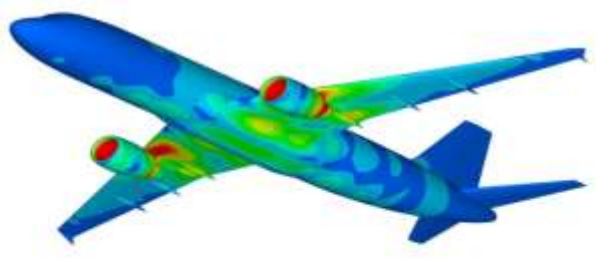
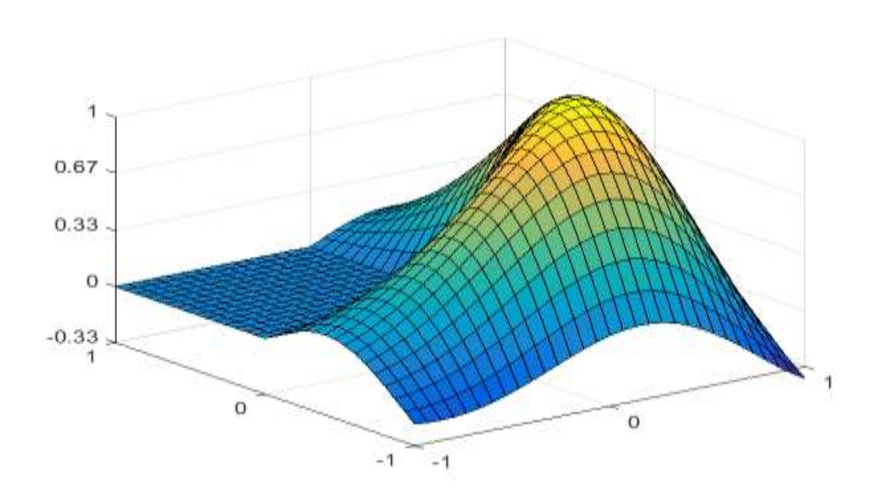
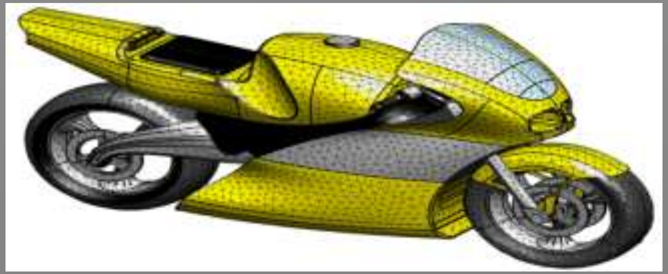
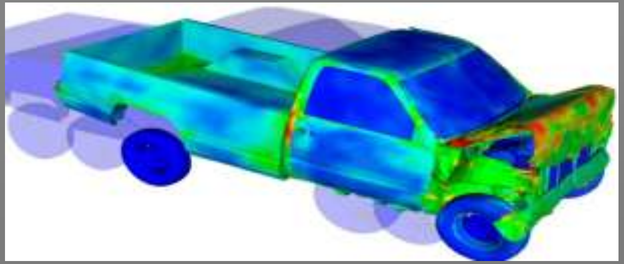
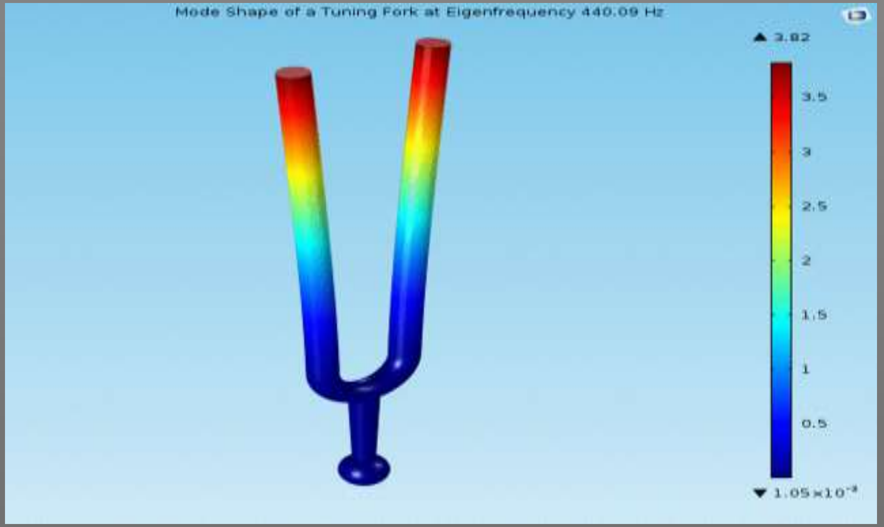
Eigen Value and Eigen Vectors











Dr. Himanshu Pathak

himanshu@iitmandi.ac.in

Introduction

- Eigenvalue problems occur naturally in the vibration analysis of mechanical structures with many degrees of freedom.

$$m\ddot{x} + kx = 0 \quad \Rightarrow \quad -kx = \varpi^2 m\ddot{x} \quad \Leftrightarrow \quad Ax = \lambda x$$

- Eigenvalue of smallest magnitude represents the natural frequency of given dynamic system.

- In a eigenvalue problem [A] {x} = λ {x} ;

  λ is eigenvalue of square matrix [A] and {x} is eigenvector for corresponding eigenvalue λ , provided x ≠ 0.

Finding Eigenvalue

$[A]\{x\} = \lambda\{x\}$ can be written by $A\,x - \lambda\,I\,x = 0$ where $\lambda\,x = \lambda\,I\,x$

- **$(A - \lambda I)x = 0$ will have non-trival solution $x \neq 0$, if and only if $A - \lambda I$ is singular.**
- **$|A - \lambda I| = 0$ --------------** *characteristics equation*
- **Solve the characteristic equation to find eigenvalue (λ).**

1. **Example 1: Eigenvalue for diagonal matrix** $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$A - \lambda I =$ $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

$$|A - \lambda I| = \left| \begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} \right| = 0 \Rightarrow (1-\lambda)(2-\lambda) = 0 \Rightarrow \lambda = 1;\quad 2$$

Finding Eigenvalue Contd..

2. Example : Eigenvalue for triangular matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$\mathbf{A - \lambda\, I} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$|A - \lambda I| = \left| \begin{bmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} \right| = 0 \Rightarrow (1-\lambda)(2-\lambda) = 0 \Rightarrow \lambda = 1; \quad 2$$

3. Example: Eigenvalue for symmetric matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{A - \lambda\, I} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \Rightarrow |A - \lambda I| = \left| \begin{bmatrix} 1-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} \right| = 0$$

$$(1-\lambda)(2-\lambda) - 1 = 0 \Rightarrow \lambda^2 - 3\lambda + 1 = 0 \Rightarrow \lambda = \frac{3}{2} + \frac{1}{2}\sqrt{5}; \quad \frac{3}{2} - \frac{1}{2}\sqrt{5}$$

Finding Eigenvalue Contd..

4. **Example 4: Eigenvalue for non-symmetric matrix** $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

$$|A - \lambda I| = \left| \begin{bmatrix} 1-\lambda & 1 \\ -1 & 2-\lambda \end{bmatrix} \right| = 0 \Rightarrow (1-\lambda)(2-\lambda)+1=0$$

$$\lambda^2 - 3\lambda + 3 = 0 \Rightarrow \lambda = \frac{3}{2} + \frac{\sqrt{3}}{2} i; \quad \frac{3}{2} - \frac{\sqrt{3}}{2} i$$

Conclusions:

- ⇨ *If A is triangular matrix (upper/lower/diagonal), eigenvalue will be diagonal elements.*
- ⇨ *If A is symmetric matrix of real elements, eigenvalue will be always real numbers.*
- ⇨ *If A is non-symmetric matrix, eigenvalue will be either real or complex conjugates.*

Finding Eigenvectors

Example 1: Eigenvector for triangular matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

[A]{x}= λ{x}

First eigenvalue $\lambda_1 = 1$ *Put this value in eigenvalue problem equation:*

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \Rightarrow \begin{aligned} x_1 + x_2 &= x_1 \\ 2x_2 &= x_2 \end{aligned} \Rightarrow \begin{aligned} x_2 &= 0 \\ x_1 &= \alpha \end{aligned}$$

Second eigenvalue $\lambda_2 = 2$ *Put this value in eigenvalue problem equation:*

$$\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 2 \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \Rightarrow \begin{aligned} x_1 + x_2 &= 2x_1 \\ 2x_2 &= 2x_2 \end{aligned} \Rightarrow x_1 = x_2 = \alpha; \quad \alpha \neq 0$$

Characteristics of Eigenvalue/Eigenvector

- **Eigenvector associated with distinct eigenvalue of symmetric matrix are always orthogonal.**
- **If system exhibits same repeated eigenvalue, will always have one set of eigenvector.**
- **If $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigenvalue of matrix A:**
  - *A^T will have same eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$.*
  - *A^{-1} will have eigenvalues $\lambda_1^{-1}, \lambda_2^{-1}, \ldots \lambda_n^{-1}$.*
  - *$A - \alpha I$ will have eigenvalues $\lambda_1 - \alpha, \lambda_2 - \alpha, \ldots \lambda_n - \alpha$.*
  - *A^k will have eigenvalues $\lambda_1^k, \lambda_2^k, \ldots \lambda_n^k$.*

Characteristics of Eigenvalue/Eigenvector

- **For any square matrix A, sum of eigenvalues will be equal to the sum of diagonal elements of A.**

$$\sum_{i=1}^{n} \lambda_n = tr(A)$$

- **For any square matrix A, the product of eigenvalues will be equal to the determinant of A.**

$$\prod_{i=1}^{n} \lambda_n = \det|A|$$

MATLAB© Script

Simplest MATLAB Program to find eigenvalue

```
clear all
clc
A = input('Enter system matrix =  ');% Define coefficient/system matrix
[n, m] = size(A);                    % Check the size of coefficient matrix
I = eye(n);                          % Define Identity matrix
syms y                               %  Define y (eigenvalue) as variable
B = A - y*I;                         % Expression for Characteristic Equation
Ch = det(B)                          % Characteristic equation
Poly = input('enter coefficient of characteristic polynomial = ');
V = roots(Poly)               % Find roots of polynomial as eigenvalue
```

Compare with inbuilt function

```
W = eig(A)                    % Find eigenvalue from inbuilt function
[X, V1] = eig(A);             % Return X as eigenvector and V1 eigenvalue by inbuilt function
```

Finding Eigenvalue (Power Method)

Iterartive method to find largest Eigen value and Eigen vector $[A]\{x\} = \lambda\{x\}$ by initial guess of Eigen vector $x^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- **Repeat $[A]\{x\} = \lambda\{x\}$; till $\lambda^{(n)} \approx \lambda^{(n-1)}$ and $x^{(n)} \approx x^{(n-1)}$.**

Example: Largest Eigenvalue for diagonal matrix $A = \begin{bmatrix} 5 & 7 \\ 3 & 2 \end{bmatrix}$

$[A]\{x\}$ $\begin{bmatrix} 5 & 7 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \end{bmatrix} = 12\begin{bmatrix} 1 \\ 0.416 \end{bmatrix}$ $\Rightarrow$ $\lambda^0 = 12; \quad x^0 = \begin{bmatrix} 1 \\ 0.416 \end{bmatrix}$

$[A]\{x^0\} =$ $\begin{bmatrix} 5 & 7 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.416 \end{bmatrix} = \begin{bmatrix} 7.912 \\ 3.832 \end{bmatrix} = 7.912\begin{bmatrix} 1 \\ 0.484 \end{bmatrix}$ $\Rightarrow$ $\lambda^1 = 7.912; \quad x^1 = \begin{bmatrix} 1 \\ 0.484 \end{bmatrix}$

$[A]\{x^1\} =$ $\begin{bmatrix} 5 & 7 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.484 \end{bmatrix} = \begin{bmatrix} 8.388 \\ 3.968 \end{bmatrix} = 8.388\begin{bmatrix} 1 \\ 0.473 \end{bmatrix}$ $\Rightarrow$ $\lambda^2 = 8.388; \quad x^2 = \begin{bmatrix} 1 \\ 0.473 \end{bmatrix}$

Finding Eigenvalue (Power Method)

$$[A]\{x^2\} = \begin{bmatrix} 5 & 7 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.473 \end{bmatrix} = \begin{bmatrix} 8.311 \\ 3.946 \end{bmatrix} = 8.311\begin{bmatrix} 1 \\ 0.474 \end{bmatrix} \Rightarrow \lambda^3 = 8.311; \quad x^3 = \begin{bmatrix} 1 \\ 0.474 \end{bmatrix}$$

$$[A]\{x^3\} = \begin{bmatrix} 5 & 7 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.474 \end{bmatrix} = \begin{bmatrix} 8.318 \\ 3.948 \end{bmatrix} = 8.318\begin{bmatrix} 1 \\ 0.474 \end{bmatrix} \Rightarrow \lambda^4 = 8.318; \quad x^4 = \begin{bmatrix} 1 \\ 0.474 \end{bmatrix}$$

$$[A]\{x^4\} = \begin{bmatrix} 5 & 7 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.474 \end{bmatrix} = \begin{bmatrix} 8.318 \\ 3.948 \end{bmatrix} = 8.318\begin{bmatrix} 1 \\ 0.474 \end{bmatrix} \Rightarrow \lambda^5 = 8.318; \quad x^5 = \begin{bmatrix} 1 \\ 0.474 \end{bmatrix}$$

$$\lambda = 8.318; \quad x = \begin{bmatrix} 1 \\ 0.474 \end{bmatrix}$$

MATLAB© Script

MATLAB Program to find eigenvalue by Power Method

```
clear all
clc
A = [5 7;3 2];
[n, m] = size(A);
xold =ones(n,1);
count =1;
tol =1e-3;
lamold = max(xold);
lam= 5;
maxitr = 1000;
while count < maxitr
x1 = A*xold;
lam = max(x1);
x = x1./lam;
if abs(lam-lamold)<tol
        disp('Eigen value and Eigen vector calculated')
        lam
        x
    count
    return
else
        lamold = lam;
        xold = x;
        count = count +1;
end
end
```

Finding Eigenvalue (Inverse Power Method)

Iterartive method to find smallest Eigen value and Eigen vector

$[A]\{x\}= \lambda\{x\}$; Multiply by $[A]^{-1}$

$\{x\}= \lambda[A]^{-1}\{x\}$;

$(1/ \lambda)\{x\}= [A]^{-1}\{x\}$

$[A]^{-1}\{x\}=(1/ \lambda)\{x\}$

$[B]\{x\}=(\beta)\{x\}$…………..Eq 1

here $[B] =[A]^{-1}$ and $(\beta)=(1/ \lambda)$

- **Apply Power method algorithm over equation (1)**
- **Repeat $[B]\{x\}= \beta\{x\}$; till $\beta^{(n)}\approx \beta^{(n-1)}$ and $x^{(n)}\approx x^{(n-1)}$.**

THANK YOU

Questions??