Sum of Gaussian Variables

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1 Proof

A variable is said to be Gaussian if its probability distribution function (PDF) is of the general form

$$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}(1)$$

To show that the sum of two Gaussian variables is Gaussian, it is sufficient to show that the convolution of the two PDFs is also of the above given form.

Let the two Gaussian variables be X and Y. Their PDFs are given by

$$F_X(x) = \frac{1}{\sigma_1\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu_1}{\sigma_1})^2}(2)$$

$$F_Y(x) = \frac{1}{\sigma_2\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu_2}{\sigma_2})^2}(3)$$

respectively.

If $Z = X + Y$, then the PDF of Z would be the convolution of $F_X(x)$ and $F_Y(x)$. That is,

$$F_Z(z) = \int_{-\infty}^{\infty} F_X(x)F_Y(z-x)dx \tag{4}$$

Substituting (2) and (3) in (4), and solving gives

$$F_z(z) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}(5)$$

where, $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$; $\mu = \mu_1 + \mu_2$

It is therefore theoretically proven that the sum of Gaussian variables is also Gaussian.