COURSE 2

2.2. Lagrange interpolation

Let $[a, b] \subset \mathbb{R}$, $x_i \in [a, b]$, $i = 0, 1, ..., m$ such that $x_i \neq x_j$ for $i \neq j$ and consider $f : [a, b] \to \mathbb{R}$.

The Lagrange interpolation problem (LIP) consists in determining the polynomial P of the smallest degree for which

$$P(x_i) = f(x_i), \ i = 0, 1, ..., m \tag{1}$$

i.e., the polynomial of the smallest degree which passes through the distinct points $(x_i, f(x_i))$, $i = 0, 1, ..., m$.

Definition 1 *A solution of (LIP) is called* **Lagrange interpolation polynomial**, *denoted by* $L_m f$.

Remark 2 *We have* $(L_m f)(x_i) = f(x_i), \ i = 0, 1, ..., m.$

$L_m f \in \mathbb{P}_m$ ($\mathbb{P}_m$ *is the space of polynomials of at most* m*-th degree*).

The Lagrange interpolation polynomial is given by

$$(L_m f)(x) = \sum_{i=0}^{m} \ell_i(x) f(x_i), \tag{2}$$

where by $\ell_i(x)$ denote **the Lagrange fundamental interpolation polynomials.**

We have

$$u(x) = \prod_{j=0}^{m} (x - x_j),$$

$$u_i(x) = \frac{u(x)}{x - x_i} = (x - x_0)...(x - x_{i-1})(x - x_{i+1})...(x - x_m) = \prod_{\substack{j=0 \\ j \neq i}}^{m} (x - x_j)$$

and

$$\ell_i(x) = \frac{u_i(x)}{u_i(x_i)} = \frac{(x - x_0)...(x - x_{i-1})(x - x_{i+1})...(x - x_m)}{(x_i - x_0)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_m)} = \prod_{\substack{j=0 \\ j \neq i}}^{m} \frac{x - x_j}{x_i - x_j}, \tag{3}$$

for $i = 0, 1, ..., m$.

Proposition 3 *We also have*

$$\ell_i(x) = \frac{u(x)}{(x - x_i)u'(x_i)}, \ \ i = 0, 1, ..., m. \tag{4}$$

Proof. We have $u_i(x) = \frac{u(x)}{x - x_i}$, so $u(x) = u_i(x)(x - x_i)$. We get $u'(x) = u_i(x) + (x - x_i)u_i'(x)$, whence it follows $u'(x_i) = u_i(x_i)$. So, as

$$\ell_i(x) = \frac{u_i(x)}{u_i(x_i)}$$

we get

$$\ell_i(x) = \frac{u_i(x)}{u'(x_i)} = \frac{u(x)}{(x - x_i)u'(x_i)}, \ \ i = 0, 1, ..., m. \tag{5}$$

■

Theorem 4 *The operator L_m is linear.*

Proof.

$$L_m(\alpha f+\beta g)(x)=\sum_{i=0}^{m}\ell_i(x)(\alpha f+\beta g)(x_i)=\sum_{i=0}^{m}[\ell_i(x)\alpha f(x_i)+\ell_i(x)\beta g(x_i)]$$
$$=\alpha(L_m f)(x)+\beta(L_m g)(x),$$

so

$$L_m(\alpha f+\beta g)=\alpha L_m f+\beta L_m g, \qquad \forall f,g:[a,b]\to\mathbb{R} \text{ and } \alpha,\beta\in\mathbb{R}.$$

■

Example 5 *1) Consider the nodes x_0, x_1 and a function f to be interpolated.*

We have $m=1$,

$$u(x)=(x-x_0)(x-x_1)$$
$$u_0(x)=x-x_1$$
$$u_1(x)=x-x_0$$

$$
\begin{aligned}
(L_1 f)(x) &= l_0(x) f(x_0) + l_1(x) f(x_1) \\
&= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1),
\end{aligned}
$$

which is the line passing through the given points $(x_0, f(x_0))$ *and* $(x_1, f(x_1))$.

Example 6 *Find the Lagrange polynomial that interpolates the data in the following table and find the approximative value of* $f(-0.5)$.

x	-1	0	3
$f(x)$	8	-2	4

Sol. *We have* $m = 2$. *The Lagrange polynomial is*

$$(L_2f)(x) = l_0(x)f(x_0) + l_1(x)f(x_1) + l_2(x)f(x_2).$$

$u(x) = (x+1)(x-0)(x-3)$ *and it follows*

$$l_0(x) = \frac{(x-0)(x-3)}{(-1-0)(-1-3)} = \frac{1}{4}x(x-3)$$
$$l_1(x) = \frac{(x+1)(x-3)}{(0+1)(0-3)} = -\frac{1}{3}(x+1)(x-3)$$
$$l_2(x) = \frac{(x+1)(x-0)}{(3+1)(3-0)} = \frac{1}{12}x(x+1),$$

The polynomial is

$$(L_2f)(x) = 2x(x-3) + \frac{2}{3}(x+1)(x-3) + \frac{1}{3}x(x+1).$$

and $(L_2f)(-0.5) = 2.25$.

Remark 7 *Disadvantages of the form (2) of Lagrange polynomial: requires many computations and if we add or substract a point we have to start with a complete new set of computations.*

Some calculations allow us to reduce the number of operations:

$$(L_m f)(x) = \frac{(L_m f)(x)}{1} = \frac{\sum\limits_{i=0}^{m} l_i(x) f(x_i)}{\sum\limits_{i=0}^{m} l_i(x)}.$$

Dividing the numerator and the denominator by

$$u(x) = \prod_{i=1}^{m} (x - x_i)$$

and denoting

$$A_i = \frac{1}{\prod\limits_{j=0, j\neq i}^{m} (x_i - x_j)} = \frac{1}{u_i(x_i)}$$

one obtains

$$(L_m f)(x) = \frac{\sum\limits_{i=0}^{m} \frac{A_i f(x_i)}{x - x_i}}{\sum\limits_{i=0}^{m} \frac{A_i}{x - x_i}}, \qquad (6)$$

called **the barycentric form** *of Lagrange interpolation polynomial*.

Remark 8 *Formula (6) needs half of the number of arithmetic operations needed for (2) and it is easier to add or substract a point.*

The Lagrange polynomial generates **the Lagrange interpolation formula**

$$f = L_m f + R_m f,$$

where $R_m f$ denotes **the remainder** (**the error**).

Theorem 9 *Let* $\alpha = \min\{x, x_0, ..., x_m\}$ *and* $\beta = \max\{x, x_0, ..., x_m\}$. *If* $f \in C^m[\alpha, \beta]$ *and* $f^{(m)}$ *is derivable on* (α, β) *then* $\forall x \in (\alpha, \beta)$, *there exists* $\xi \in (\alpha, \beta)$ *such that*

$$(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi). \tag{7}$$

Proof. Consider

$$F(z) = \begin{vmatrix} u(z) & (R_m f)(z) \\ u(x) & (R_m f)(x) \end{vmatrix}.$$

From hypothesis it follows that $F \in C^m[\alpha, \beta]$ and there exists $F^{(m+1)}$ on (α, β).

We have

$$F(x) = 0, \;\; F(x_i) = 0, \qquad i = 0, 1, ..., m,$$

as

$$u(x_i) = \prod_{j=0}^{m} (x_i - x_j) = 0$$

and

$$(R_m f)(x_i) = f(x_i) - (L_m f)(x_i) = f(x_i) - f(x_i) = 0,$$

so F has $m+2$ distinct zeros in (α, β). Applying successively the Rolle theorem it follows that: F has $m+2$ zeros in $(\alpha, \beta) \Rightarrow F'$ has at least $m+1$ zeros in $(\alpha, \beta) \Rightarrow ... \Rightarrow F^{(m+1)}$ has at least one zero in (α, β)

So $F^{(m+1)}$ has at least one zero $\xi \in (\alpha, \beta)$, $F^{(m+1)}(\xi) = 0$.

We have

$$F^{(m+1)}(z) = \begin{vmatrix} u^{(m+1)}(z) & (R_m f)^{(m+1)}(z) \\ u(x) & (R_m f)(x) \end{vmatrix},$$

with

$$u(z) = \prod_{i=0}^{m}(z - z_i) \Rightarrow u^{(m+1)}(z) = (m+1)!,$$

and

$$\begin{aligned}(R_m f)^{(m+1)}(z) &= (f - (L_m f))^{(m+1)}(z) \\ &= f^{(m+1)}(z) - (L_m f)^{(m+1)}(z) = f^{(m+1)}(z)\end{aligned}$$

(as, $L_m f \in \mathbb{P}_m$).

We have $F^{(m+1)}(\xi) = 0$, for $\xi \in (\alpha, \beta)$, so

$$F^{(m+1)}(\xi) = \begin{vmatrix} (m+1)! & f^{(m+1)}(\xi) \\ u(x) & (R_m f)(x) \end{vmatrix} = 0,$$

i.e., $(m+1)!(R_m f)(x) = u(x) f^{(m+1)}(\xi)$, whence $(R_m f)(x) = \frac{u(x)}{(m+1)!} f^{(m+1)}(\xi)$. ■

Corolar 10 *If* $f \in C^{m+1}[a,b]$ *then*

$$|(R_m f)(x)| \leq \frac{|u(x)|}{(m+1)!} \left\| f^{(m+1)} \right\|_\infty, \qquad x \in [a,b]$$

where $\|\cdot\|_\infty$ *denotes the uniform norm, and* $\|f\|_\infty = \max_{x\in[a,b]} |f(x)|$.

Example 11 *If we know that* $\lg 2 = 0.301$, $\lg 3 = 0.477$, $\lg 5 = 0.699$, *find* $\lg 76$. *Study the approximation error.*

Example 12 *Which is the limit of the error for computing* $\sqrt{115}$ *using Lagrange interpolation formula for* $f(x) = \sqrt{x}$ *and* $x_0 = 100$, $x_1 = 121$ *and* $x_2 = 144$? *Find the approximative value of* $\sqrt{115}$.

The Aitken's algorithm

Let $[a,b] \subset \mathbb{R}$, $x_i \in [a,b]$, $i = 0, 1, ..., m$ such that $x_i \neq x_j$ for $i \neq j$ and consider $f : [a,b] \to \mathbb{R}$.

Usually, for a practical approximation problem, for a given function $f : [a,b] \to \mathbb{R}$ we have to find the approximation of $f(\alpha)$, $\alpha \in [a,b]$ with an error not greater than a given $\varepsilon > 0$.

If we have enough information about f and its derivatives, we use the inequality $|(R_m f)(x)| \leq \varepsilon$ to find m such that $(L_m f)(\alpha)$ approximates $f(\alpha)$ with the given precision.

We may use the condition $\frac{|u(x)|}{(m+1)!} \left\| f^{(m+1)} \right\|_\infty \leq \varepsilon$, but it should be known $\left\| f^{(m+1)} \right\|_\infty$ or a majorant of it.

A practical method for computing the Lagrange polynomial is **the Aitken's algorithm.** This consists in generating the table:

$$
\begin{array}{c|c|c|cccc}
x_0 & f_{00} & & & & & \\
x_1 & f_{10} & f_{11} & & & & \\
x_2 & f_{20} & f_{21} & f_{22} & & & \\
x_3 & f_{30} & f_{31} & f_{32} & f_{33} & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \\
x_m & f_{m0} & f_{m1} & f_{m2} & f_{m3} & \cdots & f_{mm}
\end{array}
$$

where

$$f_{i0} = f(x_i), \quad i = 0, 1, ..., m,$$

and

$$f_{i,j+1} = \frac{1}{x_i - x_j} \begin{vmatrix} f_{jj} & x_j - x \\ f_{ij} & x_i - x \end{vmatrix}, \qquad i = 0, 1, ..., m; j = 0, ..., i-1.$$

For example,

$$
\begin{aligned}
f_{11} &= \frac{1}{x_1 - x_0} \begin{vmatrix} f_{00} & x_0 - x \\ f_{10} & x_1 - x \end{vmatrix} \\
&= \frac{1}{x_1 - x_0} [f_{00}(x_1 - x) - f_{10}(x_0 - x)] \\
&= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1) = (L_1 f)(x),
\end{aligned}
$$

so f_{11} is the value in x of Lagrange polynomial for the nodes x_0, x_1. We have

$$f_{ii} = (L_i f)(x),$$

$L_i f$ being Lagrange polynomial for the nodes $x_0, x_1, ..., x_i$.

So $f_{11}, f_{22}, ..., f_{ii}, ..., f_{mm}$ is a sequence of approximations of $f(x)$.

If the interpolation procedure is convergent then the sequence is also convergent, i.e., $\lim_{m \to \infty} f_{mm} = f(x)$. By Cauchy convergence criterion it follows

$$\lim_{i \to \infty} |f_{ii} - f_{i-1,i-1}| = 0.$$

This could be used as a stopping criterion, i.e.,

$$\left|f_{ii} - f_{i-1,i-1}\right| \leq \varepsilon, \quad \text{for a given precision } \varepsilon > 0.$$

Recommendation is to sort the nodes $x_0, x_1, ..., x_m$ with respect to the distance to x, such that

$$|x_i - x| \leq \left|x_j - x\right| \text{ if } i < j, \quad i, \quad j = 1, ..., m.$$

Example 13 *Approximate* $\sqrt{115}$ *with precision* $\varepsilon = 10^{-3}$, *using Aitken's algorithm.*