Theorem 1 :- ~~Weie~~ ~~Weire~~ Weierstrass's Theorem :-

Consider a $\boxed{\text{non-empty}}$ ~~bounded~~ $\boxed{\text{compact}}$ $S \subseteq \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}$, $\boxed{\text{continuous on } S}$, then the optimization Problem (P) : $\min \{ f(x) \; x \in S \}$ attains an optimal solution.

Compact $\to$ Closed and Bounded.

Infimum and Supremum :-

$$\inf \{ f(x) : x \in S \} = \max \{ \alpha \in \mathbb{R} : \alpha \leq f(x) \}$$

$$\sup \{ f(x) : x \in S \} = \min \{ x \in \mathbb{R} : \alpha \geq f(x) \}$$

Boundedness Theorem :-

If $S \subseteq \mathbb{R}^m$ is compact & $f : \mathbb{R}^n \to \mathbb{R}$ is continous on S, then there exists $\alpha, \beta \in \mathbb{R}$ s.t.

$$\alpha \leq f(x) \leq \beta \quad , \boxed{x \in S}$$

Proof (Weierstrass's theorem) :-

$$\exists \alpha = \inf \{ f(x) : x \in S \}$$

If, $\exists x \in S$ such that $f(x) = \alpha$, then $x = \underset{x \in S}{\operatorname{argmin}} \{ f(x) \}$

If not $\Rightarrow$ $\forall x \in S$. $f(x) > \alpha$.

Consider, $g(x) = \cancel{1/f(x)} \dfrac{1}{f(x) - \alpha} > 0 \quad \forall x \in S$

$\sup \{ g(x) : x \in S \} \leq \beta > 0$

$\Rightarrow \frac{1}{f(x)-\alpha} \le \beta \quad \forall x \in S$

$\Rightarrow f(x) \geq \alpha + \frac{1}{\beta}$, contradiction.

$\Rightarrow \exists x \in S$, s.t. $f(x) = \alpha$.

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Elements Of Convex Analysis

Vector combinations :-

Given, $x^i \in \mathbb{R}^n$, $i \in [m]$

consider $\bar{x} = \sum_{i=1}^{m} \lambda_i x^i$, $\lambda \in \mathbb{R}$.

Linear combination: $\lambda \in \mathbb{R}$

Affin combination: $\sum_{i=1}^{m} \lambda_i = 1$

Linear hull :- Set of all linear combination $= \mathbb{R}^2$

Affin hull :- ~~Straight~~ 2D :- Straight line through x^1, x^2

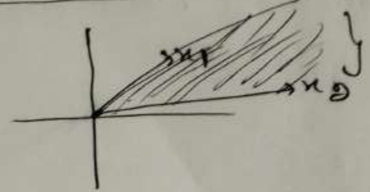
3D :- Plane through x^1, x^2, x^3

2D :- 3 Points $\Rightarrow$ Plane$(\mathbb{R}^2)$

↳ $\lambda x^1 + (1-\lambda) x^2$

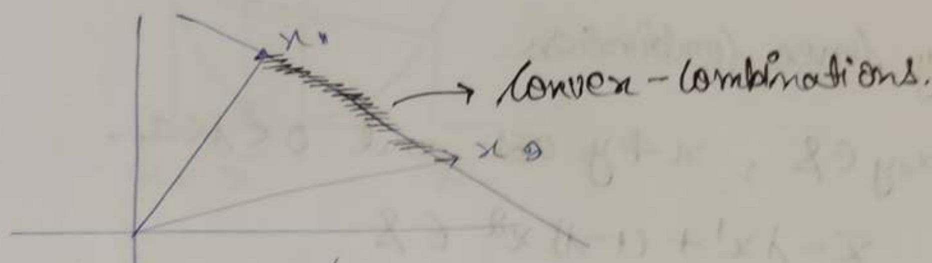
↳ $\lambda (x^1 - x^2) + 1 = \bar{x}$

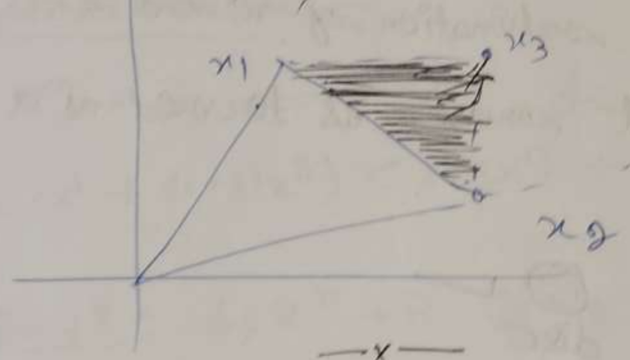
Conic combinations :- $(\lambda \geq 0)$



} → Conic combination

(*) **Convex Combinations** :— $\lambda \geq 0$, $\Sigma \lambda = 1$





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Sets :-

* **Convex Sets** :- Set $S \subseteq R^n$, is called a convex set if $\forall x^1, x^2 \in S$.

$$\bar{x} = \lambda x^1 + (1-\lambda) x^2 \in S$$

convex combination of x^1, x^2.

$$\boxed{0 \leq \lambda \leq 1}$$

* **Strictly Convex Set** :- $x^1, x^2 \neq y$

$x^1, x^2 \in S$ and $0 < \lambda < 1$