Mathematics Presentation

Euler's Equation

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Outline

Euler's Formula

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$$e^{ix} = \cos x + i \sin x \tag{1}$$

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Here, We'll use basic arithmatic and algebra to prove it.

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4. Using this assumption and the rule of complex multiplication, we'll show that we do get Euler's equation 1.

Step 1

First, we have to show that for a small number x, we have:

$$10^x = 1 + 2.3026x \tag{2}$$

To get this what we could do is that we'll start with 10 and keep taking the square root of it by usual method. Doing this a number of times will give us the following table.

Step 1 : Cont.

Table: Table 1

10^x	$\frac{10^x - 1}{x}$	x
10.0	9.0	1
3.16227766	4.32455532	1/2
1.77827941	3.11311764	1/4
1.33352143	2.66817145	1/8
1.15478198	2.47651175	1/16
1.07460782	2.38745050	1/32
1.03663292	2.34450742	1/64
1.01815172	2.32342037	1/128
1.00903504	2.31297147	1/256
1.00450736	2.30777049	1/512
1.00225114	2.30517585	1/1024
1.00112494	2.30387998	1/2048
1.00056231	2.30323241	1/4096
1.00028111	2.30290872	1/8192
1.00014054	2.30274690	1/16384
1.00007027	2.30266599	1/32768
1.00003513	2.30262554	1/65536
1.00001756	2.30260531	1/131072
1.00000878	2.30259520	1/262144
1.00000439	2.30259014	1/524288

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As we can see, as x gets smaller, the value of $\frac{10^x-1}{x}$ approaches a constant value of 2.3026. Which suggests that for a small x, we'll have:

$$\begin{aligned} \frac{10^x - 1}{x} &= 2.3026 \\ 10^x &= 1 + 2.3026x \end{aligned} \tag{3}$$

Step 2

Here we'll change the base 10 to e. We know that:

$$\begin{aligned} a^b &= e^{\ln a^b} \\ \text{So, } 10^x &= e^{\ln 10^x} \\ &= e^{x \ln(10)} \\ &= e^{2.3026x} = 1 + 2.3026x \end{aligned} \tag{4}$$

Substituting $2.3026x$ to y gives:

$$e^y = 1 + y \tag{5}$$

Step 3

To proceed further and calculate the complex powers of e we'll make an assumption that for small y the equation 5 is valid even for complex number y, say, ix. That is:

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We'll also need a rule to multiply two complex numbers. The rule is straight forward. Let $a + ib$ and $c + id$ be two complex number. Then there multiplication is:

$$\begin{aligned}(a + ib) \times (c + id) &= ac + a \times id + ib \times c + ib \times id \\ &= ac + i(ad + bc) + i^2 bd\end{aligned}$$

Using $i^2 = -1$ we get:

$$(a + ib) \times (c + id) = (ac - bd) + i(ad + bc) \tag{7}$$

Step 4

Choosing sufficiently small x, say $\frac{1}{2^{15}} = 0.00003052$ and the equation 6 We can write:

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Now, we'll repeat the step 1 in exact opposite manner. That is, we'll multiply the complex number defined in equation 8 and keep multiplying it to get another complex numbers with greater value of x and continue the process till we get x=1. For example, multiplying the number 8 with itself gives a number equivalent to $e^{\frac{i}{2^{14}}}$.

If we keep multiplying, we'll get the following table.

Step 4: Cont.

Table: Table 2

x	Real Part	Imaginary Part
1/32768	1.0	0.00003051
1/16384	0.99999999	0.00006103
1/8192	0.99999999	0.00012207
1/4096	0.99999997	0.00024414
1/2048	0.99999988	0.00048828
1/1024	0.99999953	0.00097656
1/512	0.99999812	0.00195312
1/256	0.99999243	0.00390624
1/128	0.99996960	0.00781242
1/64	0.99987817	0.01562436
1/32	0.99951223	0.03124492
1/16	0.99804846	0.06245937
1/8	0.99219955	0.12467497
1/4	0.96891611	0.24740490
1/2	0.87758925	0.47942919
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The table to the left shows the real and imaginary part of e^{ix} for the given x. First thing that we see from here is that the real part is decreasing and the imaginary part is increasing. We can keep multiplying and we'll see that the real and imaginary part are oscillating from -1 to 1.

Step 4: Cont.

To see this more clearly, we'll use $e^{\frac{i}{8}}$ as the base number and calculate $e^{\frac{2i}{8}}$, $e^{\frac{3i}{8}}$ and so on. Note that any number in the form of $e^{\frac{in}{8}}$, where n is an integer can easily be calculated using the table 2.

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x	Real Part	Imaginary Part
0	1.0	0.0
1/8	0.99219955	0.12467497
1/4	0.96891611	0.24740490
3/8	0.93051294	0.36627462
1/2	0.87758925	0.47942919
5/8	0.81097085	0.58510285
3/4	0.73169724	0.68164656
7/8	0.64100541	0.76755374
1	0.54031055	0.84148382
9/8	0.43118391	0.90228308
5/4	0.31532837	0.94900271
11/8	0.19455179	0.98091363
3/2	0.07073882	0.99751781
13/8	-0.0541784	0.99855609
7/4	-0.1782508	0.98401222
15/8	-0.2995420	0.95411307
2	-0.4161595	0.90932517

Step 4: Cont.

So it turns out that the the complex number e^{ix} is a periodic function. If we continue for even larger value of x, we find that for increasing x, the real part of e^{ix} is decreasing, reaching upto -1 and then start to increase again reaching to value of 1. The exact opposite happens for the virtual part of e^{ix} which is increasing and then start to decrease. So, we may write e^{ix} as:

$$e^{ix} = f(x) + g(x)i \tag{9}$$

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We already know some functions which behave the same way the real and imaginary part of e^{ix} do. These functions are $\cos x$ and $\sin x$ respectively. The only question we need to address is this: Are $f(x)$ and $g(x)$ these familiar functions or some other functions entirely? We can show that these functions are indeed the same.

Step 4: Cont.

This equivalence can be shown by calculating the time period of the functions $f(x)$ and $g(x)$. However, we can also show it by plotting $f(x)$ and $\cos x$ on the same graph and $g(x)$ and $\sin x$ on the same graph. The figure 1 is one of them. We see that $f(x)$ is indeed equivalent to $\sin x$.

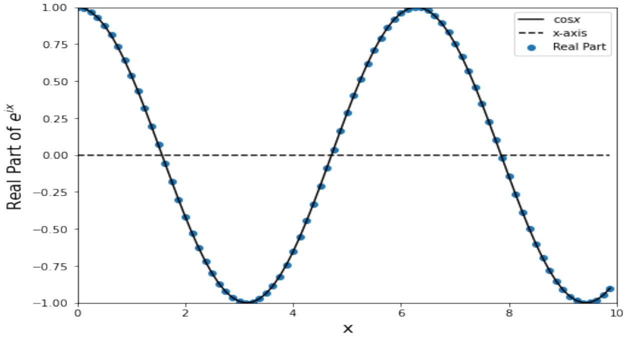

Figure: Real Part $f(x)$ of e^{ix} together with $\cos(x)$

The same can be done for $g(x)$ and $\sin x$.

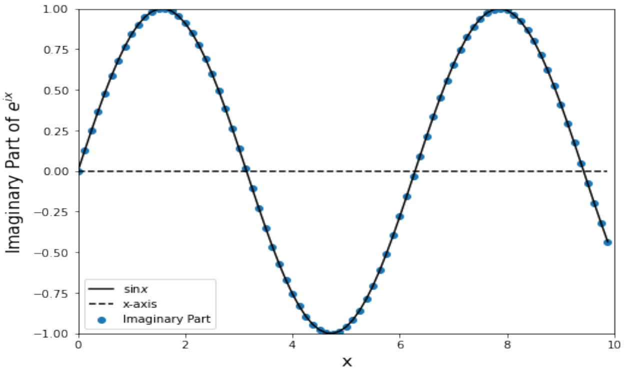

Figure: Imaginary Part $g(x)$ of e^{ix} together with $\sin(x)$

Step 4: Cont.

The same can be done for $g(x)$ and $\sin x$.

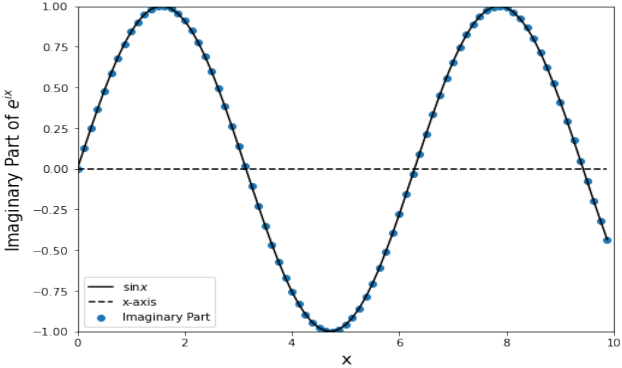

Figure: Imaginary Part $g(x)$ of e^{ix} together with $\sin(x)$

We can see that $g(x)$ is equivalent to $\sin x$.

Step 4: Cont.

Finally I'm plotting the real and imaginary part of e^{ix} together with $\cos x$ and $\sin x$ on the same axis. The plot shows a perfect fit.

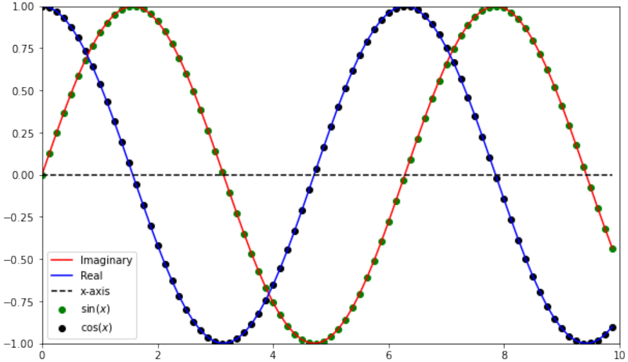

Figure: Real and Imaginary Part of e^{ix} together with $\sin(x)$ and $\cos(x)$

Result

By using simple arithmatic and algebra, we generated the tables and plotted their data. We find that even these elementary method gave us one of the most important relation in complex numbers, namely the Euler's relation:

$$e^{ix} = \cos x + i \sin x$$

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5. Complex Logarithm and General Complex Exponential

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4. Proving De Moivre's Theorem
5. Complex Logarithm and General Complex Exponential
6. Solving Some Physics Problem more Efficiently

It will be a kind of blasphamy to discuss Euler's equation but leave out Euler's identity which is given by:

$$\mathbf{e}^{\mathbf{i}\pi} + \mathbf{1} = \mathbf{0} \tag{10}$$

The equation 10 above is considered as **the most beautiful equation** in Mathematics for obvious reasons. The identity can be proved putting $x = \pi$ into Euler's equation 1.

References and Links

- References
  1. Feynman Lectures on Physics Volume I Chapter 22
  2. Math Vault
  3. Euler's formula: Wikipedia

- Source Codes
  1. Github Folder for The Source Codes
  2. Jupyter Notebook In HTML with all the Tables and Plots