inverse_problem

March 28, 2023

1 Imports

[ ]:
```
import torch
import torch.nn as nn
import numpy as np
import matplotlib.pyplot as plt
import os
```

2 The Inverse Problem

2.1 Problem Statement

Here, we'll use the PDE used above:

$$\frac{d}{dx}\left(E(x)A(x)\frac{d}{dx}u(x)\right) + p(x) = 0$$

to get the value of $EA(x)$ assuming that we know the solution. We'll use the same neural network architecture as above.

The differential equation becomes:

$$E(x)u_{xx} + E_x(x)u_x + p(x) = 0$$

now define:

$$f := E(x)u_{xx} + E_x(x)u_x + p(x)$$

We already know u, u_x and u_{xx}, so we will create a neural network to approximate $E(x)$.

> Please note that I've dropped the A, it is assumed to be absorbed in the E.

The analytical form of the functions are:

$$\begin{aligned} u(x) &= \sin(2\pi x) \\ u_x(x) &= 2\pi\cos(2\pi x) \\ u_{xx}(x) &= -4\pi^2\sin(2\pi x) \end{aligned}$$

while, we choose the following form for $E(x)$:

$$E(x) = x^2 - x^2 + 1$$

For this, the load function becomes:

$$p(x) = -2(3x^2 - 2x)\pi\cos(2\pi x) + 4(x^3 - x^2 + 1)\pi^2\sin(2\pi x)$$

2.2 Solution

Once again, we'll start by defining some variables:

[ ]:
```
#The function u(x) = sin(2*pi*x)
u_analytical = lambda x: torch.sin(2*torch.pi * x)
E_analytical = lambda x: x**3 - x**2 + 1

#The load function f(x) = 4*pi^2*sin(2*pi*x)
p = lambda x: -2*(3*x**2 - 2*x)*torch.pi*torch.cos(2*torch.pi*x) + 4*(x**3 -
 ↪x**2 + 1)*torch.pi**2*torch.sin(2*torch.pi*x)
```

[ ]:
```
def collocation_points(points):
    #Create a tensor of collocation points
    x = torch.linspace(0, 1, points, requires_grad=True, dtype=torch.float32)
 ↪#requires_grad=True to compute the gradient
    return x

X_c = collocation_points(100)
# u_c = u_analytical(X_c)
# inputs = torch.zeros((100, 2), dtype=torch.float32)
# inputs[:, 0] = X_c
# inputs[:, 1] = u_c
inputs = X_c.unsqueeze(1)
inputs.shape
```

[ ]:
```
torch.Size([100, 1])
```

We'll define the cost function here. It will take input and the model and will return the residuals.

[ ]:
```
def loss(inputs, model):
    e = model(inputs)
    e_x = torch.autograd.grad(e, inputs, grad_outputs=torch.ones_like(e),
 ↪create_graph=True, retain_graph=True)[0]

    u = u_analytical(inputs)
    u_x = torch.autograd.grad(u, inputs, grad_outputs=torch.ones_like(u),
 ↪create_graph=True, retain_graph=True)[0]
    u_xx = torch.autograd.grad(u_x, inputs, grad_outputs=torch.ones_like(u_x),
 ↪create_graph=True,  retain_graph=True)[0]
```

```
    l = torch.mean((e*u_xx + e_x*u_x + p(inputs))**2)
    return l
```

[ ]:
```
model = nn.Sequential(
    nn.Linear(1, 20),
    nn.Tanh(),
    nn.Linear(20, 1),
)
```

[ ]:
```
#create a LBFGS optimizer
optimizer = torch.optim.LBFGS(model.parameters(), lr=0.1, max_iter=1000,␣
 ↪max_eval=100, tolerance_grad=1e-05, tolerance_change=1e-06)
```

[ ]:
```
epochs = 20
losses = []
for epoch in range(epochs):
    u = model(inputs)
    loss_ = loss(inputs, model)
    optimizer.zero_grad()
    loss_.backward()
    def closure():
        optimizer.zero_grad()
        loss_ = loss(inputs, model)
        loss_.backward()
        return loss_
    optimizer.step(closure=closure)
    losses.append(loss_.item())

    print(f"Epoch {epoch+1:>4d}/{epochs} | loss={loss_.item():.4f}")
```

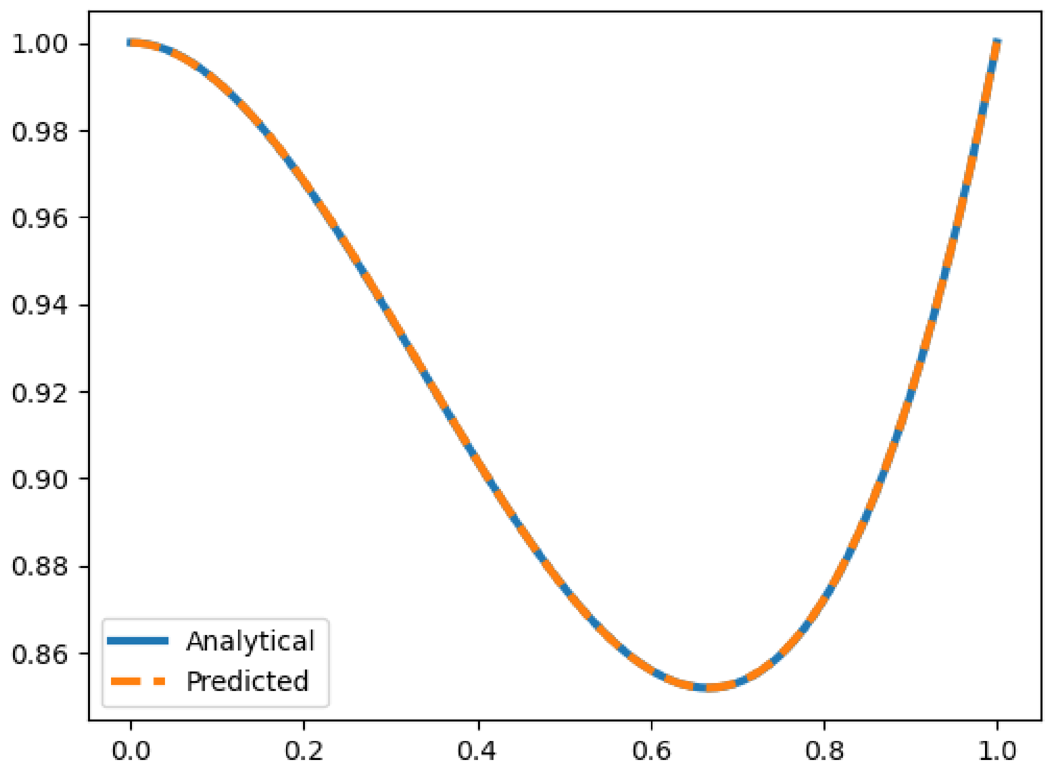
```
Epoch    1/20 | loss=488.6730
Epoch    2/20 | loss=1.8172
Epoch    3/20 | loss=0.0110
Epoch    4/20 | loss=0.0026
Epoch    5/20 | loss=0.0013
Epoch    6/20 | loss=0.0007
Epoch    7/20 | loss=0.0002
Epoch    8/20 | loss=0.0000
Epoch    9/20 | loss=0.0000
Epoch   10/20 | loss=0.0000
Epoch   11/20 | loss=0.0000
Epoch   12/20 | loss=0.0000
Epoch   13/20 | loss=0.0000
Epoch   14/20 | loss=0.0000
Epoch   15/20 | loss=0.0000
Epoch   16/20 | loss=0.0000
```

```
Epoch   17/20 | loss=0.0000
Epoch   18/20 | loss=0.0000
Epoch   19/20 | loss=0.0000
Epoch   20/20 | loss=0.0000
```

Let's plot the value of $E(x)$:

[ ]:
```
E_true = E_analytical(X_c)
E_model = model(X_c.unsqueeze(1))
```

[ ]:
```
plt.plot(X_c.detach().numpy(), E_true.detach().numpy(), label="Analytical",␣
  ↪lw=3)
plt.plot(X_c.detach().numpy(), E_model.detach().numpy(), label="Predicted",␣
  ↪linestyle="--", lw=3)
plt.legend()
plt.savefig(os.path.join("plots", "0201.png"))
plt.show()
```



Let's plot the loss.

[ ]:
```
#plot losses
plt.plot(losses)
plt.xlabel("Epochs")
```

```
plt.ylabel("Loss")
plt.savefig(os.path.join("plots", "0202.png"))
plt.show()
```