CS213/293 Data Structure and Algorithms 2024

Lecture 14: Graphs - Depth-first search

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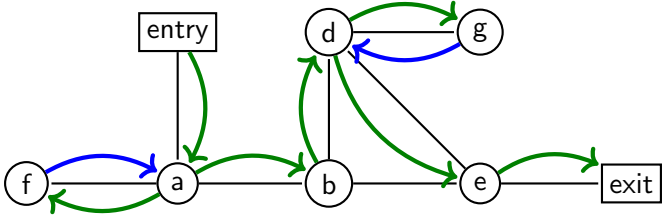
Topic 14.1

Depth-first search (DFS)

Let us solve the maze again

Breadth-first search is about considering all available options before exploring further.

We can have another strategy of search: explore a choice exhaustively before considering another choice.



Algorithm: DFS for search

Algorithm 14.1: DFS(Graph $G = (V, E)$, vertex r, Value x)

```
1  Stack S;
2  set visited;
3  S.push(r);
4  while not S.empty() do
5      v := S.pop();
6      if v.label == x then
7          return v
8      if v ∉ visited then
9          visited := visited ∪ {v};
10         for w ∈ G.adjacent(v) do
11             S.push(w)
```

Example: DFS

Green vertices in the S are already visited vertices and the first unvisited vertex is processed next.

Initially: $S = [entry]$

After visiting entry: $S = [a]$

After visiting a: $S = [f, b, entry]$

After visiting f: $S = [a, b, entry]$

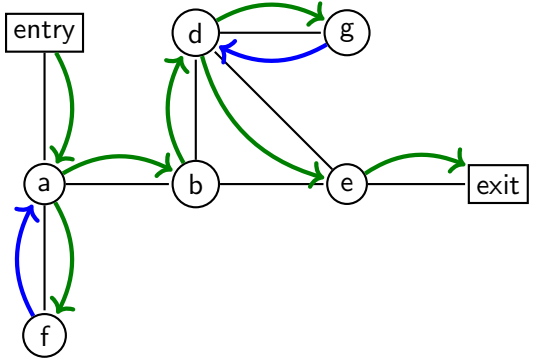
After visiting b: $S = [a, d, e, entry]$

After visiting d: $S = [b, g, e, e, entry]$

After visiting g: $S = [d, e, e, entry]$

After visiting e: $S = [exit, b, d, e, entry]$

After visiting exit: Node is found.



Exercise 14.1

Is there a bound that limits the size of S?

Algorithm: Recursive DFS

The recursive description of DFS is easier to follow.

Algorithm 14.2: DFS(graph $G = (V, E)$, vertex v)

```
1 for v ∈ V do
2     v.visited := False
3 DFSRec(G, v)
```

Algorithm 14.3: DFSRec(Graph G, vertex v)

```
1 v.visited := True;
2 for w ∈ G.adjacent(v) do
3     if w.visited == False then
4         DFSRec(G, w)
```

v can be in three possible states

- v is not visited
- v is on the call stack
- v is visited and not on the call stack

Exercise 14.2

Why is there no stack in the recursive DFS?

DFS Tree

Algorithm 14.4: DFS(graph $G = (V, E)$, vertex v)

```
1 global time := 0;
2 for v ∈ V do
3     v.visited := False
4 DFSRec(G, v)
```

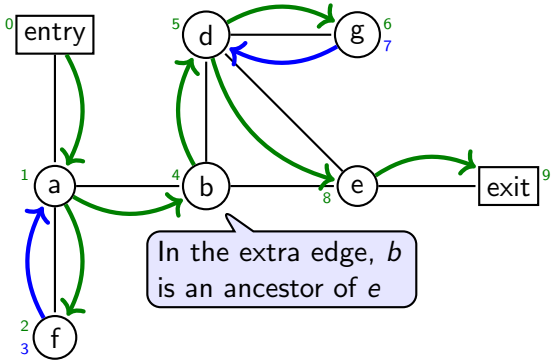
Algorithm 14.5: DFSRec(Graph G, vertex v)

```
1 v.visited := True;
2 v.arrival := time++;
3 for w ∈ G.adjacent(v) do
4     if w.visited == False then
5         w.parent := v;
6         DFSRec(G, w)
7 v.departure := time++;
```

Example: recursive execution

Green numbers are arrival times and blue numbers are the departure times.

DFSREC(G,*entry*)
  DFSREC(G,a)
    DFSREC(G,f)
    Exit DFSREC(G,f)
    DFSREC(G,b)
      DFSREC(G,d)
        DFSREC(G,g)
        Exit DFSREC(G,g)
        DFSREC(G,e)
          DFSREC(G,*exit*)

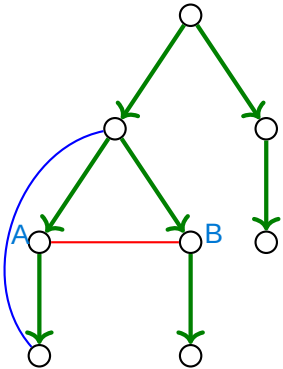


Exercise 14.3

What are the exit timings of the remaining nodes?

Non-tree edges

A run of DFS induces a tree. There are two kinds of possible extra edges.



Exercise 14.4

a. *Is blue edge possible?* *Yes.*

b. *Is red edge possible?* *No.*

Because if Red edge was present then after returning back from node A, it would have visited the node B from their only .

Reachable coverage

Theorem 14.1

Let $G = (V, E)$ be a connected graph. For each $\{v_1, v_2\} \in E$, v_1 is an ancestor of v_2 in DFS tree or vice versa.

Proof.

Without loss of generality, we assume $v_1.arrival < v_2.arrival$ at the end of DFS.

During the run of $\mathrm{DFSREC}(G, v_1)$, v_2 will be visited in one of the following two ways.

1. $\mathrm{DFSREC}(G, v_2)$ is called by $\mathrm{DFSREC}(G, v_1)$. v_1 is the parent of v_2.
2. $\mathrm{DFSREC}(G, v_2)$ has been called already, when the loop in $\mathrm{DFSREC}(G, v_1)$ reaches to v_2.

In either case, $v_2.arrival < v_1.departure$. Therefore, v_1 is an ancestor of v_2 in the DFS tree.

(Why?) □

In case 1, we call $\{v_1, v_2\}$ a tree edge. In case 2, we call $\{v_1, v_2\}$ a back edge.

Commentary: Answer to the above why: v_1 was in the call stack when v_2 arrived. The call stack is the ancestor relation.

Running time of DFS

Theorem 14.2
The running time of DFS is $O(|E| + |V|)$.

Proof.
The total number of recursive calls and iterations of initializations is $O(|V|)$.

In call $DFSRec(G, v)$, the loop iteration is bounded by $degree(v)$.

Therefore, the total number of iterations is $O(|E|)$.

Therefore, the running time is $O(|E| + |V|)$. □

Exercise 14.5
Prove that the running time besides initialization is $O(|E|)$. Use the fact that sigma(deg(v)) is bounded by 2|E|

Algorithm: DFS for not connected graph

Algorithm 14.6: DFSFULL(graph $G = (V, E)$)

```
1 global time := 0;
2 for v ∈ V do
3     v.visited := False
4 while ∃v such that v.visited == False do
5     DFSREC(G, v)
```

Parent relation is a forest

Beacuse we have a "if" condition that, recursion is called on those nodes only which are not visited , and it is connected ,

Theorem 14.3

The parent relation after the run of $\mathrm{DFSFull}$*(graph* $G = (V, E)$ *) induces spanning trees over a connected components of* G*.*

Proof.

Each call to $\mathrm{DFSRec}(G, v)$ will traverse a connected component of G that contains unvisited node v. (Why?)

If the component has k nodes, then the tree has $k - 1$ edges, because the parent of v will be Null.

Therefore, the parent relation is a tree over the component. □

Topic 14.2

Does the graph have a cycle?

Detecting cycle

If there is a back edge, there is a cycle. We modify our DFSRec as follows.

Algorithm 14.7: DFSRec(Graph G, vertex v)

```
1 v.arrival := time++;
2 for w ∈ G.adjacent(v) − {v.parent} do
3     if w.visited == False then
4         w.parent := v;
5         DFSRec(G, w)
6     else
7         raise "Found Cycle";
8 v.departure := time++;
```

Back edge == cycle

Theorem 14.4

A graph G has a cycle iff DFSFull(G, v) has a back edge for some $v \in G$.

Proof.

forward direction: Contrapositive

Due to theorem 14.3, each call to DFSRec without exception will produce a spanning tree over a connected component of G.

Since there are no extra edges besides the tree, the cycle cannot be formed within the component.

reverse direction:

If the exception "Found cycle" is raised, there are two paths between u and w.

- Edge $\{u, w\}$.
- path via the parent relation that does not contain $\{u, w\}$.

Therefore, we have a cycle. □

Topic 14.3

Checking 2-edge connected graphs

2-edge connected graph

Definition 14.1

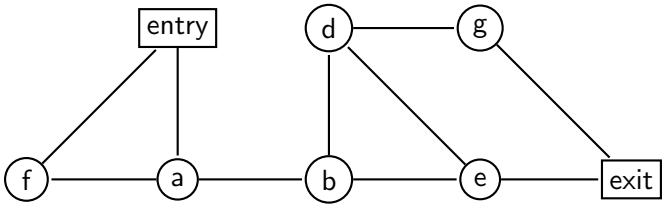
A graph $G = (V, E)$ is 2-edge connected if for each $e \in E$, $G - \{e\}$ is a connected graph.

2-edge connected graphs are useful for designing resilient networks that are tolerant of link failures.

Example: 2-edge connected graphs

Example 14.1

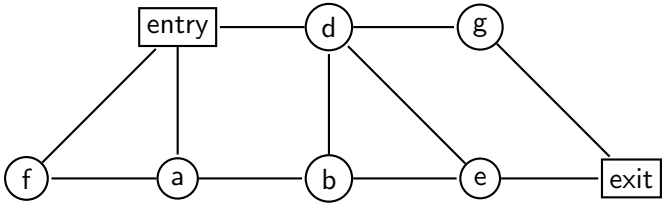
The following graph is not 2-edge connected. $\{a, b\}$ is called bridge.



Example: 2-edge connected graphs (2)

Example 14.2

The following graph is 2-edge connected.



Näive algorithm for checking 2-edge connectivity

For each edge, delete the edge and check connectedness.

The algorithm will run in $O(|E|^2)$.

We are looking for something more efficient.

Idea: 2-edge connectivity via DFS

Observation 1: If we delete any number of back edges, the graph remains connected.

Observation 2: If a tree edge is part of some cycle, the graph remains connected after its deletion.

How can we check this?

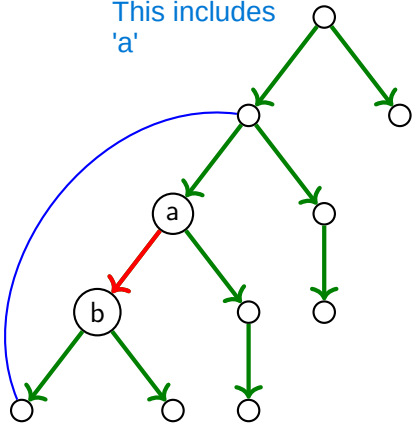
Checking participation in a cycle

Example 14.3

The red edge (a, b) in the following DFS tree is part of a cycle if there is a back edge that starts at one of the decedents of b and ends at an ancestor of a.

This also includes 'b'

This includes 'a'



Deepest back edge

We need to track the back edges that cover most edges.

Definition 14.2

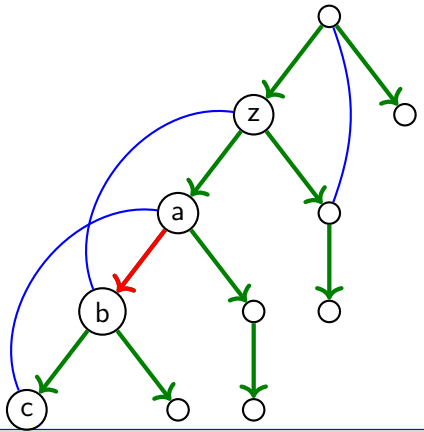
The deepest back edge for a vertex is the back edge that goes from the descendent of the vertex to an ancestor of the lowest level.

Example 14.4

In the following DFS tree, there are two back edges from the decedents of b.

$\{b, z\}$ *is deeper back edge than* $\{c, a\}$.

Since there is no other back edge from decedents of b, $\{b, z\}$ *is the deepest back edge for b.*

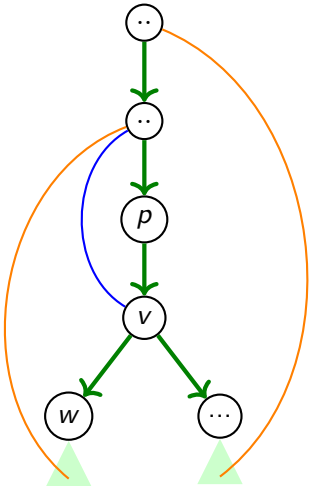


How do we identify the deepest back edge?

By comparing the arrival times of the destinations, we identify the deepest back edge.

We consider all neighbors of vertex v to find the deepest back edge. There are three possible cases.

1. child on DFS tree: recursively find the deepest edge
2. parent on the DFS tree: to be ignored
3. back edge: candidate for the deepest edge



Algorithm: 2-edge connectedness

Algorithm 14.8: int $2\mathrm{EC}$(Graph G, vertex v)

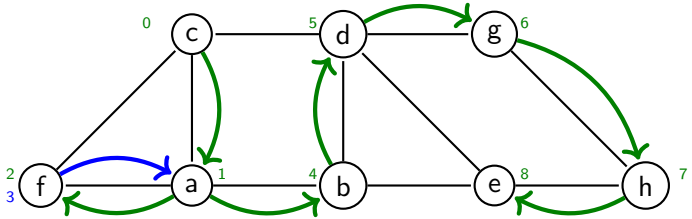
```
1  v.visited := True;
2  v.arrival := time++;
3  deepest := v.arrival;
4  for w ∈ G.adjacent(v) − {v.parent} do
5      if w.visited == False then
6          w.parent := v;
7          deepest' := 2EC(G, w);
8      else
9          deepest' := w.arrival;
10     deepest := min(deepest, deepest');
11 v.departure := time++;
12 if v.parent ≠ Null and v.arrival == deepest then raise "Bridge found!";
13 return deepest;
```

So first assume that the node where we starts, has the smallest arrival time, however if any of it's children node has a back edge, It would have been visited earlier by bfs inducing a smaller arrival time and the one with the smallest arrival time is the deepest backedge

Example: 2-edge connectedness

Exercise 14.6

Consider the following DFS run of the following graph.



What are the deepest back edges for the following nodes?

- *e* b,e
- *h* b,e
- *g* b,e
- *d*
- *b*
- *f*
- *a*
- *c*

Topic 14.4

Depth-first search for directed graph

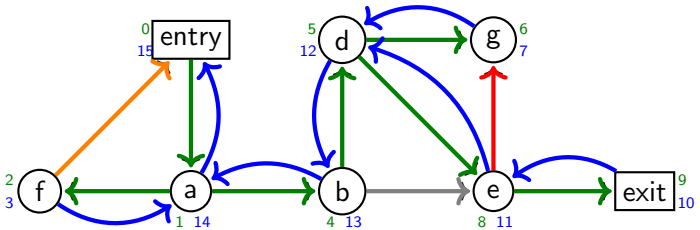
DFS for directed graph

There is no change in the code of DFSFULL for the directed graph, the code will work as it is.

However, some of the behavior concerning extra edges will change.

Example : DFS on the directed graph

Consider the following directed graph.



Now we have three kinds of extra edges.

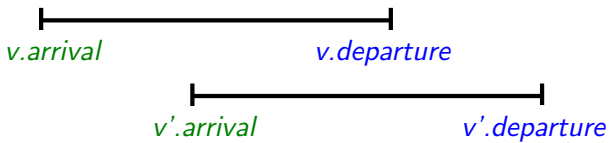
- Forward edge:(b, e), where $b.arrival < e.arrival < e.departure < b.departure$
- Back edge:$(f, entry)$, where $entry.arrival < f.arrival < f.departure < entry.departure$
- Cross edge:(e, g), where $g.arrival < g.departure < e.arrival < e.departure$

Are there other kind of edges?

Interleaved intervals are not possible

Theorem 14.5

For each $v, v' \in V$, $v.arrival < v'.arrival < v.departure < v'.departure$ is not possible.



Proof.

Let us assume $v'.arrival$ is between $v.departure$ and $v.arrival$.

Therefore, v is in the call stack when v' is put on the call stack during a run of DFSRec.

v' will leave the call stack before v.

Therefore, $v'.departure < v.departure$. The ordering of events is not possible. □

DFS always follows the available edges.

Theorem 14.6

For each $(v, v') \in E$, $v.arrival < v.departure < v'.arrival < v'.departure$ is not possible.

Proof. Because v'.arrival > v.arrival means v' is unvisited during the DFS(G,v) call => it will call DFS(G,v') => in call it will be above DFS(G,v) => v.dept > v'.dept> v'.arrival , Hence The above was not possible

Apply theorem 14.1 after replacing the undirected edges with the directed edges in the theorem. □

Exercise 14.7

a. Prove theorem 14.1 for directed graph.

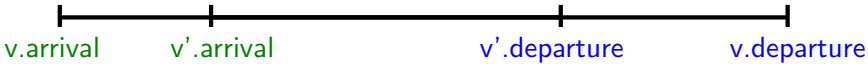
b. The theorem was proven for DFSRec. *Extend it for* DFSFull.

Commentary: We need to reword the theorem to show that v' will be on the call stack before v departs the call stack.

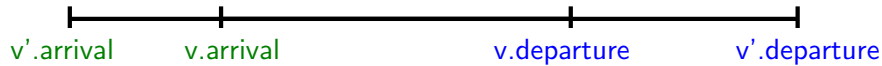
Extra edges

We are left with only the following possibilities for the extra and tree edges. Let $(v, v') \in E$.

- Forward edge/Tree edge



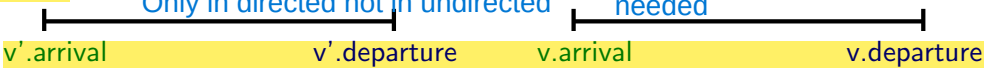
- Back edge:



- Cross edge:

For cross edge condition on both arrival and departure would be needed

Only in directed not in undirected



Exercise 14.8 (Final 2023)

By the explanation of call stack It will suffice

a. Show: If $v.departure \leq v'.departure$, (v, v') is a back edge.

b. Give the condition that identifies the back or cross edge.

Topic 14.5

Does the directed graph have a cycle?

Idea: Back edge == cyclic

If DFS finds a back edge there is a cycle in a (directed) graph.

Exercise 14.9

How can we use BFS to find cycles?

Algorithm: Has Cycle?

Algorithm 14.9: HasCycle(directed graph $G = (V, E)$)

```
1 Let v ∈ V;
2 DFSFull(G, v);
3 if ∃(v, v') ∈ E such that v.departure ≤ v'.departure then
4     return True;
5 return False;
```

Check For backedge

Back edge == Cycle

Theorem 14.7

A directed graph $G = (V, E)$ has a cycle iff DFSFull(G) has a back edge.

Proof.

forward direction:

Suppose there is no back edge. Therefore, $\forall (v, v') \in E$, $v.departure > v'.departure$.

Sort all the nodes by their departure times.

All edges will be going in one direction of the sorted sequence. Therefore, there is no cycle.

reverse direction:

Let us suppose there is a back edge $(v, v') \in E$. Therefore, $v.departure \leq v'.departure$.

Due to properties of the back edge, v' must be on the call stack when v departs.

Therefore, there is a path from v' to v. Therefore, there is a cycle. □

Commentary: Does the above argument work when v and v' are equal?

Topic 14.6

Topological sort

Topological order

Definition 14.3

For a DAG $G = (V, E)$, the topological order $<$ is an order of vertices of V such that if $(v, v') \in E$ then $v < v'$.

Algorithm: topological sort

Algorithm 14.10: TOPOLOGICALSORT(directed graph $G = (V, E)$)

```
1 DFSFULL(G, v);
2 if ∃(v, v') ∈ E such that v.departure ≤ v'.departure then
3     return "Cycle found: Sorting not possible";
4 return sorted vertices of V in the decreasing order of departure.
```

Exercise 14.10

Can we avoid sorting after the DFS run?

Just before end of DFSFull function, push the node in stack

Topic 14.7

Is strongly connected?

Strongly connected (Recall)

Consider a directed graph $G = (V, E)$.

Definition 14.4

G is strongly connected if for each $v, v' \in V$ there is a path $v, ..., v'$ in E.

Näive algorithm

Run DFS from each vertex, and check if all vertices are reached.

The running time complexity is $O(|V||E|)$.

Strongly connected via DFS

Condition 1: If $DFS(v)$ visits all vertices in G then there is a path from v to each vertex in G.

Condition 2: There is a path from every node in G to v .

We can check condition 1 using DFS 1. How can we check condition 2?

Kosaraju's algorithm

Run DFS on G and G^R (All edges of G are reversed) from some vertex v.

If BOTH DFSs cover all nodes, G is strongly connected.

The running time complexity is $O(|V| + |E|)$.

Exercise 14.11
Can we use BFS here?

Let us see if we can avoid two passes of the graph.

How can we check if we can reach the root of DFS?

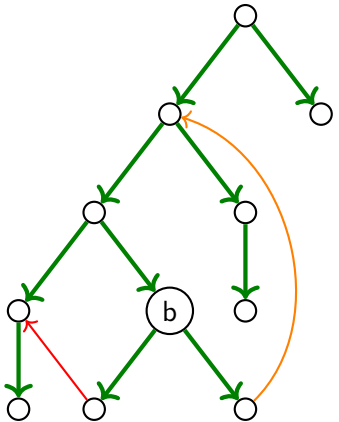
Example 14.5

Consider vertex b. We must be able to escape the subtree of b to reach the root.

There are only two ways to escape a subtree.

- Back edge
- Cross edge

Can both routes to escape be useful?



Lower arrival times

Observation: both back and cross edges take us to lower arrival times.

Induction: If we can escape from the subtree of each vertex, we are guaranteed to reach the root.

All we have to do is to show that we can escape from each vertex.

Use earliest escape edge? (Same idea as 2EC?)

Algorithm: SC

Algorithm 14.11: int SC(Graph G, vertex v)

```
1  v.visited := True;
2  v.arrival := time++;
3  earliest := v.arrival;
4  for w ∈ G.adjacent(v) do
5      if w.visited == False then
6          w.parent := v;
7          earliest' := SC(G, w);
8      else
9          earliest' := w.arrival;
10     earliest := min(earliest, earliest');
11 v.departure := time++;
12 if v.parent ≠ Null and v.arrival == earliest then raise "Not strongly connected!";
13 return earliest;
```

That is No back or cross edge from here

Topic 14.8

Tutorial problems

Exercise: 2-vertex connected graph

Definition 14.5

A graph $G = (V, E)$ is 2-vertex connected if for each $v \in V$, $G - \{v\}$ is a connected graph.

Exercise 14.12

Give an algorithm that checks if a graph is 2-vertex connected.

Exercise: Change in DFS

Suppose that we have a graph and we have run DFS starting at a vertex s and obtained a DFS tree. Next, suppose that we add an edge (u,v). Under what conditions will the DFS tree change? Give examples.

2-edge-connectedness in theory

Exercise 14.13

Let G(V,E) be a graph. We define a relation on edges as follows: two edges e and f are related if there is a cycle containing both (denoted by $e \equiv f$).

1. *Show that this is an equivalence relation. The equivalence class [e] of an edge e is called its connected component.*
2. *What is the property of the equivalence relation when we say the graph is 2-edge connected?*

Exercise: SCCs via Kosaraju's algorithm

Exercise 14.14

Modify Kosaraju's algorithm to identify all SCCs of a graph.

Exercise 14.15

Give similar modifications for the algorithm SC.

Exercise: Classification of edges via BFS

Exercise 14.16

If we run BFS on a directed graph, can we define the same classes of edges, i.e., cross, tree, back, and forward edges? Give conditions for each class.

Detecting cycle during the run for a directed graph!

Exercise 14.17

Let us modify DFSRec *to detect cycles during the run. Give the expression for the condition to detect the cycles.*

Algorithm 14.12: DFSRec(Graph G, vertex v)

```
1 v.visited := True;
2 v.arrival := time++;
3 for w ∈ G.adjacent(v) do
4     if w.visited == False then
5         DFSRec(G, w)
6     else
7         if condition then
8             throw "Found Cycle"
9 v.departure := time++;
```

Topic 14.9

Problems

Exercise: another search

Exercise 14.18

We have a new search algorithm that uses a set S for which we have two functions (i) ADD(x,S) which adds x to S, and (ii) y = SELECT(S) which returns an element of S following a certain rule and removes y from S.

Algorithm 14.13: int MYSEARCH(Graph $G = (V, E)$, vertex s)

```
1 for v ∈ V do  v.visited = False;
2 for e ∈ E do  e.found = False;
3 S=empty;ADD(s,S);nos := 1;record[nos] := s;
4 while not S.empty() do
5 |   s := select(S); nos := nos + 1; record[nos] = y;
6 |   for  w ∈ G.adjacent(v) do
7 |   |   if w.visited == False then  w.visited := True; found[{u, v}] = true; ADD(v, S) ;
```

1. Compare the bfs and dfs algorithms with the above code. Take special care in understanding visited.
2. Let us look at the sequence record[1],record[2],...,record[n]. Show that there is a path from record[i] to record[i+1] using only edges that have been found at that point.
3. Compare BFS and DFS in terms of the above path lengths.

Topic 14.10

Extra slides: Topological sort

Algorithm: Topological sort using DFS

An implementation with cycle detection and in-place sorting.

Algorithm 14.14: TopologicalSort(Directed graph $G = (V, E)$)

```
1 Stack Sorted;
2 while ∃v such that v.visited == False do visit(v) ;
```

Algorithm 14.15: Visit(Graph $G = (V, E)$, vertex v)

```
if v.visited then return;
if v.onPath then throw Cycle found ;
v.onPath = True;
for w ∈ G.adjacent(v) do
    Visit(w)
v.onPath = False;
v.visited = True;
Sorted.push(v);
```

End of Lecture 14