1 Problem Statement

Person A and Person B play a simple game of dice, as follows: Person A keeps throwing the (unbiased) die until he obtains the sequence 1,1 in two successive throws. For Person B, the rules are similar, but she throws the die until she obtains the sequence 1,2 in two successive throws. Derive the expected waiting times for Person A and Person B.

2 Solution

Expected waiting time is defined as:

$$E(X) = \sum_{i=1}^{n} x_i \cdot p_i \tag{1}$$

where,

- *E(X)* is the expected or the mean value of the random variable.
- n is the number of iterations.
- x_i represents the value of the random variable, and p_i represents it's corresponding probability.

2.1 Expected Waiting Time for A

Let:

- α be the expected *additional* waiting time if A has just tossed any number other than 1 on his previous turn(or, equivalently, if it is his first turn).
- β be the expected *additional* waiting time if A has just obtained a 1 on his previous turn.

If A has just obtained any number other than 1, then there is a $\frac{5}{6}$ probability that once again A obtains any number other than 1 on the next turn, and there is a $\frac{1}{6}$ probability that A obtains 1 on the next turn. Hence,

$$\alpha = 1 + \frac{5}{6} \cdot \alpha + \frac{1}{6} \cdot \beta \tag{2}$$

If A has just obtained 1, then there is a $\frac{1}{6}$ probability that the A ends the game on the next turn by obtaining a 1, and there is a $\frac{5}{6}$ probability that A obtains any number other than 1 on the next turn. Hence,

$$\beta = 1 + \frac{5}{6} \cdot \alpha \tag{3}$$

Solving the above 2 equations, we get α=42. This is in fact equal to *E(X)*, since the first throw of dice is equivalent to throwing the dice without having obtained 1 on the previous turn.

2.2 Expected Waiting Time for B

Let:

- α be the expected *additional* waiting time if B has just obtained any number other than 1 on her previous turn(or, equivalently, if it is her first turn).
- β be the expected *additional* waiting time if B has just obtained a 1 on her previous turn.

If B has just obtained any number other than 1, then with $\frac{5}{6}$ probability that B once again obtains any number other than 1, and a $\frac{1}{6}$ probability that B obtains 1 on the next turn. Hence,

$$\alpha = 1 + \frac{5}{6} \cdot \alpha + \frac{1}{6} \cdot \beta \tag{4}$$

If A has just obtained 1, then there is a $\frac{4}{6}$ probability that the B obtains any number other than 2 and 1 on the next turn, and there is a $\frac{1}{6}$ probability that B obtains 2 on the next turn, and there is a $\frac{1}{6}$ probability that B obtains another 1 on the next turn. Hence,

$$\beta = 1 + \frac{4}{6} \cdot \alpha + \frac{1}{6} \cdot \beta \tag{5}$$

Solving the above 2 equations, we get α=36. This is in fact equal to *E(X)*, since the first throw of dice is equivalent to throwing the dice without having obtained 1 on the previous turn.

3 Conclusion

The expected waiting time for A(42) is more than that of B(36).