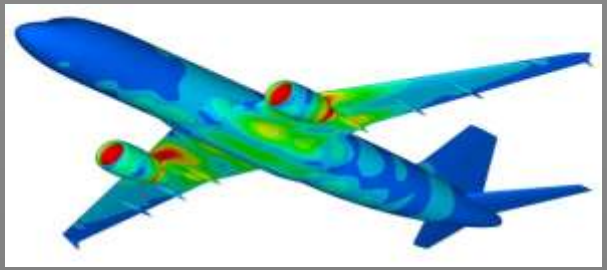
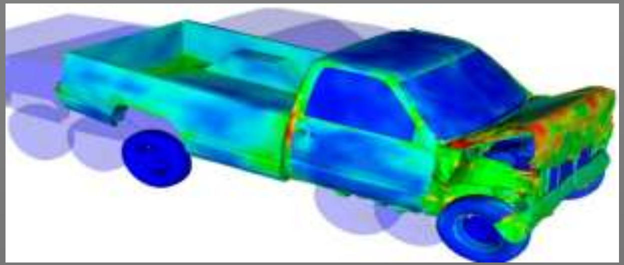
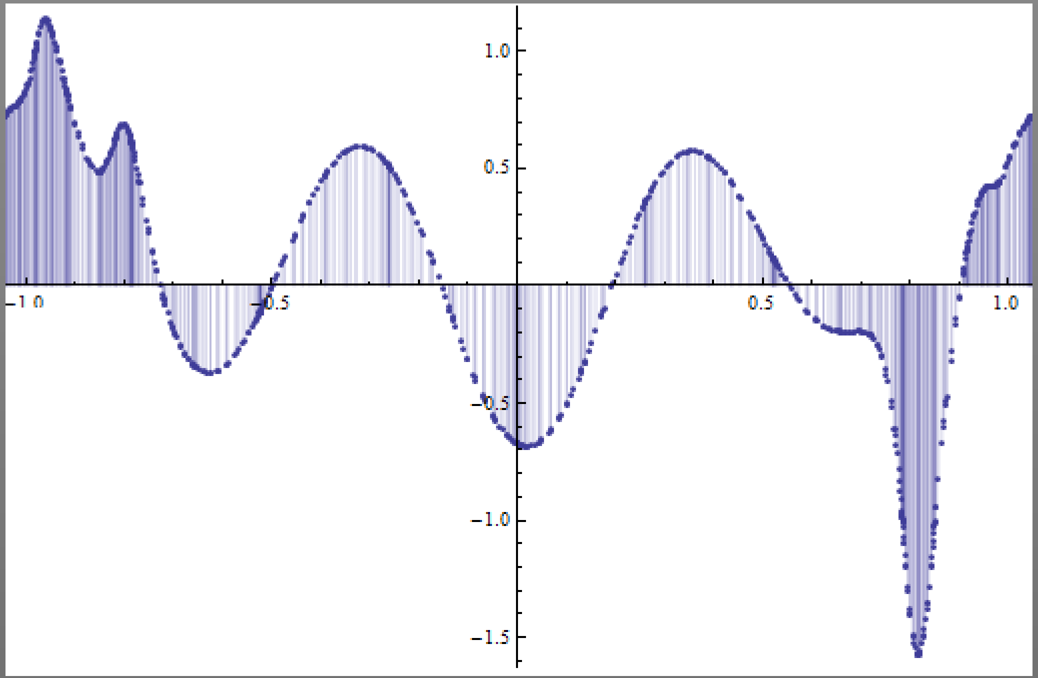
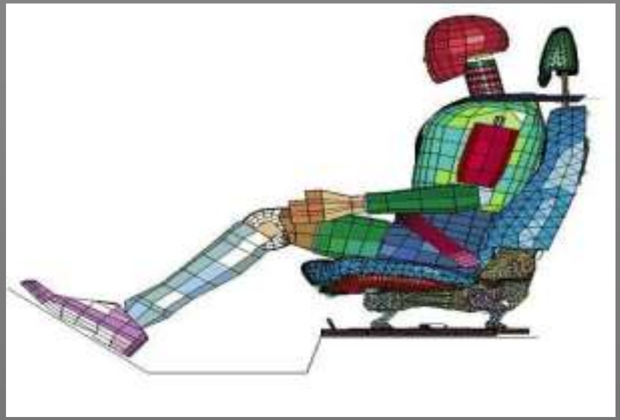
Numerical Integration









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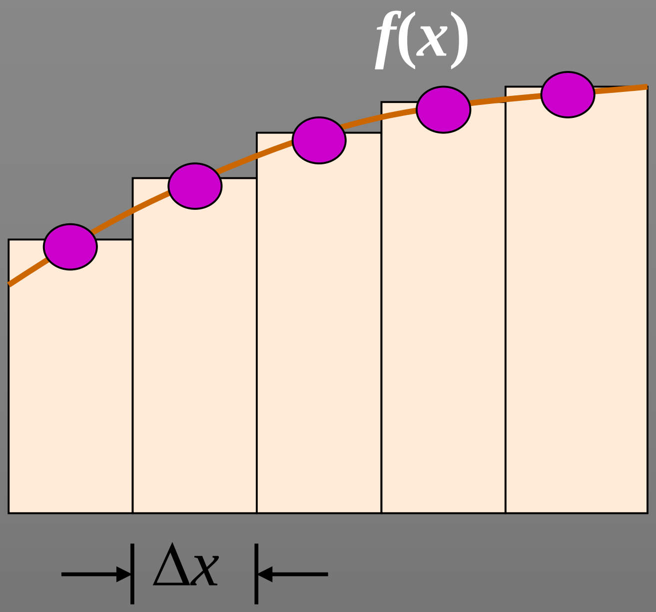
Numerical Integration

- Evaluate the integral, $I = \int_a^b f(x)dx$ without doing the calculation analytically.
- In essence, Integrand is too complicated to integrate analytically. $\int_0^2 \frac{2+\cos(1+\sqrt{x})}{\sqrt{1+0.5x}} e^{0.5x} dx$

$$\int_a^b f(x)dx = \text{area} \qquad \int_c^d \int_a^b f(x)dxdy = \text{volume}$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x$$

where $\Delta x = \dfrac{b-a}{n}$



sum of height × width

- Integration is a summing process. Thus virtually all numerical approximations can be represented by

$$I = \int_a^b f(x)dx = \sum_{i=1}^{n} w_i f(x_i) + E_t$$

w_i are the weights, x_i are the sampling points, and E_t is the truncation error.

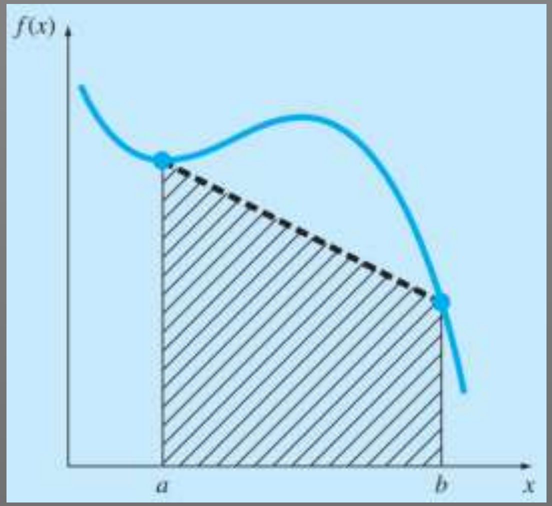
Newton-Cotes Integration

- **Newton-Cotes formulas are the most common numerical integration schemes.**
- **It replaces a complicated function with an approximating function that is easy to integrate numerically.**

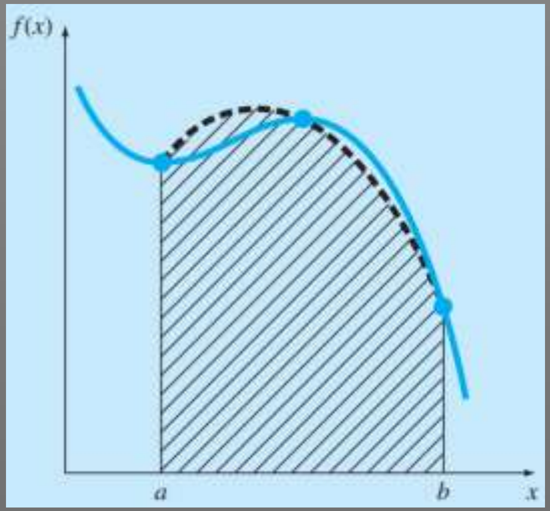
By Newton-Cotes formulas

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_n(x)\,dx$$

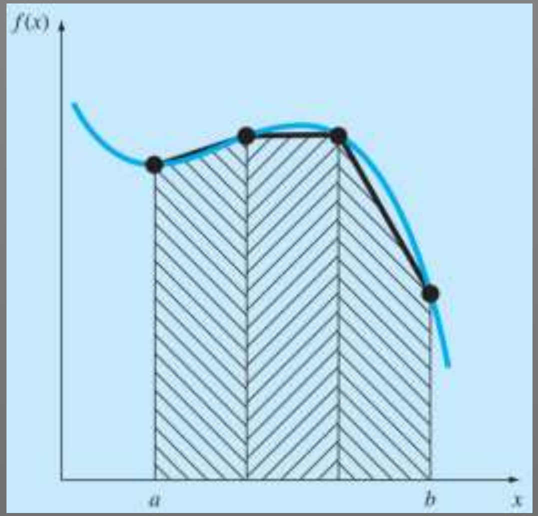
$$f_n(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$$



1st order Polynomial



2nd order Polynomial



1st order Polynomial segment wise

Trapezoidal Rule

- The trapezoidal rule is the first order example of the Newton-Cotes closed integration formulas.

Trapezoidal Rule

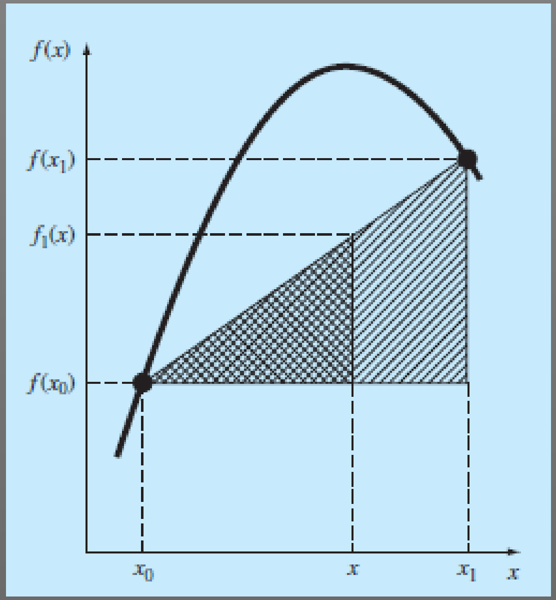
$$I = \int_a^b f(x)\,dx \cong \int_a^b f_1(x)\,dx$$

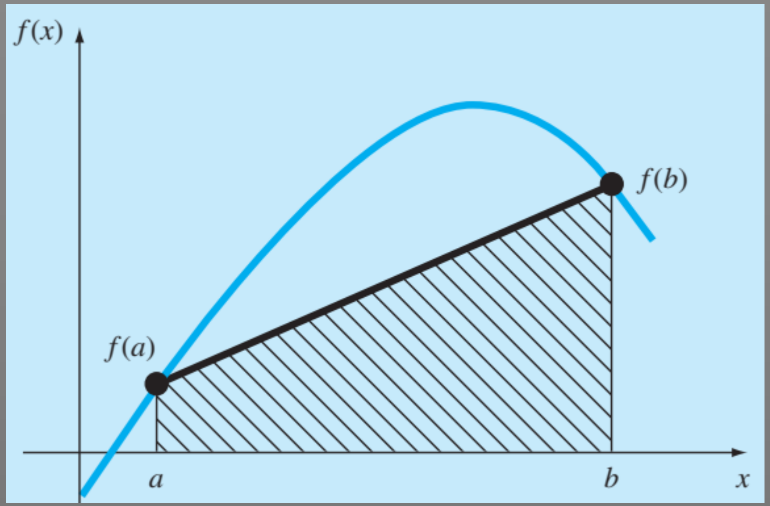
$$f_1(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

Using linear interpolation

$$\frac{f_1(x) - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$





Trapezoidal Rule

Trapezoidal Rule

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_1(x)\,dx \Leftrightarrow f_1(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

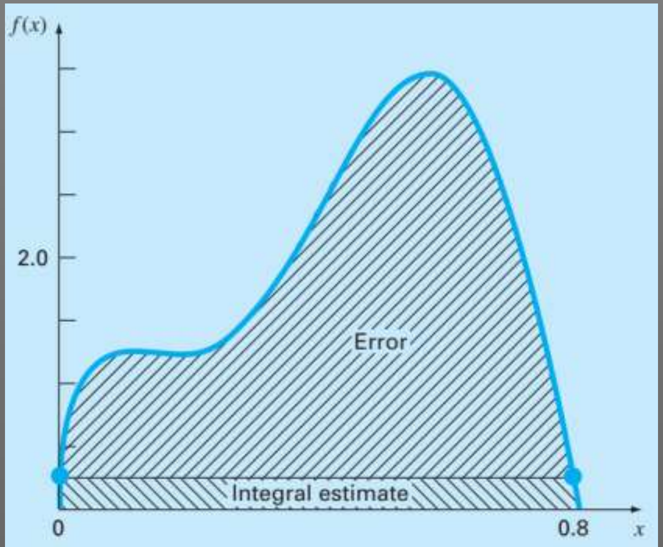
$$\Downarrow$$

$$I = \int_a^b \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right] dx$$

$$I = (b - a)\frac{f(a) + f(b)}{2} \Rightarrow I \cong \text{width} \times \text{average height}$$

$$I = \underbrace{h\frac{f(a) + f(b)}{2}}_{\text{Trapezoidal rule}} - \underbrace{\frac{1}{12}f''(\xi)h^3}_{\text{Truncation error}}$$

$$E_t = -\frac{1}{12}f''(\xi)(b - a)^3$$



Trapezoidal Rule

Example: Numerically integrate the following polynomial

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $a = 0$ to $b = 0.8$.

The exact value of the integral can be determined analytically to be 1.640533.

Trapezoidal Rule

$$I = (b - a)\frac{f(a) + f(b)}{2}$$

$$f(0) = 0.2$$
$$f(0.8) = 0.232$$

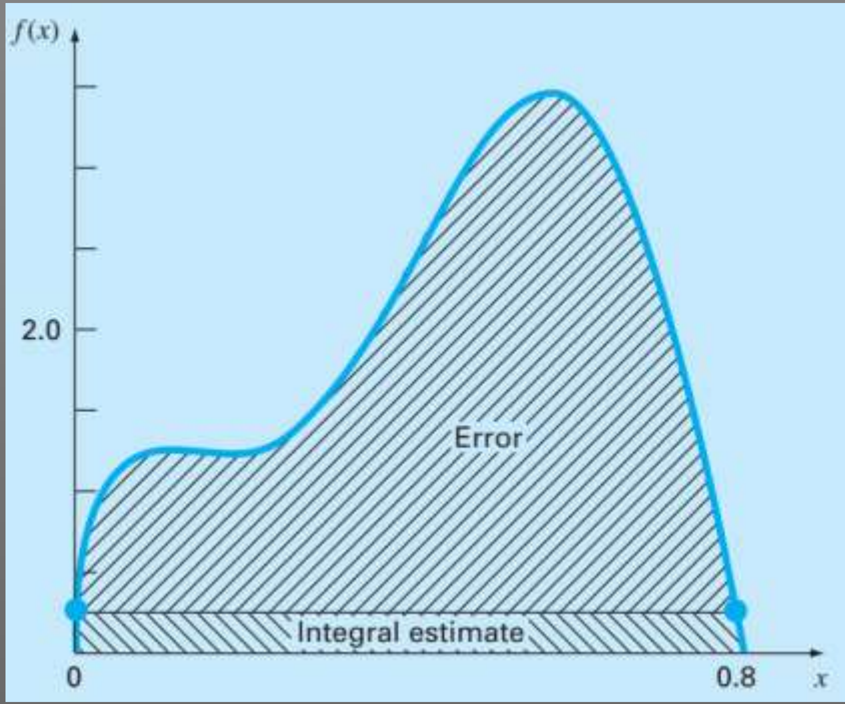
Integral

$$I \cong 0.8\frac{0.2 + 0.232}{2} = 0.1728$$

Turncation Error

$$E_t = 1.640533 - 0.1728 = 1.467733$$

$$\varepsilon_t = 89.5\%.$$



Multiple-Application Trapezoidal Rule

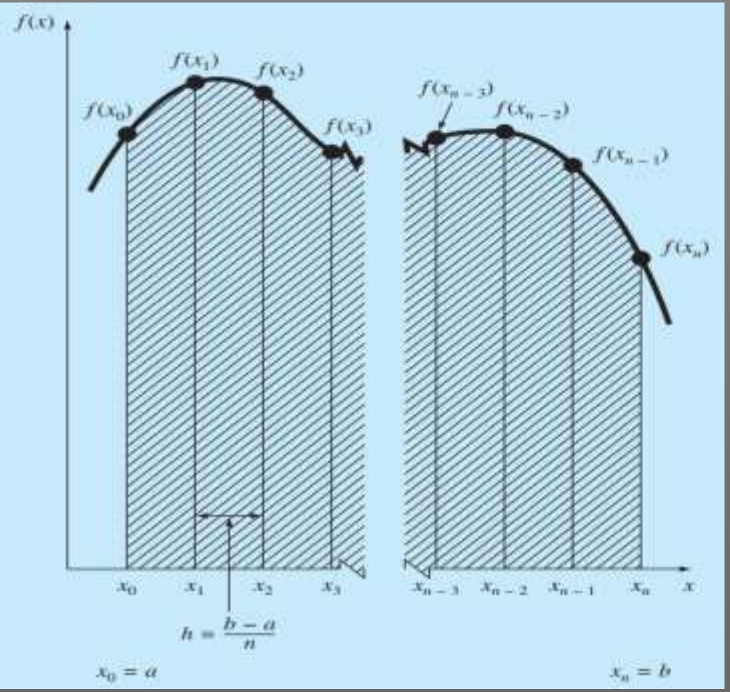
- One way to improve the accuracy of the trapezoidal rule is to divide the integration interval from a to b into a number of segments and apply the method to each segment.
- The areas of individual segments can then be added to yield the integral for the entire interval.
- The resulting equations are called multiple-application, or composite, integration formulas.

n + 1 equally spaced base points (x0, x1, x2, . . . , xn). ⇨ *n segments of equal width*

Segment Width $$h = \frac{b-a}{n}$$

Total Integrals $$I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$I = h\,\frac{f(x_0)+f(x_1)}{2} + h\,\frac{f(x_1)+f(x_2)}{2} + \cdots + h\,\frac{f(x_{n-1})+f(x_n)}{2}$$ ⇨



$$\boxed{I = \underbrace{(b-a)}_{\text{Width}}\ \underbrace{\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n}}_{\text{Average height}}}$$

$$E_t = -\frac{(b-a)^3}{12n^3}\sum_{i=1}^{n} f''(\xi_i)$$

Multiple-Application Trapezoidal Rule

Example: Numerically integrate the following polynomial

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $a = 0$ to $b = 0.8$.

The exact value of the integral can be determined analytically to be 1.640533.

Multiple-Application Trapezoidal Rule

$n = 2\ (h = 0.4)$

$$I = \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n}}_{\text{Average height}}$$

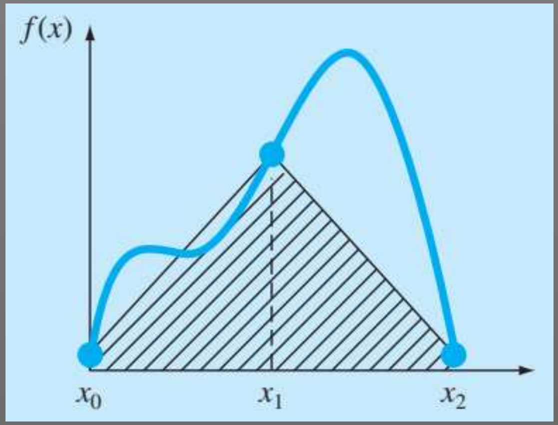
$f(0) = 0.2 \qquad f(0.4) = 2.456 \qquad f(0.8) = 0.232$

Integral $I = 0.8\frac{0.2 + 2(2.456) + 0.232}{4} = 1.0688$

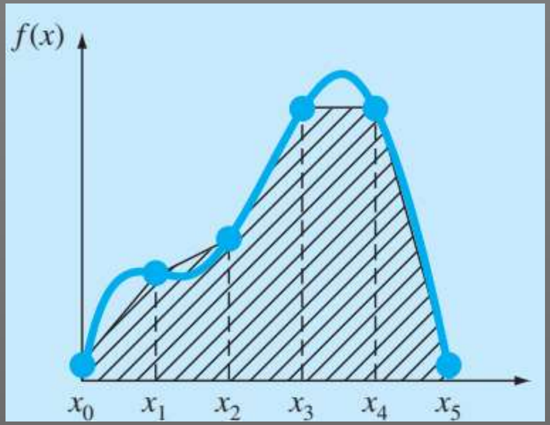
Turncation Error

$E_t = 1.640533 - 1.0688 = 0.57173 \qquad \varepsilon_t = 34.9\%$

n	h	I	ε_t (%)
2	0.4	1.0688	34.9
3	0.2667	1.3695	16.5
4	0.2	1.4848	9.5
5	0.16	1.5399	6.1
6	0.1333	1.5703	4.3
7	0.1143	1.5887	3.2
8	0.1	1.6008	2.4
9	0.0889	1.6091	1.9
10	0.08	1.6150	1.6



2 segments



5 segments

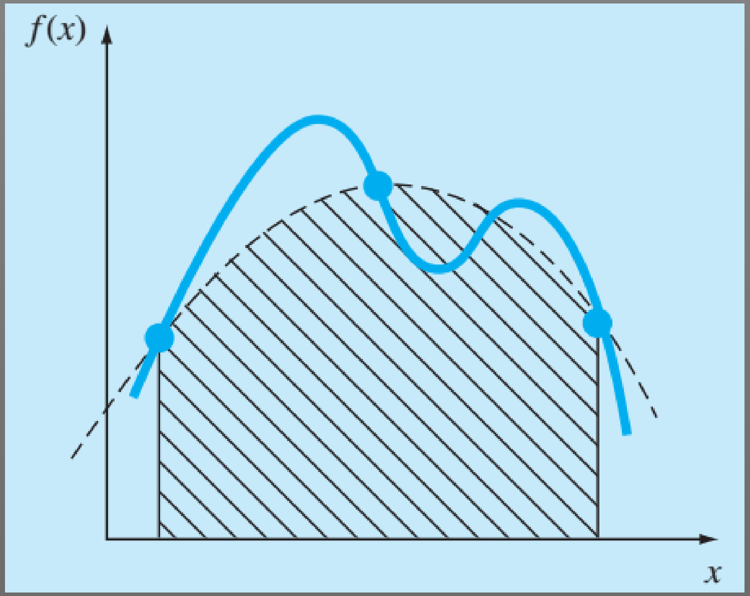
MATLAB© Script

MATLAB Program for Multiple-Application Trapezoidal Rule

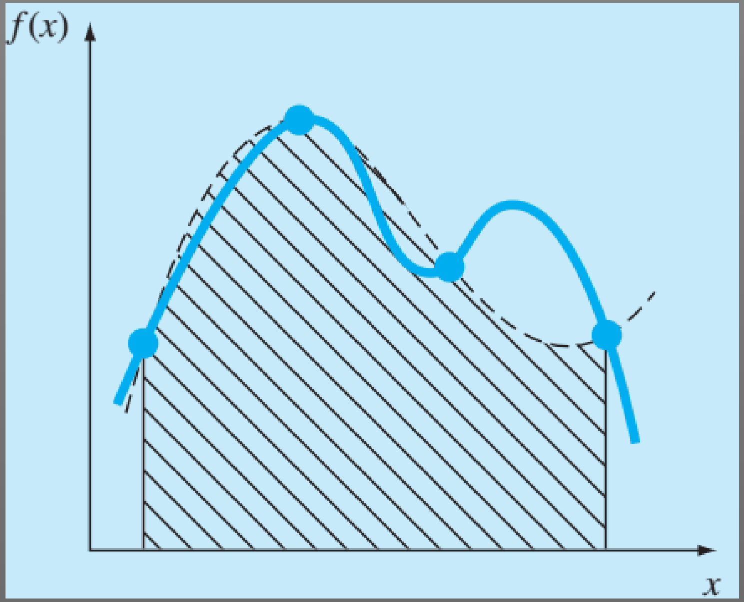
```
clear all
clc
a= 0; % lower limit
b = 0.8; % upper limit
c = 10; % No of segments
x=linspace(a,b,c+1); % point generation
n=length(x);
h = (b-a)/(n-1);
sum= 0;
for i=2:n
    X1 = int_fun(x(i-1));
    X2 = int_fun(x(i));
    sum = sum + h*((X1 + X2)/2);
end
sum
```

Simpson's Rule

- One way to improve the accuracy of the trapezoidal rule is to use higher order polynomial in function approximation.
- As compared with Trapezoidal rule (function is approximated by first order polynomial); Simpson's 1/3 rule use second-order Lagrange polynomial for each integrant segment.
- Simpson's 3/8 rule use third-order Lagrange polynomial for each integrant segment.



2nd order (Simpson's 1/3)



3rd order (Simpson's 3/8)

Simpson's 1/3 Rule

- **A second-order interpolating polynomial is substituted the integrant.**

Simpson's 1/3 Rule
$$I = \int_a^b f(x)\,dx \cong \int_a^b f_2(x)\,dx$$

If a and b are designated as x_0 and x_2 and $f_2(x)$ is represented by a second-order Lagrange polynomial.

$$I = \int_{x_0}^{x_2} \left[\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \right] dx$$

$$I \cong \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

$$I \cong \underbrace{(b - a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}}_{\text{Average height}}$$

$$E_t = -\frac{(b - a)^5}{2880} f^{(4)}(\xi)$$

Simpson's 1/3 Rule

Example: Numerically integrate the following polynomial

$f(x)= 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $a = 0$ to $b = 0.8$.

The exact value of the integral can be determined analytically to be 1.640533.

Simpson's 1/3 Rule

$$I \cong \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}}_{\text{Average height}}$$

$$f(0) = 0.2 \qquad f(0.4) = 2.456 \qquad f(0.8) = 0.232$$

Integral

$$I \cong 0.8\frac{0.2 + 4(2.456) + 0.232}{6} = 1.367467$$

Truncation Error

$$E_t = 1.640533 - 1.367467 = 0.2730667 \qquad \varepsilon_t = 16.6\%$$

Approximately 5 times more accurate than for a single application of the Trapezoidal rule

Multiple-Application Simpson's 1/3 Rule

- One way to improve the accuracy of the Simpson's 1/3 Rule is to divide the integration interval from a to b into a number of segments and apply the method to each segment.
- The areas of individual segments can then be added to yield the integral for the entire interval.

n + 1 equally spaced base points (x0, x1, x2, . . . , xn). ⇨ *n segments of equal width*

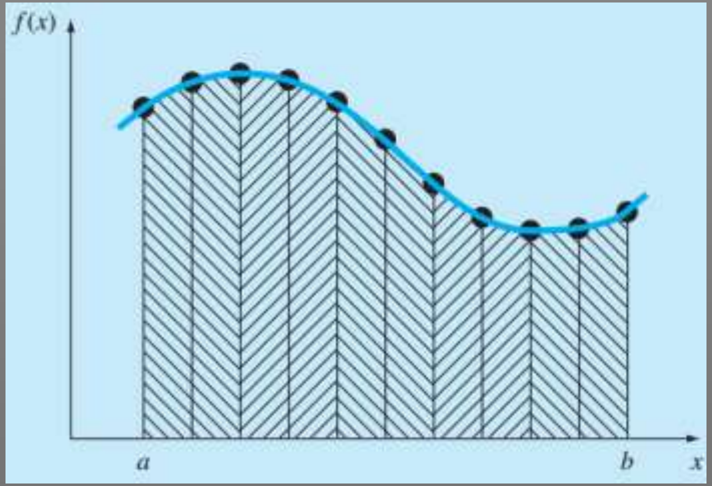
Segment Width

$$h = \frac{b-a}{n}$$

Total Integrals

$$I = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$I \cong 2h\frac{f(x_0)+4f(x_1)+f(x_2)}{6} + 2h\frac{f(x_2)+4f(x_3)+f(x_4)}{6} + \cdots + 2h\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}$$



$$I \cong \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}}_{\text{Average height}}$$

$$E_a = -\frac{(b-a)^5}{180n^4}\bar{f}^{(4)}$$

Multiple-Application Simpson's 1/3 Rule

Example: Numerically integrate the following polynomial

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $a = 0$ to $b = 0.8$.

The exact value of the integral can be determined analytically to be 1.640533.

Multiple-Application Simpson's 1/3 Rule

$n = 4 \ (h = 0.2)$

$$I \cong \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{j=2,4,6}^{n-2} f(x_j) + f(x_n)}{3n}}_{\text{Average height}}$$

$f(0) = 0.2 \qquad f(0.2) = 1.288$

$f(0.4) = 2.456 \qquad f(0.6) = 3.464$

$f(0.8) = 0.232$

Integral

$$I = 0.8\,\frac{0.2 + 4(1.288 + 3.464) + 2(2.456) + 0.232}{12} = 1.623467$$

Truncation Error

$$E_t = 1.640533 - 1.623467 = 0.017067 \qquad \varepsilon_t = 1.04\%$$

Simpson's 3/8 Rule

- **A third order Lagrange polynomial can be fit to four points and integrated.**

Simpson's 3/8 Rule

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_3(x)\,dx$$

Here, $f_3(x)$ is represented by a third-order Lagrange polynomial.

$$I \cong \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}}_{\text{Average height}}$$

$$E_t = -\frac{(b-a)^5}{6480} f^{(4)}(\xi)$$

Simpson's 3/8 Rule

Example: Numerically integrate the following polynomial

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $a = 0$ to $b = 0.8$.

(a) Use Simpson's 3/8 rule (b) Use Simpson's 1/3 and Simpson's 3/8 for total 5 segments

(a) Simpson's 3/8 Rule **Requires four equally spaced points**

$$f(0) = 0.2 \qquad f(0.2667) = 1.432724$$
$$f(0.5333) = 3.487177 \qquad f(0.8) = 0.232$$

$$I \cong \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}}_{\text{Average height}}$$

Integral

$$I \cong 0.8 \frac{0.2 + 3(1.432724 + 3.487177) + 0.232}{8} = 1.519170$$

Truncation Error

$$E_t = 1.640533 - 1.519170 = 0.1213630 \qquad \varepsilon_t = 7.4\%$$

Simpson's 3/8 Rule

Example: Numerically integrate the following polynomial

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$ from $a = 0$ to $b = 0.8$.

(a) Use Simpson's 3/8 rule (b) Use Simpson's 1/3 and Simpson's 3/8 for total 5 segments

(b) Simpson's 1/3 and Simpson's 3/8

Requires atleast 6 equally spaced points or 5 segments

$$f(0) = 0.2 \qquad f(0.16) = 1.296919$$
$$f(0.32) = 1.743393 \qquad f(0.48) = 3.186015$$
$$f(0.64) = 3.181929 \qquad f(0.80) = 0.232$$

Integral for the first two segments is obtained using Simpson's 1/3 rule

$$I \cong 0.32 \frac{0.2 + 4(1.296919) + 1.743393}{6} = 0.3803237$$

Integral for the last three segments is obtained using Simpson's 3/8 rule

$$I \cong 0.48 \frac{1.743393 + 3(3.186015 + 3.181929) + 0.232}{8} = 1.264754$$

Total Integral $I = 0.3803237 + 1.264753 = 1.645077$ **Truncation Error** $\varepsilon_t = -0.28\%$

MATLAB© Script

MATLAB Program for Simpson's 1/3 Rule

```
clear all
clc
a= 0; % lower limit
b = 0.8; % upper limit
x=linspace(a,b,3);
y = int_fun(x);
Integral = (b-a)*(y(1) + 4*y(2) + y(3))/6
```

```
clear all
clc
a= 0; % lower limit
b = 0.8; % upper limit
c = 2; % No of segments
x=linspace(a,b,2*c+1); % point generation
n=length(x);
h = (b-a)/(n-1);
for i=1:n
    y(i) = int_fun(x(i));
end
Segment=sum(y(:,1:2:end-2)+4*y(:,2:2:end-
1)+y(:,3:2:end),1)*h/3;
Integral = sum(Segment)
```

MATLAB© Script

MATLAB Program for Simpson's 3/8 Rule

```
function y = int_fun(x)
% Function for integration purpose
y = 0.2 + 25*x - 200*x.^2 + 675*x.^3 - 900*x.^4 + 400*x.^5;
```

```
clear all
clc
a= 0; % lower limit
b = 0.8; % upper limit
x=linspace(a,b,4);
y = int_fun(x);
Integral = (b-a)*(y(1) + 3*y(2) + 3*y(3) + y(4))/8
```

Integration with Unequal Segments

- **Integration scheme, if segement size is not equal.**
- **Apply trapezoidal rule to each segment and sum the results:**

Segment wise Trapezoidal Rule

$$I = h_1 \frac{f(x_0) + f(x_1)}{2} + h_2 \frac{f(x_1) + f(x_2)}{2} + \cdots + h_n \frac{f(x_{n-1}) + f(x_n)}{2}$$

Segment data:

x	f(x)	x	f(x)
0.0	0.200000	0.44	2.842985
0.12	1.309729	0.54	3.507297
0.22	1.305241	0.64	3.181929
0.32	1.743393	0.70	2.363000
0.36	2.074903	0.80	0.232000
0.40	2.456000		

$$I = 0.12 \frac{1.309729 + 0.2}{2} + 0.10 \frac{1.305241 + 1.309729}{2} + \cdots + 0.10 \frac{0.232 + 2.363}{2}$$
$$= 0.090584 + 0.130749 + \cdots + 0.12975 = 1.594801$$

$$\varepsilon_t = 2.8\%$$

Newton-Cotes Integration

Segments (n)	Points	Name	Formula	Truncation Error
1	2	Trapezoidal rule	$(b-a)\dfrac{f(x_0)+f(x_1)}{2}$	$-(1/12)h^3f''(\xi)$
2	3	Simpson's 1/3 rule	$(b-a)\dfrac{f(x_0)+4f(x_1)+f(x_2)}{6}$	$-(1/90)h^5f^{(4)}(\xi)$
3	4	Simpson's 3/8 rule	$(b-a)\dfrac{f(x_0)+3f(x_1)+3f(x_2)+f(x_3)}{8}$	$-(3/80)h^5f^{(4)}(\xi)$
4	5	Boole's rule	$(b-a)\dfrac{7f(x_0)+32f(x_1)+12f(x_2)+32f(x_3)+7f(x_4)}{90}$	$-(8/945)h^7f^{(6)}(\xi)$
5	6		$(b-a)\dfrac{19f(x_0)+75f(x_1)+50f(x_2)+50f(x_3)+75f(x_4)+19f(x_5)}{288}$	$-(275/12{,}096)h^7f^{(6)}(\xi)$

Gauss Quadrature

- Weighted sum of function values at specified points within the domain of integration.

Gauss Quadrature Approximation (two points)

$$I = \int_a^b f(x)\,dx \approx c_1 f(x_1) + c_2 f(x_2)$$

Gauss Quadrature Approximation (higher points)

$$I \cong c_0 f(x_0) + c_1 f(x_1) + \cdots + c_{n-1} f(x_{n-1})$$

Points	Weighting Factors	Function Arguments	Truncatio Error
2	$c_0 = 1.0000000$ $c_1 = 1.0000000$	$x_0 = -0.577350269$ $x_1 = 0.577350269$	$\cong f^{(4)}(\xi)$
3	$c_0 = 0.5555556$ $c_1 = 0.8888889$ $c_2 = 0.5555556$	$x_0 = -0.774596669$ $x_1 = 0.0$ $x_2 = 0.774596669$	$\cong f^{(6)}(\xi)$
4	$c_0 = 0.3478548$ $c_1 = 0.6521452$ $c_2 = 0.6521452$ $c_3 = 0.3478548$	$x_0 = -0.861136312$ $x_1 = -0.339981044$ $x_2 = 0.339981044$ $x_3 = 0.861136312$	$\cong f^{(8)}(\xi)$
5	$c_0 = 0.2369269$ $c_1 = 0.4786287$ $c_2 = 0.5688889$ $c_3 = 0.4786287$ $c_4 = 0.2369269$	$x_0 = -0.906179846$ $x_1 = -0.538469310$ $x_2 = 0.0$ $x_3 = 0.538469310$ $x_4 = 0.906179846$	$\cong f^{(10)}(\xi)$
6	$c_0 = 0.1713245$ $c_1 = 0.3607616$ $c_2 = 0.4679139$ $c_3 = 0.4679139$ $c_4 = 0.3607616$ $c_5 = 0.1713245$	$x_0 = -0.932469514$ $x_1 = -0.661209386$ $x_2 = -0.238619186$ $x_3 = 0.238619186$ $x_4 = 0.661209386$ $x_5 = 0.932469514$	$\cong f^{(12)}(\xi)$

Limit Conversion

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)dx$$

Gauss Quadrature

Example: Numerically integrate the following function: $\int_{0.1}^{1.3} 5xe^{-2x}dx$

Conversion of integration limit $$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x+\frac{b+a}{2}\right)dx$$

$$\int_{0.1}^{1.3} f(x)dx = \frac{1.3-0.1}{2}\int_{-1}^{1} f\left(\frac{1.3-0.1}{2}x+\frac{1.3+0.1}{2}\right)dx \Rightarrow \int_{0.1}^{1.3} f(x)dx = 0.6\int_{-1}^{1} f(0.6x+0.7)dx$$

Three point Gauss Quadrature

$$I = \int_a^b f(x)dx \approx c_1 f(x_1)+c_2 f(x_2)+c_3 f(x_3)$$

Points	Weighting Factors	Function Arguments	Truncatio Error
3	$c_0 = 0.5555556$ $c_1 = 0.8888889$ $c_2 = 0.5555556$	$x_0 = -0.774596669$ $x_1 = 0.0$ $x_2 = 0.774596669$	$\cong f^{(6)}(\xi)$

$$c_1 = 0.5556,\ c_2 = 0.8889,\ c_3 = 0.5556$$
$$x_1 = -0.7746,\ x_2 = 0,\ x_3 = 0.7746$$

Gauss Quadrature

$$\int_{0.1}^{1.3} 5xe^{-2x}dx$$

$$I = \int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

$$c_1 = 0.5556,\ c_2 = 0.8889,\ c_3 = 0.5556$$
$$x_1 = -0.7746,\ x_2 = 0,\ x_3 = 0.7746$$

$$0.6\int_{-1}^{1} f(0.6x+0.7)dx \approx 0.6\left[0.5556\times f(0.6\times -0.7746+0.7) + 0.8889\times f(0.6\times 0+0.7) + 0.5556\times f(0.6\times 0.7746+0.7)\right]$$

$$0.6\int_{-1}^{1} f(0.6x+0.7)dx \approx 0.6\left[0.5556\times f(0.2352) + 0.8889\times f(0.7) + 0.5556\times f(1.165)\right]$$

$$f(0.2352) = 5\times 0.2352\times e^{-2\times 0.2352} = 0.7347$$
$$f(0.7) = 5\times 0.7\times e^{-2\times 0.7} = 0.8630$$
$$f(1.165) = 5\times 1.165\times e^{-2\times 1.165} = 0.5668$$

Gauss Quadrature

$$0.6\int_{-1}^{1} f\left(0.6x+0.7\right)dx \approx 0.6\left[0.5556\times f\left(0.2352\right)+0.8889\times f\left(0.7\right)+0.5556\times f\left(1.165\right)\right]$$

$$0.6\int_{-1}^{1} f\left(0.6x+0.7\right)dx \approx 0.6\left[0.5556\times 0.7347+0.8889\times 0.8630+0.5556\times 0.5668\right]$$

$$I=\int_{0.1}^{1.3} 5xe^{-2x}dx \approx 0.8942$$

Gauss Point (n)	Integration Value (I)	Relative Error (ε)
1	1.036	15.89
2	0.9101	1.89
3	0.8942	0.03
4	0.8939	0

THANK YOU

Questions??