Dynamic Communication with Trading Commissions

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Abstract

This paper studies a dynamic cheap-talk communication game to investigate the role of trading commissions in broker-investor relationships. The investor's optimal portfolio mix depends on a fixed state which is privately known to the broker. The broker's message space includes an uninformative signal in addition the state space. Market fluctuations imply that the portfolio-mix evolves as a Brownian motion. However, it can be adjusted by the investor at a fixed cost paid to the broker. I characterize the Markov-Perfect Equilibria and show that the portfolio mix has an informational value to the investor because it influences the broker's incentives for truth-telling. I also show that the option to delay information provision restricts the set of such 'truth-telling' portfolio-mix-values. Crucially, delaying information provision is optimal for a subset of parameter values. Finally, this model illustrates a novel mechanism to explain investor over-trading.

1 Introduction

Investors, and especially large institutional investors such as pension funds or hedge funds enter into contracts with brokerage firms, who execute transactions of securities on their behalf. The brokerage firm is compensated through a commission on every transaction. In practice, brokers are observed to charge both per-trade (fixed) commissions, in which the commission charge does not vary with the size of the trade, and per-share commissions, which means that the commission charge is proportional to the size of the trade.

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Apart from execution services, brokers often also provide research and analysis services to the investors as well. For example, 'full-service' brokers can be contacted personally at the time of trading to provide investors with advice and expertise, as opposed to contracts limited to online-trading. While various leading brokerage firms have started to charge zero commissions for online trading[1], commissions for broker-assisted trades and 'full-service' contracts are charged at approximately $25-30 per trade.

Such services are often bundled with the execution services in what are called 'soft-dollar' arrangements and are not explicitly charged. For example, 'sell-side' analysts at a brokerage firm cover or analyze a certain set of securities and companies and make predictions of their performance. These may include detailed predictions such as earnings forecasts or general 'buy/sell' recommendations. Even though such information is available to the public, there is some evidence that analysts leak the contents of their reports to 'influential' or important trading clients before they are publicly released. Hence, the broker or adviser may have some discretion in releasing information to some clients who are more preferred than others.[2]

Compensation of brokers through commissions leads to certain problems and conflicts of interests. Brokers may provide biased information to generate larger trades, or commit to withhold information to induce the investors to trade and generate commission revenue. When portfolio management or trades are delegated to the broker, there may be problems of excessive churning of the investor's account at her expense.

Motivated to study the implications of commissions and particularly the ability to withhold information, in such a dynamic adviser-advisee or sender-receiver relationship, I study a model in which a decision maker's payoff depends on two states. One *state* is privately known to the sender but not to the receiver, and the other state, called a *position*, evolves stochastically but can be controlled at a cost which is paid to the sender. The receiver gets a high flow payoff if the position is held close to value of the true state.

The position is similar to an investor's portfolio position in certain assets, which depends on the stochastic price process of the assets. It can be changed by buying and selling units of the assets, but at a transaction cost paid to the broker as a commission. The hidden state is similar to some parameter which is known to the broker and which affects the risk-return tradeoff and influences the investor's optimal portfolio. In the main

[1] See `https://www.cnbc.com/2019/11/06/as-brokerage-firms-go-to-zero-commission-on-trades-advisors-w` `html`

[2] For example, see Maggio et al. (2019)

motivating example I discuss at the end of this section, I show how an optimal dynamic portfolio allocation problem can be adapted to include transaction costs and asymmetric information.

An alternative way to state the problem is that the receiver faces an optimal stochastic control problem of adjusting the position-state at a cost. If the receiver knew the state, her optimal course of action is an (s, S) policy. However, the optimal target value of the process is affected by the state, which is known privately by the sender, who earns the adjustment costs. Such models, in which an agent faces a cost and her optimal dynamic action is an (s, S) policy, are widely used in the literature. To my knowledge, I am the first to model a setting where a second agent earns the adjustment costs and has influence in choosing the adjustment thresholds.

The Sender is allowed to communicate via cheap-talk messages and I show that the Sender's incentives to truthfully reveal the state depend on the value of the position. Even if the current position is close to the true state, the sender has an incentive to misreport and convince the receiver to adjust it and incur the adjustment cost. However, for certain values of the position, the sender has no incentive to lie as the receiver would adjust no matter what the report. Thus, truth-telling can be sustained at a certain set of position values.

In any equilibrium, this influences the behavior of the receiver while uninformed, who changes positions at a cost to acquire truthful information from the sender. I focus on a refinement of equilibria in which the Sender is allowed to withhold information or send uninformative messages, but cannot commit to such messages. In such equilibria, his optimization problem can effectively be represented as an optimal stopping problem of when to reveal the truth. The equilibrium set of 'truth-telling' positions shrinks as information becomes more valuable, and when it is eventually valuable enough, the only pure strategy equilibrium is the babbling equilibrium.

The features of the equilibria suggest that the sender does not need commitment power to induce the the receiver to change his position- or analogously a broker does not need the threat of withholding information to induce excessive trades by the investor. Adjusting position values to get more reliable information, and the sender withholding information even though truth-telling is incentive compatible, turn out to be equilibrium actions even in Markov strategies. This has implications for the upper bound on the receiver's value in any equilibrium.

When the commissions charged are proportional to the change in position, the set of positions where truth-telling is incentive compatible collapses to a singleton. There is a unique position such that the sender has no incentive to lie at that value. Hence, it is this unique position which provides informational value to the receiver.

In theoretical literature on information provision, the role of commissions in financial advice is well-studied but limited to static settings, both in environments with advisers (or senders) making a cheap-talk message choice or a choice of information precision. In dynamic settings, the sender is usually assumed to have a preference on the timing of a certain action - the setting I study shows endogenous 'delays' even without such an assumption.

The results have practical implications for empirical work on brokerage advice and trading as well. They demonstrate that brokers have an incentive to 'delay' information provision to investors with certain portfolios verses others. Hence, portfolios have an informational value apart from their actual value, when the 'full-service' broker has some private information about certain parameters, and the investors would have a bias towards holding such portfolios in times of uncertainty. The equilibrium itself induces trade or portfolio adjustment by the receiver, who changes the portfolio position to eliminate the sender's bias. This implies that there may be excessive trading by an uninformed investor, which contrasts with Kyle (1985) and Glosten and Milgrom (1985), where the informed trader reduces trade to prevent the market maker from learning the state, which is the value of the asset.

I now discuss the main motivating example - a simplified version of the portfolio allocation problem studied in Merton (1969), and discuss the related literature. Then, I introduce the model and discuss the full-information case. In the Asymmetric information case, I discuss equilibria both with and without delays and present the main results. I then present a class of mixed-strategy equilibria and show related results for the model with proportional commissions and a stochastic state.

Motivating Example

Consider a setting similar to Merton (1969), in which a risk-averse agent makes a dynamic portfolio allocation decision.

We know and can show, as done here, that the optimal portfolio allocation depends on the risk-return tradeoff for the different assets available to the investor. Some risk parameters may not be known or observable to the investor or the market, but only known to a few select agents such as brokers.

Suppose an investor, with time-t wealth denoted by W_t, and an instantaneous CRRA utility function given by $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$, can invest in either a risky asset or a riskless asset. The riskless asset has a constant growth rate r, and the price of the risky asset, P_t, evolves as a geometric Brownian motion,

$$\frac{dP}{P} = \mu dt + \sigma dB$$

The investor wants to maximize utility at point in time T when she will need to consume. Assume that T occurs randomly at rate ρ. The investor's problem can be written in terms of the proportion of wealth invested in the risky asset, x. If $\theta = 0$, her wealth process follows

$$dW = rWdt + (\mu - r)xWdt + \sigma xWdB$$

I assume that the risky asset bears an additional risk apart from the stochastic price process. At consumption date T, with a probability θ, the quality of the asset is publicly revealed to be bad, and this event reduces the price P_T to 0.

Note that if θ is not known, it cannot be learned by observing the evolution of the price process before T. This assumes that the underlying market interactions are among only noise traders, who are not trading because of their private information about θ or the underlying quality of the asset, but because of other concerns such as liquidity requirements, as in Kyle (1985). I also assume that the investor cannot acquire information about the asset at a cost, as in Grossman and Stiglitz (1980). This implies that all traders in the market are equally uninformed about θ.

If $V(W)$ denotes her value function, and if $\theta = 0$, her problem can written as

$$\rho V(W) = \max_x u(W) + (rW + (\mu - r)xW) V'(W) + \frac{1}{2}\sigma^2 x^2 W^2 V''(W)$$

We can guess and verify that the solution has the form $V(W) = A\frac{W^{1-\gamma}}{1-\gamma}$.[3] The investor

[3]This is true when the value function is well-defined, i.e., when $\rho > (1-\gamma)\left(r + (\mu - r)^2/\sigma^2\gamma\right)$

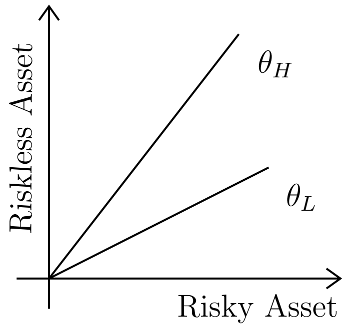


Figure 1: Optimal Portfolio Allocation

optimally maintains a constant proportion of wealth, $x^* = \frac{\mu - r}{\gamma\sigma^2}$, invested in the risky asset and her value function satisfies the form above with

$$A = 1 + (1-\gamma)\left(r + (\mu - r)x^*\right) - \frac{1}{2}(1-\gamma)\gamma\sigma^2 x^{*2}$$

When $\theta > 0$, the value function satisfies

$$\rho V(W) = \max_x (1-\theta)u(W) + \theta u((1-x)W) + \left(rW + (\mu - r)xW\right)V'(W) + \frac{1}{2}\sigma^2 x^2 W^2 V''(W)$$

The optimal fraction of wealth can be derived as the solution of the first order condition

$$\theta(1-x)^{-\gamma}\frac{W^{1-\gamma}}{1-\gamma} + (\mu - r)WV'(W) + \sigma^2 W^2 V''(W)x = 0$$

and it is decreasing in θ. For example, if $\theta \in \{\theta_H, \theta_L\}$, the optimal target portfolio ratios are shown in figure (1). If a broker knows θ and could communicate its value to the investor, that would affect the investor's optimal portfolio.

Related Literature

There are several papers which study dynamic information provision. Papers with dynamic cheap talk communication include Madsen (2019), Renault et al. (2013), Ivanov (2015), Golosov et al. (2014) and Grenadier et al. (2016). These works either assume some form of commitment on either the sender's part or the receiver's part, some restrictions on the information available to the sender or are interested in the sender's behavior for a payoff dependent on the state and the receiver's action. Golosov et al. (2014) consider the standard setting of Crawford and Sobel (1982) and construct an equilibrium with dynamic communication that is fully revealing. I explicitly model the sender as earning the commission from *changes* in the receiver's action. Orlov et al. (2016) studies a case with dynamic information provision, where the sender cannot misreport and has a preference on the timing of the receiver's action.

Papers which study models of static cheap-talk with commissions - or more generally state-independent preferences for the sender - include Inderst and Ottaviani (2012) and Lipnowski and Ravid (2018) and Chakraborty and Harbaugh (2010). Apart from cheap-talk, a strand of literature also looks at information acquisition or sale of information with or without commissions, including Brennan and Chordia (1993) and Admati and Pfleiderer (1990)

My work was motivated to add a dimension of asymmetric information to the literature on Portfolio allocation with transaction costs, which includes Merton (1969) and Davis and Norman (1990). Since the seminal work of Miller and Orr (1966), (s, S) models are widely used in the field of applied theory, to study pricing with menu costs, firm dynamics, inventory management. I study a problem where one agent faces an (s, S) problem, but with asymmetric information and a second agent earning the adjustment costs.

Empirical papers on commissions and brokerages, including Maggio et al. (2019), Malmendier and Shanthikumar (2014), Goldstein et al. (2009), have noted how the commission brokers distort or withhold information to generate trades from investors, and have commented on the broad possible reasons, including relational contracting and adding noise to forecasts.

2 Model

Two agents - a Sender (Broker, S or he) and a Receiver (Investor, R or she)- play an infinite-horizon continuous time game with a common discount rate ρ.

The investor has starting wealth W and can invest in either a Risky or Riskless asset. Both assets pay once-and-for-all at a common date s, occurring randomly at rate ρ. The investor has CARA utility over the final terminal wealth, such that $u(w_s) = -e^{-\gamma w_s}$, where $\gamma > 0$.

Asset Payoffs: If the amount invested in the risky asset is y, it generates a total return $(R+\hat{\theta})y$ at the payoff date, where

$$\begin{aligned} R \sim & \mathbf{N}(\bar{R}_s, \sigma_y) \\ d\bar{R} = & \hat{\sigma} dB \\ \hat{\theta} \in & \{\hat{\theta}_H, \hat{\theta}_L\} \end{aligned}$$

$\hat{R}_s$ is process observable to both agents, whereas θ is observable only to the broker The amount invested in the riskless asset, $W-y$, generates a return of $r(W-y)$.

The investor's problem can be written as

$$V(\bar{R}) = \max_y \int_t^\infty \rho e^{-\rho s} \mathbb{E}(-e^{-\gamma w_s}) ds = \max_y \int_t^\infty \rho e^{-\rho s} \mathbb{E}(-e^{-\gamma((W-y)r+(R+\hat{\theta})y)}) ds$$

Note that conditional on y_s and $\bar{R}_s$, $u(w_s) \sim \log \mathbf{N}(-\gamma r W - \gamma(\bar{R}_s + \hat{\theta} - r)y, \gamma^2 \sigma_y^2 y^2)$. This implies that

$$V(\bar{R}) = \max_y \int_t^\infty \rho e^{-\rho s} \mathbb{E}(-e^{-\gamma r W - \gamma(\bar{R}_s + \hat{\theta} - r)y + \gamma^2 \sigma_y^2 y^2/2}) ds$$

The optimal policy is the same as if minimizing

$$-\gamma r W - \left(\frac{\bar{R}_s + \hat{\theta} - r}{\sigma_y}\right)^2 + 2\gamma^2 \sigma_y^2 \left(y - \frac{\bar{R}_s - r}{\gamma \sigma_y^2} - \frac{\hat{\theta}}{\gamma \sigma_y^2}\right)^2$$

To simplify the problem, we can normalize $x = y - \frac{\bar{R}_s - r}{\gamma \sigma_y^2}$ and $\theta = \frac{\hat{\theta}}{\gamma \sigma_y^2}$ and consider the following adapted problem with the instantaneous payoffs such that the solution to the adapted problem will be the same as the portfolio allocation problem above.

Let the receiver's problem be given as follows. The receiver's instantaneous payoff at time t is given by the quadratic loss function $-(x_t - \theta)^2$ where θ is fixed in time, and x_t can be changed or controlled by R. The value of $\theta \in \{\theta_H, \theta_L\} \subseteq \mathbb{R}$, where $\theta_L < \theta_H$, is unknown to R. This also requires that the payoff value to the Receiver is not observed by her.[4] x_t is publicly observable and evolves according to

$$dx_t = \sigma dB_t$$

Receiver can take an action any period to change x_t, but pays a fixed adjustment cost

[4]One can also assume that the payoff is realized and observed only at a stochastic time T which occurs at rate ρ. The formulation of the value functions will then follow as specified.

$\alpha > 0$ if and when it is executed.[5] I refer to x_t as the receiver's *position* at time t, and θ as the *state*.

The above adjustment cost is a commission paid to the Sender and as such, his payoff function is the expected discounted value of the adjustment costs incurred by R over the game. The Sender privately knows θ and can reveal it to R through cheap-talk messages. p_t denotes the receiver's belief at time t of the event that $\theta = \theta_H$, p_0 denoting the initial belief. Thus, at each t R's expected payoff is

$$-p_t(x_t - \theta_H)^2 - (1 - p_t)(x_t - \theta_L)^2$$

Full information

Consider the benchmark case when R knows the value of θ, i.e. $p_t \in \{0, 1\}$, and S is just a passive agent. R optimally chooses an (s, S) policy- she chooses an inaction region (l_θ, u_θ) such that she decides to exercise control at the first time t such that x leaves the region, and chooses the target or benchmark to which she changes the value of x at t.

Denote the Receiver's and Sender's value functions by W_R and W_S, respectively. The receiver's problem can be written as solving for the value function,

$$W_R(x|\theta) = \max_{l,u,s} - \int_l^u L(\xi, x, l, u)(\xi - \theta)^2 d\xi + (\psi(x, l, u) + \Psi(x, l, u))(W_R(s|\theta) - \alpha)$$

where given thresholds l and u, $L(\xi, x, l, u)$ is the expected discounted local time spent at ξ by 4the x-process, and ψ and Ψ are the discounted hitting time probabilities (defined in the appendix) for thresholds l and u respectively.

As the problem is symmetric around θ, the target value of x is θ and $|\theta - u_\theta| = |\theta - l_\theta|$. Even though the sender is a passive agent, he still earns the commissions.

Proposition 1. *Under full information, R optimally controls x in an (s, S) policy, with an inaction region (l_θ, u_θ)*

1. *Her value function $W_R(x|\theta)$ satisfies*

$$\rho W_R(x|\theta) = -(x - \theta)^2 + \frac{1}{2}\sigma^2 W_R''(x|\theta)$$

[5] To be more concrete, assume $x_t = x_{1t} + x_{2t}$, where $dx_{1t} = \sigma dB_t$ and x_{2t} is a process controlled by the receiver and she incurs the cost α at t such that $x_{2t^-} \neq x_{2t^+}$.

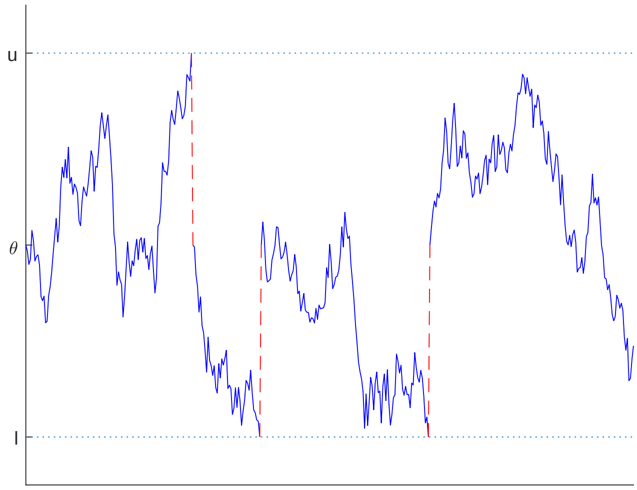


Figure 2: An (s, S) policy

for $x \in (l_\theta, u_\theta)$*, with the boundary conditions*

$$W_R(l_\theta|\theta) = W_R(u_\theta|\theta) = W_R(\theta|\theta) - \alpha$$
$$\lim_{x \downarrow l_\theta} W_R'(x|\theta) = \lim_{x \uparrow u_\theta} W_R'(x|\theta) = W_R'(x|\theta)|_{x=\theta} = 0$$

2. *The Sender's value function from R's optimal* (s, S) *policy,* $W_S(x|\theta)$*, satisfies*

$$\rho W_S(x|\theta) = \frac{1}{2}\sigma^2 W_S''(x|\theta)$$

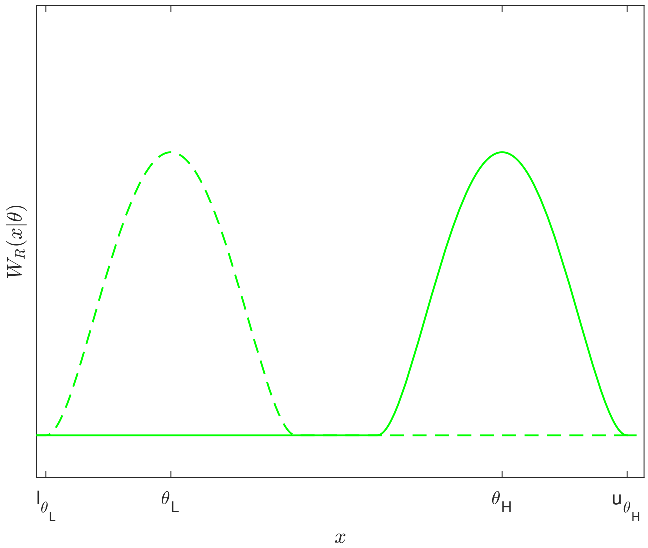
for $x \in (l_\theta, u_\theta)$*, with the boundary conditions*

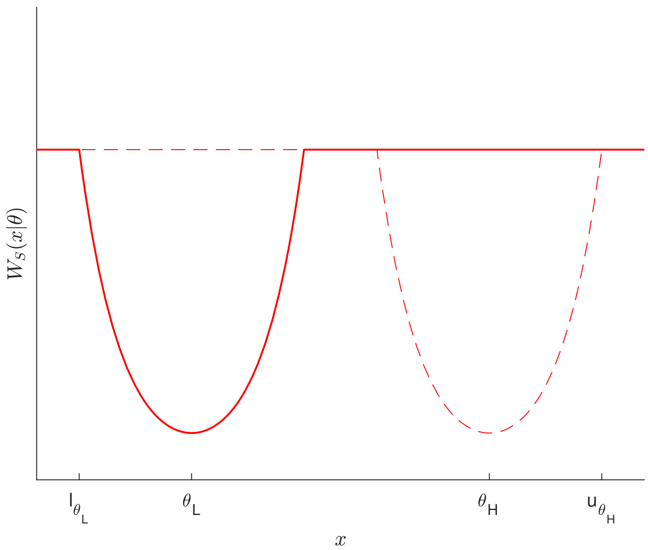
$$W_S(l_\theta|\theta) = W_S(u_\theta|\theta) = W_S(\theta|\theta) + \alpha$$

I will refer to W_R and W_S as the 'informed' value functions, shown in figures 2 and 2 respectively. All figures in the paper are presented for an example with parameters such that $[l_{\theta_L}, u_{\theta_L}]$ and $[l_{\theta_H}, u_{\theta_H}]$ are disjoint, but the results go through for general parameter values.

2.1 Asymmetric Information

Suppose the sender privately knows the true state θ. As stated earlier, the prior probability that $\theta = \theta_H$ is p_0.


$W_R(x|\theta)$
l_{θ_L}
θ_L
θ_H
u_{θ_H}
x


$W_S(x|\theta)$
l_{θ_L}
θ_L
θ_H
u_{θ_H}
x

2.1.1 No communication

In the case without a Sender and a permanently uninformed Receiver, $p_t = p_0$ for all t. The Receiver would then take her ex-ante best action, which is also to control the x-process in an (s, S) policy. Her ex-ante instantaneous payoff can also be written as

$$-(x - \bar{\theta})^2 - \mathrm{Var}(\theta)$$

where $\bar{\theta} = \mathbb{E}(\theta)$. This implies that the optimal (s, S) policy consists of the target $\bar{\theta}$ and the inaction region $(l_{\bar{\theta}}, u_{\bar{\theta}})$.

Note that the Receiver's problem here is to minimize a discounted stream of quadratic loss around $\bar{\theta}$, as under full information. Hence, the optimal inaction region is such that

$$|\theta - u_\theta| = |\theta - l_\theta| = |\bar{\theta} - u_{\bar{\theta}}| = |\bar{\theta} - l_{\bar{\theta}}|$$

Hence, the inaction interval under 'no information' is the same width as under the case of full information.

2.1.2 Case: Time-0 communication

Consider the case where the sender can communicate only at time 0, and never again. Note that, in this case, truthfully revealing the state is incentive compatible for the Sender only for certain values of x - those values where the Sender's informed value from reporting θ_L is the same as that of reporting θ_H. Let,

$$\bar{R} := \{x \mid W_S(x|\theta_H) = W_R(x|\theta_L)\}$$

2.1.3 Dynamic Case:

I focus on equilibria in pure Markov strategies. As defined earlier, p_t is a càdlàg process which denotes the belief of the receiver at time t that $\theta = \theta_H$. For each (p, x, θ), the Sender chooses a message m from a message space M. Thus, the sender's strategy in terms of messages is a function $m : [0, 1] \times \mathbb{R} \times \{\theta_H, \theta_L\} \to M$.

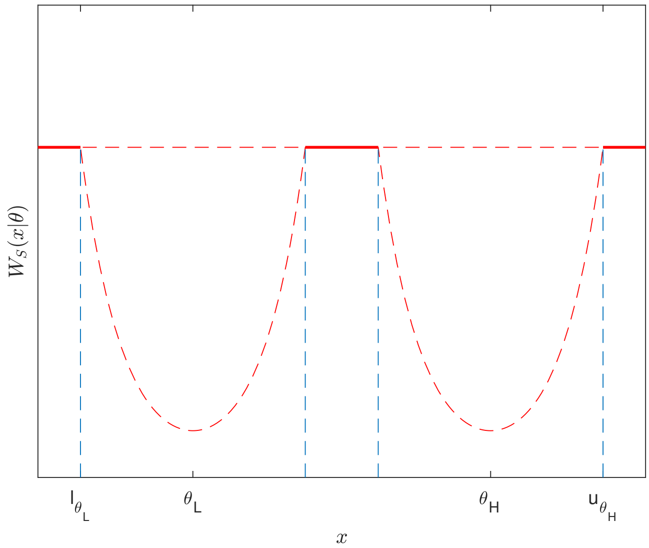


Figure 3: $\bar{R}$

Beliefs

Generally, the evolution of the Receiver's belief depends on the interpretation of the Sender's messages and is allowed to take any values between 0 and 1 over time. Let $\hat{p}(p, x, m)$ denote the updated posterior belief of the Receiver using Bayes' rule, for a prior belief p at position x, upon observing a message m. Since I consider Pure Strategies, the Sender does not send partially informative messages in any equilibrium. For all (p, x, m), $\hat{p}(p, x, m) \in \{0, p, 1\}$. Let,

$$\hat{\tau} := \inf\{t \geq 0 \mid m(p_0, x_t, \theta_H) \neq m(p_0, x_t, \theta_L)\}$$

Then,

$$p_t = \begin{cases} p_0 & \text{if } t < \hat{\tau} \\ 0 \text{ or } 1 & \text{if } t \geq \hat{\tau} \end{cases}$$

Because of the restriction to pure strategies, the game can be treated as if it is divided into two stages - 'Uninformed' and 'Informed'. In the 'informed' stage, the value functions for both players are the 'informed' value functions. The first 'uninformed' stage begins with the Receiver uninformed about θ, and it ends when credible information is sent from the Sender, which causes the posterior probability of θ_H to jump to either 0 or 1.

In the uninformed stage, there may exist a set of x-values, R, where the Receiver considers the Sender's messages to be informative. More formally,

$$R := \{x \mid \exists\, m, \hat{m} \in M \text{ where } \hat{p}(p_0, x, \hat{m}) = 0 \text{ and } \hat{p}(p_0, x, m) = 1\}$$

If $R \subseteq \bar{R}$ and the Sender sends an informative message when $x \in R$, then revealing the truth is incentive-compatible.

The Receiver's strategy in the 'uninformed' stage is to choose the threshold points at which to exercise control and 'change' x, and the value which she changes to. Denote the set of these points by B and s respectively. Whenever $x = b \in B$, the receiver pays a cost α and adjusts x to s.

Equilibrium

Let the value functions in the uninformed stage be denoted by $V_S(x|\theta)$ and $V_R(x)$. Note that the prior belief p is no longer a state variable as it remains constant in this stage. Let $\tau_A := \inf\{t \geq 0 \mid x_t \in A\}$, be the first time x_t equals a value in a Borel set A.

Define a set of states R_{R} where the Receiver believes truth is credibly revealed. That is, if the Receiver believes the Sender to follow a message strategy $m(p, x, \theta)$, then

$$R_R := \{x \mid \hat{p}(p_0, x, m(p_0, x, \theta)) \in \{0, 1\}\}$$

The *Receiver's problem* is to find a set $B \subseteq \mathbb{R}$ and target value s, such that they solve

$$\begin{aligned} V_R(x) = \sup_{B,s} \mathbb{E}_x \Bigg[\int_0^T & e^{-\rho t} u(x_t) dt + e^{-\rho T} \text{Prob}(x_T \in R_{\text{R}}) \mathbb{E}_\theta \left[W_R(x_T|\theta) \right] + \\ & e^{-\rho T} \text{Prob}(x_T \in B)(V_R(s) - \alpha) \Bigg] \end{aligned}$$

where, $T = \tau_R \wedge \tau_B$ and

$$u(x) = -p(x - \theta_H)^2 - (1 - p)(x - \theta_L)^2$$

is the ex-ante instantaneous payoff to R.

I consider a class or refinement of equilibria, where there always exists an uninformative message - a message which, if the Sender chooses to send it, does not influence or move the Receiver's belief. In other words, I consider equilibria where the Sender is allowed to 'stay silent' or 'delay' information provision. In the following section, I discuss some examples of equilibria in which the sender does not have an option to 'stay silent' and is effectively forced to communicate.

In such 'delay' equilibria, the sender's strategy is effectively to choose a stopping time $\hat{\tau}$ at which to end the stage by revealing the truth. Let R_S be the set of states where the sender credibly reveals the true state in equilibrium.

Given the sets R_R and B and the value s, the *Sender's problem* is to find a set $R_{\mathrm{S}} \subseteq R_R$, such that it solves

$$V_S(x|\theta) = \sup_{R_{\mathrm{S}} \subseteq R_R} \mathbb{E}_x \big[e^{-\rho T} \mathrm{Prob}(x_T \in R_{\mathrm{S}}) W_S(x_T|\theta) + \\ e^{-\rho T} \mathrm{Prob}(x_T \in B)(V_S(s) + \alpha) \big]$$

In a pure-strategy Markov Perfect equilibrium, the set of x-values where the Receiver expects to be informed, must coincide with the set the Sender chooses at which to reveal the true state, and truth-telling must be incentive compatible. I denote the set of such x values to be the 'truth-revealing' set.

Definition. *A **Markov Perfect Equilibrium in Pure strategies with Delay** is the collection of sets B and R and the target value s such that*

- $R_R = R_S = R \subseteq \bar{R}$,
- *they solve the Sender and Receiver's problems in the uninformed stage,*
- $\forall\ (p, x)$, $\exists\ m \in M$ *such that* $\hat{p}(p, x, m) = p$, *for all* $p \in [0, 1]$.

We will use the following lemma to derive the equilibrium value functions.

Lemma 1. *In any equilibrium, B and R are disjoint.*

When $x \in R$, the uninformed stage ends immediately, hence the receiver has no incentive to incur the adjustment cost and adjust x to s.

2.2 Equilibria with 'Forced Communication'

If the sender was forced to send an informative message when $x \in \bar{R}$, this implies for positions $x \in \bar{R}$, for all $m \in M$, either $\hat{p}(p, x, m) = 0$ or 1. For such positions, truth-telling is still incentive compatible, but the sender does not have an option to send an

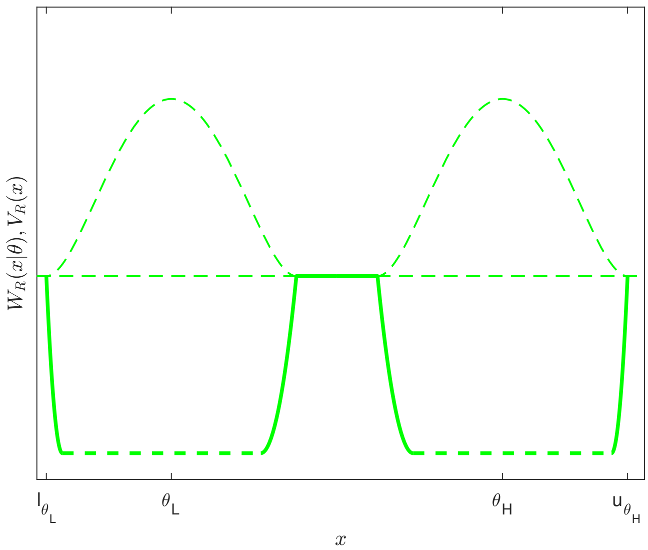


Figure 4: Uninformed Receiver's value

uninformative message. Note that I am not restricting the strategies of the sender in any way, however the nature of the equilibrium beliefs on and off path restrict the possible posterior beliefs he can induce.

Since, the receiver is best off the earliest she gets informed about the state, in the receiver-optimal equilibrium the sender is forced to communicate as soon as it is possible to send a message which is incentive-compatible with truth-telling.

Proposition 2. ***Receiver-optimal Equilibrium without delay:*** *In the uninformed stage in the Receiver-optimal Equilibrium*

- $R = \bar{R}$, *and*
- $\forall x \in R$ *and* $\forall\, m \in M,\ \hat{p}(p, x, m) = 0$ *or* 1.

Figures 4 and 5 show the value functions of the Receiver and Sender, respectively, in such an equilibrium.

In general, truthful communication can be forced and is incentive compatible over any set $R \subset \bar{R}$.

Definition. *A* ***forced communication*** *equilibrium is a tuple* (B, R, s) *such that,*

- $R_R = R_S = R \subseteq \bar{R}$,
- *they solve the Receiver's problem in the uninformed stage,*
- $\forall x \in R$ *and* $\forall\, m \in M,\ \hat{p}(p, x, m) = 0$ *or* 1.

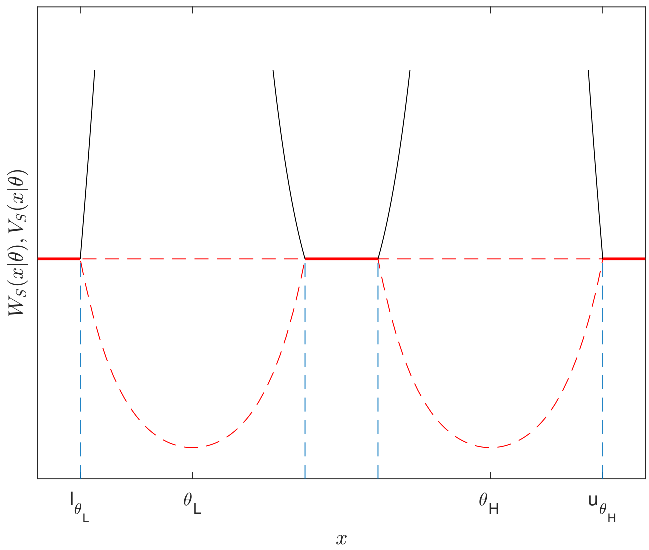


Figure 5: Sender's value in the uninformed stage with forced communication

Note that, in such equilibria, the sender does not have a choice and must communicate at τ_R.

Discussion

The sender's option to send uninformative messages can also be interpreted as an assumption about off-path beliefs about messages sent with probability 0. In any forced communication equilibrium, all messages in the message space are informative at τ_R. However, if there was a message $m \in M$ which was sent with probability 0 at τ_R, and upon receiving such a message m, $\hat{p}(p, x, m) = p$, then it would be as if the Sender had an option to not send any information. Hence, in an equilibrium with such off-path beliefs, the sender should also not have a profitable deviation in sending this uninformative message and 'delaying' information. Forcing informative communication can be interpreted as WLOG restricting the message space to two messages $\{H, L\}$, with their natural interpretations, and allowing delay can be interpreted as allowing any number of messages but restricting off-path beliefs about messages.

Value Functions

I now show how to solve for the receiver-optimal equilibrium - a 'forced communication' equilibrium- and the associated value functions for both agents in the uninformed stage. Let the tuple (B, R, s) denote such an equilibrium. Note that $R = \bar{R}$, which can be written as a union of disjoint open intervals, and the receiver's best response B and s, and the associated value functions can be solved for piece-wise, over the sets B, R and $(B \cup R)^C$.

Define $\hat{V}_R(x)$ to be the value function of the receiver when B is empty. Hence, $\hat{V}_R(x)$ satisfies the HJB equation

$$\rho V_R(x) = u(x) + \frac{1}{2}\sigma^2 V_R''(x)$$

where $u(x) = -p_0(x - \theta_H)^2 + (1 - p_0)(x - \theta_L)^2$, for any $x \notin \bar{R}$. Additionally, for all $x \in \bar{R}$, $\hat{V}_R(x) = \mathbb{E}_\theta[W_R(x|\theta)]$, as the receiver immediately becomes informed.

Receiver: If

$$\max_x \hat{V}_R(x) - \min_x \hat{V}_R(x) < \alpha$$

then the receiver optimal equilibrium is such that $B = \emptyset$, $R = \bar{R}$ and s is arbitrary. The Receiver never adjusts x while uninformed, and $V_R(x) = \hat{V}_R(x)$, for all x.

Suppose $\max_x \hat{V}_R(x) - \min_x \hat{V}_R(x) \geq \alpha$. For all $x \in R$, $V_R(x) = \mathbb{E}_\theta[W_R(x|\theta)]$, and B is non-empty and solves the Receiver's problem. For any interval $(r, b) \subset (B \cup R)^C$, such that $r \in R$ and $b \in B$, and for any $x \in (a, b)$, V_R satisfies the HJB equation

$$\rho V_R(x) = u(x) + \frac{1}{2}\sigma^2 V_R''(x)$$

where $u(x) = -p_0(x - \theta_H)^2 + (1 - p_0)(x - \theta_L)^2$. The value function also satisfies the boundary conditions,

$$\begin{aligned}
V_R(r) &= \mathbb{E}_\theta[W_R(r|\theta)] \\
V_R(b) &= V_R(s) - \alpha \\
\lim_{x \downarrow b} V_R'(x) &= 0 \\
V_R'(x)|_{x=s} &= 0 && \text{if } s \notin R \\
\max_x V_R(x) &= V_R(s) && \text{if } s \in R
\end{aligned}$$

The HJB equation and the boundary conditions pin down s, a value b for every $r \in \partial R$ and the solution for the value function over any such interval (r, b) or (b, r). They also pin down the value function over the set B, which is constant at $V_R(s) - \alpha$.

Sender: Given the equilibrium (B, R, s), $V_S(x|\theta)$ can be computed similarly. $V_S(x|\theta)$

satisfies the HJB equation

$$\rho V_S(x) = \frac{1}{2}\sigma^2 V_S''(x)$$

for any $x \notin B \cup R$,

$$V_S(x|\theta) = V_S(s|\theta) + \alpha$$

for any $x \in B$, and

$$V_S(x|\theta) = W_S(x|\theta)$$

for any $x \in R$.

2.3 Babbling or No-information Equilibrium

There always exists an equilibrium in which $R = \emptyset$, and messages are never informative. No report by the Sender is believed by the receiver, and the belief never deviates from p_0. The Receiver chooses B and s as if in the case without any communication.

As discussed in the section on 'No communication', the receiver's inaction interval is the same width for any value of the belief p_0, due to the assumed quadratic loss payoff function. Thus, the sender's 'uninformed' value function $V_S(x|\theta)$ is just a translation of the informed value function over the interval $[l_{\bar{\theta}}, u_{\bar{\theta}}]$.

$$V_S(x|\theta) = W_S(l_{\theta_H} + (x - l_{\bar{\theta}})|\theta_H$$

This also implies that for all $x \in [l_{\bar{\theta}}, u_{\bar{\theta}}]$,

$$V_S(x|\theta) < \min\{W_S(x|\theta_H), W_S(x|\theta_L)\}$$

which means that the sender would prefer to communicate some information if he could rather than no information and he does not have any incentive to keep the receiver uninformed.

3 Equilibria with 'Delay'

3.1 Equilibrium Value functions

When the sender has an option to send an uninformative message, it is as if the sender has an optimal stopping problem to decide when to reveal the truth. This imposes some boundary conditions on the sender's value function which were not required in the equilibria with forced communication.

Specifically, consider as before an equilibrium (B, R, s) and an interval $(r, b) \subset (B \cup R)^C$, such that $r \in R$ and $b \in B$. Value-matching implies

$$\lim_{x \downarrow r} V_S(x|\theta) = W_S(r|\theta)$$

For the sender, it should be not be profitable to delay communication at r. This requires the necessary condition,

$$\lim_{x \downarrow r} V_S'(x|\theta)) \leq 0 \tag{1}$$

The argument is the same as one which derives the standard 'smooth-pasting condition' for value functions in most optimal stopping or optimal option-exercise problems. If (1) holds with equality, the sender is just indifferent at point r between revealing the truth and 'staying silent'. In equilibrium, he may not be indifferent and may strictly prefer to reveal the truth, in which case (1) holds with a strict inequality.

Similarly for any interval $(b, r) \subset (B \cup R)^C$, such that $r \in R$ and $b \in B$, the analogous necessary condition is

$$\lim_{x \uparrow r} V_S'(x|\theta)) \geq 0 \tag{2}$$

Figure 3.1 shows an example how the truth-telling and adjustment boundaries, r and b respectively, are determined in a range of x position values.

For the receiver, who best-responds to the truth-revealing set R, and hence the threshold r, by choosing the set B and target value s, her value function must be smooth-pasted at b while matching the 'informed' value at r. Thus, her value functions satisfy the same boundary conditions as in the case with forced communication.

Thus we can summarize that,

Proposition 3. *(B, R, s) is an equilibrium if*

1. $R \subset \bar{R}$,

2. $B \cap R = \emptyset$,

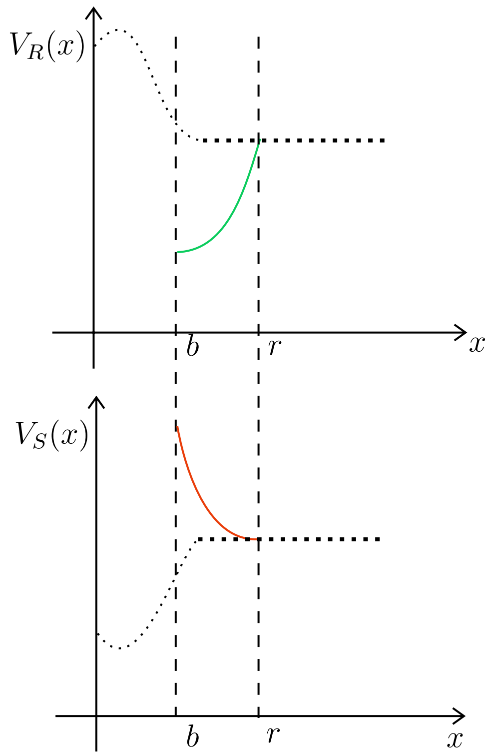


Figure 6: The 'uninformed' adjustment and 'truth-revealing' thresholds

3. *for any interval* $(r, b) \subset (B \cup R)^C$*, such that* $r \in R$ *and* $b \in B$*, the value function for the receiver,* V_R*, satisfies,*

$$\lim_{x \uparrow b} V_R'(x) = 0 \tag{3}$$

$$V_R'(x)|_{x=s} = 0 \qquad \textit{if } s \notin R \tag{4}$$

$$\max_x V_R(x) = V_R(s) \qquad \textit{if } s \in R \tag{5}$$

and the value function for the sender, V_S*, satisfies,*

$$\lim_{x \downarrow r} V_S'(x|\theta)) \leq 0 \tag{6}$$

4. *and similarly, for any interval* $(b, r) \subset (B \cup R)^C$*, such that* $r \in R$ *and* $b \in B$*, the value function for the receiver,* V_R*, satisfies,*

$$\lim_{x \downarrow b} V_R'(x) = 0 \tag{7}$$

$$V_R'(x)|_{x=s} = 0 \qquad \textit{if } s \notin R \tag{8}$$

$$\max_x V_R(x) = V_R(s) \qquad\qquad if\ s \in R \tag{9}$$

and the value function for the sender, V_S, satisfies,

$$\lim_{x \uparrow r} V_S'(x|\theta)) \geq 0 \tag{10}$$

I now present the main result of the paper regarding equilibria in which the sender has an option to delay information ('equilibria' hereafter).

Proposition 4. *$\exists\, \psi_1, \psi_2$ with $0 \leq \psi_1 < \psi_2$ such that*

- *if $\theta_H - \theta_L \in (\psi_1, \psi_2]$, $R \subsetneq \bar{R}$*
- *if $\theta_H - \theta_L > \psi_2$, R is empty and the only pure-strategy MPE is the babbling equilibrium.*

At any position x, $\theta_H - \theta_L$ affects the value of information to the receiver in the first stage. The larger this difference, the higher the Receiver's incentive to adjust the value of x to reach 'truth-revealing' states.

Suppose, for contradiction, $\bar{R}$ was the truth-revealing set. For large values of $\theta_H - \theta_L$, the boundary at which receiver adjusts x is so close to the boundary of $\bar{R}$ that the sender can profitably deviate by *not* revealing the truth at $\inf\{t|x_t \in \bar{R}\}$. Hence, $R \neq \bar{R}$, because $\partial\bar{R} \not\subset R$ and $\bar{R}$ is closed.

I show this argument graphically as well. Figures 4 and 5 show the Receiver's and the Sender's value function in the uninformed stage if the entire set $\bar{R}$ is the truth revealing set. Figure 7 shows the value to the sender in the uninformed stage if the $R_R = \bar{R}$, but $R_S \neq \bar{R}$. In other words, the dashed bold line shows the Sender's value if the Receiver expects truth to be revealed at $\partial\bar{R}$, but the sender deviates and sends an uninformative message or delays. As it is higher than the former value function, this deviation is profitable. Hence, in equilibrium, $\partial\bar{R} \subsetneq R$.

For even larger values of $\theta_H - \theta_L$, there cannot exist a truth-revealing set in pure strategies. If such a set were to exist, then for any x in the boundary of such a set, the adjusting boundary of the Receiver will be sufficiently close to it, as to incentivize the Sender to delay communication.

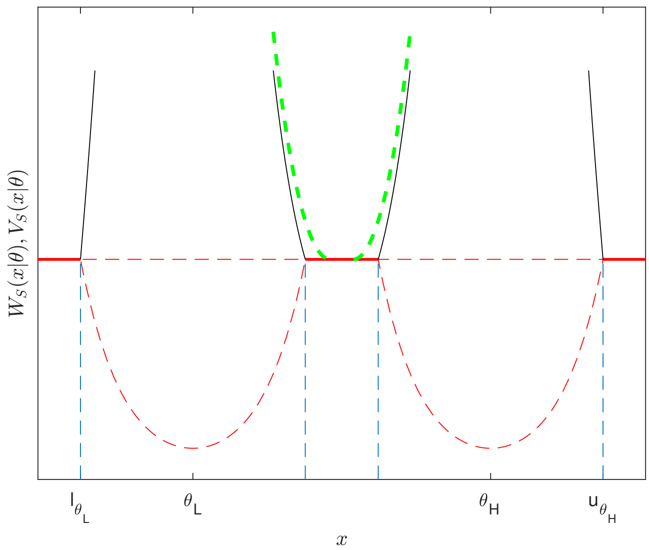


Figure 7: Sender's value from the profitable deviation

3.2 Receiver-optimal Equilibrium with Delay

Equilibrium with the earliest resolution of uncertainty, and hence the earliest transmission of a credible message from the sender, is optimal for the receiver. Hence,

Lemma 2. *An equilibrium (B, R, s) is receiver-optimal if there does not exist an equilibrium $(\hat{B}, \hat{R}, \hat{s})$, such that $R \subset \hat{R}$.*

Proposition 5. *If an equilibrium (B, R, s) exists such that (13) holds with equality, it is the unique receiver-optimal equilibrium with delay.*

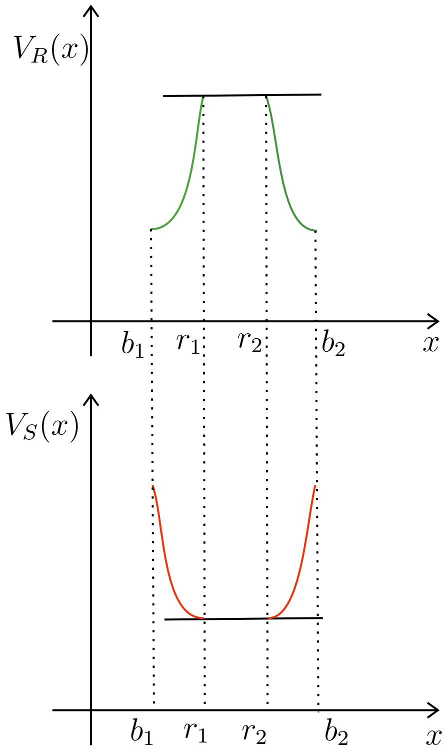
4 Mixed Strategy Equilibria

In this section, I discuss a class of mixed strategy equilibria, in which the Sender is allowed to mix between sending completely uninformative or completely informative messages. Therefore, for all $m \in M$ and for all x, $\hat{p}(p, x, m) \in \{0, p, 1\}$. However I will retain the refinement on beliefs that there always exists a message which is uninformative.

To be concrete, I study the following class of equilibria.

Definition. *A **Markov Perfect Equilibrium** with **Perfect Disclosure** is the collection of sets B and R and the target value s such that*

- $R_R = R_S = R \subseteq \bar{R}$,
- *they solve the Sender and Receiver's problems in the uninformed stage,*



- $\forall\ (p, x),\ \exists\ m \in M$ *such that* $\hat{p}(p, x, m) = p$, *for all* $p \in [0, 1]$.
- *for all* $m \in M$ *and for all* x, $\hat{p}(p, x, m) \in \{0, p, 1\}$

All MPE in pure strategies belong to this class, as the equilibrium belief process satisfies the property that that for any message, the updated posterior belief is the prior belief, 0 or 1.

When information is valuable enough, i.e. for $\theta_H - \theta_L$ large enough, the only MPE in pure strategies is the babbling equilibrium. We can, however, characterize other mixed-strategy equilibrium, particularly ones belonging to the class with Perfect Disclosure. An example is the equilibrium is when the set R is a singleton consisting of a value r, such that when the $x = r$, the sender reveals the state perfectly with probability η. The Receiver chooses the set B such that the Sender is rendered indifferent at r between revealing or concealing the true state. Such an equilibrium is the receiver-optimal equilibrium in the class of equilibria with perfect disclosure, when the only pure-strategy equilibrium is the babbling equilibrium.

The appendix provides additional details on how to characterize the associated value functions and the mixing probability, η.

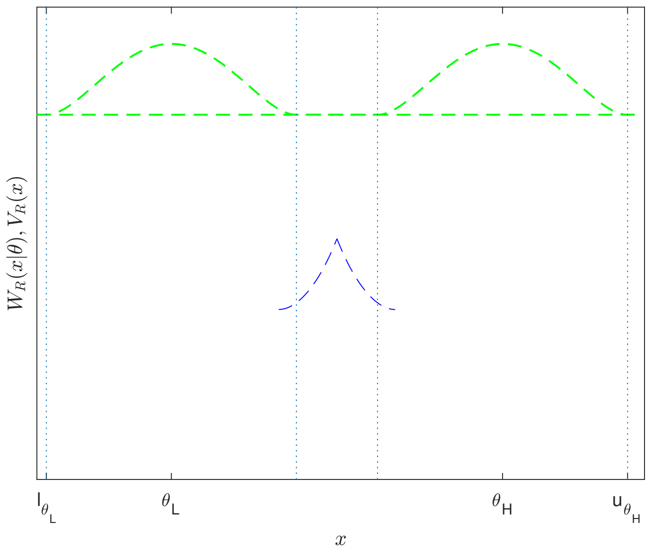


Figure 8: Receiver's value function in a 'Perfect Disclosure' equilibrium

5 Proportional Commissions

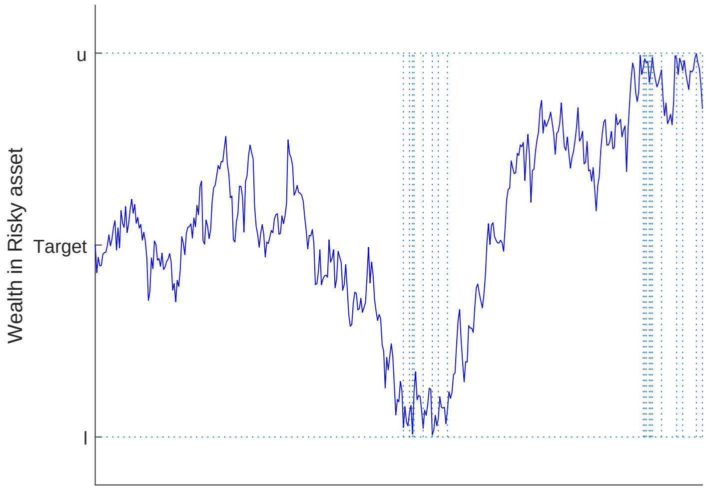
In this section, I analyze the case when the adjustment cost the receiver incurs upon a change in x is proportional to the size of the adjustment. Specifically, if at any point of time, the receiver changes the value of x from x_1 to x_2, she incurs a cost of $\alpha \times |x_2 - x_1|$, which as before is a commission paid to the sender.

5.1 Full Information

When the receiver knows the value of θ, her optimal policy is again to choose an inaction region around θ and boundaries l_θ, u_θ, such that the receiver adjusts x only when $x = l_\theta$ or u_θ. However, the behavior of the x-process at these boundaries is different as compared to the case with fixed costs. Under proportional costs, these boundaries are reflecting boundaries for the x-process. The receiver chooses the inaction region $[l_\theta, u_\theta]$ and if x is at the boundary of this region, the receiver chooses to adjust x by just enough so as to maintain x within this region. For any policy, the value functions for the receiver and the sender satisfy the associated HJB equations and smooth-pasting conditions. For the receiver's optimal policy, an additional necessary condition called the super-contact condition needs to be satisfied.[6]

Let the optimal value functions in the full-information case be denoted as before by W_R and W_S, for the receiver and sender respectively, but now recalculated tunder the

[6]Details on how this condition is derived can be found in Chapter 10 of Stokey (2009)



case of proportional costs.

Proposition 6. *Under full information, and proportional adjustment costs, R optimally controls x in an (s, S) policy, with an inaction region* (l_θ, u_θ)

1. *Her value function* $W_R(x|\theta)$ *satisfies*

$$\rho W_R(x|\theta) = -(x-\theta)^2 + \frac{1}{2}\sigma^2 W_R''(x|\theta)$$

for $x \in (l_\theta, u_\theta)$*, with the boundary conditions*

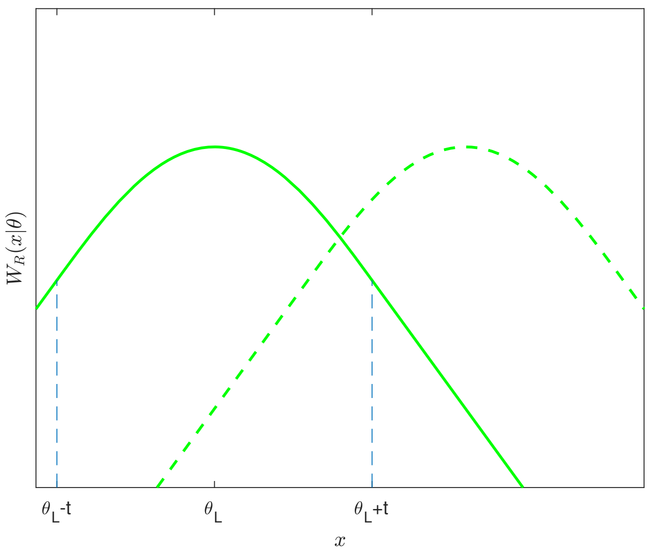
$$\begin{aligned}\lim_{x \downarrow l_\theta} W_R'(x|\theta) =& \alpha \\ \lim_{x \uparrow u_\theta} W_R'(x|\theta) =& -\alpha \\ \lim_{x \downarrow l_\theta} W_R''(x|\theta) =& \lim_{x \uparrow u_\theta} W_R''(x|\theta) = 0\end{aligned}$$

2. *The Sender's value function from R's optimal (s, S) policy,* $W_S(x|\theta)$*, satisfies*

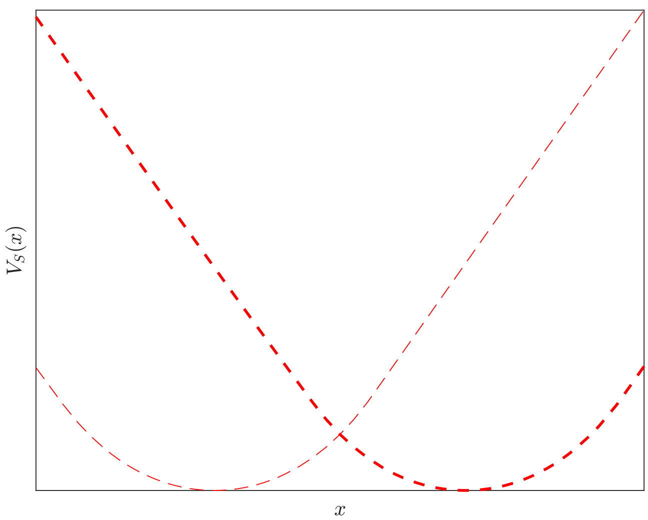
$$\rho W_S(x|\theta) = \frac{1}{2}\sigma^2 W_S''(x|\theta)$$

for $x \in (l_\theta, u_\theta)$*, with the boundary conditions*

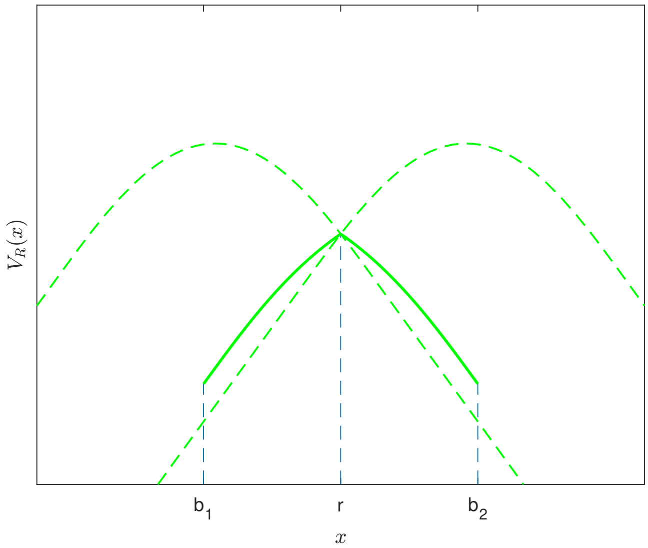
$$\begin{aligned}\lim_{x \downarrow l_\theta} W_S'(x|\theta) =& -\alpha \\ \lim_{x \uparrow u_\theta} W_S'(x|\theta) =& \alpha\end{aligned}$$



$W_R(x|\theta)$
θ_L-t
θ_L
θ_L+t
x



$V_S(x)$
x



5.2 Asymmetric Information

When the sender is privately informed of θ, there is a unique value r at which he is indifferent between both reports and has no incentive to lie. If $\bar{R}$ is defined in the same way as under fixed costs, $|\bar{R}| = 1$ and $\bar{R} = \{r\}$. Hence, in any equilibrium, $R \in \{\bar{R}, \emptyset\}$.

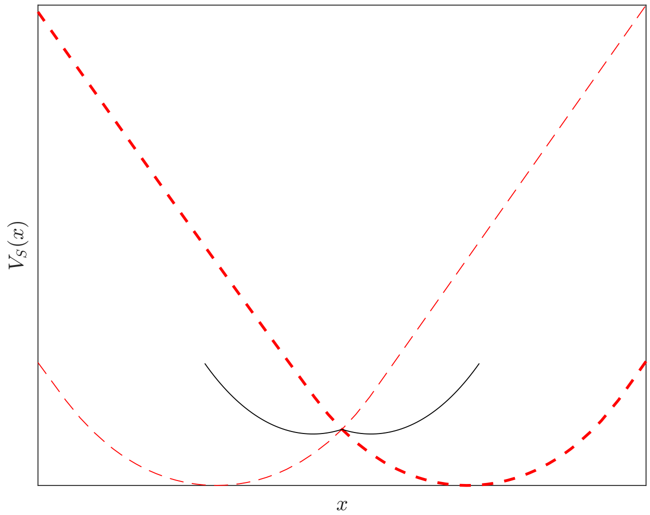
Consider the receiver's problem in a pure-strategy MPE. Under proportional costs, while there remains an active region B and an active region, the target value s might not be the same for every $x \in B$. In this case, an equilibrium is simply the set (B, R), because for any $x \in B$, the optimal target is the closest point in ∂B^C.

On equilibrium path, the game can still be divided into an 'uninformed' and an 'informed' phase. While in the uninformed phase, x remains in $(B \cup R)^C$ without the receiver adjusting x, however when x reaches ∂B, the receiver trades just enough to remain in $\partial(B^C)$. The Brownian process x reflects off of this boundary.

Proposition 7. *(B, R) is an equilibrium with delay in the uninformed stage if*

1. *$R \subset \bar{R}$,*
2. *$B \cap R = \emptyset$,*
3. *$(B \cup R)^C = (b_1, r) \cup (r, b_2)$, where $b_1, b_2 \in \partial B$.*
4. *For the interval $(r, b) \subset (B \cup R)^C$, such that $r \in R$ and $b \in B$, the value function for the receiver, V_R, satisfies,*

$$\lim_{x \uparrow b} V_R'(x) = -\alpha \tag{11}$$



$$\lim_{x \uparrow b} V_R''(x) = 0 \tag{12}$$

and the value function for the sender, V_S, satisfies,

$$\lim_{x \downarrow r} V_S'(x|\theta)) \leq 0 \tag{13}$$

5. *and similarly, for any interval $(b, r) \subset (B \cup R)^C$, such that $r \in R$ and $b \in B$, the value function for the receiver, V_R, satisfies,*

$$\lim_{x \downarrow b} V_R'(x) = \alpha \tag{14}$$

$$\lim_{x \downarrow b} V_R''(x) = 0 \tag{15}$$

and the value function for the sender, V_S, satisfies,

$$\lim_{x \uparrow r} V_S'(x|\theta)) \geq 0 \tag{16}$$

Similarly, we can state a result analogous to the main result under fixed commissions.

Proposition 8. *For large $\theta_H - \theta_L$, the only pure-strategy MPE with delay, is the babbling equilibrium.*

6 Stochastic state θ

In this section, I consider an extension where the state θ changes over time. I suppose that a shock occurs at rate ν where the value $\theta_t \in \{\theta_H, \theta_L\}$ is re-drawn with $\text{Prob}(\theta_t = \theta_H) = p$. The sender observes the new value of the state, however the receiver only observes that a shock has occurred and does not observe the new value of θ.

Essentially, while the receiver is informed, she may become uninformed at rate ν. Hence, the HJB equation denoting her informed value function can be written as

$$\rho W_R(x|\theta) = -(x-\theta)^2 + \nu(V_R(x) - W_R(x|\theta)) + \frac{1}{2}\sigma^2 W_R''(x\theta),$$

for any x in the inactive region, where W_R and V_R denote her informed and uninformed value functions, respectively. The HJB equation representing her uninformed value function remains as before,

$$\rho V_R(x) = -\mathbf{E}(x-\theta)^2 + \frac{1}{2}\sigma^2 V_R''(x)$$

Similarly for the sender, the HJB equation representing the evolution of her informed value function is

$$\rho W_S(x|\theta) = \nu(V_S(x) - W_S(x|\theta)) + \frac{1}{2}\sigma^2 W_S''(x\theta)$$

We can solve for the value functions for the mixed-strategy 'Perfect Disclosure' equilibrium in this stochastic setting, when $\theta_H - \theta_L$ is large enough. In such an equilibrium, R is a singleton, and $V_R(x)$ and $V_S(x)$ are constant for x in the inactive region while the receiver is informed. In other words, while the receiver is informed and x is in the inactive region, and if the θ-changing shock happens to occur and the receiver becomes uninformed, x is now in the active region while the receiver is uninformed. The receiver optimally adjusts x to reach R.

7 Conclusion

I study a model of asymmetric information of advising in exchange for commissions. A receiver's payoff depends on an adjustable state, called a position, and a state privately observed by the sender. The sender earns the position-adjustment costs, and both agents can communicate through cheap-talk messages.

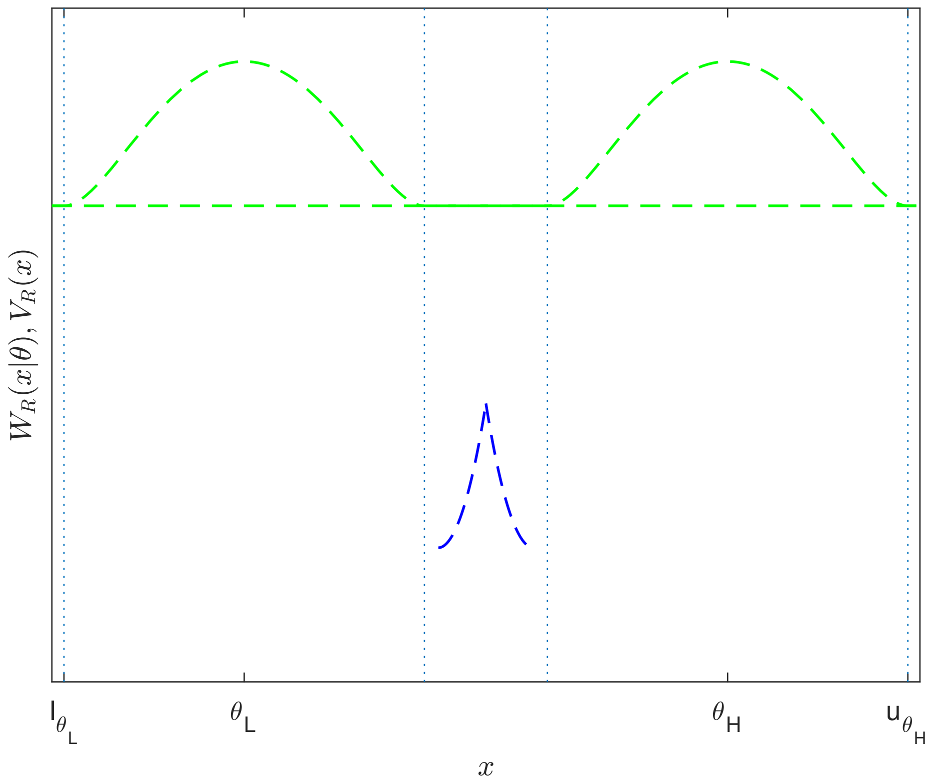


Figure 9: Receiver's informed and uninformed value functions under a stochastic state

I show that equilibria with information transmission have the feature that the sender reveals the truth only for certain positions which increases their value to the receiver. The receiver adjusts her position so that she can get informed, and this behaviour incentivizes the sender to withhold information - even at positions where truth-telling is incentive-compatible - in order to induce a larger adjustment region.

I show that the adjustment set of position grows larger while the informative set shrinks if the uncertainty is more acute; as the difference between the two possible values of the privately known state grows larger, information becomes more valuable to the receiver. Eventually, any informative equilibria must be in mixed strategies, and hence exhibit on-path delay in information transmission. I characterize an example of such a mixed-strategy equilibrium.

These results speak to relationships between investors and brokers who provide investment advice in exchange for trading commissions. They demonstrate the issue of inducement of trades by brokers remains despite the lack of commitment power and even for portfolio values where there is no incentive to misreport the truth. The model shows that such inducement can be an equilibrium feature, as the investor trades and adjusts her portfolio to remove any potential bias of the sender, whereas the sender withholds information to induce larger adjustments.

I extend the setting to study the case with a stochastic state, and note that the informative equilibrium involves the receiver adjusting the position to a 'truth-telling' position whenever the value of the state becomes uncertain. I also consider the case of commissions which are proportional to the change in positions, and show that there is a unique position at which truth-telling is incentive compatible. For information valuable enough, the only informative equilibrium is a mixed-strategy equilibrium.

References

Admati, A. R. and P. Pfleiderer (1990): "Direct and indirect sale of information," *Econometrica: Journal of the Econometric Society*, 901–928.

Brennan, M. J. and T. Chordia (1993): "Brokerage commission schedules," *The Journal of Finance*, 48, 1379–1402.

Chakraborty, A. and R. Harbaugh (2010): "Persuasion by cheap talk," *American Economic Review*, 100, 2361–82.

Crawford, V. P. and J. Sobel (1982): "Strategic information transmission," *Econometrica: Journal of the Econometric Society*, 1431–1451.

Davis, M. H. and A. R. Norman (1990): "Portfolio selection with transaction costs," *Mathematics of operations research*, 15, 676–713.

Glosten, L. R. and P. R. Milgrom (1985): "Bid, ask and transaction prices in a specialist market with heterogeneously informed traders," *Journal of financial economics*, 14, 71–100.

Goldstein, M. A., P. Irvine, E. Kandel, and Z. Wiener (2009): "Brokerage commissions and institutional trading patterns," *The Review of Financial Studies*, 22, 5175–5212.

Golosov, M., V. Skreta, A. Tsyvinski, and A. Wilson (2014): "Dynamic strategic information transmission," *Journal of Economic Theory*, 151, 304–341.

Grenadier, S. R., A. Malenko, and N. Malenko (2016): "Timing decisions in organizations: Communication and authority in a dynamic environment," *American Economic Review*, 106, 2552–81.

Grossman, S. J. and J. E. Stiglitz (1980): "On the impossibility of informationally efficient markets," *The American economic review*, 70, 393–408.

Inderst, R. and M. Ottaviani (2012): "Competition through commissions and kickbacks," *American Economic Review*, 102, 780–809.

Ivanov, M. (2015): "Dynamic information revelation in cheap talk," *The BE Journal of Theoretical Economics*, 15, 251–275.

Kyle, A. S. (1985): "Continuous auctions and insider trading," *Econometrica: Journal of the Econometric Society*, 1315–1335.

Lipnowski, E. and D. Ravid (2018): "Cheap talk with transparent motives," *Available at SSRN 2941601.*

Madsen, E. (2019): "Optimal project termination with an informed agent," .

Maggio, M. D., F. Franzoni, A. Kermani, and C. Sommavilla (2019): "The relevance of broker networks for information diffusion in the stock market," *Journal of Financial Economics*, 134, 419 – 446.

Malmendier, U. and D. Shanthikumar (2014): "Do security analysts speak in two tongues?" *The Review of Financial Studies*, 27, 1287–1322.

Merton, R. C. (1969): "Lifetime portfolio selection under uncertainty: The continuous-time case," *The review of Economics and Statistics*, 247–257.

Miller, M. H. and D. Orr (1966): "A Model of the Demand for Money by Firms," *The Quarterly journal of economics*, 80, 413–435.

Orlov, D., A. Skrzypacz, and P. Zryumov (2016): "Persuading the regulator to wait," *Available at SSRN*, 2767234.

Renault, J., E. Solan, and N. Vieille (2013): "Dynamic sender–receiver games," *Journal of Economic Theory*, 148, 502–534.

Stokey, N. L. (2009): *The Economics of Inaction: Stochastic Control models with fixed costs*, Princeton University Press.

Appendices

A Preliminaries

Let $X(s)$ be a Brownian motion on the filtered space $(\Omega, \mathbb{F}, P)$. The discounted local time of sample path $X(s,\omega)$ at x is denoted be

$$l(x,t,\omega) := \lim_{\varepsilon \to 0} \int_A \int_0^t e^{-\rho s} \frac{1}{2\varepsilon} \mathbf{1}_A(X(s,\omega)) ds dx$$

where $A = (x-\varepsilon, x+\varepsilon)$. Define $\tau_x := \inf\{t \geq 0 \mid X(t) = x\}$. For any $b < B$, let

$$L(\xi, x, b, B) := \mathbb{E}[l(\xi, \tau_b \wedge \tau_B), .)|X(0) = x]$$

denote the expected discounted local time at ξ before either threshold b or B is reached by X, given the initial state $X(0) = x$. Define the terms,

$$\begin{aligned}\psi(x,b,B) :=& \mathbb{E}_x[e^{-\rho T}|X(T) = b]\mathrm{Pr}_x[X(T) = b] \\ \Psi(x,b,B) :=& \mathbb{E}_x[e^{-\rho T}|X(T) = B]\mathrm{Pr}_x[X(T) = B]\end{aligned}$$

where $T = \tau_b \wedge \tau_B$. They are the expected discounted values of an indicator function for the event of reaching b before B, and B before b, respectively - given the initial state x.

B Proof of Proposition 4

I show the proof for the case when $\theta_H - \theta_L$ is large enough for $u_{\theta_L} < l_{\theta_H}$. In other words, the receiver's s, S bands under full-information, in the two possible states, are disjoint. Let

$$\bar{W} := W_S(x|\theta)$$

for any $x \notin (l_\theta, u_\theta)$. Hence, $\bar{R} = (-\infty, l_{\theta_L}] \cup [u_{\theta_L}, l_{\theta_H}] \cup [u_{\theta_H}, \infty)$
Let the tuple (B, R, s) constitute an equilibrium. The sender has no incentive to send an informative message while position $x \in B$, hence $B \cap R = \emptyset$.

Case: $s \in R$

Consider the interval $[r, b]$, where $[r, b] \cap B = b$ and $[r, b] \cap B = r$.
Suppose $x_t \in (r, b)$. If the sender is to choose a threshold $r \in [\bar{R}$ so as to reveal the truth at τ_r , and end the uninformed stage, the following necessary condition must hold.

$$\lim_{x \downarrow r} V_S'(x|\theta)|_{x=r} \leq 0 \tag{17}$$

This is a necessary smooth-pasting condition, because revealing the truth is essentially similar to exercising an option for the Sender - one which secures him the informed value at r. This option must be exercised optimally, because the alternative option - to send an uninformative message and continue the uninformed stage- is available.

Additionally,

$$\lim_{x \downarrow r} V_S(x|\theta)|_{x=r} = \bar{W}, \tag{18}$$

$$V_S(b|\theta) = V_S(s|\theta) + \alpha = \bar{W} + \alpha \tag{19}$$

and V_S satisfies the HJB equation

$$\rho V_S(x|\theta) = \frac{1}{2}\sigma^2 V_S''(x|\theta)$$

for $x \in (r, b)$. Hence, I can pin down the value function of the sender over the interval (r, b).

$$V_S(x|\theta) = k_1 e^{Jx} + k_2 e^{-Jx}$$

where $J = \frac{\sqrt{2\rho}}{\sigma}$, and

$$k_1 = \frac{\alpha e^{bJ} + \bar{W} e^{bJ} - \bar{W} e^{Jr}}{e^{2bJ} - e^{2Jr}} \tag{20}$$

$$k_2 = \frac{e^{bJ+Jr}\left(-\bar{W} e^{bJ} + \alpha e^{Jr} + \bar{W} e^{Jr}\right)}{e^{2Jr} - e^{2bJ}} \tag{21}$$

(17) implies

$$\begin{aligned} k_1 J e^{Jr} - k_2 J e^{-Jr} &\leq 0 \qquad \Rightarrow \\ k_1 - k_2 e^{-2Jr} &\leq 0 \end{aligned}$$

This and (18) and (19) imply the condition,

$$-(\alpha e^{bJ} + \bar{W}e^{bJ} - \bar{W}e^{Jr}) - e^{J(b-r)}\left(-\bar{W}e^{bJ} + \alpha e^{Jr} + \bar{W}e^{Jr}\right) \geq 0 \tag{22}$$

$$-(\alpha e^{rJ} + \bar{W}e^{rJ} - \bar{W}e^{2Jr-bJ}) - \left(-\bar{W}e^{bJ} + \alpha e^{Jr} + \bar{W}e^{Jr}\right) \geq 0 \tag{23}$$

$$-2 - \frac{2\alpha}{\bar{W}} + e^{J(b-r)} + e^{-J(b-r)} \geq 0 \tag{24}$$

Hence, equations 17-19 imply $b - r > \psi > 0$ for some $\psi > 0$ independent of θ_H or θ_L.

Let $W_R(x|\theta) = w$ for $x \in \bar{R}$. Now consider $[u_{\theta_L}, l_{\theta_H}] \subset \bar{R}$. Suppose, for contradiction, $[u_{\theta_L}, l_{\theta_H}] \subset R$. Receiver chooses $b > l_{\theta_H}$ at which to exercise control and adjust x. Consider the policy $b = l_{\theta_H} + \psi$ and the value to the receiver.

$$V_R(x) = -\int_r^b L(\xi, x, r, b)\mathbb{E}_\theta[(\xi - \theta)^2]d\xi + \psi(x, r, b)w + \Psi(x, r, b)(w - \alpha)$$

Differentiating w.r.t. x,

$$V_R'(x) = -\int_r^b L_x'(\xi, x, r, b)\mathbb{E}_\theta[(\xi - \theta)^2]d\xi + \psi_x'(x, r, b)w + \Psi_x'(x, r, b)(w - \alpha)$$

Using results proved in Stokey (2009), Chapter 5, For large $\theta_H - \theta_L$, $\lim_{x\uparrow b} V_R(x) > 0$, indicating that the optimal $b < l_{\theta_H} + \psi$.

Heuristically, b is the adjusting boundary and r is the truth-telling boundary. For the Sender to be incentivized to reveal the truth at r, b and r must be sufficiently far apart. However, if $r = l_{\theta_H}$, the receiver's optimal adjusting boundary in $[l_{\theta_H}, u_{\theta_H}]$ is too close to r, for the Sender to optimally reveal the true state at r.

For the second part of the proposition, suppose for contradiction R is non-empty. Then for any $r \in \partial R$, the condition (17) must hold for the Sender's value function. This implies a lower bound on $b - r$, as in the first part. For large enough $\theta_H - \theta_L$, the optimal b chosen by the receiver will violate this lower bound, no matter what the value of r is.

C Characterizing a mixed-strategy equilibrium

Consider the class of MPE with Perfect Disclosure, as defined in the main text. Here, I show how to characterize such an equilibrium -in which at the Sender mixes between disclosing the true value of θ and concealing it at $x = r \in R$.

The equilibrium I characterize is such that the positions where truth is disclosed, R, is a singleton. Let the tuple (B, r, s, η) constitute an equilibrium. B and s, as before, solve the receiver's problem, r solves the receiver's problem, and η is the probability with which the sender reveals the true state at $x = r$. Formally, during the time spent by the x-process at r, a truthful message is sent by the sender at rate η.

Consider an interval $(b, r) \subset (B \cup r)^C$, where $b \in B$. Firstly, because the sender is indifferent between disclosing or concealing θ, the 'uninformed' value function of the sender satisfies the two conditions:

$$\lim_{x \uparrow r} V_S(x) = W_S(r)$$

$$\lim_{x \uparrow r} V_S'(x) = 0$$

and

$$\lim_{x \downarrow b} V_S(x) = V_S(r) + \alpha$$

The above three boundary conditions characterize the value function and pin down the width of the interval, $|b - r|$, to a value $\bar{\psi}$. Note that similarly, for any such interval (r, b), the width is pinned down as the same value. Thus, $\{r - \bar{\psi}, r + \bar{\psi}\} \subset \partial B$. If we knew r, the conditions which pin down the receiver's value function over $(r - \bar{\psi}, r)$ and $(r, r + \bar{\psi})$, are

$$\lim_{x \downarrow r - \bar{\psi}} V_R(x) = V_R(r) - \alpha$$

$$\lim_{x \uparrow r + \bar{\psi}} V_R(x) = V_R(r) - \alpha$$

$$\lim_{x \downarrow r - \bar{\psi}} V_R'(x) = 0$$

$$\lim_{x \uparrow r + \bar{\psi}} V_R'(x) = 0$$

The continuity of the value function at $x = r$ implies that $r = E(\theta)$. (This is also a result of the receiver's value function beingsymmetric around r and the inactive regions being the same width around r). Thus we characterize the receiver's 'uninformed' value function with a kink at $x = r$.

To identify η, note that $V_R(x)$ can be written as the weighted average of two value functions. Consider a hypothetical value of the receiver if he were to secure $W_R(r)$ at $x = r$, earn the flow quadratic-loss payoff over $(r - \bar{\psi}, r)$, and immediately adjusts x to

$x = r$ at $\tau_{r-\bar{\psi}}$. One can characterize this value function, $\bar{V}_R(x)$ as if the receiver becomes informed with certainty at $x = r$. Similarly, one can identify a value function where she remains uninformed with certainty - $\underline{V}_R(x)$. η can be identified from the equation

$$V_R(r) = \eta W_R(r) + (1 - \eta)\underline{V}_R(r)$$