show that the deflection is given by $3wR^4(5+\nu)(1-\nu)/16Et^3$ where R is the radius and t, the thickness. Find the maximum values of radial and tangential stresses and where do these occur?

16.2 Deduce expressions for the deflection and the maximum stresses for a flat circular plate acted upon by a uniformly distributed load when the plate is rigidly clamped at the edges.

16.3 Show that in a flat circular plate simply supported at the edges and under the action of a point load at the centre, the stresses are infinite at the centre theoretically. How do you justify them practically?

16.4 Find expressions for the deflection and the stresses for a flat circular plate clamped at the edges when a concentrated load acts at the centre.

Numerical Problems

16.1 A 15-mm thick plate of 300-mm diameter sustains a uniformly distributed load of 600 kN/m^2. Find the deflection at the centre if the plate is simply supported at the edges. What are the maximum values of the radial and tangential stresses? E = 204 GPa and Poisson's ratio = 0.3.

(0.307 mm, $\sigma_x = \sigma_z$ = 74.25 MPa)

16.2 A flat circular plate of 600-mm diameter and 15-mm thickness has a uniformly distributed load of 400 kN/m^2. If the plate is clamped at the edges, sketch the radial and tangential stresses. E = 200 GPa and Poisson's ratio = 0.3.

16.3 Calculate the central deflection and the radial and the tangential stresses at a radial distance of 5 mm from the centre of a solid circular plate of 400-mm diameter and 8-mm thickness when acted upon by a point load of 3 kN at the centre of the plate. The plate is simply supported at the edges. E = 204 GPa and Poisson's ratio = 0.3. (0.633 mm, 107.3 MPa, 123 MPa)

16.4 A circular plate is 5-mm thick and 200-mm in diameter. It is acted upon by a concentrated load of 2 kN at the centre of the plate. Sketch the distribution of deflection of the plate under the load and the radial and tangential stresses. Assume the plate to be simply supported at the edges. E = 200 GPa and Poisson's ratio = 0.28.

16.5 A solid circular plate freely supported at the edges has a load of 20 kN distributed round a circle of 200-mm diameter. The plate is of 600-mm diameter and is 15-mm thick. Determine the central deflection and the maximum values of the radial and tangential stresses. Take E = 202 GPa and Poisson's ratio = 0.3. (1.154 mm; $\sigma_x = \sigma_z$ = 73.8 MPa)

Chapter **17**

Plastic Bending and Torsion

If the strain caused in a test specimen due to load disappears when the load is removed, the material is said to behave elastically and the maximum value of stress is the elastic limit of the material. However, if the material is loaded beyond the maximum limit of stress and the yield takes place, the stress and the strain decrease in a linear way parallel to the straight-line portion of the main curve on removal of the load. Figure 17.1 shows this decrease along line *AB* parallel to the straight-line portion *OP* of the main curve. The fact that the strain does not completely disappear indicates that a *permanent set* or *plastic deformation* of the material has taken place. If after being loaded and unloaded once in the above way, the test specimen is again loaded, the new stress–strain curve will closely follow the earlier unloading curve *AB* and then follow the original curve to the right. For most of the materials, the plastic deformation is found to be dependent not only upon the maximum value of the stress, but also upon the time elapsed before removal of the load. The stress due to plastic deformation is referred as *slip* whereas that due to time period as *creep*. The temperature also influences the creep.

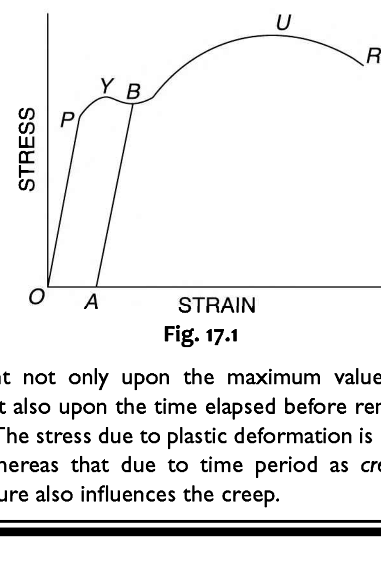


Fig. 17.1

17.1 PLASTIC THEORY OF BENDING

In the elastic theory, the designing of structures is based upon the assumption of a linear stress–strain relationship, i.e., it is assumed that the proportional limit of the material is never exceeded. As the proportional limit for all practical purposes coincides with the elastic limit and the yield strength of the material, it is assumed that the material remains elastic throughout the loading and regains the original shape after unloading. However, if the stresses in any part of the material exceed the yield strength, plastic deformation takes place and the results obtained by the elastic theory are not valid.

Though the actual stress–strain relationship in the plastic range is complex, an idealised elastoplastic material such as mild steel may be considered to have a stress–strain diagram as shown in Fig. 17.2, consisting of two straight-line segments. As long as the loading is in elastic range, the value of stress is less than the yield stress value σ_y and the material obeys the Hooke's law. Once the stress reaches the value of

σ_y, the material starts deforming plastically under a constant load. On unloading, the stress follows the line *AB* parallel to initial portion *OP* of the loading curve. The segment *OA* indicates the strain corresponding to the permanent set or plastic deformation due to loading and unloading of the material.

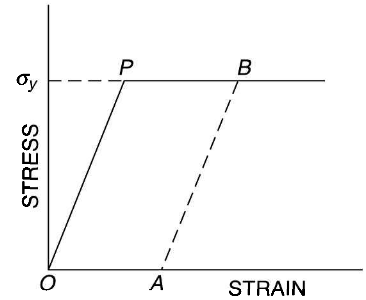


Fig. 17.2

Practically, a structure does not fail as soon as the stress at an edge of the weakest cross-section reaches the yield point. The structure continues to sustain the load as long as a central *core section* remains in the elastic state. If the load is increased gradually, the yielding first occurs at the extreme fibre of the weakest section and then the plastic state of the material commences. Further increase in loading increases the strain and thus the deflection considerably. When the whole cross-section at any point in a structure becomes plastic, the moment of resistance of the structure is maximum. The complete yielding across the section of a structure is termed as *plastic hinge* and the corresponding load required to produce complete yielding as the *collapse load.*

The assumptions made in the plastic theory are as follows:

1. The material has a single yield point and can undergo considerable strain at yield without further increase in stress.
2. The value of yield stress is the same in tension and compression.
3. Transverse sections remain plane throughout the loading so that the strain is proportional to the distance from the neutral axis.
4. The collapse of the whole structure does not take place till the formation of the plastic hinges at all the required points is not complete.
5. As a plastic hinge develops at any cross-section, the moment of resistance remains constant at that point till the collapse of the whole structure takes place.

17.2 MOMENT OF RESISTANCE AT PLASTIC HINGE

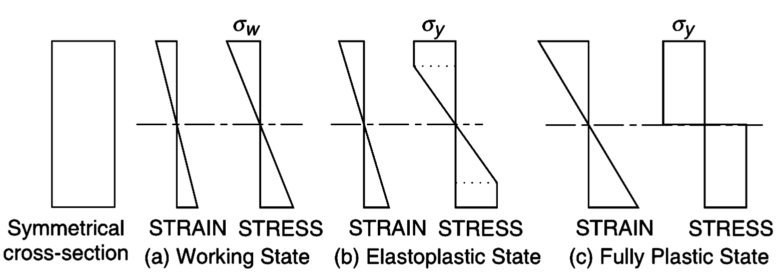


Fig. 17.3

Figure 17.3 shows the variation of stress and strain in a symmetrical cross-section under increasing load conditions. Figure 17.3a shows the strain and stress under the working load by simple theory of bending. Figure 17.3b shows the elastoplastic state on increasing the load in which a portion of the cross-section becomes in the plastic state and the rest in the elastic state. Figure 17.3c is the fully plastic state on further increase of load. It can be observed that whereas the strain goes on increasing on increasing the load, the stress is limited to its yield value.

17.3 SYMMETRICAL BENDING

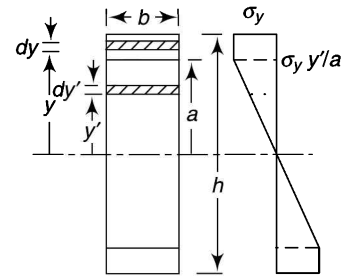


Fig. 17.4

Let a rectangular beam of width b and depth h subjected to bending moment be in the elastoplastic state at a cross-section under the load. It is also assumed that it has yielded beyond a distance a from the neutral axis (Fig. 17.4).

Let σ_y = yield point stress

M_y = moment of resistance at the commencement of yield

M_p = moment of resistance in the fully plastic state

Total bending moment at the section will be the sum of the bending moment due to plastic resistance and the elastic resistance. Thus considering a small area of depth dy at distance y in the plastic range and dy' at a distance y' in the elastic range,

Moment of resistance, $$M = 2\int_a^{h/2} \sigma_y \cdot (b \cdot dy) \cdot y + 2\int_0^a \frac{\sigma_y \cdot y'}{a} \cdot (b \cdot dy') \cdot y'$$

$$= 2\sigma_y \cdot b\int_a^{h/2} y \cdot dy + 2\frac{\sigma_y \cdot b}{a}\int_0^a y'^2 \cdot dy'$$

$$= 2\sigma_y \cdot b\left[\left(\frac{y^2}{2}\right)_a^{h/2} + \frac{1}{a}\left(\frac{y'^3}{3}\right)_0^a\right]$$

$$= 2\sigma_y \cdot b\left(\frac{h^2}{8} - \frac{a^2}{2} + \frac{a^2}{3}\right)$$

$$= \sigma_y \cdot b\left(\frac{h^2}{4} - \frac{a^2}{3}\right) \tag{17.1}$$

- When the yielding just commences, $a = h/2$, $M_y = \dfrac{bh^2}{6} \cdot \sigma_y$ (17.2)
- In fully plastic state, $a = 0$, $M_p = \dfrac{bh^3}{4} \cdot \sigma_y$ (17.3)
- The ratio of M_p/M_y is known as the *shape factor (S)*. Thus

  Shape factor for a rectangular section, $S = \dfrac{M_p}{M_y} = 1.5$ (17.4)

  Also note that as the working moment, $M_w = \sigma_w \cdot Z = \sigma_w \cdot \dfrac{bh^2}{6}$ (17.5)

  From 17.2, $M_y = \dfrac{\sigma_y}{\sigma_w} \cdot M_w$ (17.6)

  And thus $M_p = S \cdot M_y = S \cdot \dfrac{\sigma_y}{\sigma_w} \cdot M_w = S \cdot \dfrac{\sigma_y}{\sigma_w} \cdot \sigma_w \cdot Z = SZ\sigma_y$ (17.7)
- The value of the bending moment that causes the failure of the member is known as *ultimate bending moment* (M_u). This value can be determined by inserting $\sigma_y = \sigma_u$ in Eq. 17.3. However, in practice the value of M_u is determined experimentally for a specimen of a material and assuming a linear distribution of stresses, corresponding stress value is obtained using the usual bending equation.

Thus $\sigma_u = \dfrac{M_u}{I} \cdot \dfrac{h}{2}$

The fictitious stress σ_u is known as the *modulus of rupture in bending* of the given material. Thus *modulus of rupture in bending may be defined as maximum stress value as obtained from the bending formula by using an experimentally found ultimate bending moment value required to rupture a shaft.* This is useful in determining the ultimate bending moment of members made of the same material and of the same shape but of different dimensions by using the above relation and solving for M_u.

17.4 UNSYMMETRICAL BENDING

In a symmetrical section, it is easy to find the neutral axis as it coincides with the geometrical axis. But in unsymmetrical sections, the neutral axis is displaced. However, in the fully plastic state, the neutral axis can be determined from the fact that for the equilibrium of forces, at any cross-section, the tensile forces on one side of the neutral axis must be equal to the compressive forces on the other side. If the tensile and compressive stresses are taken to be equal, the areas on the two sides of the neutral axis must be equal. Thus, the neutral axis divides the area into equal halves.

If A is the total area of the cross-section and G_1 and G_2 the centroids of the two halves at a distance $y_1 + y_2$ apart (Fig. 17.5).

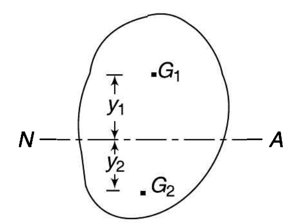


Fig. 17.5

Then $M_p = \dfrac{A}{2} \cdot \sigma_y \cdot y_1 + \dfrac{A}{2} \cdot \sigma_y \cdot y_2 = \dfrac{A}{2} \cdot \sigma_y (y_1 + y_2)$ (17.8)

At first yield, $M_y = Z \cdot \sigma_y$ where Z is the section modulus.

$\therefore$ shape factor, $S = \dfrac{M_p}{M_y} = \dfrac{A(y_1 + y_2)}{2Z}$ (17.9)

Example 17.1 A steel bar of rectangular cross-section 90 mm × 36 mm and of 1.5-m length is used as a simply supported beam. If the yield stress is 240 MPa, determine the load at the midspan at the occurring of the first yield. Take the long edges of the section to be vertical.

Also find the load required to cause yielding for a depth of 15 mm at the top and bottom of the section at the midspan. What is the length of beam over which yielding occurs?

Solution

Given $b = 36$ mm $h = 90$ mm

$l = 1.5$ m $a = 45 - 15 = 30$ mm

To find

— Load at midspan at first yield

— Load to cause yield at depth of 30 mm from neutral axis

— Length over which yield occurs

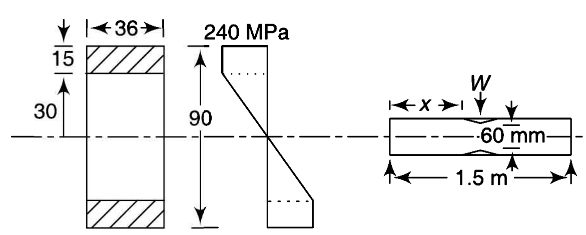


Fig. 17.6

Refer Fig. 17.6.

Load at first yield

Let W_y be the load at first yield at the midspan,

$$M_y = \frac{bh^2}{6} \cdot \sigma_y$$

$$= \frac{36 \times 90^2}{6} \times 240 = 11\,664 \times 10^3 \text{ N} \cdot \text{mm} \quad \text{or} \quad 11\,664 \text{ N} \cdot \text{m}$$

Thus, $\dfrac{W_y \times 1.5}{4} = 11\,664$ or $W_y = 31\ 104$ N

Load to cause yield at 15 mm from top

For elastoplastic state, moment of resistance, $M = \sigma_y \cdot b\left(\dfrac{h^2}{4} - \dfrac{a^2}{3}\right)$

$$= 240 \times 36\left(\frac{90^2}{4} - \frac{30^2}{3}\right) = 14\ 904 \times 10^3\ \text{N} \cdot \text{mm or } 14\ 904\ \text{N} \cdot \text{m}$$

or $$\frac{W_y \times 1.5}{4} = 14\ 904 \text{ or } W_y = 39\ 744\ \text{N}$$

Length over which yield occurs

For this required depth of yield, let the yield occur beyond a cross-section at distance x m from one end of the beam. Then this cross-section will be having the first yield and thus the moment of resistance will be equal to the first yield at the midspan, i.e., 11 664 × 10^3 N · mm.

Reaction at each end support = 39 744/2=19 872 N

Bending moment at distance x from the support = 19 872 x

Thus 19 872x = 11 664 × 10^3 or x = 587 mm

Length over which the yield occurs = 1.5 – 2 × 0.587 = 0.326 m

Example 17.2 Find the shape factor for the beam of *I*-section having dimensions of web 76 mm × 10 mm and of the flanges 80 mm × 12 mm.

Solution

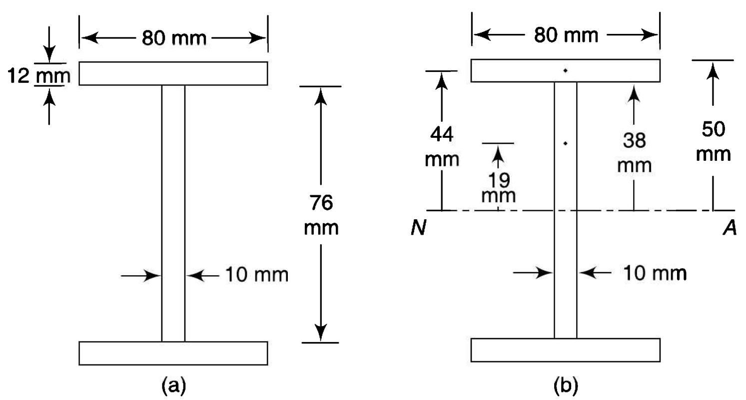


Fig. 17.7

Given A beam of *I*-section as shown in Fig. 17.7a

To find Shape factor

Moment of inertia of the section = $\dfrac{80 \times 100^3}{12} - \dfrac{(80-10) \times (100-24)^3}{12} = 4.106 \times 10^6\ \text{mm}^4$

Determination of shape factor

Moment of resistance at first yield, $M_y = \dfrac{4.106 \times 10^6}{50} \cdot \sigma_y = 82\ 119\sigma_y$

In fully plastic state, $M_p = 2(80 \times 12 \times 44 + 10 \times 38 \times 19) \cdot \sigma_y$ ...(Refer Fig. 17.7b)

$$= 98\,920\sigma_y$$

$$\text{Shape factor, } S = \frac{M_p}{M_y} = \frac{98\,920}{82\,119} = 1.205$$

Example 17.3 Find the shape factor for a *T*-beam of overall size 80 mm × 100 mm and thickness of web and the flange of 10 mm.

Solution

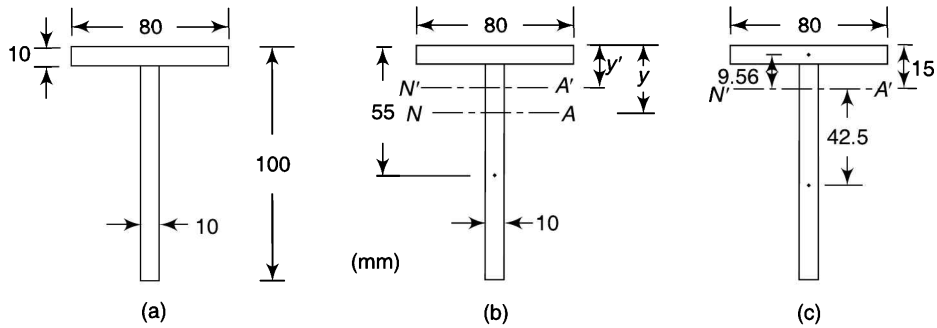


Fig. 17.8

Given A beam of *T*-section as shown in Fig. 17.8a

To find Shape factor

Refer Fig. 17.8 b.

Let *N*-*A* and *N'*-*A'* be the elastic and plastic neutral axes respectively.

$$y = \frac{80 \times 10 \times 5 + 90 \times 10 \times 55}{80 \times 10 + 90 \times 10} = 31.47 \text{ mm}$$

$$I = \frac{80 \times 10^3}{12} + 80 \times 10 \times (31.47 - 5)^2 + \frac{10 \times 90^3}{12} + 10 \times 90 \times (55 - 31.47)^2 = 1.673 \times 10^6 \text{ mm}^4$$

Determination of shape factor

Yield will start at the bottom edge,

Moment of resistance at first yield,

$$M_y = \frac{I}{y} \cdot \sigma_y = \frac{1.673 \times 10^6}{(100 - 31.47)} \cdot \sigma_y = 24\,413\sigma_y$$

In the fully plastic state, the neutral axis divides the total area of the beam into two equal parts.

$$\text{Total area} = 80 \times 10 + 10 \times 90 = 1700 \text{ mm}^2$$

$$y' = 100 - \frac{1700/2}{10} = 15 \text{ mm}$$

$$M_p = [80 \times 10 \times (15 - 5) + 10 \times 5 \times 2.5 + 85 \times 10 \times 42.5]\sigma_y = 44\,250\sigma_y$$

$$\text{Shape factor,} \quad S = \frac{M_p}{M_y} = \frac{44\,250}{24\,413} = 1.813$$

Alternate Solution

Alternatively, shape factor can be found from the relation, $S = \dfrac{A(y_1 + y_2)}{2Z}$

where $A = 1700\ \text{mm}^2$, $Z = \dfrac{I}{y} = 24\,413\ \text{mm}^3$

$$y_1 = \frac{80 \times 10 \times 10 + 10 \times 5 \times 2.5}{80 \times 10 + 10 \times 5} = 9.56\ \text{mm} \text{ and } y_2 = \frac{100 - 15}{2} = 42.5\ \text{mm (Fig. 17.8c)}$$

or $$S = \frac{1700(y_1 + y_2)}{2Z} = \frac{1700(9.56 + 42.5)}{2 \times 24\,413} = 1.813$$

Example 17.4 || Find the shape factor for a *T*-beam of web size 88 mm × 10 mm and a flange size 80 mm × 12 mm.

Solution

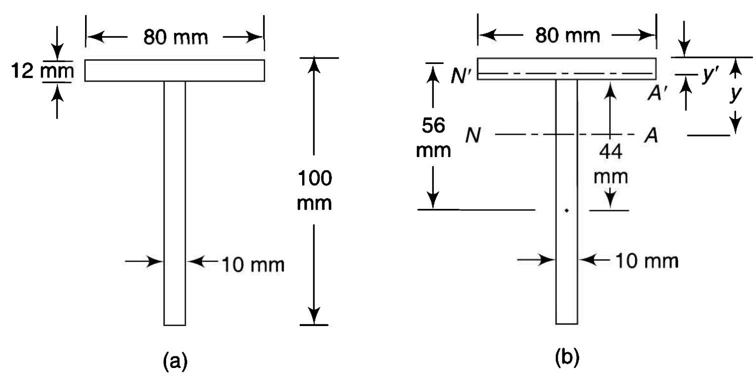


Fig. 17.9

Given A beam of *T*-section as shown in Fig. 17.9a
To find Shape factor

Refer Fig. 17.9,
Let *N-A* and *N'-A'* be the elastic and plastic neutral axes respectively.

$$y = \frac{80 \times 12 \times 6 + 88 \times 10 \times 56}{80 \times 12 + 88 \times 10} = 29.91\ \text{mm}$$

$$I = \frac{80 \times 12^3}{12} + 80 \times 12 \times (29.91 - 6)^2 + \frac{10 \times 88^3}{12} + 10 \times 88 \times (56 - 29.91)^2$$
$$= 1.727 \times 10^6\ \text{mm}^2$$

Determination of shape factor

Moment of resistance at first yield, $M_y = \dfrac{1.727 \times 10^6}{(100 - 29.91)} \cdot \sigma_y = 24\,643\sigma_y$

In the fully plastic state, the neutral axis divides the total area of the beam into two equal parts.
Total area = $80 \times 12 + 10 \times 88 = 1840\ \text{mm}^2$

As the area of flange is more than the half of the total area (= 80×12),

$$y' = \frac{1840/2}{80} = 11.5\ \text{mm}$$

$$M_p = [80 \times 11.5 \times (11.5/2) + 80 \times 0.5 \times (0.5/2) + 88 \times 10 \times (44 + 0.5)]\sigma = 44\,460\sigma$$

Shape factor, $S = \dfrac{M_p}{M_y} = \dfrac{44\,460}{24\,643} = 1.804$

Example 17.5 Find the shape factor for the channel shape beam shown in Fig. 17.10.

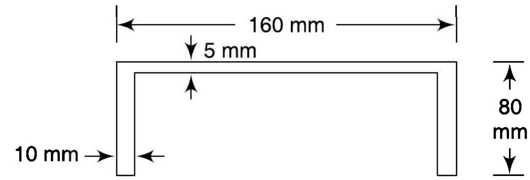


Fig. 17.10

Solution

Given A channel shape.
To find Shape factor

Refer Fig. 17.11.

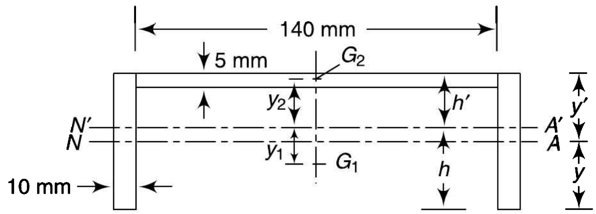


Fig. 17.11

Determination of section modulus

Let NA be the elastic neutral axis of the section.

$$y' = \frac{140 \times 5 \times 2.5 + 2 \times 80 \times 10 \times 40}{140 \times 5 + 2 \times 80 \times 10} = 28.6 \text{ mm}$$

$$y = 80 - 28.6 = 51.4 \text{ mm}$$

$$I = \frac{140 \times 5^3}{12} + 140 \times 5 \times (28.6 - 2.5)^2 + 2\left(\frac{10 \times 80^3}{12} + 10 \times 80 \times (40 - 28.6)^2\right)$$

$$= 1.539 \times 10^6 \text{ mm}^4$$

$$Z = \frac{I}{y} = \frac{1.539 \times 10^6}{51.4} = 29\,950 \text{ mm}^3$$

Determination of shape factor

Let $N'A'$ be the neutral axis under fully plastic conditions. This neutral axis, therefore, divides the total area in two halves.

Total area = $140 \times 5 + 2 \times 80 \times 10 = 2300 \text{ mm}^2$

Thus, $h = \dfrac{2300/2}{10 \times 2} = 57.5$ mm and $h' = 80 - 57.5 = 22.5$ mm

Let the centroids of the two areas G_1 and G_2 be at distances y_1 and y_2 respectively from the neutral axis.

$$y_1 = \frac{h}{2} = \frac{57.5}{2} = 28.75 \text{ mm}$$

$$y_2 = \frac{(140 \times 5)(22.5 - 2.5) + 2 \times 10 \times 22.5 \times (22.5/2)}{140 \times 5 + 2 \times 10 \times 22.5} = 16.58 \text{ mm}$$

Shape factor, $S = \dfrac{A(y_1 + y_2)}{2Z} = \dfrac{2300(28.75 + 16.58)}{2 \times 29\,950} = 1.74$

Example 17.6 The load at the end of a 1.6-m long cantilever of *T*-section is increased so that the top of the flange just yields. Find the position of the neutral axis and the load if the yield stress is 240 MPa. The web is 55 mm × 10 mm and the flange is 80 mm × 10 mm.

Solution

Given A cantilever of T-section as shown in Fig. 17.12a.

$l = 1.6$ m $\qquad\qquad \sigma_y = 240$ MPa

To find

— Position of neutral axis

— Load

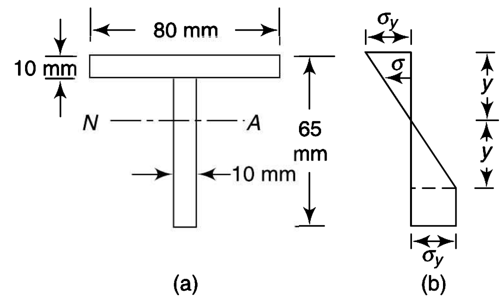


Fig. 17.12

Position of neutral axis

Let the neutral axis be at a distance y from the top edge. The yield starts at the bottom of the web and will cover some portion when the top of the flange will just yield. The stress distribution in the beam section is shown in Fig. 17.12b. Let the stress at the bottom of the flange be σ when the top of flange just yields.

Then $\quad \sigma = \dfrac{y-10}{y}\cdot\sigma_y$

$$\text{Positive force} = \frac{\sigma_y+\sigma}{2}\cdot(80\times 10) + \frac{\sigma}{2}\cdot(y-10)10 = 400\sigma_y + 350\sigma + 5\sigma_y$$

$$\text{Negative force} = \frac{\sigma_y}{2}\cdot 10y + \sigma_y(65-2y)\times 10 = -15\sigma_y\cdot y + 650\sigma_y$$

Equating the two, $400\sigma_y + 350\sigma + 5\sigma_y = -15\sigma_y\cdot y + 650\sigma_y$

$$350\sigma + 5\sigma_y = -15\sigma_y\cdot y + 250\sigma_y$$

$$(350+5y)\left(\frac{y-10}{y}\right)\sigma_y = (-15y+250)\sigma_y$$

$$(350+5y)(y-10) = (-15y+250)\,y$$

$$20y^2 + 50y - 3500 = 0$$

$$y^2 + 2.5y - 175 = 0$$

$$y = \frac{-2.5\pm\sqrt{2.5^2+700}}{2} = 12.04 \text{ mm}$$

Determination of load

$$\sigma = \frac{y-10}{y}\cdot\sigma_y = \frac{12.04-10}{12.04}\cdot\sigma_y = 0.17\sigma_y$$

$$\text{Distance of CG of the force in the flange from its top} = \frac{10}{3}\left(\frac{1+2\times 0.17}{1+0.17}\right) = 3.82 \text{ mm}$$

Distance of CG of the force in the flange from neutral axis = 12.04 – 3.82 = 8.22 mm

Moment of resistance

$$= \sigma_y\left[\begin{aligned}&\frac{1+0.17}{2}\times(80\times 10)\times 8.22 + \frac{0.17}{2}\times(12.04-10)\times 10\left(2.04\times\frac{2}{3}\right)\\ &+\frac{0.17}{2}\times 10\times 12.04\times\left(12.04\times\frac{2}{3}\right) + (65-2\times 12.04)\times 10\times\left(y+\frac{65-2y}{2}\right)\end{aligned}\right]$$

$$= 240[3847 + 2.36 + 82 + 26\,598] = 7327\times 10^3 \text{ N}\cdot\text{mm} = 7327 \text{ N}\cdot\text{m}$$

If W is the load at the end of the cantilever,

$$1.6\ W = 7327 \text{ or } W = 4579 \text{ N or } 4.579 \text{ kN}$$

Example 17.7 The load at the end of a 2-m long cantilever is increased so that the yield spreads to within 32 mm of the lower edge of the web. Determine the moment of resistance. The yield stress is 240 MPa. The web is 130 mm × 10 mm and the flange 80 mm × 10 mm.

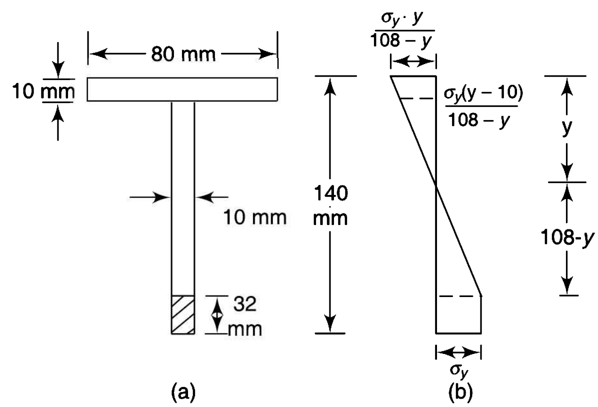


Fig. 17.13

Solution

Given A cantilever of T-section as shown in Fig. 17.13a.

$l = 2$ m $\qquad \sigma_y = 240$ MPa

To find Moment of resistance

Position of neutral axis

As the load increases and the yield spreads up to 32 mm from the lower edge of the web, assume that the top of the flange is still in the elastic state. Let the neutral axis be at a distance y from the top edge. The stress distribution in the beam section will be as shown in Fig. 17.13b.

Stress at the top of flange, $\sigma_t = \dfrac{y}{108 - y} \cdot \sigma_y$

Stress at the bottom of flange, $\sigma_b = \dfrac{y - 10}{108 - y} \cdot \sigma_y$

$$\text{Total tensile stress force} = \frac{\sigma_t + \sigma_b}{2} \cdot (80 \times 10) + \frac{\sigma_b}{2} \cdot (y - 10)10$$

$$= 400\sigma_y \left(\frac{y}{108 - y} + \frac{y - 10}{108 - y} \right) + 5\sigma_y \left(\frac{y - 10}{108 - y} \right)(y - 10)$$

$$= 5\sigma_y \left(\frac{80y + 80y - 800 + y^2 - 20y + 100}{108 - y} \right)$$

$$= 5\sigma_y \left(\frac{y^2 + 140y - 700}{108 - y} \right)$$

$$\text{Compressive force} = \frac{\sigma_y}{2} \cdot 10 \times (108 - y) + \sigma_y \times 32 \times 10 = \sigma_y\,(860 - 5y)$$

Equating the two, $5(y^2 + 140y - 700) = (108 - y)(860 - 5y)$

or $\quad 5y^2 + 700y - 3500 = 92\,880 + 5y^2 - 1400y$

or $\quad 2100y = 96\,380$

or $\quad y = 45.9$ mm

Determination of CG of force in the flange

$$\sigma_t = \frac{y}{108 - y} \cdot \sigma_y = \frac{45.9}{108 - 45.9} \cdot \sigma_y = 0.739\sigma_y$$

As the value of σ_t is less than σ_y, the top of the flange is in elastic state as assumed.

$$\sigma_b = \frac{y - 10}{108 - y} \cdot \sigma_y = \frac{45.9 - 10}{108 - 45.9} \cdot \sigma_y = 0.578\sigma_y$$

Distance of CG of the force in the flange from neutral axis

$$= 45.9 - \frac{10}{3}\left(\frac{0.739 + 2 \times 0.578}{0.739 + 0.578}\right) = 41.1$$

Moment of resistance

Moment of resistance

$$= \sigma_y \begin{bmatrix} \dfrac{0.739 + 0.578}{2} \cdot (80 \times 10) \times 41.1 + \dfrac{0.578}{2} \cdot (45.9 - 10) \times 10 \times (45.9 - 10) \cdot \dfrac{2}{3} \\ + \dfrac{1}{2} \cdot 10 \times (108 - 45.9) \times (108 - 45.9) \cdot \dfrac{2}{3} + 32 \times 10 \times (108 - 45.9 + 16) \end{bmatrix}$$

$= 240[21\,651 + 2483 + 12\,855 + 24\,992] = 14.88 \times 10^3$ N · mm = 14.88 kN · m

17.5 COLLAPSE LOAD

As stated earlier, *plastic hinge* is the complete yielding across a section of a structure and the corresponding load required to produce this complete yielding is the *collapse load*. Thus, for a structure loaded under working conditions, the failure will occur if the load continues to increase till the collapse load. The ratio of the *collapse load* to the *working load* is termed as the *load factor*.

Let W and W_c be the working load and the collapse load respectively for a structure.

Load factor = W_c / W (17.10)

Consider a simply supported beam in which the load divides the length in the ratio $a:b$ (Fig. 17.14). The maximum bending moment will be at the point of load and the collapse condition reaches when the plastic hinge is formed at this point or when the whole of the section will become plastic.

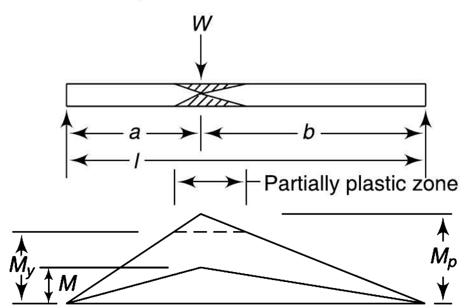


Fig. 17.14

We have, $M_w = Wab / l$ and $M_p = W_c ab / l$

And thus, the load factor, $\dfrac{W_c}{W} = \dfrac{M_p}{M_w} = \dfrac{S(\sigma_y / \sigma_w) \cdot M_w}{M_w} = S(\sigma_y / \sigma_w)$ (17.11)

As (σ_y / σ_w) = Factor of safety, the load factor is the product of the *shape factor* and the *factor of safety*. Due to the shape factor component which depends upon different shapes of the section, load factors for different sections are also different under the same system of loading.

The load factor obtained above is also valid for cantilevers and for distributed loads.

Built-in beam with Uniformly Distributed Load For this type of loading, the working bending moment diagram as well as the plastic bending moment diagrams are shown in Fig. 17.15 (refer Example 8.2). For collapse, three plastic hinges are required to be formed, one at the centre and two at the fixed ends. By symmetry, reactions at the ends are $W_c/2$. Let the plastic bending moment at each end be M_p.

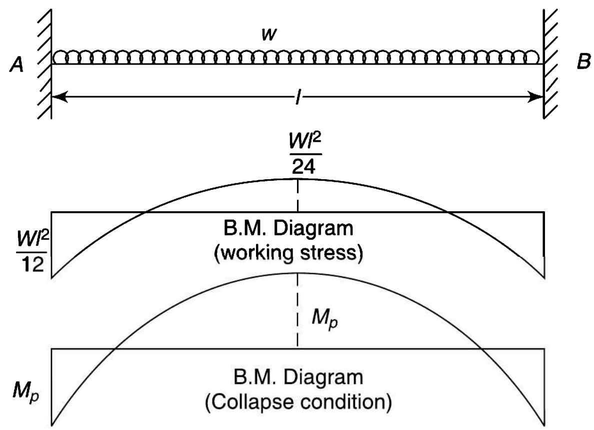


Fig. 17.15

Plastic bending moment at the centre = $\dfrac{W_c}{2}\cdot\dfrac{l}{2}-\dfrac{W_c}{2}\cdot\dfrac{l}{4}-M_p$

Equating the plastic bending moment at the centre with that at each end,

$$\frac{W_c}{2}\cdot\frac{l}{2}-\frac{W_c}{2}\cdot\frac{l}{4}-M_p=M_p \text{ or } M_p=\frac{W_c l}{16} \tag{17.12}$$

Maximum bending moment under working conditions, $M_w=\dfrac{Wl}{12}$

$$\text{Load factor, } \frac{W_c}{W}=\frac{16M_p/l}{12M_w/l}=\frac{4M_p}{3M_w}=\frac{4S(\sigma_y/\sigma_w)M_w}{3M_w}=\frac{4}{3}\cdot S(\sigma_y/\sigma_w) \tag{17.13}$$

i.e. an increase of 4/3 over that for simply supported beam.

Propped Cantilever There will be two plastic hinges in this case.

Consider a propped cantilever with a central load W and propped at the free end as shown in Fig. 17.16. The reaction R at the prop can be calculated as under:

Downward deflection at free end due to load W when prop is not there = $\dfrac{5Wl^3}{48EI}$ (Eq. 7.7a)

Upward deflection at free end due to load R when load W is not there = $\dfrac{Rl^3}{3EI}$ (Eq. 7.6a)

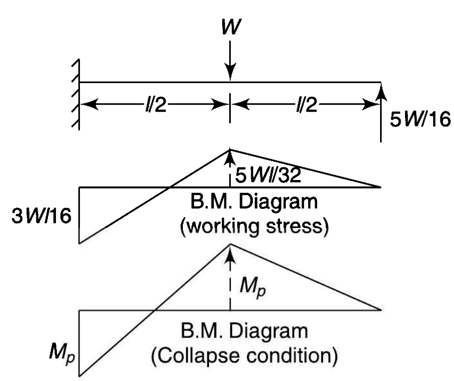


Fig. 17.16

If the prop is rigid, these must be equal i.e. $\dfrac{5Wl^3}{48EI}=\dfrac{Rl^3}{3EI}$ or $R=\dfrac{5}{16}W$ (17.14)

Bending moment at load point = $R \cdot \frac{l}{2} = \frac{5}{16}\frac{Wl}{2} = \frac{5Wl}{32}$

Bending moment at fixed end = $R \cdot l - \frac{Wl}{2} = \frac{5Wl}{16} - \frac{Wl}{2} = -\frac{3Wl}{16}$

The bending moment diagram for the working stress is shown underneath the load diagram. A gradual increase in the load will cause a plastic hinge to form first at the fixed end as the value of bending moment at this point is somewhat higher than at the load point. However, due to support at the free end, collapse condition is not reached until a second plastic hinge is formed at the load point. When this is attained, the numerical values of the bending moment at the two points will be M_p each and the shape as shown in the diagram. This diagram is not similar to first one.

Let P be the load on the prop at the collapse. Then

Plastic bending moment at the fixed end = $W_c \cdot \frac{l}{2} - P \cdot l$ (clockwise)

Plastic bending moment at the load point = $P \cdot \frac{l}{2}$ (counter-clockwise)

Equating the two, $W_c \cdot \frac{l}{2} - P \cdot l = \frac{Pl}{2}$ or $P = \frac{W_c}{3}$

or Plastic bending moment, $M_p = \frac{Pl}{2} = \frac{W_c l}{6}$ (17.15)

Maximum bending moment under working conditions, $M_w = \frac{3}{16}Wl$

Load factor, $\frac{W_c}{W} = \frac{6M_p/l}{16M_w/3l} = \frac{18M_p}{16M_w} = \frac{18S(\sigma_y/\sigma_w)M_w}{16M_w} = \frac{9}{8} \cdot S(\sigma_y/\sigma_w)$ (17.16)

i.e. an increase of 9/8 over that for simply supported beam.

17.6 TORSION OF CIRCULAR SHAFTS

The analysis for the circular shafts transmitting torque can be made in the same manner as for beams. It can be assumed that the stress is proportional to strain upto the yield point and constant thereafter, and the plane cross-sections remain plane after torsion.

Consider a hollow circular shaft with inner and outer radii as r_1 and r_2 respectively (Fig. 17.17). Assume that the outer portion of the shaft is deformed plastically beyond radius a under the action of external torque T.

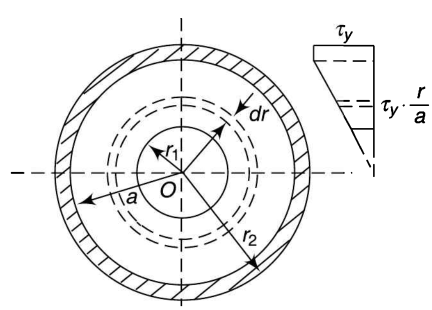


Fig. 17.17

Let τ_y = yield point shear stress

Considering an elementary ring of the shaft in the elastic region,

Shear stress in this region = $\frac{\tau_y \cdot r}{a}$

Torque = $\int_{r_1}^{a} (2\pi r \cdot dr \cdot \frac{\tau_y \cdot r}{a})r$

Similarly, considering an elementary ring of the shaft in the plastic region,

$$\text{Torque} = \int_a^{r_2} (2\pi r \cdot dr \cdot \tau_y) r$$

$$\text{Total torque, } T = \int_a^{r_2} (2\pi r \cdot dr \cdot \tau_y) r + \int_{r_1}^{a} \left(2\pi r \cdot dr \cdot \frac{\tau_y \cdot r}{a}\right) r$$

$$= 2\pi \cdot \tau_y \int_a^{r_2} r^2 \cdot dr + \frac{2\pi\tau_y}{a} \int_{r_1}^{a} r^3 \cdot dr = 2\pi\tau_y \cdot \left(\frac{r^3}{3}\right)_a^{r_2} + \frac{2\pi\tau_y}{a}\left(\frac{r^4}{4}\right)_{r_1}^{a}$$

$$= 2\pi\tau_y \left(\frac{r_2^3}{3} - \frac{a^3}{3}\right) + \frac{2\pi\tau_y}{a}\left(\frac{a^4}{4} - \frac{r_1^4}{4}\right) = 2\pi\tau_y \left(\frac{r_2^3}{3} - \frac{a^3}{3} + \frac{a^3}{4} - \frac{r_1^4}{4a}\right)$$

$$= \frac{\pi\tau_y}{6a}(4ar_2^3 - a^4 - 3r_1^4) \tag{17.17}$$

- When the yield commences, $a = r_2$, $T_y = \frac{\pi\tau_y}{6r_2}(4r_2^4 - r_2^4 - 3r_1^4) = \frac{\pi\tau_y}{2r_2}(r_2^4 - r_1^4)$ (17.18)
- For a fully plastic torque, $a = r_1$, $T_p = \frac{\pi\tau_y}{6r_1}(4r_1 r_2^3 - r_1^4 - 3r_1^4) = \frac{2\pi\tau_y}{3}(r_2^3 - r_1^3)$ (17.19)
- For solid shaft, $r_1 = 0$, $T = \frac{\pi\tau_y}{6}(4r_2^3 - a^3)$

and $T_y = \frac{\pi\tau_y}{6r_2}(4r_2^4 - r_2^4) = \frac{\pi r_2^3 \tau_y}{2}$; $T_p = \frac{\pi\tau_y}{6r_1}(4r_1 r_2^3) = \frac{2\pi\, r_2^3\tau_y}{3}$ (17.20)

- Shape factor, for hollow shaft

$$S = \frac{T_p}{T_y} = \frac{2\pi\tau_y(r_2^3 - r_1^3)/3}{\pi\tau_y(r_2^4 - r_1^4)/(2r_2)} = \frac{4}{3}\left[\frac{r_2^3 - r_1^3}{r_2^3 - r_1^4/r_2}\right] = \frac{4r_2^3}{3r_2^3}\left[\frac{1-(r_1/r_2)^3}{1-(r_1/r_2)^4}\right] \tag{17.21}$$

For solid shaft, $S = \frac{T_p}{T_y} = \frac{2\pi\, r_2^3\tau_y/3}{\pi\, r_2^3\tau_y/2} = \frac{4}{3}$ (17.22)

- ***Angle of Twist*** Angle of twist can be found on the assumption that the outer plastic region does not offer any resistance to torque.

$$\frac{T}{J} = \frac{\tau_y}{a} = \frac{G\theta}{l} \quad \text{or } \theta = \frac{\tau_y \cdot l}{aG} = \frac{T \cdot l}{JG} \tag{17.23}$$

where J = Polar moment of inertia of the elastic core $= \frac{\pi}{2}(a^4 - r_1^4)$

- The value of the torque that causes the failure of the shaft is known as *ultimate torque* (T_u). This value can be determined by inserting $\tau_y = \tau_u$ in Eq. 17.20. However, in practice the value of T_u is determined experimentally by twisting a specimen of a material till it breaks and assuming a linear distribution of stresses, corresponding stress value is obtained using the usual torsion equation.

Thus $\tau_u = \frac{T_u}{J} \cdot R$

The fictitious stress σ_u is known as the *modulus of rupture in torsion* of the given material. *Thus modulus of rupture in torsion may be defined as maximum shear stress value as obtained from the torsion formula by using an experimentally found ultimate torque value required to rupture a shaft.*

This is useful in determining the ultimate torque of shafts made of the same material but of different radii by using the above relation and solving for T_u.

Example 17.8 A hollow circular shaft with 80-mm outer diameter and 60-mm inner diameter is applied a varying torque. Determine the value of the torque

(i) at the first yield
(ii) when the plastic zone extends up to a depth of 5 mm
(iii) the section is fully plastic
(iv) the shape factor

Assume yield shear stress as 125 MPa.

Solution

Given A hollow circular shaft

$$r_1 = 30 \text{ mm} \qquad r_2 = 40 \text{ mm}$$
$$\sigma_y = 125 \text{ MPa}$$

To find Value of torque

— at first yield
— when plastic zone is upto a depth of 5 mm
— section is fully plastic
— shape factor

Torque at first yield

$$T_y = \frac{\pi\tau_y}{2r_2}(r_2^4 - r_1^4) = \frac{\pi \times 125}{2 \times 40}(40^4 - 30^4)$$
$$= 8.59 \times 10^6 \text{ N} \cdot \text{mm or } 8.59 \text{ kN} \cdot \text{m}$$

When plastic zone upto a depth of 5 mm

$$a = 40 - 5 = 35 \text{ mm}$$
$$T = \frac{\pi\tau_y}{6a}(4ar_2^3 - a^4 - 3r_1^4)$$
$$= \frac{\pi \times 125}{6 \times 35}(4 \times 35 \times 40^3 - 35^4 - 3 \times 30^4)$$
$$= 9.405 \times 10^6 \text{ N} \cdot \text{mm or } 9.405 \text{ kN} \cdot \text{m}$$

When the section is fully plastic

$$T_p = \frac{2\pi\tau_y}{3}(r_2^3 - r_1^3)$$
$$= \frac{2\pi \times 125}{3}(40^3 - 30^3) = 9.687 \times 10^6 \text{ N} \cdot \text{mm or } 9.687 \text{ kN} \cdot \text{m}$$

Shape factor

$$\text{Shape factor} = \frac{T_p}{T_y} = \frac{9.687}{8.59} = 1.128$$

Example 17.9 A 500-mm long steel shaft has a diameter of 60 mm. It is subjected to an increasing torque. Find

(i) the torque and the angle of twist at the first yield
(ii) the torque when the angle of twist increases to double that at first yield

Take τ_y = 125 MPa and G = 80 GPa

Solution

Given A steel shaft

$r_2 = 30$ mm $\qquad l = 500$ mm

$\tau_y = 125$ MPa $\qquad G = 80$ GPa

To find

— Torque and angle of twist at first yield

— When angle of twist doubles

Torque and angle of twist at first yield

$$T_y = \frac{\pi r_2^3 \tau_y}{2} = \frac{\pi \times 30^3 \times 125}{2} = 5.3 \times 10^6 \text{ N} \cdot \text{mm or } 5.3 \text{ kN} \cdot \text{m}$$

Angle of twist, $\theta = \dfrac{\tau_y \cdot l}{aG} = \dfrac{125 \times 500}{30 \times 80\,000} = 0.026\,04$ rad or 1.492°

When angle of twist doubles

When the angle of twist is double, a can be found from the above relation.

$$2 \times 0.026\,04 = \frac{125 \times 500}{a \times 80\,000} \text{ or } a = 15 \text{ mm}$$

and $$T = \frac{\pi \tau_y}{6}(4r_2^3 - a^3) = \frac{\pi \times 125}{6}(4 \times 30^3 - 15^3) = 6.848 \times 10^6 \text{ N} \cdot \text{mm or } 6.868 \text{ kN} \cdot \text{m}$$

17.7 COMBINED DIRECT AND BENDING STRESS

In case of combined direct and bending stress on a beam or column in the elastic range, the neutral axis is displaced to one side of the centroid axis (Sec. 5.8). A similar pattern is observed in the plastic range also. Let a column be acted upon by a thrust F, the line of action of which cuts the cross-section at a point on the x-axis at a distance a from the centroid O (Fig. 17.18a). Figure 17.18b shows the variation of the stress on the column in the working state when the stress is within the elastic limit. As the load increases, the stress on one side reaches the yield point value. Gradually, this spreads over the section as shown in Fig. 17.18c. On further increase of the load, fully plastic state is attained (Fig. 17.18d).

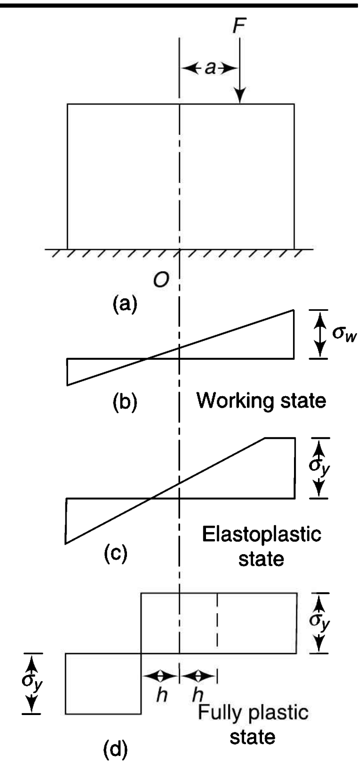


Fig. 17.18

Let h be the displacement of the neutral axis from the centroid axis in the fully plastic state. In the absence of any direct load and under the action of bending moment only, the neutral axis would have coincided with the centroid axis. Thus the displacement is due to the direct load acting on the section on the area ($2h \times b$), where b is the width of the section.

Load acting in the fully plastic state = area $\times$ yield stress = $(2h \times b) \cdot \sigma_y$

But load acting is also equal to the working load multiplied by the load factor or equal to FL.

$$\therefore \quad FL = (2h \times b) \cdot \sigma_y \text{ or } h = \frac{FL}{2b\sigma_y} \tag{17.24}$$

The plastic moment of resistance will be reduced by an amount equivalent to moment due to direct stress, i.e.,

$$M_p = SZ\sigma_y - 2(b \cdot h \cdot \sigma_y)\frac{h}{2} = SZ\sigma_y - b \cdot \sigma_y \cdot h^2 = SZ\sigma_y - b \cdot \sigma_y \cdot \left(\frac{FL}{2b\sigma_y}\right)$$

$$= SZ\sigma_y - \frac{F^2L^2}{4b\sigma_y} \quad (17.25)$$

Dividing throughout by load factor L, permissible working moment of resistance can be obtained,

$$M = \frac{SZ\sigma_y}{L} - \frac{F^2L}{4b\sigma_y} \quad (17.26)$$

Example 17.10 A rectangular mild steel bar has a cross-section of 96 mm × 64 mm. An axial load is applied eccentrically to the bar in such a way that it cuts all the sections at a point midway between both the larger sides and 32 mm from a shorter side. Determine the maximum load which can be applied. Load factor is 1.8 and the yield stress is 260 MPa.

Solution

Given A rectangular mild steel bar as shown in Fig. 17.19

$L = 1.8$ $\quad\quad$ $\sigma_y = 260$ MPa

To find Maximum load

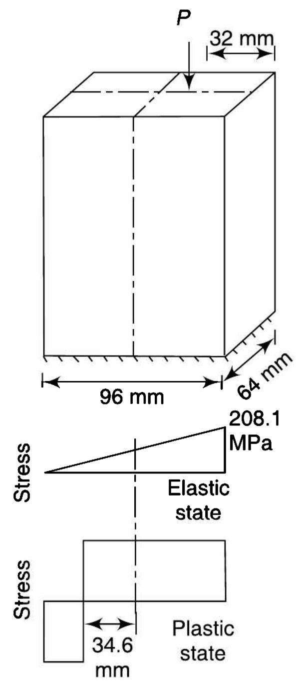


Fig. 17.19

Position of neutral axis

The bar with the loading is shown in Fig. 17.19. Load factor, $L = 1.8$

Now, Shape factor for rectangular bar, $S = 1.5$; (Eq. 17.4)

and section modulus $Z = \dfrac{bd^2}{6} = \dfrac{64 \times 96^2}{6} = 98\,304$ mm^3

$$M = \frac{SZ\sigma_y}{L} - \frac{F^2L}{4b\sigma_y} = \frac{1.5 \times 98\,304 \times 260}{1.8} - \frac{F^2 \times 1.8}{4 \times 64 \times 260}$$

$$= 21.299 \times 10^6 - 27.043 \times 10^{-6} P^2$$

As eccentricity, $e = 48 - 32 = 16$ mm; $M = F \times 16$

Thus $\quad F \times 16 = 21.299 \times 10^6 - 27.043 \times 10^{-6} P^2$

or $\quad F \times 16 \times 10^6 = 21.299 \times 10^{12} - 27.043P^2$

$$F^2 + 591\,650F - 0.7876 \times 10^{12} = 0$$

$$F = \frac{-591\,650 + \sqrt{591\,650^2 + 4 \times .7876 \times 10^{12}}}{2} = 639\,650 \text{ N}$$

The working stress,

$$\frac{F}{A} + \frac{Fe}{Z} = \frac{639\,650}{64 \times 96} + \frac{639\,650 \times 16}{98\,304} = 104.1 + 104.1 = 208.2 \text{ MPa compression}$$

and $\quad \dfrac{F}{A} - \dfrac{Fe}{Z} = 104.1 - 104.1 = 0$ tension

Thus the neutral axis or zero stress is at the smaller edge of the section.

For Plastic state, $h = \dfrac{FL}{2b\sigma_y} = \dfrac{639\,650 \times 1.8}{2 \times 64 \times 260} = 34.6$ mm from the centre of the section.

Summary

1. If a material is loaded beyond the maximum value of stress and the yield takes place, the strain does not completely disappear on removal of the load and plastic deformation takes place.
2. The ratio of M_p/M_y is known as the shape factor.
3. For a rectangular section, shape factor = 1.5.

   Moment of resistance at first yield, $M_y = (\sigma_y / \sigma_w) \cdot M_w$

   Moment of resistance in fully plastic state, $M_p = SZ\sigma_y$
4. For unsymmetrical sections, shape factor, $S = \dfrac{A(y_1 + y_2)}{2Z}$
5. The complete yielding across a section of a structure is known as *Plastic hinge* and the corresponding load required to produce this complete yielding is the *collapse load.*
6. The ratio of the *collapse load* to the *working load* is termed as the *load factor.*
7. Load factor, $L = W_c / W = S(\sigma_y / \sigma_w)$ = Shape factor × Factor of safety
8. For hollow circular shafts transmitting torque,

$$T_y = \frac{\pi\tau_y}{2r_2}(r_2^4 - r_1^4); T_p = \frac{2\pi\tau_y}{3}(r_2^3 - r_1^3); S = \frac{4r_2^3}{3r_2^3}\left[\frac{1-(r_1/r_2)^3}{1-(r_1/r_2)^4}\right]$$

9. For solid shaft, $T_y = \dfrac{\pi r_2^3 \tau_y}{2}$; $T_p = \dfrac{2\pi\, r_2^3\tau_y}{3}$; $S = 4/3$
10. Angle of twist, $\theta = \dfrac{T \cdot l}{JG}$
11. In case of eccentric loading, $M_p = SZ\sigma_y - \dfrac{F^2L^2}{4b\sigma_y}$

Objective Type Questions

1. In a rectangular beam of width b and depth h subjected to bending moment, when the yielding just commences, the moment of resistance is given by

(a) $\dfrac{bh}{6}\cdot\sigma_y$ (b) $\dfrac{bh^3}{4}\cdot\sigma_y$ (c) $\dfrac{bh^2}{4}\cdot\sigma_y$ (d) $\dfrac{bh^2}{6}\cdot\sigma_y$

2. In a rectangular beam of width b and depth h subjected to bending moment, the moment of resistance in the fully plastic state is given by

(a) $\dfrac{bh}{6}\cdot\sigma_y$ (b) $\dfrac{bh^3}{4}\cdot\sigma_y$ (c) $\dfrac{bh^2}{4}\cdot\sigma_y$ (d) $\dfrac{bh^2}{6}\cdot\sigma_y$

3. The ratio of M_p/M_y is known as

(a) elastic factor (b) plastic factor (c) shape factor (d) resistance factor

4. For a rectangular section, the value of the shape factor is

(a) 0.5 (b) 1 (c) 1.5 (d) 2

5. Load factor is

(a) Shape factor × Factor of safety (b) Shape factor/Factor of safety

(c) Load × Factor of safety (d) Load/Factor of safety

6. The ratio of the *collapse load* to the *working load* is termed as

(a) load factor (b) shape factor (c) yield factor (d) factor of safety

7. For the solid circular shafts transmitting torque, the shape factor is
(a) 1/2 (b) 2/3 (c) 4/3 (d) 3/2

8. Load factor of a built-in beam loaded with a uniformly distributed load is ______ times over the simply supported beam for the same working stress conditions.
(a) 4/5 (b) 5/4 (c) 8/9 (d) 9/8

Answers

1. (d) 2. (b) 3. (c) 4. (c) 5. (a) 6. (a)
7. (c) 8. (d)

Review Questions

17.1 What do you mean by plastic deformation of a material? Discuss the behaviour of the material when loaded beyond the elastic limit.
17.2 State the assumptions taken in the theory of plastic bending.
17.3 Define the terms: plastic hinge, collapse load and load factor.
17.4 Derive an expression for moment of resistance of a rectangular beam subjected to bending moment in the elastoplastic range. Deduce expressions for the same at the first yield and in the fully plastic state.
17.5 Find an expression to find shape factor of beams of unsymmetrical sections.
17.6 Show that for plastic bending, the load factor is the product of shape factor and factor of safety.
17.7 Derive an expression for torque of a hollow circular shaft in the elastoplastic state. Deduce expressions for the same at the first yield, in the fully plastic state and for a solid shaft.
17.8 Find an expression for the plastic moment of resistance of a column subjected to a combined direct and bending stress.

Numerical Problems

17.1 Find the shape factor for an 80 mm × 60 mm *T*-section with a uniform thickness of 10 mm throughout. (1.8)
17.2 Determine the shape factor for an 120 mm × 80 mm *I*-beam, the web and the flange thickness of which is 10 mm. (1.155)
17.3 A 140 mm × 60 mm channel with a web thickness of 6 mm and a flange thickness of 10 mm is used in pure bending with the plane of bending being perpendicular to the web of the channel. Determine the shape factor. (1.8)
17.4 A simply supported beam of rectangular steel bar of section 72 mm × 30 mm has a span of 1.2 m in such a way that the long edges are vertical. The beam is loaded at the midspan. Determine the load on the beam at the first yield if the yield stress is 280 MPa. What is the load when the yield is 12 mm at the top and bottom of the section at the midspan. Also, calculate the length of the beam over which yielding occurs. (24.2 kN; 30.9 kN; 260 mm)
17.5 A *T*-section beam of size 200 mm × 120 mm with a uniform thickness of 15 mm is loaded in such a way that the yielding occurs at the lower part of the web over a depth of 50 mm. The yield stress of 240 MPa may be assumed constant over the yielded area. Find the moment of resistance of the section. (46.66 kN · m)
17.6 The load at the end of a 1.8-m long cantilever and of *T*-section is increased so that the top of the flange just yields. Find the value of the end load. The width of the flange is 100 mm and the length of the web is 70 mm with a uniform thickness of 10 mm throughout. Take yield stress value as 220 MPa. (3.389 kN)

17.7 The internal and external diameters of a 1.75-mm long steel shaft are 25 mm and 50 mm respectively. Find the torque and the angle of twist at the first yield, when the plastic zone extends up to a depth of 5 mm and when the whole section has yield. $\tau_y = 125$ MPa and $G = 80$ GPa.

(2.88 kN · m, 0.0109 rad; 3.33 kN · m, 0.0137 rad; 3.58 kN · m, 0.0219 rad)

Chapter 18

Plane Frame Structures

A *plane frame structure* or *truss* is made up of a number of members connected to one another at several joints and is used to support external loads. The members usually are bars of angle, channel or *T*-sections. Frame members are connected to each other through gusset plates by means of rivets, bolts or welding. However, as the effect of joint rigidity is small on the member forces, for analytical purposes, the members are assumed to be connected through ideal frictionless hinged joints so that only axial forces are assumed to occur. The analysis of frames is based on the equations of equilibrium and accordingly systematic methods have been developed. Several configurations of the frames are possible which mainly depend upon the span, loading, material, economy and from practical considerations. Frames are mainly used for roof systems and bridges.

18.1 PERFECT FRAMES

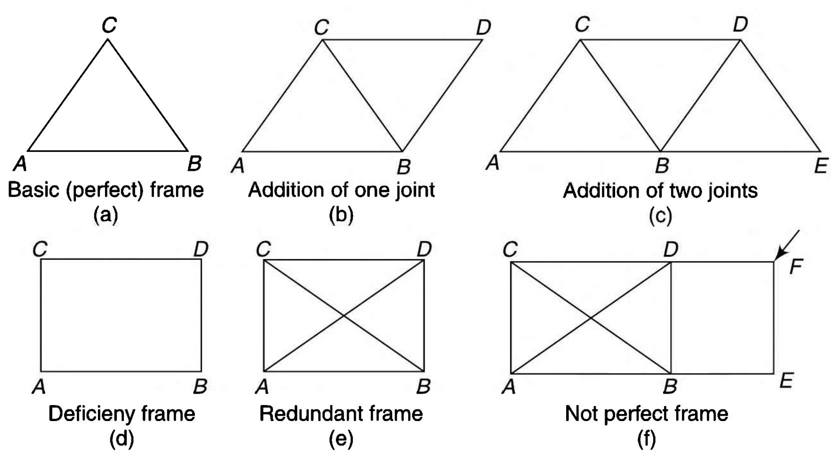


Fig. 18.1

The main function of a frame is to transmit loads to the supports. A basic configuration of frame is a triangle consisting of three members joined together (Fig. 18.1a). It is a shape, the profile of which cannot be altered without changing the dimensions of its sides and is stable under any loading conditions. Such a frame is known as a *perfect frame*. Additional joint in a frame can be obtained by adding two more bars. Figure 18.1b shows the addition of two more bars *BD* and *CD* to the basic frame *ABC* to obtain one more joint. In a similar way, more number of joints can be added (Fig. 18.1c). The relationship between the number of members and the number of joints can be expressed as

$$m = 2j - 3$$

where m and j are the number of members and joints respectively in a perfect frame.

Now consider the frame of Fig. 18.1d. This frame does not satisfy the above equation and thus is not a perfect frame. It is also not stable as its configuration can be altered. Such a frame in which the number of members is less than that required for a perfect frame is known as *deficient frame*. Figure 18.1e shows a frame in which the number of members is more than the required number of members in a perfect frame. Such a frame is known as *redundant frame*. Such frames are stable but not statically determinate.

It is to be noted that the equation for a perfect frame mentioned above is necessary but not a sufficient condition of a perfect frame. Figure 18.1f shows a frame which has nine members and six joints thus satisfying the equation for a perfect frame. However, this frame is not a perfect frame as it will not be able to retain its shape if loaded at F.

18.2 REACTIONS AT THE SUPPORTS

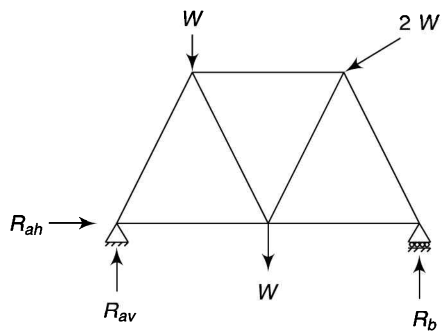


Fig. 18.2

A frame may have a roller or hinged support. At a roller support, the translation is possible along the roller base and the reaction is normal to the roller base (Fig. 18.2). In a hinged support, no translation is possible and the direction and the line of action depend upon the load system on structure. At such a support, there are horizontal and vertical components of the reaction for general system of loading.

18.3 STATICALLY DETERMINATE FRAMES

At each joint in a plane frame, two independent equations of static equilibrium can be formed along the two directions of force components. Thus, in all $2j$ number of equations of equilibrium are obtained in a frame having j number of joints. For a general system of loading of frame, there are three support reaction components in all, one at the roller support and two at the hinged support. Also, force in each member is unknown. Thus there are $m + 3$ unknown forces including reactions at the supports if m is the number of members in a frame. A frame will be *statically determinate* if the number of equations is equal to the number of unknowns, i.e., if the relation $m + 3 = 2j$ is satisfied for a frame, it is *statically determinate*. As the equation of a perfect frame is also the same, it may be said that perfect frames are also *statically determinate* frames. The frames in which the number of equations is not equal to the number of unknowns will be a *statically indeterminate frame*.

18.4 ASSUMPTIONS IN THE ANALYSIS OF FRAMES

The following assumptions are made while analysing frames and trusses:

1. The frames are perfect and statically determinate.
2. All members of the frame lie in one plane.
3. The frame work is made up of pinned-joints and the external loads are applied only at the joints. This makes the members to carry axial forces only.
4. The weight of the members is negligible as compared to the external forces.
5. The frame work is made up of rigid members and there is no geometrical distortion under the action of external forces.
6. The conditions of static equilibrium are applicable to each member of the frame.

18.5 SIGN CONVENTION

The axial forces in the members of a frame can either be tensile of compressive. Figure 18.3a shows a member under the action of tensile forces at each end. The internal reactive forces act in the opposite directions. Thus a member in tension is shown by marking arrows which point towards each other (Fig.18.3b). Similarly, a member in compression is shown by arrows pointing towards the joints indicating the internal reactive forces to resist the external compressive forces (Fig.18.3c and d).

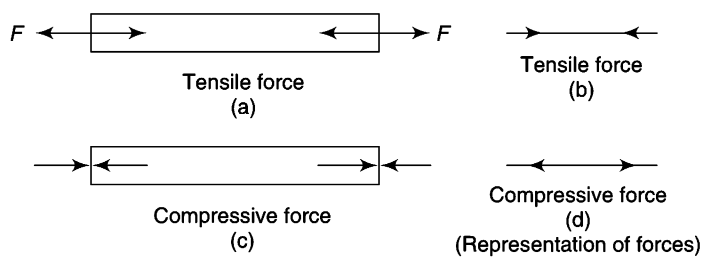


Fig. 18.3

While writing the equations of equilibrium at a joint of a frame, assume the force in the unknown member to be tensile, i.e., in a direction away from the connecting joint. In case, the force is found to be negative, it can be taken as compressive. The force is denoted by F_{ij}, the subscripts i and j indicating the connected joints of the member. The subscripts can be written in any manner, i.e., $F_{ij} = F_{ji}$.

18.6 METHODS OF ANALYSIS

There are several methods of analysis of plane perfect frames. However, here only two analytical methods will be discussed:

1. Method of joints
2. Method of sections

   Graphical methods can be referred in any book on graphic statics.

18.7 METHOD OF JOINTS

Method of joints is useful when it is desired to know the forces in all the members of a frame. It is based on the principle of static equilibrium. The procedure is as follows:

1. Determine the reactions at the supports.

2. Consider each joint for the static equilibrium assuming the joint to be a free body under the action of external forces acting on the body.
3. Calculate the unknown forces at a joint from the conditions of static equilibrium, i.e., $\Sigma F_v = 0$ and $\Sigma F_H = 0$ where ΣF_v and ΣF_H are the sum of the vertical and horizontal components of the forces acting at the joint respectively. Since only two unknowns can be calculated from two equations, maximum number of unknowns at a joint should not be more than two.
4. After determining the force in a member, the same should be marked in the diagram before considering the forces at the next joint.
   While analysing, note that:
   - If two forces are acting at a joint, they must be equal and opposite along the same line of action.
   - If three forces are acting on a joint and two of them are perpendicular to the third, then the force in the third member is zero.
   - While considering the equilibrium along any direction, the forces in the members which are perpendicular to that direction are not considered as their components along the direction under consideration are zero.
   - Sometimes it is convenient to consider the equilibrium of forces along the direction of a member instead of x- or y-directions.

Example 18.1 Determine the forces in all the members of the frame shown in Fig. 18.4. *D* and *E* are the mid-points of *AC* and *BC* respectively.

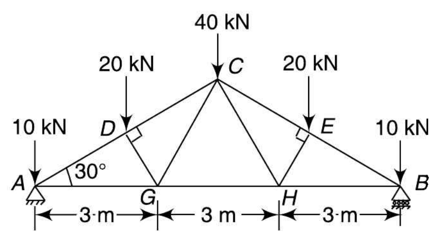


Fig. 18.4

Solution

Given A frame as shown in Fig. 18.4.
To find Forces in all the members

By symmetry, each reaction at A and B is $\dfrac{10 + 20 + 40 + 20 + 10}{2} = 50$ kN

Right angled triangles ADG and CDG are congruent as $AD = CD$ and DG is common. Thus $\angle DCG = \angle DAG = 30°$ and $CG = AG = 3$ m.

The horizontal reaction at A, $R_{ah} = 0$ as there is no horizontal component of the external loads.

Joint A

Consider free body of the joint (Fig. 18.5a). Assuming the force in the unknown member AD as tensile, $\Sigma F_V = 0$, (horizontal force in the member AG is not to be considered)

Thus $F_{AD} \sin 30° + 50 - 10 = 0$ or $F_{AD} = -80$ kN,

Negative sign indicates that the force in the member AD is compressive.

For $\Sigma F_H = 0$, (Vertical forces are not to be considered). F_{AD} is already known.

$\therefore\ F_{AG} - F_{AD} \cos 30° = 0$ or $F_{AG} - 80 \cos 30° = 0$ or $F_{AG} = 69.3$ kN (tensile)

Mark the direction of forces in the members AG and AD (Fig. 18.5b)

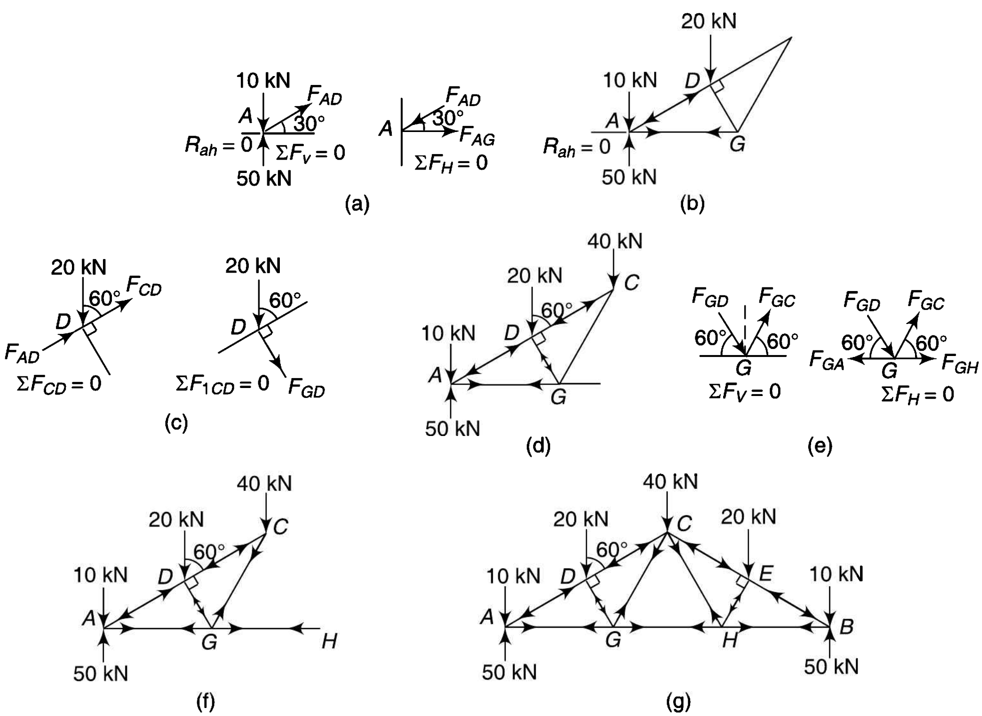


Fig. 18.5

Joint D

Refer Fig. 18.5c. Resolving the forces along AD instead of x- and y-directions as in that case there will be two unknowns in each equation and the two simultaneous equations will have to be solved.

Thus $F_{CD} + F_{CD} - 20 \cos 60° = 0$ or $F_{CD} + 80 - 20 \cos 60° = 0$ or $F_{CD} = -70$ kN

i.e., force in the member CD is compressive.

Resolving the forces $\perp$ AC, $F_{GD} + 20 \sin 60° = 0$ or $F_{GD} = -17.3$ kN (compressive)

Mark the direction of forces in the members DC and DG (Fig. 18.5d).

Joint G

Considering $\Sigma F_V = 0$, (Fig. 18.5e)

$$F_{GC} \sin 60° - F_{GD} \sin 60° = 0 \text{ or } F_{GC} = F_{GD} = 17.3 \text{ kN (tensile).}$$

Considering $\Sigma F_H = 0$, $F_{GH} + F_{GC} \cos 60° + F_{GD} \cos 60° - F_{GA} = 0$

or $$F_{GH} + 17.3 \cos 60° + 17.3 \cos 60° - 69.3 = 0 \text{ or } F_{GH} = 52 \text{ kN (tensile).}$$

Mark the direction of forces in the members GC and GH (Fig. 18.5f).

The forces in the members on the right half may be written from the symmetry of the figure. Fig. 18.5g indicates the type of forces in all the members.

Example 18.2 || Determine the forces in all the members of the frame shown in Figure 18.6. D and E are the mid-points of AC and BC respectively.

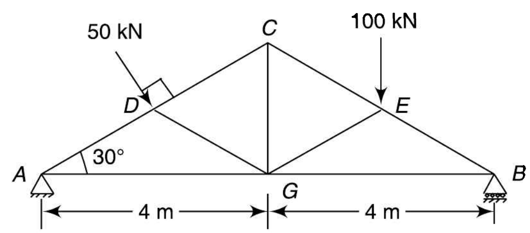


Fig. 18.6

Solution

Given A frame as shown in Fig. 18.6.
To find Forces in all the members

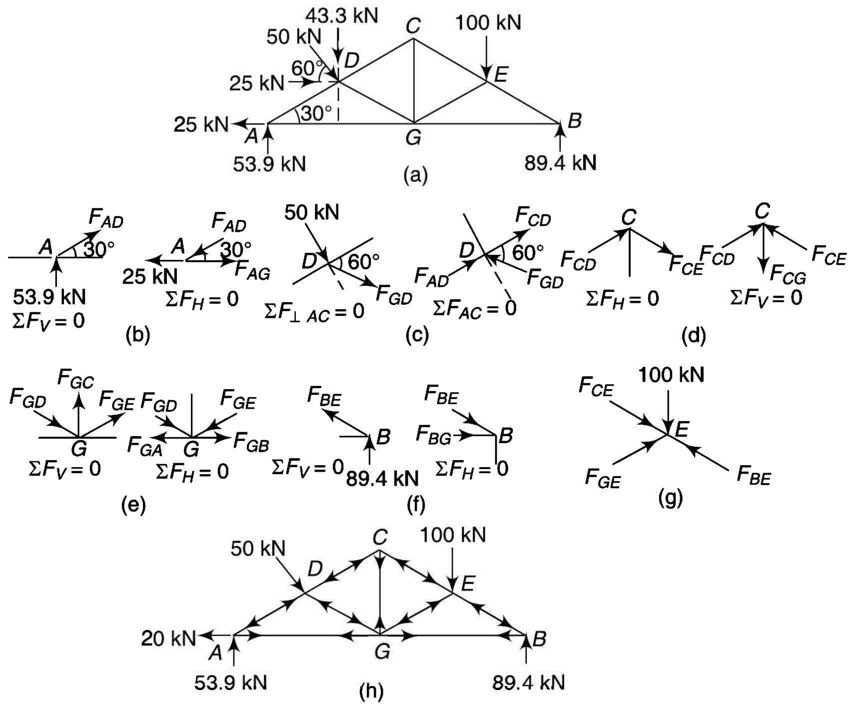


Fig. 18.7

Horizontal and vertical components of load at D (Fig.18.7a),

$$R_{dv} = 50 \sin 60° = 43.3 \text{ kN and } R_{dh} = 50 \cos 60° = 25 \text{ kN}$$

To find reactions at the supports, take moments about A,

$$R_b \times 8 = 100 \times 6 + 43.3 \times 2 + 25 \times 2 \tan 30° \text{ or } R_b = 89.4 \text{ kN}$$

$$R_{av} = 100 + 43.3 - 89.4 = 53.9 \text{ kN and } R_{ah} = 25 \text{ kN}$$

Joint A

Refer Fig. 18.7b,

Considering $\Sigma F_V = 0$, $53.9 + F_{AD} \sin 30° = 0$ or $F_{AD} = -107.8$ kN (compressive).

Considering $\Sigma F_H = 0$, $F_{AG} - F_{AD} \cos 30° - 25 = 0$

or $\quad -107.8 \cos 30° - 25 + F_{AG} = 0$ or $F_{AG} = 118.4$ kN (tensile).

Joint D

Resolving the forces $\perp$ AC (Fig. 18.7c),

$$F_{GD} \cdot \sin 60° + 50 = 0 \text{ or } F_{GD} = -57.7 \text{ kN (compressive).}$$

Resolving the forces along AC, $F_{AD} + F_{CD} - F_{GD} \cos 60° = 0$

or $\quad 107.8 + F_{CD} - 57.7 \cos 60° = 0$ or $F_{CD} = -79$ kN (compressive)

Joint C

Considering $\Sigma F_H = 0$ (Fig. 18.7d),

$$F_{CD} \cos 30° + F_{CE} \cos 30° = 0$$

or $F_{CE} = -F_{CD} = -79$ kN (compressive).

Considering $\Sigma F_V = 0$, $F_{CG} - F_{CD} \sin 30° - F_{CE} \sin 30° = 0$

or $F_{CG} - 79 \sin 30° - 79 \sin 30° = 0$ or $F_{CG} = 79$ kN (tensile)

Joint G

Refer Fig. 18.7e,

Considering $\Sigma F_V = 0$, $F_{GC} - F_{GD} \cdot \sin 30° + F_{GE} \cdot \sin 30° = 0$

or $79 - 57.7 \sin 30° + F_{GE} \cdot \sin 30° = 0$, $F_{GE} = -100.3$ kN (compressive)

Considering $\Sigma F_H = 0$, $F_{GB} + F_{GD} \cdot \cos 30° - F_{GA} - F_{GE} \cdot \cos 30° = 0$

$F_{GB} + 57.7 \cos 30° - 118.4 - 100.3 \cos 30° = 0$ or $F_{GB} = 155.3$ kN (tensile)

Joint B

Considering $\Sigma F_V = 0$ (Fig. 18.7f),

$F_{BE} \sin 30° + 89.4 = 0$ or $F_{BE} = -178.8$ kN (compressive)

Considering $\Sigma F_H = 0$, $F_{BE} \cos 30° - F_{BG} = 0$

or $178.8 \cos 30° - 155.3 \approx 0$ which verifies the values.

Verification at Joint E

Considering $\Sigma F_V = 0$ (Fig. 18.7g),

$F_{GE} \cos 60° + F_{BE} \cos 60° = 100 + F_{CE} \cdot \cos 60°$

or $100.3 \cos 60° + 178.8 \cos 60° \approx 100 + 79 \cos 60°$ which verifies the values.

Considering $\Sigma F_H = 0$, $F_{GE} \sin 60° + F_{CE} \sin 60° = F_{BE} \cdot \sin 60°$

or $100.3 + 79 \approx 178.8$ which verifies the values.

Fig. 18.7(h) shows the whole frame indicating the type of forces in all the members.

Example 18.3 Determine the forces in all the members of the frame shown in Fig. 18.8.

Solution

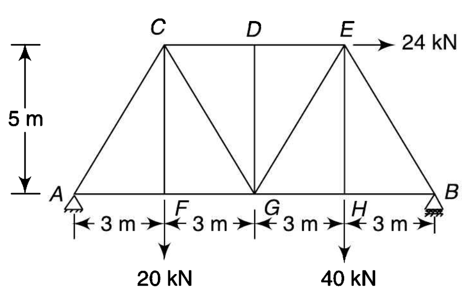


Fig. 18.8

Given A frame as shown in Fig. 18.8.
To find Forces in all the members

To find the reaction at support B, take moments about A,

$R_b \times 12 = 24 \times 5 + 20 \times 3 + 40 \times 9$ or $R_b = 45$ kN

$R_{av} = 40 + 20 - 45 = 15$ kN and $R_{ah} = 24$ kN

$\angle FAC = \tan - (5 / 3) = 59°$

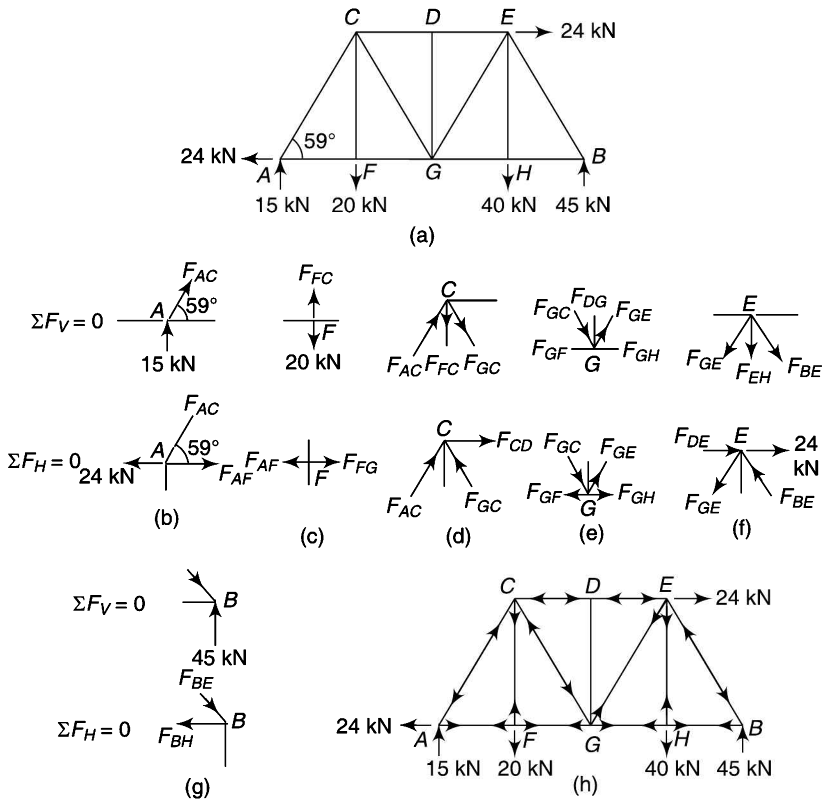


Fig. 18.9

Figure 18.9a shows the reactions at the supports.

Joint A

Refer Fig. 18.9b

Considering $\Sigma F_V = 0$, $15 + F_{AC} \sin 59° = 0$ or $F_{AC} = -17.5$ kN (compressive)

Considering $\Sigma F_H = 0$, $F_{AF} - F_{AC} \cos 59° - 24 = 0$ or $F_{AF} - 17.5 \cos 59° - 24 = 0$

or $\quad F_{AF} = 33$ kN (tensile)

Joint F

Considering $\Sigma F_V = 0$ (Fig. 18.9c),

$$F_{FC} = 20 \text{ kN (tensile)}$$

Considering $\Sigma F_H = 0$, $F_{FG} = F_{AF} = 33$ kN (tensile)

Joint C

Considering $\Sigma F_V = 0$ (Fig. 18.9d)

$$F_{GC} \sin 59° + F_{FC} - F_{AC} \sin 59° = 0$$

or $\quad F_{GC} \sin 59° + 20 - 17.5 \sin 59° = 0$, $F_{GC} = -5.83$ kN (compressive)

Considering $\Sigma F_H = 0$, $17.5 \cos 59° - 5.83 \cos 59° + F_{CD} = 0$ or $F_{CD} = -6$ kN compressive

Joint D

Considering $\Sigma F_V = 0$, $F_{DG} = 0$

Considering $\Sigma F_H = 0$, $F_{DE} = F_{CD} = 6$ kN compressive

Joint G

Considering $\Sigma F_V = 0$ (Fig. 18.9e)

$$F_{GE} \sin 59° - F_{GC} \sin 59° = 0 \text{ or } F_{GE} = 5.83 \text{ kN tensile ..... } (F_{DG} = 0)$$

Considering $\Sigma F_H = 0$, $F_{GH} - F_{GF} + F_{GC} \cos 59° + F_{GE} \cos 59° = 0$

or $$F_{GH} - 33 + 5.83 \cos 59° + 5.83 \cos 59° = 0 \text{ or } F_{GH} = 27 \text{ kN tensile}$$

Joint H

Considering $\Sigma F_V = 0$, $F_{HE} = 40$ kN tensile

Considering $\Sigma F_H = 0$, $F_{HB} = F_{GH} = 27$ kN tensile

Joint E

Considering $\Sigma F_V = 0$ (Fig. 18.9f)

$$F_{BE} \sin 59° + F_{GE} \sin 59° + F_{HE} = 0$$

or $$F_{BE} \sin 59° + 5.83 \sin 59° + 40 = 0 \text{ or } F_{BE} = -52.5 \text{ kN (compressive)}$$

Considering $\Sigma F_H = 0$,

$$-5.83 \cos 59° - 52.5 \cos 59° + 6 + 24 = 0 \text{ which verifies the result.}$$

Joint B

Considering $\Sigma F_V = 0$ (Fig. 18.9g)

$$52.5 \sin 59° - 45 = 0 \text{ which verifies the result}$$

Considering $\Sigma F_H = 0$, $52.5 \cos 59° - 27 = 0$ which verifies the result.

Figure 18.9(h) shows the whole frame indicating the type of forces in all the members.

18.8 METHOD OF SECTIONS

The method of sections also known as the *method of moments* is useful when forces in a limited number of members are required to be found. This method does not require determining the stresses in most of the other members to find the stress in a particular member. An imaginary section is passed through a number of members including the one in which the stress is to be found in such a way that the frame is divided into two parts. Then the equations of static equilibrium can be applied to any one part to determine the stress in the required member.

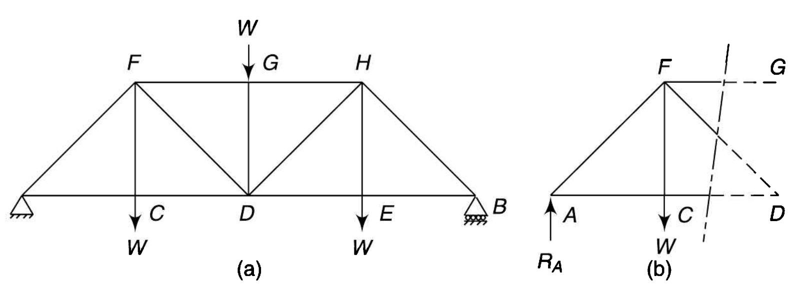


Fig. 18.10

Consider a frame as shown in Fig. 18.10a. Now, if it is required to find the stress in the member *FD* of the frame, a section can be passed through *FD* across the depth of the frame. This divides the frame into two parts. The left-hand cut portion of the frame will be as shown in Fig. 18.10b. As such this portion of the frame will collapse under the external force at *C* and the reaction at *A*. However, if forces are exerted by right-hand portion through the members *FG*, *FD* and *CD*, the left-hand portion can remain in static equilibrium.

Thus, the left-hand portion will be in static equilibrium under the action of external forces at C and A and the internal forces F_{FG}, F_{FD} and F_{CD}. The equations of equilibrium, i.e., $\Sigma F_v = 0$, $\Sigma F_H = 0$ and $\Sigma M = 0$ can be applied to find the force in the desired member. The moments are taken about such a point that only one unknown force is there. For example, to find the force in member CD, moments should be taken about point F. This will eliminate the unknown forces F_{FG} and F_{FD}. Similarly, to find force F_{FG} moments should be taken about D to eliminate forces F_{FD} and F_{CD}. To find the force F_{FD}, moments can be taken about A provided the force F_{FG} is known beforehand. Alternatively, all the forces can be resolved vertically on the left hand portion. The forces will be vertical component of force F_{FD} and the forces at A and C.

Example 18.4 || Determine the forces in all the members of frame shown in Fig. 18.11 by method of sections.

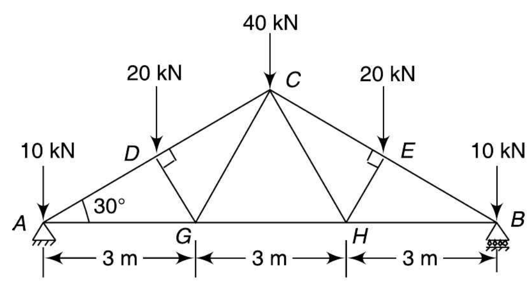


Fig. 18.11

Solution

Given A frame as shown in Fig. 18.11.
To find Forces in all the members

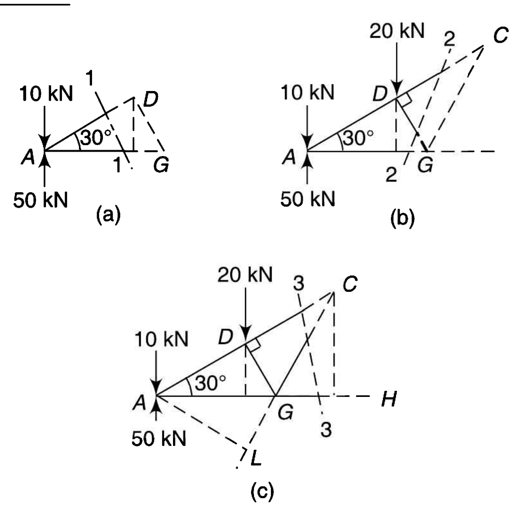


Fig. 18.12

The frame is the same as of Example 18.1. Reactions at each support can be found in the same way. Thus R_a = 50 kN.

Forces in AD and AG

Pass a section 1-1 through members AD and AG (Fig. 18.12a).

$$GD = 3 \sin 30° = 1.5 \text{ m and } AD = 3 \cos 30° = 2.6 \text{ m}$$

For free-body diagram of the left portion,
Taking moments about G,

$$F_{AD} \cdot GD + 40 \times 3 = 0 \text{ or } F_{AD} \times 1.5 = -120$$

or $$F_{AD} = -80 \text{ kN (compressive)}$$

Taking moments about D,

$$F_{AG} \cdot AD \sin 30° - (50 - 40)\, AD \cos 30° = 0$$

or $$F_{AG} = 69.3 \text{ kN (tensile)}$$

Forces in DC and DG

Pass a section 2-2 through members DC, DG and AG (Fig. 18.12b). Considering the free body diagram of the left portion and taking moments about G,

$$F_{DC} \cdot GD + 40 \times 3 - 20GD \cos 60° = 0$$

or $$F_{DC} \times 1.5 + 40 \times 3 - 20 \times 1.5 \cos 60° = 0$$

or $$F_{DC} = 70 \text{ kN, i.e., compressive}$$

Taking moments about A,

$$F_{GD} \cdot AD = 20\, AD \cos 30°$$

or $$F_{GD} = 17.3 \text{ kN (compressive)}$$

Forces in GC and GH

Pass a section 3-3 through members CD, CG and GH (Fig. 18.12c). For free-body diagram of the left portion, taking moments about A,

$$F_{GC} \cdot AL = 20\, AD \cos 30° \text{ or } F_{GC} \times AD = 20 \times AD \cos 30°$$

or $$F_{GC} = 17.3 \text{ kN (tensile)}$$

($AL = AD$, as triangles ALG and ADG are congruent)
Taking moment about C,

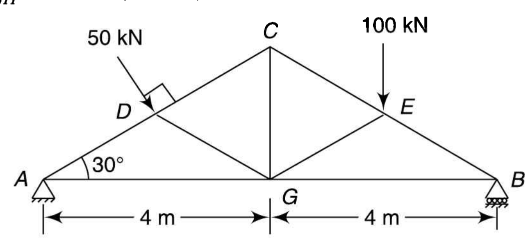
$$F_{GH} \cdot AC \sin 30° = 40\, AC \cos 30° - 20\, DC \cos 30°$$

or $$F_{GH} \cdot 2DC \sin 30° = 40 \times 2DC \cos 30° - 20DC \cos 30°$$

or $$F_{GH} = 40 \times 2 \cos 30° - 20 \cos 30° \text{ or } F_{GH} = 52 \text{ kN (tensile)}$$

Forces in the right half can be written from symmetry.

Example 18.5 || Determine the forces in the members CE, GE and GB of the frame shown in Fig. 18.13 by method of sections.



(a)
Fig. 18.13

Solution

Given A frame as shown in Fig. 18.13.
To find Forces in members CE, GE and GB.

The frame is the same as of Example 18.2. Reactions at each support can be found in the same way. Thus R_b = 89.4 kN.

Determination of forces

Pass a section through members CE, GE and GB as shown in Fig. 18.14. Consider the right hand portion for equilibrium. Taking moments about G,

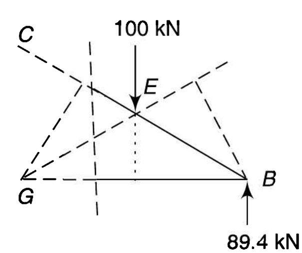


Fig. 18.14

$$100 \times 2 - 89.4 \times 4 + F_{CE} \times 4 \sin 30° = 0$$

or $$F_{CE} = 78.8 \text{ kN (compressive)}$$

Taking moments about B,

$$F_{GE} \times 4 \sin 30° = 100 \times 2 \text{ or } F_{GE} = 100 \text{ kN} \qquad \text{(compressive)}$$

Taking moments about E,

$$F_{GB} \times 2 \tan 30° = 89.4 \times 2 \text{ or } F_{GB} = 154.8 \text{ kN} \qquad \text{(tensile)}$$

Example 18.6 Determine the forces in the members *DE*, *GH* and *GE* of frame shown in Fig. 18.15 by method of sections.

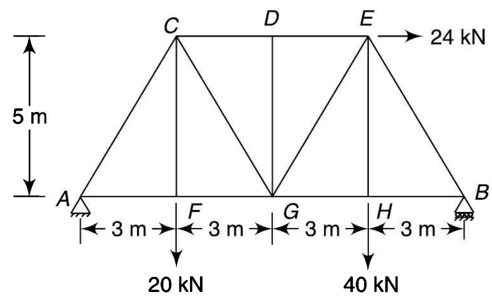


Fig. 18.15

Solution

Given A frame as shown in Fig. 18.15.
To find Forces in members *DE, GH* and *GE.*

The frame is the same as of Example 18.3. Reactions at each support can be found in the same way. Thus $R_b = 45$ kN.

Forces in DE and GH

To find the force in the member *DE*, *GH* and *GE*, take a section as shown in Fig. 18.16a. Taking moments about G,

$$F_{DE} \times 5 - 40 \times 3 + 45 \times 6 - 24 \times 5 = 0$$

or $\quad F_{DE} = 6$ kN (compressive)

To find the force in the member *GH*, Taking moments about E,

$$F_{GH} \times 5 + 45 \times 3 = 0$$

or $\quad F_{GH} = -27$ kN (compressive)

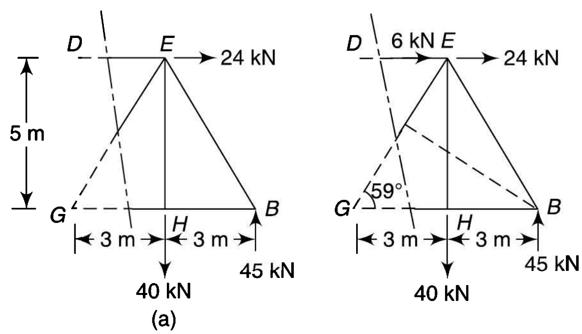


Fig. 18.16

Forces in GE

To find the force in the member *GE*, Taking moments about B (Fig. 18.16b),

$$F_{GE} \times 6 \sin 59° + 24 \times 5 - 40 \times 3 + F_{DE} \times 5 = 0 \qquad .....(\angle HGE = \tan^{-}(5/3) = 59°)$$

or $\quad F_{GE} \times 6 \sin 59° + 24 \times 5 - 40 \times 3 + 6 \times 5 = 0$ or $F_{GE} = 5.83$ kN (tensile)

- Alternatively, resolve all the forces acting on the right hand side of the section vertically (assuming the force in the unknown member *GE* as tensile force i.e. downwards),

i.e. $\quad -F_{GE} \sin 59° + 45 - 40 = 0$

or $\quad F_{GE} = 5.83$ kN

Example 18.7 Determine the value of *W* which produces a force of 280 kN in the member *AB* of the frame shown in Fig. 18.17. Also, find the forces in the members *GH* and *AG*.

Solution

Given A cantilever frame as shown in Fig. 18.17.
To find
- Force *W*
- Forces in *GH* and *AG*

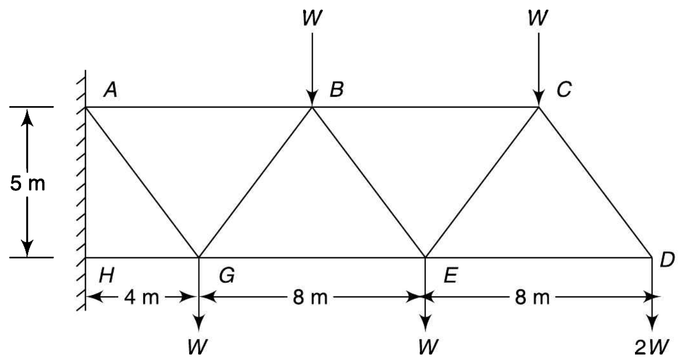


Fig. 18.17

Force W

In case of cantilevers, usually, it is not necessary to find the reactions at the supports. Pass a section through the members AB, AG and GH as shown in Fig. 18.18. Take moments about G (The force in the member AB must be counter-clockwise as all the moments due to loads are to be clockwise indicating the moments in the member AB to be tensile),

$$4W + 8W + 12W + 16 \times 2W = 280 \times 5 \text{ or } W = 25 \text{ kN (tensile)}$$

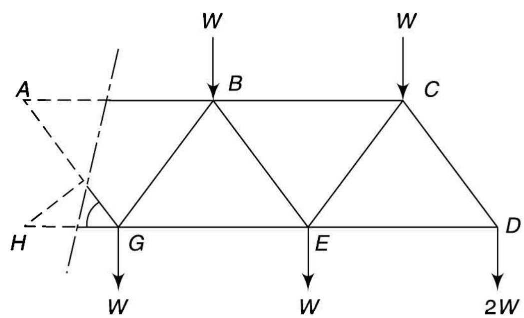


Fig. 18.18

Forces in GH and AG

To find the force in the member GH, Take moments about A,

$$(4 + 8 + 12 + 16 + 20 \times 2) \times 25 = F_{GH} \times 5 \text{ or } F_{GH} = 400 \text{ kN (compressive)}$$

To find the force in the member AG, Take moments about H,

$$(\angle AGH = \cos^{-}(4/5) = 36.9°)$$

$$(4 + 8 + 12 + 16 + 20 \times 2)W - F_{AB} \times 5 = F_{AG} \times 4 \sin 36.9°$$

$$(4 + 8 + 12 + 16 + 20 \times 2) \times 25 - 280 \times 5 = F_{AG} \times 4 \sin 36.9°$$

or $\qquad F_{AG} = 249.8$ kN tensile.

- Alternatively, resolve all the forces acting on the right hand side of the section vertically,

  i.e. $\qquad F_{AG} \sin 36.9° = 6 \times 25$

  or $\qquad F_{AG} = 249.8$ kN

Summary

1. A plane frame structure consists of a number of members connected to one another at several joints and is used to support external loads.
2. Frames are mainly used for roof systems and bridges.
3. The analysis of frames is based on the equations of equilibrium.
4. Frames whose profiles cannot be changed and are stable under any load conditions are known as *perfect frames*.

5. A frame in which the number of members is less than that required for a perfect frame is known as *deficient frame.*
6. A frame in which the number of members is more than that required for a perfect frame is known as *redundant frame.*
7. The frames which can be analysed using the equations of static equilibrium, i.e., $\Sigma F_V = 0$, $\Sigma F_H = 0$ and $\Sigma M = 0$, are known as *statically determinate* frames.
8. Two analytical methods to analyse the frames are the method of joints and the method of sections.
9. The method of joints is useful when it is desired to know the forces in all the members of a frame.
10. The method of sections, also known as the *method of moments,* is useful when forces in a limited number of members are required to be found.
11. In the method of joints, each joint is considered as a free body in equilibrium and equations of static equilibrium are applied at each joint to determine the forces in the members.
12. In the method of sections, a section is passed through members to divide the frame into two parts. Then equations of static equilibrium, $\Sigma F_v = 0$, $\Sigma F_H = 0$ and $\Sigma \mathrm{M} = 0$ can be applied to any part.

Objective Type Questions

1. A plane frame structure consists of
(a) rectangular spaces (b) triangular spaces
(c) rectangular and triangular spaces (d) any shapes

2. A plane frame structure is a system having
(a) riveted joints (b) welded joints (c) rigid joints (d) pin joints

3. A triangle is the geometric figure for a plane frame structure because it is
(a) strong (b) rigid (c) flexible (d) none of these

4. Hinged joints are used in frames to ensure ______ forces in the members.
(a) axial (b) transverse
(c) inclined (d) horizontal and vertical

5. The relationship between the number of members and the number of joints of a perfect frame can be expressed as
(a) $m = 2j - 3$ (b) $j = 2m - 3$ (c) $m = 3j - 2$ (d) $2m = j - 3$

6. A frame in which the number of members is less than that required for a perfect frame is known as a
(a) redundant frame (b) deficient frame (c) plane frame (d) complex frame

7. A frame in which the number of members is more than that required for a perfect frame is known as a
(a) redundant frame (b) deficient frame (c) plane frame (d) complex frame

8. Method of joints is preferred in the analysis of plane frames if forces are required to be determined in
(a) one member only (b) two members only (c) a few members (d) in all the members

9. Method of sections is preferred in the analysis of plane frames if forces are required to be determined in
(a) one member only (b) two members only
(c) in a few members (d) in all the members

10. In a plane frame, the forces in collinear members at a joint are equal if the joint is
(a) loaded (b) not loaded
(c) loaded and has three members only (d) not loaded and has three members only

Answers

1. (b) 2. (d) 3. (b) 4. (a) 5. (a) 6. (b)
7. (a) 8. (d) 9. (c) 10. (d)

Review Questions

18.1 What is meant by plane frame structures? What are perfect frames?
18.2 What do you mean by deficient and redundant frames?
18.3 What are *statically determinate* and *statically indeterminate frames?*
18.4 Enumerate various assumptions made in the analysis of frames and trusses.
18.5 What are the analytical methods available to find the forces in the members of a frame?
18.6 Explain the method of joints to analyse a frame and its limitations.
18.7 How is the method of sections useful in determining the forces in members of a frame? What are its limitations?

Numerical Problems

18.1 Determine the forces in all the members of the frame shown in Fig. 18.19.

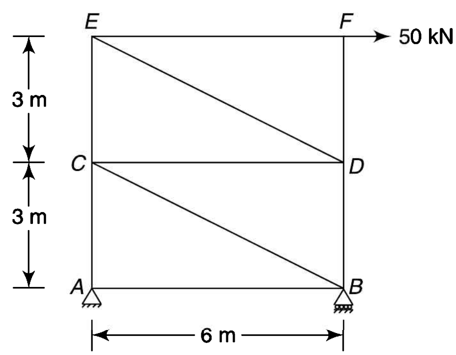


Fig. 18.19

Answer

Member	*AB*	*BC*	*CD*	*DE*	*EF*	*AC*	*CE*	*BD*	*DF*
Force (kN)	50 T	55.9 C	50 T	55.9 C	50 T	50 T	25 T	25 C	0

18.2 Determine the forces in all the members of the frames shown in Fig. 18.20.
(Hint: Resolve the force at joint *G* along *GD*)

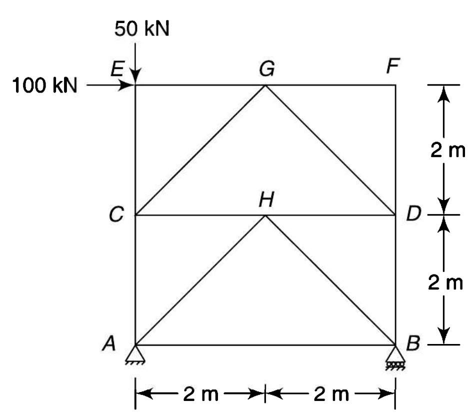


Fig. 18.20

Answer

Member	*AB*	*BD*	*DF*	*FG*	*GE*	*CE*	*AC*	*AH*	*BH*	*CH*	*DH*	*CG*	*DG*
Force (kN)	50 T	50 C	0	0	100 C	50 C	0	70.7 T	70.7 C	50 C	50 T	70.7 T	70.7 C

18.3 Analyse the frame shown in Fig. 18.21 to determine the forces in all the members of the frame.

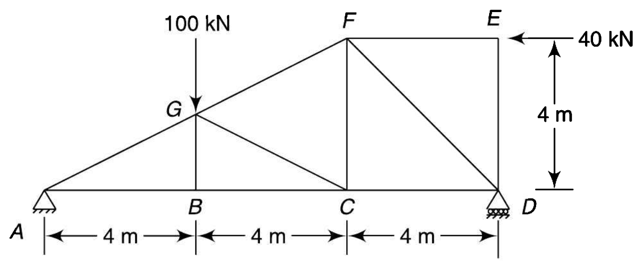


Fig. 18.21

Answer

Member	*AB*	*BC*	*CD*	*DE*	*EF*	*FG*	*AG*	*DF*	*CF*	*CG*	*BG*
Force (kN)	120 T	120 T	20 T	0	40 C	67 C	178.9 C	28.28 C	50 T	111.8 C	0

18.4 Determine the forces in all the members of the frame shown in Fig. 18.22 by method of sections.

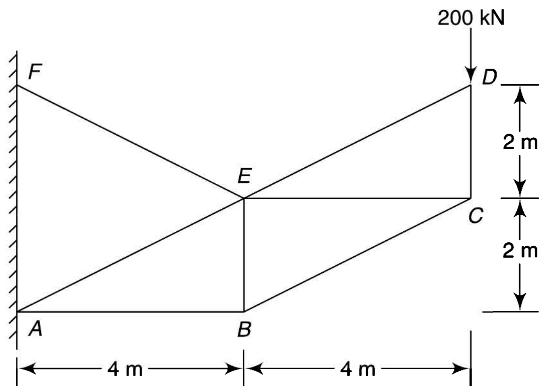


Fig. 18.22

Answer

Member	*AB*	*BC*	*CD*	*DE*	*EF*	*AE*	*BE*	*CE*
Force (kN)	400 C	447.2 C	200 C	0	447.2 T	0	200 T	400 T

18.5 Determine the forces in all the members of the frame shown in Fig. 18.23 using any method of analysis.

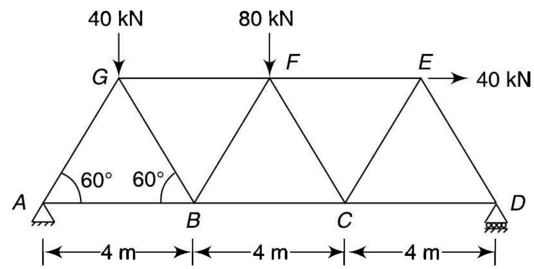


Fig.18.23

Answer

Member	*AB*	*BC*	*CD*	*DE*	*EF*	*FG*	*AG*	*BG*	*BF*	*CF*	*CE*
Force (kN)	75.7 T	100.9 T	33.6 T	67.2 C	27.2 C	48.3 C	71.4 C	25.2 T	25.2 C	67.2 C	67.2 T

Chapter 19

Properties and Testing of Materials

The primary objective of study of strength of materials is to provide an engineer with tools of analysing and designing components of various machines and load-bearing structures. To determine the suitability of a material under the given loading conditions requires the knowledge of various types of stresses incurring on the components as well as the mechanical properties of the materials. No material can possess all the desired properties in one material. Depending upon the requirements, sometimes a compromise may have to be made while choosing a particular material for a part of a machine. This chapter describes some of the properties most commonly desired and the way of testing the materials to determine these properties.

19.1 MECHANICAL PROPERTIES

The main properties of materials are as follows:

1. Ductility It is the property of a material by virtue of which it can be drawn into wires under the action of tensile force. A ductile material must have a high degree of plasticity and strength so that large deformations can take place without failure or rupture of the material. In ductile extension, a material exhibits a certain amount of elasticity along with a high degree of plasticity.

2. Brittleness It is opposite to ductility, i.e., when a material cannot be drawn out by tension to smaller sections. A brittle material fails instantly under the load without exhibiting any significant deformation. Examples of brittle materials are cast iron, concrete, glass, stone, etc.

3. Malleability This property of a material allows it to expand in all directions without rupture. A malleable material has to be highly plastic, though it may not possess high strength. This property is of great use in processes such as forging, hot rolling, etc.

4. Hardness The resistance of a material to indentation including scratching or surface abrasion is termed as hardness.

5. Strength It may be defined as the capability of a material to withstand load. It is obtained by dividing the load by the area. The *ultimate strength* of a material is the load required to cause fracture divided by the area of the test specimen.

6. Toughness It is the capacity of a structure to withstand an impact load, i.e., the capacity to absorb energy without fracture. It depends upon the ductility of a material and its ultimate strength. *Toughness* is represented by the area under the stress–strain diagram and is the energy per unit volume required to cause the material to rupture. This property is highly desirable in components subject to shock or cyclic loading. A general test for toughness is the *bend test* in which the capability of a material is tested for angular bending.

7. Fatigue When loadings are repeated thousands or millions of times, rupture occurs at a stress much below the static breaking strength. This phenomenon is known as fatigue. Consideration of fatigue is an integral part of design if the structural or machine components are subjected to fluctuating or repeated loads.

8. Creep If the stress in a material exceeds the yield point, the strain caused in the material by the application of load does not disappear totally on the removal of load. The plastic deformation caused to the material is known as *creep*. At high temperatures, the strain due to creep is quite appreciable.

19.2 FACTOR OF SAFETY

It is defined as the ratio of the ultimate load to the allowable or working load. The factor of safety for a component or structural part has to be chosen very carefully. If chosen too small, the possibility of failure becomes dangerously large and if chosen very large, the design can become uneconomical. Generally, the choice of factor of safety for a component is influenced by the following considerations:

- ***Material Properties*** Material used for a component may not be homogeneous, i.e., it may not have the same properties throughout. The composition, strength and its dimensions may have small variations during its manufacture. The properties are also altered when a material undergoes various heat treatments.
- ***Type of Loadings*** A designer may keep in mind the type of loading (static, impact, reversal, etc.) a structure is going to withstand; yet there are possibilities of inaccuracy in his estimates. There can also be alterations of use of a structure with time.
- ***Number of Loadings*** The actual number of loadings during the life of a component may be different from the expected loading at the time of designing. The ultimate stress usually decreases with increase in the number of loadings. This phenomenon is known as *fatigue* and plays a key role in designing.
- ***Type of Expected Failure*** Sometimes, the collapse of a structure may occur suddenly and there is no prior indication as in case of brittle materials. In such cases where there is possibility of sudden failure, a higher factor of safety has to be used than where there is possibility of pre-warning in the form of yielding of the material.
- ***Method of Analysis*** As all relations and analysing methods are based on certain assumptions, it is appropriate to include some factor of safety for such type of assumptions and simplifications.
- ***Maintenance*** All mechanical components and structures require good maintenance during their lifetime. However, to take into account poor maintenance, rusting, corrosion, decay and other natural uncertainties in future, some margin has to be included.

19.3 TENSILE TESTING

The behaviour of a ductile material (such as steel), on being subjected to an increased tensile load was described in Section 1.15. It is observed that in the elastic range, the stress is proportional to strain and the material regains its original size on removal of the load. As the load is increased, the plastic range sets in and the material yields. At this point, the test specimen begins to form a *neck* at some position along its length till the specimen fractures. Salient features of the test are discussed below:

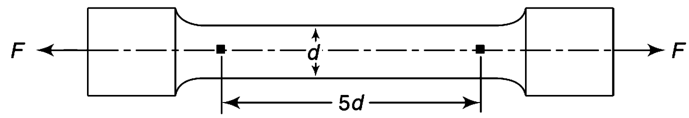
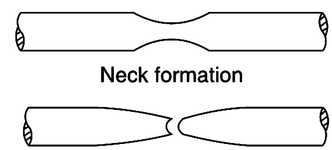


Fig. 19.1

- The cross-section of the specimen for the test can be circular, square or rectangular. At each end, the cross-section is enlarged for a suitable length to have a proper grip. The ends may be screwed into the grips or may be made roughened to have tight grips in the wedges. The grips should be self-centering to ensure that the load is applied axially and evenly on the specimen. A test specimen of circular cross-section is shown in Fig. 19.1. Usually, the shape and dimensions are standardised. The standard gauge length is taken as $5.65\sqrt{a}$ where a is the cross-sectional area of the specimen. For a circular section, the length becomes $5d$ where d is the diameter of the circular section.
- In the elastic range, the strain has to be measured by *extensometers* which are able to measure very small changes in length. In the plastic range, the strains are larger and can be measured by pairs of dividers and scale rule.
- Beyond the yield point, an appreciable increase in strain with a relatively small increase in the applied load is observed. The elongation can be even 200 times than that before the yield point. This deformation is found to be due to slippage of the material along oblique surfaces and thus is mainly due to shear stresses. It is seen that the fracture occurs along a cone-shaped surface with an approximate angle of 45° with the original surface of the specimen and the ends of the failed specimen form a cup and a cone as shown in Fig. 19.2. This indicates that the shear is mainly responsible for the failure of ductile materials. This also confirms that under axial loads, shear stresses are maximum at angles of 45° with the load.



Neck formation

Fracture

Fig. 19.2

- After the yield, the stress remains constant over a large range of values of the strain. Later, the stress has to be increased to elongate the specimen till the ultimate value is reached. This is due to a property of the steel known as *strain hardening*.
- Some alloys of steel and light alloys of aluminium and magnesium do not indicate well defined yield points. In such cases, the stress–strain diagram may be curved even at the origin. The yield point for such materials is determined from the criterion of *proof stress* which is defined as the stress inducing a residual strain of approximately 0.2% in a material. To find the proof stress of such a material, a line parallel to the tangent at the origin from a 0.2% strain on the x-axis is drawn and the intersection of this line with the stress–strain curve gives the value of the proof stress. The procedure is shown in Fig. 19.3.

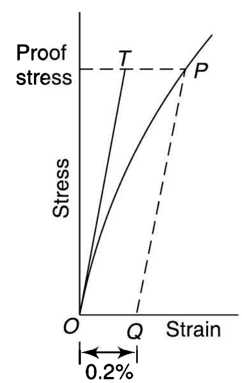
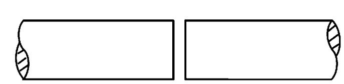


Fig. 19.3

- As the ductility of a material is the ability of the material to be drawn into thin wires and the material can sustain large strains at fracture; the larger the plastic range, the greater will be the ductility of a material. Usually, the elastic range of a material is much smaller as compared to its plastic range (mild steel has an elastic range of 0.12% and a plastic range of 35%); therefore, ductility of a material may be expressed as the percentage elongation at fracture. It may be noted that the



Fracture of a brittle specimen

Fig. 19.4

ductility of materials reduces with increase in strength. For example, mild steel, high strength deformed steel and high carbon steels can sustain strains upto 35%, 18% and 10% respectively whereas there ultimate strengths are 410 MPa, 550 MPa and 1000 MPa in that order.

- Materials which have less than 5% elongation at fracture on a gauge length of 100 are classified as brittle materials, e.g., cast iron, stone and glass, etc. In such types of materials, the fracture occurs without any noticeable change in the rate of elongation which implies that there is no difference between the ultimate strength and the rupture strength. Figure 19.4 shows the rupture of a brittle material. It can be noted that there is no necking of the specimen and the rupture occurs along a surface perpendicular to the load. This indicates that the shear strength of brittle materials is more than half of their tensile strength.

19.4 COMPRESSION TESTING

Compression testing of materials is similar to tensile testing. However, the following problems are experienced during the compression testing:

- It is difficult to apply truly axial loads; therefore, bending stresses due to buckling effect accompanies the compressive stresses.
- To avoid the buckling of the test specimen, the length has to be kept small which makes taking of measurements difficult.
- For a ductile material like mild steel, lateral expansion takes place on applying the compressive force. But due to restraining effect of friction at the end surfaces, the maximum enlargement of the cross-section is at the centre. This affects the results.

Ductile materials do not rupture under compressive loads as such. As the load increases, the material starts enlarging in cross-section, maximum at the centre. When the load is increased excessively, some cracks appear on the surface which slowly spreads inwards. For brittle materials, the shear strength is less than half of the compressive strength and thus the shear plane angle should be at 45°. However, due to friction effects, brittle materials generally fail by shearing along the planes inclined at 50° to 60° to the longitudinal axis (Fig. 19.5).

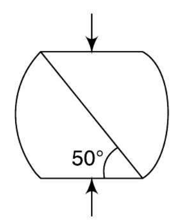


Fig. 19.5

19.5 TORSION TESTING

Torsion testing is carried out to determine the modulus of rigidity of a given material. From Eq. 10.1, $\theta = Tl / JG$, which indicates that within the elastic range, the angle of twist θ is proportional to the applied torque T. This provides a convenient method for finding the modulus of rigidity of a material from plots of T vs. θ. A specimen of the material of the shape of a cylindrical rod of known diameter and length is fixed in a torsion testing machine and torques of increasing magnitude are applied to the specimen. Values of the corresponding angle of twist for the length l are then noted. The points will be lying on a straight line on the T vs. θ plot till the yield stress of the material is not reached. The slope of this line represents the quantity JG/l from which G may be calculated if J and l are known. A typical torsion test piece is shown in Fig. 19.6. The ends of the specimen are flattened to have a firm grip in the testing machine and to avoid slipping during twisting.

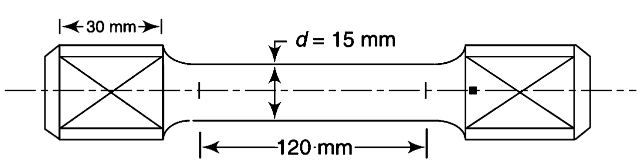


Fig.19.6

In case a shaft consists of several portions of different cross-sections and of different materials, it can be divided into component parts and then the above relation will be applicable to each part. Thus, in case of a shaft with multiple cross-sections, the rotation of one end relative to the other is obtained by adding algebraically the angles of twist of each part, i.e., $\theta = \Sigma TL / JG$

19.6 HARDNESS TESTING

Hardness represents the resistance of a material to indentation, penetration and scratching. In hardness testing, a loaded ball or diamond is pressed against the surface of a material which causes the plastic deformation of the same. This deformation is measured by one of the following methods:

- ***Brinell Method*** In this method, a steel hardened ball is pressed into the surface of the material under a specified load (Fig. 19.7). The load is held in position for a fixed period and then released. This leaves a permanent impression in the surface of the material. Then either the diameter or the depth of the impression is measured. The *Brinell Hardness Number* (BHN) is defined as the ratio of the applied load to the spherical area of the impression. Conversion tables are also available to determine the hardness number.

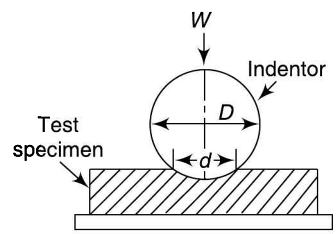


Fig. 19.7

While testing the specimen, ensure that the test specimen is smooth and the test spot is free from oil or dirt. Apply the load slowly and progressively at right angle to the surface. The full load must be maintained for a fixed period. Take care to have the ratio of the diameter of impression to the diameter of ball between 0.3 and 0.5. The centre of indentation from the edge of the specimen should not be less than 2.5 times the diameter of indentation and the distance between the centres of two indentations should be at least 4 times the diameter of indentation. The diameter of indentation is measured in two perpendicular directions and then the mean is taken.

The Brinell method is useful with hardness numbers below 500. Above this value, the readings are appreciably affected by the distortion of the steel ball used for indentation. The thickness of the test piece for this test should be at least 10 times the depth of impression.

The Brinell hardness of metals increases with the tensile strength.

- ***Vicker Pyramid Diamond Method*** This method is also similar to the Brinell method except that the indenter is a 136° pyramid diamond on a square base (Fig.19.8). As hardness of diamond is excessively high, it can be used for the whole range of materials. A linear relationship is found to exist between the depth of impression and the hardness number. The *Vicker Pyramid Number* (VPN) is defined as the ratio of applied load to the impressed area. The area is calculated by measuring the length of the diagonal of the square impression on the surface of the material. The thickness of the test piece should be at least 1.5 times the diagonal of the impression (d) for this test. The centre of indentation from the edge of the test piece should be at least 2.5 d. The same is also to be the distance between two indentations.

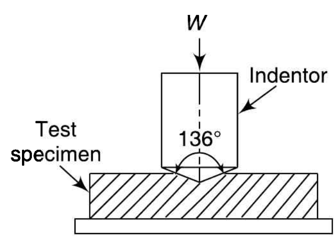


Fig. 19.8

- ***Rockwell Hardness Method*** It is similar to the above methods. In this method, depth of impression is used as the criterion of hardness.

19.7 IMPACT TESTING

Static tests are useful only when the loads are static in nature. These tests do not indicate the resistance of a material against shock or impact loads to which usually the automobile parts are subjected to. In such cases, an impact test has to be undertaken. An impact test indicates the toughness of a material which is defined as the energy absorbed by the specimen without fracture. As the energy absorbed by a specimen during its failure is the product of its deformation and the average stress, a highly stressed and greatly deformed material is able to withstand a high blow and is said to have more toughness. In an impact test, a notched specimen of the material is fractured by a single blow of a heavy hammer. The energy used in fracturing the specimen is the measure of resistance to impact. The following are the main types of impact tests undertaken:

Izod Impact Test Figure 19.9a shows an Izod impact testing machine. It consists of an anvil in which a notched specimen can be fixed. The specimen is taken of some standard dimensions. While fixing, care is to be taken to have the notch on the side of the falling hammer and the level with the level of top face of the hammer (Fig. 19.9b). The hammer is released from a fixed position. It strikes the specimen and after breaking it continues for some distance on the other side. Usually, the scale has zero along the vertical line and graduated on both sides. If the hammer is released from one side from some known angle, it will swing to the other side through the same angle in the absence of any obstruction in the way if the bearing friction is ignored. But as the specimen is kept in the path of the hammer, some energy is absorbed by the hammer during its breaking; the angle through which the hammer will reach the other side will be lesser. The difference between the two angles indicates the measure of energy absorbed by the specimen. By means of a calibrated pointer that moves over a scale, the energy absorbed in fracturing the test piece is noted down.

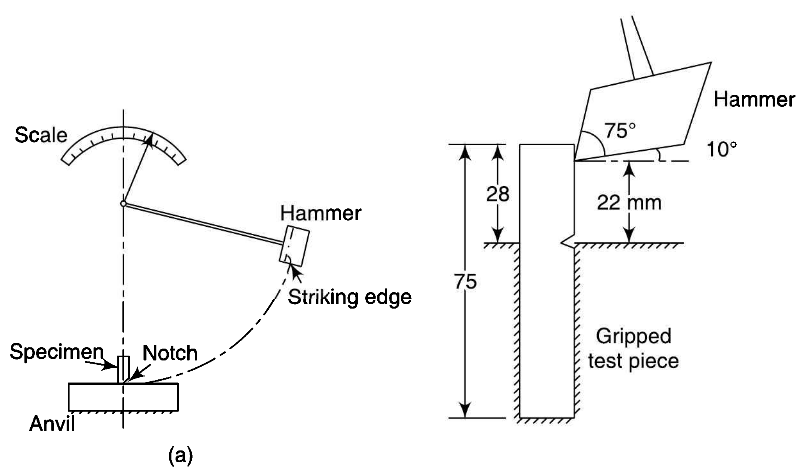


Fig. 19.9

Charpy Impact Test This test is similar to the Izod impact test except that instead of fixing the notched specimen in the anvil, it is supported at each end as a beam as shown in Fig. 19.10. The hammer strikes at notch in the centre.

Impact tests are important as they can reveal the temper brittleness in heat treated materials such as nickel chrome steels. As stress concentration conditions prevail at the notch which can lead to cracks, the impact test also provides the information regarding resistance to stress concentration.

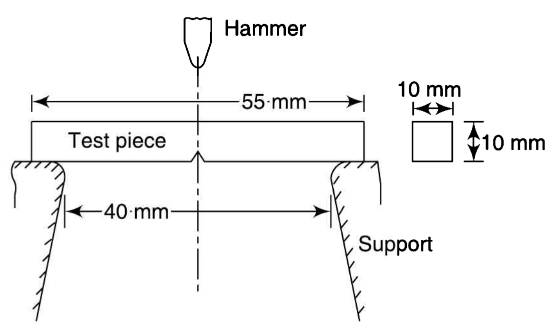


Fig. 19.10

19.8 COLUMN TESTING

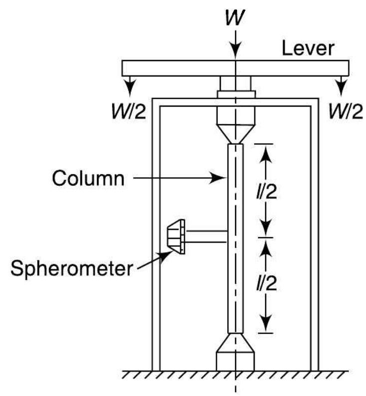


Fig. 19.11

Column testing is carried out to determine the buckling load on a column. Figure 19.11 shows the set up of a column testing machine. It consists of a lever to apply the load on the column. The provision of different types of end supports, e.g., fixed or hinged may be available and used accordingly. The diameter of the column is taken between 6 and 12 mm and the length is 100 mm 120 mm. The deflection at the centre of the column is determined by a spherometer. The initial reading is taken by touching the column at the mid length. A lighting system ensures the proper contact. While loading the lever, care is taken to apply equal loads on both sides. At a particular loading, the deflection of the column can be determined by turning the spindle forward and rotating the spherometer base in horizontal plane. On the lighting of the lamp, reading may be taken. Difference of this reading and the previous reading provides the deflection of the column for the applied load. After each reading, the spherometer spindle is taken back. A number of readings can be taken by increasing the load on the column till the deflection becomes excessive. A graph of load vs. deflection can be plotted from which buckling load can be found. The result can be compared with empirical relations of *Euler* or *Rankine*.

19.9 CREEP TESTING

Creep is the property by which, at elevated temperatures, metals continue to elongate continuously under a constant stress. The stress may be much less than the ultimate tensile stress. Creep is measured by the rate of strain per hour at a certain temperature under a given stress. When a tensile test is carried out in a shorter period of time on a metal specimen at some specific temperature, a definite value of ultimate tensile stress can be obtained. However, under creep a material can also fail even at lower stress values but at higher temperatures if sufficient time is allowed to pass. It is also observed that there is limiting stress value below which creep does not take place at a certain temperature and the material does not fracture even if the stress is applied for indefinite time. Thus, it becomes necessary that the materials which are exposed to high temperatures for longer periods at certain stress values, the design must be based on limiting creep stress.

To determine the limiting creep stress requires very long periods which may be months or even years for investigations. It is also observed that the rate of creep diminishes with time. Thus to have a fair value of limiting creep stress at a certain temperature, effort is made to find the stress at which, after a shorter period of time, a definite very small rate of creep takes place. Usually, a typical stress value is obtained which causes a creep of 1 millionth per hour after 40 days.

Some creep resistant alloy steels have been developed by adding small percentages of molybdenum, cobalt, vanadium and tungsten for high-temperature applications such as gas turbines and high-pressure steam fittings.

19.10 FATIGUE TESTING

In previous sections, it was said that if the maximum stress in a material does not exceed its elastic limit, it regains its initial condition on removal of the load. From this, it may be concluded that if the stresses in a material remain within the elastic range, the given loading may be repeated several times. Of course, this conclusion is correct if the loading is repeated for up to a few hundred times. However, it is observed that this is not correct if the repetition of loading is done thousands or million of times. In situations when a machine part is subjected to a varying or fluctuating stresses, the failure of the part will take place at much lower stress than the static stress. This is known as *fatigue* failure. This indicates that fatigue must be considered in the design of all machine components which are subjected to repeated or fluctuating loads. For example, the stress in the members of a bridge is due to its weight only when there is no traffic. But as the traffic passes, the stress level will increase and then again lowering to its previous value after that. Another case can be of the axle of an automobile in which there is complete reversal of stress after each half revolution of the crank.

A fatigue failure is always of a brittle nature even though the materials are ductile. Usually, the failure under fatigue is initiated at a microscopic crack or at some imperfection. Then at each loading, it is enlarged very slightly. At one stage, the undamaged material becomes insufficient to bear the maximum load and an abrupt brittle failure takes place. While designing, the number of loading cycles expected during the useful life of a component has to be ascertained. For example, if a crane beam has 200 loadings per working day, it will be loaded more than one million times in 20 years. An automobile crankshaft may be loaded about a million times if driven for 600 000 km.

Let σ_v = variable stress i.e. the alternating or variable component of stress.

σ_{max} = maximum stress

σ_{min} = minimum stress (treating compressive stress as negative)

σ_m = mean value of stress

Then

$$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2} \quad \text{and} \quad \sigma_v = \frac{\sigma_{max} - \sigma_{min}}{2}$$

- If σ_{max} and σ_{min} both are positive (or negative), it is known as *fluctuating stress* (Fig. 19.12a).
- If σ_{min} is zero, the stress ranges from zero to σ_{max} value. It is known as *repeated stress* (Fig. 19.12b).
- If σ_m is zero, the maximum and the minimum stress values are equal, but of opposite sign. It is known as *reversed stress* (Fig. 19.12c).

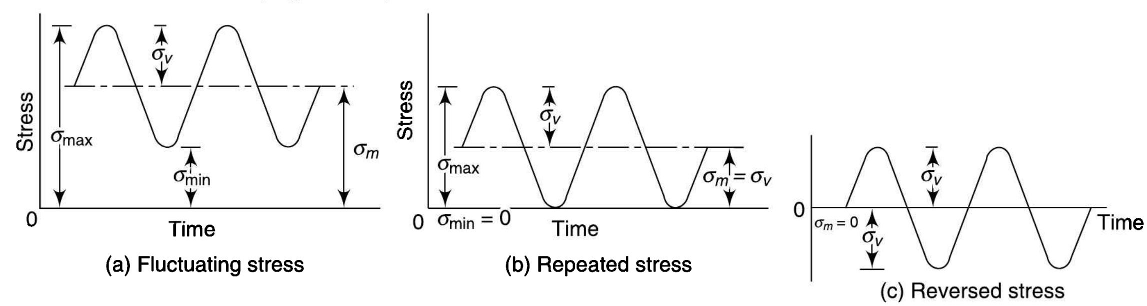


Fig. 19.12

The following relations proposed by Gerber and Goodman are quite useful in the calculation work (Fig. 19.13):

- Gerber relation, $\dfrac{1}{FS} = \left(\dfrac{\sigma_m}{\sigma_u}\right)^2 \cdot FS + \dfrac{\sigma_v}{\sigma_e}$

where σ_u = ultimate stress

σ_e = endurance limit

FS = factor of safety

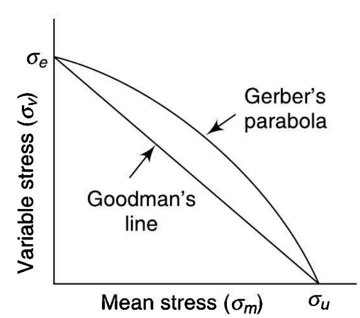


Fig. 19.13

- Goodman relation, $\dfrac{1}{FS} = \dfrac{\sigma_m}{\sigma_u} + \dfrac{\sigma_v}{\sigma_e}$

To find the fatigue limit, i.e., the number of cycles (n) necessary for the failure of a specimen for any given maximum stress level, a series of tests have to be conducted. The resulting data so obtained is then plotted in the form of σ vs. n curve. Owing to large number of cycles required for rupture, number of cycles is plotted on a logarithmic scale along the x-axis and stress σ along the y-axis as shown in Fig. 19.14. It can be observed that as the range of stress decreases, the number of cycles of stress necessary to fracture the specimen increase until a stress known as the fatigue limit is reached.

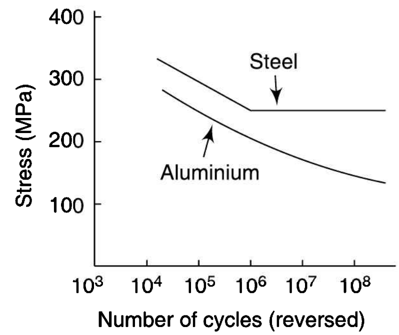


Fig. 19.14

Endurance or Fatigue Limit For a given mean stress, it is the limiting range of stress below which no fracture takes place for an indefinite number of cycles. It is found to be around 25 per cent higher in reversed bending and about 45 per cent lower in reversed torsion than in reversed tension or compression. For a low-carbon steel like structural steel, it is about one-half of the ultimate strength of steel. For non-ferrous metals like copper and aluminium, it is observed that the stress continues to decrease as the number of cycles increase. In such cases, from practical considerations, endurance limit is arbitrarily defined as the stress corresponding to 500 million cycles.

Fatigue strength depends upon the following:

- *Surface condition of specimen* The fatigue limit for smooth and polished surfaces is higher as compared to rough and corroded surfaces.
- *Stress concentration* Fatigue failures are found to initiate at cracks at the points in the stress concentration region.
- *Surface treatment* Surface hardening is found to resist the formation of fatigue cracks and thus improve the fatigue strength.

Summary

1. Ductility is the property of a material by virtue of which it can be drawn into wires under the action of tensile force.
2. Brittle materials cannot be drawn out to smaller sections by tension.
3. Malleability is the property of a material by which it can be expanded in all directions without rupture.
4. Resistance of a material to indentation is due to its property of hardness.
5. Strength is the capability of a material to withstand the load.
6. The capacity of a structure to withstand an impact load is due to its property of toughness.
7. Fatigue is the phenomenon due to which rupture of a material occurs at a stress much below the static breaking strength if loadings are repeated thousands or millions of times.
8. Creep is the property by which, at elevated temperatures, metals continue to elongate continuously under a constant stress.
9. Factor of safety is the ratio of the ultimate load to the allowable or working load.

Objective Type Questions

1. The strength of a material is its ability to resist
   (a) fracture due to impact loads (b) deformation under stress
   (c) external forces without yielding (d) none of these
2. A ductile material can be drawn into
   (a) sheets (b) wires (c) none of these
3. A brittle material can be drawn into
   (a) sheets (b) wires (c) none of these
4. Malleability is the property of a material which allows it to expand in ______ directions without rupture.
   (a) one (b) two (c) all
5. Hardness is the resistance to
   (a) indentation (b) scratching (c) abrasion (d) all of these
6. The capacity of a structure to withstand an impact load is due to its property of
   (a) ductility (b) toughness (c) rigidity (d) strength
7. Endurance limit is the stress at which failure
   (a) takes place for a given number of cycles
   (b) does not take place for a given number of cycles
   (c) does not take place for any number of cycles
8. Work done on a unit volume of material on increasing the tensile force gradually from zero to the rupture value is called
   (a) modulus of toughness (b) modulus of resilience
   (c) modulus of endurance
9. Brittle materials generally fail by shearing along the planes inclined at ______ to the longitudinal axis.
   (a) 30° to 40° (b) 50° to 60° (c) 70° to 80° (d) 90°

Answers

1. (c) 2. (b) 3. (c) 4. (c) 5. (d) 6. (b)
7. (c) 8. (a) 9. (b)

Review Questions

19.1 Enumerate the mechanical properties of materials. What is their significance?
19.2 What do you mean by the term *factor of safety*? What are the considerations which influence its value?
19.3 State the manner in which ductile and brittle materials fail when subjected to tensile testing.
19.4 What is *proof stress*? How is it useful in defining the yield point of some materials?
19.5 What problems are experienced during the compression testing?
19.6 What are the methods available for hardness testing of a material? Compare them.
19.7 What is *Izod* impact test? How is it different from *Charpy* impact test?
19.8 Discuss the salient features of fatigue testing of a material.

Appendix

Important Relations and Results

1. Elongation of a bar, $\Delta = \dfrac{PL}{AE}$
2. Temperature stress in bar, $\sigma = \alpha\, t\, E = \alpha\, t\, \sigma/\varepsilon$
3. Net strain in the direction of σ_1, $\quad \varepsilon_1 = \sigma_1/E - v\sigma_2/E - v\sigma_3/E$
4. Relation between elastic constants, $E = 2G(1 + v) = 3K(1 - 2\,v) = \dfrac{9KG}{3K + G}$
5. Normal stress on an inclined plane = $\sigma \cos^2 \theta$
6. Shear stress on an inclined plane = $\frac{1}{2}\sigma \sin 2\theta$
7. Strain energy stored in a bar = $\dfrac{P^2L}{2AE} = \dfrac{\sigma^2}{2E} \times$ volume = $\dfrac{1}{2} \times$ stress $\times$ strain $\times$ volume
8. Shear strain energy per unit volume = $\dfrac{\tau^2}{2G}$
9. Maximum bending moments in standard cases:
   - In a cantilever with a point load at free end = Wl
   - In a cantilever with a uniformly distributed load = $\dfrac{wl^2}{2} = \dfrac{Wl}{2}$
   - In a simply supported beam with a point load at mid-span = $\dfrac{Wl}{4}$
   - In a simply supported beam with a point load W at a distance a from one end of the span $l = \dfrac{Wa(l - a)}{l}$
   - In a simply supported beam carrying a uniformly distributed load = $\dfrac{wl^2}{8} = \dfrac{Wl}{8}$
10. The moment of inertia of a rectangular lamina of sides b and d about centroidal axis parallel to side $b = \dfrac{bd^3}{12}$

11. The moment of inertia of circular lamina $= \dfrac{\pi d^4}{64}$
12. The relation governing the simple bending of beam is $\dfrac{\sigma}{y} = \dfrac{M}{I} = \dfrac{E}{R}$
13. The diameter of kern of a circular cross-section of diameter d is $d/4$.
14. Shear centre of a semi-circular arc is at $4r/\pi$
15. The flexural rigidity of a beam is EI
16. Deflection and slope of standard cases:

    Cantilevers:

    - A point load at the free-end: Deflection of free end $= \dfrac{Wl^3}{3EI}$ (maximum)

      Slope of free end $= \dfrac{Wl^2}{2EI}$ (maximum)

      A point load at the mid-span, deflection of free end $= \dfrac{5Wl^3}{48EI}$
    - A uniformly distributed load, deflection of free end $= \dfrac{wl^4}{8EI}$ (maximum)

      Slope of free end $= \dfrac{wl^3}{6EI}$ (maximum)

    Simply supported beam:

    - Central point load, Deflection at mid-span $= \dfrac{Wl^3}{48EI}$ (maximum)

      Slope at mid-span = Zero

      Slope at the support $= \dfrac{Wl^2}{16EI}$ (maximum)
    - Uniformly distributed load, Deflection at mid-span $= \dfrac{5Wl^4}{384EI}$ (maximum)

      Slope at the supports $= \dfrac{wl^3}{24EI}$

    Fixed beam:

    - Central point load, Maximum deflection $= \dfrac{Wl^3}{192EI}$
    - Uniformly distributed load, Maximum deflection $= \dfrac{wl^4}{384EI}$
17. The relation governing the torsional torque in circular shafts is $\dfrac{T}{J} = \dfrac{\tau}{r} = \dfrac{G\theta}{l}$
18. Torsional rigidity of a shaft $= GJ = Tl/\theta$
19. Maximum shear stress of a solid shaft is given by $\dfrac{16T}{\pi d^3}$

20. Deflection of a closely coiled helical spring under axial load = $\dfrac{32WR^2l}{G\pi d^4} = \dfrac{8WD^3n}{Gd^4}$
21. Shear stress in a closed-coiled helical spring under an axial load = $\dfrac{8WD}{\pi d^3}$
22. The angle of twist of a closely coiled helical spring = $\dfrac{64TDn}{Ed^4}$
23. The equivalent stiffness of two springs joined in series, $s = \dfrac{s_1 s_2}{s_1 + s_2}$
24. The equivalent stiffness of two springs joined in parallel, $s = s_1 = s_2$
25. The equivalent lengths of columns for different types of end conditions:
    - both ends hinged, $l_e = l$
    - one end fixed and the other free, $l_e = 2l$
    - both ends fixed, $l_e = l/2$
    - one end fixed, other hinged, $l_e = l/\sqrt{2}$
26. Euler crippling load for columns with both ends hinged = $\dfrac{\pi^2 EI}{l^2}$
27. In a thin cylinder, hoop stress = $\dfrac{pd}{2t}$; longitudinal stress = $\dfrac{pd}{4t}$
28. In a thin spherical shell, hoop stress = $\dfrac{pd}{4t}$
29. The volumetric strain in a thin spherical shell = $\dfrac{3pd}{4tE}(1 - \nu)$
30. Hoop stress induced in a rotating ring, $\sigma_\theta = \rho \cdot r^2\omega^2 = \rho \cdot v^2$
31. In a solid rotating disc, at the centre of the disc, $\sigma_r = \sigma_\theta = \dfrac{3+\nu}{8}\rho\omega^2 R^2$

    At the outer surface, $\sigma_\theta = \dfrac{1-\nu}{4}\rho\omega^2 R^2$, $\sigma_r = 0$
32. In a hollow rotating disc,
    - Radial stresses are zero at inside and outside radii.
    - σ_r is maximum at $r = \sqrt{R_i R_o}$ and is $\dfrac{3+\nu}{8}\rho\omega^2(R_o - R_i)^2$
    - σ_θ is maximum at inner radius and is $\dfrac{\rho\omega^2}{4}[(1-\nu)R_i^2 + (3+\nu)R_o^2]$
    - At outer radius, $\sigma_\theta = \dfrac{\rho\omega^2}{4}[(3+\nu)R_i^2 - (1-\nu)R_o^2]$
33. In a long rotating solid cylinder, the radial stress at the centre = $\dfrac{(3-2\nu)}{8(1-\nu)}\rho\omega^2 R^2$
34. In plastic bending:
    - Moment of resistance at first yield, $M_y = (\sigma_y/\sigma_w) \cdot M_w$
    - Moment of resistance in fully plastic state, $M_p = SZ\sigma_y$
    - Load factor, $L = W_c/W = S(\sigma_y/\sigma_c)$ = Shape factor × Factor of safety

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