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Oblique incidence of a strong electromagnetic wave on a cold inhomogeneous electron plasma. Relativistic effects

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A method which simplifies the investigation of the form of an obliquely incident wave on a plane parallel layered plasma by reducing it to the case where the wave is normally incident is presented. The resonance at $N_0 = N_c$ (critical density) is discussed. Another resonance at $N_0 = 4N_c$ is described.

I. INTRODUCTION

Our goal is to study the propagation of an obliquely incident wave taking into account relativistic effects.[1-3] We present a method which simplifies this problem by introducing a frame in which the wave is normally incident. To do so, we use a Lorentz transformation. In the present paper we only describe and test our method.

We consider a medium of plane layers, whose density gradient is in the z direction. The electric field $\mathbf{E}$ of the incident wave is assumed to be in the plane of incidence; we consider that the wave vectors $\mathbf{k}_0$ and $\mathbf{E}$ are in the xz plane (Fig. 1). Ions are regarded as a stationary background $(m_e/m_i \ll 1)$. The electron velocity induced by the wave is supposed to be much greater than that due to thermal motion.

We obtain a solution through a perturbation method which is complementary to Ginzburg's.[4] In the part of this theory which concerns the oblique incidence case he uses approximations which imply that for a given value of the density gradient his solution is only valid for sufficiently large angles of incidence. On the contrary our solution is valid for very small values of θ [angle of incidence (Fig. 1)]; when $\theta \to 0$, it tends towards Ginzburg's solution obtained in the normal incidence problem.

An advantage of our method is that it leads to an analytic solution which yields an accurate value of the absorption due to the z component of the electric field along the density gradient when it is at its maximum.

We also show that there is a resonance taking place at four times the critical density which exists even when $\theta = 0$. This resonance appears as we have taken into account nonlinear terms due to relativistic effects.

II. LORENTZ TRANSFORMATION TO A NORMAL INCIDENCE FRAME

Before going further into our problem let us introduce a simplifying Lorentz transformation. To do so we study the trajectory of one photon of our incident wave in a vacuum; this has a meaning as we consider a linear layer of plasma confined in a half-space for which $z > 0$ (Fig. 1). We introduce a new frame (l') which moves uniformly relative to (l)

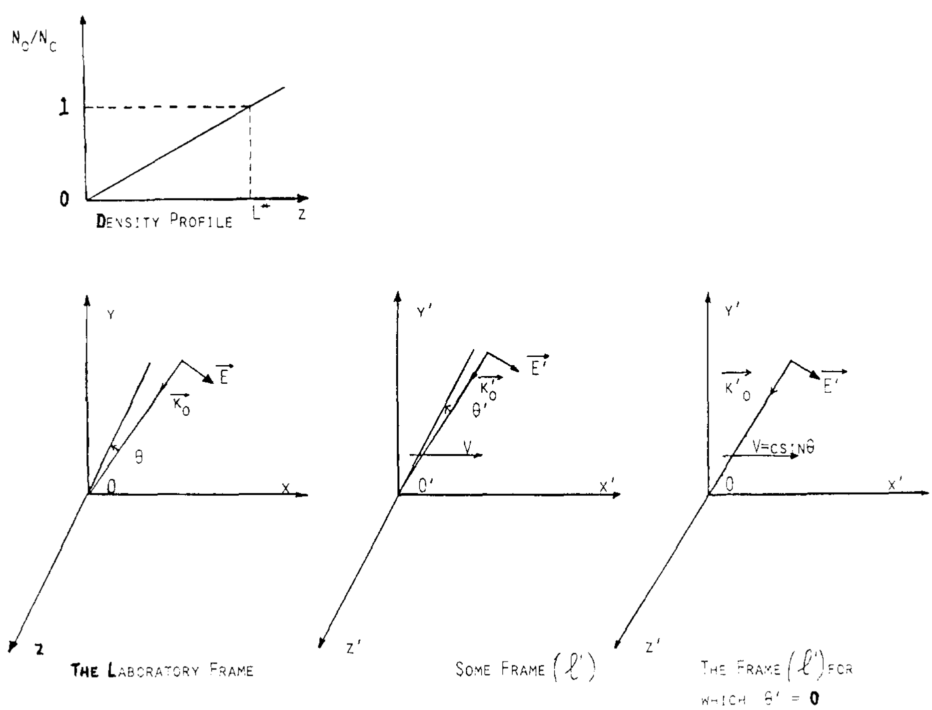

FIG. 1. The frames (l) and (l').

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(laboratory frame) along the x axis with the velocity $\mathbf{V}$ (Fig. 1). The velocity of such a photon in (l) and (l') is, respectively,

$$\mathbf{c} = c(\sin\theta,0,\cos\theta), \tag{1a}$$

$$\mathbf{c}' = c(\sin\theta',0,\cos\theta'). \tag{1b}$$

Using the formulas of transformation for the velocity of a particle,[5] we obtain a set of two equations between θ and θ'. We find that there is a particular frame (l') for which $\theta' = 0$. The velocity of this frame along the x axis is defined by

$$\sin\theta = V/c, \tag{2}$$

when θ is in the $(-\pi/2,\pi/2)$ range. In what follows (l') will be this special frame.

The fact that our wave is normally incident in (l') can be verified by using the law of transformation of the incident wave four-vector[5]

$$k'_{0x} = \frac{k_{0x} - (\omega/c^2)V}{(1 - V^2/c^2)^{1/2}} \tag{3a}$$

$$k'_{0y} = k_{0y}, \tag{3b}$$

$$k'_{0z} = k_{0z}, \tag{3c}$$

$$\omega'/c = \frac{\omega/c - (V/c)k_{0x}}{(1 - V^2/c^2)^{1/2}}. \tag{3d}$$

On the one hand, we know that $k_{0y} = 0$ as the wave vector $\mathbf{k}_0$ is in the plane of incidence, in consequence $k'_{0y} = 0$ [Eq. (3b)]. On the other hand Eq. (3a) shows that $k'_{0x} = 0$ as we consider Eq. (2) is satisfied. So, in (l'), the wave vector $\mathbf{k}'_0$ is parallel to the z' direction, that is to say, parallel to the density gradient

$$\mathbf{k}'_0 = (0,0,k_0\cos\theta). \tag{4}$$

Using the Lorentz transformation formulas for the field and for coordinates,[5] we can show that in (l') the incident electric field has no component along the z' direction and that if, in (l), it does not depend on space and time separately but only on the combination $\omega t - k_{0x}x$, it will depend only on $\omega' t'$ in (l), where

$$\omega' = \omega\cos\theta. \tag{5}$$

Using again the fact that $(\omega/c,\mathbf{k})$ is a four-vector, it can easily be shown that Snell's law is invariant with respect to our Lorentz transformation. As a consequence, in (l') a normally incident wave traveling through our plasma has a wave vector which remains parallel to the density gradient. So, we shall seek a solution for a wave in (l') since it is much simpler to deal with a one-dimensional problem, and we shall use the Lorentz transformation for the field to determine the form of our wave in (l).

III. WAVE EQUATION

First, we have to find the equation in (l'). It can be shown that we can use the following equations:[2,3]

$$e\mathbf{E}' = -\frac{\partial\mathbf{P}'}{\partial t'} - \nabla' c P'_0, \quad P'_0 = (P'^2 + m^2c^2)^{1/2}, \tag{6a}$$

$$e\mathbf{B}' = c\nabla'\wedge\mathbf{P}', \tag{6b}$$

where $\mathbf{P}'$ is the electron momentum. When Eqs. (6) are satisfied, the following two Maxwell equations remain to be verified in order to describe an oscillation:

$$\nabla'\wedge\mathbf{B}' = \frac{1}{c}\frac{\partial\mathbf{E}'}{\partial t'} + \frac{4\pi e}{c}(N'_0\mathbf{V} - n'\mathbf{v}), \tag{7}$$

$$\nabla\cdot\mathbf{E}' = 4\pi e(n' - N'_0), \tag{8}$$

where N'_0 is the ion density, and n' and $\mathbf{v}'$ are, respectively, the density and the velocity of the electrons. The ion current which appears in Eq. (7) is due to the fact that our plasma has a drift velocity in (l').

Using expansions with respect to $\overline{P} = /mc$ ($\overline{P}'$ is the amplitude of $\mathbf{P}'$) and neglecting $\overline{P}'^3/m^3c^3$ terms, we obtain from Eqs. (6)–(8) the following wave equation in $\mathbf{P}'$:

$$\frac{1}{c^2}\frac{\partial^2\mathbf{P}'}{\partial t'^2} + \nabla'(\nabla'\cdot\mathbf{P}') - \Delta'\mathbf{P}' + \frac{\omega_e'^2}{c^2}(\mathbf{P}' - m\mathbf{V})$$
$$= -\frac{1}{2mc^2}\nabla'\frac{\partial}{\partial t'}\mathbf{P}'^2 + \frac{\omega_e'^2}{2c^2}\left(\frac{P'}{mc}\right)^2\mathbf{P}'$$
$$-\frac{\mathbf{P}'}{mc^2}\frac{\partial}{\partial t}(\nabla'\cdot\mathbf{P}') - \frac{\mathbf{P}'}{2m^2c^2}\Delta'\mathbf{P}'^2. \tag{9}$$

The density profile is assumed to be in the form

$$1 - \omega_e'^2/\omega'^2 = 1 - z'/L' + \alpha^{(2)}, \tag{10}$$

where

$$\omega_e'^2 = 4\pi e^2 N'_0/m, \tag{11}$$

and where $\alpha^{(2)}$ is a relativistic term which is chosen further not to have a secular term in our solution.

IV. SMALL ANGLE OF INCIDENCE AND ELECTROMAGNETIC FIELD SOLUTION

We use the Lindstedt–Poincaré method[6] to solve the wave equation considering that in the nonrelativistic approximation (when all the nonlinear terms are neglected[1–3]) we take the exact solution which can be written as an Airy integral. In order to simplify, and as we only wish to test our method, we have neglected the terms in $(\Lambda/mc)(V/c)$ and in Λ^2/m^2c^2 (Λ is the amplitude of the electron momentum due to the wave); on the other hand, the terms in V^2/c^2 are taken into consideration (that is, we do not consider a very strong field). This way we find a solution for $\mathbf{P}'$ from which we calculate the electromagnetic field in (l') using Eqs. (6).

Then we determine the electric field in (l) using the Lorentz transformation formulas for the field and for coordinates. We obtain

$$E_x = -(\omega/e)(1 - \tfrac{1}{2}\sin^2\theta)\Lambda\,\mathrm{Ai}(-\zeta)$$
$$\times\cos(\omega t - k_0\sin\theta x) + E_x^{(2)}, \tag{12a}$$

$$E_y = 0, \tag{12b}$$

$$E_z = (k_0L)^{-1/3}\frac{\omega}{e}\Lambda\left(\frac{1}{\omega_e^2/\omega^2 - 1}\sin\theta\,\mathrm{Ai}'(-\zeta)\right.$$
$$\times\sin(\omega t - k_0\sin\theta x) - \frac{\omega_e^2/\omega^2}{2(4 - \omega_e^2/\omega^2)}\frac{\Lambda}{mc}$$
$$\times\mathrm{Ai}(-\zeta)\mathrm{Ai}'(-\zeta)\cos 2(\omega t - k_0\sin\theta x)$$
$$\left.-\frac{\Lambda}{2mc}\mathrm{Ai}(-\zeta)\mathrm{Ai}'(-\zeta)\right). \tag{12c}$$

when $\zeta\to\infty$, $E_x^{(2)}\to 0$ rapidly, and when $\omega_e^2/\omega^2\sim 1$, we have

$$E_x^{(2)} \simeq -(\omega/e)(k_0L)^{2/3}A \sin^2\theta \times \log|(k_0L)^{2/3}[1-(\omega_e^2/\omega^2)]| \times \mathrm{Ai}'(-\zeta)\cos(\omega t - k_0 \sin\theta x), \quad (13)$$

where Ai is the Airy function, $\mathrm{Ai}' = [d/d(-\zeta)]\mathrm{Ai}$,

$$\zeta = (k_0L)^{2/3}(\cos^2\theta - z/L), \; k_0 = \omega/c,$$

and

$$L = L'/\cos^2\theta.$$

Here A is determined by using the boundary conditions. To do so we consider a point ζ_1 such as $\zeta = \zeta_1$ which is chosen near $z = 0$ and satisfies the condition $\mathrm{Ai}'(-\zeta_1) = 0$. We let

$$\mathbf{E}(\zeta_1) \simeq [2E_0(1-\tfrac{1}{2}\sin^2\theta)\cos(\omega t - k_0 \sin\theta x),0,0], \quad (14)$$

where the constant E_0 is the amplitude of the field at $z = 0+$ near vacuum. This way we obtain

$$A \simeq -(2eE_0/\omega)(\pi)^{1/2}(k_0L)^{1/6}(1-\tfrac{1}{4}\sin^2\theta). \quad (15)$$

Our solution [Eqs. (12)] corresponds to the density profile

$$1 - \omega_e^2/\omega^2 = 1 - z/L^*, \quad (16)$$

with

$$L^* = L(1 + \sin^2\theta), \quad (17)$$

and with

$$\omega_e^2 = 2\pi e^2 N_0/m, \quad (18)$$

where N_0 is the ion density in (l).

V. DISCUSSION

Equation (13) shows that E_x has a logarithmic singularity at the critical density. Equation (12c) contains another singularity which also takes place at the critical density. This singularity already exists in (l'); it is due to the $e(\mathbf{V}/c)\wedge\mathbf{B}$ Lorentz force, that is, to the fact the plasma has a drift velocity $\mathbf{V}$. In (l), this resonance is a consequence of the fact that the incident field has a component along the density gradient. We have checked that in our solution the $\omega-$ frequency term in Eq. (12c) ($E_{z\omega}$) resembles that of Ginzburg[4]; to do so, we have considered a case for which $q = (k_0L^*)^{2/3}\sin^2\theta \sim 1$ (Fig. 2). We know that his approximation only leads to very good results when $q \gtrsim 2$; still we can consider that when $q \sim 1$ his solution is satisfactory. Moreover, our numerical results show that when q becomes greater than unity our results become less and less accurate. Here we must mention that taking more terms into account in our perturbation method (terms in V^3/c^3 for instance) would improve our solution in a neighborhood of the resonance, enlarging its domain of validity.

We can calculate the absorption due to the ω-frequency term ($E_{z\omega}$) in Eq. (12c). To do so, we consider a very small damping and take the numerator of $E_{z\omega}$ as a high-frequency driver.[9] Then we know that the absorption is independent of the damping.[9,10] In those conditions the absorption is $A = \frac{1}{2}\phi^2(q)$, where $\phi(q)$ is the field multiplication factor.[11] In our case, this quantity almost only depends on q in the weak angle of incidence limit. We can compare it to the one obtained by Speziale and Catto[12] through their small-q analytic solution, and to numerical results of Forslund *et al.*[13] Figure 3 shows that our solution leads to an accurate absorption

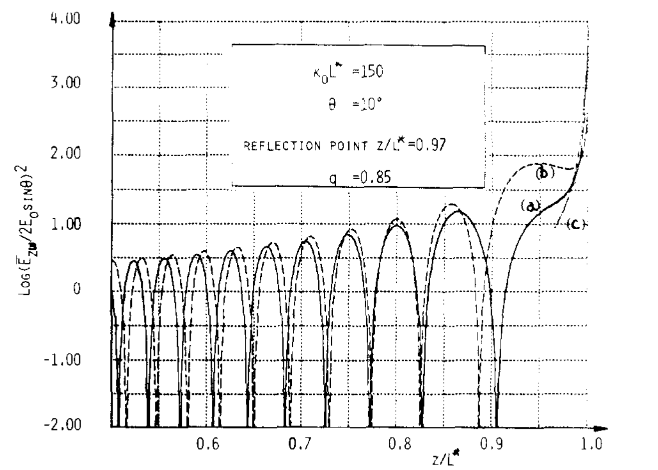

FIG. 2. Comparison between the spatial dependence of the ω-frequency term in E_z [Eq. (12c)] (a) and the two approximate solutions given by Ginzburg (b) and (c). (b) is valid everywhere outside a small neighborhood of the point $z/L^* = 1$. (c) is valid to the right of the reflection point.

when $q \sim 0.5$ (that is to say, when absorption is maximum). In Fig. 4 we have plotted the spatial dependence of E_z vs z/L in a case where $q \sim 0.5$.

A third singularity appears in Eqs. (12), it corresponds to a resonance which takes place at $N_0 = 4N_c$ and which is due to the $e(\mathbf{v}/c)\wedge\mathbf{B}$ Lorentz force ($\mathbf{v}$ is the electron velocity due to the field). This resonance does not depend on θ and therefore exists in the normal incident case. It might influence absorption for very long wavelengths.

The last two resonances are due to what we have called semirelativistic terms.[1-3]

Equation (12c) contains a term which only depends on space: it corresponds to a force maintaining the plasma at rest on an average. This force cancels the effect of the ponderomotive force one could calculate through the oscillating part of the field.

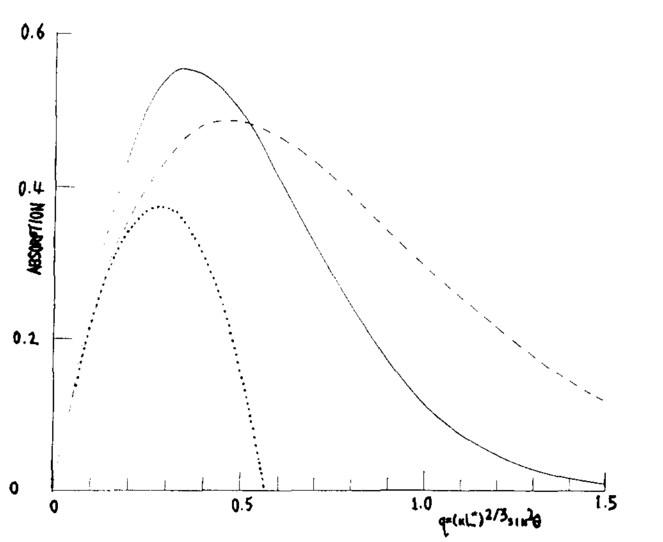

FIG. 3. The absorption coefficient obtained through our theory as a function of q compared to the small-q solution of Ref. 9 (dotted line) and to numerical results of Ref. 10 (dashed line).

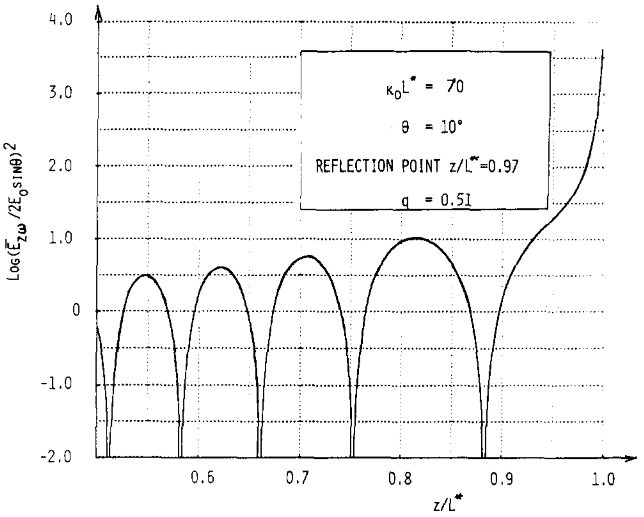

FIG. 4. The evolution of the spatial dependence of the ω-frequency component of E_z vs (z/L^*) when $q = 0.51$.

VI. SUMMARY

In summary we have presented a method which reduces a two-dimensional problem to a one-dimensional one and, as a consequence simplifies its solution. Applying the method to a simple case (we did not consider a very strong field), we have again found the resonance at $N_0 = N_c$ and improved its description. We have also described another resonance taking place at $N_0 = 4\,N_c$, which exists even when the wave is normally incident.

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