Birthday Paradox

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1 Introduction

The Birthday Paradox is a mathematical problem which is based on finding the probability of 2 people having the same birthday from a set of n people. It claims that from a set of 23 random people, there's a 50% chance that 2 people will have the same birthday. Here we assume a non-leap year.

2 Theoretical Calculations

Consider a set of n people. The probability of 2 people from the set of n people not having their birthdays overlap is = $\overline{p}$(n).

Total no. of available days = 365 no. of different days we need = n

and each person can have any of the 365 birthdays.
$\therefore$ total no. of birthdays = 365^n

$\implies \overline{p}(\text{n}) = \frac{365!}{365^n(365-n)!}$

$\therefore$ the probability of 2 people have the same birthday among a group of n people:

$p(n)$ = 1 - $\overline{p}$(n)

hence the required probability = 1 - $\frac{365!}{365^n(365-n)!}$

2.1 Approximation for $\overline{p}$(n):

$$\overline{p}(n) = \left(\frac{365}{365}\right)\left(\frac{364}{365}\right).....\left(\frac{365-n+1}{365}\right)$$

$$= (1)\left(1 - \frac{1}{365}\right)\left(1 - \frac{2}{365}\right).....\left(1 - \frac{n-1}{365}\right)$$

$\therefore$ now using taylor's expansion

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

as x $\to 0$

$e^x \approx 1 + x$

Replacing x as $\frac{-1}{365}, \frac{-2}{365}, ... \frac{-(n-1)}{365}$ we get:

$$p(n) = 1 - 1 \cdot e^{\frac{-1}{365}} \cdot e^{\frac{-2}{365}} ...... \cdot e^{\frac{-(n-1)}{365}}$$

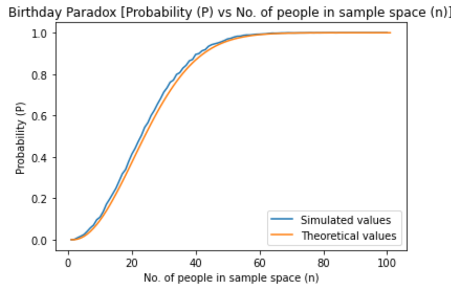
$$p(n) = 1 - e^{\frac{-n(n-1)}{720}}$$

3 Monte Carlo Simulation

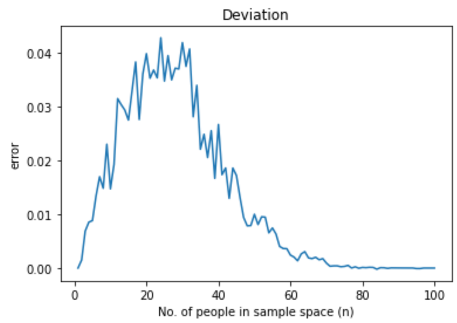
We use Monte Carlo Simulations to plot a graph between expected probability of a particular value of n, vs n. Using a python program, we collect random values between 1 and 365, n number of times and calculate the probability of duplicate birthdays (consider them as common birthdays). Taking multiple iterations of the corresponding probabilities we refine the plot and make it as accurate to the theoretical values as possible.

4 References

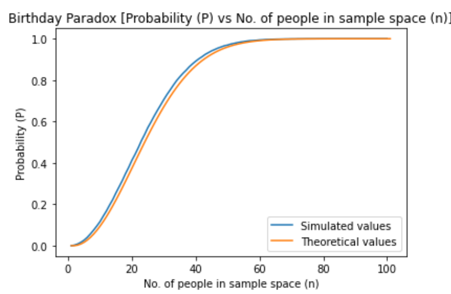
https://www.geeksforgeeks.org/birthday-paradox/
https://en.wikipedia.org/wiki/Birthday_problem
https://muthu.co/birthday-problem-and-monte-carlo-simulation/



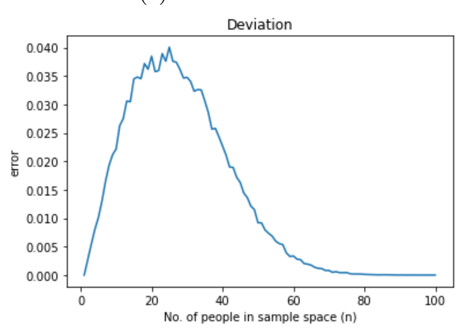
(a) 10 iterations



(b) Error Plots for 10 iterations



(c) 100 iterations



(d) Error Plots for 100 iterations