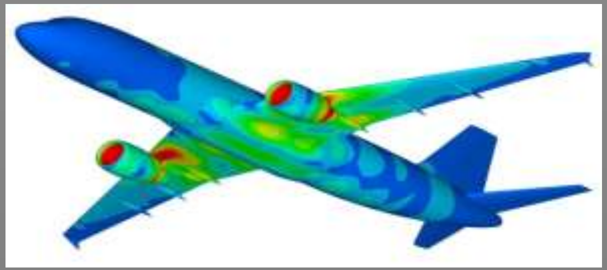
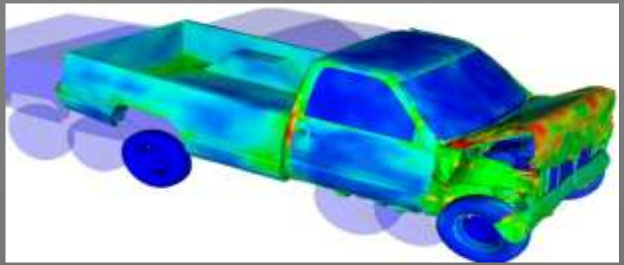
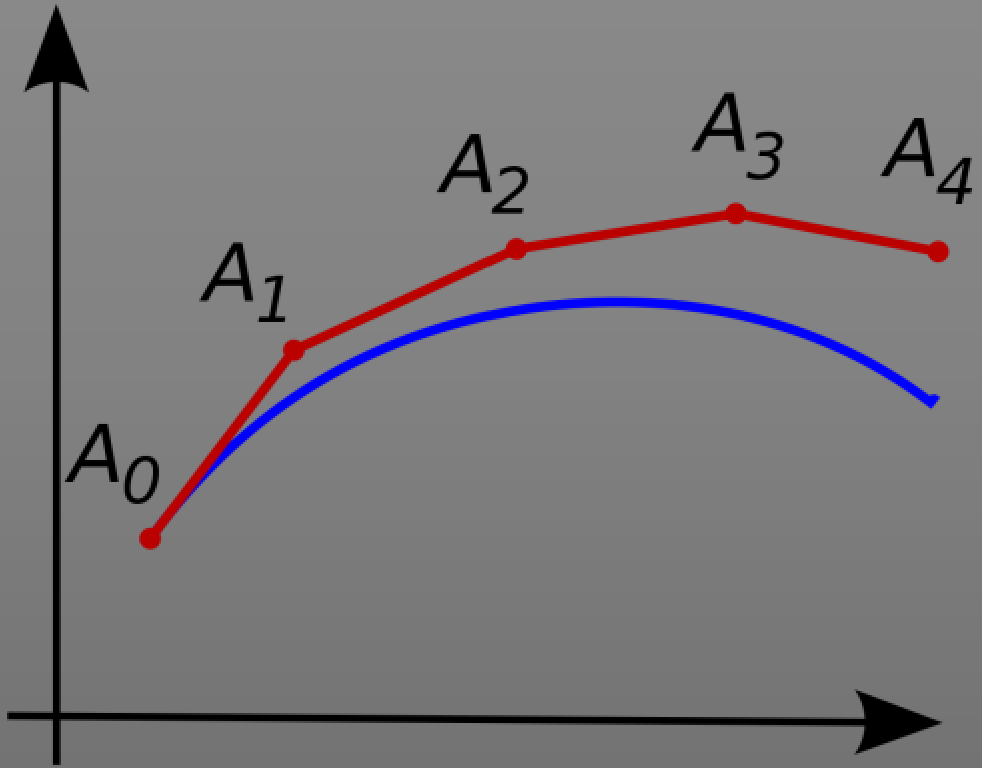
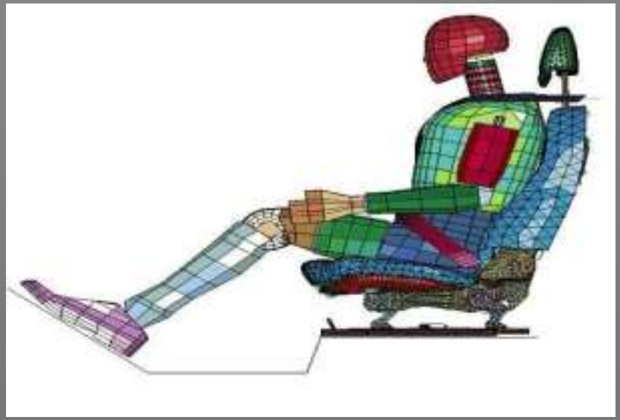
ODE









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ODE vs PDE

ODE: One independent variable

$$\frac{dy}{dx} = ay + q(y) \qquad\qquad m\frac{d^2y}{dt^2} + a\frac{dy}{dt} + ky = f(t)$$

PDE: Multiple independent variable

Heat Equation $m\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$

Wave Equation $\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$

Laplace Equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

Initial and boundary value problems

Boundary Value Problems:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu=f; \qquad 0\le x\le L$$

$$u(0)=u_0, \qquad \left(a\frac{du}{dx}\right)_{x=L}=q_0$$

Initial Value Problems:

$$a\frac{du}{dt}+cu=f; \qquad 0\le t\le T$$

$$u(0)=u_0$$

Boundary and Initial Value Problems:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)+cu=f; \qquad 0\le x\le L$$

$$u(0)=u_0, \qquad \left(a\frac{du}{dx}\right)_{x=L}=q_0$$

Eigen Value Problems:

$$-\frac{d}{dx}\left(a\frac{du}{dx}\right)-\lambda u=0; \qquad 0\le x\le L$$

$$u(0)=u_0, \qquad \left(a\frac{du}{dx}\right)_{x=L}=0$$

Taylor Series

- Taylor series are expansions of a function f(x) by some finite distance dx to f(x+dx).
- In essence, the Taylor series provides a means to predict a function value at one point in terms of the function value and its derivatives at another point.
- In particular, the theorem states that any smooth function can be approximated as a polynomial.

$$f(x_{i+1}) \cong f(x_i)$$

0^{th} order Approximation

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i)$$

1^{st} order Approximation

$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''(x_i)}{2!}(x_{i+1} - x_i)^2$$

2^{nd} order Approximation

$$\boxed{f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f^{(3)}(x_i)}{3!}h^3 + \cdots + \frac{f^{n}(x_i)}{n!}h^n + R_n}$$

N^{th} order Approximation

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}h^{n+1}$$

Remainder

Taylor Series

Solve the differential equation using Taylor series $\frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0$

Expand y(x) using Taylor series: $\frac{dy}{dx} = x - y^2; \quad y(0) = 1; \quad find\ y(0.1)$

$$y(x) = y(x_0) + \frac{(x - x_0)}{1!} y'(x_0) + \frac{(x - x_0)^2}{2!} y''(x_0) + \frac{(x - x_0)^3}{3!} y'''(x_0) + \ldots.$$

$$y(x) \approx y(x_0) + xy'(x_0) + \frac{x^2}{2!} y''(x_0) + \frac{x^3}{3!} y'''(x_0)$$

$$here, \quad y' = x - y^2; \; y'' = 1 - 2yy'; \; y''' = -2yy'' - 2(y')^2;$$

$$y_0' = x_0 - y_0{}^2; \; y_0'' = 1 - 2y_0 y_0'; \; y_0''' = -2y_0 y_0'' - 2(y_0')^2;$$

$$y_0' = 0 - 1^2 = -1; \; y_0'' = 1 - 2 \times 1 \times -1 = 3; \; y_0''' = -2 \times 1 \times 3 - 2(-1_0)^2 = -8$$

$$y(x) \approx 1 + x \times -1 + \frac{x^2}{2} \times 3 + \frac{x^3}{6} \times -8 \approx 1 - x + \frac{3}{2} x^2 - \frac{8}{6} x^3$$

$$y(0.1) \approx 1 - 0.1 + \frac{3}{2} 0.1^2 - \frac{8}{6} 0.1^3 \approx 0.9138$$

PICARD'S Method

- **Integration is applied to solve ODE.** $\frac{dy}{dx} = f(x, y)$ $\Rightarrow$ $dy = f(x, y)dx$

Integrating: $\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)dx$

Integral Equ^n $y = y_0 + \int_{x_0}^{x} f(x, y)dx$

Approximations:

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)dx;$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(1)}\right)dx;$$

$$y^{(3)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(2)}\right)dx;$$

$$y^{(4)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(3)}\right)dx;$$

and so on

PICARD'S Method

Solve the differential equation using PICARD'S method:

$$\frac{dy}{dx} = 2x - y; \quad y(0) = 0.9; \quad find\ \ y(0.2).$$

Integral Equ^n

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx$$

Approximations:

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)dx;$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(1)}\right)dx;$$

$$y^{(3)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(2)}\right)dx;$$

$$y^{(4)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(3)}\right)dx;$$

and so on

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)dx = 0.9 + \int_{0}^{x}(2x - 0.9)dx = 0.9 + x^2 - 0.9x$$

$$y^{(2)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(1)}\right)dx = 0.9 + \int_{0}^{x}\left[2x - \left(0.9 + x^2 - 0.9x\right)\right]dx = 0.9 - 0.9x + 2.9\frac{x^2}{2} - \frac{1}{3}x^3$$

$$y^{(3)} = y_0 + \int_{x_0}^{x} f\left(x, y^{(2)}\right)dx = 0.9 + \int_{0}^{x}\left[2x - 0.9 + 0.9x - 2.9\frac{x^2}{2} + \frac{1}{3}x^3\right]dx$$

$$= 0.9 - 0.9x + 1.45x^2 - 1.45\frac{x^3}{3} + \frac{x^4}{12}$$

$$y^{(3)}(0.2) = 0.9 - 0.9\times 0.2 + 1.45\times 0.2^2 - 1.45\frac{0.2^3}{3} + \frac{0.2^4}{12} = 0.7742$$

Euler's Method / First Order Runge Kutta Method

- **Stepwise approximation is used to find numerical solution of differential equation.**

Approximations: $y_{n+1} = y_n + h \times f(x_n, y_n)$

Example: Solve the differential equation using Euler's method: $\frac{dy}{dx} = x + y; \quad y(0) = 1; \quad find \ y(0.1).$

Find step size: $h = \frac{x - x_0}{n} = \frac{0.1 - 0}{5} = 0.02$

n	x	y
0	$x_0=0$	$y_0=1$
1	$x_1=0.02$	$y_1= y_0 + h \times f(x_0, y_0) = 1+0.02\times(0+1) = 1.02$
2	$x_2=0.04$	$y_2= y_1 + h \times f(x_1, y_1) = 1.02+0.02\times(0.02+1.02) = 1.0408$
3	$x_3=0.06$	$y_3= y_2 + h \times f(x_2, y_2) = 1.0408+0.02\times(0.04+1.0408) = 1.0624$
4	$x_4=0.08$	$y_4= y_3 + h \times f(x_3, y_3) = 1.0624+0.02\times(0.06+1.0624) = 1.0848$
5	$x_5=0.10$	$y_5= y_4 + h \times f(x_4, y_4) = 1.0848+0.02\times(0.08+1.0848) = 1.1081$

Euler's Modified Method

- **Stepwise approximation with predictor-corrector is used to find numerical solution of differential equation.**

Two step Predictor-Corrector Approximation

$$y^*_{n+1} = y_n + h \times f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{h}{2} \times \left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1}) \right]$$

Example: Solve the differential equation using modified Euler's method:

$$\frac{dy}{dx} = x^2 + y; \quad y(0) = 1; \quad find \ \ y(0.02), y(0.04)$$

step size: $h = x_2 - x_1 = 0.04 - 0.02 = 0.02$

n	x	y*	y
0	$x_0=0$	$y^*_0=1$	$y_0=1$
1	$x_1=0.02$	$y^*_1= y_0 + h \times f(x_0, y_0) = 1+0.02\times(0^2+1) = 1.02$	$y_1= y_0 + h/2 \times [f(x_0, y_0)+ f(x_1, y^*_1)$ $= 1 + 0.02/2 \times [(0^2+1) + (0.02^2+1.02)] = 1.0202$
2	$x_2=0.04$	$y^*_2= y_1 + h \times f(x_1, y_1)$ $= 1.0202+0.02\times(0.02^2+1.0202) = 1.0406$	$y_2= y_1 + h/2 \times [f(x_1, y_1)+ f(x_2, y^*_2)$ $= 1.0202 + 0.02/2 \times [(0.02^2+1.0202) + (0.04^2+1.0406)]$ $= 1.0408$

Runge Kutta Method

- **The method achieve accuracy of a Taylor series approach without requiring the calculation of higher order derivatives.**

R-K Method $y_{n+1} = y_n + \phi(x_i, y_i, h) \times h$

Incremental function $\phi = a_1 k_1 + a_2 k_2 + a_3 k_3 + \ldots + a_n k_n$

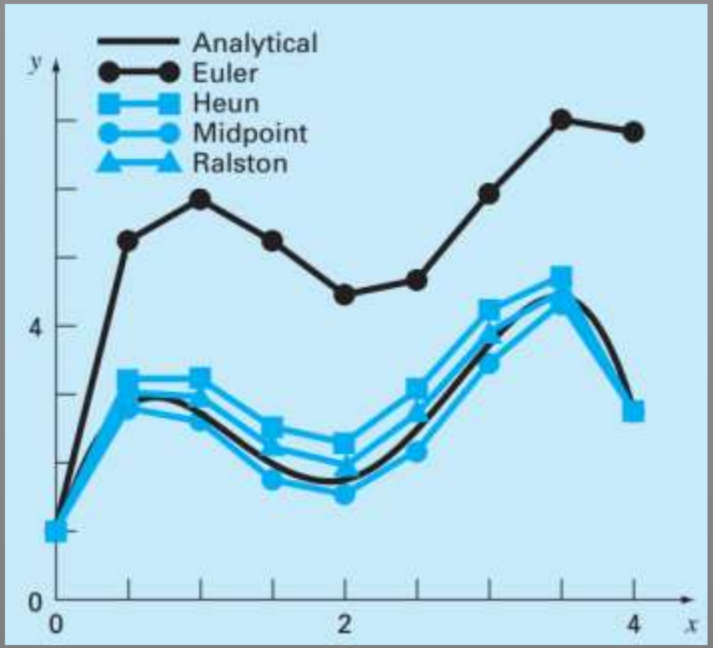
Incremental slope factor

$$
\begin{aligned}
k_1 &= f(x_i, y_i) \\
k_2 &= f(x_i + p_1 h,\ y_i + q_{11} k_1 h) \\
k_3 &= f(x_i + p_2 h,\ y_i + q_{21} k_1 h + q_{22} k_2 h) \\
&\vdots \\
k_n &= f(x_i + p_{n-1} h,\ y_i + q_{n-1,1} k_1 h + q_{n-1,2} k_2 h + \ldots + q_{n-1,n-1} k_{n-1} h)
\end{aligned}
$$

Second-Order Runge Kutta Method

R-K Method

$$y_{i+1} = y_i + (a_1 k_1 + a_2 k_2) \times h$$
$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + p_1 h,\ y_i + q_{11} k_1 h)$$



Constant values:

$$a_1 + a_2 = 1$$
$$a_2 p_1 = 1/2$$
$$a_2 q_{11} = 1/2$$

Heun method

$$a_2 = 1/2$$

$$y_{i+1} = y_i + \left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right) \times h$$
$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + h,\ y_i + k_1 h)$$

Midpoint method

$$a_2 = 1$$

$$y_{i+1} = y_i + k_2 \times h$$
$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1 h\right)$$

Ralston's method

$$a_2 = \frac{2}{3}$$

$$y_{i+1} = y_i + \left(\frac{1}{3}k_1 + \frac{2}{3}k_2\right) \times h$$
$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \frac{3}{4}h,\ y_i + \frac{3}{4}k_1 h\right)$$

Runge Kutta Method

Third Order R-K Method

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 4k_2 + k_3)\times h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f(x_i + h,\ y_i - k_1 h + 2k_2 h)$$

Fourth Order R-K Method

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)\times h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}k_2 h\right)$$

$$k_4 = f(x_i + h,\ y_i + k_3 h)$$

Fifth Order R-K method (Butcher's Method)

$$y_{i+1} = y_i + \frac{1}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6)\times h$$

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{4}h,\ y_i + \frac{1}{4}k_1 h\right)$$

$$k_3 = f\left(x_i + \frac{1}{4}h,\ y_i + \frac{1}{8}k_1 h + \frac{1}{8}k_2 h\right)$$

$$k_4 = f\left(x_i + \frac{1}{2}h,\ y_i - \frac{1}{2}k_2 h + k_3 h\right)$$

$$k_5 = f\left(x_i + \frac{3}{4}h,\ y_i + \frac{3}{16}k_1 h + \frac{9}{16}k_4 h\right)$$

$$k_6 = f\left(x_i + h,\ y_i - \frac{3}{7}k_1 h + \frac{2}{7}k_2 h + \frac{12}{7}k_3 h - \frac{12}{7}k_4 h + \frac{8}{7}k_5 h\right)$$

MATLAB© Script

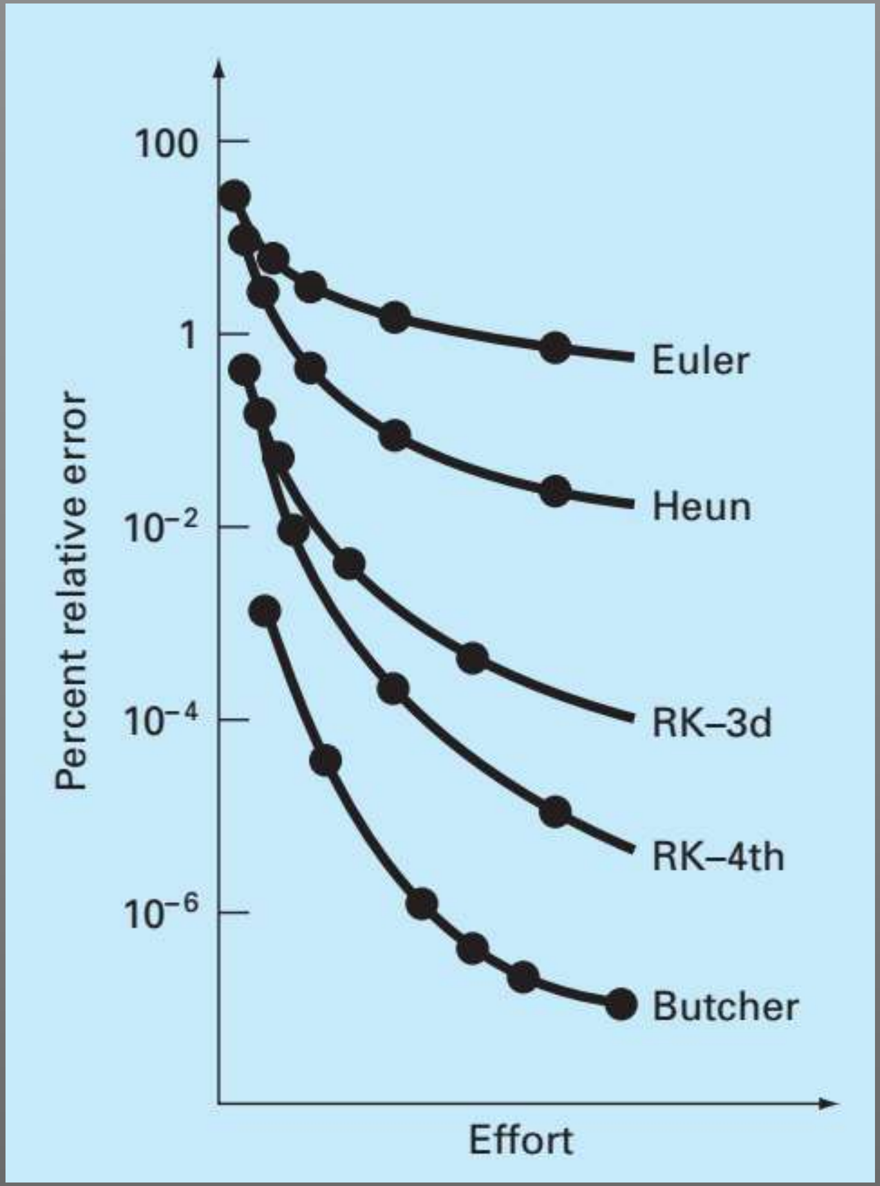
MATLAB Program for 4th Order R-K Method

```
clear all
close all
clc
h=0.5; % step size
x = 0:h:100;
Y = zeros(1,length(x));
y(1) = 0.5;     % initial condition
F_xy = @(t,r) 2.*exp(-0.5*t)-r;          %  the function
for i=1:(length(x)-1)
    k_1 = F_xy( x(i), y(i) );
    k_2 = F_xy( x(i)+0.5*h, y(i)+0.5*h*k_1);
    k_3 = F_xy( (x(i)+0.5*h), (y(i)+0.5*h*k_2));
    k_4 = F_xy( (x(i)+h), (y(i)+k_3*h));
    y(i+1) = y(i) + (1/6)*(k_1+2*k_2+2*k_3+k_4)*h;
end
```

```
% Plot data with in-built function
tspan = [0,100];
y0 = 0.5;
[tx, yx] = ode45(F_xy, tspan, y0);
plot(x, y, 'o-',  tx,  yx, '--')
```

Runge Kutta Method

Accuracy of R-K Method



THANK YOU

Questions??