2.3. Hermite interpolation (continuation)

The Hermite interpolation formula is

$$f = H_n f + R_n f,$$

where $R_n f$ denotes the remainder term (the error).

Theorem 1 *If $f \in C^n[\alpha, \beta]$ and $f^{(n)}$ is derivable on (α, β), with $\alpha = \min\{x, x_0, ..., x_m\}$ and $\beta = \max\{x, x_0, ..., x_m\}$, then there exists $\xi \in (\alpha, \beta)$ such that*

$$(R_n f)(x) = \frac{u(x)}{(n+1)!} f^{(n+1)}(\xi). \qquad (1)$$

Proof. Consider

$$F(z) = \begin{vmatrix} u(z) & (R_n f)(z) \\ u(x) & (R_n f)(x) \end{vmatrix}.$$

$F \in C^n[\alpha, \beta]$ and there exists $F^{(n+1)}$ on (α, β).

We have

$$F(x) = 0, \quad F^{(j)}(x_k) = 0, \qquad k = 0, ..., m; \; j = 0, ..., r_k;$$

because

$$u(x) = \prod_{k=0}^{m} (x - x_k)^{r_k + 1} \Rightarrow u^{(j)}(x_k) = 0, \; j = 0, ..., r_k$$

and

$$(R_m f)^{(j)}(x_k) = f^{(j)}(x_k) - (H_n f)^{(j)}(x_k) = f^{(j)}(x_k) - f^{(j)}(x_k) = 0.$$

So, F and its derivatives have $n + 2$ distinct zeros in (α, β). Applying successively Rolle's theorem it follows that F' has at least $n + 1$ zeros in $(\alpha, \beta) \Rightarrow ... \Rightarrow F^{(n+1)}$ has at least one zero $\xi \in (\alpha, \beta)$, $F^{(n+1)}(\xi) = 0$.

We have

$$F^{(n+1)}(z) = \begin{vmatrix} u^{(n+1)}(z) & (R_n f)^{(n+1)}(z) \\ u(x) & (R_n f)(x) \end{vmatrix},$$

with $u(z) = \prod\limits_{k=0}^{m} (z - z_k)^{r_k+1} \in \mathbb{P}_{n+1} \Rightarrow u^{(n+1)}(z) = (n+1)!$, and $(R_n f)^{(n+1)}(z) = f^{(n+1)}(z) - (H_n f)^{(n+1)}(z) = f^{(n+1)}(z)$ (as, $H_n f \in \mathbb{P}_n$). We get

$$F^{(n+1)}(\xi) = \begin{vmatrix} (n+1)! & f^{(n+1)}(\xi) \\ u(x) & (R_n f)(x) \end{vmatrix} = 0,$$

whence it follows (1). ∎

Corolar 2 *If $f \in C^{n+1}[a, b]$ then*

$$|(R_n f)(x)| \leq \frac{|u(x)|}{(n+1)!} \left\| f^{(n+1)} \right\|_{\infty}, \qquad x \in [a, b]$$

where $\|\cdot\|_{\infty}$ denotes the uniform norm ($\|f\|_{\infty} = \max\limits_{x \in [a,b]} |f(x)|$).

Remark 3 *In case of $m = 0$, i.e., $n = r_0$, (HIP) becomes* **Taylor interpolation problem**. *Taylor interpolation polynomial is*

$$(T_n f)(x) = \sum_{j=0}^{n} \frac{(x - x_0)^j}{j!} f^{(j)}(x_0).$$

Example 4 *Find the Hermite interpolation formula for the function $f(x) = xe^x$ for which we know $f(-1) = -0.3679$, $f(0) = 0$, $f'(0) = 1$, $f(1) = 2.7183$, (equivalent with $x_0 = -1$ simple, $x_1 = 0$ multiple of order 2 and $x_2 = 1$ simple). Which is the limit of the error for approximating $f(\frac{1}{2})$?*

Hermite interpolation with double nodes

Example 5 *In the following table there are some data regarding a moving car. We may estimate the position (and the speed) of the car when the time is $t = 10$ using Hermite interpolation.*

Time	0	3	5	8	13
Distance	0	225	383	623	993
Speed	75	77	80	74	72

Consider $f : [a, b] \to \mathbb{R}$, $x_0, x_1, ..., x_m \in [a, b]$

and the values $f(x_0), f(x_1), ..., f(x_m), f'(x_0), f'(x_1), ..., f'(x_m)$.

The Hermite interpolation polynomial with double nodes, H_{2m+1}, satisfies the interpolation properties:

$$H_{2m+1}(x_i) = f(x_i), \quad i = \overline{0, m},$$
$$H'_{2m+1}(x_i) = f'(x_i), \quad i = \overline{0, m}.$$

It is a polynomial of $n = 2m + 1$ degree.

For computation: use Lagrange polynomial written in Newton form, with divided differences table having each node x_i written twice.

Consider $z_0 = x_0$, $z_1 = x_0$, $z_2 = x_1$, $z_3 = x_1$, $\ldots$, $z_{2m} = x_m$, $z_{2m+1} = x_m$.

Form divided differences table: each node appear twice, in the first column write the values of f for each node twice; in the second column, at the odd positions put the values of the derivatives of f; the other elements are computed using the rule from divided differences.

We obtain the following table:

z_0	$f(z_0)$	$(\mathcal{D}^1 f)(z_0) = f'(x_0)$	$(\mathcal{D}^2 f)(z_0)$		$(\mathcal{D}^{2m} f)(z_0)$	$(\mathcal{D}^{2m+1} f)(z_0)$
z_1	$f(z_1)$	$(\mathcal{D}^1 f)(z_1)$	$\vdots$		$(\mathcal{D}^{2m} f)(z_1)$	
z_2	$f(z_2)$	$(\mathcal{D}^1 f)(z_2) = f'(x_1)$				
z_3	$f(z_3)$	$\vdots$				
$\vdots$	$\vdots$	$(\mathcal{D}^1 f)(z_{2m-1})$	$(\mathcal{D}^2 f)(z_{2m-1})$	$\ddots$		
z_{2m}	$f(z_{2m})$	$(\mathcal{D}^1 f)(z_{2m}) = f'(x_m)$		$\ldots$		
z_{2m+1}	$f(z_{2m+1})$			$\ldots$		

Newton interpolation polynomial for the nodes $x_0, ..., x_n$ is

$$(N_n f)(x) = f(x_0) + \sum_{i=1}^{n} (x - x_0)...(x - x_{i-1})(\mathcal{D}^i f)(x_0),$$

and similarly, Hermite interpolation polynomial is

$$(H_{2m+1} f)(x) = f(z_0) + \sum_{i=1}^{2m+1} (x - z_0)...(x - z_{i-1})(\mathcal{D}^i f)(z_0),$$

where $(\mathcal{D}^i f)(z_0)$, $i = 1, ..., 2m + 1$ are the elements from the first line and columns $2, ..., 2m + 1$.

Example 6 *Consider the double nodes $x_0 = -1$ and $x_1 = 1$, and $f(-1) = -3, f'(-1) = 10, f(1) = 1, f'(1) = 2$. Find the Hermite interpolation polynomial, that approximates the function f, in both forms, using the classical formula and using divided differences.*

Sol. We present here the method with divided diffrences. We have
$$m = 1, r_0 = r_1 = 1 \Rightarrow n = 3$$

$z_0 = -1$	$f(-1) = -3$	$f'(-1) = 10$	$\dfrac{\frac{f(1)-f(-1)}{2}-f'(-1)}{z_2-z_0} = -4$	$\dfrac{0-(-4)}{z_3-z_0} = 2$
$z_1 = -1$	$f(-1) = -3$	$\dfrac{f(1)-f(-1)}{z_2-z_1} = 2$	$\dfrac{f'(1)-\frac{f(1)-f(-1)}{2}}{z_3-z_1} = 0$	
$z_2 = 1$	$f(1) = 1$	$f'(1) = 2$		
$z_3 = 1$	$f(1) = 1$			

The Hermite interpolation polynomial is

$$(H_3 f)(x) = f(z_0) + \sum_{i=1}^{3} (x - z_0)...(x - z_{i-1})(\mathcal{D}^i f)(z_0)$$
$$= f(z_0) + (x - z_0)(\mathcal{D}^1 f)(z_0) + (x - z_0)(x - z_1)(\mathcal{D}^2 f)(z_0)$$
$$+ (x - z_0)(x - z_1)(x - z_2)(\mathcal{D}^3 f)(z_0)$$

i.e.,

$$(H_3 f)(x) = f(-1) + (x + 1)f'(-1) + (x + 1)^2 \frac{f(1) - f(-1) - 2f'(-1)}{4}$$
$$+ (x + 1)^2 (x - 1)\frac{2f'(1) - f(1) + f(-1)}{4}$$
$$= -3 + 10(x + 1) - 4(x + 1)^2 + 2(x + 1)^2(x - 1)$$
$$= 2x^3 - 2x^2 + 1.$$

Example 7 *Considering the the following data*

$$
\begin{array}{c|ccc}
x & 0 & 2 & 3 \\
f(x) & 0 & 10 & 12 \\
f\prime(x) & 5 & 3 & 7
\end{array}
$$

find the corresponding Hermite interpolation polynomial.