IIT MADRAS
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Monte-Carlo Simulation of Complex Systems

Mathematics Club

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Fundamental Idea

Monte-Carlo Simulations are a class of algorithms which use **repeated random sampling** to obtain numerical solutions to complicated problems.

Monte-Carlo algorithms base their working on the essence of a fundamental interpretation of probability. Let us consider the toss of an *unbiased coin*. Obtaining two heads in two tosses does not make the coin biased. **Probability theory does not predict the result of a small amount of experiments.** If we were to toss a coin thousands of times, it is more natural to expect 500 heads.

As the number of experiments we perform tends to infinity, we **expect an equal number** of heads and tails. Monte-Carlo Simulations are an application of this concept. If we wanted to calculate the probability of obtaining a head on a coin toss, we would simulate these experiments multiple times (balancing out computational complexity and accuracy of the results) the number of heads would be roughly *half* of the number of experiments. This answer would improve by increasing the number of experiments conducted.

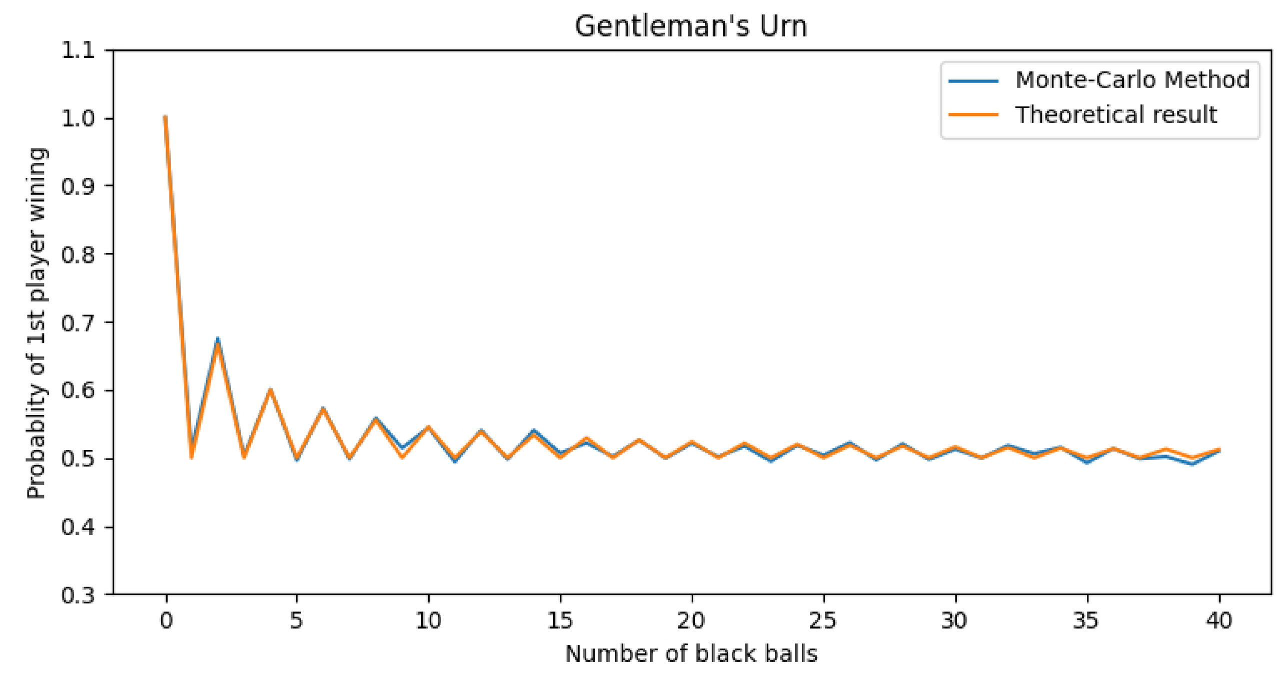
Monte-Carlo methods in general require the ingenuity of the programmer as depending on the experiment conducted, the *computation time* can increase by a lot. Special techniques and theoretical tools might be needed in order to make the experiment tractable when performed several times. Apart from this, in general Monte-Carlo methods require a lot of *computing power* and *optimisation methods* such as parallel processing, might be helpful in order to make the most out of the available *memory constraints* in the system.

Monte-Carlo method is one of the most basic tools available in order to calculate probability and in some cases, it might be the only method *feasible* for the programmer in order to implement an algorithm for estimation. In complicated events, it is even sometimes simpler to model *highly interconnected events* with constraints for *practical applications*, using Monte-Carlo as explicit calculation might be theoretically very time consuming,

Monte-Carlo simulations are simple yet powerful tools in tackling otherwise complicated problems.

Drawing Balls from an Urn

An urn contains k black balls and 1 red ball. A and B draw balls without replacement from this urn, alternating until the red ball is drawn. The game is won by the player who happens to draw the single red ball. Should B start or not?

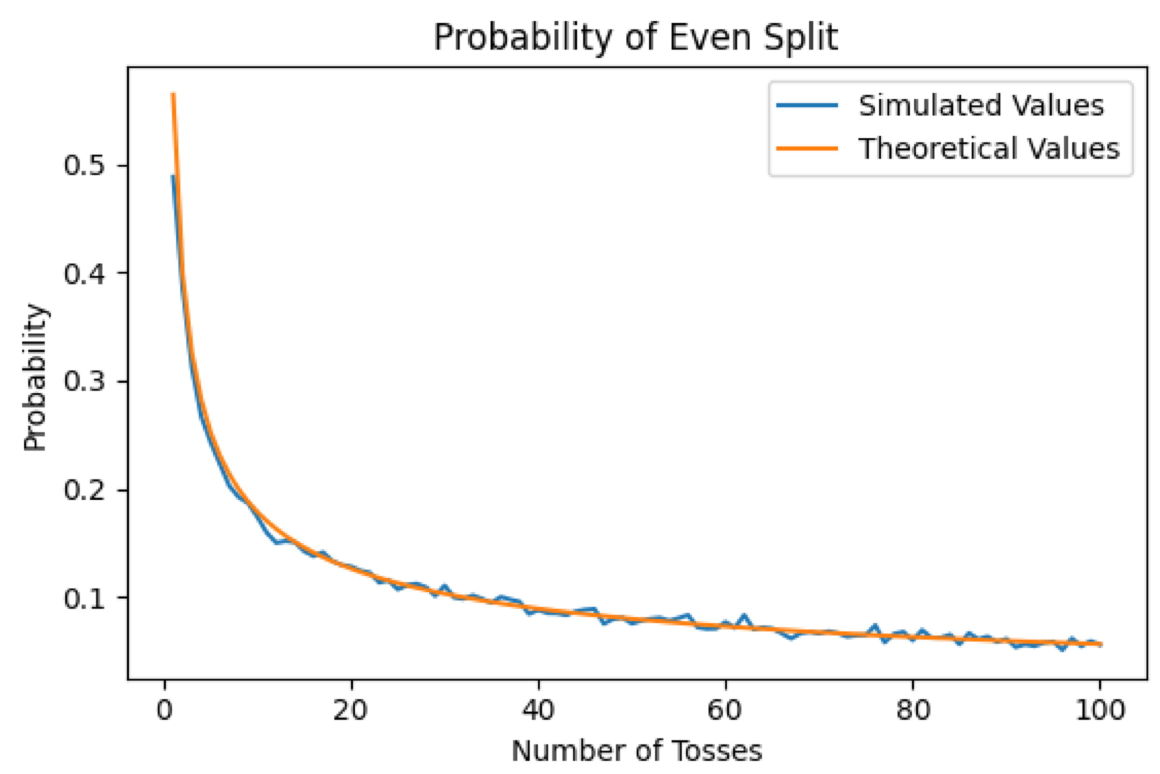


By theoretical calculation, Probability of winning P_{win} was found to be: $P_{\text{win}} = \begin{cases} \frac{1}{2} & \text{if } k \text{ is odd} \\ \frac{k+2}{2(k+1)} & \text{if } k \text{ is even} \end{cases}$

This matches our Monte-Carlo simulation values as expected.

Intuition fails in Probability?

Consider that $2n$ fair and independent coins are thrown at a time. What is the probability of an even $n : n$ split for head and tail?

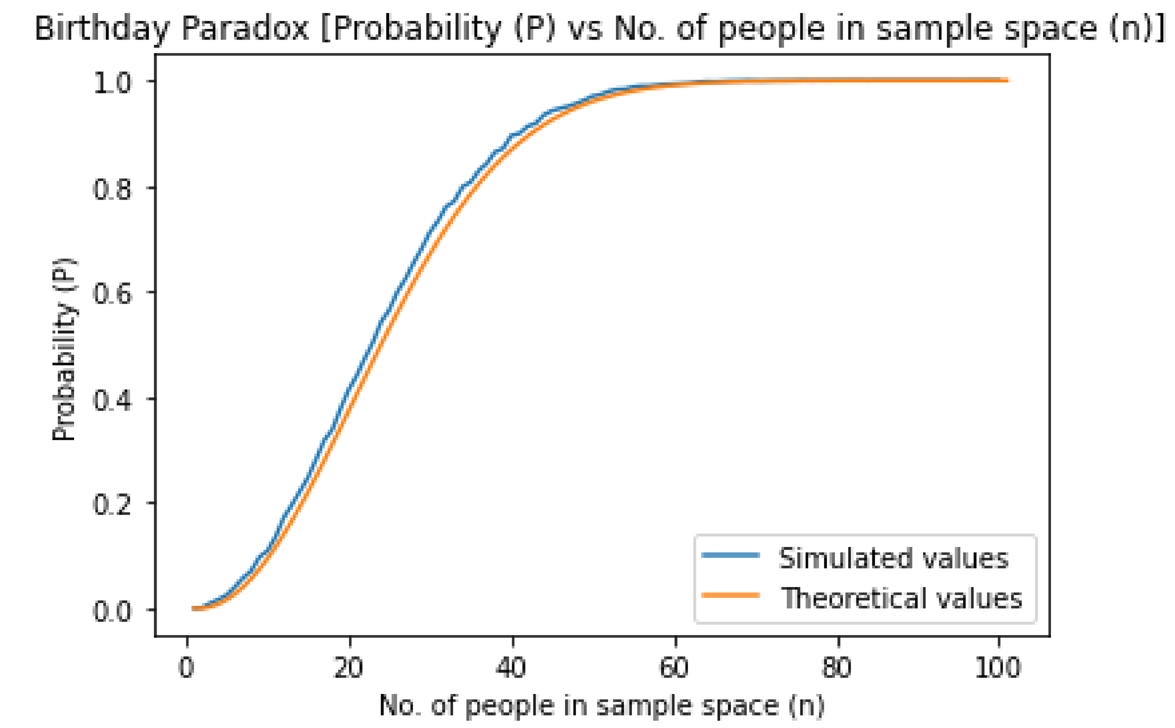


$P(n) = {}^{2n}C_n \left(\frac{1}{2}\right)^{2n} \approx \frac{1}{\sqrt{\pi n}}$, so as we toss more coins, probability of an even split decreases, which is very non-intuitive.

Birthday Paradox

What is the probability that in a set of n randomly chosen people, at least two will share a birthday?

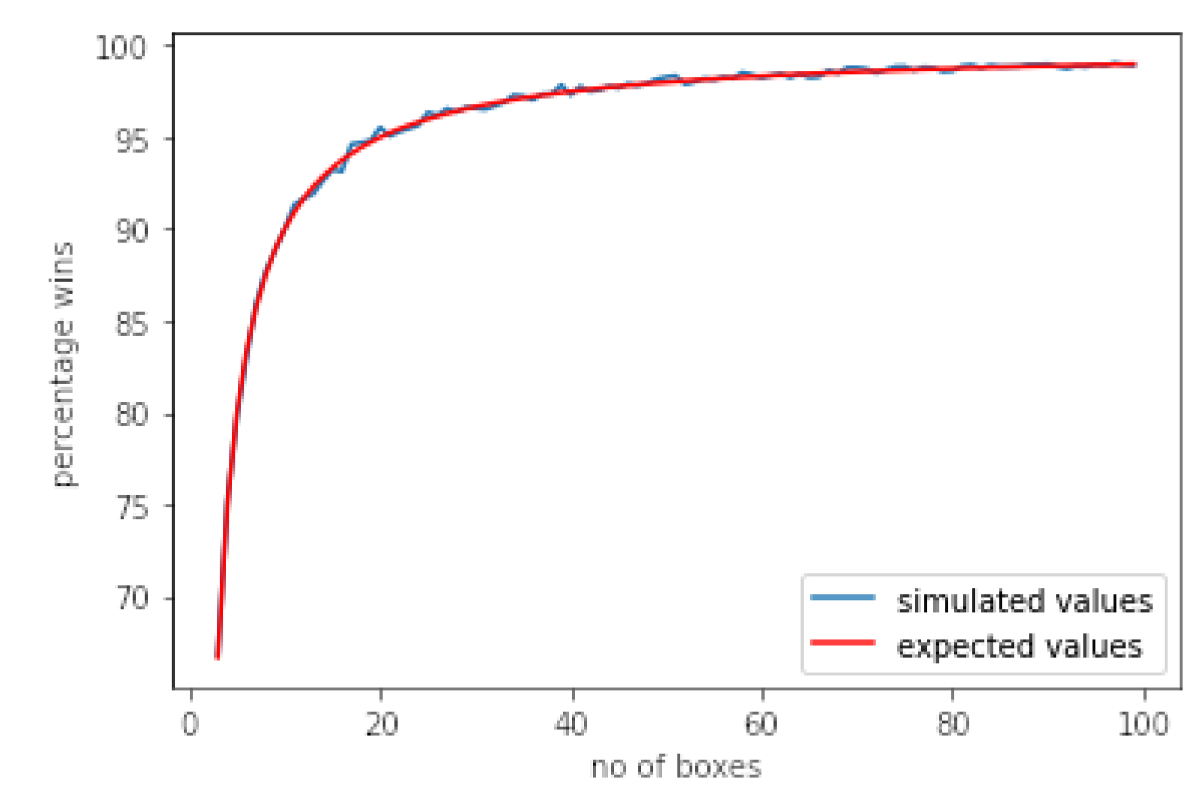
Interestingly, the probability that two people share a birthday exceeds 50% for 23 people itself!



The theoretical value of $p(n), n \leq 365$ is $p(n) = 1 - \frac{^{365}P_n}{365^n}$, while for $n > 365$, it is 1, by the pigeonhole principle.

Monty Hall Problem

On a game show, you're given the choice of n doors: behind one door is a car; behind the others are goats. You pick a door, and the host opens another door which has a goat. Is it to your advantage to switch your choice now?



It turns out that it is always better to switch, because probability of winning by switching is $\frac{n-1}{n}$, while probability of winning by not switching is $\frac{1}{n}$.

Analysis of the Cauchy Density Function

The Cauchy Density function has a probability distribution function $f(x, x_0, \gamma) = \frac{\gamma}{\pi(x-x_0)^2 + \pi\gamma^2}$. For Cauchy distribution the CDF is,

$$F(x) = \int_{-\infty}^{x} \frac{\gamma}{\pi(x'-x_0)^2 + \pi\gamma^2} dx' = \frac{1}{2} + \frac{1}{\pi} tan^{-1}\left(\frac{x - x_0}{\gamma}\right)$$

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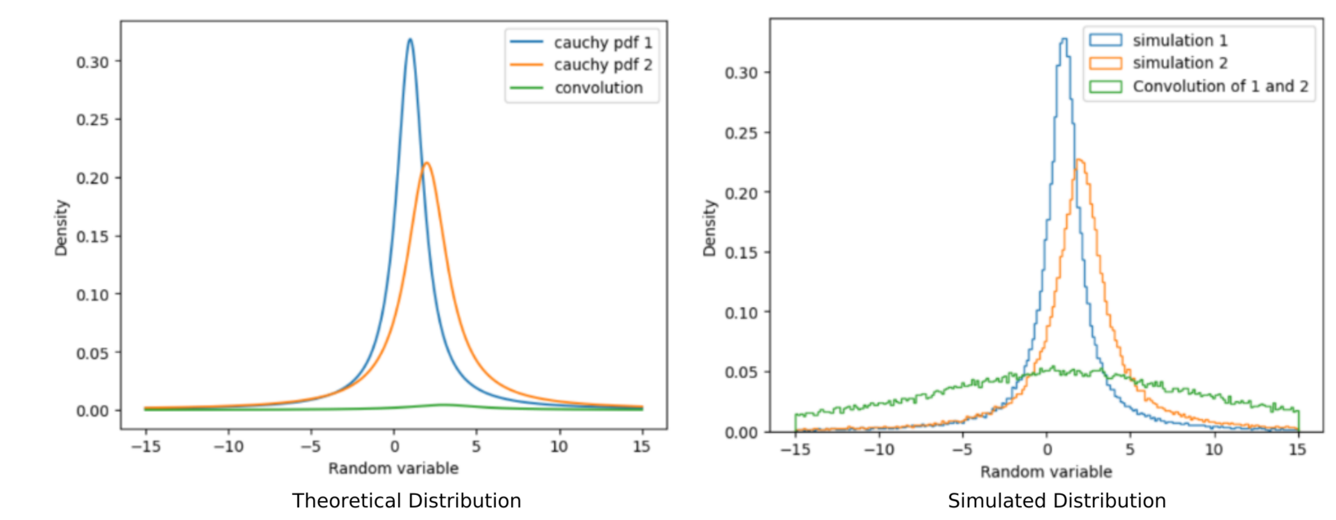

The above image shows two Cauchy PDFs, with $x_0 = 2, \gamma = 1$, and $x_0 = 2, \gamma = 1.5$.

The simulation demonstrates that the sum of two Cauchy random variables is also a Cauchy random variable.

Expected Number of Dice Throws

A and B play a simple game of dice, as follows. A keeps throwing the unbiased die until he obtains the sequence 1 − 1 in two successive throws. B throws the die until she obtains the sequence 1 − 2 in two successive throws. On average, how many times do they have to throw the dice?

Person A:

Let α be the expected *additional* waiting time if A has just tossed any number other than 1 on his previous turn.

Let β be the expected *additional* waiting time if A has just obtained a 1 on his previous turn.

Solving the equations, $\alpha = 1 + \frac{5}{6} \cdot \alpha + \frac{1}{6} \cdot \beta$ and $\beta = 1 + \frac{5}{6} \cdot \alpha$. We get an expected waiting time of 42.

We got the simulated value to be 42.026605.

Person B:

A similar calculation yields a waiting time of 36.

We got the simulated value to be 36.001937.

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