

CS213/293 Data Structure and Algorithms 2024

Lecture 13: Graphs - Breadth-first search

Instructor: Ashutosh Gupta

IITB India

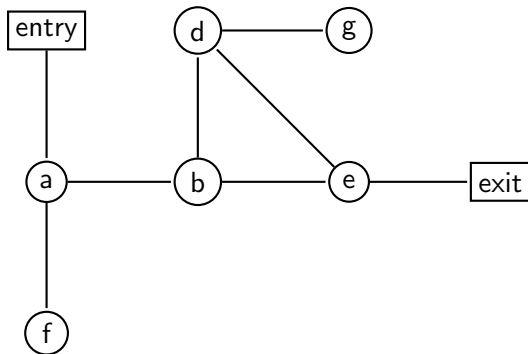
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Topic 13.1

Breadth-first search (BFS)

Solving a maze

What is a good way of solving a maze?



- ▶ Every choice point is a vertex
- ▶ Paths connecting the points are edges
- ▶ Problem: find the exit node

Breadth-first search



Definition 13.1

A *breadth-first search (BFS)* traverses a connected component in the following order.

- ▶ *BFS starts at a vertex, which is at level 0.*
- ▶ *BFS traverses the unvisited adjacent vertices of level $n - 1$ vertices, which are the vertices at level n .*

The above traversal defines a spanning tree of the graph.

In the algorithm, we need to keep track of the already visited vertices and visit vertices at lower level first.

Algorithm: BFS for search

Algorithm 13.1: BFS(Graph $G = (V, E)$, vertex r , Value x)

```
1 Queue Q;  
2 set  $visited := \{r\}$ ;  
3  $Q.enqueue(r)$ ;  
4 while not  $Q.empty()$  do  
5      $v := Q.dequeue()$ ;  
6     if  $v.label == x$  then Marks exit  
7         return  $v$   
8     for  $w \in G.adjacent(v)$  do  
9         if  $w \notin visited$  then  
10              $visited := visited \cup \{w\}$ ;  
11              $Q.enqueue(w)$ 
```

A vertex can be in three possible states

- ▶ Not visited
- ▶ Visited and in queue
- ▶ Visited and not in queue

Exercise 13.1

How do we maintain the visited set?

Commentary: We are only inserting and checking membership. There is no delete. We may mix ideas from hashing and BST. Bloom filter is a classic technique to store such objects. https://en.wikipedia.org/wiki/Bloom_filter

Example : BFS

Initially: $Q = [entry]$

After visiting entry: $Q = [a]$

After visiting a: $Q = [f, b]$

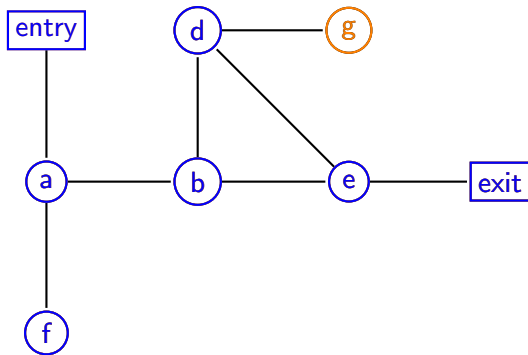
After visiting f: $Q = [b]$

After visiting b: $Q = [e, d]$

After visiting e: $Q = [d, exit]$

After visiting d: $Q = [exit, g]$

After visiting exit: the return is triggered



We not only want to find the exit node **but also the path** to the exit node.

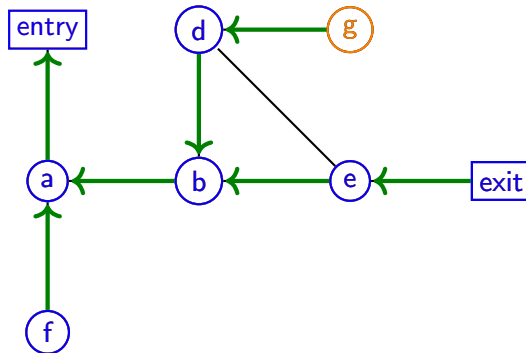
Algorithm: BFS for a path to the found node

Algorithm 13.2: BFS(Graph $G = (V, E)$, vertex r , Value x)

```
1 Queue Q;  
2 visited := { $r$ };  
3  $Q.enqueue(r)$ ;  
4 while not  $Q.empty()$  do  
5      $v := Q.dequeue()$ ;  
6     if  $v.label == x$  then  
7         return  $v$   
8     for  $w \in G.adjacent(v)$  do  
9         if  $w \notin visited$  then  
10              $visited := visited \cup \{w\}$ ;  
11              $Q.enqueue(w)$ ;  
12              $w.parent = v$ 
```

Example : BFS with parent relation

Green edges point at the parents.



Algorithm: BFS for rooted spanning tree

We do not stop at some node but traverse the entire graph.

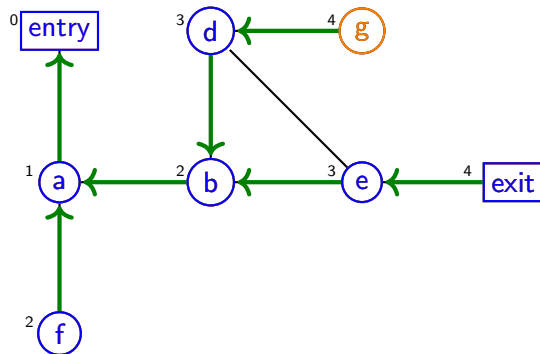
We also keep track of levels, which allow us to record **auxiliary information** about the algorithm.

Algorithm 13.3: BFSSPANNING(Graph $G = (V, E)$, vertex r)

```
1 Queue Q;  
2  $visited := \{r\}$ ;  
3  $Q.enqueue(r)$ ;  
4  $r.level := 0$ ;  
5 while not  $Q.empty()$  do  
6    $v := Q.dequeue()$ ;  
7   for  $w \in G.adjacent(v)$  do  
8     if  $w \notin visited$  then  
9        $visited := visited \cup \{w\}$ ;  
10       $Q.enqueue(w)$ ;  
11       $w.parent := v$ ;  
12       $w.level := v.level + 1$ 
```

Example : Spanning tree from BFS

Superscripts are level of the nodes.



Topic 13.2

Analysis of BFS

Running time of BFS

- ▶ For Each node there is an enqueue and a dequeue. Therefore, $O(|V|)$ queue operations.
- ▶ For each node adjacent nodes are enumerated. Therefore, the inner loop will have $O(|E|)$ iterations.

Therefore, the running time is $O(|V| + |E|)$

The pattern in the content of Q

Theorem 13.1

Q has vertices with level sequence $k \dots k \underbrace{(k+1) \dots (k+1)}_{\text{possibly empty}}$ for some k .

Proof.

We prove it by induction.

Base case:

Initially, Q has level sequence 0.

Induction step:

Let us suppose at a given time the level sequence is $k \dots k \underbrace{(k+1) \dots (k+1)}$.

We will dequeue a k and possibly add a number of $k+1$ s. Hence proved. □

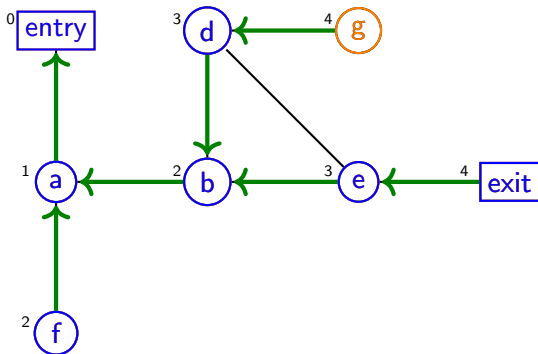
Example: contents of Q

Example 13.1

Let us look at the content of Q in our running example.

Different states of Q

Q	Level sequence in Q
$[entry]$	0
$[a]$	1
$[f, b]$	2,2
$[b]$	2
$[e, d]$	3,3
$[d, exit]$	3,4
$[exit, g]$	4,4



Level difference of adjacent nodes.

Theorem 13.2

For any edge $\{v, v'\} \in E$, $|v.level - v'.level| \leq 1$.

Proof.

Let us suppose v was added to Q before v' .

Due to the previous theorem, the vertices will enter Q with increasing levels.

We have two possible cases.

- ▶ v' entered Q at the iteration for the dequeue of v . Therefore, $v'.level = v.level + 1$.
- ▶ v' entered Q before dequeue of v . $v'.level$ must be either $v.level + 1$ or $v.level$. (Why?)



Commentary: In the last case, v is in Q when v' is entering, therefore $v'.level$ cannot enter Q with a value greater than $v.level + 1$.

BFS finds the shortest path

Theorem 13.3

For each $v \in V$, the path from v to r via the parent field is a path with the shortest length.

Proof.

Since $r.level = 0$, the path from v to r has $v.level$ edges.

Due to the previous theorem, no edge can reduce *level* more than one, the lengths of all paths to r from v cannot be smaller than $v.level$. □

Topic 13.3

Finding connected components

Detect a component

Algorithm 13.4: BFSCONNECTED(Graph $G = (V, E)$, Vertex r , int id)

```
1 Queue Q;  
2 visited := { $r$ };  
3 Q.enqueue( $r$ );  
4 r.component :=  $id$ ;  
5 while not Q.empty() do  
6      $v$  := Q.dequeue();  
7     for  $w \in G.adjacent(v)$  do  
8         if  $w \notin visited$  then  
9             visited := visited  $\cup$  { $w$ };  
10            Q.enqueue( $w$ );  
11            w.component :=  $id$ 
```

Find all connected components

Algorithm 13.5: CC(Graph $G = (V, E)$)

```
1 for  $v \in V$  do  
2    $v.component := 0$   
3  $componentId := 1$ ;  
4 while  $r \in V$  such that  $r.component == 0$  do  
5   BFSCONNECTED( $G, r, componentId$ );  
6    $componentId := componentId + 1$ ;
```

This works , assuming that the Queue and set visited and global

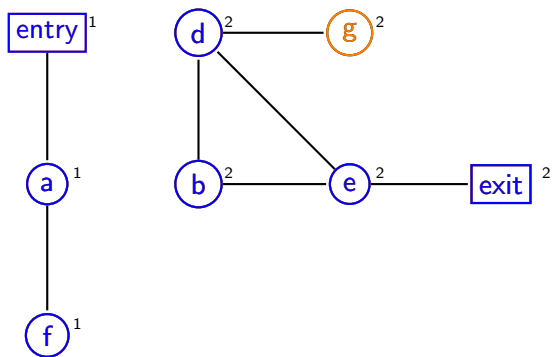
Exercise 13.2

- What is the cost of evaluating the condition at line 4?
- What is the running time of the above procedure?

Doubt

Commentary: We should not evaluate the condition at line 4 from the start. We should try to reuse the previous runs of the condition.

Example: find all connected components



Topic 13.4

Checking bipartite graph

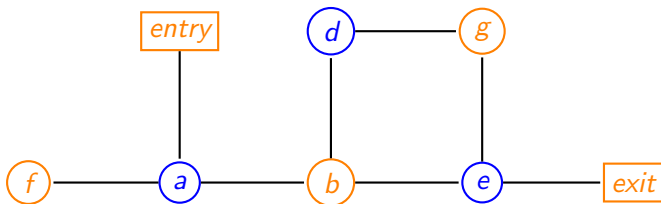
Bipartite graphs

Definition 13.2

A graph $G = (V, E)$ is bipartite if there are V_1 and V_2 such that $V = V_1 \uplus V_2$ and for all $e \in E$, $e \not\subseteq V_1$ and $e \not\subseteq V_2$.

Example 13.2

The following is a bipartite graph. Where $V_1 = \{\text{entry}, f, b, g, \text{exit}\}$ and $V_2 = \{a, d, e\}$.



Theorem 13.4

A bipartite graph does not contain cycles of odd length. Done in tutorial.

Checking Bipartite graph

Algorithm 13.6: ISBIPARTITE(Graph $G = (V, E)$)

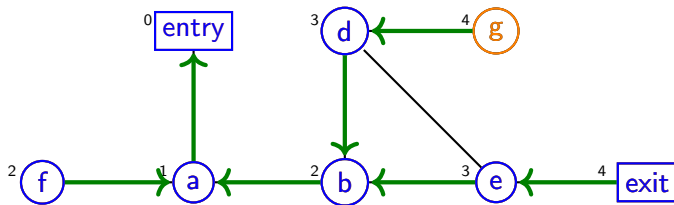
```
1 ASSUME(graph is connected);
2 Choose  $r \in V$ ;
3 BFS_SPANNING( $G, r$ );
4 if for each  $\{v, v'\} \in E$   $v.level \neq v'.level$  then
5     return True;
6 return False;
```

Exercise 13.3

- Modify the above algorithm to support not-connected graph.
- What is the cost of evaluating the condition at line 4?
- What is the running time of the above procedure?

Example : IsBIPARTITE

We ran BFS on the following graph.

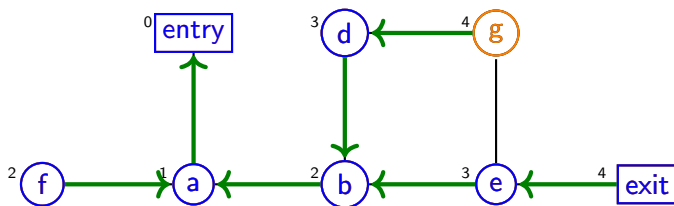


Since $d.level = b.level$, $\{d, e\}$ edge causes the if condition to fail.

Therefore, the graph is not bipartite.

Example : another example for IsBIPARTITE

We ran BFS on the following graph.



BFS spanning tree edges will naturally satisfy the if condition.

Since $g.level \neq e.level$, the extra edge (g, e) also does not cause the if condition to fail.

Therefore, the graph is bipartite.

Correctness of IsBIPARTITE

Theorem 13.5

If $\text{IsBIPARTITE}(G = (V, E))$ returns true, G is bipartite.

Proof.

Let $V_1 = \{v | v.\text{level} \% 2 = 0\}$ and $V_2 = \{v | v.\text{level} \% 2 = 1\}$.

Due to the condition in IsBIPARTITE , there are no edges connecting the same level.

Due to theorem 13.2, we only have edges connecting neighboring levels.

There are no edges that are inside V_1 or V_2 .



Correctness of IsBIPARTITE

Theorem 13.6

If $\text{IsBIPARTITE}(G = (V, E))$ returns false, G is not bipartite.

Proof.

Since false is returned, there exists $\{v_1, v_2\} \in E$ such that $v_1.\text{level} = v_2.\text{level}$.

Let v be the least common ancestor of v_1 and v_2 in the spanning tree induced by the run of BFS.

The lengths of paths v_1, \dots, v and v_2, \dots, v are the same. (Why?)

Therefore, path $v_1..v...v_2v_1$ is an odd cycle.

Therefore, G is not bipartite.



Topic 13.5

Diameter of a graph

Diameter of a graph G

Definition 13.3

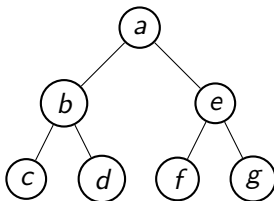
For a graph G , let the *distance*(v, v') be the length of one of the shortest paths between v and v' .

Definition 13.4

For a graph $G = (V, R')$, *diameter*(G) = $\max\{\text{distance}(v, v') \mid \{v, v'\} \in E\}$.

Diameter is only defined for a connected graph.

Example 13.3



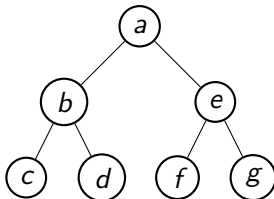
The diameter of the above graph is 4.

Can we use BFS for diameter?

Let us run $\text{BFSSpanning}(G = (V, E), r)$ for some $r \in V$.

Let *maxlevel* be the maximum level assigned to a node in the above graph.

Example 13.4



In the above graph, let $r = a$. The maxlevel is 2.

BFS diameter relation.

Theorem 13.7

Let maxlevel be the maximum level assigned to a node in $G = (V, E)$ after running BFS from node r . $\text{maxlevel} \leq \text{diameter}(G) \leq 2 * \text{maxlevel}$

Proof.

Since there are nodes maxlevel distance away from r , $\text{maxlevel} \leq \text{diameter}(G)$.

Let $v_1, v_2 \in V$. $\text{distance}(r, v_1) \leq \text{maxlevel}$ and $\text{distance}(r, v_2) \leq \text{maxlevel}$.

Therefore, $\text{distance}(r, v_1) + \text{distance}(r, v_2) \leq 2 * \text{maxlevel}$.

Therefore, $\text{distance}(v_1, v_2) \leq 2 * \text{maxlevel}$.

This holds for all v_1, v_2
Hence holds for diameter

Therefore, $\text{diameter}(G) \leq 2 * \text{maxlevel}$.



All runs of BFS

Algorithm 13.7: DIAMETER(Graph $G = (V, E)$)

```
1 ASSUME(graph is connected);
2  $maxlevel := 0$ ;
3 for  $r \in V$  do
4     BFSSPANNING( $G, r$ );
5      $maxlevel' :=$  maximum level assigned to a node;
6      $maxlevel := \max(maxlevel, maxlevel')$ 
7 return  $maxlevel$ ;
```

Exercise 13.4

Is this the best algorithm for Diameter computation? Ask Search engines, LLMs, etc.

Topic 13.6

Tutorial problems

Exercise: shortest path

Exercise 13.5 Doubt

There are many variations of BFS to solve various needs. For example, suppose that every edge $e=(u,v)$ also has a weight $w(e)$ (say the width of the road from u to v). Assume that the set of values that $w(e)$ can take is small. For a path $p=(v_1,v_2,\dots,v_k)$, let the weight $w(p)$ be the minimum of the weights of the edges in the path. We would like to find a shortest path from a vertex s to all vertices v . If there are multiple such paths, we would like to find a path whose weight is maximum. Can we adapt BFS to detect this path?

Exercise: correctness of BFS

Exercise 13.6

Write an induction proof to show that if vertex r and v are connected in a graph G , then v will be visited in call `BFSCONNECTED(Graph $G = (V, E)$, Vertex r , int id)`.

Exercise: Graph with two kinds of edges

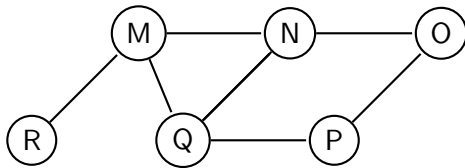
Exercise 13.7

Suppose that there is an undirected graph $G(V,E)$ where the edges are colored either red or blue. Given two vertices u and v . It is desired to (i) find the shortest path irrespective of color, (ii) find the shortest path, and of these paths, the one with the fewest red edges, (iii) a path with the fewest red edges. Draw an example where the above three paths are distinct. Clearly, to solve (i), BFS is the answer. How will you design algorithms for (ii) and (iii)?

Exercise: order of traversal (quiz 23)

Exercise 13.8

Is MNRQPO a possible BFS traversal for the following graph?



End of Lecture 13