COURSE 11

Hermite inverse interpolation

Let $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}$. Consider the equation

$$f(x) = 0, \quad x \in \Omega. \tag{1}$$

Assume that α is a solution of equation $f(x) = 0$ and $V(\alpha)$ is a neighborhood of α. If $y_k = f(x_k)$, where $x_k \in V(\alpha)$, $k = 0, ..., m$, are approximations of α, $r_k \in \mathbb{N}$, then if there exist $g^{(j)}(y_k) = (f^{-1})^{(j)}(y_k), j = 0, ..., r_k$, one considers the Hermite type interpolation problem.

Theorem 1 *Let α be a solution of equation $f(x) = 0$, $V(\alpha)$ a neighborhood of α and $x_0, x_1 ..., x_m \in V(\alpha)$. For $n = r_0 + ... + r_m + m$, where r_k represents the multiplicity order of the nodes x_k, $k = 0, ..., m$, if $f \in C^{n+1}(V(\alpha))$ and $f'(x) \neq 0$ for $x \in V(\alpha)$, we have the following*

Hermite approximation method for α :

$$\alpha \approx F_n^H(x_0, ..., x_m) = \tag{2}$$

$$= (H_n g)(0) = \sum_{k=0}^{m} \sum_{j=0}^{r_k} \sum_{\nu=0}^{r_k - j} \frac{(-1)^{j+\nu}}{j!\nu!} f_k^{j+\nu} v_k(0) (\frac{1}{v_k(y)})^{(\nu)}_{y=f_k} g^{(j)}(f_k),$$

where $f_k = f(x_k), k = 0, ..., m, g = f^{-1}$, *and*

$v_k(y) = (y - f_0)^{r_0+1} ... (y - f_{k-1})^{r_{k-1}+1} (y - f_{k+1})^{r_{k+1}+1} ... (y - f_m)^{r_m+1}.$

For $g = f^{-1}$ the corresponding Hermite polynomial is

$$(H_n g)(y) = \sum_{k=0}^{m} \sum_{j=0}^{r_k} h_{kj}(y) g^{(j)}(y_k),$$

and it satisfies the conditions:

$$(H_n g)^{(j)}(y_k) = g^{(j)}(y_k), \quad j = 0, ..., r_k; \quad k = 0, ..., m,$$

where h_{kj} are the fundamental interpolating polynomials, i.e.,

$$h_{kj}^{(p)}(y_\nu) = 0,\ k \neq \nu,\ \ p = 0, ..., r_\nu$$
$$h_{kj}^{(p)}(y_k) = \delta_{pj},\ p = 0, ..., r_k$$

and the corresponding interpolation formula is

$$g = H_n g + R_n g,$$

where $R_n g$ is the remainder term.

Taking into account that

$$\alpha = g(0) \approx (H_n g)(0),$$

defines a new approximation to α, we have that

$$F_n^H(x_0, ..., x_m) = (H_n g)(0)$$

is an approximation method for α.

Regarding the order of Hermite-type inverse interpolation method F_n^H, we have two results, first for the case of equal information (the same multiplicity order for all the nodes x_k, $k = 0, ..., m$) and then for different multiplicities.

Theorem 2 *(Equal information) The order* $ord(F_n^H)$ *is the unique positive root of the equation:*

$$t^{m+1} - (r+1) \sum_{j=0}^{m} t^j = 0,$$

where r *is the multiplicity order of the points* x_k, $\forall k = 0, ..., m$.

Theorem 3 *(Unequal information) The order of* F_n^H *is the unique positive and real root of the equation:*

$$t^{m+1} - (r_m + 1)t^m - (r_{m-1} + 1)t^{m-1} - .... - (r_1 + 1)t - (r_0 + 1) = 0,$$

where $r_0, ..., r_m$ *are real numbers, permutation of the multiplicity orders of the nodes* x_k, $k = 0, ..., m$ *satisfying the conditions:*

$$r_0 + r_1 + ... + r_m > 1 \tag{3}$$

and

$$r_m \geq r_{m-1} \geq ... \geq r_1 \geq r_0. \tag{4}$$

Remark 4 *The order of the Taylor-type inverse interpolation method, can be expressed as the solution of equation*

$$t - (r_0 + 1) = 0,$$

where r_0 is the multiplicity order of the node x_0.

Particular cases.

1) For $x_0,\ x_1 \in V(\alpha)$ with $r_0 = 0,\ r_1 = 1$, we have the following approximation method:

$$F_2^H(x_0, x_1) = x_1 - \left[\frac{f(x_1)}{f(x_0)-f(x_1)}\right]^2 (x_1 - x_0) - \frac{f(x_1)}{f(x_0)-f(x_1)} \frac{f(x_0)}{f\,'(x_1)}.$$

The *order* of this method is the solution of the equation:

$$t^2 - r_1 t - r_0 = 0,$$

so,

$$t^2 - 2t - 1 = 0,$$

and $p = 1 + \sqrt{2}$.

2) For nodes $x_0,\ x_1,\ x_2 \in V(\alpha)$ with $r_0 = r_1 = 0;\ r_2 = 1$, the method is:

$$F_4^H(x_0, x_1, x_2) =$$
$$= \frac{f(x_2)^2}{f(x_1)-f(x_0)} \left[\frac{x_0 f(x_1)}{[f(x_0)-f(x_2)]^2} - \frac{x_1 f(x_0)}{[f(x_1)-f(x_2)]^2}\right]$$
$$+ \frac{f(x_0)f(x_1)}{[f(x_2)-f(x_0)][f(x_2)-f(x_1)]} \left[1 + \frac{f(x_2)}{[f(x_2)-f(x_0)][f(x_2)-f(x_1)]}\right] \left[x_2 - \frac{f(x_2)}{f'(x_2)}\right].$$

The *order* of this method is the solution of the equation:

$$t^3 - 2t^2 - t - 1 = 0,$$

so $p = 2.548$.

3) For x_0, $x_1 \in V(\alpha)$ with $r_0 = r_1 = 1$, (double nodes), the approximation method is

$$F_3^H(x_0, x_1) = x_1 - \frac{3f(x_0)f(x_1)^2 - f(x_1)^3}{[f(x_0) - f(x_1)]^3}(x_0 - x_1) + \frac{f(x_0)f(x_1)}{[f(x_0) - f(x_1)]^2}\left[\frac{f(x_1)}{f'(x_0)} - \frac{f(x_0)}{f'(x_1)}\right].$$

The *order* of this method is the solution of the equation:

$$t^{m+1} - (r+1)\sum_{j=0}^{m} t^j = 0,$$

so, $t^2 - 2t - 2 = 0$, and $p = 1 + \sqrt{3}$.

Birkhoff inverse interpolation

Assume that α is a solution of equation $f(x) = 0$ and $V(\alpha)$ is a neighborhood of α. If $y_k = f(x_k)$, where $x_k \in V(\alpha)$, $k = 0, ..., m$, are approximations of α, $r_k \in N$ and $I_k \subset \{0, ..., r_k\}$, then if there exist $g^{(j)}(y_k) = (f^{-1})^{(j)}(y_k), j \in I_k$, one considers the Birkhoff type interpolation problem.

The Birkhoff polynomial

$$(B_n g)(y) = \sum_{k=0}^{m} \sum_{j \in I_k} b_{kj}(y) g^{(j)}(y_k),$$

satisfies the conditions:

$$(B_n g)^{(j)}(y_k) = g^{(j)}(y_k), \;\; j \in I_k, \; k = 0, ..., m,$$

where b_{kj} are the fundamental interpolating polynomials, i.e.,

$$b_{kj}^{(p)}(y_\nu) = 0, \; k \neq \nu, \; p \in I_\nu$$
$$b_{kj}^{(p)}(y_k) = \delta_{pj}, \; p \in I_k$$

and the corresponding interpolation formula is

$$g = B_n g + R_n g,$$

where $R_n g$ is the remainder term.

Taking into account that

$$\alpha = g(0) \approx (B_n g)(0),$$

defines a new approximation to α we have that

$$F_n^B(x_0, ..., x_m) = (B_n g)(0)$$

is an approximation method for α.

Particular case.

1) Let $x_0, x_1 \in V(\alpha)$, $I_0 = \{0\}$, $I_1 = \{1\}$ and $y_0 = f(x_0), y_1 = f(x_1)$. Find the corresponding $F-$method for approximating the solution of equation $f(x) = 0$.

Sol. Taking

$$F_1^B(x_0, x_1) = (B_1 g)(0),$$

we obtain the method defined by

$$F_1^B(x_0, x_1) = x_0 - \frac{f(x_0)}{f'(x_1)}.$$

5.3. Numerical methods for solving nonlinear systems

Let $D \subseteq \mathbb{R}^n$, $f_i : D \to \mathbb{R}$, $i = 1, ..., n$ and the system

$$f_i(x_1, ..., x_n) = 0,\ i = 1, ..., n;\ (x_1, ..., x_n) \in D. \tag{5}$$

The system (5) can be written as

$$f(x) = 0,\ x \in D,\ \text{with } f = (f_1, ..., f_n).$$

5.3.1. Successive approximation method

We rewrite the system (5) as

$$x_i = \varphi_i(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n), \qquad i = 1, ..., n;\ (x_1, ..., x_n) \in D$$

or

$$x = \varphi(x), \quad \text{with } x = (x_1, ..., x_n) \in D \text{ and } \varphi = (\varphi_1, ..., \varphi_n), \tag{6}$$

where $\varphi_i : D \to \mathbb{R}$ are continuous functions on D such that for any point $(x_1, ..., x_n) \in D$ to have $(\varphi_1(x_1, ..., x_n), ..., \varphi_n(x_1, ..., x_n)) \in D$.

Considering the starting point $x^{(0)}$ we generate the sequence

$$x^{(0)}, x^{(1)}, ..., x^{(n)}, ... \tag{7}$$

with

$$x^{(m+1)} = \varphi(x^{(m)}),\ \ m = 0, 1, ...$$

If the sequence (7) is convergent and x^* is its limit, then x^* is the solution of system (5). We have

$$\lim_{m\to\infty} x^{(m+1)} = \varphi(\lim_{m\to\infty} x^{(m)}),$$

namely,

$$x^* = \varphi(x^*).$$

The convergence of the method, using Picard-Banach theorem: if $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ verifies the contraction condition

$$\|\varphi(x) - \varphi(y)\| \leq \alpha \|x - y\|,\ \ x, y \in \mathbb{R}^n; 0 < \alpha < 1,$$

then there exists an unique element $x^* \in \mathbb{R}^n$, which is solution of equation (6) and limit of the sequence (7). The approximation error is:

$$\left\| x^* - x^{(n)} \right\| \leq \frac{\alpha^n}{1-\alpha} \left\| x^{(1)} - x^{(0)} \right\|.$$

Example 5 *Choosing* $x^{(0)} = (0.1, 0.1, -0.1)$, *solve the system*

$$\begin{cases} 3x_1 - \cos(x_2 x_3) - \frac{1}{2} & = & 0 \\ x_1^2 - 81(x_2 + 0.1)^2 + \sin x_3 + 1.06 & = & 0 \\ e^{-x_1 x_2} + 20 x_3 + \frac{10\pi - 3}{3} & = & 0 \end{cases}.$$

Sol. We have

$$\begin{cases} x_1^{(1)} = \frac{1}{3}\cos(x_2^{(0)} x_3^{(0)}) + \frac{1}{6} \\ x_2^{(1)} = \frac{1}{9}\sqrt{(x_1^{(0)})^2 + \sin x_3^{(0)} + 1.06} - 0.1 \\ x_3^{(1)} = -\frac{1}{20} e^{-x_1^{(0)} x_2^{(0)}} - \frac{10\pi - 3}{60} \end{cases},$$

and

$$\begin{cases} x_1^{(m)} = \frac{1}{3}\cos(x_2^{(m-1)}x_3^{(m-1)}) + \frac{1}{6} \\ x_2^{(m)} = \frac{1}{9}\sqrt{(x_1^{(m-1)})^2 + \sin x_3^{(m-1)} + 1.06} - 0.1 \\ x_3^{(m)} = -\frac{1}{20}e^{-x_1^{(m-1)}x_2^{(m-1)}} - \frac{10\pi-3}{60} \end{cases}.$$

The sequence converges to (0.5,0,-0.5236).

5.3.2. Newton's method for solving nonlinear systems

Consider the system (5) written as

$$f(x) = 0,\ x \in D, D \subseteq \mathbb{R}^n.$$

Let $x^* \in D$ be a solution of this equation and $x^{(p)}$ an approximation of it.

The Newton's method for nonlinear systems:

$$x^{(p+1)} = x^{(p)} - J^{-1}(x^{(p)})f(x^{(p)}),\ \ p = 0, 1, ... \tag{8}$$

where

$$J(x^{(p)}) = f'(x^{(p)}) = \left(\frac{\partial f_i}{\partial x_j}(x^{(p)})\right)_{i,\ j=1,...,n}$$

is the Jacobian matrix.

If the sequence $(x^{(p)})_{p\in\mathbb{N}}$ is convergent and x^* is its limit then by (8) it follows that x^* is solution of the system. Regarding the convergence of the sequence we have:

Theorem 6 *Let* $f : D \subseteq R^n \to R^n$ *and consider a given norm* $\|\cdot\|$ *on* R^n. *If*

- *there exists a solution* $x^* \in D$, *such that* $f\,(x^*) = 0$;

- f *is differentiable on* D, *with* f' *Lipschitz continuous, i.e.,* $\exists L > 0$ *s.t.*

$$\left\| f'(x) - f'(y) \right\| \leq L \|x - y\|, \forall x, y \in D;$$

- *the Jacobian of* f *is nonsingular at* x^**:* $\exists f'(x^*)^{-1} : R^n \to R^n$,

then there exists an open neighborhood $D_0 \subseteq D$ *of* x^* *such that for any initial approximation* $x_0 \in D_0$ *the sequence generated by the Newton's method remains in* D_0*, converges to the solution* x^* *and there exists a constant* $K > 0$ *such that*

$$\left\| x_{k+1} - x^* \right\| \leq K \|x_k - x^*\|^2, \ \forall k \geq 0.$$

Example 7 *Solve the system*

$$\begin{cases} x_1^3 + 3x_2^2 - 21 = 0 \\ x_1^2 + 2x_2 + 2 = 0 \end{cases}$$

using $x^{(0)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\varepsilon = 10^{-6}$.

Sol. We have

$$x^{(p+1)} = x^{(p)} - J^{-1}(x^{(p)})f(x^{(p)}),\ \ p = 0, 1, ... \tag{9}$$

with

$$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} x_1^3 + 3x_2^2 - 21 \\ x_1^2 + 2x_2 + 2 \end{pmatrix}.$$

We compute

$$J(x) = f'(x) = \left(\frac{\partial}{\partial x_j} f_i(x)\right)_{i,\ j=1,\dots,n} = \begin{pmatrix} 3x_1^2 & 6x_2 \\ 2x_1 & 2 \end{pmatrix}$$

and

$$x^{(p+1)} = x^{(p)} - J^{-1}(x^{(p)})f(x^{(p)}), \tag{10}$$

i.e.,

$$\begin{aligned} x^{(1)} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 & -6 \\ 2 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 + 3 - 21 \\ 1 - 2 + 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0.(1) & 0.(3) \\ -0.(1) & 0.1(6) \end{pmatrix} \begin{pmatrix} -17 \\ 1 \end{pmatrix} \approx \begin{pmatrix} 2.55 \\ -3.05 \end{pmatrix}. \end{aligned}$$

Continuing in this way we obtain the approx. solution $\begin{pmatrix} 1.64 \\ -2.35 \end{pmatrix}$